Periodic Orbits on Discrete Dynamical Systems

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Abstract—In this paper, we discuss the discrete dynamical system

$$x_{n+1} = \beta x_n - g(x_n), \qquad n = 0, 1, \ldots, \tag{*}$$

arising as a discrete-time network of single neuron, where β is the internal decay rate, g is a signal function. First, we consider the case where g is of McCulloch-Pitts nonlinearity. Periodic orbits are discussed according to different range of β. Moreover, we can construct periodic orbits. Then, we consider the case where g is a sigmoid function. Sufficient conditions are obtained for (*) has periodic orbits of arbitrary periods and an example is also given to illustrate the theorem. © 2003 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

There are a lot of problems which can be described, at least to a crude approximation, by a simple first-order difference equation

$$x_{n+1} = f(x_n), \qquad n = 0, 1, \ldots. \tag{1.1}$$

Studies of the dynamical properties of such models can provide us with many useful messages. There are many works in this area [1–5]. Since Li and York [6] and May [7] noticed that even simple mathematical models can demonstrate very complicated dynamics, people have paid more attention to the periodic solutions and "chaotic" behavior of discrete dynamical systems; we refer to [8–10]. However, most works of (1.1) are based on the assumption that f is continuous. But in reality, for example, in some models of neural networks, f may be not continuous [11–13].

In this paper, we consider the following discrete-time equation:

$$x_{n+1} = \beta x_n - g(x_n), \qquad n = 0, 1, \ldots, \tag{1.2}$$

where $\beta \in (0, \infty)$, g is a nonlinear function.

Equation (1.2) arises as a discrete-time network of single neuron where β is the internal decay rate, g is a signal function [11]. We consider two cases of the signal function:

  (I) g is of McCulloch-Pitts nonlinearity;
  (II) g is a sigmoid function.

Now we recall some definitions [8].

DEFINITION 1.1. *A point x_s is called a stationary state of (1.1) if*

$$f(x_s) = x_s.$$

DEFINITION 1.2. *A stationary state x_s of (1.1) is stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$|x_0 - x_s| \leq \delta \text{ implies that } |x_n - x_s| \leq \epsilon \text{ for all } n \geq 1.$$

A stationary state x_s that is not stable is said to be unstable.

DEFINITION 1.3. *An orbit $O(x_0) = \{x_0, x_1, x_2, \dots\}$ of (1.1) is said to be periodic of period $p \geq 2$ if*

$$x_p = x_0 \quad \text{and} \quad x_i \neq x_0, \qquad \text{for } 1 \leq i \leq p - 1.$$

DEFINITION 1.4. *A periodic orbit $\{x_0, x_1, \dots, x_{p-1}, \dots\}$ of period p is stable if each point x_i, $i = 0, 1, \dots, p - 1$, is a stable stationary state of dynamical system $x_{n+1} = f^p(x_n)$. A periodic orbit $\{x_0, x_1, \dots, x_{p-1}, \dots\}$ of period p which is not stable is said to be unstable.*

DEFINITION 1.5. *A point z is said to be a limit point of $O(x_0)$ if there exists a subsequence $\{x_{n_k} : k = 0, 1, \dots\}$ of $O(x_0)$ such that $|x_{n_k} - z| \to 0$ as $k \to \infty$. The limit set $L(x_0)$ of the orbit $O(x_0)$ is the set of all limit points of the orbit.*

DEFINITION 1.6. *An orbit $O(x_0)$ is said to be asymptotically periodic if its limit set is a periodic orbit. An orbit $O(x_0)$ such that $x_{n+p} = x_n$ for some $n \geq 1$ and some $p \geq 2$ is said to be eventually periodic.*

In Section 2, we shall discuss the periodic orbits of (1.2) when g is of McCulloch-Pitts nonlinearity. We shall show that (1.2) has a unique stable periodic orbit of period 2 when $\beta \in (0, 1)$; equation (1.2) has infinitely many stable periodic orbits of period 2 when $\beta = 1$; equation (1.2) has an unstable stational state and unstable periodic orbits of period 2,4 when $\beta \in (1, \sqrt{2})$; equation (1.2) has unstable periodic orbits of all even periods when $\beta \in [\sqrt{2}, (1 + \sqrt{5})/2)$, and (1.2) has periodic orbits of arbitrary periods when $\beta \in [(1 + \sqrt{5})/2, \infty)$. Moreover, we can construct the periodic orbits of (1.2). In Section 3, we consider (1.2) when g is a sigmoid function. By using Li and York's theorem [6], we get a sufficient condition for (1.2) to have periodic orbits of arbitrary periods. We also give an example to illustrate the result.

