FNCE 924 — Problem Set 2

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March 27, 2019

Problem 1: Neoclassical Growth Model

1. What is the long run growth rate of GDP or income?

Since in steady state (= long run) we have $k^\star = k_{t+1} = k_t \forall t$ and $N^\star = N_{t+1} = N_t \forall t$ we have

$$\frac{A_{t+1}P_{t+1}}{A_t P_t} = \gamma_A \gamma_P \equiv \gamma$$

as the long run growth rate.

2. Write down the first order conditions for the problem

We first de-trend the lagrangian by dividing the budget constraint as well as the level-consumption in the utility by $A_t P_t$ using our usual standardization that $P_0 = 1$ and $A_0 = 1$ such that $P_t = \gamma_P^t$ and $A_t = \gamma_A^t$. We denote de-trended variables by their lower-case letters:

$$\begin{aligned}
\mathcal{L} &= \sum_{t=0}^{\infty} \beta^t P_t \frac{\left[(A_t P_t)^\theta \left(\frac{C_t}{A_t P_t}\right)^\theta (1-N_t)^{1-\theta}\right]^{1-\sigma} - 1}{1-\sigma} + \lambda_t \left[k_t^{1-\alpha}(ZN_t)^\alpha - c_t - qk_{t+1}\gamma + q(1-\delta)k_t - g_t\right] \\
&= \sum_{t=0}^{\infty} [\beta^\star]^t \frac{\left[c_t^\theta (1-N_t)^{1-\theta}\right]^{1-\sigma} - (A_t P_t)^{\theta(\sigma-1)}}{1-\sigma} + \lambda_t \left[k_t^{1-\alpha}(ZN_t)^\alpha - c_t - qk_{t+1}\gamma + q(1-\delta)k_t - g_t\right] \\
&\text{where } \beta^\star = \beta\gamma_P(\gamma_A\gamma_P)^{\theta(1-\sigma)}
\end{aligned}$$

$\Rightarrow$ FOCs:

$$\begin{aligned}
[c_t] &\quad \lambda_t = [\beta^\star]^t \theta c_t^{\theta-1}(1-N_t)^{1-\theta}[c_t^\theta(1-N_t)^{1-\theta}]^{-\sigma} \\
[N_t] &\quad \alpha\left(\frac{k_t}{ZN_t}\right)^{1-\alpha}\lambda_t = [\beta^\star]^t(1-\theta)c_t^\theta(1-N_t)^{-\theta}[c_t^\theta(1-N_t)^{1-\theta}]^{-\sigma} \\
[k_{t+1}] &\quad 1 = \frac{\lambda_{t+1}}{\lambda_t}\frac{1}{q\gamma}[\underbrace{(1-\alpha)k_{t+1}^{-\alpha}(ZN_{t+1})^\alpha}_{\equiv \text{ net return to capital } r_k} + q(1-\delta)] \\
[\lambda_t] &\quad k_t^{1-\alpha}(ZN_t)^\alpha = c_t + qk_{t+1}\gamma - q(1-\delta)k_t + g_t
\end{aligned}$$

3. Calibrate the model

$$\gamma_P = 1.01 \tag{1}$$
$$\gamma = 1.03/\gamma_P = 1.02 \tag{2}$$
$$\delta = 0.08 \tag{3}$$
$$r_k = 0.05 \tag{4}$$
$$\alpha = 2/3 \tag{5}$$
$$qi^\star = g^\star = 1/6y_t \tag{6}$$
$$N^\star = 0.35 \tag{7}$$
$$\sigma = 4 \tag{8}$$

