CS 33

Data Representation (Part 2)

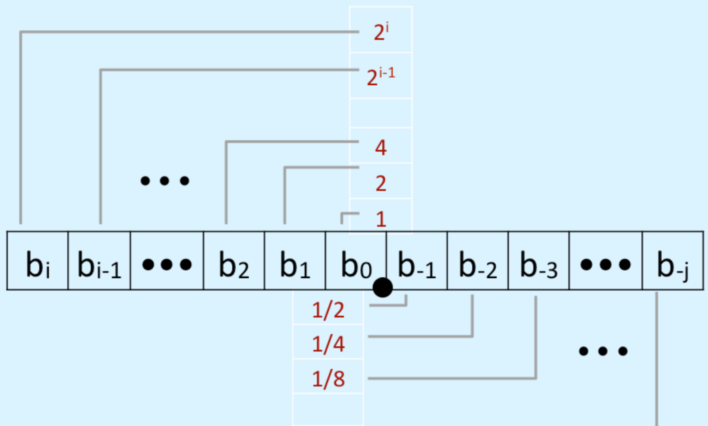
Many of the slides in this lecture are either from or adapted from slides provided by the authors of the textbook "Computer Systems: A Programmer's Perspective." 2nd Edition and are provided from the website of Carnegie-Mellon University, course 15-213, taught by Randy Bryant and David O'Hallaron in Fall 2010. These slides are indicated "Supplied by CMU" in the notes section of the slides.

Fractional binary numbers

- What is 1011.101_2?

Supplied by CMU.

Fractional Binary Numbers



- **Representation**
  - **bits to right of "binary point" represent fractional powers of 2**
  - **represents rational number:** $\sum_{k=-j}^{i} b_k \times 2^k$

Supplied by CMU.

Representable Numbers

- **Limitation #1**
  - – **can exactly represent only numbers of the form $n/2^k$**
    - » **other rational numbers have repeating bit representations**
  - – **value representation**
    - » **1/3** 0.0101010101[01]…$_2$
    - » **1/5** 0.001100110011[0011]…$_2$
    - » **1/10** 0.0001100110011[0011]…$_2$

- **Limitation #2**
  - – **just one setting of decimal point within the *w* bits**
    - » **limited range of numbers (very small values? very large?)**

Supplied by CMU.

IEEE Floating Point

- IEEE Standard 754
  - established in 1985 as uniform standard for floating point arithmetic
    - before that, many idiosyncratic formats
  - supported by all major CPUs
- Driven by numerical concerns
  - nice standards for rounding, overflow, underflow
  - hard to make fast in hardware
    - numerical analysts predominated over hardware designers in defining standard

Supplied by CMU.

