Unit-III

Algebraic Structures

If there exists a system that it consists of a non-empty set and one or more operations on that set, then that system is called an algebraic system. It is generally denoted by $(A, op_1, op_2, ----, op_n)$ where A is a non-empty set and $op_1, op_2, ----, op_n$ are operations on A.

An algebraic system is also called an algebraic structure because the operations on the set A define a structure on the elements of A.

n-ary operation:-

A function $f: A \times A \times ---- A \rightarrow A$ is called an n-ary operation.

Binary Operation:-

Consider a non-empty set A and a function f such that $f: A \times A \longrightarrow A$ is called a binary operation on A. If $*$ is a binary operation on A, then it may be written as $a * b$.

A binary operation can be denoted by any of the symbols $+, -, *, \oplus, \Delta, \vee, \wedge$ etc.

The value of the binary operation is denoted by placing the operator b/w two operands.

e.g:- (i) The operation of addition is a binary operation on the set of natural numbers

(ii) The operation of subtraction is a binary operation on set of integers. But, the operation of subtraction is not a binary operation on the set of natural numbers because the subtraction of two natural numbers may or may not be a natural numbers.

$$a - b \neq \text{binary operation.}$$

(III) The operation of multiplication is a binary operation on the set of natural numbers, set of integers and set of complex numbers.

(IV) The operation of set union is a binary operation on the set of subsets of a universal set. Similarly, the operation of set intersection is a binary operation on the set of subsets of a universal set.

Tables of Operation :-

Consider a non-empty finite set $A = \{a_1, a_2, a_3, ----, a_n\}$. A binary operation $*$ on A can be described by means of a table as

$*$	a_1	a_2	a_3		a_n
a_1	$a_1 * a_1$	$a_1 * a_2$	$a_1 * a_3$		$a_1 * a_n$
a_2	$a_2 * a_1$	$a_2 * a_2$	$a_2 * a_3$		$a_2 * a_n$
a_3	$a_3 * a_1$	$a_3 * a_2$	$a_3 * a_3$		$a_3 * a_n$
a_n	$a_n * a_1$	$a_n * a_2$	$a_n * a_3$		$a_n * a_n$

Example

Consider the set $A = \{1, 2, 3\}$ and a binary operation $*$ on the set A defined by

$$a * b = 2a + 2b$$

Represent the operation $*$ as a table on A

Sol^n The table of the operation is as

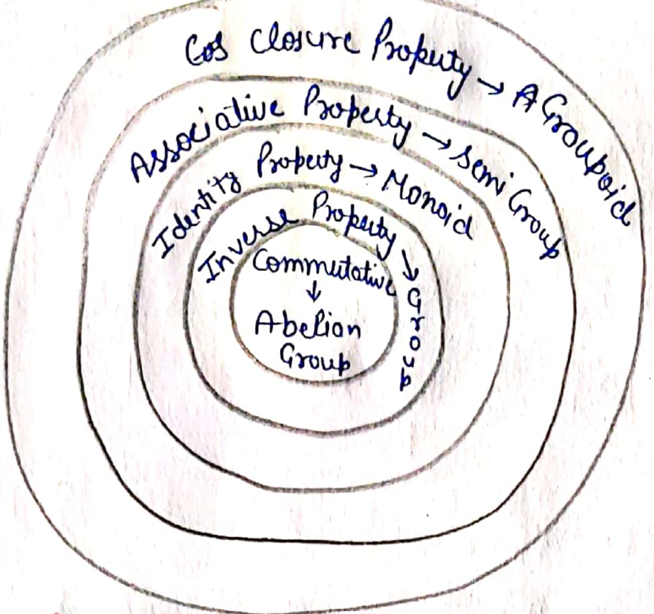
$a \downarrow \; * \; b \rightarrow$	1	2	3
1	4	6	8
2	6	8	10
3	8	10	12

Group:-

An algebraic structure $(G, *)$ where G is a non-empty set and $*$ is a binary operation defined on G, is called a group, if it satisfies the following Properties:-

1. Closure Property
2. Associative Property
3. Existence of Identity
4. Existence of inverse
5. Commutative Property

① Property Satisfy ⟶ Groupoid
①+② " " ⟶ Semi group
①+②+③ " " ⟶ Monoid Group
①+②+③+④ Property Satisfy ⟶ Group
①+②+③+④+⑤ " " ⟶ ~~Abli~~ Abelian Group



1. <u>Closure Property</u> :-

Consider a non-empty set A and a binary operation * on A. Then A is closed under the operation *, if $a * b \in A$ when a and b are elements of A.

