Partially ordered Set (POSET)

Set P
↓
R be a Relation on set P
↓
Partially ordered relation if satisfies
- → R is Reflexive
- → R is Anti-Symmetric
- → Transitive

If given f^n satisfy above three properties then we can say that given set is POSET

Partial ordering Relation:-

A Relation 'R' on a set 'p' is said to be partial ordering (or) partial ordering relation if and only if R is reflexive, Anti-symmetric and transitive.

→ Partial ordering is denoted by the symbol '$\preccurlyeq$'

Partial ordered set on POSET:-

If $\preccurlyeq$ is a partial ordering on a set 'P', Then the order pair $(P, \preccurlyeq)$ is called a "Partial ordered set or "POSET".

Totally ordered Relation:-

Let $(P, \preccurlyeq)$ is a partially ordered set if ~~∀~~ for every two elements $a, b \in P$, we have either $a \preccurlyeq b$ or $b \preccurlyeq a$ (comparable), then '$\preccurlyeq$' is called a simple ordering (or) linear ordering on 'P' and $(P, \preccurlyeq)$ is called a totally ordered set / simply ordered set or a chain.

Ques:- A Set $S = \{a, b, c\}$ together with the relation of set inclusion $\subseteq$ is a partial order on $P(S)$, where $P(S)$ is the power set of S

Sol^n Given that $S = \{a, b, c\}$

The power set of S is given as :-

$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$.

then $(P(S), \subseteq)$ is a poset if it satisfies the following Conditions:-

① Reflexivity:- Since $A \subseteq A \ \forall \ A \in P(S)$ hence it is reflexive

② Antisymmetry :- If $A \subseteq B$ and $B \subseteq A \Rightarrow A = B$

Hence, it is antisymmetry.

③ Transitivity :- If $A \subseteq B$, $B \subseteq C \Rightarrow A \subseteq C$

hence it is transitive

$\therefore$ $(P(S), \subseteq)$ is poset.

Ques:- Let $A = \{2, 3, 6, 12, 24, 36\}$ and R be a relation in A which is defined by "a divides b" then R is poset in A.

Sol^n $A = \{2, 3, 6, 12, 24, 36\}$

① Reflexivity:- Since $a/a \ \forall \ a \in A$

$\therefore$ It is reflexive

② Antisymmetry:- If a/b and $b/a \ \forall \ a, b \in A \Rightarrow a = b$

③ Transitivity:- Let $a, b, c \in A$ then

if $a/b = b/c$ then a/c

$\therefore$ it is transitive

hence $(A, '/')$ is a POSET.

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Comparable Two elements a and b in a POSET $(S, \leq)$ are said to be comparable if either $a \leq b$ or $b \leq a$. Otherwise it is called incomparable.

HASSE DIAGRAM:-

Graphical representation of POSET is known as Hasse diagram.

Procedure for Drawing Hasse Diagram:-

① Draw the directed graph of given relation

② Delete all loop at all vertices i.e

③ Eleminate all edges that are implied by the transitive relation

④ Draw the digraph of a partial order with edges pointing upward so that arrows may be omitted from edges.

⑤ Replace the circles representing the vertices by dots.

Ques:- Draw the Hasse Diagram of the following :-

(a) Draw the Hasse Diagram of $(A, \leq)$ where $A = \{3, 4, 12, 24, 48, 72\}$ and the relation $\leq$ be s.t $a \leq b$ if a divides b.

(b) Draw the Hasse Diagram of the relation S defined as "divides" on the set B when $B = \{2, 3, 4, 6, 12, 36, 48\}$

Sol^n The Hasse diagram is given as:-

(a) $R = \{ \cancel{(3,4)}, (3,12), (3,24), (3,48), (3,72), (4,12),$
$(4,24), (4,48), (4,72), (12,24), (12,48), (12,72)$
$(24,48), (24,72) \}$

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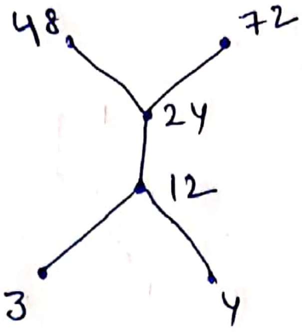
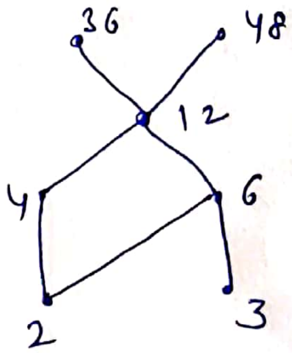
$R = \{(3,12), (4,12)\ (12,24), (24,48), (24,72)\}$



11 (b)



Component of Poset :-

① **Maximal Element:-** a is maximal in $(A, \leq)$ if there is no $b \in A$ s.t $a \leq b$ i.e; (all top elements of the Has Diagram.

② **Minimal Element:-** a is minimal in $(A, \leq)$ if there is no $b \in S$ s.t $b \leq a$

i.e; (all the bottom elements of the Hasse Diagram)

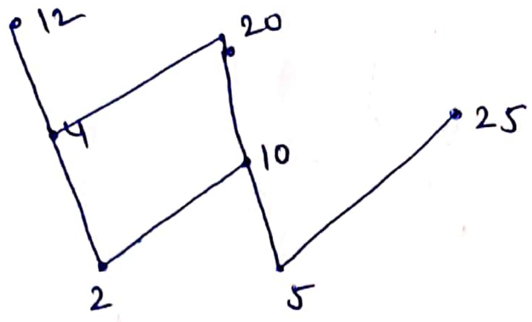
③ **Greatest Element:-** a is the greatest element of $(A, \leq)$ if $b \leq a$ for all $b \in S$. It must be unique.

If more than one maximal elements in a Hasse diagram, then there is no. greatest element in the POSET.

④ **Least Element:-** a is the least element of $(A, \leq)$ if $a \leq b$ for all $b \in S$. It must be unique.