Floating-Point Representation

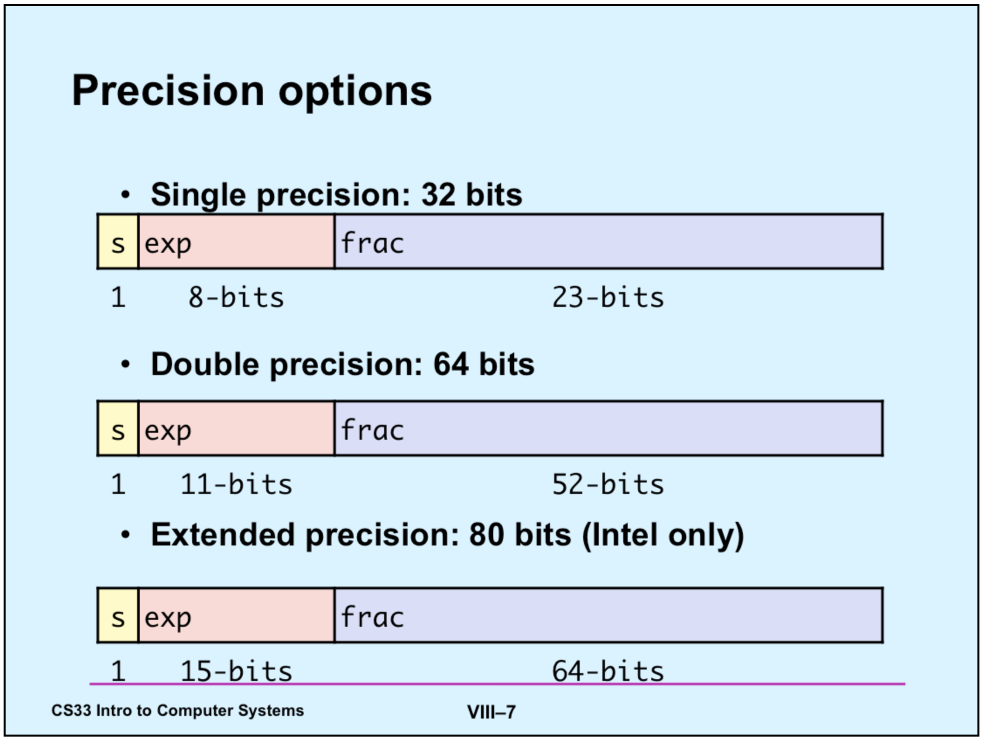
- **Numerical Form:**

$$(-1)^s \; M \; 2^E$$

  - sign bit **s** determines whether number is negative or positive
  - significand **M** normally a fractional value in range [1.0,2.0)
  - exponent **E** weights value by power of two
- **Encoding**
  - MSB `s` is sign bit **s**
  - `exp` field encodes **E** (but is not equal to E)
  - `frac` field encodes **M** (but is not equal to M)

s	exp	frac

Supplied by CMU.



Supplied by CMU.

On x86 hardware, all floating-point arithmetic is done with 80 bits, then reduced to either 32 or 64 as required.

“Normalized” Values

- When: exp ≠ 000…0 and exp ≠ 111…1
- Exponent coded as biased value: E = Exp – Bias
  - exp: unsigned value exp
  - bias = 2^{k-1} - 1, where k is number of exponent bits
    - » single precision: 127 (Exp: 1…254, E: -126…127)
    - » double precision: 1023 (Exp: 1…2046, E: -1022…1023)
- Significand coded with implied leading 1: M = $1.xxx…x_2$
  - xxx…x: bits of frac
  - minimum when frac=000…0 (M = 1.0)
  - maximum when frac=111…1 (M = 2.0 – ε)
  - get extra leading bit for “free”

Supplied by CMU.

Normalized Encoding Example

- **Value:** `float F = 15213.0;`
  - $15213_{10} = 11101101101101_2$
    $= 1.1101101101101_2 \times 2^{13}$

- **Significand**

  M = `1.1101101101101`$_2$

  `frac` = `11011011011010000000000`$_2$

- **Exponent**

  E = **13**

  bias = **127**

  exp = **140** = `10001100`$_2$

- **Result:**

0	10001100	11011011011010000000000
`s`	`exp`	`frac`

Supplied by CMU.

Denormalized Values

- **Condition: exp = 000…0**
- **Exponent value: E = –Bias + 1 (instead of E = 0 – Bias)**
- **Significand coded with implied leading 0: M = $0.xxx…x_2$**
  - **`xxx…x`: bits of `frac`**
- **Cases**
  - **`exp` = 000…0, `frac` = 000…0**
    - » **represents zero value**
    - » **note distinct values: +0 and –0 (why?)**
  - **`exp` = 000…0, `frac` ≠ 000…0**
    - » **numbers closest to 0.0**
    - » **equispaced**

Supplied by CMU.

