Unit :- V

Mathematical Induction

Let $P(n)$ be a statement involving the natural no. n
To Prove that $P(n)$ is true for all natural numbers, we proceed as :-

(I) Verify $P(n)$ is true for $n=1$

(II) Assume the result is true for $n = k \geq 1$

(III) Using (I) & (II) Prove that $P(k+1)$ is true

that is called mathematical induction

Ques:- Show that $1+2+3+\cdots+n = \frac{n(n+1)}{2}$

Sol^n Let $S(n) = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$ — (I)

For $n=1$

$$S(1) = 1 = \frac{1(1+1)}{2}$$

$$= 1 \Rightarrow S(n) \text{ is true for } n=1$$

Let $S(n)$ is true for $n=k$ i.e;

$$S(k) = 1+2+\cdots+k = \frac{k(k+1)}{2} \quad \text{— (II)}$$

Now, we have to show that $S(n)$ is true for $n=k+1$. Put $n=k+1$ into $S(n)$ then

$$S(k+1) = 1+2+\cdots+k+(k+1) = \frac{(k+1)(k+2)}{2}$$

Now, L.H.S for $S(k+1) = \underbrace{1+2+\cdots+k}+(k+1)$

$$\frac{k(k+1)}{2} + k+1 \quad \longrightarrow \text{by (II)}$$

$$(k+1)\left(\frac{k}{2}+1\right) \Rightarrow \frac{(k+1)(k+2)}{2}$$

$$L.H.S = R.H.S$$

therefore $S(n)$ is true for $n=k+1$

So, By Mathematical induction Principle $S(n)$ is true $\forall$ $n \in N$

Ques:- Show $1^3 + 2^3 + 3^3 + \dots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

Sol^n Let $P(n) = 1^3 + 2^3 + 3^3 + \dots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

for $n = 1$

$$P(1) = 1 = \left(\frac{1(1+1)}{2}\right)^2 \quad \text{—— (I)}$$

$$1 = 1 \qquad L.H.S = R.H.S$$

$P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$ i.e;

$$P(k) = 1^3 + 2^3 + 3^3 + \dots + k^3 = \left(\frac{k(k+1)}{2}\right)^2 \quad \text{—— (II)}$$

Now we have to show that $P(n)$ is true for $n = k+1$. Put $n = (k+1)$ into $P(n)$ then

$$P(k+1) = 1^3 + 2^3 + 3^3 + \dots + k^3 + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2$$

Now. L.H.S for

$$P(k+1) = \underbrace{1^3 + 2^3 + 3^3 + \dots + k^3}_{} + (k+1)^3$$

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$= (k+1)^2\left(\frac{k^2}{4} + k+1\right) \Rightarrow (k+1)^2\left(\frac{k^2+4k+4}{4}\right)$$

$$= \frac{(k+1)^2 (k+2)^2}{4} \Rightarrow \left(\frac{(k+1)(k+2)}{2}\right)^2$$

$$L.H.S = R.H.S$$

therefor $P(n)$ is true for $n = k+1$

So, By Mathematical induction Principle $P(n)$ is true $\forall n \in N$.

Ques: Show that for all $n \in N$, $7^n - 3^n$ is divisible by 4.

Sol^n Let $P(n) = 7^n - 3^n$

for $n = 1$ $P(1) = 7^1 - 3^1 = 4 =$ multiple of 4 —— (I)

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Let $S(n)$ is true for $n=k$ i.e;

$$S(k) = 7^k - 3^k = 4m \quad \text{— (ii)}$$

Now, we have to show that $S(n)$ is true for $n=k+1$. Put $n=k+1$ into $S(n)$ then

$$S(k+1) = 7^{k+1} - 3^{k+1}$$

$$= 7 \cdot 7^k - 3^{k+1}$$

$$= 7(7^k - 3^k) + 7\,3^k - 3^{k+1}$$

$$= 7(4m) + 3^k(7-3) \quad \text{— by (ii)}$$

$$= 7(4m) + 3^k\, 4 \Rightarrow 4(7m + 3^k)$$

$m, k \in N$

therefor $S(n)$ is true for $n=k+1$

So, By Mathematical induction Principle $S(n)$ is true for all $n \in N$.

Recurrence Relations

Recursive formula :- A formula which defines any terms of a sequence in terms of any number of its previous or which express any term of a sequence as a f^n of its previous terms is called Recursive and the relation is called Recurrence Relation.

e.g.:- 1 Fibonacci Sequence

$$\langle \underset{a_0}{1}, \underset{a_1}{1}, \underset{a_2}{2}, \underset{a_3}{3}, \underset{a_4}{5}, \underset{a_5}{8}, \ldots \rangle$$

$$\boxed{a_n = a_{n-1} + a_{n-2}} \quad , n \geq 2$$

Let $n = 2$

$$a_2 = a_1 + a_0$$
$$= 1 + 1$$
$$2 = 2$$

2 Take a sequence of no.'s

$$\underset{a_1}{2}, \underset{a_2}{4}, \underset{a_3}{6}, \underset{a_4}{8}, \underset{a_5}{10}, \underset{a_6}{12}, \ldots$$

$$\boxed{a_n = a_{n-1} + a_{n-2}} \quad n > 2$$

Let $n = 3$

$$a_3 = a_{3-1} + a_{3-2}$$
$$a_3 = a_2 + a_1$$
$$6 = 4 + 2$$

Illy

$$\underset{a_0}{2}, \underset{a_1}{4}, \underset{a_2}{6}, \underset{a_3}{8}, \underset{a_4}{10}, \ldots$$

$$a_n = a_{n-1} + a_{n-2} \quad n \geq 2$$

$n = 2$

$$a_2 = a_1 + a_0$$
$$6 = 4 + 2$$
$$= 6$$

The equation $a_r + 3a_{r-1} = 0$ is of first order recurrence relation

The eqⁿ $7f(x) + f(x+1) + 5f(x+2) = K(x)$ is a second order Difference eqⁿ or recurrence relation.

Degree :-

The degree of a recurrence relation or Difference equation is defined to be the highest power of $f(x)$ or a_r or y_k.

e.g :-

① $y_{k+3}^3 + 2y_{k+2}^2 + 2y_{k+1} = 0$ ③ $a_r^4 + a_{r-1}^3 + 5a_{r-2}^2 + 3a_{r-3} = 0$

② $y_{k+4} + y_{k+1} + y_k = 0$ ④ $f(x+2h) + 4f(x+h) + 2f(x) = 0$

The degree of ① is 3, degree of ② is 1

the degree of ③ is 4, " " ④ is 1

Linear :- The recurrence relation is called linear, if it is of degree one

Quadratic :-

If it is called Quadratic if its degree is 2.

Homogeneous :-

A recurrence relation is called Homogeneous, if it contains no terms that depends only on the variable n

Non-homogeneous :-

The recurrence relation which is not homogeneous is called non-homogeneous or inhomogeneous

OR

If recurrence relation contain variable n

e.g: $a_{n-1}^2 + n^3$

This formula is called Recursive formula and relation is called recurrence relation, some times it is called Difference Equation. The beginning equation $a_1 = 2$ is called Initial Condition or Boundary Condⁿ for the sequence.

Linear Recurrence relation with Constant Coefficients:-

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \cdots + C_k a_{r-k} = f(r)$$

where C_i's are Constant

① $f(r) = 0 \longrightarrow$ homogeneous eqⁿ

② $f(r) = r^2, r^2+1 \longrightarrow$ non-homogeneous eqⁿ.

Order:- The order of recurrence relation or difference eqⁿ is defined to be difference between highest and lowest subscripts of $f(x)$, a_r or y_n (numeric fⁿ)

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \cdots + C_k a_{r-k} = f(r)$$

order $r - (r-k)$

$= r - r + k \Rightarrow k$ order

e.g: ① $a_r + 2a_{r-1} = r^2$

order $\rightarrow r-(r-1)$

$= r - r + 1$

$= 1$

order $-$ 1 & non-homogeneous eqⁿ

② $a_r + 3a_{r-2} = 0$

order $\rightarrow$ $r-(r-2)$

$r - r + 2$

order $\rightarrow 2$ & homogeneous eqⁿ

Solving a Recurrence Relation

- Iteration Method
- Characteristic Roots
- Generating functions

Iterative Method (IM)

e.g.: $T(P) = T(P-1) + 3 \;, \; P \geq 1$

$T(0) = 2$

find $T(3)$ by use of I.M

Sol^n $T(P) = T(P-1) + 3$ — (I)

when $P = 1$ in eq^n (I)

$T(1) = T(1-1) + 3$

$= T(0) + 3$

$= 2 + 3$

$T(1) = 5$

Put $P = 2$ in eq^n (I)

$T(2) = T(1) + 3$

$= 5 + 3$

$= 8$

Put $P = 3$ in eq^n (I)

$T(3) = T(2) + 3$

$= 8 + 3$

$= 11$

Put $P = 3$

$T(3) = 2 + 9 \Rightarrow 11$

find $T(100)$ by use of I.M

Sol^n $T(P) = T(P-1) + 3$ — (I)

Now, Put $P = P-1$, $P-2$ in eq^n (I)

$T(P-1) = T(P-2) + 3$
$T(P-2) = T(P-3) + 3$ — (II)

Substitute $T(P-1)$ in eq^n (I)

$T(P) = T(P-2) + 3 + 3$

$T(P) = T(P-2) + 6$ — (III)

$T(P) = T(P-3) + 3 + 6$

$= T(P-3) + 9$ — (IV)

$$\boxed{\begin{aligned} T(P) &= T(P-1) + 1\cdot 3 \quad \text{— (I)} \\ T(P) &= T(P-2) + 2\cdot 3 \quad \text{— (III)} \\ T(P) &= T(P-3) + 3\cdot 3 \quad \text{— (IV)} \end{aligned}}$$

$T(P) = T(P-K) + K\cdot 3$

General Term

Put $K = P$

$T(P) = T(P-P) + P\cdot 3$

$= T(0) + P\cdot 3$

$T(P) = 2 + 3P \rightarrow$ Closed form

Put $P = 100$

$T(100) = 2 + 300 \Rightarrow 302$

Characteristic Root Method

Step I Write Characteristic Equation (Put $a_n = r^n$)