If there is more than one minimal elements in a Hasse Diagram, then there is no least element in POSET

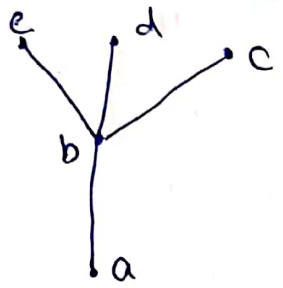
Ques:. which element of POSET are $\{\{2,4,5,10,12,25\}, /\}$ Maximal, minimal, Greatest and Least elements.



Maximal = 12, 20, 25
Minimal = 2, 5
Greatest = None
Least = None

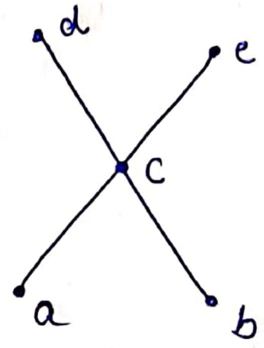
Ques: Find the maximal, minimal, Greatest & Lowest elements of the following diagram :-

(i)



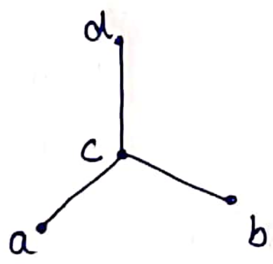
Maximal = b, c, d
Minimal = a
Greatest = None
Least = a

(ii)



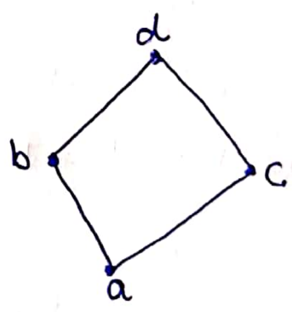
Maximal = d, e
Minimal = a, b
Greatest = None
Least = None

(iii)



Maximal = d
Minimal = a, b
Greatest = d
Least = None

(iv)



Maximal = d
Minimal = a
Greatest = d
Least = a

Ques:- Let A be the set of factors of a particular +ve integer D and let $\leq$ be the relation "divide" i.e,

$\leq = \{(x, y) : x \in A, y \in A \text{ and } x \text{ '/' } y\}$

Draw the Hasse diagram for

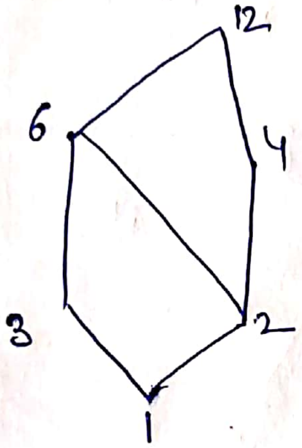
(i) D = 12 (ii) D = 30 (iii) D = 45

OR

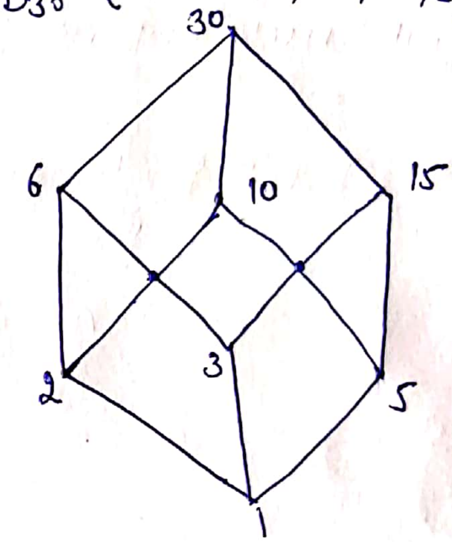
(i) D_{12} (ii) D_{30} (iii) D_{45}

Sol^n Hasse diagram for D_{12}

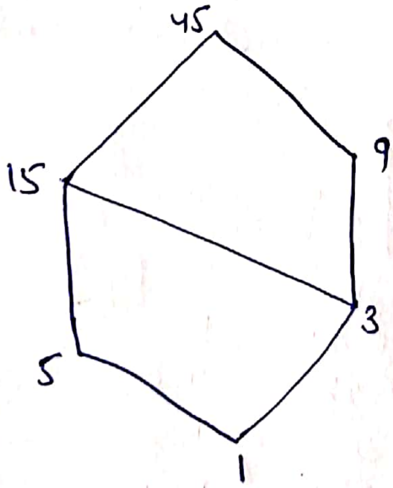
(i) Now $A = \{1, 2, 3, 4, 6, 12\}$



(ii) $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$



(iii) $D_{45} = \{1, 3, 5, 9, 15, 45\}$



Ques: {A "/"}

when $A = \{2, 4, 5, 10, 12, 20, 25\}$

Do your self.

Lattice

A lattice is a POSET in which any two element a + b have a GLB called as meet $(a \wedge b)$ and a LUB called as join $(a \vee b)$

OR

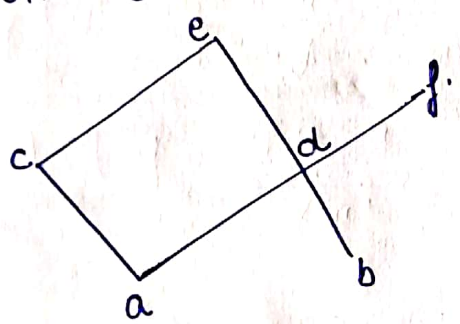
Every pair of element has glb as well as lub

$\therefore$ $(A, \leq)$ is a lattice.

OR

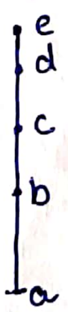
Join-Semi Lattice :- In a poset if LUB/JOIN/Supremum/$\vee$ exist for every pair of elements, then poset is called JOIN-Semi-Lattice.

①



If there is no single top then, it is not a join-semi lattice.

