

FNCE Problem Set 1

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February 5, 2019

Problem 1

a) Consumer maximizes

$$U = E_0 \left[\sum_{t=0}^{\infty} \beta^t (u(a + \theta_t)c_t - bc_t^2) \right]$$

subject to $c_t = Rk_t + y_t - k_{t+1}$. The objective function and budet constraint imply the Lagrangian is

$$\mathcal{L} = u(c) + \beta E[V(k', y', R')] + \lambda [Rk + y - c' - k'] \quad (1)$$

First order conditions and the envelope condition (3rd line) imply

$$\begin{aligned} u_c(c) - \lambda &= 0 \\ \beta E_{y,R}[V_k(k', y', R')] - \lambda &= 0 \\ V_k(k, y, R) &= R\lambda \end{aligned} \quad (2)$$

The FOCs with respect to c and the Envelope condition with respect to k imply the Euler equation

$$u_c(c) = \beta E_{y,R}V_k(k', y', R) \quad (3)$$

Iterating forward one time period on the Envelope condition gives

$$\begin{aligned} u_c(c) &= \beta E_{y,R}Ru_c(c') \\ &= E_y u_c(c') \end{aligned} \quad (4)$$

where $u_c(c) = (a + \theta) + 2bc$, so our Euler equation is

$$a + \theta_t - 2bc_t = E_t[a + \theta_{t+1} - 2bc_{t+1}] \quad (5)$$

b) Assuming $\theta_t = \phi\theta_{t-1} + \epsilon_t$ and for convergence assuming also $|\phi| < 1$, we can rewrite the Euler equation as

$$\begin{aligned}
c_t &= E_t c_{t+1} + \left(\frac{(1-\phi)\theta_t}{2b} \right) \\
&= E_t \left[c_{t+2} + \left(\frac{(1-\phi)\theta_{t+1}}{2b} \right) \right] + \left(\frac{(1-\phi)\theta_t}{2b} \right) \\
&= E_t [c_{t+s}] + \left(\frac{(1-\phi)}{2b} \right) \sum_{s=1}^{\infty} \phi^{s-1} \theta_t \\
&= E_t [c_{t+s}] + \left(\frac{(1-\phi)}{2b} \right) \frac{\theta_t}{1-\phi} \\
&= E_t [c_{t+s}] + \frac{\theta_t}{2b}
\end{aligned} \tag{6}$$

which implies $E_0 [c_t] = c_0 - \frac{\theta_0}{2b}$. We now consider the lifetime budget constraint to characterize the process for optimal consumption.

$$\begin{aligned}
Rk_0 + \sum_{j=0}^{\infty} R^{-t} E_0 [y_t] &= \sum_{t=0}^{\infty} R^{-t} E_0 [c_t] \\
\implies Rk_0 + \sum_{t=0}^{\infty} R^{-t} E_0 [y_t] &= \frac{R}{R-1} \left(c_0 - \frac{\theta_0}{2b} \right) \\
\implies \frac{R-1}{R} \left(Rk_0 + \sum_{t=0}^{\infty} R^{-t} E_0 [y_t] \right) + \frac{\theta_0}{2b} &= c_0
\end{aligned} \tag{7}$$

where the second line holds from the Euler equation. If we assume the process of y satisfies the Markov property, we have characterized consumption as a function of current state variables.

c) We now assume θ follows a random walk, $\theta_{t+1} = \theta_t + \epsilon_{t+1}$. We can rewrite the Euler equation as

$$\begin{aligned}
a + \theta_t - 2bc_t &= E_t [a + \theta_{t+1} - 2bc_{t+1}] \\
&= E_t [a + \theta_t + \epsilon_{t+1} - 2bc_{t+1}] \\
\implies c_t &= E_t [c_{t+1}]
\end{aligned} \tag{8}$$

Thus, when θ follows a random walk, we see that we are in the case that c is a random walk. Plugging this condition into the budget constraint, we get

$$\begin{aligned}
Rk_0 + \sum_{t=0}^{\infty} R^{-t} E_0 [y_t] &= \sum_{t=0}^{\infty} R^{-t} E_0 [c_t] \\
Rk_0 + \sum_{t=0}^{\infty} R^{-t} E_0 [y_t] &= \frac{R}{R-1} c_0 \\
c_0 &= \frac{R-1}{R} \left(Rk_0 + \sum_{t=0}^{\infty} R^{-t} E_0 [y_t] \right)
\end{aligned} \tag{9}$$

