Unit - III

* Propositional Logic

• Propositional → True / → False

→ It is a declearative statement that is either true or false, but not both.

→ It is denoted by lower case letters p, q, r ... etc.

→ The truth value of a proposition is true and denoted by T or 1.
False denoted by F or 0.

* Logical connectives (operations)

→ These are words, letter or phrase or connections or links used to combine two or more propositions.

(I) Disjunction (OR) $\vee$
(II) Conjunction (AND) $\wedge$
(III) Negation (NOT) $\neg$ or $\sim$
(IV) Exclusive OR (XOR) $\oplus$

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(5) Implication (if-then) $\rightarrow$ or $\Rightarrow$
(6) Bi-Conditional (if and only if) $\longleftrightarrow$ or $\Longleftrightarrow$

→ Truth Table can be used to show how these operations can be combined the propositions to compound propositions.

(1) Disjunction (OR) "∨"

→ Let p and q be propositions.

→ The disjunction of p and q denotes by '$p \vee q$' is the proposition 'p or q'.

→ The disjunction $p \vee q$ is false when p and q are false and otherwise true.

Example :-

p : Today is friday
q : It is raining today.

$p \vee q$: Today is friday or it is raining today.

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Truth Table

p	q	p ∨ q
1	1	1
1	0	1
0	1	1
0	0	0

(2) Conjunction (AND) "∧"

→ The conjunction of p and q denoted by $p \wedge q$ is the proposition 'p and q'.

→ The conjunction of p and q is true when both p and q is true otherwise false.

Example :-

p: Today is friday.
q: It is raining today.

$p \wedge q$: Today is friday and it is raining today.

Truth Table

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

③ Negation (NOT) '$\sim$'

→ The negation of p is denoted by $\neg p$ ($\sim p$).

example :-

p: Today is friday.

$\sim p$: Today is not friday.

Truth Table

p	$\sim p$
1	0
0	1

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(4) Exclusive OR (XOR) "$\oplus$"

→ The exclusive OR of p and q is denoted by $p \oplus q$.

→ The exclusive or, $p \oplus q$ is true, when exactly one of p and q is true otherwise false.

Example :-

p : Neha ~~was the~~ will pass the exam.

q : Neha will fail the exam.

$p \oplus q$: Neha will either pass or fail the exam.

Truth Table

p	q	$p \oplus q$
1	1	0
1	0	1
0	1	1
0	0	0

(5) Implication (if-then) ' $\Rightarrow$ '

→ An implication $A \rightarrow B$ is the proposition " If A, then B ".

→ It is false if A is true and B is false, otherwise True.

Truth Table

p	q	$p \rightarrow q$
1	1	1
1	0	0
0	1	1
0	0	1

Example :- Assume that your father gave you following promise.

→ If you recieve marks of 90 or better in this semester exam then you will get a ~~gar~~ Car.

p → You got 90% marks.
q → You got car.

- Mark > 90, Car ✓ (T T → T)
- Mark > 90, Car ✗ (T F → F)
- Mark < 90, Car ✓ (F T → T)
- Mark < 90, Car ✗ (F F → T).

⑥ **Biconditional (if and only if) '$\longleftrightarrow$'**

→ $p \leftrightarrow q$ is biconditional logical connective which is true when p and q are same i.e both true or both false.

p	q	$p \leftrightarrow q$
1	1	1
1	0	0
0	1	0
0	0	1

→ $p \longleftrightarrow q$ is true whenevers the truth value of p & q are same.

→ Not only p is sufficient for q but p is necessary for q as well.

Representation

→ p is necessary and sufficient for q and vice-versa.

→ If p then q, and conversely.

→ p iff q.

Example:- Let p be a proposition "You get promotion" and q be "You have connections".

Then $p \longleftrightarrow q$ is the statement.

"You get promotion if and only if you have connections".

(7) **Tautology**:- A statement is said to be tautology if it is always true irrespective (identity) of the truth table values of its component statements.

→ Thus, a statement is a tautology if each entry in the last column of its truth table is 'T'.

Example :- The statement 'p' or 'not p' ; $p \vee \sim p$ is a tautology.

Truth table

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

→ Thus, the statement $p \vee \sim p$ has its truth values T ∀ its entries in the truth tables.

→ Hence, it is a tautology. This is called the law of exclusive middle.

Either p is true or false. There is no middle possibility.

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eg:- If p is the statement 'Ram is honest'; then the statement 'Ram is honest' or 'Ram is dishonest' is a tautology.

Contradiction

A statement is said to be identically false or a contradiction if it is always false irrespective of the truth values of its component statements.

Thus, a statement is a contradiction if each entry in the last column of its truth table is F.