Step II Find the roots and let roots are $\alpha_1, \alpha_2 \ldots \alpha_k$

Step III

	Roots	General sol^n
1.	Roots are distinct $\alpha_1, \alpha_2, \alpha_3 \ldots, \alpha_k$	$a_n = c_1\alpha_1^n + c_2\alpha_2^n + c_3\alpha_3^n + \ldots + c_n\alpha_k^n$
2.	If two roots are same $\alpha_1 = \alpha_2, \alpha_3, \alpha_4$	$a_n = (c_1 + c_2 n)\alpha_1^n + c_3\alpha_3^n + c_4\alpha_4^n$
3.	If two roots are Complex $\alpha \pm i\beta$	$a_n = \rho^n(c_1 \cos n\theta + c_2 \sin n\theta)$ $\alpha + i\beta = \rho e^{i\theta} = \rho(\cos\theta + i\sin\theta)$ $\alpha - i\beta = \rho e^{-i\theta} = \rho(\cos\theta - i\sin\theta)$ $\rho^2 = \alpha^2 + \beta^2 \Rightarrow \alpha = \sqrt{\alpha^2+\beta^2}$ $\tan\theta = \beta/\alpha \Rightarrow \theta = \tan^{-1}\beta/\alpha$
4.	If two roots are Complex $\alpha \pm i\beta, \alpha \pm i\beta$	$a_n = \rho^n[(c_1 + c_2 n)\cos n\theta + (c_3 + c_4 n)\sin n\theta]$ $\rho^2 = \alpha^2 + \beta^2 \Rightarrow \rho = \sqrt{\alpha^2+\beta^2}$ $\tan\theta = \beta/\alpha$ $\theta = \tan^{-1}\beta/\alpha$

Ques. 1 $a_n - 6a_{n-1} + 8a_{n-2} = 0$ solve this by Char. root method. $a_0 = 4$ & $a_1 = 10$

Sol^n

$$a_n - 6a_{n-1} + 8a_{n-2} = 0$$

Put $a_n = r^n$

$$r^n - 6r^{n-1} + 8r^{n-2} = 0$$

Divide both sides by r^{n-2}

$$r^2 - 6r + 8 = 0$$

$$r^2 - 4r - 2r + 8 = 0$$

$$r = 2, 4$$

$$\boxed{a_n = C_1 2^n + C_2 4^n} \rightarrow \text{General sol}^n \text{ — (I)}$$

Put $n = 0$ in eq^n (I)

$$a_0 = C_1 + C_2 \Rightarrow 4 = C_1 + C_2 \text{ — (II)}$$

Put $n = 1$

$$a_1 = C_1 2 + C_2 4 \Rightarrow 10 = 2C_1 + 4C_2$$

$$5 = C_1 + 2C_2 \text{ — (III)}$$

By solving (II) & (III) $\boxed{C_1 = 3}$, $\boxed{C_2 = 1}$

Substitute C_1 & C_2 values in eq^n (I)

$$a_n = 3\,2^n + 4^n$$

Ques. $a_n - 2a_{n-1} + a_{n-2} = 0$, $a_0 = 2$, $a_1 = 5$

Put $a_n = r^n$

$$r^n - 2r^{n-1} + r^{n-2} = 0$$

$$r^2 - 2r + 1 = 0 \Rightarrow r = 1, 1$$

$$a_n = (C_1 + C_2 n) 1^n \text{ — (I)}$$

Put $n = 0$

$a_0 = C_1 + C_2 \cdot 0 \Rightarrow \boxed{C_1 = 2}$

Put $n = 1 \Rightarrow a_1 = C_1 + C_2 \Rightarrow 5 = C_1 + C_2 \Rightarrow \boxed{C_2 = 3}$

Substitute a_1 & a_2 in eqn (1).

$a_n = 2 + 3 \cdot n$

Ques: Solve the following recurrence relations

(1) $a_r - 4a_{r-1} + 13a_{r-2} = 0 \Rightarrow a_r = (13)^{r/2}\{C_1 \cos r(\tan^{-1} 3/2) + C_2 \sin r(\tan^{-1} 3/2)\}$

(2) $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with initial condn's

$a_0 = 1, \; a_1 = -2, \; a_2 = -1$

$a_n = (1 + 3n - 2n^2)(-1)^n$

(3) $a_{r+1} - 1.5\, a_r = 0 \quad r \geq 0 \qquad a_r = C_1 (1.5)^r$

(4) $a_r = 5a_{r-1} - 6a_{r-2} \;, r \geq 2 \;, a_0 = a_1 = 3$

$a_r = 6(2)^r - 3(3^r)$

(5) $a_r + 6a_{r-1} + 12a_{r-2} + 8a_{r-3} = 0$

$a_r = (C_1 + C_2 r + C_3 r^2)(-2)^r$

Characteristic root Method in Non-homogeneous f^n

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = f(n)$$

General sol^n (G.S) is $a_n = a_n^h + a_n^p$

a_n^h → homogeneous f^n

a_n^p → Particular f^n

	$f(n)$	a_n^p
Case I	Constant (C_0)	d_0
	$C_0 + C_1 n$	$d_0 + d_1 n$
	$C_0 + C_1 n + C_2 n^2$	$d_0 + d_1 n + d_2 n^2$
	$\vdots$	
	$C_0 + C_1 n + C_2 n^2 + \cdots + C_m n^n$	$d_0 + d_1 n + d_2 n^2 + \cdots + d_m n^n$
Case II :-	$C\, r^n$	$d\, r^n$
	IIIly $\cancel{r^n}$	

For d_0:
- Find d_0 by Putting $a_n = d_0$ in given eq^n
- If fail ① then Put $a_n = n \cdot d_0$ " "
- Again " ① " " $a_n = n^2 d_0$ " "
- $\vdots$
- $a_n = n^3 d_0$ " "

\# If homogeneous sol^n contains a term containing r^k then $a_n = n\,(d \cdot r^k)$

IIIly ~~root~~ homogeneous sol^n $r^k \cdot r^k$

$$a_n = n^2\,(d \cdot r^k)$$

$A, B, C, D, \rightarrow$

(2)

Ques:- Find the General sol^n of the f^n $a_n - 2a_{n-1} + a_{n-2} = 1$ by characteristic root Method also given $a_0 = 2$ & $a_1 = 5$

Sol^n

$$a_n - 2a_{n-1} + a_{n-2} = 1 \quad \text{—— (I)}$$

$$G.S = a_n = a_n^h + a_n^p \quad \text{—— (II)}$$

Homogeneous sol^n

$$a_n - 2a_{n-1} + a_{n-2} = 0$$

Put $a_n = r^n$

$$r^n - 2r^{n-1} + r^{n-2} = 0$$

$$r^2 - 2r + 1 = 0 \Rightarrow r = 1, 1$$

$$a_n^h = (c_1 + c_2 n)\, 1^n$$

P.S

Put $a_n = d_0$ in eq^n (I)

$$\Rightarrow d_0 - 2d_0 + d_0 = 1$$

$$0 \neq 1$$

fail case

Put $a_n = n d_0$ in eq^n (I)

$$\Rightarrow n d_0 - 2(n-1) d_0 + (n-2) d_0 = 1$$

$$\Rightarrow n d_0 - 2n d_0 + 2d_0 + n d_0 - 2d_0 = 1$$

$$\Rightarrow 0 \neq 1$$

Again fail

Put $a_n = n^2 d_0$ in eq^n (I)

$$n^2 d_0 - 2(n-1)^2 d_0 + (n-2)^2 d_0 = 1$$

$$n^2 d_0 - 2(n^2+1-2n) d_0 + (n^2+4-4) d_0 = 1$$

$$d_0 = 1/2$$

$$\Rightarrow a_n^p = \frac{1}{2} n^2$$

$a_n^p =$

$G.S = a_n = (c_1 + c_2 n)\, 1^n + \frac{1}{2} n^2$ —— (III)

Put $n = 0$

$a_0 = c_1 \Rightarrow \boxed{c_1 = 2}$

Put $n = 1$

$a_1 = c_1 + c_2 + \frac{1}{2} \Rightarrow \boxed{c_2 = 5/2}$

Substitute c_1 & c_2 in eqn (III)

$$\boxed{G.S = a_n = \left(2 + \frac{5}{2} n\right) 1^n + \frac{1}{2} n^2}$$

Ques: find $a_n - 8a_{n-1} = 5 + 14n$ by characteristic root Method

Soln $a_n - 8a_{n-1} = 5 + 14n$ —— (I)

For Homogeneous soln

$a_n - 8a_{n-1} = 0$

Put $a_n = r^n$

$r^n - 8r^{n-1} = 0 \Rightarrow r - 8 = 0 \Rightarrow r = 8$

$a_n^h = c_1 8^n$

P.S Let $a_n^p = d_0 + n d_1$ —— (II)

Substitute in eqn (I)

$d_0 + n d_1 - 8(d_0 + (n-1)d_1) = 5 + 14n$

$d_0 + n d_1 - 8d_0 - 8nd_1 + 8d_1 = 5 + 14n$

$-7d_0 + 8d_1 - 7nd_1 = 5 + 14n$

Comparing both sides

$-7d_1 = 14$

$\boxed{d_1 = -2}$

$-7d_0 + 8d_1 = 5$

$-7d_0 - 16 = 5$

$-7d_0 = 21$

$\boxed{d_0 = -3}$

Substitute in eq^n (II) $d_0 = -3$ & $d_1 = -2$

$$a_n^{p} = -3 - 2n$$

$$G.S = a_n = C_1 8^n - 3 - 2n$$

Que:- Find the G.S $a_{n+2} + 3a_{n+1} + 2a_n = 3^n, n \geq 0$
$a_0 = 0, a_1 = 1$ by

Characteristic Root Method.

Sol^n $a_{n+2} + 3a_{n+1} + 2a_n = 3^n$ —— (I)

$$G.S = a_n = a_n^h + a_n^p \quad \text{—— (II)}$$

Homogeneous Sol^n

$$a_{n+2} + 3a_{n+1} + 2a_n = 0$$

Put $a_n = r^n$

$$r^{n+2} + 3r^{n+1} + 2r^n = 0$$

$$r^2 + 3r + 2 = 0 \Rightarrow r = -1, -2$$

$$a_n^h = C_1(-1)^n + (-2)^n C_2$$

(III)

P.S $a_n = d_0 3^n$ substitute in eq^n (I)

$$d_0 3^{n+2} + 3 d_0 3^{n+1} + 2 d_0 3^n = 3^n$$

$$3^n (d_0 3^2 + 3 d_0 3 + 2 d_0) = 3^n$$

$$d_0 9 + 9 d_0 + 2 d_0 = 1$$

$$\boxed{d_0 = 1/20}$$

Substitute d_0 value in eq^n (III)

$$a_n = \frac{1}{20} 3^n$$

$$G.S \Rightarrow a_n = C_1(-1)^n + (-2)^n C_2 + \frac{1}{20} 3^n \quad \text{—— (A)}$$

Put $n=0$

$$a_0 = c_1 + c_2 = \frac{1}{20} \Rightarrow 0 + c_1 + c_2 + \frac{1}{20}$$

$$\boxed{c_1 + c_2 = -\frac{1}{20}} \quad \text{(IV)}$$

Put $n=1$

$$a_1 = c_1(-1)^1 + (-2)^1 c_2 + \frac{3}{20} \Rightarrow 1 = -c_1 - 2c_2 + \frac{3}{20}$$

$$-c_1 - 2c_2 = 1 - \frac{3}{20}$$

$$\boxed{c_1 + 2c_2 = -\frac{17}{20}} \quad \text{(V)}$$

on solving (IV) & (V) then we get

$$c_1 = \frac{3}{4} \quad \& \quad c_2 = -\frac{4}{5}$$

Substitute c_1 & c_2 values in eq^n (A)

$$a_n = \frac{3}{4}(-1)^n + (-2)^n\left(-\frac{4}{5}\right) + \frac{1}{20}3^n$$

Que: Find the G.S of $a_n - 8a_{n-1} = 5 \cdot 2^n$ by Characteristic root Method.

