FNCE 924 Problem Set 1

Patrick Shultz

February 12, 2019

Part I: Consumption Choices

Problem 1

a) Consumer maximizes

$$U = E_0 \left[\sum_{t=0}^{\infty} \beta^t (u(a+\theta_t)c_t - bc_t^2) \right]$$

subject to $c_t = Rk_t + y_t - k_{t+1}$ and where $R = 1/\beta$ and the shock θ captures taste shocks to the marginal utility of consumption. Denoting the value of future periods with "′", the objective function and budget constraint imply the Lagrangian is

$$\mathcal{L} = u(c) + \beta E\left[V(k', y', R')\right] + \lambda\left[Rk + y - c - k'\right]$$

First order conditions for c and k':

$$\begin{aligned} u_c(c) - \lambda &= 0 \\ \beta E_{y,R}\left[V_k(k', y', R')\right] - \lambda &= 0 \end{aligned}$$

Differentiating the value function with respect to k gives the Envelope condition

$$V_k(k, y, R) = R\lambda$$

The FOCs with respect to c and the Envelope condition with respect to k imply the Euler equation

$$u_c(c) = \beta E_{y,R} V_k(k', y', R)$$

Iterating forward one time period on the Envelope condition gives

$$\begin{aligned} u_c(c) &= \beta E_{y,R} R u_c(c') \\ &= E_y u_c(c') \end{aligned}$$

where $u_c(c) = (a+\theta) + 2bc$, so our Euler equation is

$$a + \theta_t - 2bc_t = E_t\left[a + \theta_{t+1} - 2bc_{t+1}\right] \tag{1}$$

b) Assuming θ_t follows an AR(1) process with mean 0, we can characterize the process for optimal consumption as follows. Let $\theta_t = \phi\theta_{t-1} + \epsilon_t$ where$|\phi| < 1$, so the series converges. Solving for c_t, we can then rewrite the Euler equation as

$$
\begin{aligned}
c_t &= E_t c_{t+1} + \left(\frac{(1-\phi)\theta_t}{2b}\right) \\
&= E_t\left[c_{t+2} + \left(\frac{(1-\phi)\theta_{t+1}}{2b}\right)\right] + \left(\frac{(1-\phi)\theta_t}{2b}\right) \\
&= E_t\left[c_{t+s}\right] + \left(\frac{(1-\phi)}{2b}\right)\sum_{s=1}^{\infty}\phi^{s-1}\theta_t \\
&= E_t\left[c_{t+s}\right] + \left(\frac{(1-\phi)}{2b}\right)\frac{\theta_t}{1-\phi} \\
&= E_t\left[c_{t+s}\right] + \frac{\theta_t}{2b}
\end{aligned}
\tag{2}
$$

By iterating back to the initial condition, equation (2) implies $E_0\left[c_t\right] = c_0 - \frac{\theta_0}{2b}$. We now consider the lifetime budget constraint to characterize the process for optimal consumption.

$$
\begin{aligned}
Rk_0 + \sum_{j=0}^{\infty} R^{-t}E_0\left[y_t\right] &= \sum_{t=0}^{\infty} R^{-t}E_0\left[c_t\right] \\
\implies Rk_0 + \sum_{t=0}^{\infty} R^{-t}E_0\left[y_t\right] &= \frac{R}{R-1}\left(c_0 - \frac{\theta_0}{2b}\right) \\
\implies \frac{R-1}{R}\left(Rk_0 + \sum_{t=0}^{\infty} R^{-t}E_0\left[y_t\right]\right) + \frac{\theta_0}{2b} &= c_0
\end{aligned}
$$

where the second line holds from the Euler equation. If we assume the process of y satisfies the Markov property, we have characterized consumption as a function of current state variables. This is the same solution as the deterministic case, but with an additional factor to take into account initial tastes and one of the parameters of the quadratic utility function.