2. g IS OF MCCULLOCH-PITTS NONLINEARITY

Throughout this section, we will assume that g is of McCulloch-Pitts nonlinearity and

$$g(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ -1, & \text{if } x < 0. \end{cases} \tag{2.1}$$

First, we consider (1.2) when $\beta \in (0, 1)$, and we have the following theorem.

THEOREM 2.1. *Assume that $\beta \in (0, 1)$. Then the periodic orbit $O(1/(\beta+1))$ is a stable periodic orbit with period 2. And for every $x_0 \in R$, the orbit $O(x_0)$ is asymptotically periodic with $L(x_0) = \{1/(\beta + 1), -1/(\beta + 1)\}$.*

PROOF. Let $h(x) = \beta x - g(x)$. Then (1.2) can be rewritten

$$x_{n+1} = h(x_n), \qquad n = 0, 1, \dots. \tag{2.2}$$

Clearly,

$$h\left(\frac{1}{\beta + 1}\right) = -\frac{1}{\beta + 1}, \qquad h\left(-\frac{1}{\beta + 1}\right) = \frac{1}{\beta + 1};$$

this implies $O(1/(\beta+1)) = \{1/(\beta+1), -1/(\beta+1), \dots\}$ is a periodic orbit of (1.2) with period 2.

Now we show that $O(1/(\beta+1))$ is stable. In fact, for each $\epsilon > 0$, let $\delta = \min\{1/(\beta+1), \epsilon\}$. Then

$$\left| x_0 - \frac{1}{\beta+1} \right| < \delta \quad \text{implies} \quad \left| h^{2n}(x_0) - \frac{1}{\beta+1} \right| = \beta^{2n} \left| x_0 - \frac{1}{\beta+1} \right| < \epsilon.$$

This shows that $1/(\beta+1)$ is a stable stationary state of the dynamical $x_{n+1} = h^2(x_n)$. Similarly, $-1/(\beta+1)$ is a stable stationary state of the dynamical $x_{n+1} = h^2(x_n)$.

For every $x_0 \in R$, there is no harm in assuming that $x_0 \geq 0$. Since $x_{n+1} = \beta x_n - 1 \leq x_n - 1$ for $x_n \geq 0$, there exists a nonnegative integer n_0 such that $x_i \geq 0$ for $0 \leq i \leq n_0$ and $x_{n_0+1} < 0$. Thus,

$$x_{n_0+2} = \beta x_{n_0+1} + 1 = \beta^2 x_{n_0} + 1 - \beta.$$

This leads to

$$x_{n_0+2} - \frac{1}{\beta+1} = \beta^2 \left(x_{n_0} - \frac{1}{\beta+1} \right).$$

In general, we have

$$x_{n_0+2n} - \frac{1}{\beta+1} = \beta^{2n} \left(x_{n_0} - \frac{1}{\beta+1} \right).$$

Therefore, $\lim_{n\to\infty} x_{n_0+2n} = 1/(\beta+1)$.

Similarly, we get $\lim_{n\to\infty} x_{n_0+1+2n} = -1/(\beta+1)$. Thus, $L(x_0) = \{1/(\beta+1), -1/(\beta+1)\}$, and the proof is complete.

Similar to proof of Theorem 2.1, we can show the following theorem.

THEOREM 2.2. *Assume $\beta = 1$. Then, for every $x_0 \in R$, the orbit $O(x_0)$ of (1.2) is eventually periodic with period 2.*

Now we consider the case when $\beta > 1$.

THEOREM 2.3. *Let $\beta \in (1, \infty)$. Then (1.2) has a stationary state, periodic orbits of period 2,4. Suppose there is a positive integer k such that*

$$\beta^{2k} - 2\beta^{2k-2} + 1 > 0. \tag{2.3}$$

Then, (1.2) has a periodic orbit of period $2k$. Suppose there is a positive integer k such that

$$\beta^{2k+1} - 2\beta^{2k-1} - 1 \geq 0. \tag{2.4}$$

Then, (1.2) has a periodic orbit of period $2k + 1$.

PROOF. $1/(\beta-1)$ is a stationary state of (1.2); $O(1/(\beta+1))$ is a 2-periodic orbit of (1.2) and $O((\beta+1)/(\beta^2+1))$ is a 4-periodic orbit of (1.2).

