FNCE Problem Set 1

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Part I: Consumption Choices

Problem 1

a) Consumer maximizes

$$U = E_0\left[\sum_{t=0}^{\infty}\beta^t(u(a+\theta_t)c_t - bc_t^2)\right]$$

subject to $c_t = Rk_t + y_t - k_{t+1}$. The objective function and budet constraint imply the Lagrangian is

$$\mathcal{L} = u(c) + \beta E\left[V(k', y', R')\right] + \lambda\left[Rk + y - c' - k'\right] \tag{1}$$

First order conditions and the envelope condition (3rd line) imply

$$\begin{aligned} u_c(c) - \lambda &= 0 \\ \beta E_{y,R}\left[V_k(k', y', R')\right] - \lambda &= 0 \\ V_k(k, y, R) &= R\lambda \end{aligned} \tag{2}$$

The FOCs with respect to c and the Envelope condition with respect to k imply the Euler equation

$$u_c(c) = \beta E_{y,R} V_k(k', y', R) \tag{3}$$

Iterating forward one time period on the Envelope condition gives

$$\begin{aligned} u_c(c) &= \beta E_{y,R} R u_c(c') \\ &= E_y u_c(c') \end{aligned} \tag{4}$$

where $u_c(c) = (a+\theta) + 2bc$, so our Euler equation is

$$a + \theta_t - 2bc_t = E_t\left[a + \theta_{t+1} - 2bc_{t+1}\right] \tag{5}$$

b) Assuming $\theta_t = \phi\theta_{t-1} + \epsilon_t$ and for convergence assuming also $|\phi| < 1$, we can rewrite the Euler equation as

$$\begin{aligned}
c_t &= E_t c_{t+1} + \left(\frac{(1-\phi)\theta_t}{2b}\right) \\
&= E_t\left[c_{t+2} + \left(\frac{(1-\phi)\theta_{t+1}}{2b}\right)\right] + \left(\frac{(1-\phi)\theta_t}{2b}\right) \\
&= E_t\left[c_{t+s}\right] + \left(\frac{(1-\phi)}{2b}\right)\sum_{s=1}^{\infty}\phi^{s-1}\theta_t \\
&= E_t\left[c_{t+s}\right] + \left(\frac{(1-\phi)}{2b}\right)\frac{\theta_t}{1-\phi} \\
&= E_t\left[c_{t+s}\right] + \frac{\theta_t}{2b}
\end{aligned} \tag{6}$$

which implies $E_0\left[c_t\right] = c_0 - \frac{\theta_0}{2b}$. We now consider the lifetime budget constraint to characterize the process for optimal consumption.

$$\begin{aligned}
Rk_0 + \sum_{j=0}^{\infty} R^{-t}E_0\left[y_t\right] &= \sum_{t=0}^{\infty} R^{-t}E_0\left[c_t\right] \\
\implies Rk_0 + \sum_{t=0}^{\infty} R^{-t}E_0\left[y_t\right] &= \frac{R}{R-1}\left(c_0 - \frac{\theta_0}{2b}\right) \\
\implies \frac{R-1}{R}\left(Rk_0 + \sum_{t=0}^{\infty} R^{-t}E_0\left[y_t\right]\right) + \frac{\theta_0}{2b} &= c_0
\end{aligned} \tag{7}$$

where the second line holds from the Euler equation. If we assume the process of y satisfies the Markov property, we have characterized consumption as a function of current state variables.

c) We now assume θ follows a random walk, $\theta_{t+1} = \theta_t + \epsilon_{t+1}$. We can rewrite the Euler equation as

$$\begin{aligned}
a + \theta_t - 2bc_t &= E_t\left[a + \theta_{t+1} - 2bc_{t+1}\right] \\
&= E_t\left[a + \theta_t + \epsilon_{t+1} - 2bc_{t+1}\right] \\
\implies c_t &= E_t\left[c_{t+1}\right]
\end{aligned} \tag{8}$$

Thus, when θ follows a random walk, we see that we are in the case that c is a random walk. Plugging this condition into the budget constraint, we get

$$\begin{aligned}
Rk_0 + \sum_{t=0}^{\infty} R^{-t}E_0\left[y_t\right] &= \sum_{t=0}^{\infty} R^{-t}E_0\left[c_t\right] \\
Rk_0 + \sum_{t=0}^{\infty} R^{-t}E_0\left[y_t\right] &= \frac{R}{R-1}c_0 \\
c_0 &= \frac{R-1}{R}\left(Rk_0 + \sum_{t=0}^{\infty} R^{-t}E_0\left[y_t\right]\right)
\end{aligned} \tag{9}$$

Once again, we assume y satisfies the Markov property.