c) We now assume our process for θ is given by $\theta_{t+1} = \theta_t + \epsilon_{t+1}$, a random walk. Solving for c_t, we can rewrite the Euler equation as

$$
\begin{aligned}
a + \theta_t - 2bc_t &= E_t\left[a + \theta_{t+1} - 2bc_{t+1}\right] \\
&= E_t\left[a + \theta_t + \epsilon_{t+1} - 2bc_{t+1}\right] \\
\implies c_t &= E_t\left[c_{t+1}\right]
\end{aligned}
$$

Thus, when taste shocks follow a random walk, we are back in the case where consumption is a random walk.The lifetime budget constaint is

$$\begin{aligned}
Rk_0 + \sum_{t=0}^{\infty} R^{-t} E_0\left[y_t\right] &= \sum_{t=0}^{\infty} R^{-t} E_0\left[c_t\right] \\
Rk_0 + \sum_{t=0}^{\infty} R^{-t} E_0\left[y_t\right] &= \frac{R}{R-1} c_0 \\
c_0 &= \frac{R-1}{R}\left(Rk_0 + \sum_{t=0}^{\infty} R^{-t} E_0\left[y_t\right]\right)
\end{aligned}$$

Once again, we assume y satisfies the Markov property.

d) Assume $\beta R < 1$ and θ is a random walk. The Euler equation gives

$$\begin{aligned}
a + \theta_t + 2bc_t &= \beta R E_t\left[a + \theta_{t+1} + 2bc_{t+1}\right] \\
\implies -2bc_t &= a(\beta R - 1) + \theta_t(\beta R - 1) - 2bE_t\left[c_{t+1}\right] \\
\implies c_t &= \left(\frac{1-\beta R}{2b}\right)(a + \theta_t) + E_t\left[c_{t+1}\right]
\end{aligned}$$

Iterating forward on $c_t = \left(\frac{1-\beta R}{2b}\right)(a + \theta_t) + E_t\left[c_{t+1}\right]$ yields

$$c_t = E_t\left[c_{t+s}\right] + s\left(\frac{1-\beta R}{2b}\right)(a + \theta_t)$$

Intuitively, this implies that when $\beta R < 1$ the rate of return on savings is too low, so an agent will consume relatively more than in comparison to the case when $\beta R = 1$. Plugging into the lifetime budget constraint, we get

$$\begin{aligned}
Rk_0 + \sum_{t=0}^{\infty} R^{-t} E_0\left[y_t\right] &= \sum_{t=0}^{\infty} R^{-t} E_0\left[c_t\right] \\
Rk_0 + \sum_{t=0}^{\infty} R^{-t} E_0\left[y_t\right] &= \sum_{t=0}^{\infty} R^{-t}\left(c_0 - t\left(\frac{1-\beta R}{2b}\right)(a + \theta_0)\right) \\
Rk_0 + \sum_{t=0}^{\infty} R^{-t} E_0\left[y_t\right] &= \frac{R}{R-1} c_0 - \sum_{t=0}^{\infty} R^{-t}\left(t\left(\frac{1-\beta R}{2b}\right)(a + \theta_0)\right)
\end{aligned}$$

We can solve this expression for consumption and see that our optimal consumption process is given by the following

$$\begin{aligned}
c_0 &= \frac{R-1}{R}\left[Rk_0 + \sum_{t=0}^{\infty} R^{-t} E_0\left[y_t\right] + \sum_{t=0}^{\infty} R^{-t}\left(t\left(\frac{1-\beta R}{2b}\right)(a + \theta_0)\right)\right] \\
&= \frac{R-1}{R}\left[Rk_0 + \sum_{t=0}^{\infty} R^{-t} E_0\left[y_t\right] + \frac{R}{(R-1)^2}\left(\left(\frac{1-\beta R}{2b}\right)(a + \theta_0)\right)\right]
\end{aligned} \tag{3}$$

Problem 2

We now consider the optimization problem of an individual that can invest in human capital to get a return rate, A_t, in units of the consumption good. The agent's problem is

$$V = E_0 \left[\sum_{t=0}^{\infty} \beta^t \log(c_t) \right]$$

subject to the budget constraint $c_t + h_{t+1} = A_t h_t$.