Now suppose (2.3) holds. We will construct a periodic orbit $O(x_0)$ of period $2k$ such that

$$\begin{aligned} &x_0 > 0, \quad x_1 \geq 0, \quad x_2 < 0, \quad x_3 < 0, \quad (-1)^i x_i > 0, \\ &\text{for } i = 4, \ldots, 2k-1, \quad x_{2k} = x_0. \end{aligned} \tag{2.5}$$

Then

$$\begin{aligned}
x_1 &= \beta x_0 - 1 \geq 0, \\
x_2 &= \beta^2 x_0 - \beta - 1 < 0, \\
x_3 &= \beta^3 x_0 - \beta^2 - \beta + 1 < 0, \\
x_4 &= \beta^4 x_0 - \beta^3 - \beta^2 + \beta + 1 > 0, \\
x_5 &= \beta^5 x_0 - \beta^4 - \beta^3 + \beta^2 + \beta - 1 < 0, \\
&\vdots \\
x_{2k-1} &= \beta^{2k-1} x_0 - \beta^{2k-2} - \beta^{2k-3} + \beta^{2k-4} + \beta^{2k-5} - \beta^{2k-6} + \cdots - 1, \\
x_{2k} &= \beta^{2k} x_0 - \beta^{2k-1} - \beta^{2k-2} + \beta^{2k-3} + \beta^{2k-4} - \beta^{2k-5} + \beta^{2k-6} - \cdots + 1.
\end{aligned}$$

Since $x_{2k} = x_0$, we get

$$x_0 = \frac{\beta^{2k-1} + \beta^{2k-2} - \beta^{2k-3} - \beta^{2k-4} + \beta^{2k-5} - \beta^{2k-6} + \cdots - 1}{\beta^{2k} - 1}. \tag{2.6}$$

By (1.2), we see that

$$
\begin{aligned}
\min\{x_0, x_1, x_4, x_6, \ldots, x_{2k-2}\} = x_1 &= \frac{\beta^{2k-1} - \beta^{2k-2} - \beta^{2k-3} + \beta^{2k-4} - \beta^{2k-5} + \beta^{2k-6} - \cdots + 1}{\beta^{2k} - 1} \\
&= \frac{(\beta - 1)\left(\beta^{2k-2} - \beta^{2k-4} - \beta^{2k-6} - \cdots - 1\right)}{\beta^{2k} - 1} \\
&= \frac{\beta^{2k} - 2\beta^{2k-2} + 1}{(\beta^{2k} - 1)(\beta + 1)} > 0
\end{aligned}
$$

and

$$
\begin{aligned}
\max\{x_2, x_3, x_5, x_7, \ldots, x_{2k-1}\} &= x_{2k-1} \\
&= \frac{-\beta^{2k-1} + \beta^{2k-2} + \beta^{2k-3} - \beta^{2k-4} - \beta^{2k-5} + \beta^{2k-6} - \cdots + 1}{\beta^{2k} - 1} \\
&< 0.
\end{aligned}
$$

Thus, if (2.3) holds, the orbit $O(x_0)$ where x_0 is defined by (2.6) satisfies (2.5) and is a periodic orbit of period $2k$ for (1.2).

Similarly, if (2.4) holds, we can show that the orbit $O(x_0)$ where

$$x_0 = \frac{\beta^{2k} - \beta^{2k-1} - \beta^{2k-2} + \beta^{2k-3} - \beta^{2k-4} + \beta^{2k-5} - \beta^{2k-6} + \cdots - 1}{\beta^{2k+1} - 1} \tag{2.7}$$

is a periodic orbit of period $2k + 1$ for (1.2) and satisfies $x_0 \geq 0$, $x_1 < 0$, $x_2 < 0$, $(-1)^i x_i < 0$ for $i = 3, \ldots, 2k$.

REMARK 2.1. If $\beta > 1$, all periodic orbits of (1.2) are unstable and (2.3) always holds for $k = 1, 2$.

By a simple computation, we see that the equation

$$x^{2k} - 2x^{2k-2} + 1 = 0 \tag{2.8}$$

has a unique positive real root α_k for $k \geq 3$. Clearly, (2.3) holds for $\beta > \alpha_k$. Let $\alpha_1 = \alpha_2 = 1$ since (2.3) always holds for $k = 1$ and $k = 2$. Then, $\{\alpha_k\}_{k \geq 2}$ is an increasing sequence in $[1, \sqrt{2})$ and $\lim_{k \to \infty} \alpha_k = \sqrt{2}$.

