FNCE 921
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Assignment #2

Due Monday March 25^{th}, 2019

1. The environment is as follows. See Mehra and Prescott (1985) for many more details.

   - The economy consists of a representative agent with preferences:

   $$U(c) = E_0 \sum_{t=0}^{\infty} \beta^t u(c_t), \quad u(c) = \frac{c^{1-\alpha}}{1-\alpha}$$

   where β and α are the agent's discount factor and risk aversion parameter respectively.

   - There is no production. The aggregate endowment each period follows the stochastic process

   $$y_{t+1} = \lambda_{t+1} y_t$$

   where the growth rate λ takes on one of two values, λ_1, λ_2 with probabilities given by the first order Markovian transition matrix:

   $$\Pi = \begin{bmatrix} (1+\rho)/2 & (1-\rho)/2 \\ (1-\rho)/2 & (1+\rho)/2 \end{bmatrix}$$

   Furthermore let $\lambda_1 = \mu + \sigma$, and $\lambda_2 = \mu - \sigma$.

   - In equilibrium $c_t = y_t$
   - Choose numerical values for the model's parameters such that the equilibrium consumption growth process matches that of US *annual* per-capita consumption growth, net of durables.
     (a) First use the original Mehra and Prescott numbers (they used data from 1889-1978).
     (b) Downlowad Non-durable and service consumption data from WRDS. Generate real per-capita consumption growth for two samples (a) one that starts at 1929-2008 (b) post-war, starting at 1950 and ending at 2008.

     For each one of these samples find the corresponding μ, σ and ρ.
   - Also set $\alpha = 2$ but leave β as a variable for now.

   (a) *Markov Chains*

i. Compute the conditional moments of the Markov chain which describes the evolution of the λ process.

ii. Compute the stationary distribution Π^*, which satisfies $\Pi^* = \Pi \times \Pi^*$.

iii. Confirm that the unconditional mean, standard deviation, and first order autocorrelation coefficient for the λ process are μ, σ and ρ respectively.

(b) *Term Structure of interest rates*

In this economy, like other real economies n period bond is a sure claim to a single unit of risk free consumption n periods hence,

i. Use the agent's first order condition to compute the price b_i^1 and the return R_i^1, on a one-period bond in each state i. Choose β to produce a mean real interest rate of 5 percent (i.e., $R = 1.05$), and use this value in what follows,

ii. Consider the *risk neutral probability* defined by

$$p_{ij} = \frac{\pi_{ij}\beta\lambda_j^{-\alpha}}{b_i^1}$$

where $\pi_{ij} \equiv$ the i, j element of the transition matrix Π. Show that the p's are in fact legitimate probabilities. Use the agent's first order condition to show that an asset with dividends d_j in state j, one period hence has current value,

$$q_i = b_i^1 \sum_j p_{ij} d_j = b_i^1 E_p(d)$$

where the expectation on the far right hand side is taken with respect to the risk neutral probability measure.

iii. Use the agent's first-order condition or the risk neutral pricing formula to compute the prices and expected one period holding returns on a two period bond in each state i.

iv. *Implied Forward Rates* are a useful way to express the term-structure of interest rates. Define the implied two-period ahead forward rate in state i as

$$(1 + f_i) = b_i^1 / b_i^2$$

Interpret this expression. What are the risk premiums on the two period bond?. Can you relate your results to the classical *expectations theory of the term structure of interest rates?* This theory states that forward rates are unbiased predictors (i.e., conditional means) of future expected spot rates. Under what conditions might this theory work well in this model?

v. By trying out alternative parameter values or possibly even deriving a theoretical relation, describe the effect of α on the risk premium. Are negative risk premia possible? Is the choice of ρ required to generate positive risk premiums reasonable given what we know about the dynamics of the interest rates?

(c) *Equity Premium*

i. A share of equity in this economy entitles the holder to the aggregate dividend in each period. Compute the equilibrium price dividend ratio's in state i.

ii. Compute the equilibrium returns on equity for each state.

iii. Using your results from the bond pricing section, compute the equilibrium excess returns on equity in each state.

iv. Finally, by using the stationary distribution for the Markov chain, compute the unconditional mean, standard deviation, and coefficient of first order autocorrelation for the excess returns on equity and the one period risk free interest rate.

v. The unconditional mean of the excess return on the US stock market over the last 100 years is about 6% per annum. The unconditional mean of the risk free rate is close to zero (recall these are real interest rates). Do your results confirm the *equity premium puzzle* and the *risk free rate puzzle*? In what sense are your findings 'puzzling'?

vi. Derive the standard deviation of the risk free rate, the market rate, and the price-consumption ratio. How do these quantities fare relative to their data counterparts?

2. GMM. Download annual data for the years 1929-2015 from the BEA on consumption of nondurables and services, and from CRSP download the Value Weighted CRSP return with and without dividends as well as the return on the 1-year Treasury. Construct the real growth rate of consumption and dividends using the CPI as well as convert the returns to real ones. In addition construct the log price-dividend ratio for the VWRET.