②



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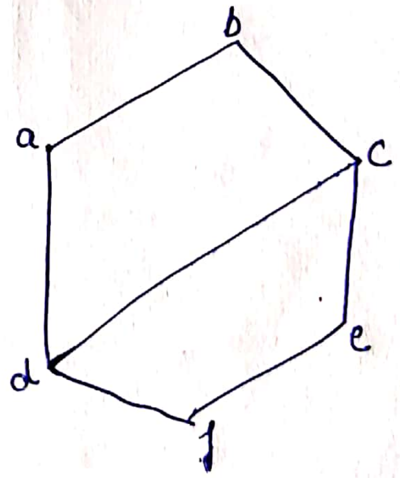

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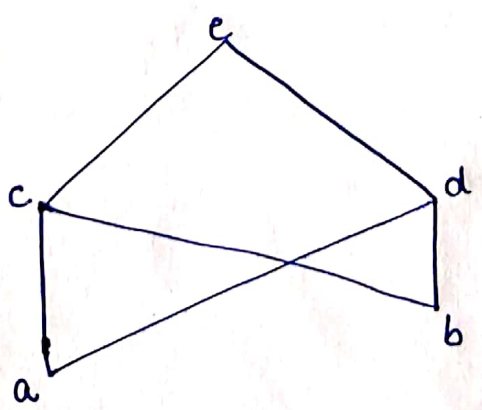
④

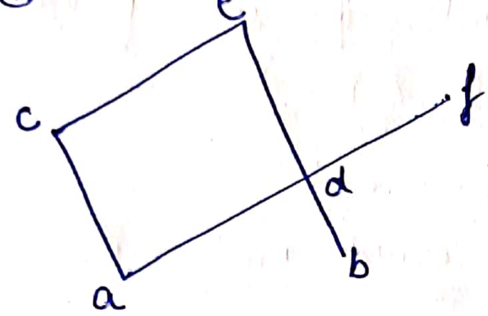


x

$(a \vee b = ?)$ $\because$ 2 values, can't decide minimum/LUB

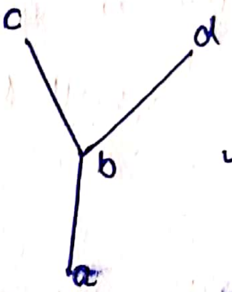
Meet - Semi Lattice:- In a POSET if GLB / MEET / infimum / $\wedge$ exist for every pair of elements, then POSET is called Meet - Semi Lattice. [there should be single bottom first & no multiple GLB]

①



X

②



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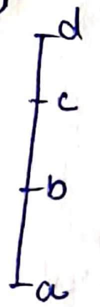

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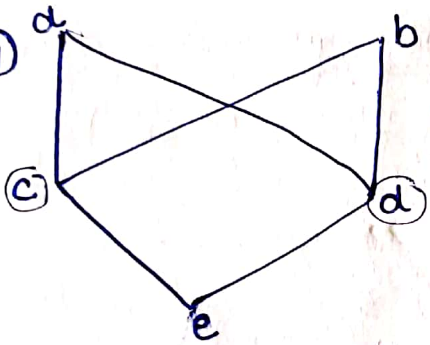

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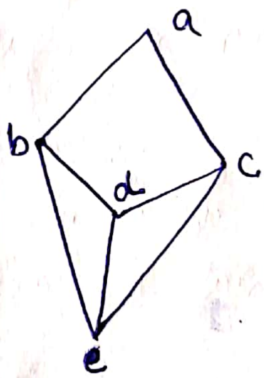

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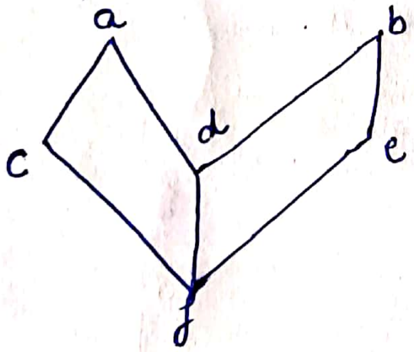

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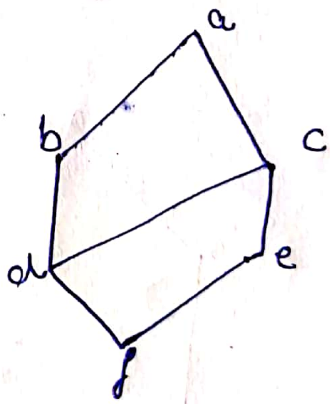

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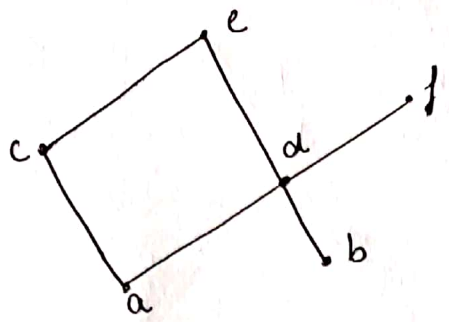
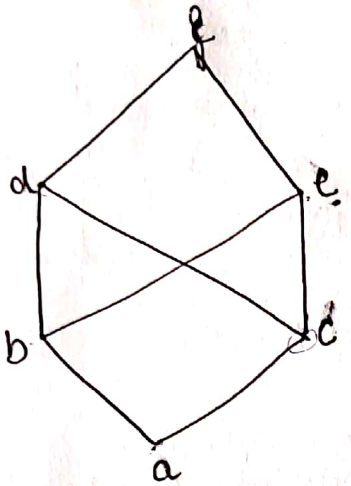
(3)



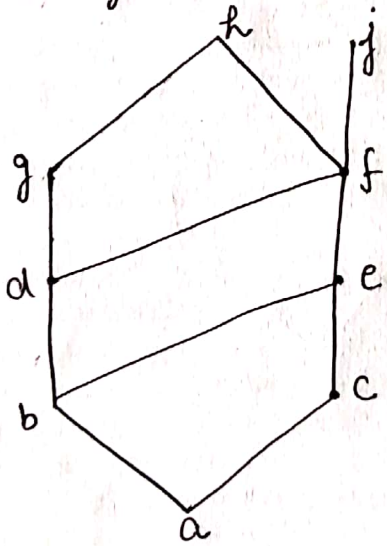
Subset	B = {c, d}	B = {a, b}	B = {e, f}
UB(B)	e	d, e, f	ϕ
LUB(B)	e	d	ϕ
LB(B)	a	ϕ	d, a, b
GLB(B)	a	ϕ	d

(4)