d) Assume $\beta R < 1$ and θ is a random walk. The Euler equation gives

$$\begin{aligned} a + \theta_t + 2bc_t &= \beta R E_t\left[a + \theta_{t+1} + 2bc_{t+1}\right] \\ \implies -2bc_t &= a(\beta R - 1) + \theta_t(\beta R - 1) - 2bE_t\left[c_{t+1}\right] \\ \implies c_t &= \left(\frac{1 - \beta R}{2b}\right)(a + \theta_t) + E_t\left[c_{t+1}\right] \end{aligned} \tag{10}$$

Iterating forward on $c_t = \left(\frac{1-\beta R}{2b}\right) + E_t\left[c_{t+1}\right]$ yields

$$c_t = E_t\left[c_{t+s}\right] + s\left(\frac{1 - \beta R}{2b}\right)(a + \theta_t) \tag{11}$$

Intuitively, this implies that when $\beta R < 1$ the rate of return on savings is too low, so an agent will consume rather than invest. In our expression for c_t, we can see that consumption in later periods is lower than consumption in earlier periods. Plugging into the lifetime budget constraint, we get

$$\begin{aligned} Rk_0 + \sum_{t=0}^{\infty} R^{-t}E_0\left[y_t\right] &= \sum_{t=0}^{\infty} R^{-t}E_0\left[c_t\right] \\ Rk_0 + \sum_{t=0}^{\infty} R^{-t}E_0\left[y_t\right] &= \sum_{t=0}^{\infty} R^{-t}\left(c_0 - t\left(\frac{1 - \beta R}{2b}\right)(a + \theta_0)\right) \\ Rk_0 + \sum_{t=0}^{\infty} R^{-t}E_0\left[y_t\right] &= \frac{R}{R-1}c_0 - \sum_{t=0}^{\infty} R^{-t}\left(t\left(\frac{1 - \beta R}{2b}\right)(a + \theta_0)\right) \end{aligned}$$

Thus, we get that our consumption process is given by

$$\begin{aligned} c_0 &= \frac{R-1}{R}\left[Rk_0 + \sum_{t=0}^{\infty} R^{-t}E_0\left[y_t\right] + \sum_{t=0}^{\infty} R^{-t}\left(t\left(\frac{1 - \beta R}{2b}\right)(a + \theta_0)\right)\right] \\ &= \frac{R-1}{R}\left[Rk_0 + \sum_{t=0}^{\infty} R^{-t}E_0\left[y_t\right] + \frac{R}{(R-1)^2}\left(\left(\frac{1 - \beta R}{2b}\right)(a + \theta_0)\right)\right] \end{aligned} \tag{12}$$

Problem 2

a) Recursive formulation via Bellman equation:

$$V(h_t; A_t) = \max_{c_t, h_{t+1}} \left\{log(c_t) + \beta E_t\left[V((h_{t+1}; A_{t+1}))\right]\right\} \text{ s.t. } c_t = A_t h_t - h_{t+1} \tag{13}$$

b) We ought to guess/ verify the following form of the value function: $V(h; A) = a/log(h) + v(A)$. We start by assuming it holds for $t+1$ and then show it also holds for t:

$$\begin{aligned}
V(h_t; A_t) &= \max_{c_t, h_{t+1}} \{log(c_t) + \beta E_t [a\ log(h_{t+1}) + v(A_{t+1}))]\} \\
&= \max_{c_t, h_{t+1}} \{log(c_t) - log(h_t) + log(h_t) + \beta E_t [a\ (log(h_{t+1}) - log(h_t) + log(h_t)) + v(A_{t+1}))]\} \\
&= (1 + \beta a) log(h_t) + \max_{\hat{c}_t, \hat{h}_{t+1}} \{log(\hat{c}_t) + \beta E_t \left[a\ log(\hat{h}_{t+1}) + v(A_{t+1}))\right]\} \qquad (\star) \\
&= \underbrace{(1 + \beta a)}_{\equiv a} log(h_t) + \underbrace{\beta E_t[v(A_{t+1})] + \underbrace{\max_{\hat{c}_t, \hat{h}_{t+1}} \{log(\hat{c}_t) + \beta E_t \left[a\ log(\hat{h}_{t+1}))\right]\}}_{\equiv g(A_t)}}_{\equiv v(A_t)}
\end{aligned}$$

$(\star)$ introducing the following changes of variables: $\hat{c}_t = \frac{c_t}{h_t}$ and $\hat{h}_{t+1} = \frac{h_{t+1}}{c_t}$
We clearly see that a is a constant: $a = \frac{1}{1-\beta}$. Now, we conjecture that $v(A_t) = E_t \left[\sum_{s=0}^{\infty} \beta^s g(A_{t+s})\right]$ where, as indicated above, $g(A_t) = \max_{\hat{c}_t, \hat{h}_{t+1}} \{log(\hat{c}_t) + \beta E_t \left[a\ log(\hat{h}_{t+1}))\right]\}$. We again verify this by assuming that it holds for $t+1$ and show it holds for t:

$$\begin{aligned}
\beta E_t(v(A_{t+1})) + g(A_t) &= \beta E_t \left[E_{t+1}\left[\sum_{s=0}^{\infty} \beta^s g(A_{t+1+s})\right]\right] + g(A_t) \\
&= \beta E_t \left[\beta^{-1} g(A_t) + \sum_{s=0}^{\infty} \beta^s g(_{t+1+s})\right] \\
&= E_t \left[g(A_t) + \sum_{s=0}^{\infty} \beta^{s+1} g(A_{t+1+s})\right] \\
&= E_t \beta^s \sum_{s=0}^{\infty} g(A_{t+s}) \\
&= v(A_t) \ \blacksquare
\end{aligned}$$