Sol^n $\quad a_n - 8a_{n-1} = 5\,2^n$ — (1)

Homogeneous sol^n

$$a_n - 8a_{n-1} = 0 \Rightarrow \text{Put } a_n = r^n$$

$$r^n - 8r^{n-1} = 0 \Rightarrow r - 8 = 0 \Rightarrow r = 8$$

$$a_n^h = c_1 8^n$$

P.S $\quad$ Put $a_n = d\,2^n$ in eq^n (1)

$$d\,2^n - 8\,d\,2^{n-1} = 5\,2^n$$

$$\cancel{2^n}\left(d - \frac{\cancel{8}^4 d}{\cancel{2}}\right) = 5\cancel{2^n} \Rightarrow -3d = 5$$

$$d = -5/3$$

therefore $a_n^p = -\frac{5}{3}2^n$

$$G.S = c_1 8^n - \frac{5}{3}2^n$$

Generating Function

let $\{A\}$ be any sequence with terms $a_0, a_1, a_2 ---$ the G.F $G(A,Z)$ of a sequence A is infinite series

$$\boxed{G(A,Z) = \sum_{n=0}^{\infty} a_n z^n}$$

$$= a_0 + a_1 z + a_2 z^2 + ---- \infty$$

Ques:- ① $a_n = c \;, n \geq 0$

$$G(s,z) = \sum_{n=0}^{\infty} a_n z^n$$

$$= \sum_{n=0}^{\infty} c\, z^n$$

$$= c \sum_{n=0}^{\infty} z^n$$

$$= c\left(1 + z + z^2 + --- \infty\right)$$

with $a = 1$, $r = z$

$$S_\infty = \frac{a}{1-r} \quad r<1$$

$$= c \cdot \frac{1}{1-z}$$

$$= \frac{c}{1-z}$$

② $a_n = b^n \quad n \geq 0$

$$G(s,z) = \sum_{n=0}^{\infty} a_n z^n$$

$$= \sum_{n=0}^{\infty} b^n z^n$$

$$= \sum_{n=0}^{\infty} (bz)^n$$

$$= 1 + bz + (bz)^2 + ---$$

G.P with $a = 1$, $r = bz$

$$= \frac{1}{1-bz}$$

③ $a_n = c \cdot b^n \quad n \geq 0$

$$G(s,z) = \sum_{n=0}^{\infty} a_n z^n$$

$$= \sum_{n=0}^{\infty} c\, b^n z^n$$

$$= c \sum_{n=0}^{\infty} b^n z^n$$

$$= c\left[1 + bz + (bz)^2 + ---\right]$$

$$= c\left[\frac{1}{1-bz}\right]$$

$$= \frac{c}{1-bz}$$

④ $a_n = n \;, n \geq 0$

$$G(s,z) = \sum_{n=0}^{\infty} a_n z^n$$

$$= \sum_{n=0}^{\infty} n\, z^n$$

$$= 0 + z + 2z^2 + 3z^3 + --$$

$$= z\left(1 + 2z + 3z^2 + ---\right)$$

$$= z(1-z)^{-2} \text{ by using Binomial theorem} \Rightarrow \frac{z}{(1-z)^2}$$

IIIly $(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + --$

General Binomial expansion.

Method to Solve Recurrence Relation by using Generating function:-

Method

Let given R.R is $C_0 a_n + C_1 a_{n-1} + \cdots + C_k a_{n-k} = 0$ for $n \geq k$

① Multiply both sides by z^n / z^r and sum up from $n=k$ to ∞

② Write each term in form of $G(A, z)$

③ Solve $G(A, z)$, then using standard Generating function sequence can be found

① $a_n = c$	② $a_n = b^n$	③ $a_n = cb^n$	④ $a_n = n$
$G(s,z) = \frac{c}{1-z}$	$G(s,z) = \frac{1}{1-bz}$	$G(s,z) = \frac{c}{1-bz}$	$G(s,z) = \frac{z}{(1-z)^2}$

$$G(A,z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$$

Ques:- Find G.F and sequence of R.R $a_n + 2a_{n-1} = 0$ with $a_0 = 5$

Sol^n ①

$$a_n z^n + 2a_{n-1} z^n = 0 \cdot z^n$$

$$a_n z^n + 2a_{n-1} z^n = 0$$

$$\sum_{n=1}^{\infty} a_n z^n + 2\sum_{n=1}^{\infty} a_{n-1} z^n = 0$$

$$\sum_{n=0}^{\infty} a_n z^n - a_0 + 2\sum_{n=\phi}^{\infty} a_{n-1} z^{n-1} z = 0$$

$$\sum_{n=0}^{\infty} a_n z^n - a_0 + 2z\sum_{n=1}^{\infty} a_{n-1} z^{n-1} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n z^n - a_0 + 2z \sum_{n=0}^{\infty} a_n z^{n-1}$$

$$\Rightarrow G(A,z) - 5 + 2z\, G(A,z) = 0$$

$$G(A,z)\,((1+2z)) = 5$$

$$G(A,z) = \frac{5}{1+2z}$$

$$\boxed{G(A,z) = \frac{5}{1-(-2z)}} \rightarrow \text{Generating } f^n$$

$$\boxed{a_n = 5(-2)^n} \rightarrow \text{Corresponding sequence.}$$

Numeric function for the closed form expression of generating f^n

$$A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots$$

Generating function	Numeric f^n
$A(z) = \frac{1}{1-az}$	a^n
$A(z) = \frac{1}{(1-az)^2}$	$(n+1)\, a^r$
$A(z) = \frac{1}{(1-z)^2}$	$(n+1)$
$A(z) = \frac{z}{(1-z)^2}$	n
$A(z) = \frac{az}{(1-az)^2}$	$n\, a^n$
$A(z) = e^z$	$\frac{1}{n!}$
$A(z) = (1+z)^n$	$\begin{cases} {}^nC_r & 0 \leq r \leq n \\ 0 & r > n \end{cases}$

Ques: Solve recurrence relation by generating f^n

$$a_r - 2a_{r-1} - 3a_{r-2} = 0 \quad , r \geq 2$$

$$a_0 = 3, \quad a_1 = 1$$

$$a_r z^r - 2a_{r-1} z^r - 3a_{r-2} z^r = 0$$

$$a_r z^r - 2a_{r-1} z^{r-1} z - 3a_{r-2} z^{r-2} z^2 = 0$$

$$\sum_{r=2}^{\infty} a_r z^r - 2z\sum_{r=2}^{\infty} a_{r-1} z^{r-1} - 3z^2\sum_{r=2}^{\infty} a_{r-2} z^{r-2} = 0$$

① $\sum_{r=2}^{\infty} a_r z^r = a_2 z^2 + a_3 z^3 + a_4 z^4 + \dots$

$$A(z) = a_0 + a_1 z + \underbrace{a_2 z^2 + a_3 z^3 + \dots}_{\sum_{r=2}^{\infty} a_r z^r}$$

$$G(A,z) = A(z) \qquad a_0 + a_1 z$$

$$G(A,z) - a_0 - a_1 z - 2z\left(G(A,z) - a_0\right) - 3z^2\left(G(A,z)\right) = 0$$

$$G(A,z)\left[1 - 2z - 3z^2\right] = a_0 + a_1 z - 2z a_0$$

$$G(A,z) = \frac{a_0 + a_1 z - 2z a_0}{(1 - 2z - 3z^2)}$$

$$G(A,z) = \frac{3 + z - 6z}{(1-3z)(1+z)}$$

$$= \frac{3 - 5z}{(1-3z)(1+z)}$$

$$G(A,z) = \frac{1}{1-3z} + \frac{2}{1+z}$$ (By Partial fraction)

$$\boxed{a_r = A(z) = 3^r + 2(-1)^r}$$

Unit-5

Chapter 14

Combinatories

14.1 Counting Techniques

14.1.1 Permutation

The different arrangement which can be made out of a given number of things by taking some or all at a time, are called ***Permutations***.

For example, consider arranging the digits 1, 2 and 3. The possible arrangements as follows:

123, 132, 231, 213, 312, 321.

These are 6 permutations. Here the same three digits 1, 2 and 3 have been used but the number changes when the order of digits is changed.

Thus, forming numbers with given digits means arranging the digits and hence it is the problem of permutation.

► **Notations**

Let r and n be positive integers such that $1 \le r \le n$.

Then, the number of all permutations of n things taking r at a time is denoted by $P(n, r)$ or ${}^nP_r$

Theorem 1: *Let $1 \le r \le n$. Then the number of all permutations of n dissimilar things taken r at a time is given by*

$$ {}^nP_r = n(n-1)(n-2)\ldots\ldots(n-r+1) = \frac{\lfloor n}{\lfloor n-r}. $$

Theorem 2: *The number of all permutations of n different things taken all at a time is given by ${}^nP_n = \lfloor n$.*

Proof: We have, ${}^nP_r = \frac{\lfloor n}{\lfloor n-r}$ Put $r = n$, we find ${}^nP_n = \frac{\lfloor n}{\lfloor n-n} = \frac{\lfloor n}{\lfloor 0} = \frac{\lfloor n}{1} = \lfloor n$.

