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Mehra and Prescott

1 The economy consists of a representative agent with preferences:

$$U(c) = E_0 \sum_{t=0}^{\infty} \beta^t u(c_t), \quad u(c) = \frac{c^{1-\alpha}}{1-\alpha}$$

There is no production. Aggregate endowment each period follows the stochastic process

$$y_{t+1} = \lambda_{t+1} y_t$$

where the growth rate λ takes on one of two value, λ_1 or λ_2, with probabilities given by the first order Markovian transition matrix:

$$\Pi = \begin{bmatrix} (1+\rho)/2 & (1-\rho)/2 \\ (1-\rho)/2 & (1+\rho)/2 \end{bmatrix} = \begin{bmatrix} \phi & 1-\phi \\ 1-\phi & \phi \end{bmatrix}$$

Let $\lambda_1 = \mu + \sigma$, and $\lambda_2 = \mu - \sigma$. In equilibrium $y_t = c_t$.

Download Non-durable and service consumption data from WRDS. Generate real-per capita consumption growth for two sample (a) one that starts at 1929, (b) one that starts at 1950.

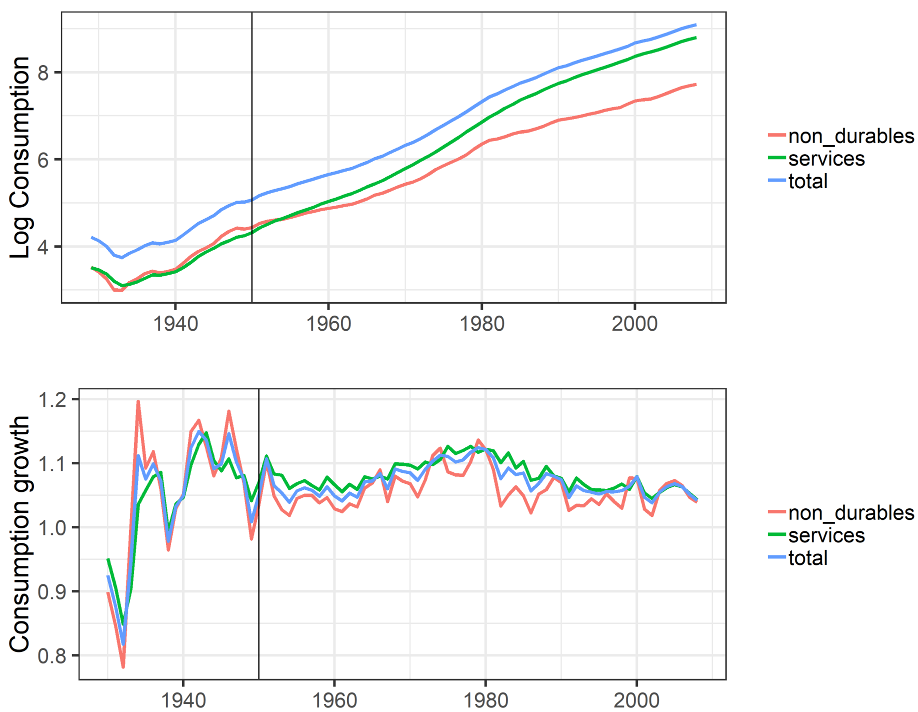

Figure 1: Log Consumption and Consumption Growth (Gross). Vertical line represents starting point of sample (b)

Mehra and Prescott find 0.018, 0.036, and -0.14, are the mean, standard deviation and autocorrelation of continuous mean consumption growth in their sample, respectively. We can then calculate the parameters of the Markov process as follows:

- **Persistence:** To compute the persistence of the process, consider the indicator function, denoted as ξ_t, which is equal to 1 in the high growth state and zero otherwise. Given our transition matrix Π, we then have

$$\begin{bmatrix} \xi_t \\ 1-\xi_t \end{bmatrix} = \begin{bmatrix} \phi & 1-\phi \\ 1-\phi & \phi \end{bmatrix} \begin{bmatrix} \xi_{t-1} \\ 1-\xi_{t-1} \end{bmatrix} + \begin{bmatrix} v_{1,t} \\ v_{2,t} \end{bmatrix}$$

which implies $\xi_{t+1} = (1-\phi) + \xi_t(\phi - 1 + \phi) + v_{1,t+1}$. Taking the unconditional expectation implies the persistence in the process is given by $2\phi - 1 = 2(\frac{1}{2}(1+\rho)) - 1 = \rho$. Thus, $2\phi - 1 = -0.014$, and $\phi = 0.43$. Further note that $E\left[\xi_{t+1}\right] = \frac{1-\phi}{2(1-\phi)}$. Thus, the unconditional probability $\pi_1 = \pi_2 = 0.5$.