4. Compute the steady-state level of consumption and capital as a function of your parameter choices and Z

Solving sequentially for ratios which are constant functions of the model parameters in steady state:

$$\begin{aligned}
via\ [k_{t+1}]: \quad & q = \frac{r_k}{\gamma - 1 + \delta} = 0.5000 \\
via\ [k_{t+1}]: \quad & \frac{k^\star}{N^\star} = \left[\frac{(1-\alpha)}{q(\gamma - 1 + \delta)}\right]^{1/\alpha} Z = 0.0172 * Z \\
& \Rightarrow k^\star = N^\star * \frac{k^\star}{N^\star} = 0.0060 * Z \\
via\ [\lambda_t]\ and\ Eq.\ (6): \quad & \frac{c^\star}{k^\star} = \left(\frac{k^\star}{N^\star}\right)^{-\alpha} Z^\alpha - q\frac{i^\star}{k^\star} - \frac{g^\star}{k^\star} \\
& = \left(\frac{k^\star}{N^\star}\right)^{-\alpha} Z^\alpha - 2q(\gamma - 1 + \delta) = 14.9000 \\
& \Rightarrow c^\star = \frac{c^\star}{k^\star} * k^\star = 0.0898 * Z
\end{aligned}$$

5. Suppose the productivity parameter Z increases by 10%. What is the percent increase in consumption and capital?

As we see above both steady state policies consumption and capital are linear functions in Z and thus a 10% increase in the latter entails a 10% increase in the policies.

6. Suppose country X has the same economic parameters as the US, expect for productivity Z and the cost of investment goods q and the share of government spending in GDP which is 25%

6.i GDP per capita in X 25% of US and Z is 50% lower

With $N_{US} = N_X$ we get:

$$\frac{GDPPC_{US}}{GDPPC_X} = \frac{\gamma_A^t y_{US}}{\gamma_A^t y_X} = \frac{y_{US}}{y_X} = 4$$

$$4 = \left(\frac{k_{US}^\star}{k_X^\star}\right)^{1-\alpha} \left(\frac{Z_{US}}{0.5Z_{US}}\right)^{\alpha} \left(\frac{N_{US}}{N_X}\right)^{\alpha}$$

$$= \left[\frac{N_{US}^\star Z_{US}\left(\frac{1-\alpha}{q_{US}(\gamma-1+\delta)}\right)}{N_X^\star 0.5 Z_{US}\left(\frac{1-\alpha}{q_X(\gamma-1+\delta)}\right)}\right]^{1-\alpha} 2^\alpha \left(\frac{N_{US}}{N_X}\right)^{\alpha}$$

$$\Rightarrow 2 = \frac{q_X}{q_{US}}$$

6.ii Local government of X lowers q by 10%

$$\frac{y_{US}}{y_X} = \left[\frac{N_{US}^\star Z_{US}\left(\frac{1-\alpha}{q_{US}(\gamma-1+\delta)}\right)}{N_X^\star 0.5 Z_{US}\left(\frac{1-\alpha}{0.9*2q_{US}(\gamma-1+\delta)}\right)}\right]^{1-\alpha} 2^\alpha \left(\frac{N_{US}}{N_X}\right)^{\alpha}$$

$$= 2 * \frac{1.8q_{US}}{q_{US}} = 3.6$$

So now the GDPPC of X is 27.78% of the US, a 2.78p.p. increase.

Problem 2: Long Run Growth and the Rate of Return on Capital

1. What is the growth rate of wage income and thus the stock of human capital in this economy?

Consider the firm's problem

$$\mathcal{L} = \sum_{s=0}^{\infty} \beta^s \frac{\lambda_{t+s}}{\lambda_t} \left(K_{t+s}^{1-\alpha} (ZA_{t+s} P_{t+s} N_{t+s})^{\alpha} - N_{t+s} W_{t+s} - I_{t+s} - q_{t+s} \left[K_{t+s+1} - (1-\delta) K_{t+s} - I_{t+s} \right] \right) \quad (9)$$

The first order condition with respect to N_{t+s} implies

$$\frac{\lambda_{t+s}}{\lambda_t} \alpha \left[\frac{K_{t+s}}{ZA_{t+s} P_{t+s} N_{t+s}} \right]^{1-\alpha} (ZA_{t+s} P_{t+s}) = W_{t+s}$$

shifting back s periods gives

$$\alpha \left[\frac{K_t}{ZA_t P_t N_t} \right]^{1-\alpha} (ZA_t P_t) = W_t$$
$$\alpha \left[\frac{k}{ZN_t} \right]^{1-\alpha} (ZA_t P_t) = W_t$$

since we know that the detrended capital to labor ratio is constant, we see that $\gamma_W = \gamma_A \gamma_P \equiv \gamma$.