Theorem 3: *Prove that $\lfloor 0 = 1$.*

Proof: We have,

$$ {}^nP_r = \frac{\lfloor n}{\lfloor n-r} \qquad \text{[put } n = r\text{]} $$

$$ \Rightarrow \quad {}^nP_n = \frac{\lfloor n}{\lfloor 0} \qquad ({}^nP_n = \lfloor n) $$

$$ \Rightarrow \quad \lfloor n = \frac{\lfloor n}{\lfloor 0} \quad \Rightarrow \quad \lfloor 0 = \frac{\lfloor n}{\lfloor n} = 1. $$

Example 1: *Evaluate the following:*

(i) $^{12}P_4$ (ii) $^{6}P_4$ (iii) $^{8}P_8$ (iv) $^{30}P_2$

Solution: (i) $^{12}P_4 = \dfrac{\lfloor 12}{\lfloor 12-4} = \dfrac{\lfloor 12}{\lfloor 8} = \dfrac{12\times 11\times 10\times 9\times \lfloor 8}{\lfloor 8} = 11{,}880$

(ii) $^{6}P_4 = \dfrac{\lfloor 6}{\lfloor 6-4} = \dfrac{\lfloor 6}{\lfloor 2} = \dfrac{6\times 5\times 4\times 3\times 2}{2} = 360$ or $^{6}P_4 = 6\times 5\times 4\times 3 = 360$

(iii) $^{8}P_8 = \lfloor 8 = 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1 = 40{,}320$

(iv) $^{30}P_2 = \dfrac{\lfloor 30}{\lfloor 30-2} = \dfrac{\lfloor 30}{\lfloor 28} = \dfrac{30\times 29\times \lfloor 28}{\lfloor 28} = 870$

Example 2: *If* $^{20}P_r = 6{,}840$, *find r.*

Solution: We have, $^{20}P_r = 6{,}840$

$$\Rightarrow \quad \frac{\lfloor 20}{\lfloor 20-r} = 6{,}840$$

$$\Rightarrow \quad \frac{\lfloor 20}{\lfloor 20-r} = \frac{20\times 19\times 18\times \lfloor 17}{\lfloor 17}$$

$$\Rightarrow \quad \frac{\lfloor 20}{\lfloor 20-r} = \frac{\lfloor 20}{\lfloor 17} \quad \Rightarrow \quad 20-r = 17 \text{ or } r = 3$$

Example 3: *If* $^{n}P_4 = 20\times {}^{n}P_2$, *find n.*

Solution: We have, $^{n}P_4 = 20\times {}^{n}P_2$

$$\Rightarrow \quad n(n-1)(n-2)(n-3) = 20n(n-1)$$

$$(n-2)(n-3) = 20 \text{ or } n^2 - 5n - 14 = 0 \text{ or } (n-7)(n+2) = 0 \text{ or } n = 7 \quad (\text{as } n \neq -2)$$

Example 4: $^{(n+5)}P_{n+1} = \dfrac{11(n-1)}{2} \times {}^{(n+3)}P_n$, *find n.*

Solution: We have,

$$^{n+5}P_{n+1} = \frac{11(n-1)}{2} \times {}^{(n+3)}P_n$$

$$\Rightarrow \quad \frac{\lfloor n+5}{\lfloor (n+5)-(n+1)} = \frac{11(n-1)}{2}\,\frac{\lfloor n+3}{\lfloor n+3-n}$$

$$\Rightarrow \quad \frac{(n+5)(n+4)\lfloor n+3}{4\times \lfloor 3} = \frac{11(n-1)}{2} \times \frac{\lfloor n+3}{\lfloor 3}$$

$$\Rightarrow \quad (n+5)(n+4) = 22(n-1)$$

$$\Rightarrow \quad n^2 + 9n + 20 = 22n - 22$$

$$\Rightarrow \quad n^2 - 13n + 42 = 0 \text{ or } (n-6)(n-7) = 0 \text{ or } n = 6, 7.$$

Example 5: *If ${}^nP_4 = 2 \times {}^5P_3$, find n.*

Solution:

$$^nP_4 = 2 \times {}^5P_3$$

$$\Rightarrow \quad n(n-1)(n-2)(n-3) = 2 \times 5 \times 4 \times 3$$

$$\Rightarrow \quad n(n-1)(n-2)(n-3) = 120$$

$$\Rightarrow \quad (n^2 - 3n)(n^2 - 3n + 2) = 120$$

$\Rightarrow m(m+2) = 120$ where $m = n^2 - 3n$

$$\Rightarrow \quad m^2 + 2m - 120 = 0$$

$$\Rightarrow \quad (m+12)(m-10) = 0$$

$$\Rightarrow \quad m = -12, \text{ or } m = 10$$

$$\Rightarrow \quad n^2 - 3n = -12 \quad \text{or} \quad n^2 - 3n = 10$$

$$\Rightarrow \quad n^2 - 3n + 12 = 0 \quad \text{or} \quad n^2 - 3n - 10 = 0$$

$$\Rightarrow \quad n = \frac{3 \pm \sqrt{9 - 48}}{2} \quad \text{or} \quad (n-5)(n+2) = 0$$

$$\Rightarrow \quad n = \frac{3 \pm i\sqrt{39}}{2} \quad \text{or} \quad n = 5 \quad \text{or } n = -2$$

$\Rightarrow \quad n = 5$ [neglecting negative and imaginary value]

Example 6: *Find n if ${}^9P_5 + 5 \cdot {}^9P_4 = {}^{10}P_n$.*

Solution: We have, ${}^9P_5 + 5 \cdot {}^9P_4 = {}^{10}P_n$

$$\Rightarrow \quad \frac{\lfloor 9}{\lfloor 9-5} + 5 \cdot \frac{\lfloor 9}{\lfloor 9-4} = \frac{\lfloor 10}{\lfloor 10-n}$$

$$\Rightarrow \quad \frac{\lfloor 9}{\lfloor 4} + 5 \cdot \frac{\lfloor 9}{\lfloor 5} = \frac{\lfloor 10}{\lfloor 10-n}$$

$$\Rightarrow \quad \frac{\lfloor 9}{\lfloor 4} + \frac{\lfloor 9}{\lfloor 4} = \frac{\lfloor 10}{\lfloor 10-n}$$

$$\Rightarrow \quad 2 \times \frac{\lfloor 9}{\lfloor 4} = \frac{10 \times \lfloor 9}{\lfloor (10-n)}$$

$$\Rightarrow \quad 5 \times \lfloor 4 = \lfloor 10-n$$

$$\Rightarrow \quad \lfloor 10-n = \lfloor 5 \quad \Rightarrow \quad 10 - n = 5 \quad \Rightarrow \quad n = 5.$$

Example 7: *Prove that*

$$^nP_r = {}^{n-1}P_r + r \cdot {}^{(n-1)}P_{r-1}$$

Solution: R.H.S. $= {}^{(n-1)}P_r + r\cdot {}^{(n-1)}P_{r-1} = \dfrac{\lfloor\underline{n-1}}{\lfloor\underline{n-1-r}} + r\cdot\dfrac{\lfloor\underline{n-1}}{\lfloor\underline{(n-1)-(r-1)}} = \lfloor\underline{n-1}\left[\dfrac{1}{\lfloor\underline{n-r-1}} + \dfrac{r}{\lfloor\underline{n-r}}\right]$

$$= \lfloor\underline{n-1}\left[\frac{1}{\lfloor\underline{n-r-1}} + \frac{r}{(n-r)\lfloor\underline{n-r-1}}\right] = \lfloor\underline{n-1}\left[\frac{n-r+r}{(n-r)\lfloor\underline{n-r-1}}\right]$$

$$= \frac{n\lfloor\underline{n-1}}{(n-r)\lfloor\underline{n-r-1}} = \frac{\lfloor\underline{n}}{\lfloor\underline{n-r}} = {}^nP_r = \text{L.H.S.}$$

Example 8: *Prove that*

(*i*) ${}^nP_n = {}^nP_{(n-1)}$ (*ii*) ${}^nP_r = n\cdot {}^{n-1}P_{(r-1)}$

Solution: (i) ${}^nP_{n-1} = \dfrac{\lfloor\underline{n}}{\lfloor\underline{n-(n-1)}} = \dfrac{\lfloor\underline{n}}{\lfloor\underline{1}} = \lfloor\underline{n} = {}^nP_n$ (ii) ${}^nP_r = \dfrac{\lfloor\underline{n}}{\lfloor\underline{n-r}} = \dfrac{n\lfloor\underline{n-1}}{\lfloor\underline{(n-1)-(r-1)}} = n\cdot {}^{(n-1)}P_{r-1}$

Exercise

1. Prove that ${}^9P_3 + 3\cdot {}^9P_2 = {}^{10}P_3$.
2. Prove that ${}^nP_n = 2\cdot {}^nP_{(n-2)}$.
3. Find all permutations of 7 objects taken 3 at a time.
4. (i) If ${}^nP_4 : {}^nP_5 = 1:2$, find n. (ii) If ${}^{(n-1)}P_3 : {}^{(n+1)}P_3 = 5:12$, find n.
   (iii) If ${}^{2n+1}P_{n-1} : {}^{(2n-1)}P_n = 3:5$, find n.
5. If ${}^{11}P_r = {}^{12}P_{r-1}$, find r.
6. (i) If ${}^{2n}P_3 = 100 \times {}^nP_2$, find n. (ii) If $16\,{}^nP_3 = 13\,{}^{n+1}P_3$, find n. (iii) If ${}^nP_5 = 20 \times {}^nP_3$, find n.

Answers

3.	210	**4.**	(i) 6	(ii) 8	(iii) 4	**5.**	9	**6.**	(i) 13	(ii) 15	(iii) 8

14.1.2 Permutation with Repetition

If out of n objects in a set, p objects are exactly alike of one kind, q objects exactly alike of second kind and r objects exactly alike of third kind and remaining objects are all different then the number of permutation of n objects taken all at a time is

$$= \frac{\lfloor\underline{n}}{\lfloor\underline{p}\,\lfloor\underline{q}\,\lfloor\underline{r}}$$

Theorem 4: *The number of permutations of n different objects taken r at a time when each object may be replaced any number of times in each arrangement is n^r.*

Proof: The number of permutations of n objects taken r at a time is the same as the number of ways of filling r places with n different objects.

1st	2nd	3rd	------	rth	Place
↓	↓	↓		↓	No. of ways
n	n	n		n	

Number of ways of filling first place = n,

Number of ways of filling second place = n, (Since the object used in filling the first place can be repeated)

Number of way of filling third place = n,

....

....

....

Number of way of filling rth place = n

Therefore, by the fundamental principle of counting, the required number of permutations

$$= n \cdot n \cdot n \ldots\ldots r \text{ (times)} = n^r$$

Simple Practical Problems on Permutation

Example 9: *How many ways are there to arrange the nine letters in the word ALLAHABAD.*

Solution: Since the word ALLAHABAD contains 4A's and 2L's, therefore, the total number of arrangement of nine letters in the word ALLAHABAD $= \dfrac{\lfloor 9}{\lfloor 4 \lfloor 2} = 7{,}560$ ways.