- **Mean:** Given that the mean gross growth rates in the high and low growth regimes are given by $\mu + \sigma$ and $\mu - \sigma$, we know that the unconditional mean is given by $\mu = 1.018$.
- **Standard Deviation:** Once again using the unconditional probability derived above, we have that the unconditional variance of the process is given by $\sigma^2 = 0.5\sigma^2 + 0.5\sigma^2$, which implies $\sigma = 0.036$.

We can follow the same the process to calculate the parameters for sample (a) and sample (b).

a) Markov Chains

I. Compute the conditional moments of the Markov chain which describes the evolution of the λ process.

Letting $s_t = h/l$ denote whether we are currently in the high or low state, the conditional means are given by

$$\begin{aligned} E\left[\lambda_{t+1}|s_t = h\right] &= \pi_{hh}\lambda_h + \pi_{hl}\lambda_l \\ &= \phi(\mu+\sigma) + (1-\phi)(\mu-\sigma) \\ &= \mu + \sigma(2\phi - 1) \\ E\left[\lambda_{t+1}|s_t = l\right] &= \pi_{lh}\lambda_h + \pi_{ll}\lambda_l \\ &= (1-\phi)(\mu+\sigma) + \phi(\mu-\sigma) \\ &= \mu + \sigma(1 - 2\phi) \end{aligned}$$

The conditional variance is generally given by

$$\begin{aligned} Var(\lambda_{t+1}|s_t = i) &= E\left[\lambda_{t+1}^2|s_t = i\right] - E\left[\lambda_{t+1}|s_t = i\right]^2 \\ &= \sum_j \pi_{ij}\lambda_j^2 - \left(\sum_j \pi_{ij}\lambda_j\right)^2 \end{aligned}$$

Thus, we have

$$\begin{aligned} Var(\lambda_{t+1}|s_t = h) &= \pi_{hh}\lambda_h^2 + \pi_{hl}\lambda_l^2 - (\pi_{hh}\lambda_h + \pi_{hl}\lambda_l)^2 \\ &= \phi(\mu+\sigma)^2 + (1-\phi)(\mu-\sigma)^2 + (\phi(\mu+\sigma) - (1-\phi)(\mu-\sigma))^2 \\ &= 4\sigma^2\phi(1-\phi) \\ Var(\lambda_{t+1}|s_t = l) &= \pi_{lh}\lambda_h^2 + \pi_{ll}\lambda_l^2 - (\pi_{lh}\lambda_h + \pi_{ll}\lambda_l)^2 \\ &= 4\sigma^2\phi(1-\phi) \end{aligned}$$

Hence, we see that the conditional variances are equal to each other.

II. Compute the stationary distribution Π^*, which satisfies $\Pi^* = \Pi \times \Pi^*$.

We know that $\Pi^*\iota = 1$, where ι denotes the vector of ones. Using these two restrictions, we can solve for Π^*. Define the matrix A as

$$A = \begin{bmatrix} I_2 - \Pi \\ \iota' \end{bmatrix}$$

Then we have $A\Pi^* = [0 \quad 0 \quad 1]'$. The solution for Π^* is

$$\begin{aligned} A\Pi^* &= [0 \quad 0 \quad 1]' \\ A'A\Pi^* &= A'\,[0 \quad 0 \quad 1]' \\ \Pi^* &= (A'A)^{-1}A'\,[0 \quad 0 \quad 1]' \end{aligned}$$

Plugging in our parameters for ϕ, we get

$$\Pi^* = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

III. Confirm that the unconditional mean, standard deviation, and first order autocorrelation coeffcient for the λ process are μ, σ, and ρ respectively.

We can calculate the unconditional mean and standard deviation using Π^*

$$\begin{aligned} E\left[\lambda\right] &= \sum_i \pi_i^* \lambda_i \\ &= 0.5(\mu+\sigma) + 0.5(\mu-\sigma) \\ &= \mu \\ Var(\lambda) &= \sum_i \pi_i^* \lambda_i^2 - \left(\sum_i \pi_i^* \lambda_i\right)^2 \\ &= 0.5(\mu+\sigma)^2 + 0.5(\mu-\sigma)^2 - (0.5(\mu+\sigma)+0.5(\mu-\sigma))^2 \\ &= 0.5((\mu+\sigma)^2 + (\mu-\sigma)^2) - \mu^2 \\ &= 0.5(2(\mu^2+\sigma^2)) - \mu^2 \\ &= \sigma^2 \end{aligned}$$

Thus, we have verified that the unconditional mean and standard deviation are μ and σ, respectively.

b) The Term Structure of Interest Rates: In this economy, like other real economies, an n period bond is a sure claim to a single unit of risk free consumption n periods hence.