The equation

$$x^{2k+1} - 2x^{2k-1} - 1 = 0 \tag{2.9}$$

has a unique positive real root β_k for $k \geq 1$ and $\beta_1 = (1 + \sqrt{5})/2$. Clearly, (2.4) holds for $\beta \geq \beta_k$, $\{\beta_k\}$ is a decreasing sequence in $(\sqrt{2}, 2)$, and $\lim_{k \to \infty} \beta_k = \sqrt{2}$.

Based on the above discussion and Theorem 2.3, we have the following corollary.

COROLLARY 2.4. *If (2.3) holds, (1.2) has periodic orbits of period $2k, 2k - 2, \ldots, 4, 2$ and a stationary state. If (2.4) holds, (1.2) has periodic orbits of period $2k + 1, 2k + 3, \ldots$. Especially, for $\beta \in [\sqrt{2}, \infty)$, (1.2) has periodic orbits of arbitrary even periods; for $\beta \in [(1 + \sqrt{5})/2, \infty)$, (1.2) has periodic orbits of arbitrary periods.*

REMARK 2.2. $\beta \in [(1 + \sqrt{5})/2, \infty)$ is a necessary and sufficient condition for (1.2) has a periodic orbit of period 3.

3. g IS A SIGMOID FUNCTION

In this section, motivated by [14], we assume that there exist positive constants r, ϵ such that

$$
\begin{aligned}
|g(x) - 1| &\leq \epsilon, && \text{if } x \geq r, \\
|g(x) + 1| &\leq \epsilon, && \text{if } x \leq -r.
\end{aligned}
\tag{3.1}
$$

THEOREM 3.1. *Assume that* $\beta \in ((1+\sqrt{5})/2, \infty)$, $r \in (0, (\beta^2 - \beta - 1)/(\beta^3 - 1))$, $g(x)$ *is continuous in* R *and satisfies* (3.1). *If*

$$
\frac{1}{\beta^2}\left(1 + \frac{1}{\beta^2} + \frac{1}{\beta^4}\right)\left(\frac{\beta^2 - \beta - 1}{\beta^3 - 1} - r\right)^2 + \frac{\epsilon^2}{\beta^2}\left(3 + \frac{4}{\beta} + \frac{4}{\beta^2} + \frac{2}{\beta^3} + \frac{1}{\beta^4}\right)
$$
$$
+ \frac{2\epsilon}{\beta^2}\left(1 + \frac{1}{\beta} + \frac{2}{\beta^2} + \frac{1}{\beta^3} + \frac{1}{\beta^4}\right)\left(\frac{\beta^2 - \beta - 1}{\beta^3 - 1} - r\right) \leq \left(\frac{\beta^2 - \beta - 1}{\beta^3 - 1} - r\right)^2,
\tag{3.2}
$$

for every $m = 1, 2, \ldots$, (1.2) *has a periodic orbit of period* m.

PROOF. Since $\beta x - g(x)$ is continuous in R, by Theorem 1 of [6], it suffices if we can prove that (1.2) has a periodic orbit of period 3. We rewrite (1.2) as

$$
x_n = \frac{1}{\beta}x_{n+1} + \frac{1}{\beta}g(x_n).
\tag{3.3}
$$

Now, we show that

$$
y_{n+1} = \frac{1}{\beta}y_n + \frac{1}{\beta}g(y_{n-2})
\tag{3.4}
$$

has a 3-periodic solution $\{y_n^*\}$. Then $O(y_0^*)$ is a 3-periodic orbit of (1.2).

If y_{-2}, y_{-1}, y_0 are given, by (3.4), we have

$$
\begin{aligned}
y_1 &= \frac{1}{\beta}y_0 + \frac{1}{\beta}g(y_{-2}), \\
y_2 &= \frac{1}{\beta^2}y_0 + \frac{1}{\beta^2}g(y_{-2}) + \frac{1}{\beta}g(y_{-1}), \\
y_3 &= \frac{1}{\beta^3}y_0 + \frac{1}{\beta^3}g(y_{-2}) + \frac{1}{\beta^2}g(y_{-1}) + \frac{1}{\beta}g(y_0).
\end{aligned}
\tag{3.5}
$$

We define a map H from R^3 to R^3 by $H(y_1, y_2, y_3) = (z_1, z_2, z_3)$ where

$$
\begin{aligned}
z_1 &= \frac{1}{\beta}y_3 + \frac{1}{\beta}g(y_1), \\
z_2 &= \frac{1}{\beta^2}y_3 + \frac{1}{\beta^2}g(y_1) + \frac{1}{\beta}g(y_2), \\
z_3 &= \frac{1}{\beta^3}y_3 + \frac{1}{\beta^3}g(y_1) + \frac{1}{\beta^2}g(y_2) + \frac{1}{\beta}g(y_3).
\end{aligned}
\tag{3.6}
$$