Subset	B = {d, e}	B = {b, c}
UB(B)	f	d, f, e
LUB(B)	f	ϕ
LB(B)	b, c, a	a
GLB(B)	ϕ	a

Quis:- Find the UB, LB, LUB and GLB of the following Hasse diagram given below for {a, b, c}



Solⁿ A = {a, b, c, d, e, f, g, h, j}

B = {a, b, c}

UB → for finding Upper Bound we consider B → A

aRa, aRb, aRc, aRd, aRe, aRf, aRg, aRh, aRj

bRa, bRb, bRc, bRd, bRe, bRf, bRg, bRh, bRj

cRa, cRb, cRc, cRd, cRe, cRf, cRg, cRh, cRj

UB = {e, f, g, h, j}

Now LUB = {e} for {a, b, c}

Components:-

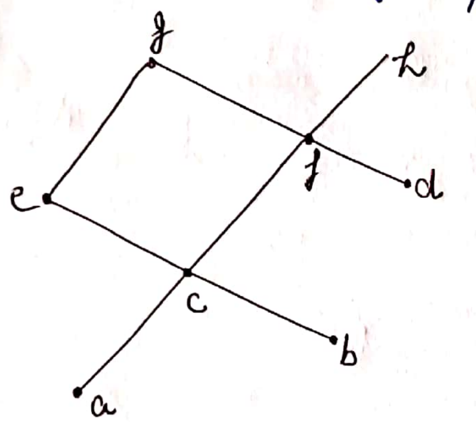
① **Upper Bound**:- Consider B → A & compare it and find
where A → All Elements contains in Hasse diagram (main set)
B → subset of A.

② **Lower Bound**:-
Consider A → B and compare it and find
where A → All elements contains in Hasse diagram (main set)
B → subset of A.

③ **Greatest lower Bound (GLB)**:-
GLB / meet / $\wedge$ → infimum → inf.

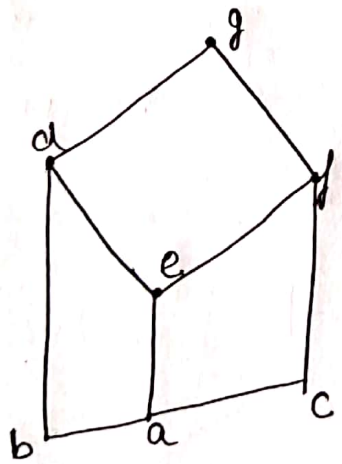
④ **Least Upper Bound (LUB)**:-
LUB / join / $\vee$ → supremum → sup.

eg:- ①



subset	B = {e, c}	B = {c, f, d}
~~UB~~ LB	{a, b, c}	{φ}
UB	{g, e}	{g, h, f}

②



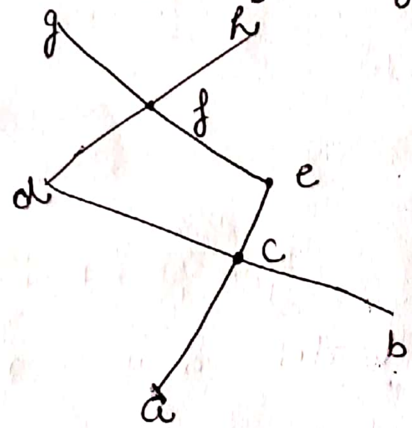
Subset	B = {d, g}	B = {e, f}
UB(B)	{g}	{g, f}
LB(B)	{a, b, e, d}	{a, e}

for LB :- $(A \to B)$

aRa, aRb, aRc
$b\not R a$, bRb, $b\not R c$
$c\not R a$, $c\not R b$, cRc
dRa, dRb, dRc
$e\not R a$, $e\not R b$, $e\not R c$
$f\not R a$, $f\not R b$, $f\not R c$
$g\not R a$, $g\not R b$, $g\not R c$
$h\not R a$, $h\not R b$, $h\not R c$
jRa, $j\not R b$, $j\not R c$

$LB = \{a\}$
$GLB = \{a\}$

Que: Find the UB, LB, LUB and GLB of the given Hasse diagram given below for $\{c, d, e\}$



$A = \{a, b, c, d, e, f, g, h\}$
$B = \{c, d, e\}$

UB → $B \to A$

$c\not R a$, $c\not R b$, cRc, cRd, cRe, cRf, cRg, cRh
$d\not R a$, $d\not R b$, $d\not R c$, dRd, $d\not R e$, dRf, $d\not R g$, dRh
$e\not R a$, $e\not R b$, $e\not R c$, $e\not R d$, eRe, eRf, eRg, eRh

$UB = \{f, g, h\}$
$LUB = \{f\}$

LB $(A \to B)$

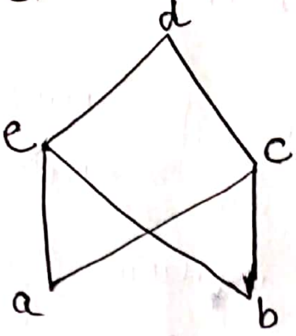
aRc, aRd, aRe
bRc, bRd, bRe
cRc, cRd, cRe
$d\not R c$, dRd, $d\not R e$
$e\not R c$, $e\not R d$, eRe
$f\not R c$, $f\not R d$, $f\not R e$
$g\not R c$, $g\not R d$, $g\not R h$
$h\not R c$, $h\not R d$, $h\not R h$