Example 10: *Find the number of different messages that can be represented by sequences of four dashes and three dots.*

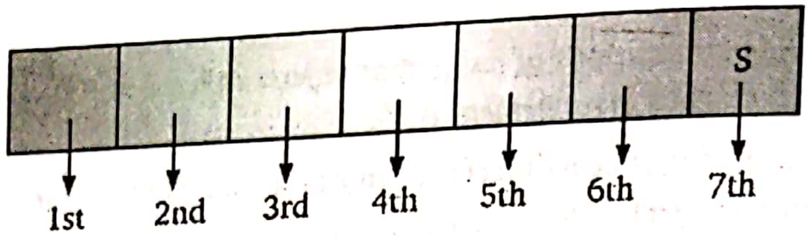
Solution: The number of different message sent by 4 deshes and 3 dots $= \dfrac{\lfloor 7}{\lfloor 4 \lfloor 3} = 35$

Example 11: *Find the number of ways to point 12 offices so that 3 of them are green, 2 of them pink, 2 of them yellow and the remaining one white.*

Solution: The required number of ways $= \dfrac{\lfloor 12}{\lfloor 3 \lfloor 2 \lfloor 2 \lfloor 5} = 1{,}66{,}320$

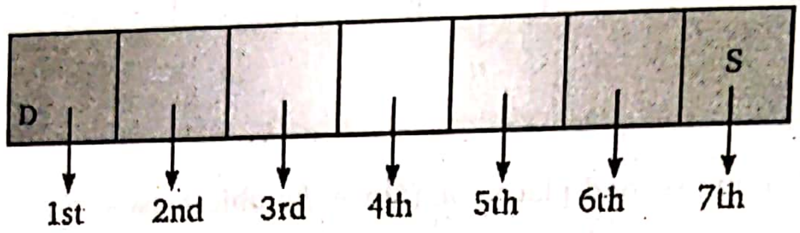
Example 12: *The letters of the word "TUESDAY" are arranged in a line each arrangement ending with letter S. How many different arrangements are possible? How many of them start with letter D?*

Solution: There are seven different letter in the word "TUESDAY"



when last place is filled with letter S then remaining 6 places can be filled by 6 letters in

$$^6P_6 = 6! \text{ ways} \qquad = 720 \text{ ways}$$



Now, when the letter 'D' is also fixed, then remaining 5 places can be filled by 5 letters in

$$^5P_5 = 5! \text{ ways} \qquad = 120 \text{ ways.}$$

Example 13: *How many signals can be made by using 6 flags of different colours when any number of them may be hoisted at a time?*

Solution: Number of signals that can be made using 1 flag $= {}^6P_1 = 6$

Number of signals that can be made using 2 flags $= {}^6P_2 = \dfrac{6!}{(6-2)!} = 6 \times 5 = 30$

Number of signals that can be made using 3 flags $= {}^6P_3 = \dfrac{6!}{(6-3)!} = 6 \times 5 \times 4 = 120$

Number of signals that can be made using 4 flags $= {}^6P_4 = \dfrac{6!}{(6-4)!} = 6 \times 5 \times 4 \times 3 = 360$

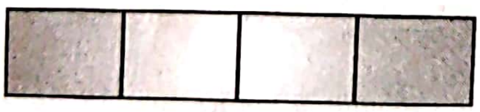
Number of signals that can be made using 5 flags $= {}^6P_5 = \dfrac{6!}{(6-5)!} = 6 \times 5 \times 4 \times 3 \times 2 = 720$

Number of signals that can be made using 6 flags $= {}^6P_6 = 6! = 720$

Therefore the total signals that can be made $= 6 + 30 + 120 + 360 + 720 + 720 = 1{,}956$

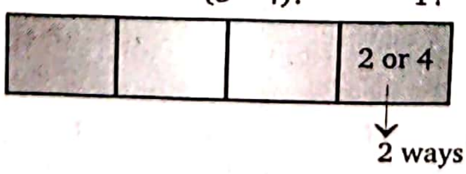
Example 14: *Find the number of 4 digit numbers that can be formed using the digits 1, 2, 3, 4 and 5 if no digit is used more than once in a number. How many of these number will be even?*

Solution: (i)



4 places can be filled by 5-digits in $^5P_4$ ways $= \dfrac{5!}{(5-4)!}$ ways $= \dfrac{5!}{1!}$ ways $= 120$ ways

(ii)



For even numbers, unit's place can be filled in 2 ways (either digit 2 or digit 4)

If units place is filled by any one digit, then remaining 3 places can be filled by remaining 4 digit in

$$^4P_3 = \frac{4!}{(4-3)!} \quad = 4! = 24 \text{ ways}$$

Hence, total number of ways $= 2 \times 24 = 48$ ways.

Example 15: *How many 4 letter words, with or without meaning, can be formed out of the letters of the word 'LOGARITHMS' if repetition of letters is not allowed?*

Solution: The word 'LOGARITHMS' contains different letters

$\therefore$ the number of required words = number of arrangements of 10 letters, taken 4 at a time

$$= {}^{10}P_4 \quad = 10 \times 9 \times 8 \times 7 = 5{,}040$$

Example 16: *It is required to seat 5 men and 4 women in a row so that the women occupy the even places. How many such arrangement are possible?*

Solution: In a row of 9 positions, the 2nd, 4th, 6th and 8th are the even places.

These 4 places can be occupied by 4 women in ${}^4P_4 = \lfloor\underline{4} = 24$ ways

The remaining positions can be occupied by 5 men in ${}^5P_5 = \lfloor\underline{5} = 120$ ways

$\therefore$ the total number of seating arrangements $= 24 \times 120 = 2{,}880$.

Example 17: *How many words, with or without meaning, can be founded by using all letters of the word 'DELHI', using each letter exactly once?*

Solution: The word 'DELHI' contains 5 different letters.

$\therefore$ the number of required words = number of arrangements of 5 letters taken all at a time

$$= {}^5P_5 = \lfloor\underline{5} = 5 \times 4 \times 3 \times 2 = 120.$$

Example 18: *In an examination hall, there are four rows of chairs. Each row has 8 chairs, one behind the other. There are two classes appearing for the examination with 16 students in each class. It is desired that in each row all students belong to the same class and that no two adjacent rows are allotted to the same class. In how many way can these 32 students be seated?*

Solution: There are 2 ways of selecting the rows for a class *i.e.*, either I and III or II and IV.

Now, 16 students of one class can be arranged in 16 chairs in ${}^{16}P_{16} = \lfloor\underline{16}$ ways

and 16 students of another class can be arranged in 16 remaining chairs in ${}^{16}P_{16} = \lfloor\underline{16}$ ways

$\therefore$ the required number of ways $= [2 \times \lfloor\underline{16} \times \lfloor\underline{16}\,]$

Example 19: *How many odd numbers greater than 80,000 can be formed using the digits 2, 3, 4, 5 and 8. If each digit is used only once in a number?*

Solution:

Ten Thousand	Thousand	Hundreds	Ten's	Unit's
8				3 or 5
1 way				2 way

Since, it is an odd number greater than 80,000, therefore unit's place can be filled by 2 ways (either by digit 3 or by digit 5) and ten thousand's can be filled by only 1 digit *i.e.*, 8.

Also other 3 places can be filled by remaining 3 digits in ${}^3P_3 = 3!$ ways

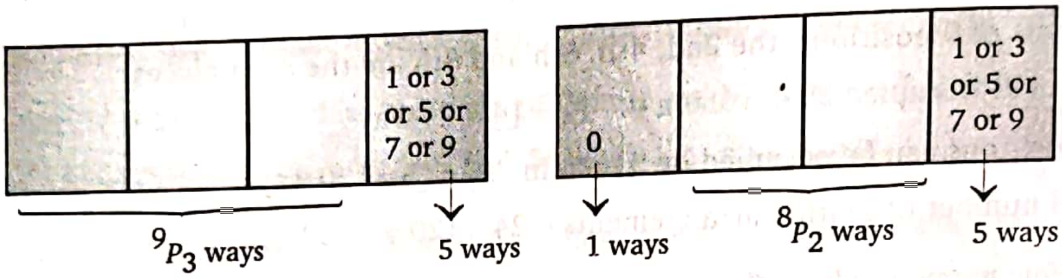
Total number of ways $= 2 \times 1 \times 3! = 12$ (By F.P.C.).

Example 20: *How many four-digit numbers (between 1,000 and 10,000) are there with distinct 2 of these, how many are odd numbers ?*

Solution: (i) Total number of arrangements of ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 taking 4 at a time is ${}^{10}P_4$. But these arrangements, also include those numbers which has 0 at thousand's place. Such numbers are not the four digit numbers. So, we have to exclude those three digit numbers when 0 is in thousand's place, we have to arrange remaining 9 digits by taking 3 digits at a time. The number of such arrangements is ${}^{9}P_3$.

Hence, the total number of four digit numbers $= {}^{10}P_4 - {}^{9}P_3 = 5{,}040 - 504 = 4{,}536.$

(ii) Now, since we want the odd number, therefore unit's place can be filled in 5 (1 or 3 or 5 or 7 or 9) ways.



The required number of odd numbers = number of all 4 digited number (including 0) having an odd digit in thousand's place – number of these 4 digited odd numbers which have 0 in thousand's place

$$= 5 \times {}^{9}P_3 - 1 \times {}^{8}P_2 \times 5 = 5 \times 9 \times 8 \times 7 - 8 \times 7 \times 5 = 2{,}240.$$

Example 21: *Find the number of permutations of n different things taken r at a time each of which includes 3 particular things ?*

Solution: Given r places, 3 places by particular things can be filled in ${}^{r}P_3$ ways.

Now, we have to fill up the remaining $(r-3)$ places with the remaining $(n-3)$ things in ${}^{n-3}P_{r-3}$ ways.

Total number of permutations $= {}^{r}P_3 \times {}^{n-3}P_{r-3}$.

Example 22: *Find the number of permutations of n different things taken r at a time from which three particular things are to be excluded.*

Solution: We exclude the 3 particular things from the total number of things and form permutations of the remaining $(n-3)$ things taken r at a time.

Therefore, the required number of permutations $= {}^{n-3}P_r$.

Example 23: *How many permutation can be made out of the letters of the word 'TRIANGLE'. How many of these will begin with T and end with E ?*

Solution: The word TRIANGLE contains 8 different letters.

The number of all permutation made out of 8 letters $= {}^{8}P_8 = \underline{|8} = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40{,}320.$

If the word starts with the letter T and ends with letter E then, the number of ways to form the words $= {}^{8}P_6$

$$= \frac{\underline{|8}}{\underline{|8-6}} = \frac{\underline{|8}}{\underline{|2}} = 20{,}160$$

Example 24: *In how many ways can 10 books be arranged on a shelf so that a particular pair of books shell be (i) always together (ii) never together ?*

Solution: Since a particular pair of books is always together, let us tie these two books together and consider the pair as one book. Then, we shall have to arrange 9 books on the shelf.

