# FNCE 924 — Problem Set 2

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## Problem 1: Neoclassical Growth Model

### 1. What is the long run growth rate of GDP or income?

Since in steady state (= long run) we have $k^\star = k_{t+1} = k_t \forall t$ and $N^\star = N_{t+1} = N_t \forall t$ we have

$$\frac{A_{t+1}P_{t+1}}{A_t P_t} = \gamma_A \gamma_P \equiv \gamma$$

as the long run growth rate.

### 2. Write down the first order conditions for the problem

We first de-trend the lagrangian by dividing the budget constraint as well as the level-consumption in the utility by $A_t P_t$ using our usual standardization that $P_0 = 1$ and $A_0 = 1$ such that $P_t = \gamma_P^t$ and $A_t = \gamma_A^t$. We denote de-trended variables by their lower-case letters:

$$\begin{aligned}
\mathcal{L} &= \sum_{t=0}^{\infty} \beta^t P_t \frac{\left[(A_t P_t)^\theta \left(\frac{C_t}{A_t P_t}\right)^\theta (1-N_t)^{1-\theta}\right]^{1-\sigma} - 1}{1-\sigma} + \lambda_t \left[k_t^{1-\alpha}(ZN_t)^\alpha - c_t - qk_{t+1}\gamma + q(1-\delta)k_t - g_t\right] \\
&= \sum_{t=0}^{\infty} [\beta^\star]^t \frac{\left[c_t^\theta (1-N_t)^{1-\theta}\right]^{1-\sigma} - (A_t P_t)^{\theta(\sigma-1)}}{1-\sigma} + \lambda_t \left[k_t^{1-\alpha}(ZN_t)^\alpha - c_t - qk_{t+1}\gamma + q(1-\delta)k_t - g_t\right] \\
&\text{where } \beta^\star = \beta \gamma_P (\gamma_A \gamma_P)^{\theta(1-\sigma)}
\end{aligned}$$

$\Rightarrow$ FOCs:

$$\begin{aligned}
&[c_t] \quad \lambda_t = [\beta^\star]^t \theta c_t^{\theta-1} (1-N_t)^{1-\theta} [c_t^\theta (1-N_t)^{1-\theta}]^{-\sigma} \\
&[N_t] \quad \alpha \left(\frac{k_t}{ZN_t}\right)^{1-\alpha} \lambda_t = [\beta^\star]^t (1-\theta) c_t^\theta (1-N_t)^{-\theta} [c_t^\theta (1-N_t)^{1-\theta}]^{-\sigma} \\
&[k_{t+1}] \quad 1 = \frac{\lambda_{t+1}}{\lambda_t} \frac{1}{q\gamma} [\underbrace{(1-\alpha)k_{t+1}^{-\alpha}(ZN_{t+1})^\alpha}_{\equiv \text{ net return to capital } r_k} + q(1-\delta)] \\
&[\lambda_t] \quad k_t^{1-\alpha}(ZN_t)^\alpha = c_t + qk_{t+1}\gamma - q(1-\delta)k_t + g_t
\end{aligned}$$

## 3. Calibrate the model

$$\gamma_P = 1.01 \tag{1}$$
$$\gamma = 1.03/\gamma_P = 1.02 \tag{2}$$
$$\delta = 0.08 \tag{3}$$
$$r_k = 0.05 \tag{4}$$
$$\alpha = 2/3 \tag{5}$$
$$qi^\star = g^\star = 1/6 y_t \tag{6}$$
$$N^\star = 0.35 \tag{7}$$
$$\sigma = 4 \tag{8}$$

## 4. Compute the steady-state level of consumption and capital as a function of your parameter choices and Z

Solving sequentially for ratios which are constant functions of the model parameters in steady state:

$$\begin{aligned}
via\ [k_{t+1}]: \quad & q = \frac{r_k}{\gamma - 1 + \delta} = 0.5000 \\
via\ [k_{t+1}]: \quad & \frac{k^\star}{N^\star} = \left[\frac{(1-\alpha)}{q(\gamma - 1 + \delta)}\right]^{1/\alpha} Z = 0.0172 * Z \\
& \Rightarrow k^\star = N^\star * \frac{k^\star}{N^\star} = 0.0060 * Z \\
via\ [\lambda_t]\ and\ Eq.\ (6): \quad & \frac{c^\star}{k^\star} = \left(\frac{k^\star}{N^\star}\right)^{-\alpha} Z^\alpha - q\frac{i^\star}{k^\star} - \frac{g^\star}{k^\star} \\
& = \left(\frac{k^\star}{N^\star}\right)^{-\alpha} Z^\alpha - 2q(\gamma - 1 + \delta) = 14.9000 \\
& \Rightarrow c^\star = \frac{c^\star}{k^\star} * k^\star = 0.0898 * Z
\end{aligned}$$