a) We use a Bellman equation to specify the recursive formulation of this problem:

$$V(h_t, A_t) = \max_{c_t, h_{t+1}} \{\log(c_t) + \beta E_t [V((h_{t+1}, A_{t+1}))]\} \text{ s.t. } c_t = A_t h_t - h_{t+1}$$

Since we are given that A_t follows a first order Markov process, we see that this is recursive problem.

b) We first conjecture the solution to our problem can be written as

$$V(h_t, A_t) = E_t \left[\sum_{t=s}^{\infty} \beta^{s-t} \log(c_t^*) \right]$$

where c_t^* denotes optimal consumption and show that this expression is satisfied by $V(h, A) = \alpha \log(h) + v(A)$ by showing it is a fixed point for our problem.

$$\begin{aligned}
V(h_t, A_t) &= \max_{c_t, h_{t+1}} \{\log(c_t) + \beta E_t [V(h_{t+1}, A_{t+1})]\} \\
&= \max_{c_t, h_{t+1}} \{\log(c_t) + \beta E_t [\alpha \log(h_{t+1}) + v(A_{t+1}))]\} \\
&= \max_{c_t, h_{t+1}} \{\log(c_t) - \log(h_t) + \log(h_t) + \beta E_t [\alpha(\log(h_{t+1}) - \log(h_t) + \log(h_t)) + v(A_{t+1}))]\} \\
&= (1 + \beta a) \log(h_t) + \max_{\hat{c}_t, \hat{h}_{t+1}} \{\log(\hat{c}_t) + \beta E_t \left[\alpha \log(\hat{h}_{t+1}) + v(A_{t+1}))\right]\} \qquad (\star) \\
&= \underbrace{(1 + \beta\alpha)}_{\equiv \alpha} \log(h_t) + \underbrace{\beta E_t[v(A_{t+1})] + \max_{\hat{c}_t, \hat{h}_{t+1}} \{\log(\hat{c}_t) + \beta E_t \left[\alpha \log(\hat{h}_{t+1})\right]\}}_{\equiv v(A_t)} \\
&= \alpha \log(h_t) + v(A_t)
\end{aligned}$$

where $(\star)$ introduces the change of variables where variables are normalized by human capital investment: $\hat{c}_t = \frac{c_t}{h_t}$ and $\hat{h}_{t+1} = \frac{h_{t+1}}{h_t}$ and the constraint of the maximization problem is

$$\hat{c}_t + \hat{h}_{t+1} = A_t$$

Thus we have $\alpha = \frac{1}{1-\beta}$ and since the distribution of A_{t+1} only depends on A_t, we can define $v(A_t)$ as the solution to the problem

$$v(A_t) = \beta E_t[v(A_{t+1})] + \max_{\hat{c}_t, \hat{h}_{t+1}} \{\log(\hat{c}_t) + \beta E_t \left[\alpha \log(\hat{h}_{t+1})\right]\}$$

Thus, we have found a fixed point to our problem and have shown that the conjectured form holds.

c) To find optimal consumption and human capital choices, we take the first order conditions with respect to c_t and h_t to get

$$0 = u_c(c_t) - \lambda \tag{4}$$

$$V_h(h_t, A_t) = \lambda A_t \tag{5}$$

Equations (4), (5) and the value function imply

$$\begin{aligned} u_c(c_t) &= V_h(h_t, A_t)\frac{1}{A_t} \\ &= \frac{\partial}{\partial h_t}(\alpha \log(h_t) + v(A_t))\frac{1}{A_t} \\ \frac{1}{c_t} &= \alpha \frac{1}{h_t}\frac{1}{A_t} \end{aligned}$$