$\therefore LB = \{a, b, c\}$

$GLB = \{c\}$

example

①



$B = \{a, b\}$

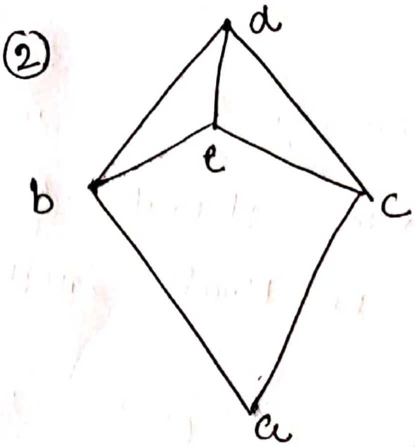
$UB = \{e, c, d\}$

$LUB = \phi$

$LB = \phi$

$GLB = \phi$

②



$B = \{e, c\}$

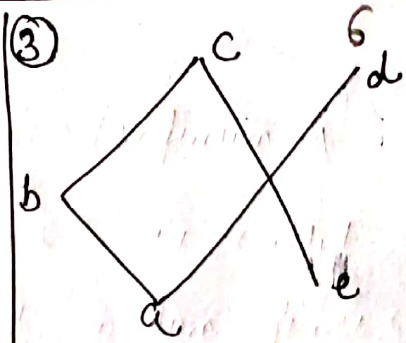
$UB = \{d, e\}$

$LUB = \cancel{d}\ e$

$LB = a$

$GLB = a$

③



$B = \{d, c\}$

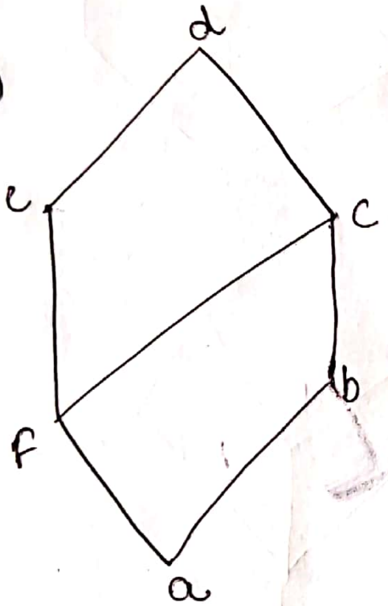
$UB = \phi$

$LUB = \phi$

$LB = a$

$GLB = a$

④



$B = \{f, c\}$

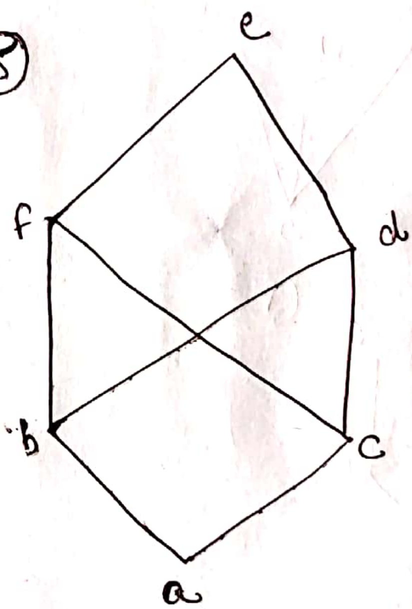
$UB = \{c, d\}$

$LUB = c$

$LB = a, f$

$GLB = f$

⑤



$B = \{b, d\}$

$UB = e, d$

$LUB = d$

$LB = b, a$

$GLB = b$

$B = \{f, c\}$

$UB = f, e$

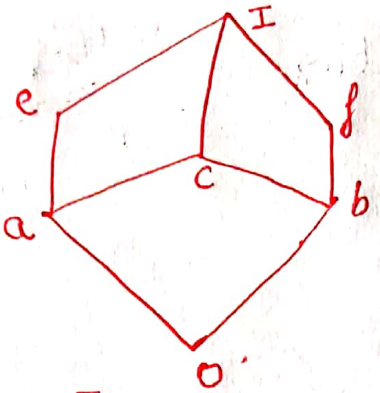
$LUB = f$

$LB = c, a$

$GLB = c$

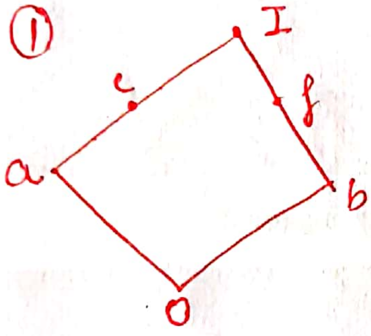
Sub lattice:- Let $(L, \leq)$ be a lattice. A non-empty subset S of L is called a sublattice of L if $a \vee b$ in S and $a \wedge b$ in S whenever a and b in S.

Que: Consider the lattice $(L, \leq)$ as shown



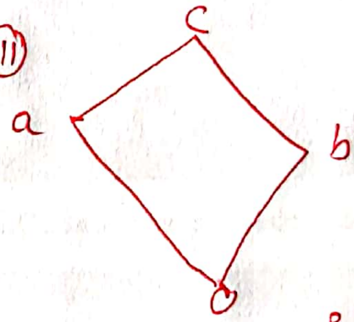
Determine which of the following is sublattice of

(i)



S_1

(ii)



S_2

Soln $S_1 = \{0, a, b, e, f, I\}$

LUB

$\vee$	0	a	b	e	f	I
0	0	a	b	e	f	I
a	a	a	c	e	I	I
b	b	c	b	I	f	I
e	e	e	I	e	I	e
f	f	I	f	I	f	f
I	I	I	I	e	I	I

$\notin S_1$

GLB

$\wedge$	0	a	b	e	f	I
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	b	a	b	b
e	0	a	a	e	0	e
f	0	0	b	0	f	f
I	0	a	b	e	f	I

$\in S_1$

S_1 is not a sub lattice because LUB table ~~to~~ exist an element which is not belong to S_1 i.e; c

(ii) $S_2 = \{0, a, b, c\}$

LUB

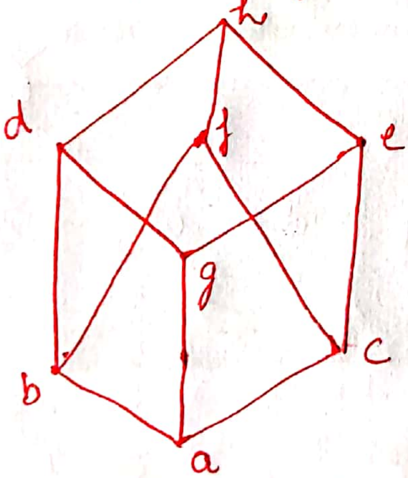
$\vee$	0	a	b	c
0	0	a	b	c
a	a	a	c	c
b	b	c	b	c
c	c	c	c	c

GLB

$\vee$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

S_2 is a sublattices, because LUB and GLB of every pair belongs to S_2

Que: Consider a lattice $(L, \leq)$ under the relation $\leq$ by Hasse Diagram given below



Determine whether or not following are sublattices of L

(i) $L_1 = \{a, b, d, f\}$

(ii) $L_2 = \{c, e, g, h\}$

(iii) $L_3 = \{a, b, d, h\}$

Soln (i) In the given lattice

$d \vee f = h \notin L_1$

$\therefore$ L_1 is not a sublattice.