## 5. Suppose the productivity parameter Z increases by 10%. What is the percent increase in consumption and capital?

As we see above both steady state policies consumption and capital are linear functions in Z and thus a 10% increase in the latter entails a 10% increase in the policies.

## 6. Suppose country X has the same economic parameters as the US, expect for productivity Z and the cost of investment goods q and the share of government spending in GDP which is 25%

### 6.i GDP per capita in X 25% of US and Z is 50% lower

With $N_{US} = N_X$ we get:

$$\frac{GDPPC_{US}}{GDPPC_X} = \frac{\gamma_A^t y_{US}}{\gamma_A^t y_X} = \frac{y_{US}}{y_X} = 4$$
$$4 = \left(\frac{k^\star_{US}}{k^\star_X}\right)^{1-\alpha} \left(\frac{Z_{US}}{0.5Z_{US}}\right)^\alpha \left(\frac{N_{US}}{N_X}\right)^\alpha$$
$$= \left[\frac{N^\star_{US} Z_{US}\left(\frac{1-\alpha}{q_{US}(\gamma-1+\delta)}\right)}{N^\star_X 0.5 Z_{US}\left(\frac{1-\alpha}{q_X(\gamma-1+\delta)}\right)}\right]^{1-\alpha} 2^\alpha \left(\frac{N_{US}}{N_X}\right)^\alpha$$
$$\Rightarrow 2 = \frac{q_X}{q_{US}}$$

### 6.ii Local government of X lowers q by 10%

$$\frac{y_{US}}{y_X} = \left[\frac{N^\star_{US} Z_{US}\left(\frac{1-\alpha}{q_{US}(\gamma-1+\delta)}\right)}{N^\star_X 0.5 Z_{US}\left(\frac{1-\alpha}{0.9*2q_{US}(\gamma-1+\delta)}\right)}\right]^{1-\alpha} 2^\alpha \left(\frac{N_{US}}{N_X}\right)^\alpha$$
$$= 2 * \frac{1.8q_{US}}{q_{US}} = 3.6$$

So now the GDPPC of X is 27.78% of the US, a 2.78p.p. increase.

# Problem 2: Long Run Growth and the Rate of Return on Capital

## 1. What is the growth rate of wage income and thus the stock of human capital in this economy?

Consider the firm's problem

$$\mathcal{L} = \sum_{s=0}^{\infty} \beta^s \frac{\lambda_{t+s}}{\lambda_t} \left( K_{t+s}^{1-\alpha} (ZA_{t+s}P_{t+s}N_{t+s})^{\alpha} - N_{t+s}W_{t+s} - I_{t+s} - q_{t+s}\left[K_{t+s+1} - (1-\delta)K_{t+s} - I_{t+s}\right]\right) \tag{9}$$

The first order condition with respect to $N_{t+s}$ implies

$$\frac{\lambda_{t+s}}{\lambda_t}\alpha \left[\frac{K_{t+s}}{ZA_{t+s}P_{t+s}N_{t+s}}\right]^{1-\alpha} (ZA_{t+s}P_{t+s}) = W_{t+s}$$

shifting back $s$ periods gives

$$\begin{aligned} \alpha \left[\frac{K_t}{ZA_tP_tN_t}\right]^{1-\alpha} (ZA_tP_t) &= W_t \\ \alpha \left[\frac{k}{ZN_t}\right]^{1-\alpha} (ZA_tP_t) &= W_t \end{aligned}$$

since we know that the detrended capital to labor ratio is constant, we see that $\gamma_W = \gamma_A\gamma_P \equiv \gamma$.