(a) Report the mean, standard deviation and first autocorrelation of the consumption growth, dividend growth, log price dividend ratio, the VWRET and Risk free rate. For the mean and standard deviation report also the Robust HAC standard errors.

(b) Hansen and Singleton (1982). Assume there is a representative agent with time separable CRRA felicity function. That is the felicity function is $U(C_t) = \frac{{C_t}^{1-\gamma}}{1-\gamma}$ and the life time utility is $\sum_{t=0}^{\infty} \beta^t U(C_t)$. Estimate the parameters β and γ for the following specifications:

i. Use the VWRET with the instruments = [1, lagged consumption growth, and lagged VWRET].

ii. Use the VWRET and the risk free rate with the instrument = [1, lagged consumption growth, lagged VWRET, and lagged risk free rate].

iii. Same as above, but adding the lagged price-dividend ratio to the list of instruments.

iv. Report the point estimate as well as their standard errors

v. Report the overall J statistic and the degrees of freedom and corresponding $P-value$. Is the model rejected?

(c) Adapt your code to solving and simulating the Internal Habit Model. Simulate and solve the economy for the two configurations: (1) $\gamma = 5$ $\beta = .97$, $\theta = -0.5$, and (2) same as (1) but with $\theta = -0.7$.

(d) Find the equity premium on the aggregate dividend claim for these two parameter configurations.

(e) Simulate the economy with a time series of length (i) 100 observation (ii) 5000 observation. If you use GMM to estimate the parameters do your recover the true ones? Is the model rejected according to the J stat?

3. Long-Run Risk Consider the specification of the long-run risks model. Consumption and dividend dynamics are given by

$$\begin{aligned}
\Delta c_{t+1} &= \mu_g + x_t + \sigma_t \eta_{t+1}, \\
x_{t+1} &= \rho x_t + \varphi_e \sigma_t e_{t+1}, \\
\sigma^2_{t+1} &= \sigma_0^2 + \nu(\sigma_t^2 - \sigma_0^2) + \sigma_w w_{t+1}, \\
\Delta d_{t+1} &= \mu_d + \phi x_t + \pi_d \sigma_t \eta_{t+1} + \varphi_d \sigma_t u_{d,t+1}.
\end{aligned}$$

where all the shocks are i.i.d. uncorrelated standard Normal.

The investor has recursive Epstein-Zin preferences over the future consumption,

$$U_t = \left[(1-\delta) C_t^{1-\frac{1}{\psi}} + \delta \left(E_t U_{t+1}^{1-\gamma} \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right]^{\frac{1}{1-\frac{1}{\psi}}}.$$

Theoretical Model Solution:

(a) Conjecture that price-consumption ratio is linear in expected growth,

$$pc_t = A_0 + A_x x_t + A_\sigma \sigma_t^2,$$

and use Euler equation on consumption asset and log-linearization of consumption return to solve for A_0, A_1, A_2 and log-linearization coefficient κ_1 in terms of the fundamental model parameters. Under what conditions asset valuations respond positively to expected growth? negatively to consumption volatility? What do those conditions mean, economically?

(b) Express the stochastic discount factor in terms of the primitive state variables and parameters of the model:

$$m_{t+1} = m_0 + m_x x_t + m_\sigma \sigma_t^2 - \lambda_c \eta_{t+1} - \lambda_x \varphi_e \sigma_t e_{t+1} - \lambda_w \sigma_w w_{t+1}.$$

What are the signs of market prices of risks? Under what conditions the market prices of expected growth and volatility risk are equal to zero?

(c) Conjecture that the log price of an $n-$period zero-coupon risk-free bond satisfies

$$p_{n,t} = -B_{0,n} - B_{x,n}x_t - B_{\sigma,n}\sigma_t^2,$$

so that (monthly) yields are given by $y_{n,t} = -p_{n,t}/n$. Set up equations for $B_{x,n}$ and $B_{\sigma,n}$. How do risk-free rates respond to expected growth and volatility shocks?

(d) Define $rx_{n,t+1}$ a one-period log excess return on an $n-$period bond. That is, it is the excess return on buying an $n-$period bond now and selling it tomorrow as an $(n-1)$ period bond:

$$rx_{n,t+1} = -p_{n,t} + p_{n-1,t+1} - y_{t,1}.$$

Show that the risk-premia on bonds is time-varying and driven by stochastic volatility:

$$E_t rx_{n,t+1} + \frac{1}{2}Var_t rx_{n,t+1} = r_{n,0} + r_{n,1}\sigma_t^2.$$

What is the sign of $r_{n,1}$? Show that the real bond risk premium *decreases* in high uncertainty times. Why does it happen in the model, from an economic point of view?

(e) Conjecture that price-dividend ratio is linear in expected growth,

$$pd_t = H_0 + H_x x_t + H_\sigma \sigma_t^2,$$

and use Euler equation on a dividend-paying asset and log-linearization of the return to solve for H_0, H_x, H_σ and log-linearization coefficient $\kappa_{1,d}$ in terms of the fundamental model parameters.