This can be done $^9P_9 = \lfloor 9$ ways

Now, in each of these arrangements, the two books comparising the particular pair can be arranged among themselves in $\lfloor 2$ *i. e.*, 2 ways.

$\therefore$ the required number of ways $= 2 \times (\lfloor 9)$

(ii) We know that 10 books can be arranged on the shelf in $^{10}P_{10} = \lfloor 10$ ways

Also, the number of ways of arranging 10 books so that a particular pair is always together $= 2 \times (\lfloor 9)$

$\therefore$ the number of ways of arranging 10 books so that a particular pair is never together

$$= (\lfloor 10 - 2\lfloor 9) \quad = (10 \times \lfloor 9 - 2 \times \lfloor 9) \quad = (8 \times \lfloor 9)$$

Example 25: *There are 6 English, 4 Sanskrit and 5 Hindi books. In how many ways can they be arranged on a shelf so as to keep all the books of the same language together?*

Solution: Let us make one packet for each of the books on the same language.

Now, 3 packets can be arranged in $^3P_3 = \lfloor 3 = 6$ ways

6 books on English can be arranged in $\lfloor 6 = 720$ ways

4 books on Sanskrit can be arranged in $\lfloor 4 = 24$ ways

5 books on Hindi can be arranged in $\lfloor 5 = 120$ ways

$\therefore$ the required number of ways $= 6 \times 720 \times 24 \times 120$.

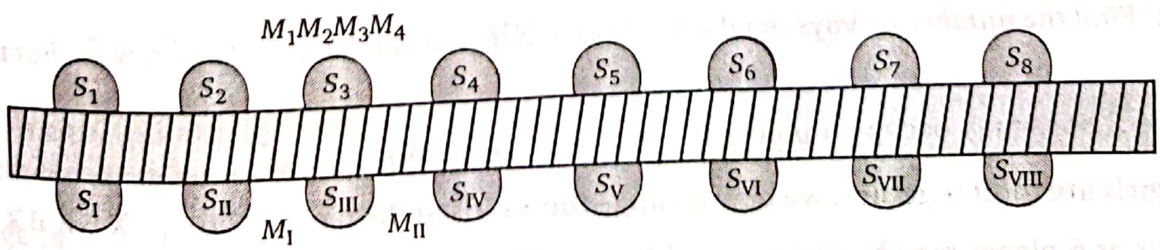
Example 26: *In how many ways 9 pictures can be hung of 6 picture nails on a wall (in a row) ?*

Solution: Here, it is obvious that only 6 pictures out of 9 will be hung and 3 will be left out. The number of ways in which pictures can be hung on 6 picture nails on a wall is the same as the number of arrangements of 9 things, taking 6 at a time.

Hence, the required number $= {}^9P_6 \;= 9 \times 8 \times 7 \times 6 \times 5 \times 4 = 60{,}480$.

Example 27: *A tea party is arranged for 16 people along the two sides of a long table with 8 chairs on each side. Four men wish to sit on one particular side and two on the other side. In how many ways can they be seated ?*

Solution: The order of seating arrangement is as below,



(i) 4 persons (M_1, M_2, M_3, M_4) are to sit one-one side in any one side in any of 8 chairs in $^8P_4$ ways.

(ii) 2 persons (M_1 and M_2) wist to sit at other side on any of the 8 chairs. This can be in $^8P_2$ ways.

(iii) Rest 10 persons can sit on any of the remaining 10 chairs in 10 chairs in $^{10}P_{10}$ ways.

$\therefore$ Total number of ways of sitting $= {}^8P_4 \times {}^8P_2 \times {}^{10}P_{10}$ (by using F.P.C.)

Example 28: *In how many ways can 6 balls of different colours, namely, white, black, blue, red, green and yellow be arranged in a row in such a ways that the white and black balls are never together ?*

Solution: Let x be the number of all arrangements, each one of which contains white and black balls together. And, let y be the number of arrangements, none of which contains white and black balls together.

Then, $$x + y = {}^6P_6 = \lfloor 6 = 720.$$

Now, considering a white and a black ball tied together as one ball.

The number of ways of arranging the 5 balls $= {}^5P_5 = \lfloor 5 = 120.$

Also, these 2 balls may be arranged among themselves in $\lfloor 2 = 2$ ways.

$\because$ $x = 2 \times 120 = 240,$ So, $y = 720 - 240 = 480$

Hence, the requisite number of ways $= 480.$

Example 29: *In how many ways can the word 'PENCIL' be arranged so that N is always next to E ?*

Solution: Keeping E and N together and considering then as one letter, we have to arrange 5 letters at 5 places. This can be done in $^5P_5 = \lfloor 5 = 120$ ways.

Example 30: *The principal wants to arrange 5 students on the platform such that the boy Salim occupies the second position and such that the girl Sita is always adjacent to the girl Rita. How many such arrangements are possible?*

Solution: Let the seats be arranged as shown

O	S	O	O	O
I	II	III	IV	V

Keep Salim fixed at the second position. Since, Sita and Rita are to sit together, none of the two can occupy the first seat. This seat can be occupied by any of the remaining two students in 2 ways. Now, two seats, namely, III , IV, V, may be occupied by Sita and Rita in 4 ways. The remaining seat may now be occupied by the 5th student in one way only.

$\therefore$ the number of required arrangement $= 2 \times 4 \times 1 = 8.$

Example 31: *Find the number of ways in which 5 boys and 3 girls can be seated in a row so that two girls are together.*

Solution: The 5 boys may occupy 5 places in $^5P_5 = \lfloor 5 = 120$ ways.

Since no. two girls are to sit together, we may arrange the 3 girls at the 6 places. *i. e.*, X B X B X B X B X B

Now, the 3 girls at 6 places may be seated in $^6P_3 = 6 \times 5 \times 4 = 120$ ways.

$\therefore$ The required number of ways $= 120 \times 120 = 14{,}400.$

Example 32: *How many permutations of the letters of the word 'APPLE' are there ?*

Solution: Here there are 5 objects, two of which are of same kind and the others are each of its own kind.

$\therefore$ The required number of permutations $= \dfrac{\lfloor 5}{\lfloor 2 \times \lfloor 1 \times \lfloor 1 \times \lfloor 1} = 60.$

Example 33: *(i) Find how many arrangements can be made with the letters of the word 'MATHEMATICS'.*

(ii) In how many of them are the vowels together?

Solution: (i) There are 11 letters in the word 'MATHEMATICS'

Out of these letters M occurs twice. A occurs twice, T occurs twice and the rest are all different.

$\therefore$ Required number of arrangement $= \dfrac{\lfloor 11}{\lfloor 2 \times \lfloor 2 \times \lfloor 2} = 49{,}89{,}600.$

(ii) The given word consists of 4 vowels namely A, E, A and I. Treating these four vowels as one letter, we have to arrange 8 letters (MTHMTCS) + (AEAI);

out of which M occurs twice, T occurs twice and the rest are all different.

$\therefore$ The number of such arrangements $= \dfrac{\lfloor 8}{(\lfloor 2) \times (\lfloor 2)} = 10{,}080$

Corresponding to each such arrangement, the four vowels in which A occurs twice and the rest are all different, can be arranged amongest themselves in $\dfrac{\lfloor 4}{\lfloor 2} = 12$ ways.

$\therefore$ Total number of arrangements in which vowels are always together $= 10080 \times 12 = 1{,}20{,}960$

Example 34: *In the different permutations of the word 'EXAMINATION' are listed as in a dictionary, how many items are there in the list before the first word starting with E ?*

Solution: Starting with E, we have to find the number of arrangements of the remaining 10 letters, EXAMINATION, out of which there are 2I's, 2N's, 2 A's and the rest of the letters are distinct.

The number of such arrangements is $= \dfrac{\lfloor 10}{\lfloor 2 \times (\lfloor 2) \times (\lfloor 2)} = 4{,}53{,}600$

Clearly, we will have to start the next word with E, so the required numbers of items = 4,53,600.

Example 35: *How many numbers greater than a million can be formed with digits 2, 3, 0, 3, 4, 2, 3?*

Solution: Any number greater than one million will contain all the seven digits.

Now, we have to arrange seven digits, out of which 2 occurs twice, 3 occurs thrice and the rest are distinct.

The number of such arrangements $= \dfrac{\lfloor 7}{(\lfloor 2) \times \lfloor 3} = 420$

These arrangements will also include those which contain 0 at the million's place. Keeping 0 fixed at the million's place, the remaining 6 digits out of which 2 occurs twice, 3 occurs thrice and the rest are distinct can be arrange in $= \dfrac{\lfloor 6}{(\lfloor 2) \times \lfloor 3} = 60$ ways.

$\therefore$ The number of required numbers $= 420 - 60 = 360.$

Example 36: *In a shipment, there are 40 floppy disks of which 5 are defective, determine:*

(i) *In how many ways can we select five floppy disks?*

(ii) *In how many ways can we select five non-defective floppy disks?*

(iii) *In how many ways can we select five floppy disks containing exactly three defective floppy disks?*

(iv) *In now many ways can we select five disks containing at least 1 defective floppy disk?*

[U.P.T.U. (B.Tech.) 2004]

Solution: (i) The required number of ways

$$= 40C_5 = \frac{\lfloor 40}{\lfloor 5 \lfloor 35} = \frac{40\times 39\times 38\times 37\times 36\times \lfloor 35}{5\times 4\times 3\times 2\times \lfloor 35} = 39\times 38\times 37\times 12 = 6580088$$

(ii) The non-defective floppy are 35

The number of ways to select five non-defective floppy $= 35C_5 = \dfrac{35}{\lfloor 5 \lfloor 30}$

$$= \frac{35\times 34\times 33\times 32\times 31\times \lfloor 30}{5\times 4\times 3\times 2\times 1\times \lfloor 30} = 324632$$

(iii) The number of ways to select five floppy contain exactly 3 defective floppy disks $= \dfrac{35C_2\times 5C_3}{40C_5}$

(iv) The required selection $= \dfrac{35C_4\times 5C_1 + 35C_3\times 5C_2 + 35C_2\times 5C_3 + 35C_1\times 5C_4}{40C_5}$

Example 37: *Given a cube and four identical balls. In how many way can the balls be arranged on the corners of the cube?. Two arrangements are considered the same if by any rotation of the cube they can be transformed into each other.* **[U.P.T.U. (B.Tech.) 2005]**

Solution: The balls can be arranged on the eight corners of the cube in the number of ways.

$$= 8C_4 = \frac{\lfloor 8}{\lfloor 4 \lfloor 4} = \frac{8\times 7\times 6\times 5\times \lfloor 4}{\lfloor 4\times 4\times 3\times 2} = 70 \text{ ways}$$

Now, two arrangements are same iff the two corners between the two arrangements become common.