Once again, we assume y satisfies the Markov property.

d) Assume $\beta R < 1$ and θ is a random walk. The Euler equation gives

$$\begin{aligned}
a + \theta_t + 2bc_t &= \beta R E_t [a + \theta_{t+1} + 2bc_{t+1}] \\
\implies -2bc_t &= a(\beta R - 1) + \theta_t(\beta R - 1) - 2bE_t [c_{t+1}] \\
\implies c_t &= \left(\frac{1 - \beta R}{2b} \right) (a + \theta_t) + E_t [c_{t+1}]
\end{aligned} \tag{10}$$

Iterating forward on $c_t = \left(\frac{1 - \beta R}{2b} \right) + E_t [c_{t+1}]$ yields

$$c_t = E_t [c_{t+s}] + s \left(\frac{1 - \beta R}{2b} \right) (a + \theta_t) \tag{11}$$

Intuitively, this implies that when $\beta R < 1$ the rate of return on savings is too low, so an agent will consume rather than invest. In our expression for c_t , we can see that consumption in later periods is lower than consumption in earlier periods. Plugging into the lifetime budget constraint, we get

$$\begin{aligned}
Rk_0 + \sum_{t=0}^{\infty} R^{-t} E_0 [y_t] &= \sum_{t=0}^{\infty} R^{-t} E_0 [c_t] \\
Rk_0 + \sum_{t=0}^{\infty} R^{-t} E_0 [y_t] &= \sum_{t=0}^{\infty} R^{-t} \left(c_0 - t \left(\frac{1 - \beta R}{2b} \right) (a + \theta_0) \right) \\
Rk_0 + \sum_{t=0}^{\infty} R^{-t} E_0 [y_t] &= \frac{R}{R - 1} c_0 - \sum_{t=0}^{\infty} R^{-t} \left(t \left(\frac{1 - \beta R}{2b} \right) (a + \theta_0) \right)
\end{aligned}$$

Thus, we get that our consumption process is given by

$$\begin{aligned}
c_0 &= \frac{R - 1}{R} \left[Rk_0 + \sum_{t=0}^{\infty} R^{-t} E_0 [y_t] + \sum_{t=0}^{\infty} R^{-t} \left(t \left(\frac{1 - \beta R}{2b} \right) (a + \theta_0) \right) \right] \\
&= \frac{R - 1}{R} \left[Rk_0 + \sum_{t=0}^{\infty} R^{-t} E_0 [y_t] + \frac{R}{(R - 1)^2} \left(\left(\frac{1 - \beta R}{2b} \right) (a + \theta_0) \right) \right]
\end{aligned} \tag{12}$$

Problem 2

a) Recursive formulation via Bellman equation:

$$V(h_t; A_t) = \max_{c_t, h_{t+1}} \{ \log(c_t) + \beta E_t [V((h_{t+1}; A_{t+1}))] \} \text{ s.t. } c_t = A_t h_t - h_{t+1} \tag{13}$$

b) We ought to guess/ verify the following form of the value function: $V(h; A) = a/\log(h) + v(A)$. We start by assuming it holds for $t + 1$ and then show it also holds for t :

$$\begin{aligned}
V(h_t; A_t) &= \max_{c_t, h_{t+1}} \{ \log(c_t) + \beta E_t [a \log(h_{t+1}) + v(A_{t+1})] \} \\
&= \max_{c_t, h_{t+1}} \{ \log(c_t) - \log(h_t) + \log(h_t) + \beta E_t [a (\log(h_{t+1}) - \log(h_t) + \log(h_t)) + v(A_{t+1})] \} \\
&= (1 + \beta a) \log(h_t) + \max_{\hat{c}_t, \hat{h}_{t+1}} \{ \log(\hat{c}_t) + \beta E_t [a \log(\hat{h}_{t+1}) + v(A_{t+1})] \} \tag{*} \\
&= \underbrace{(1 + \beta a) \log(h_t)}_{\equiv a} + \underbrace{\beta E_t [v(A_{t+1})] + \max_{\hat{c}_t, \hat{h}_{t+1}} \{ \log(\hat{c}_t) + \beta E_t [a \log(\hat{h}_{t+1})] \}}_{\equiv g(A_t)} \\
&\quad \underbrace{\hspace{10em}}_{\equiv v(A_t)}
\end{aligned}$$