(f) Show that the risk-premia on the stock market is time-varying and driven by stochastic volatility:

$$E_t r_{d,t+1} + \frac{1}{2}Var_t r_{d,t+1} = r_{d,0} + r_{d,1}\sigma_t^2.$$

What is the sign of $r_{d,1}$? Under what conditions market premia goes up in high volatility times?

Calibration and Asset-Pricing Implications:

i. What is the average conditional volatility of the stochastic discount factor?

Decompose the volatility into components related to short-run, long-run and volatility news.

ii. Extend the code to solve for the term-structure of real rates, and the price-dividend ratio on the dividend-paying asset.

iii. What are the model implications for the levels of the risk-free rates from 1 month to 5 years in maturity?

iv. What are the average log price-consumption and log price-dividend ratios? How does the market pd ratio compare with the data?

v. What is the average equity premium on consumption asset? on dividend-paying asset? Decompose the premia into components related to short-run, long-run and volatility risks.

vi. What is the volatility of equity returns and risk-free rates in the model? How do they compare to the data?

vii. Comment on the ability of the model to solve equity premium, risk-free rate, and return volatility puzzles.

4. Campbell-Cochrane

Consider an environment very similar to the external habits Campbell-Cochrane, 1999.

- Investors have utility over consumption relative to a reference point X_t :

$$u_t = \frac{(C_t - X_t)^{1-\gamma} - 1}{1-\gamma}$$

- Consumption growth is i.i.d Normal:

$$g_{t+1} = g + v_{t+1}, \quad Var(v_{t+1}) = \sigma_v^2$$

- Define surplus consumption S_t :

$$S_t = \frac{C_t - X_t}{C_t}$$

- Log surplus consumption is driven by consumption news:

$$s_{t+1} = \bar{s} + \phi(s_t - \bar{s}) + \lambda(s_t)v_{t+1},$$

where the sensitivity function is specified as in CC, 99:

$$\lambda(s_t) = \frac{1}{\bar{S}}\sqrt{1 - 2(s - \bar{s})} - 1, \quad \text{when } s_t < s_{max} \text{ and 0 otherwise.}$$

- The only difference between CC, 99 is the specification of $\bar{s} = \log \bar{S}$:

$$\bar{S} = \sigma_v \sqrt{\frac{\gamma}{1 - \phi - b/\gamma}},$$

where b is the preference parameter.

(a) Show that the one-period real risk-free rates are now time-varying, and are linear in the consumption surplus s_t.

(b) Consider the case of $b > 0$ — this is the case studied in Wachter, 2006. How do the interest rates vary with the consumption surplus ratio? Are they low or high in "good times"?

(c) Compare this specification to the one used in Adrien Verdelhan "A Habit-Based Explanation of the Exchange Rate Risk Premium," Journal of Finance, February 2010, Vol. 65, No 1, pp 123-145. Compare the implications relative to part b.

(d) Compare the behavior of interest rates in version (b) and (c) with the predictions in Long-Run Risk model. In long-run risks model, do real rates fall or rise in "good" times (think about "good" times as times of high expected growth and/or low conditional volatility).

(e) Can you use this model predictions to test the two asset-pricing theories? How would you distinguish between them empirically?

(f) Compare stock return predictability in both models with the empirical evidence in HW1.

By the way, if you are interested on how the Markov process was derived here are some background details.

It is often useful to estimate continuous variables and then convert them into a discrete Markov process. To do that Tauchen and Hussey, 1991, Quadrature-Based Methods for Obtaining Approximate Solutions to Nonlinear Asset Pricing Models, *Econometrica*, 59, 371-396) provide a very useful tool which convert a system of VAR into discrete Markov process. To utilize the code you input the parameters of the VAR process. The code supplies the states and the corresponding transition matrix. The number of states is user-supplied.

I estimated the following bivariate VAR for Dividend $\zeta_{t+1} = \frac{D_{t+1}}{D_t}$ and Consumption growth $\lambda_{t+1} = \frac{c_{t+1}}{c_t}$ as follows:

$$\begin{bmatrix} \ln \zeta_{t+1} \\ \ln \lambda_{t+1} \end{bmatrix} = \begin{bmatrix} .0012 \\ .0019 \end{bmatrix} + \begin{bmatrix} -.1768 & .1941 \\ .0267 & -.2150 \end{bmatrix} \begin{bmatrix} \ln \zeta_t \\ \ln \lambda_t \end{bmatrix} + \epsilon_{t+1} \tag{0.1}$$

where

$$E(\epsilon_{t+1}\epsilon'_{t+1}) = \begin{bmatrix} .1438 & .0001 \\ .0001 & .0145 \end{bmatrix} \times 10^{-3} \tag{0.2}$$

The system used 16 state Markov process, so the state variable s_t, representing the state $\{\zeta, \lambda\}$ is 16x2 vector with a transition matrix Π which is 16×16.