Quad-Rotor Dynamical Model with Euler-Lagrange Approach

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Euler-Lagrange equation

Euler Lagrange equation

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] - \frac{\partial L}{\partial q} = \mathbf{Q}'$$

L :lagrangian
$\boldsymbol{q}$:generalized coordinate
$\mathbf{Q}'$:generalized force

$$M(\boldsymbol{\eta})\ddot{\boldsymbol{\eta}} + C(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}})\dot{\boldsymbol{\eta}} = \mathbf{Q}'$$

Lagrangian

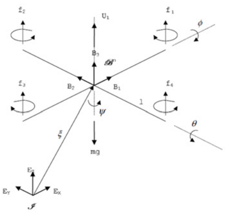
$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = T - U$$

T :kinetic energy
U :potential energy

Lagrangian

Generalized coordinate

$$\boldsymbol{q} = (x, y, z, \phi, \theta, \psi)^T \in \mathbb{R}^6$$
$$\boldsymbol{\xi} = (x, y, z)^T \in \mathbb{R}^3$$
$$\boldsymbol{\eta} = (\phi, \theta, \psi)^T \in \mathbb{R}^3$$



Translational kinetic energy

$$T_{trans} = \frac{m}{2}\dot{\boldsymbol{\xi}}^T\dot{\boldsymbol{\xi}}$$ m:the mass of the quad-rotor

Rotational kinetic energy

$$T_{rot} = \frac{1}{2}\boldsymbol{\Omega}^T\boldsymbol{I}\boldsymbol{\Omega}$$ $\boldsymbol{\Omega}$:angular velocity $I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$:inertia matrix

Potential energy

$$U = mgz$$ g:acceleration of gravity z:rotorcraft altitude

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{m}{2}\dot{\boldsymbol{\xi}}^T\dot{\boldsymbol{\xi}} + \frac{1}{2}\boldsymbol{\Omega}^T\boldsymbol{I}\boldsymbol{\Omega} - mgz$$

Generalized force

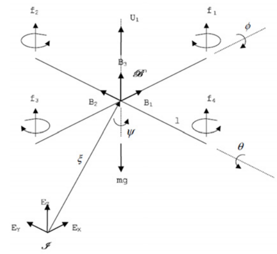
Angular velocity

$$\boldsymbol{\Omega} = \boldsymbol{W}_\eta\dot{\boldsymbol{\eta}} \qquad \boldsymbol{W}_\eta = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \cos\theta\sin\phi \\ 0 & -\sin\phi & \cos\theta\cos\phi \end{bmatrix} \qquad \mathbb{J}(\boldsymbol{\eta}) = \boldsymbol{W}_\eta^T\boldsymbol{I}\boldsymbol{W}_\eta$$

$$T_{rot} = \frac{1}{2}\dot{\boldsymbol{\eta}}^T\mathbb{J}\dot{\boldsymbol{\eta}}$$

Generalized force

$$\mathbf{Q}' = \begin{bmatrix} \boldsymbol{F}_\xi \\ \boldsymbol{\tau} \end{bmatrix}$$



$$\boldsymbol{F}_\xi = \boldsymbol{R}\hat{F} \in \mathbb{R}^3 \qquad \hat{F} = \begin{bmatrix} 0 \\ 0 \\ \sum_{i=1}^4 f_i \end{bmatrix}$$

$$\boldsymbol{R} = \begin{bmatrix} C\psi C\theta & C\psi S\theta S\phi - S\psi C\phi & C\psi S\theta C\phi + S\psi S\phi \\ S\psi C\theta & S\psi S\theta S\phi + C\psi C\phi & S\psi S\theta C\phi - C\psi S\phi \\ -S\theta & C\theta S\phi & C\theta C\phi \end{bmatrix}$$

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_\phi \\ \tau_\theta \\ \tau_\psi \end{bmatrix} \triangleq \begin{bmatrix} (f_3 - f_1)l \\ (f_2 - f_4)l \\ \sum_{i=1}^4 \tau_{M_i} \end{bmatrix}$$