Special Values

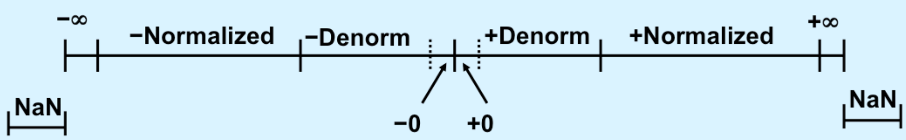
- **Condition: `exp` = 111…1**

- **Case: `exp` = 111…1, `frac` = 000…0**
  - represents value ∞ (infinity)
  - operation that overflows
  - both positive and negative
  - e.g., 1.0/0.0 = −1.0/−0.0 = +∞, 1.0/−0.0 = −∞

- **Case: `exp` = 111…1, `frac` ≠ 000…0**
  - not-a-number (NaN)
  - represents case when no numeric value can be determined
  - e.g., sqrt(–1), ∞ − ∞, ∞ × 0

Supplied by CMU.

Visualization: Floating-Point Encodings



Supplied by CMU.

Tiny Floating-Point Example

s	exp	frac
1	4-bits	3-bits

- **8-bit Floating Point Representation**
  - the sign bit is in the most significant bit
  - the next four bits are the exponent, with a bias of 7
  - the last three bits are the `frac`

- **Same general form as IEEE Format**
  - normalized, denormalized
  - representation of 0, NaN, infinity

Supplied by CMU.

Dynamic Range (Positive Only)

	s	exp	frac	E	Value	
Denormalized numbers	0	0000	000	-6	0	
	0	0000	001	-6	1/8*1/64 = 1/512	closest to zero
	0	0000	010	-6	2/8*1/64 = 2/512	
	…					
	0	0000	110	-6	6/8*1/64 = 6/512	
	0	0000	111	-6	7/8*1/64 = 7/512	largest denorm
Normalized numbers	0	0001	000	-6	8/8*1/64 = 8/512	smallest norm
	0	0001	001	-6	9/8*1/64 = 9/512	
	…					
	0	0110	110	-1	14/8*1/2 = 14/16	
	0	0110	111	-1	15/8*1/2 = 15/16	closest to 1 below
	0	0111	000	0	8/8*1 = 1	
	0	0111	001	0	9/8*1 = 9/8	closest to 1 above
	0	0111	010	0	10/8*1 = 10/8	
	…					
	0	1110	110	7	14/8*128 = 224	
	0	1110	111	7	15/8*128 = 240	largest norm
	0	1111	000	n/a	inf	

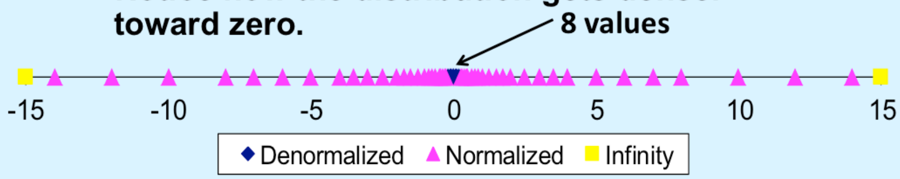
Supplied by CMU.

Distribution of Values

- **6-bit IEEE-like format**
  - **e = 3 exponent bits**
  - **f = 2 fraction bits**
  - **bias is $2^{3-1}-1 = 3$**

s	exp	frac
1	3-bits	2-bits

- **Notice how the distribution gets denser toward zero.**

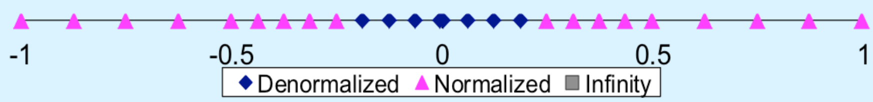


Supplied by CMU.

Distribution of Values (close-up view)

- **6-bit IEEE-like format**
  - **e = 3 exponent bits**
  - **f = 2 fraction bits**
  - **bias is 3**

s	exp	frac
1	3-bits	2-bits



Supplied by CMU.