<u>e.g</u>:- The operation of addition on the set of integers is closed operations.

Ques:- Consider the set $A = \{-1, 0, 1\}$. Determine whether A is closed under (i) addition (ii) multiplication.

<u>Solⁿ</u> (i) The sum of the elements $(-1) + (-1) = -2$ and $1 + 1 = 2$ does not belong to given set A. Hence A is not closed under addition.

(ii) The multiplication of every two elements of the set are

$-1 * 0 = 0$; $-1 * 1 = -1$; $-1 * -1 = 1$.

$0 * -1 = 0$; $0 * 0 = 0$; $0 * 1 = 0$

$1 * -1 = -1$; $1 * 0 = 0$; $1 * 1 = 1$.

Since, each multiplication belongs to A hence A is closed under multiplication

Ques:- Consider the set $A = \{1, 3, 5, 7, 9, ...\}$, the set of odd +ve integers. Determine whether A is colsed under
(i) addition (ii) Multiplication

Sol^n (i) → No
(ii) → Yes

2. Associative Property:-

Consider a non-empty set A and a binary operation * on A. Then the operation * on A is associative if for every $a, b, c \in A$, we have

$$(a * b) * c = a * (b * c)$$

Ques:- Consider the binary operation * on Q, the set of rational numbers, defined by

$$a * b = a + b - ab \quad \forall \; a, b \in Q$$

Determine whether * is associative.

Sol^n Let us assume some elements $a, b, c \in Q$, then by definition

$$(a * b) * c = (a + b - ab) * c$$
$$= (a + b - ab) + c - (a + b - ab)c$$
$$= a + b - ab + c - ac - bc + abc$$
$$= a + b + c - ab - ac - bc + abc$$

Similarly

$$a * (b * c) = a * (b + c - bc)$$
$$= a + (b + c - bc) - a(b + c - bc)$$
$$= a + b + c - bc - ab - ac + abc$$
$$= a + b + c - ab - ac - bc + abc$$

Therefore, $(a * b) * c = a * (b * c)$

Hence * is associative.

Ques:- Consider the binary operation $*$ on R, the set of real numbers, defined by

$$a * b = a + b + ab \quad \forall\ a, b \in R$$

Determine whether $*$ is associative.

3. Commutative Property:-

Consider a non-empty set A and a binary operation $*$ on A. Then operation $*$ on A is Commutative if, for every $a, b \in A$, we have $a * b = b * a$

Ques:- Consider the binary operation $*$ on Q, the set of rational numbers, defined by

$$a * b = a^2 + b^2 \quad \forall\ a, b \in Q$$

Determine whether $*$ is commutative.

Sol^n Let us assume some elements $a, b \in Q$, then by definition

$$a * b = a^2 + b^2 = b * a$$

Hence $*$ is commutative.

Ques: Consider the binary operation $*$ and Q, the set of rational numbers defined by

$$a * b = \frac{ab}{2} \quad \forall\ a, b \in Q$$

Determine whether $*$ is (i) associative (ii) Commutative

Sol^n Let $a, b, c \in Q$, then by definition, we have

$$(a * b) * c = a * (b * c)$$

L.H.S $\quad (a * b) * c = \left(\frac{ab}{2}\right) * c$

$$= \frac{\left(\frac{ab}{2}\right) * c}{2}$$

$$= \frac{abc}{4}$$

Illy R.H.S $a * (b * c) = a * \left(\frac{bc}{2}\right)$

$$= \frac{a\left(\frac{bc}{2}\right)}{2} = \frac{abc}{4}$$

Therefore, $a * (b * c) = (a * b) * c$

Hence, $*$ is associative.

(ii) Let $a, b \in Q$, then we have

$$a * b = \frac{ab}{2} = b * a$$

Hence, $*$ is Commutative.

4. Identity:-

Consider a non-empty set A and a binary operation $*$ on A. Then the operation $*$ has an identity property if there exists an element e, in A such that

$$a * e \text{ (right identity)} = e * a \text{ (left ~~idemity~~ identity)}$$
$$= a \ \forall \ a \in A$$

Note:-

$+$ operation $= 0$ identity element

$*$ operation $= 1$ " "

Ques:- Consider the binary operation $*$ on I_+ the set of +ve integers defined by

$$a * b = \frac{ab}{2}$$

Determine the identity for the binary operation $*$, if exists.