## 2. What is the growth rate of capital?

To calculate the growth rate of capital, we consider the first order condition of Equation 9 with respect to capital ($K_{t+s+1}$)

$$\frac{\lambda_{t+s}}{\lambda_t} q_{t+s} = \frac{\lambda_{t+s+1}}{\lambda_{t+s}}\left((1-\alpha)\left[\frac{ZA_{t+s+1}P_{t+s+1}N_{t+s+1}}{K_{t+s+1}}\right]^{\alpha} + (1-\delta)q_{t+s+1}\right)$$

Noting that the first order condition with respect to investment implies $q_t = 1\ \forall t$, we can use a similar approach to solving for next periods wage. We see that next period's capital is given by

$$K_{t+1} = (\delta\frac{\lambda_t}{\lambda_{t+1}})^{-\frac{1}{\alpha}}(\frac{1}{1-\alpha})^{-\frac{1}{\alpha}} ZA_{t+1}P_{t+1}N_{t+1}$$

$A_{t+1}$ and $P_{t+1}$ are the only variables in this expression that are growing, so we see that $\gamma_K = \gamma_A\gamma_P$. The $\lambda$ term is constant because it is the ratio of growing variables that are also constant in steady state.

## 3. Derive an expression for the equilibrium growth rate, $r$, in this economy? What is the sign of $r - g$? Can it be negative?

Start with the detrended production function $y_t = k_t^{1-\alpha}(ZN_t)^{\alpha}$ and take the derivative with respect to $k_t$ to get

$$r_t = (1-\alpha)k_t^{-\alpha}(ZN_t)^{\alpha}$$

plugging in the steady state of capital from question 1 part 4, we get

$$\begin{aligned} r^{\star} &= (1-\alpha)\left[NZ\left(\frac{1-\alpha}{q(\gamma-1+\delta)}\right)^{\frac{1}{\alpha}}\right]^{-\alpha}(ZN)^{\alpha} \\ &= q(\gamma - 1 + \delta) \end{aligned}$$

which implies

$$r - g = q(\gamma - 1 + \delta) - \gamma + 1 \tag{10}$$

Setting the expression above equal to zero, we can solve for a condition for when it is negative versus positive

$$q^* = \frac{\gamma - 1}{\gamma - 1 + \delta} = \frac{g}{g + \delta}$$

Thus, when $q > q^*$ the sign of $r - g$ is positive and when $q < q^*$ the sign of $r - g$ is negative.

Considering the optimal condition with respect to consumption from problem 1

$$c_t = k_t^{1-\alpha}(ZN_t)^\alpha - qk_{t+1}\gamma + q(1-\delta)k_t - g_t$$

and taking the derivative with respect to capital for maximum steady state consumption

$$\frac{dc^*}{dk^*} = (1-\alpha)k^{-\alpha}(ZN_t)^\alpha - q_c\gamma + q_c(1-\delta) = 0 \quad \text{By the FOC}$$

which implies

$$(1-\alpha)k^{-\alpha}(ZN_t)^\alpha = r = q_c(\gamma - 1 + \delta)$$

Thus, the optimal price of capital for the optimal consumption is given by

$$q_c^* = \frac{r_c}{\gamma - 1 + \delta}$$

Observing that $\frac{d^2c}{dk^2} = -\alpha(1-\alpha)k^{-\alpha-1}(ZN_t)^\alpha < 0$, we know $q < q_c^*$ is inefficient. Hence, $q$ will always be sufficiently high for $r - g$ to remain non negative.

### 4. Suppose population growth declines. What is the impact on the steady-state value of $r - g$?

Substituting in $\gamma = \gamma_A\gamma_P$ into Equation 10 we get

$$(r - g) = q(\gamma - 1 + \delta) - \gamma + 1 = q(\gamma_A\gamma_P - 1 + \delta) - \gamma_A\gamma_P + 1$$

which implies

$$\frac{\partial(r-g)}{\partial\gamma_P} = q\gamma_A - \gamma_A$$

so if $q > 1$ then a decrease in $\gamma_A$ decreases $r - g$ and if $q < 1$ then a decrease in $\gamma_A$ increases $r - g$. Since we have previously established that in an efficient (steady state consumption) economy we have $\gamma - 1 < r_c$ we see that assuming $\delta > 0$ we will always have the numerator of $q$ bigger than the denominator and thus we are in the latter case.

### 5. Suppose average growth rates of population and productivity each decline by 0.5% per year from the previous steady state. What is the quantitative impact of these declines on the steady-state level of the real interest rates?

Considering our expression for $r$, we can measure the decline of 0.5% as

$$r' = q(0.995^2\gamma - 1 + \delta)$$

Since we have restricted $q \leq 1$, $r'$ decreases by more than just the $0.995^2$ factor.