Since $\alpha = \frac{1}{1-\beta}$, optimal consumption is a function of current state variables and known parameters

$$c_t^* = A_t h_t (1-\beta) \tag{6}$$

Plugging c_t^* into the budget constraint yields optimal human capital investment as a function of the current state variables and known parameters

$$h_{t+1}^* = A_t h_t \beta \tag{7}$$

d) Assuming $\log A \sim^{iid} N(\mu_A, \sigma^2)$ and using 7, we can solve for expected growth of human capital investments as follows

$$\begin{aligned} h_{t+1} &= \beta A_t h_t \\ \frac{h_{t+1}}{h_t} &= \beta A_t \\ \log\left[\frac{h_{t+1}}{h_t}\right] &= \log\beta + \log A_t \\ E_t \log\left[\frac{h_{t+1}}{h_t}\right] &= \log\beta + E_t \log A_t \\ E_t \log\left[\frac{h_{t+1}}{h_t}\right] &= \log\beta + \mu_A \end{aligned}$$

To solve for the consumption growth process, we use of the Euler equation, $u_c(c_t) = \beta E_t[A_{t+1} u_c(c_{t+1})]$

$$
\begin{aligned}
\frac{1}{c_t} &= \beta E_t \left[A_{t+1} \frac{1}{c_{t+1}} \right] \\
1 &= \beta E_t \left[A_{t+1} \frac{c_t}{c_{t+1}} \right] \\
&= \beta E_t \left[A_{t+1} \right] E_t \left[\frac{c_t}{c_{t+1}} \right] \\
E_t \left[\frac{c_{t+1}}{c_t} \right] &= \beta E_t \left[A_{t+1} \right] \\
\log E_t \left[\frac{c_{t+1}}{c_t} \right] &= \log \beta + \mu_A + \frac{1}{2}\sigma^2
\end{aligned}
$$

where we can split the expectation from lines two to three because A is distributed iid.

Part II: Numerical Methods
We consider the following problem

$$
U = E_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]
$$

subject to $c_t = Rk_t + y_t - k_{t+1}$ and $k_0 > 0$. Assume that $\log(y)$ follows $\log(y_{t+1}) = 0.05 + 0.95 \log(y_t) + 0.1\epsilon$.

Problem 1

a) Construct a 5 and 9 point discrete Markov Chain approximation to the process, using a grid with equally spaced points that is centered around the long run mean.

See code attached to problem set for the implementation.

b) Simulate your Markov chains. Compute the long run mean, serial correlation, and volatility of this simulated process and check the accuracy of your approximation with $1,000, 5,000, 10,000$ periods.

We first simulate the given AR(1) process for $1000, 5000$,and 10000 steps. This provides a baseline for our Markov Simulations of the process. The mean, serial correlation, and volatility of the process are reported in Table 1.

After simulating the $AR(1)$ process we simulate the Markov chains for 5 grid points and 1,000, 5,000, and 10,000 simulations then for 9 grid points and 1,000, 5,000, and 10,000 simulations. The results are reported in Table 2. We see that our Markov simulations match the mean of the $AR(1)$ process well. However, the serial correlation is low relative to our

	Mean	Serial Correlation	Volatility
1000 Simulations	1.079	0.947	0.317
5000 Simulations	1.056	0.953	0.331
10000 Simulations	1.002	0.949	0.318

Table 1: AR(1) Summary statistics

$AR(1)$ process. When we increase the number of grid points of y, our Markov chain does a better job matching the autocorrelation. We see a similar pattern for the volatility of the process.

	Mean	Serial Correlation	Volatility
5 grid points, 1000 simulations	1.025	0.876	0.186
5 grid points, 5000 simulations	1.011	0.864	0.186
5 grid points, 10000 simulations	1.002	0.874	0.189
9 grid points, 1000 simulations	1.013	0.926	0.247
9 grid points, 5000 simulations	1.004	0.926	0.253
9 grid points, 10000 simulations	1.005	0.922	0.250

Table 2: Markov Simulation Summary Statistics

Problem 2

Assume preferences are CES/CRRA with an IES of 0.5, $\beta = 0.95$, and $R = 1.04$.

a-b) Compute the value function for this problem assuming a discrete grid of 51 then 101 equally spaced points for k. Plot the value function as a function of current k, holding y at its mean and when y is one standard deviation above its mean.