(*) introducing the following changes of variables: $\hat{c}_t = \frac{c_t}{h_t}$ and $\hat{h}_{t+1} = \frac{h_{t+1}}{c_t}$. We clearly see that a is a constant: $a = \frac{1}{1-\beta}$. Now, we conjecture that $v(A_t) = E_t [\sum_{s=0}^{\infty} \beta^s g(A_{t+s})]$ where, as indicated above, $g(A_t) = \max_{\hat{c}_t, \hat{h}_{t+1}} \{ \log(\hat{c}_t) + \beta E_t [a \log(\hat{h}_{t+1})] \}$. We again verify this by assuming that it holds for $t + 1$ and show it holds for t :

$$\begin{aligned}
\beta E_t (v(A_{t+1})) + g(A_t) &= \beta E_t \left[E_{t+1} \left[\sum_{s=0}^{\infty} \beta^s g(A_{t+1+s}) \right] \right] + g(A_t) \\
&= \beta E_t \left[\beta^{-1} g(A_t) + \sum_{s=0}^{\infty} \beta^s g(A_{t+1+s}) \right] \\
&= E_t \left[g(A_t) + \sum_{s=0}^{\infty} \beta^{s+1} g(A_{t+1+s}) \right] \\
&= E_t \beta^s \sum_{s=0}^{\infty} g(A_{t+s}) \\
&= v(A_t) \blacksquare
\end{aligned}$$

Note that this representation of $v(A_t)$ is only true if $E_t \beta^s \sum_{s=0}^{\infty} g(A_{t+s}) < \infty$ which depends on the distribution of A .

c) For the optimal consumption and human capital investment choices we deploy the FOC w.r.t. c_t , see eq. 14, and the envelope condition, see eq. 15, where λ is the Lagrange multiplier from rewriting eq. 13 as a constrained maximization:

$$\frac{\partial V}{\partial c_t} : 0 = u_c(c_t) - \lambda \tag{14}$$

$$\frac{\partial V}{\partial h_t} = \lambda A_t \tag{15}$$

Combining eq. 14 and 15 and using our newly confirmed form of the value function we then get:

$$\begin{aligned} u_c(c_t) &= V_h(h_t; A_t) \frac{1}{A_t} \\ &= \frac{\partial}{\partial h_t} (a \log(h_t) + v(A_t)) \frac{1}{A_t} \\ \frac{1}{c_t} &= a \frac{1}{h_t} \frac{1}{A_t} \end{aligned}$$

Which then with the fact that $a = \frac{1}{1-\beta}$ gives the optimal consumption as a function of current state variables and known parameters:

$$c_t^* = A_t h_t (1 - \beta) \quad (16)$$

Using c_t^* in the budget constraint then yields optimal next period human capital investment as a function of the current state variables and known parameters:

$$h_{t+1}^* = A_t h_t \beta \quad (17)$$

d) Knowing that $\log A \sim^{iid} N(\mu_A, \sigma^2)$ we can use our expression for next period's optimal human capital investment to find the expected (gross) growth rate of human capital investments:

$$\begin{aligned} h_{t+1} &= \beta A_t h_t \\ \frac{h_{t+1}}{h_t} &= \beta A_t \\ \log \left[\frac{h_{t+1}}{h_t} \right] &= \log \beta + \log A_t \\ E_t \log \left[\frac{h_{t+1}}{h_t} \right] &= \log \beta + E_t \log A_t \\ E_t \log \left[\frac{h_{t+1}}{h_t} \right] &= \log \beta + \mu_A \end{aligned}$$

For the consumption growth process we make use of the Euler equation $u_c(c_t) = \beta E_t[A_{t+1} u_c(c_{t+1})]$:

$$\begin{aligned} \frac{1}{c_t} &= \beta E_t \left[A_{t+1} \frac{1}{c_{t+1}} \right] \\ 1 &= \beta E_t \left[A_{t+1} \frac{c_t}{c_{t+1}} \right] \\ &= \beta E_t [A_{t+1}] E_t \left[\frac{c_t}{c_{t+1}} \right] \quad \text{due to iid'nes of A} \\ E_t \left[\frac{c_{t+1}}{c_t} \right] &= \beta E_t [A_{t+1}] \\ \log E_t \left[\frac{c_{t+1}}{c_t} \right] &= \log \beta + \mu_A + \frac{1}{2} \sigma^2 \end{aligned}$$