If a statement is a contradiction its negation will be a tautology and vice-versa.

Example:- The statement 'p' and 'not p' i.e $p \wedge \sim p$ is a contradiction.

→ Truth Table

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

→ Thus, the statement $p \wedge \sim p$ has its truth table F ∀ its entries in the truth table.
Hence it is a contradiction.

(9) Contingency :- A proposition that is neither a tautology nor contradiction is called contingency.

→ Thus, the statement is a contingency if some truth table values are true and some are false.

Example :- The statement $p \vee q$ is a contingency.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

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→ Thus, the statement $p \vee q$ has True & false both entries in the last column.
Hence it is a Contingency.

⑩ <u>Logical equivalence</u> ($\equiv$) :- Two proposition are said to be logically equivalent or equal if they have same truth Table / values.

eg :- $\sim(p \vee q)$, $\sim p \wedge \sim q$

①

p	q	$\sim(p \vee q)$
1	1	0
1	0	0
0	1	0
0	0	1

⑪

p	q	$\sim p \wedge \sim q$
1	1	0
1	0	0
0	1	0
0	0	1

So, $\sim(p \vee q) \equiv \sim p \wedge \sim q$.

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* Equivalent forms.

(1) $\sim\sim p \equiv p$

(2) $p \vee p \equiv p$

(3) $(p \wedge \sim p) \vee q \equiv q$

(4) $p \vee \sim p \equiv q \vee \sim q$

(5) $(p \rightarrow q) \equiv (\sim p \vee q)$

(6) $p \rightarrow (q \rightarrow r) \equiv p \rightarrow (\sim q \vee r)$

(7) $q \leftrightarrow r \equiv (q \rightarrow r) \wedge (r \rightarrow q)$

(8) $p \wedge q \equiv \sim(\sim p \vee \sim q)$

↳ De Morgan's Law.

(9) $p \vee q \equiv \sim(\sim p \wedge \sim q)$

This can be used to remove either conjunction or disjunction.

(10) $p \Rightarrow q \equiv \sim q \Rightarrow \sim p$

(11) $q \Rightarrow p \equiv \sim p \Rightarrow \sim q$.

* Law of algebra of propositions

① Commutative law :-

$$p \vee q \equiv q \vee p$$

$$p \wedge q \equiv q \wedge p.$$

② Associative law :-

$$(p \vee q) \vee r \equiv p \vee (q \vee r)$$

$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$

③ Distributive law :-

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

④ Identity law :-

$$p \vee F \equiv p \;,\; p \wedge F \equiv F$$

$$p \wedge T \equiv p \;,\; p \vee T \equiv T$$

(5) Negation law or Complement law :-

$$p \wedge \sim p \equiv F$$
$$p \vee \sim p \equiv T$$

(6) Idempotent law :-

$$p \vee p \equiv p$$
$$p \wedge p \equiv p.$$

(7) Absorption law :-

$$p \wedge (p \vee q) \equiv p$$
$$p \vee (p \wedge q) \equiv p.$$

* (8) De-Morgan's law :-

$$\sim (p \vee q) \equiv (\sim p) \wedge (\sim q)$$
$$\sim (p \wedge q) \equiv (\sim p) \vee (\sim q)$$

(9) Conditional law :-

$$p \rightarrow q \equiv \sim p \vee q$$

(10) Bi-Conditional law :-

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow$$

Q.) Solve the following without using truth table.

(1) $\sim(\sim p \wedge q) \wedge (p \vee q) \equiv p.$

$\rightarrow \sim(\sim p) \wedge \sim q \wedge (p \vee q)$

$\rightarrow (p \wedge \sim q) \wedge (p \vee q)$

$\rightarrow \boxed{p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)}$

On comparing,

$p \rightarrow p, \; q \rightarrow \sim q, \; r \rightarrow q$

$\rightarrow p \vee (\sim q \wedge q)$

$\rightarrow p \vee F \equiv p$ proved.

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① Negation of conditional statement :-

$$\sim(p \to q) \equiv p \wedge \sim q$$

② Conditional rules :-

$$(p \to q) \equiv ((\sim p) \vee q)$$

③ Contrapositive law :-

$$(p \to q) \equiv \sim q \to \sim p.$$

④ Bi-conditional or equivalence rules :-

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$
$$p \leftrightarrow q \equiv ((p \wedge q) \vee ((\sim p) \wedge (\sim q)))$$

⑤ Exportion law :-

$$(p \wedge q) \to r \equiv p \to (q \to r)$$

⑥ Absurdity law :-

$$(p \to q) \wedge (p \to \sim q) \equiv \sim p$$

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7. Negation of bi-conditional statement

$\sim(p \leftrightarrow q) \equiv \sim\{(p \to q) \wedge (q \to p)\}$

8. Negation of disjunction :-

$\sim(p \vee q) \equiv \sim p \wedge \sim q$

9. Negation of conjunction :-

$\sim(p \wedge q) \equiv \sim p \vee \sim q$.