See code attached for the Matlab implementation. The results of the value function iteration are shown in Figures 1 and 2

c) Plot optimal consumption and saving decisions as a function of k when y is at its mean and one standard deviation above its mean.

We use our value function to solve for optimal consumption and saving as a function of k. We compute this for 50 and 101 grid points for k as shown in figures (3) and (4). We see that these plots are kinked, but are becoming smoother as we increase the number of grid points for k as shows in Figure (5). We also see that draws into the state with higher income

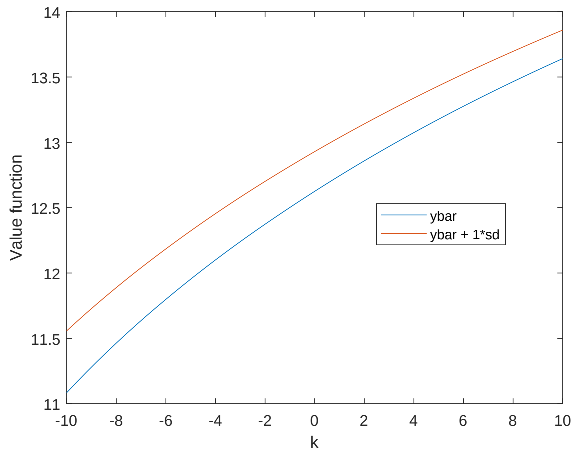

Figure 1: 51 gridpoints

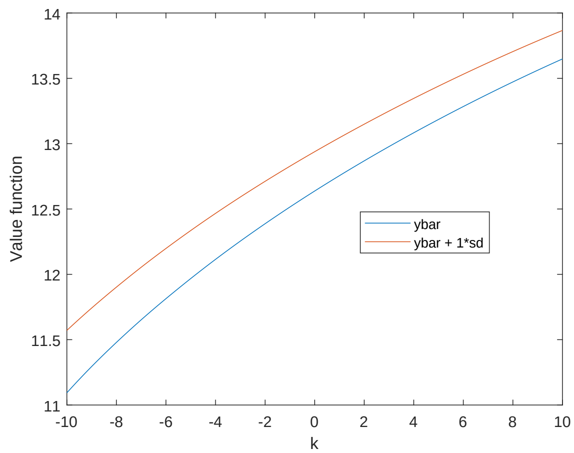

Figure 2: 101 gridpoints

shift the functions up (although this is somewhat difficult to see on the scale of kp).