I. Use the agent's first order condition to compute the price b_i^1 and the return R_i^1, on a one-period bond in each state i. Choose β to produce a mean real interest rate of 5 percent (i.e. $R = 1.05$).

Let i denote a state in Ω (the set of all states) and $q_t(i)$ denote the time t price of an Arrow-Debreu security that pays off in state $i \in \Omega$. Then b_i^1 at time t is given by

$$b_i^1 = \sum_{i\in\Omega} q_t(i) = \sum_{i\in\Omega} \pi_t(i)\beta\frac{u'(c_{t+1})}{u'(c_t)} = \beta E_t\left[\frac{u'(c_{t+1})}{u'(c_t)}\right] = \beta E\left[(c_{t+1}/c_t)^{-\alpha}\right] = \beta E\left[\lambda_{t+1}^{-\alpha}\right]$$

where λ_{t+1} is the growth rate in consumption from time t to $t+1$. We can use our Markov chain to compute this expectation, so our general expression for the one period bond price in state i is

$$b_i^1 = \beta \sum_{j=1}^{N} \pi_{ij}\lambda_j^{-\alpha} \tag{1}$$

Thus, when $i = 1$ (the high state), we have the following bond price

$$\begin{aligned} b_1^1 &= \beta \sum_{j=1}^{N} \pi_{1j}\lambda_j^{-\alpha} \\ &= \beta\left(\pi_{11}\lambda_1^{-\alpha} + \pi_{12}\lambda_2^{-\alpha}\right) \\ &= \beta\left(\phi\lambda_1^{-\alpha} + (1-\phi)\lambda_2^{-\alpha}\right) \\ &= \beta\left(\frac{1+\rho}{2}(\mu+\sigma)^{-\alpha} + \frac{1-\rho}{2}(\mu-\sigma)^{-\alpha}\right) \end{aligned}$$

Similarly, when $i = 2$ (the low state), we have the following bond price

$$\begin{aligned} b_2^1 &= \beta \sum_{j=1}^{N} \pi_{2j} \lambda_j^{-\alpha} \\ &= \beta \left(\pi_{21} \lambda_1^{-\alpha} + \pi_{22} \lambda_2^{-\alpha} \right) \\ &= \beta \left(\frac{1-\rho}{2} (\mu + \sigma)^{-\alpha} + \frac{1+\rho}{2} (\mu - \sigma)^{-\alpha} \right) \end{aligned}$$

For a one period bond we define the return as $R_i^1 = 1/b_i^1$

II. Consider the *risk neutral probability* defined by

$$p_{ij} = \frac{\pi_{ij} \beta \lambda_j^{-\alpha}}{b_i^1}$$

where $\pi_{ij} \equiv$the i, j element of Π. Show that the p's are legitimate probabilities.

Substituting in Equation 1, the risk neutral probabilities are given by

$$\begin{aligned} p_{ij} &= \frac{\pi_{ij} \beta \lambda_j^{-\alpha}}{\beta \sum_{j=1}^{N} \pi_{ij} \lambda_j^{-\alpha}} \\ &= \frac{\pi_{ij} \lambda_j^{-\alpha}}{\sum_{j=1}^{N} \pi_{ij} \lambda_j^{-\alpha}} \end{aligned}$$

Since p_{ij} represents the probability of going from state i to state j, and we must end up in a state in the next period, $\sum_j p_{ij} = 1$ for these to be legitimate probabilities.

$$\sum_j p_{ij} = \sum_j \frac{\pi_{ij} \lambda_j^{-\alpha}}{\sum_j \pi_{ij} \lambda_j^{-\alpha}} = \frac{1}{\sum_j \pi_{ij} \lambda_j^{-\alpha}} \sum_j \pi_{ij} \lambda_j^{-\alpha} = 1$$

where the second equality holds, because once we have summed over all the j's in the denominator, the term is only dependent on i. Noting that p_{ij} is clearly less than one and positive (the numerator is the price of a state contingent claim, so it is positive by no arbitrage), so the risk neutral probabilities are in fact legitimate.

Shoe that an asset with dividends d_j in state j, one period hence has current value given by

$$q_i = b_i^1 E_p \left[d \right]$$

where the expectation is taken with respect to the risk neutral probability measure.