Hence, the total ways $= 70 - 65 = 65$

Example 38: (a) *How many selections any number at a time may be made from three white balls, four green balls, one red ball one black ball, if at least one must be chosen.*

(b) *There are twelve students in the class. Find the number of ways that the twelve students take three different tests if four students are to take each test.* **[U.P.T.U. (B.Tech.) 2008]**

Solution: (a) Total number of balls $= 9$.

The probability that no ball is chosen $= 1$

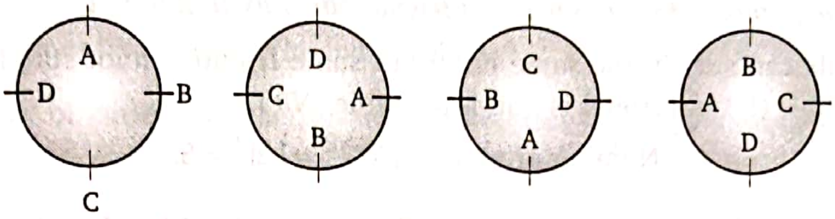
Then, number of selection $= \lfloor 9 - 1 = 362880 - 1 = 362879$

(b) The required number of ways $= \dfrac{\lfloor 12}{\lfloor 3 \lfloor 3 \lfloor 3 \lfloor 3} = 369600$

14.1.3 Circular Permutations

So far we have been considering the arrangements of objects in a line. Such permutation are known as linear permutations. In circular permutations, what really matters is the position of an object relative to the others. If we consider the linear permutations ABCD, BCDA, CDAB and DABC. Then, clearly they are distinct.

Now, we arrange A, B, C, D along the circumference of a circle as shown



If we consider the position of an object retative to others then we find that the above four arrangements are the same.

Theorem 5: *The number of circular permutations of n different objects, is* $\underline{|n-1}$.

Proof : Fixing the position of an object can be done in n ways, as the position of anyone of them may be fixed. Thus, each circular permutation corresponding to n linear permutations depending upon where from we start.

Since, there are $\underline{|n}$ linear permutations, it follows that there are $\frac{\underline{|n}}{n}$ *i. e.*, $\underline{|(n-1)}$ circular permutations.

Theorem 6: *The number of ways in which n persons can be seated round a table is* $\underline{|n-1}$.

Proof: Let us fix the position of one person and then arrange the remaining $(n-1)$ persons in all possible ways. Clearly, this can be done in $\underline{|(n-1)}$ ways. Hence, the required number of ways = $\underline{|n-1}$.

Theorem 7: *Show that the number of ways in which n different beads can be arranged to form a necklace is* $\frac{1}{2}\underline{|(n-1)}$.

Proof : Fixed the position of one bead, the remaining $(n-1)$ beads can be arranged in $\underline{|n-1}$ ways.

In case of arranging the beads, there is no distinction between the clockwise and anticlockwise arrangements. So, the required number of ways $= \frac{1}{2}\underline{|(n-1)}$.

Example 39: *A gentleman has 6 friends to invites. In how many ways can he send invitation cards to them, if he has 3 servants to carry the cards ?*

Solution: Two friends can be invited by the same servant. Hence, in this problem, the servant is repeatable (R) and the friend is non-repeatable (NR).

$$\text{Number of ways } = [R]^{NR} = 3^6 = 729.$$

Example 40: *A telegraph has 5 arms and each arm is capable of 4 distinct positions, including the position of rest. Find the total signals that can be made ?*

Solution: Two arms may have the same position, but same arms cannot have two positions at a time. Hence, position is repeatable (R), but arm is non-repeatable (NR)

$$\text{Number of ways } = [R]^{NR} = 4^5 = 1{,}024$$

But, in one case, when all the 5 arms will be in rest position, no signal will be made.
Hence, the required number of signals = 1,024 – 1 = 1,023.

Example 41: *Find the number of way in which 3 friends can stay in 2 hotels ?*

Solution: Two friends can stay in the same hotel but same friend cannot stay in two hotel at a time. Hence, hotel is repeatable (R) and friend is non-repeatable (NR).

$$\text{Number of ways } = [R]^{NR} = [2]^3 = 8.$$

Example 42: *Find the number of ways in which Dr. Bansal can give 2 hotels to his 3 sons?*

Solution: A son can get 2 hotels but a hotel cannot be given again and again.

Hence, son is repeatable (R) but hotel is non-repeatable (NR).

$$\text{Number of ways } = [R]^{NR} = (3)^2 = 9.$$

Example 43: *Find the number of ways in which 6 pictures can be hung on 9 picture nails (in a row)?*

Solution: A picture can not hung on two picture nails at a time, but two pictures can hung on one picture nail. Hence, picture nail is repeatable (R) but picture is non-repeatable (NR).

$$\text{Number of ways } = [R]^{NR} = (9)^6.$$

Example 44: *In how many ways can 8 students be arranged in*

(i) a line *(ii) a circle.*

Solution: (i) The number of ways in which 8 students can be arranged in a line = $\lfloor 8$

$$= 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40{,}320.$$

(ii) The number of ways in which 8 students can be arranged in a circle = $\lfloor 8-1 = \lfloor 7$

$$= 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5{,}040.$$

Example 45: *In how many ways can a party of 4 men and 4 women be seated at a circular table so that no two women are adjacent ?*

Solution: The 4 men can be seated at the circular table such that there is a vacant seat between every pair of men.

The number of ways in which these 4 men can be seated at the circular table = $\lfloor 3 = 6$.

Now, the 4 vacant seats may be occupied by 4 women in ${}^4P_4 = \lfloor 4 = 24$ ways.

$\therefore$ The required number of ways = $6 \times 24 = 144$.

Example 46: *In how many ways can 7 persons sit around a table so that all shall not have the same neighbours in any arrangements.*

Solution: 7 persons set at a round table in $\lfloor 6$ ways.

But, in clockwise and anticlockwise arrangements, each person will have the same neighbours.

So, the required number of ways = $\dfrac{1}{2} \times \lfloor 6 = 360$.

14.2 Combinations

Combinations: Each of the different groups or selections which can be formed by taking some or all of a number of objects, irrespective of their arrangements, is called a combination. The total number of combinations of n distinct objects taking $r\ (1 \le r \le n)$ at a time is denoted by $C(n, r)$ or ${}^nC_r$ or $\binom{n}{r}$.

where, ${}^nC_r$ is defined only when n and r integers such that $n \ge r$ and $n > 0,\ r \ge 0$.

Illustration: *The combinations of 4 objects a, b, c, d taking 2 at a time are ab, ac, ad, bc, bd, cd*

14.2.1 Difference between a Permutation and a Combination

In combination, only a group is made and the order in which the objects are arranged is immaterial. In permutation, not only a group is formed, but also an arrangement in definite order is considered.

Note: We used the word 'arrangement' for permutation and selections for combinations.

Theorem 8: *The number of all combinations of n distinct objects, taken r at a time, is given by*

$$ {}^nC_r = \frac{\lfloor n}{\lfloor r \lfloor n-r} $$

Proof: Let the number of all combinations of n objects, taken r at a time, be x. Then, ${}^nC_r = x$.

Now, each combination contains r objects, which may be arranged amongst themselves is $\lfloor r$ ways.

Thus, each combination gives rise to $\lfloor r$ permutations.

$\therefore$ x combinations will give rise to $x \times \lfloor r$ permutations.

So, the number of permutations of n things, taken r at a time is $x \times \lfloor r$

Hence, $$ {}^nP_r = x \times \lfloor r = {}^nC_r \times \lfloor r $$

$\therefore$ $$ {}^nC_r = \frac{{}^nP_r}{\lfloor r} = \frac{\lfloor n}{\lfloor r \lfloor n-r} \qquad \left[\because {}^nP_r = \frac{\lfloor n}{\lfloor n-r}\right] $$

Note: We can write $$ {}^nC_r = \frac{n\,(n-1)\,(n-2)\ldots\ n \text{ factors}}{\lfloor r}. $$

Theorem 9: *Let $0 \le r \le n$. Prove that* $\boxed{{}^nC_r = {}^nC_{n-r}.}$

Proof: We have
$${}^nC_{n-r} = \frac{\underline{|n}}{\underline{|n-r}\,\underline{|n-(n-r)}} = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}} = {}^nC_r.$$

Theorem 10: *To prove that* $\boxed{{}^nC_r + {}^nC_{r-1} = {}^{(n+1)}C_r.}$

Proof: We have,
$${}^nC_r + {}^nC_{r-1} = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}} + \frac{\underline{|n}}{\underline{|r-1}\,\underline{|n-(r-1)}} = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}} + \frac{\underline{|n}}{\underline{|r-1}\,\underline{|n-r+1}} = \frac{\underline{|n}\cdot(n-r+1)}{\underline{|r}\cdot\underline{|(n-r+1)}} + \frac{\underline{|n}\cdot r}{\underline{|r}\,\underline{|n-r+1}}$$
$$= \left\{\frac{\underline{|n}}{\underline{|r}\,\underline{|n-r+1}}\right\}\cdot\{n-r+1+r\} = \frac{(n+1)\,\underline{|n}}{\underline{|r}\,\underline{|n-r+1}} = \frac{\underline{|n+1}}{\underline{|r}\,\underline{|n+1-r}} = {}^{n+1}C_r$$

Theorem 11: *If $1 \le r \le n$, prove that $n \times {}^{(n-1)}C_{r-1} = (n-r+1) \times {}^nC_{r-1}$.*

Proof: We have
$$n \times {}^{(n-1)}C_{r-1} = n \times \frac{\underline{|n-1}}{\underline{|r-1} \times \underline{|(n-1)-(r-1)}} = \frac{n \times \underline{|n-1}}{\underline{|r-1}\,\underline{|n-r}}$$
$$= \frac{\underline{|n}\,(n-r+1)}{\underline{|r-1} \times \underline{|n-r} \times (n-r+1)} = (n-r+1) \times \frac{\underline{|n}}{\underline{|r-1} \times \underline{|n-r+1}} = (n-r+1) \times {}^nC_{r-1}$$

Theorem 12: *If n and r are positive integers such that $1 \le r \le n$ then* $\boxed{\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}.}$

Proof : We know that
$${}^nC_r = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}} \quad \text{and} \quad {}^nC_{r-1} = \frac{\underline{|n}}{\underline{|r-1} \times \underline{|n-r+1}}$$
$$\therefore \quad \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}} \times \frac{\underline{|r-1}\,\underline{|n-r+1}}{\underline{|n}} = \frac{\underline{|r-1} \times (n-r+1)\,\underline{|n-r}}{r \times \underline{|(r-1)}\,\underline{|n-r}} = \left(\frac{n-r+1}{r}\right)$$

Theorem 13: *If $1 \le r \le n$, prove that* $\boxed{{}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}.}$

Proof:
$${}^nC_r + {}^nC_{r+1} = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}} + \frac{\underline{|n}}{\underline{|(r+1)}\,\underline{|n-r-1}}$$
$$= \frac{\underline{|n}}{\underline{|r} \times (n-r)\,\underline{|n-r-1}} + \frac{\underline{|n}}{(r+1)\,\underline{|r} \times \underline{|n-r-1}}$$
$$= \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r-1}}\left(\frac{1}{n-r} + \frac{1}{r+1}\right)$$
$$= \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r-1}} \times \frac{(n+1)}{(n-r)(r+1)} = \frac{(n+1)\,\underline{|n}}{(r+1)\,\underline{|r} \times (n-r) \times \underline{|n-r-1}}$$
$$= \frac{\underline{|n+1}}{\underline{|r+1}\,\underline{|n-r}} = \frac{\underline{|n+1}}{\underline{|r+1}\,\underline{|(n+1)-(r+1)}} = {}^{n+1}C_{r+1}$$

Theorem 14: *Prove that* ${}^nC_p = {}^nC_q \Rightarrow p = q$ *or* $p + q = n$.