(ii) In the given lattice

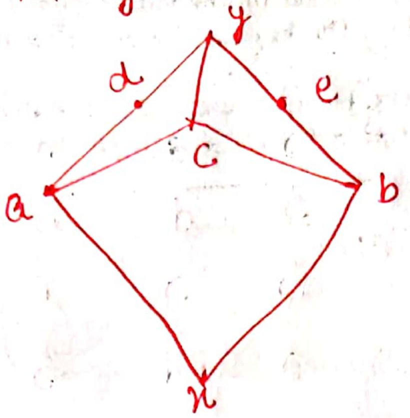
$c \wedge g = a \notin L_2$

$\therefore$ L_2 is not a sublattice.

(iii) In the given lattice ~~ava~~ ~~is~~ $a \vee b \in L_3$ & $a \wedge b \in L_3$

So L_3 is sublattice of L

③ Find the given str & its subsets are lattice or NOT?



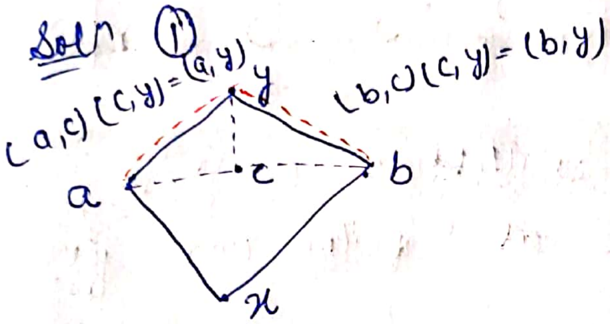
(1) $L_1 = \{x, a, y, b\}$

(2) $L_2 = \{x, a, c, y\}$

(3) $L_3 = \{x, c, d, y\}$

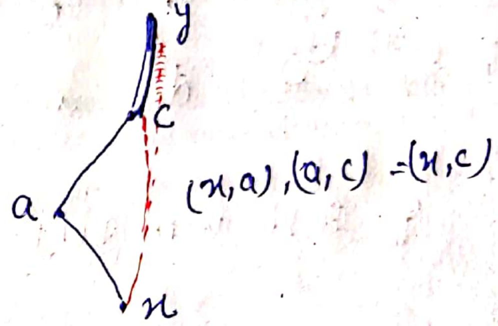
(4) $L_4 = \{x, b, d, y\}$

Soln: (i)



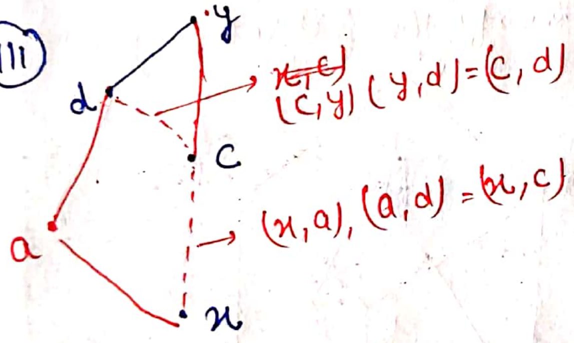
Yes.

(ii)



Yes.

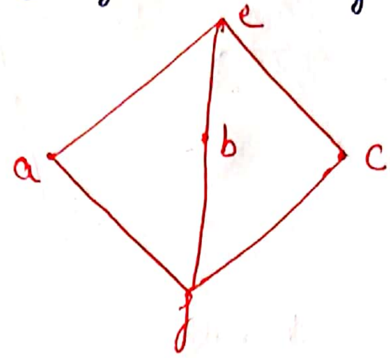
(iii)



yes.

Properties of Lattices

Let $(L, \wedge, \vee, \leq)$ be any Lattice. The Lattice L for any a, b and c satisfies the following properties:-



① **Idempotent Law:-**

$A \cup A = A$
$A \cap A = A$ } set theory

Illy $a \vee a = a \longrightarrow LUB(B)$

$a \wedge a = a \longrightarrow GLB(B)$

So, we can say that Lattice holds idempotent law.

② ~~Asso~~ **Commutative Law:-**

$A \cup B = B \cup A \Rightarrow a \vee b = b \vee a$

$A \cap B = B \cap A \Rightarrow a \wedge b = b \wedge a$

$a \vee b = b \vee a$	$a \wedge b = b \wedge a$
$e = e$	$f = f$

$\therefore$ order of elements in a pair doesn't matter

$(a,b) \Leftrightarrow (b,a)$

③ **Associative Law:-**

Set theory $\rightarrow$ $A \cup (B \cup C) = (A \cup B) \cup C$	$A \cap (B \cap C) = (A \cap B) \cap C$
$a \vee (b \vee c) = (a \vee b) \vee c$	$a \wedge (b \wedge c) = (a \wedge b) \wedge c$
$a \vee (e) = e \vee c$	$a \wedge f = f \wedge c$
$e = e$	$f = f$