Quiz 1

- **6-bit IEEE-like format**
  - **e = 3 exponent bits**
  - **f = 2 fraction bits**
  - **bias is 3**

s	exp	frac
1	3-bits	2-bits

What number is represented by 0 011 10?

a) 12
b) 1.5
c) .5
d) none of the above

Floating-Point Operations: Basic Idea

- $x +_f y = \mathrm{Round}(x + y)$

- $x \times_f y = \mathrm{Round}(x \times y)$

- Basic idea
  - first compute exact result
  - make it fit into desired precision
    - » possibly overflow if exponent too large
    - » possibly round to fit into `frac`

Supplied by CMU.

Rounding

- Rounding modes (illustrated with $ rounding)

	$1.40	$1.60	$1.50	$2.50	–$1.50
towards zero	$1	$1	$1	$2	–$1
round down (−∞)	$1	$1	$1	$2	–$2
round up (+∞)	$2	$2	$2	$3	–$1
nearest even (default)	$1	$2	$2	$2	–$2

Supplied by CMU.

Closer Look at Round-To-Nearest-Even

- Default rounding mode
  - hard to get any other kind without dropping into assembly
  - all others are statistically biased
    - » sum of set of positive numbers will consistently be over- or under-estimated

- Applying to other decimal places / bit positions
  - when exactly halfway between two possible values
    - » round so that least significant digit is even
  - e.g., round to nearest hundredth

1.2349999	1.23	(less than half way)
1.2350001	1.24	(greater than half way)
1.2350000	1.24	(half way—round up)
1.2450000	1.24	(half way—round down)

Supplied by CMU.

Rounding Binary Numbers

- Binary fractional numbers
  - "even" when least significant bit is `0`
  - "half way" when bits to right of rounding position = $100\ldots_2$
- Examples
  - round to nearest 1/4 (2 bits right of binary point)

Value	Binary	Rounded	Action	Rounded Value
2 3/32	10.00011_2	10.00_2	(<1/2—down)	2
2 3/16	10.00110_2	10.01_2	(>1/2—up)	2 1/4
2 7/8	10.11100_2	11.00_2	(1/2—up)	3
2 5/8	10.10100_2	10.10_2	(1/2—down)	2 1/2

Supplied by CMU.

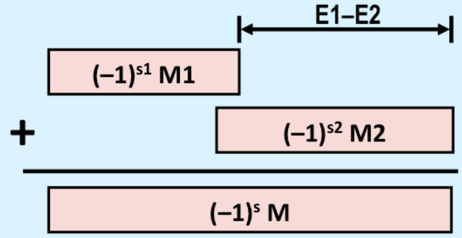
Floating-Point Multiplication

- $(-1)^{s1}$ M1 2^{E1} x $(-1)^{s2}$ M2 2^{E2}
- Exact result: $(-1)^{s}$ M 2^{E}
  - sign s: s1 ^ s2
  - significand M: M1 x M2
  - exponent E: E1 + E2
- Fixing
  - if M ≥ 2, shift M right, increment E
  - if E out of range, overflow
  - round M to fit `frac` precision
- Implementation
  - biggest chore is multiplying significands

Supplied by CMU.

Floating-Point Addition

- $(-1)^{s1}\ M1\ \ 2^{E1}\ \ +\ \ (-1)^{s2}\ M2\ \ 2^{E2}$
  - –assume E1 > E2



- **Exact result:** $(-1)^{s}\ M\ \ 2^{E}$
  - –sign s, significand M:
    - » result of signed align & add
  - –exponent E: E1

- **Fixing**
  - –if M ≥ 2, shift M right, increment E
  - –if M < 1, shift M left k positions, decrement E by k
  - –overflow if E out of range
  - –round M to fit `frac` precision

Supplied by CMU.