Sol^n Let us assume that e be a +ve integer no., then

$$e * a = a \;,\; a \in I_+$$

$$\frac{ea}{2} = a \Rightarrow e = 2 \qquad \text{— (i)}$$

Similarly, $a * e = a \;,\; a \in I_+$

$$\frac{ae}{2} = a \quad \text{or} \quad e = 2 \qquad \text{— (ii)}$$

from (i) & (ii) for $e = 2$, we have $e * a = a * e = a$

Therefore, 2 is the identity element for $*$.

5. **Inverse:-** Consider a non-empty set A and a binary operation $*$ on A. Then operation $*$ has the inverse property if for each $a \in A$, there exists an element b in A such that

$a * b$ (right inverse) $= b * a$ (left inverse) $= e$, where b is called an inverse of a.

Note 1 inverse element $= -1$

-1 " " $= 1$

Ques:- Consider an algebraic system $(A, *)$, where $A = \{1, 3, 5, 7, 9, ...\}$ the set of all +ve odd integers and $*$ is a binary operations means multiplication. Determine whether $(A, *)$ is a semi-group.

Sol^n Semi group < Closure Property, Associative Property } satisfy

$$A = \{1, 3, 5, 7, 9, ----\}$$

(1) Closure Property :- The operation $*$ is an ~~associative~~ closed

operation because multiplication of two +ve odd integers is a +ve odd number.

(2) Associative Property: The operation $*$ is an associative operation on set A. Since for every $a, b, c \in A$, we have

$$(a * b) * c = a * (b * c)$$

e.g.: $1, 3, 5$ (a, b, c) $\Rightarrow (1 * 3) * 5 = 1 * (3 * 5)$

$$15 = 15$$

Hence, the algebraic system $(A, *)$ is a semigroup because both the properties are satisfied.

Ques: Consider the algebraic system $(\{0,1\}, *)$ where $*$ is a multiplication operation. Determine whether $(\{0,1\}, *)$ is a semi-group.

Soln (1) Closure Property :- The operation $*$ is a closed one on the give set since

$$0 * 0 = 0 ; \quad 0 * 1 = 0 ;$$

$$1 * 0 = 0 ; \quad 1 * 1 = 1 .$$

(2) Associative Property :- The operation $*$ is associative since we have

$$(a * b) * c = a * (b * c) \quad \forall \ a, b, c$$

Since, the algebraic system is closed and associative, Hence, it is a semi-group.

Ques:- Let $(A,*)$ be semi-group. Show that a,b,c in A, if $a*c = c*a$ and $b*c = c*b$ then $(a*b)*c = c*(a*b)$

$(a*b)*c = c*(a*b)$

Sol^n Take L.H.S

$= (a*b)*c \Rightarrow a*(b*c)$ [$\because *$ is associative]

$\Rightarrow a*(c*b)$ [Given]

$\Rightarrow (a*c)*b$ [$\because *$ is associative]

$= \cancel{a}*(c*a)*b$ [Given]

$= c*[a*b]$

Which is equal to R.H.S

Hence $(a*b)*c = c*(a*b)$

Subsemigroup:-

Consider a semigroup $(A,*)$ and let $B \subseteq A$. Then the system $(B,*)$ is called a subsemigroup, if the set B is closed under the operation $*$.

e.g.:- Consider a semigroup $(N,+)$, where N is the set of all natural numbers and $+$ is an addition operation. The algebraic system $(E,+)$ is a sub-semigroup of $(N,+)$ where E is a set of all +ve even integers.

Ques:- Consider an algebraic system $(N,+)$, where the set $N = \{0,1,2,3,---\}$ the set of Natural numbers and $+$ is an addition operation. Determine whether $(N,+)$ is a monoid.

Sol^n

Monoid — Closure Property, Associative Property, Identity Property $\}$ Satisfy.

Sol^n ① Closure Property :- The operation + is closed since sum to two natural numbers is a natural number.

② Associative Property :- The operation + is ~~closed since sum of~~ an associative since we have

$$(a+b)+c = a+(b+c) \quad \forall \ a,b,c \in N$$

③ Identity Property :- There exist an identity element in set N w.r.t the operation + . The element 0 is an identity element w.r.t the operation +.

Since, the operation + is a closed, associative and there exist an identity.

Hence, the algebraic system $(N,+)$ is a monoid.