H.P

(4) **Absorption Law**

① $a \wedge (a \vee b) = a$ | ② $a \vee (a \wedge b) = a$

$a \wedge e = a$ | $a \vee f = a$

H.P

(5) **Distributed Law:-** (Not always true) so lattice doesn't hold this law

① $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$a \vee f = e \wedge e$

$f\,a \neq e$

② $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

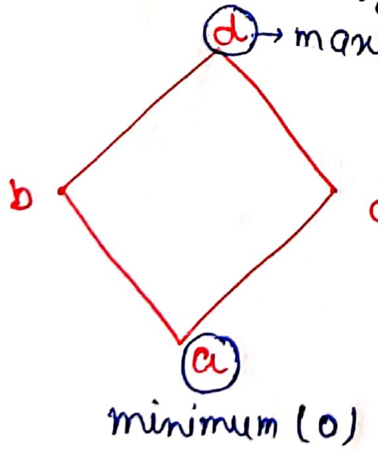
$a \wedge (e) = f \vee f$

$a \neq f$

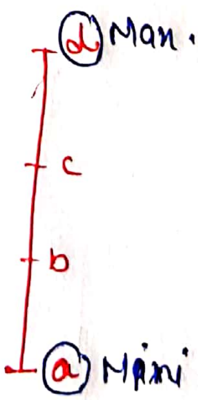
Upper Bound and lower bound of a Lattice (Maximum & Minimum)

Upper Bound:- In Lattice L, if there is an element I which satisfy $a R I \;\forall\; a \in L$, then I is called upper bound of Lattice

Lower Bound:- In Lattice L, if there is an element 'O' which satisfy $O R b \;\forall\; a \in L$, O is called lower bound.



→ d (I)
a (0)

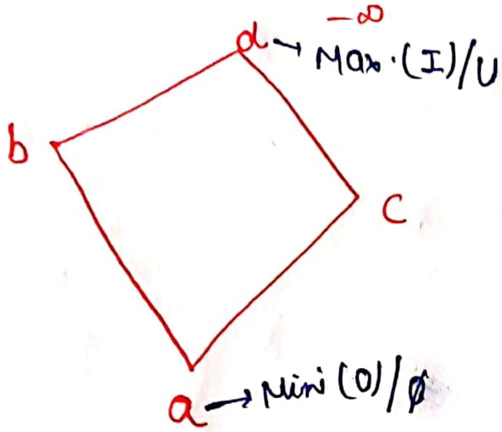


d (I)
→ a (0)

Finite lattices (only finite lattices) can have upper & lower bound and also called bounded lattices.

In Infinite lattice of integers (but not bounded, because we can't find max. and mini / upper & lower bound)

#



$\left.\begin{array}{l} I = d \\ 0 = a \end{array}\right\} \rightarrow$ relate

$A \cup U = U$	$A \cap U = A$
$A \cup \phi = A$	$A \cap \phi = \phi$

} set theory

$a \vee I = d$ $a \vee 0 = a$ $a \wedge I = a$ $a \wedge 0 = 0$	$A \cup U = U$ $A \cup \phi = A$ $A \cap U = A$ $A \cap \phi = \phi$
$a \vee a' = I$ $a \wedge a' = 0$	$A \cup A' = U \rightarrow$ universal set $A \cap A' = \phi$

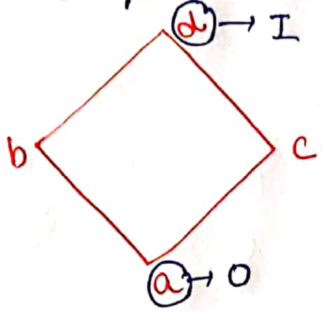
$b \vee c = I \longrightarrow b' = c$

Illy $c' = b$

we know that
$A \cup B = B \cup A$

Complement of an element in a Lattice :-

In a bounded Lattice L for any element $a \in L$, if there exist an element $b \in L$, such that $a \vee b = I$, $a \wedge b = 0$ that b is called complement of a, we can also say that 'a' & 'b' are complements of each other.



In set theory, complement of U (universal set) is ϕ

Illy " Lattice , " " I is 0

$\therefore \left[\begin{array}{l} U^c = \phi \iff I^c = 0 \\ \phi^c = U \iff 0^c = I \end{array} \right]$

e.g.! ① for (a, d)

$a \vee d = d$ (max.)

$a \wedge d = a$ (min)

then $a^c = d$

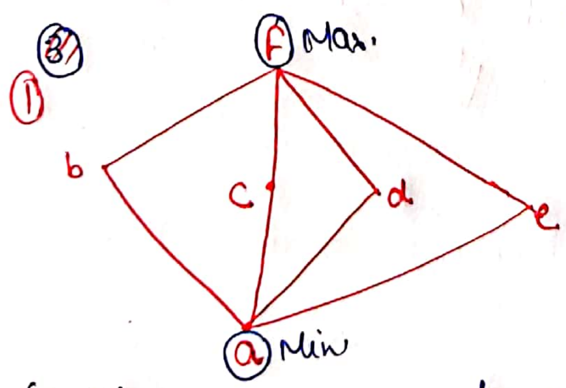
$d^c = a$

② for (b, c)

$b \vee c = d \rightarrow$ Max. $\Rightarrow b^c = c$

$b \wedge c = a \rightarrow$ Min $\quad c^c = b$

③



for (a, f)

$a \vee f = f \rightarrow I$ (Max.) $\quad a^c = f$

$a \wedge f = a \rightarrow 0$ (Min). $\quad f^c = a$

for (b, c)

$\left. \begin{array}{l} b \vee c = f - I \\ b \wedge c = a \rightarrow 0 \end{array} \right\} \begin{array}{l} b^c = c \\ c^c = b \end{array}$

for (b, e)

$\left. \begin{array}{l} b \vee e = f - I \\ b \wedge e = a \rightarrow 0 \end{array} \right\} \begin{array}{l} b^c = e \\ e^c = b \end{array}$

~~Illy (c, d) → $c \vee d = f$, $c \wedge d = a$~~

for (b, d)

$\left. \begin{array}{l} b \vee d = f \rightarrow I \\ b \wedge d = a \rightarrow 0 \end{array} \right\} \begin{array}{l} b^c = d \\ d^c = b \end{array}$

$\left\{ \begin{array}{l} b^c = c, d, e \\ c^c = b, \cancel{d} \\ d^c = b, c, e \\ e^c = b \end{array} \right.$

Illy $f(c, e)$

$c \vee e = f - I$, $c \wedge e = a - 0$ $\Big\}$ $c^c = e$, $e^c = c$

f· Illy (c, d)