* Types of conditional

→ Let p and q are any two statements. then some other conditional related to $p \to q$ are given as :-

(a) $p \to q$:-
It is called direct conditional statement.

(b) $q \to p$:-
It is called Converse condition statement i.e; converse

implication of $p \to q$.

(3) $\sim p \to \sim q$

It is called inverse or opposite implication i.e it is if not p then not q.

(4) $\sim q \to \sim p$

It is called contrapositive implication, in other words it is said that if not q then not p.

→ Let us consider the following truth table which clearly explain the different types of condition as:-

p	q	$\sim p$	$\sim q$	$p \to q$	$q \to p$	$\sim p \to \sim q$	$\sim q \to \sim p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T.

(c) Negation law :-

(i) $\sim(p \rightarrow q) \equiv p \wedge \sim q$

(ii) $p \rightarrow q \equiv (\sim p \vee q)$

(iii) $p \rightarrow q \equiv \sim q \rightarrow \sim p$

(iv) $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

(v) $p \leftrightarrow q \equiv ((p \wedge q) \vee (\sim p \wedge \sim q))$

(vi) $(p \wedge q) \rightarrow r \equiv p \rightarrow (q \rightarrow r)$

(vii) $(p \rightarrow q) \wedge (p \rightarrow \sim q) \equiv \sim p$

(viii) $\sim(p \leftrightarrow q) \equiv \sim\{(p \rightarrow q) \wedge (q \rightarrow p)\}$

(ix) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

(x) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

(d) Influence laws

(i) Modus Ponens

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

(ii) Modus tollens

$$\begin{array}{l} p \rightarrow q \\ \sim q \\ \hline \therefore \sim p \end{array}$$

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(iii) Hypothetical syllogism

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

(iv) Disjunctive syllogism

$$\begin{array}{l} p \vee q \\ \sim p \\ \hline \therefore q \end{array}$$

(v) Constructive delima

$$\begin{array}{l} (p \to q) \wedge (r \to s) \\ p \vee r \\ \hline \therefore q \vee s \end{array}$$

(vi) Absorption

$$\begin{array}{l} p \to q \\ \hline \therefore p \to (p \wedge q) \end{array}$$

(vii) Conjunction

$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

(viii) Simplification

$$p \wedge q$$
$$\therefore p$$

(ix) Addition

$$p$$
$$\therefore p \vee q$$

(x) Resolution

$$p \vee q$$
$$\underline{p \vee r}$$
$$\therefore q \vee r$$

Q8. What do you understand by an arguments?

→ The knowledge of tautology and implication can now used to describe a valid argument.

→ Terms such as theorem, premises, conclusion used in the definition of an argument.

* **Theorem**:- It is a true statement derived from the axioms of a mathematical structure or system.

* **Conclusion**:- It is the statement that is asserted on the basis of other propositions.

* **Premise (or hypotheses)**:- It is a statement which is assumed to be true for accepting a conclusion.

→ An argument is an assertion that a finite number of statements $P_1, P_2 \ldots P_n$ called the premises yeilds another statement q, called conclusion.

→ In other words, if each of $P_1, P_2 \ldots \ldots P_n$ is a true statement, then an implication of the form:

$$P_1 \wedge P_2 \wedge P_3 \ldots \wedge P_n \Rightarrow q$$

→ In predicate calculus, the sentence is breakdown into terms, predicates and quantifiers to get the conclusion.

→ The indivisual variables and constants either in the form of proper names or descriptions are classified as terms.

→ A predicate is a word or words, in a sentence which express the nature of the subject.

eg:- Every rational number is a real number.

Here, 'is a real number' is a predicate.

→ Quantifiers is a declarative sentence which contains one or more variable in it ~~are replaced by certain~~ and is not a statement. but becomes a statement when the variables in it are replaced by certain allowable choices is called an open statement.

eg:- $P(x)$: The number $x+4 \leq 10$ is an even number.

Quantifiers

- Universal quantifiers
- Existential quantifiers.

(a) Universal quantifiers :- The symbol $\forall$ (for all) is used to represent universal quantifier and is read in either ways:-

- $\forall x \in A : P(x)$
- $P(x), \forall x \in A$

→ The variable x in an open statement is called its free variable because as x varies over the universe for an open statement the truth or false of the statement may vary.