Group (Inverse Property)

Let us consider an algebraic system $(G,*)$ where $*$ is a binary operation on G. Then the system $(G,*)$ is said to be a group if it satisfies following properties :-

(i) The operation $*$ is a closed operation

(ii) The operation $*$ is an associative operation

(iii) There exists an identity element w.r.t the operation $*$

(iv) for every $a \in G$, there exist an element $a^{-1} \in G$ such that $a^{-1} * a = a * a^{-1} = e$

e.g :- The algebraic system $(I,+)$, where I is the set of all integers and + is an addition operation, is a group. The element 0 is the identity element w.r.t the operation +. The inverse of ~~ele~~ every element $a \in I$ is $-a \in I$

Ques:- Determine whether the algebraic system $(Q,+)$ is a group when Q is the set of all rational numbers and $+$ is an addition operation.

Sol^n

Group →
- Closure Property
- Associative Property
- Identity "
- Inverse "

} satisfies

① **Closure Property:-** The set Q is closed under operation $+$. Since the addition of two rational number is a rational number.

② **Associative Property:-** The operation $+$ is associative since

$(a+b)+c = a+(b+c) \ \forall \ a,b,c \in Q$

③ **Identity:-** The element 0 is the identity element. Hence

$a+0 = 0+a = a \ \forall \ a \in Q$

④ **Inverse:-** The inverse of every element $a \in Q$ is $-a \in Q$.
Hence the inverse of every element exists.

Since, the algebraic system $(Q,+)$ satisfy all properties of a group. Hence $(Q,+)$ is a group.

Ques:- Consider an algebraic system $(Q,*)$ where Q is the set of rational numbers and $*$ is a binary operation defined by

$a*b = a+b-ab \ \forall \ a,b \in Q$

Determine whether $(Q,+)$ is a group.

Sol^n

Group →
- Closure Property
- Associative "
- Identity "
- Inverse "

} satisfy

Sol^n ① **Closure Property:-** Since the element $a, b \in Q$ for every $a, b \in Q$. Hence, the set Q is closed under the operation $*$.

② **Associative Property:-** Let us assume $a^*, b, c \in Q$, then we have

$$(a * b) * c = (a+b-ab) * c$$
$$= (a+b-ab) + c - (a+b-ab)c$$
$$= a+b+c-ab-ac-bc+abc$$

$$a * (b * c) = a * (b+c-bc)$$
$$= a+b+c-bc - a(b+c-bc)$$
$$= a+b+c-ab-bc-ac+abc$$

Therefore, $(a*b)*c = a*(b*c)$

$\therefore$ $*$ is associative.

③ **Identity:-** Let e is an identity element. Then we have

$a * e = a \quad \forall a \in Q$

$\therefore \quad a+e-ae = a \quad$ or $\quad e - ae = 0$

$= e(1-a) = 0 \quad$ or $\quad e = 0$

Similarly $\quad e * a = a \; \forall a \in Q$

Therefore, for $e=0$, we have $a * e = e * a = a$

Thus 0 is the identity element.

④ **Inverse:-** Let us assume an element $a \in Q$. Let a^{-1} is an inverse of a, where $a^{-1} \in Q$. Then we have

$$a * a^{-1} = 0$$
$$a + a^{-1} - aa^{-1} = 0$$
$$a^{-1}(1-a) = -a \Rightarrow a^{-1} = \frac{a}{a-1}$$

Now, $\frac{a}{a-1} \in Q$ if $a \neq 1/a$

Therefore, every element has inverse such that $a \neq 1$

Since, the algebraic system $(Q, *)$ satisfy all the properties of a group.

Hence, $(Q, *)$ is a group.

Finite and Infinite Group:-

→ A group $(G, *)$ is called a finite group if G is a finite set

→ A group $(G, *)$ is called an infinite group if G is an infinite set.

e.g: The group $(I, +)$ is an infinite group as the set I of integers is an infinite set.

e.g: The group $G = \{0, 1, 2, 3, 4, 5, 6, 7\}$ under multiplication modulo 8 ($a *_m b$ → non-ve integers) is a finite group as the set G is a finite set.

Addition Modulo (m):-

Denoted by $a +_m b$ → non-negative integer

Integers

$a +_m b = (r)$ → remainder.

→ when $a+b$ divided by m.

① $7 +_3 3 = \frac{10}{3} = 3(3) + (1)$ → remainder

② $8 +_3 11 = \frac{19}{3} = 3(6) + (1)$ → remainder

③ $8 +_3 10 = \frac{18}{3} = 3(6) + 0$ → "

④ $10 +_7 11 = \frac{21}{7} = 3(7) + 0$ → "

⑤ $10 +_7 7 = \frac{27}{7} = 21 + (6)$ → remainder