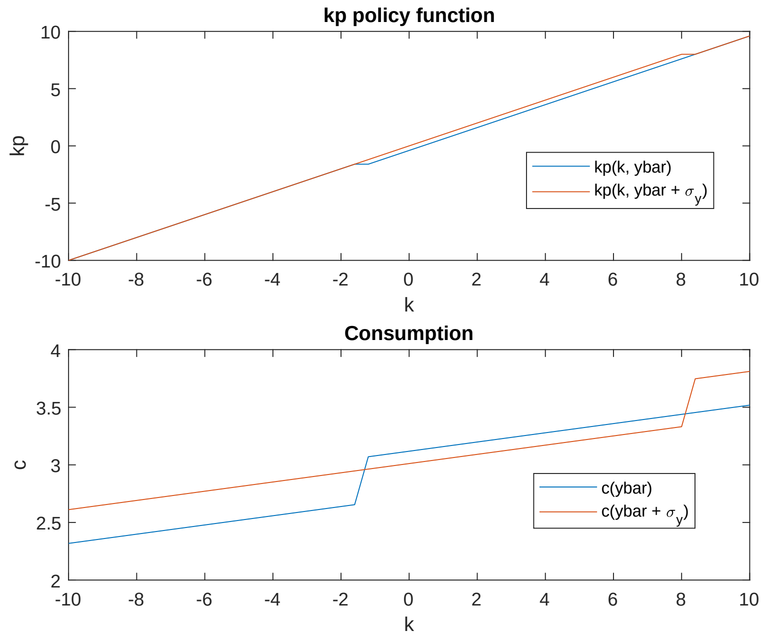

Figure 3: 51 gridpoints

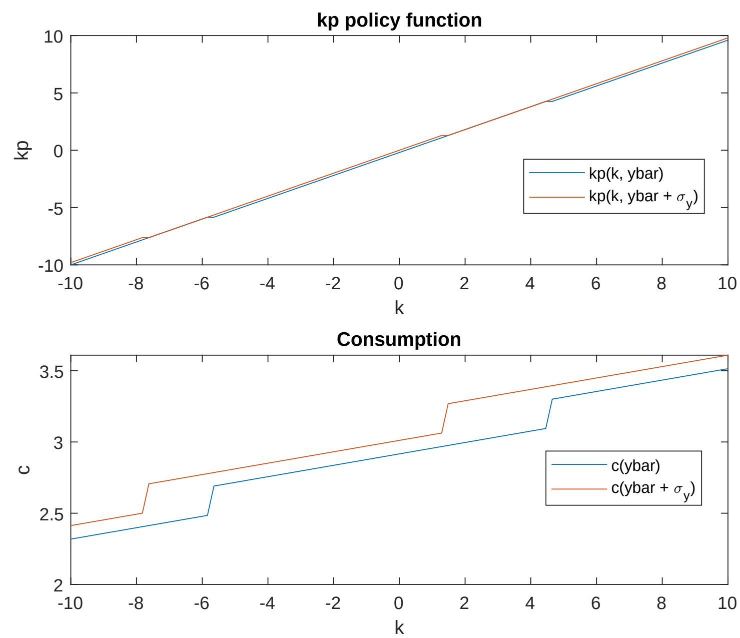

Figure 4: 101 gridpoints

d) Simulate this economy for 5500 and 10500 periods. Drop the first 500 observations in each simulation. Compare the time series average, standard deviation, and persistence of

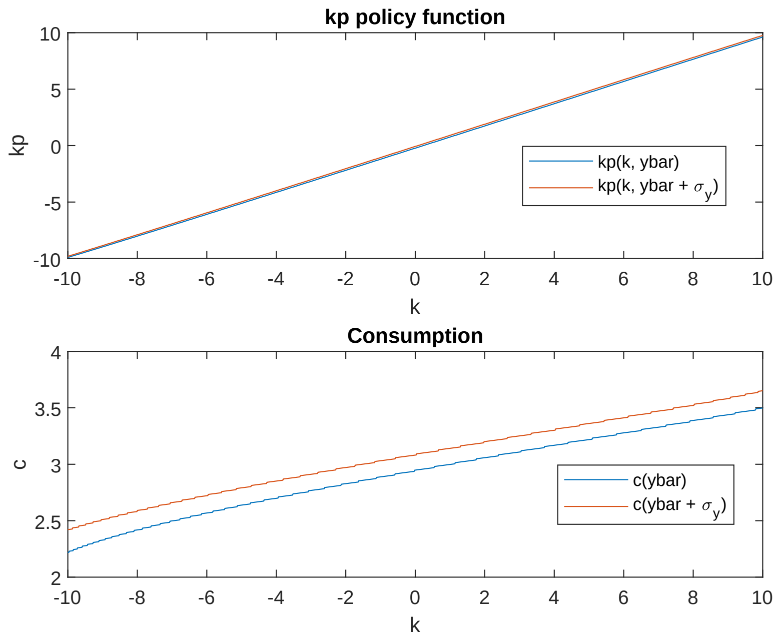

Figure 5: 2001 gridpoints

consumption growth across the two simulations.