Proof: We have, ${}^nC_p = {}^nC_q = {}^nC_{n-q}$

$$\Rightarrow \qquad p = q \quad \text{or} \quad p = n - q \quad \text{or} \quad p + q = n.$$

Evaluate

Example 47: *If* ${}^nP_r = 720$ *and* ${}^nC_r = 120$, *find r.*

Solution: We know that $${}^nC_r = \frac{{}^nP_r}{\lfloor r}$$

$$\therefore \qquad 120 = \frac{720}{\lfloor r} \Rightarrow \lfloor r = \frac{720}{120} = \lfloor 3$$

Hence, $$r = 3.$$

Example 48: *Prove that* ${}^{2n}C_n = \dfrac{2^n \times [1 \cdot 3 \cdot 5 \ldots . (2n-1)]}{\lfloor n}$.

Solution: $${}^{2n}C_n = \frac{\lfloor 2n}{\lfloor n \lfloor 2n - n} = \frac{\lfloor 2n}{(\lfloor n)^2}$$

$$= \frac{(2n)(2n-1)(2n-2) \ldots \ldots 4 \cdot 3 \cdot 2 \cdot 1}{(\lfloor n)^2}$$

$$= \frac{[2n(2n-2)(2n-4) \ldots . 4 \cdot 2] \times [(2n-1)(2n-3) \ldots . 5 \cdot 3 \cdot 1]}{(\lfloor n)^2}$$

$$= \frac{2^n [n(n-1)(n-2) \ldots . 2 \cdot 1][(2n-1)(2n-3) \ldots . 5 \cdot 3 \cdot 1]}{(\lfloor n)^2}$$

$$= \frac{2^n \lfloor n [1 \cdot 3 \cdot 5 \ldots . (2n-3)(2n-1)]}{(\lfloor n)^2}$$

$$= \frac{2^n \times [1 \cdot 3 \cdot 5 \ldots . (2n-3)(2n-1)]}{\lfloor n}.$$

Example 49: *If* ${}^nC_{r-1} : {}^nC_r : {}^nC_{r+1} :: 3 : 4 : 5$ *find n and r.*

Solution: $$\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{3}{4} \text{ and } \frac{{}^nC_r}{{}^nC_{r+1}} = \frac{4}{5} \quad \Rightarrow \quad \frac{r}{n-r+1} = \frac{3}{4} \quad \text{and} \quad \frac{r+1}{n-r} = \frac{4}{5}$$

$$\Rightarrow \qquad 3n - 7r = -3 \text{ and } 4n - 9r = 5$$

$$\Rightarrow \qquad r = 27, n = 62$$

14.5 Pigeonhole Principle

This principle is also known as ***Dirilhlet Drawer Principle*** or ***the Shoe Box Principle***. It can be stated as follows.

14.5.1 Theorem of Pigeonhole Principle [U.P.T.U. (B.Tech.) 2005, 2006, 2007; U.P.T.U. (M.C.A.) 2008]

If n pigeons are assigned to m pigeonholes, and $m<n$, then at least one pigeonhole contains two or more pigeons.

***Proof*:** Let $H_1, H_2 \ldots H_m$ denote m pigeonholes and $P_1, P_2, \ldots P_m, P_{m+1}, \ldots P_n$ denote n pigeons. We consider the assignment of n pigeons to m pigeonholes as follows:

Let the pigeon P_1 is assigned to pigeonhole H_1, pigeon P_2 is assigned to pigeonhole H_2... and pigeon P_m to pigeonhole H_m. This assigns as many as pigeons possible to individuals pigeonholes. Since $m < n$, there are $(n-m)$ pigeons that have not yet been assigned. At least one pigeonhole will be assigned to a second pigeon.

Example 77: *Show that among 13 people, there are at least two people who were born in the same month.*

Solution: We consider 13 peoples as pigeons and 12 months (January, February, ... December) as the pigeonholes. Here the number of pigeons is grater than the number of pigeonholes, therefore by pigeonhole principle there will be atleast two people who were born in the same month.

Example 78: *If eight people are chosen in any way from some group, at least two of them will have been born on the same day of the week.* [Delhi (B.E.) 2005]

Solution: Here each person (pigeon) is assigned to the day of the week (pigeonhole) on which he or she was born. Since there are eight people and only seven days of the week, then by pigeon hole principle at least two people must be assigned to the same day of the week.

Example 79: *Show that if we choose 9 single shoes out of 8 distinct pairs of shoes, then we are sure to have a pair.* [Rohtak (B.E.) 2008]

Solution: Here 8 distinct pairs correspond to 8 pigeonholes and 9 single shoes correspond to pigeons. It means $m = 8$ and $n = 9$, $m < n$ then by pigeonhole principle, there must be one pigeonhole with 2 shoes which means that we have chosen atleast one pair.

Example 80: *Find the minimum number of students in a class so that three of them are born in the same month.* [Kurukshetra (B.E.) 2007; Osmania (B.E.) 2009]

Solution: Here $n = 12$ (months) which represent the number of pigeonholes.

$$m = k+1 = 3 \text{ or } k = 3,\ k \text{ is positive integer}$$

Hence, the minimum number of pigeons $= kn+1 = 2\times 12+1 = 25$ students.

Example 81: *How many students must there be in a class to guarantee that at least two students receive the same score in the final examination if the examination is graded on a scale from zero to one hundred points.*

Solution: Here $m = 2$ But $m = k+1$, k is any integer

then $$k+1 = 2 \Rightarrow k = 1$$

$$m = 101$$

So, total number of students in class to guarantee that atleast two students receive the same score.

$$= kn+2 \quad = 1\times 101+1 \quad = 102.$$

Example 82: *Show that if any five integers from 1 to 8 are chosen, then atleast two of them will have a sum 9.*

Solution: Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, \}$

The different sets, each containing two numbers whose sum is equal to 9 are

$$A_1 = \{1, 8\},\ A_2 = \{2, 7\},\ A_3 = \{3, 6\},\ A_4 = \{4, 5\}.$$

Each of the five number chosen from 1 to 8 must belong to one of these sets. The four sets are consider as pigeonholes and five chosen numbers are taking as pigeon. Since $m = 4$, $n = 5$ i.e. $4 < 5$ Then by pigeonhole principle we can say that the two of the selected numbers must belong to the same set whose sum is 9.

Example 83: *Show that in any room of people who have been doing handshaking there will always be at least two people who have shaken hands the same number of times.*

Solution: Let the pigeonholes be labelled with the different numbers of hands shaken and we put the people (pigeons) into their correct pigeonhole. Suppose there are n people, then since only shake hands with each person at most once, the lables on the pigeonholes will go from o to $(n-1)$ *i.e.* we have n holes and n people. It is not possible for the o^{th} and $(n-1)^{th}$ the holes both to be occupied, because if one person has shaken hands with nobody then there can not be any one person who has shaken hands with every other person. Thus, we have at most $(n-1)$ holes occupied at any one time. Hence, by the pigeonholes principle at least one of the holes has two occupants, which shows that there are at least two people who have shaken hands the same number of time.

Example 84: *What shall be the minimum number of words that must begin with the same alphabet at 27 English words.*

Solution: Here, $m = kn+1 = 27$ (Pigeonholes), $n = 26$ (number of alphabet) = (pigeons) -1, where k is any integer

$$\therefore \qquad k\times 26+1=27 \Rightarrow k=1 \qquad \text{and} \qquad m=k+1=2$$

Hence at least two words must begin with the same alphabet.

Example 85: *If there are 13 boys in a class, then find minimum number of boys born in the same months.*

Solution: Consider month as pigeons *i.e.* $n=12$ then find pigeonholes m. But $n=12$, $kn+1=13$

$$\Rightarrow \qquad k\times 12+1=13 \Rightarrow k=1, \qquad \text{then} \quad m=k+1=2$$

Example 86: *Show that if seven numbers from 1 to 12 are chosen, then two of them will add up to 13.*

Solution: There are six sets each containing two number that add up to 13 as follows:

$$A_1=\{1, 12\},\ A_2=\{2, 11\},\ A_3=\{3, 10\},\ A_4=\{4, 9\}\ \ A_5=\{5, 8\},\ A_6=\{6, 7\}$$

Each of the seven numbers belongs to one of these six sets. Since there are 6 sets, then by pigeonholes principle two of the numbers chosen belong to the same set. These numbers add upto 13.

Example 87: *If 20 candidates appear in a competitive examination then show that there exists at least two among them whose Roll number differ by a multiple of 19.* **[U.P.T.U. (B.Tech.) 2004]**

Solution: Let X be the set of Roll number of 20 candidates and Y be the set of remainder when any positive integer is divided by 19.

i.e., $$Y=\{0, 1, 2, \ldots, 18\}.$$

Here we define a function $f: X \to Y$ as $f(x)$ = the remainder obtained when any positive integer is divided by 19. Since $|X|=20$ and $|Y|=19$ *i. e.*, $|X|>|Y|$, therefore by the pigeonhole principle function f cannot be one-one. This shows that there exists two distinct Roll numbers x_1 and x_2 such that $f(x_1)=f(x_2)$

x_1, x_2 can be written as $x_1=19n_1+f(x_1)$ and $x_2=19n_2+f(x_2)$

where n_1 and n_2 are positive integers.

This gives $$x_1-x_2=19(n_1-n_2). \qquad [\because f(x_1)=f(x_2)]$$

Now n_1-n_2 is an integer. Therefore x_1-x_2 is a multiple of 19.

Hence there are at least two candidate whose Roll numbers differ by a multiple of 19.