Floating Point in C

- **C guarantees two levels**
  - `float` **single precision**
  - `double` **double precision**
- **Conversions/casting**
  - **casting between `int`, `float`, and `double` changes bit representation**
  - `double`/`float` → `int`
    - » **truncates fractional part**
    - » **like rounding toward zero**
    - » **not defined when out of range or NaN: generally sets to TMin**
  - `int` → `double`
    - » **exact conversion, as long as `int` has ≤ 53-bit word size**
  - `int` → `float`
    - » **will round according to rounding mode**

Supplied by CMU.

Creating Floating-Point Numbers

s	exp	frac
1	4-bits	3-bits

- **Steps**
  - normalize to have leading 1
  - round to fit within fraction
  - postnormalize to deal with effects of rounding

- **Case study**
  - convert 8-bit unsigned numbers to tiny floating point format

example numbers

128	10000000
13	00001101
33	00010001
35	00010011
138	10001010
63	00111111

Supplied by CMU.

Normalize

s	exp	frac
1	4-bits	3-bits

- **Requirement**
  - set binary point so that numbers of form 1.xxxxx
  - adjust all to have leading one
    - » decrement exponent as shift left

Value	Binary	Fraction	Exponent
128	10000000	1.0000000	7
13	00001101	1.1010000	3
17	00010001	1.0001000	4
19	00010011	1.0011000	4
138	10001010	1.0001010	7
63	00111111	1.1111100	5

Supplied by CMU.

Rounding

1.BBGRXXX

Guard bit: LSB of result

Round bit: 1st bit removed

Sticky bit: OR of remaining bits

- **Round-up conditions**
  - round = 1, sticky = 1 ⇒ > 0.5
  - guard = 1, round = 1, sticky = 0 ⇒ round to even

Value	Fraction	GRS	Incr?	Rounded
128	1.0000000	000	N	1.000
13	1.1010000	100	N	1.101
17	1.0001000	010	N	1.000
19	1.0011000	110	Y	1.010
138	1.0001010	011	Y	1.001
63	1.1111100	111	Y	10.000

Supplied by CMU.

Postnormalize

- **Issue**
  - rounding may have caused overflow
  - handle by shifting right once & incrementing exponent

Value	Rounded	Exp	Adjusted	Result
128	1.000	7		128
13	1.101	3		13
17	1.000	4		16
19	1.010	4		20
138	1.001	7		134
63	10.000	5	$1.000*2^6$	64

Supplied by CMU.

Quiz 2

Suppose *f*, declared to be a `float`, is assigned the largest possible floating-point positive value (other than $+\infty$). What is the value of *g = f+1.0*?

a) f

b) $+\infty$

c) NAN

d) 0

Float is not Rational …

- Floating addition
  - commutative: a $+^f$ b = b $+^f$ a
    - » yes!
  - associative: a $+^f$ (b $+^f$ c) = (a $+^f$ b) $+^f$ c
    - » no!
      - 2 $+^f$ (1e10 $+^f$ -1e10) = 2
      - (2 $+^f$ 1e10) $+^f$ -1e10 = 0

Note that the floating-point numbers in this and the next two slides are expressed in base 10, not base 2.

Float is not Rational …

- **Multiplication**
  - commutative: a $*^f$ b = b $*^f$ a
    - » yes!
  - associative: a $*^f$ (b $*^f$ c) = (a $*^f$ b) $*^f$ c
    - » no!
      - 1e20 $*^f$ (1e20 $*^f$ 1e-20) = 1e20
      - (1e20 $*^f$ 1e20) $*^f$ 1e-20 = +∞

Float is not Rational …

- More …
  - multiplication distributes over addition:
    $a *^f (b +^f c) = (a *^f b) +^f (a *^f c)$
    - » no!
    - » $1e20 *^f (1e20 +^f -1e20) = 0$
    - » $(1e20 *^f 1e20) +^f (1e20 *^f -1e20) = NaN$
  - loss of significance:
    x=y+1
    z=2/(x-y)
    z==2?
    - » not necessarily!
      - consider y = 1e20