Euler-Lagrange equation

Translational lagrangian

$$L_{trans} = \frac{m}{2}\dot{\boldsymbol{\xi}}^T\dot{\boldsymbol{\xi}} - mgz$$

Translational equation

$$\frac{d}{dt}\left[\frac{\partial L_{trans}}{\partial \dot{\boldsymbol{\xi}}}\right] - \frac{\partial L_{trans}}{\partial \boldsymbol{\xi}} = \boldsymbol{F}_\xi$$

$$m\ddot{\boldsymbol{\xi}} + mg\boldsymbol{E}_z = \boldsymbol{F}_\xi$$

Rotational lagrangian

$$L_{rot} = \frac{1}{2}\dot{\boldsymbol{\eta}}^T\mathbb{J}\dot{\boldsymbol{\eta}}$$

Rotational equation

$$\frac{d}{dt}\left[\frac{\partial L_{rot}}{\partial \dot{\boldsymbol{\eta}}}\right] - \frac{\partial L_{rot}}{\partial \boldsymbol{\eta}} = \boldsymbol{\tau}$$

$$\frac{\partial L_{rot}}{\partial \dot{\boldsymbol{\eta}}} = \frac{1}{2}\left(\mathbb{J} + \mathbb{J}^T\right)\dot{\boldsymbol{\eta}} = \mathbb{J}\dot{\boldsymbol{\eta}}$$

$$\mathbb{J}\ddot{\boldsymbol{\eta}} + \left(\dot{\mathbb{J}} - \frac{1}{2}\frac{\partial}{\partial \boldsymbol{\eta}}\left(\dot{\boldsymbol{\eta}}^T\mathbb{J}\right)\right)\dot{\boldsymbol{\eta}} = \boldsymbol{\tau}$$

$$C(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}) = \dot{\mathbb{J}} - \frac{1}{2}\frac{\partial}{\partial \boldsymbol{\eta}}\left(\dot{\boldsymbol{\eta}}^T\mathbb{J}\right)$$

$$\mathbb{J}\ddot{\boldsymbol{\eta}} + C(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}})\dot{\boldsymbol{\eta}} = \boldsymbol{\tau}$$

Rotational equation

$$\mathbb{J} = \begin{bmatrix} I_{xx} & 0 & -I_{xx}S\theta \\ 0 & I_{yy}C^2\phi + I_{zz}S^2\phi & (I_{yy} - I_{zz})C\phi S\phi C\theta \\ -I_{xx}S\theta & (I_{yy} - I_{zz})C\phi S\phi C\theta & I_{xx}S^2\theta + I_{yy}S^2\phi C^2\theta + I_{zz}C^2\phi C^2\theta \end{bmatrix}$$

$$C(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}) = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{31} & c_{33} \end{bmatrix}$$

$$c_{11} = 0$$
$$c_{12} = (I_{yy} - I_{zz})(\dot{\theta}C\phi S\phi + \dot{\psi}S^2\phi C\theta) + (I_{zz} - I_{yy})\dot{\psi}C^2\phi C\theta - I_{xx}\dot{\psi}C\theta$$
$$c_{13} = (I_{zz} - I_{yy})\dot{\psi}C\phi S\phi C^2\theta$$
$$c_{21} = (I_{zz} - I_{yy})(\dot{\theta}C\phi S\phi + \dot{\psi}S^2\phi C\theta) + (I_{yy} - I_{zz})\dot{\psi}C^2\phi C\theta + I_{xx}\dot{\psi}C\theta$$
$$c_{22} = (I_{zz} - I_{yy})\dot{\phi}C\phi S\phi$$
$$c_{23} = -I_{xx}\dot{\psi}S\theta C\theta + I_{yy}\dot{\psi}S^2\phi C\theta S\theta + I_{zz}\dot{\psi}C^2\phi S\theta C\theta$$
$$c_{31} = (I_{yy} - I_{zz})\dot{\psi}C\phi S\phi C^2\theta - I_{xx}\dot{\theta}C\theta$$
$$c_{32} = (I_{zz} - I_{yy})(\dot{\theta}C\phi S\phi S\theta + \dot{\phi}S^2\phi C\theta) + (I_{yy} - I_{zz})\dot{\phi}C^2\phi C\theta$$
$$+I_{xx}\dot{\psi}C\theta S\theta - I_{yy}\dot{\psi}S^2\phi C\theta S\theta - I_{zz}\dot{\psi}C^2\phi C\theta S\theta$$
$$c_{33} = (I_{yy} - I_{zz})\dot{\phi}C\phi S\phi C^2\theta - I_{yy}\dot{\theta}S^2\phi C\theta S\theta - I_{zz}\dot{\theta}C^2\phi C\theta S\theta + I_{xx}\dot{\theta}C\theta S\theta$$