2. What is the growth rate of capital?

To calculate the growth rate of capital, we consider the first order condition of Equation 9 with respect to capital (K_{t+s+1})

$$\frac{\lambda_{t+s}}{\lambda_t} q_{t+s} = \frac{\lambda_{t+s+1}}{\lambda_{t+s}} \left((1-\alpha) \left[\frac{ZA_{t+s} P_{t+s} N_{t+s}}{K_{t+s+1}} \right]^{\alpha} + (1-\delta) q_{t+s+1} \right)$$
$$\frac{\lambda_{t+s}}{\lambda_t} q_{t+s} = \frac{\lambda_{t+s+1}}{\lambda_{t+s}} \left((R_{t+1}^k - 1) + (1-\delta) q_{t+s+1} \right)$$

where r^k denotes the net return to capital. Shifting back s periods

$$q_t = \frac{\lambda_{t+1}}{\lambda_t} \left((R_{t+1}^k - 1) + (1-\delta) q_{t+1} \right)$$

The first order condition with respect to investment gives $\frac{\lambda_{t+s}}{\lambda_t}(q_{t+s} - 1) = 0$, which implies $q_t = 1 \ \forall t$. Thus, Equation becomes

$$1 = \frac{\lambda_{t+1}}{\lambda_t} \left((R_{t+1}^k - 1) + (1-\delta) \right)$$

solving for $(R_{t+1}^k - 1)$ gives

$$(R_{t+1}^k - 1) = \frac{\lambda_t}{\lambda_{t+1}} - (1-\delta) = \delta - 1$$

so the growth rate of capital is constant and is dependent on the depreciation rate in the economy.

3. Derive an expression for the equilibrium growth rate, r, in this economy? What is the sign of $r - g$? Can it be negative?

Start with the detrended production function $y_t = k_t^{1-\alpha}(ZN)^{\alpha}$ and take the derivative with respect to k_t to get

$$r_t = (1-\alpha) k_t^{-\alpha} (ZN)^{\alpha}$$

plugging in the steady state of capital from question 1 part 4, we get

$$
\begin{aligned}
r^\star &= (1-\alpha)\left[NZ\left(\frac{1-\alpha}{q(\gamma-1+\delta)}\right)^{\frac{1}{\alpha}}\right]^{-\alpha}(ZN)^{\alpha} \\
&= q(\gamma-1+\delta)
\end{aligned}
$$

which implies

$$
r - g = q(\gamma-1+\delta) - \gamma + 1
$$

Setting the expression above equal to zero, we can solve for a condition for when it is negative versus positive

$$
q^* = \frac{\gamma-1}{\gamma-1+\delta} = \frac{g}{g+\delta}
$$

Thus, when $q > q^*$ the sign of $r-g$ is positive and when $q < q^*$ the sign of $r-g$ is negative.

4. Suppose population growth declines. What is the impact on the steady-state value of $r-g$?

5. Suppose average growth rates of population and productivity each decline by 0.5% per year from the previous steady state. What is the quantitative impact of these declines on the steady-state level of the real interest rates?

Problem 3: Solow Growth Model

Consider the following deterministic version of the Solow model:

$$
c_t + k_{t+1} - (1-\delta)k_t \leq Ak_t^{\alpha}
$$

where $k_0 > 0$, consumption is 75% of national income, $\delta = 0.1$, and $\alpha = 1/3$.

1. Compute the steady state level of capital as a function of A. For what follows pick A to normalize the steady-state of capital to 1.

