Homomorphisms

Let $(G_1, *)$ and (G_2, o) be two algebraic systems where $*$ and o both are binary operations. Then, the mapping $f: G_1 \to G_2$ is said to be homomorphism from $(G_1, *)$ to (G_2, o) such that $\forall\ a, b \in G$, we have

$$f(a * b) = f(a) \, o \, f(b)$$

where $f(a)$ & $f(b) \in G_2$

Properties:- ① $f(e) = e'$, $e \in G_1$ & $e' \in G_2$

② $[f(a^{-1})] = [f(a)]^{-1}$, $a, a^{-1} \in G$

Proof: ① $a, b \in G_1 \Rightarrow f(a) \circ f(b) \in G_2$

$$f(a)\underset{\substack{\uparrow \\ \text{Identity element}}}{e'} = f(a)$$

$f(a)\, e' = f(a * e)$ $[a \circ e = a]$

$f(a)\, e' = f(a) \circ f(e)$ [By homomorphism def.]

$e' = f(e)$ – by cancellation law

Cancellation Law

$a * b = a * c \Rightarrow b = c$ Left cancellation

$b * a = c * a \Rightarrow b = c$ Right "

② $f(a), f(a^{-1}) \in G_2$

$f(a) \circ f(a^{-1}) = f(a * a')$ $\because f$ is homomorphism

$f(a) \circ f(a^{-1}) = f(e) = e'$

$f(a^{-1}) = [f(a)]^{-1}$ $\Rightarrow a a' = I$

Isomorphism

Let $(G_1, *)$ and (G_2, O) be two algebraic systems where $*$ and O both are binary operations. The systems $(G_1, *)$ and (G_2, O) are said to be isomorphic if there exists an isomorphic mapping $f : G_1 \to G_2$

When two algebraic systems are isomorphic, the system are structurally equivalent and one can be obtained from another by simply renaming the elements and the operation.

OR

Let $(G_1, *)$ and (G_2, O) are two groups. A Mapping $f : G_1 \to G_2$ is called isomorphism if

1. f is one-one
2. f is onto
3. f is homomorphism $(f(a*b) = f(a) \, O \, f(b) \; \forall a, b \in G$

Ques:- $f : G_1 \to G_2$ where G_1 & G_2 both are additive Group ~~and~~ G_1 is of Integers and even integers respectively. and f defined as

$$f(x) = 2x \quad \forall x \in G$$

Sol^n ① f is one-one $\to$ Let $f(x_1)$ & $f(x_2) \in G_2$

s.t $\quad f(x_1) = f(x_2)$

$\Rightarrow \cancel{2}x_1 = \cancel{2}x_2$

$\Rightarrow x_1 = x_2$

$\Rightarrow f$ is one-one

② f is onto

Let any $y \in G_2$ (y is even $\exists$ a element $x \in G$ s.t

Ques:- Define Ring.

$$x = y/2$$

Since $f(y/2) = 2(y/2) = y$

$f(x) = y \quad \forall\ y \in G_2$

we have $x \in G$ s.t $f(x) = y$

(3) f is homomorphism :-

$x_1, x_2 \in G_1$

$$f(x_1 + x_2) = 2(x_1 + x_2)$$
$$= 2x_1 + 2x_2$$
$$= f(x_1) + f(x_2)$$

f is homomorphism

$\Rightarrow f$ is Isomorphism Group.

Ques:- Let $(A_1, *)$ and $(A_2, \Box)$ be two algebraic system as shown in given fig. Determine whether the two algebraic system are isomorphic.

$*$	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

$\Box$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

Solⁿ The two algebraic systems $(A_1, *)$ and $(A_2, \Box)$ are isomorphic and $(A_2, \Box)$ is an isomorphic image of A_1

s.t

$$f(a) = 1$$
$$f(b) = \omega$$
$$f(c) = \omega^2$$

Ques:- Define Ring.

Ans:- **Ring:-** An algebric system $(R, +, \cdot)$ is called ring if the binary operation '+' and '·' in R, satisfies the following properties.

① $(R, +)$ is an abelian group

(ii) $(R, \cdot)$ is a semi-group

(iii) The operation '·' is distributive over '+':

Let $a, b, c \in R$ then

ⓐ $a \cdot (b+c) = a \cdot b + a \cdot c$

ⓑ $(b+c) \cdot a = b \cdot a + c \cdot a$

Ques:- Define field.

Ans:- **Field:-** A ring R with atleast two elements is called a field if it is

① Commutative ring

② has unit element

③ Each non-zero element having multiplication inverse.

Ques:- Prove that a ring R is commutative iff $(a+b)^2 = a^2 + 2ab + b^2 \quad \forall a, b \in R$

Soln Let R be a commutative ring, then

$$
\begin{aligned}
(a+b)^2 &= (a+b)(a+b) \\
&= a(a+b) + b(a+b) \\
&= a^2 + ab + ba + b^2 \\
&= a^2 + ab + ab + b^2 \quad (\because ba = ab) \\
&= a^2 + 2ab + b^2
\end{aligned}
$$

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Conversely:-

Let $(a+b)^2 = a^2 + 2ab + b^2$, then we have to show that R is commutative i.e;

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$\Rightarrow a^2 + ab + ba + b^2 = a^2 + 2ab + b^2$$

$$\Rightarrow ab + ba = 2ab \quad (\text{by cancellation law})$$

$$\Rightarrow ba = ab \quad \forall a, b \in R$$

$\Rightarrow R$ is commutative ring.

Ques:- Prove that $(-1)(-1) = 1$

Soln we know that

$$(-1)a = -a \quad \forall a \in R$$

Put $a = -1$, in above, we have

$$(-1)(-1) = -(-1) = 1$$

Hence

$(-1)(-1) = 1$ proved.