Note that this representation of $v(A_t)$ is only true if $E_t \beta^s \sum_{s=0}^{\infty} g(A_{t+s}) < \infty$ which depends on the distribution of A.

c) For the optimal consumption and human capital investment choices we deploy the FOC w.r.t. c_t, see eq. 14, and the envelope condition, see eq. 15, where λ is the Lagrange multiplier from rewriting eq. 13 as a constrained maximization:

$$\frac{\partial V}{\partial c_t} : 0 = u_c(c_t) - \lambda \qquad (14)$$

$$\frac{\partial V}{\partial h_t} = \lambda A_t \qquad (15)$$

Combining eq. 14 and 15 and using our newly confirmed form of the value function we then get:

$$\begin{aligned} u_c(c_t) &= V_h(h_t; A_t)\frac{1}{A_t} \\ &= \frac{\partial}{\partial h_t}(a\ log(h_t) + v(A_t))\frac{1}{A_t} \\ \frac{1}{c_t} &= a\frac{1}{h_t}\frac{1}{A_t} \end{aligned}$$

Which then with the fact that $a = \frac{1}{1-\beta}$ gives the optimal consumption as a function of current state variables and known parameters:

$$c_t^\star = A_t h_t(1-\beta) \tag{16}$$

Using $c_t^\star$ in the budget constraint then yields optimal next period human capital investment as a function of the current state variables and known parameters:

$$h_{t+1}^\star = A_t h_t \beta \tag{17}$$

d) Knowing that $logA \sim^{iid} N(\mu_A, \sigma^2)$ we can use our expression for next period's optimal human capital investment to find the expected (gross) growth rate of human capital investments:

$$\begin{aligned} h_{t+1} &= \beta A_t h_t \\ \frac{h_{t+1}}{h_t} &= \beta A_t \\ log\left[\frac{h_{t+1}}{h_t}\right] &= log\beta + logA_t \\ E_t log\left[\frac{h_{t+1}}{h_t}\right] &= log\beta + E_t logA_t \\ E_t log\left[\frac{h_{t+1}}{h_t}\right] &= log\beta + \mu_A \end{aligned}$$

For the consumption growth process we make use of the Euler equation $u_c(c_t) = \beta E_t[A_{t+1}u_c(c_{t+1})]$:

$$\begin{aligned} \frac{1}{c_t} &= \beta E_t\left[A_{t+1}\frac{1}{c_{t+1}}\right] \\ 1 &= \beta E_t\left[A_{t+1}\frac{c_t}{c_{t+1}}\right] \\ &= \beta E_t\left[A_{t+1}\right] E_t\left[\frac{c_t}{c_{t+1}}\right] && \text{due to iid'nes of A} \\ E_t\left[\frac{c_{t+1}}{c_t}\right] &= \beta E_t\left[A_{t+1}\right] \\ logE_t\left[\frac{c_{t+1}}{c_t}\right] &= log\beta + \mu_A + \frac{1}{2}\sigma^2 \end{aligned}$$

Part II: Numerical Methods

$$U = E_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

subject to $c_t = Rk_t + y_t - k_{t+1}$ and $k_0 > 0$.

1 Problem 1

Assume that $log(y)$ follows $\log(y_{t+1}) = 0.05 + 0.95 \log(y_t) + 0.1\epsilon$. The theoretical moments for the process are given by

$$\begin{aligned} \log(y_{t+1}) &= (0.05 + \epsilon_t) + 0.95(0.05 + \epsilon_{t-1}) + 0.95^2(0.05 + \epsilon_{t-2}) + ... \\ &= [0.05/(1 - 0.05)] + \epsilon_t + \phi\epsilon_{t-1} + ... \end{aligned}$$

Thus, $E[\log(y_t)] = 1$. Variance is given by

$$\begin{aligned} E(\log(y_t) - E[\log(y_t)])^2 &= E(\epsilon_t + 0.05\epsilon_{t-1} + ...)^2 \\ &= (1 + 0.05^2 + 0.05^4 + ...)\sigma \\ &= \sigma^2/(1 - 0.05^2) \end{aligned}$$

When $\sigma = 0.1$, we get $0.1/(1 - 0.05^2)$.