   Solution: Using the fact that $c_t = \frac{3}{4}Ak_t^{\alpha}$, imposing equilibrium and noting that $k_{t+1} = k_t$ at steady state, we get the following equation for capital accumulation

   $$
   k - (1-\delta)k = \frac{1}{4}Ak^{\alpha}
   $$

   solving this equation for k, we see that

   $$
   k^\star = \left[\frac{A}{4\delta}\right]^{\frac{1}{1-\alpha}}
   $$

   where $\star$ denotes the steady state. To calibrate A such that steady state is normalized to 1, we solve the following expression for A

   $$
   1 = \left[\frac{A}{4\delta}\right]^{\frac{1}{1-\alpha}} \tag{10}
   $$

   which implies $A = \frac{2}{5}$.

2. Start the economy with an initial capital stock of 0.5. Compute and plot the equilibrium level of capital over time. How long does it take for the capital stock to reach 0.9? **Solution:** We use the following capital accumulation equation to simulate the growth path of capital

   $$
   k_{t+1} - (1-\delta)k_t = \frac{1}{4}Ak_t^{\alpha} \tag{11}
   $$

   The results are shown in Figure 1. We see that it takes 25 periods for capital to reach 0.9.

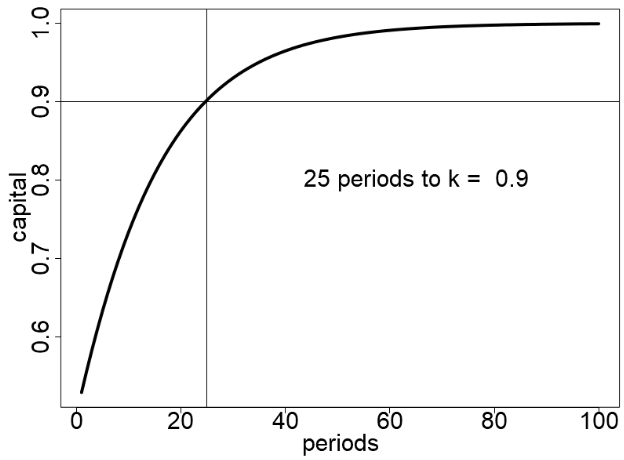

Figure 1: Capital accumulation with $\alpha = 1/3$

3. Now suppose $\alpha = 2/3$. Recompute the steady-state and again normalize A to ensure that $k = 1$ at the steady-state.

   Solution: Inspecting Equation 10, we see that changing α does not change the level of A required to normalize the steady-state of capital to 1. We once again use Equation 11 to specify the growth of capital in our economy. The results are shown in Figure 2. Notably, we see that it takes about twice as

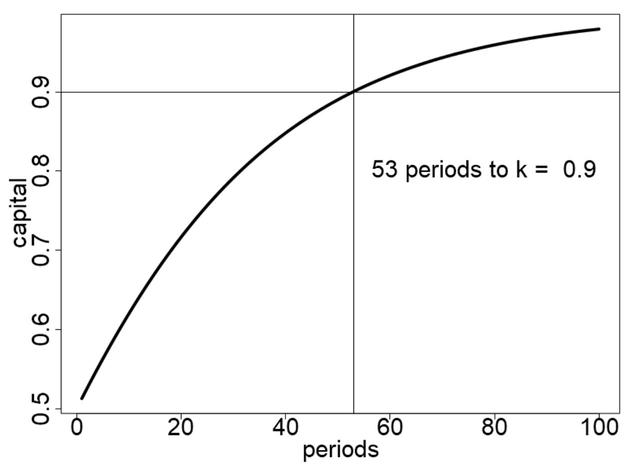

Figure 2: Capital accumulation with $\alpha = 1/3$

long for this economy to reach $k = 0.9$ than the economy with $\alpha = 1/3$. Intuitively, this is because a higher level of α indicates that the production technology is more relatively more capital intensive, so it takes a longer amount of time to go from a low level of capital to $k = 0.9$.