With our optimal consumption and savings decision rules we can simulate this economy using the AR(1) process specified above. We simulate the economy for 51 grid points for 5500 and 10500 periods (dropping the first 500 observations) then do the same for 101 grid points. Table 3 compares the time series average, standard deviation, and persistence of consumption growth across the simulations. Once again, the code to implement the simulations is attached. Figures 6 and 7 show the simulations.

	Mean	Std. Deviation	Persistence
51 grid points, 5000 simulations	0.000	0.0781	-0.0528
51 grid points, 10000 simulations	0.000	0.0771	-0.0113
101 grid points, 5000 simulations	-0.001	0.0777	-0.0264
101 grid points, 10000 simulations	0.000	0.0775	-0.0219

Table 3: Economy Simulation Summary Statistics

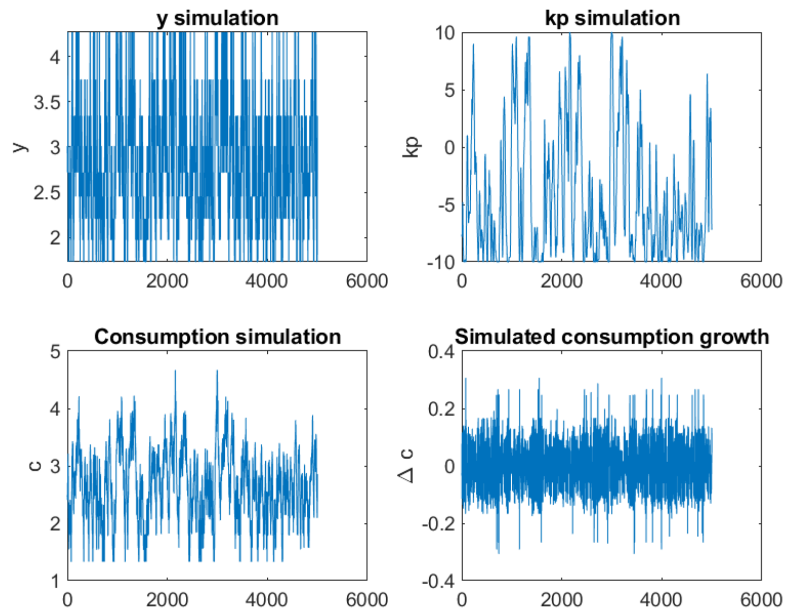

Figure 6: 5000 simulations

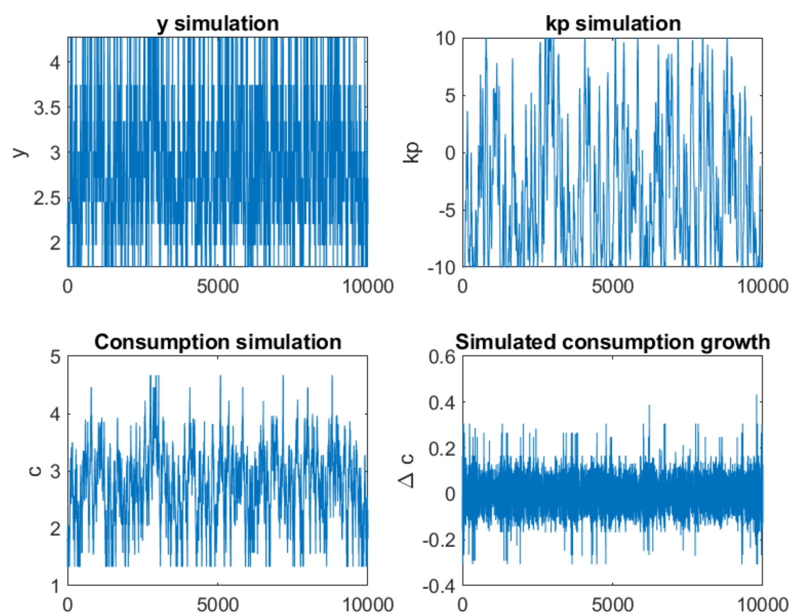

Figure 7: 10000 simulations