## Problem 3: Solow Growth Model

Consider the following deterministic version of the Solow model:

$$c_t + k_{t+1} - (1-\delta)k_t \leq Ak_t^\alpha$$

where $k_0 > 0$, consumption is 75% of national income, $\delta = 0.1$, and $\alpha = 1/3$.

## 1. Compute the steady state level of capital as a function of $A$. For what follows pick $A$ to normalize the steady-state of capital to 1.

Using the fact that $c_t = \frac{3}{4}Ak_t^{\alpha}$, imposing equilibrium and noting that $k_{t+1} = k_t$ at steady state, we get the following equation for capital accumulation

$$k - (1-\delta)k = \frac{1}{4}Ak^{\alpha}$$

solving this equation for $k$, we see that

$$k^{\star} = \left[\frac{A}{4\delta}\right]^{\frac{1}{1-\alpha}}$$

where $\star$ denotes the steady state. To calibrate $A$ such that steady state is normalized to 1, we solve the following expression for $A$

$$1 = \left[\frac{A}{4\delta}\right]^{\frac{1}{1-\alpha}} \tag{11}$$

which implies $A = \frac{2}{5}$.

## 2. Start the economy with an initial capital stock of 0.5. Compute and plot the equilibrium level of capital over time. How long does it take for the capital stock to reach 0.9?

We use the following capital accumulation equation to simulate the growth path of capital

$$k_{t+1} - (1-\delta)k_t = \frac{1}{4}Ak_t^{\alpha} \tag{12}$$

The results are shown in Figure 1. We see that it takes 25 periods for capital to exceed 0.9.

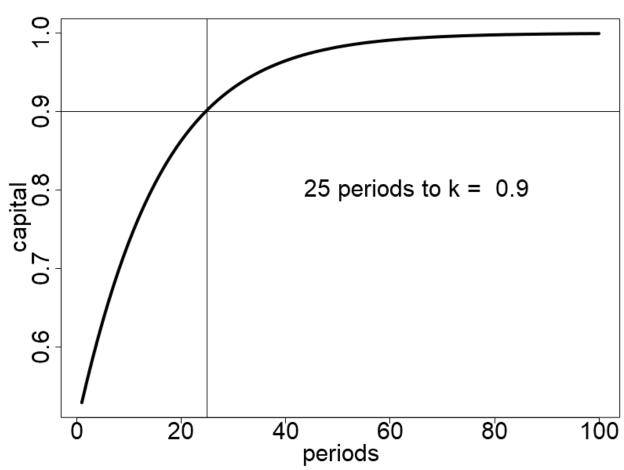


Figure 1: Capital accumulation with $\alpha = 1/3$

## 3/4. Now suppose $\alpha = 2/3$. Recompute the steady-state and again normalize $A$ to ensure that $k = 1$ at the steady-state

**Solution:** Inspecting Equation 11, we see that changing $\alpha$ does not change the level of $A$ required to normalize the steady-state of capital to 1. We once again use Equation 12 to specify the growth of capital in our economy. The results are shown in Figure 2. Notably, we see that it takes about twice as long for this economy to reach $k = 0.9$ than the economy with $\alpha = 1/3$. Intuitively, this is because a higher level of $\alpha$ indicates that the production technology is relatively more capital intensive, so it takes a longer amount of time to go from $k = 0.5$ to $k = 0.9$.

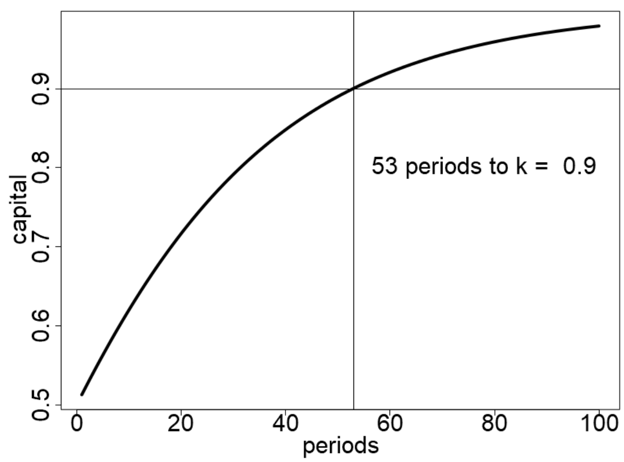


Figure 2: Capital accumulation with $\alpha = 1/3$