$c \vee d = f - I$, $c \wedge d = a - 0$ $\Big\}$ $c^c = d$, $d^c = c$

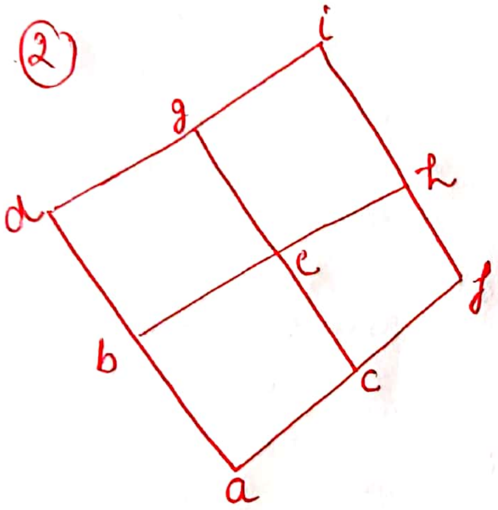
$b^c = c, d, e$

$c^c = b, d, e$

$d^c = b, c, \cancel{d}\ e$

$e^c = b, d$

(2)



for (a, i) $a^c = I$

$a \vee i = i \to I$

$a \wedge i = a \to 0$

for (d, f)

$d \vee f = i \to I$

$d \wedge f = a \to 0$ $d^c = f$

for (d, e)

$d \vee e = g \xrightarrow{\times} I$

$d \wedge e = b \xrightarrow{\times} 0$

for (a, d)

$a \vee d = d \xrightarrow{\times} I$

$a \wedge d = a$ $\therefore a^c \neq d$

for (d, c)

$d \vee c = g$

$d \wedge c = a$ $d^c \neq c$

for d, h

$d \vee h = i$ $d^c = h$

$d \wedge h = a$

$\therefore$ so no complements of g, b, c & e here.

Distributive Lattice:- A lattice L is said to be distributive if $\forall a, b \in L$, must satisfies the following properties:-

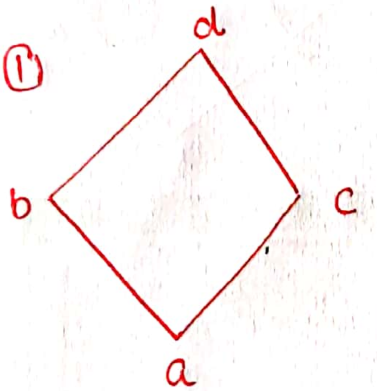
① $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

② $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

\# If lattice is distributive then for any element single complement would exist only

\# If every element of lattice has atmost (either one or nothing (no complement)) one complement, is distributive lattice.

①



$a' = d$
$b' = c$
$c' = b$
$d' = a$

for (a, d)
$a \vee d = d \rightarrow I$ $\quad a^c = d$
$a \wedge d = a \rightarrow 0$ $\quad d^c = a$

for (b, c)
$b \vee c = d \rightarrow I$ $\quad b^c = c$
$b \wedge c = a \rightarrow 0$ $\quad c^c = b$

Every element have unique complement (one complement / single complement). So, we can say that it is distributive lattice.

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

L.H.S $\qquad$ R.H.S

$a \vee (a) \qquad b \wedge c$

$a = a$

Illy

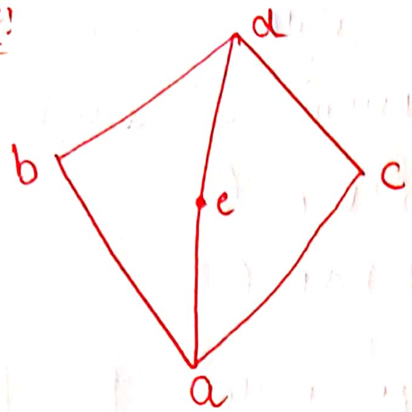
$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

$$a \wedge (d) = a \vee a$$

$$a = a$$

If every element have unique complement then distributive law hold.

e.g.:



$a' = d$

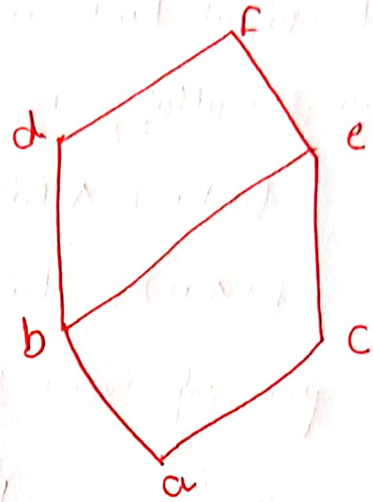
$b' = e, c$

$c' = b, e$

$d' = a$

$e' = b, c$

So, Lattice is not distributive hence, it is a Complemented Lattice.



$a' = f$

$b' = c$

$c' = d$

$d' = c$

$e' = b$

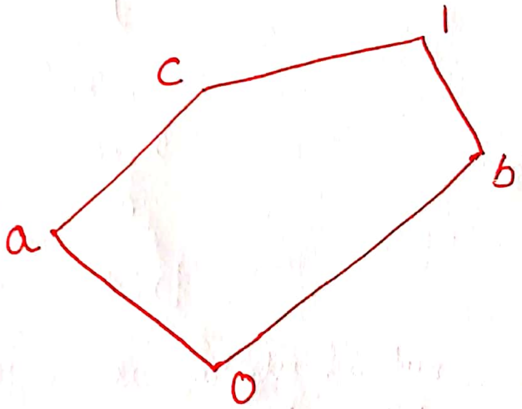
$f' = a$

Distributive & Lattice

Modular Lattice :- A Lattice L is said to be modular Lattice if for an element a, b, c $\in$ L s.t

(i) $a \vee (b \wedge c) = (a \vee b) \wedge c$ when $a \leq c$

(ii) $a \wedge (b \vee c) = (a \wedge b) \vee c$ when $a \geq c$



$a \vee (b \wedge c) = (a \vee b) \wedge c$

L.H.S

$a \vee (b \wedge c)$

$= a \vee (0)$

$= a$

R.H.S

$(a \vee b) \wedge c$

$1 \wedge c$

c

L.H.S $\neq$ R.H.S

So, ~~This~~ it is not a modular Lattice.