Let $\bar{y} = (\bar{y}_1, \bar{y}_2, \bar{y}_3)$ where

$$
\bar{y}_1 = \frac{-\beta^2 + \beta + 1}{\beta^3 - 1}, \qquad \bar{y}_2 = \frac{\beta^2 - \beta + 1}{\beta^3 - 1}, \qquad \bar{y}_3 = \frac{\beta^2 + \beta - 1}{\beta^3 - 1}.
$$

Here $\bar{y}_3$, $\bar{y}_2$, $\bar{y}_1$ are the first three terms of the 3-periodic orbit $O((\beta^2 + \beta - 1)/(\beta^3 - 1))$ in Theorem 2.3. Clearly, $\bar{y}_1 < 0$, $\bar{y}_2 > 0$, $\bar{y}_3 > 0$ and we have

$$
\begin{aligned}
\bar{y}_1 &= \frac{1}{\beta}\bar{y}_3 - \frac{1}{\beta}, \\
\bar{y}_2 &= \frac{1}{\beta^2}\bar{y}_3 - \frac{1}{\beta^2} + \frac{1}{\beta}, \\
\bar{y}_3 &= \frac{1}{\beta^3}\bar{y}_3 - \frac{1}{\beta^3} + \left(\frac{1}{\beta^2} + \frac{1}{\beta}\right).
\end{aligned}
\tag{3.7}
$$

If $\sqrt{(y_1 - \bar{y}_1)^2 + (y_2 - \bar{y}_2)^2 + (y_3 - \bar{y}_3)^2} \le \delta_0 = (\beta^2 - \beta - 1)/(\beta^3 - 1) - r$,

$$y_1 \le \bar{y}_1 + \delta_0 = -r,$$

$$y_2 \ge \bar{y}_2 - \delta_0 = \frac{\beta^2 - \beta + 1}{\beta^3 - 1} - \frac{\beta^2 - \beta - 1}{\beta^3 - 1} + r \ge r,$$

$$y_3 \ge \bar{y}_3 - \delta_0 = \frac{\beta^2 + \beta - 1}{\beta^3 - 1} - \frac{\beta^2 - \beta - 1}{\beta^3 - 1} + r \ge r.$$

In view of (3.6),(3.7), we get

$$z_1 - \bar{y}_1 = \frac{1}{\beta}(y_3 - \bar{y}_3) + \frac{1}{\beta}(g(y_1) + 1).$$

Noticing (3.1), we have

$$|z_1 - \bar{y}_1| \le \frac{1}{\beta}|y_3 - \bar{y}_3| + \frac{1}{\beta}|g(y_1) + 1| \le \frac{1}{\beta}\delta_0 + \frac{1}{\beta}\epsilon.$$

Similarly, we get

$$|z_2 - \bar{y}_2| \le \frac{1}{\beta^2}\delta_0 + \left(\frac{1}{\beta^2} + \frac{1}{\beta}\right)\epsilon, \qquad |z_3 - \bar{y}_3| \le \frac{1}{\beta^3}\delta_0 + \left(\frac{1}{\beta^3} + \frac{1}{\beta^2} + \frac{1}{\beta}\right)\epsilon.$$

Therefore, by (3.2),

$$\begin{aligned}(z_1 - \bar{y}_1)^2 + (z_2 - \bar{y}_2)^2 + (z_3 - \bar{y}_3)^2 &\le \frac{1}{\beta^2}\left(1 + \frac{1}{\beta^2} + \frac{1}{\beta^4}\right)\delta_0^2 + \frac{\epsilon^2}{\beta^2}\left(3 + \frac{4}{\beta} + \frac{4}{\beta^2} + \frac{2}{\beta^3} + \frac{1}{\beta^4}\right) \\ &\quad + \frac{2\delta_0\epsilon}{\beta^2}\left(1 + \frac{1}{\beta} + \frac{2}{\beta^2} + \frac{1}{\beta^3} + \frac{1}{\beta^4}\right) \\ &\le \left(\frac{\beta^2 - \beta - 1}{\beta^3 - 1} - r\right)^2 = \delta_0^2.\end{aligned}$$

Thus, H maps $N(\bar{y}, \delta_0) = \{(y_1, y_2, y_3) \in R^3 : (y_1 - \bar{y}_1)^2 + (y_2 - \bar{y}_2)^2 + (y_3 - \bar{y}_3)^2 \le \delta_0^2\}$ into $N(\bar{y}, \delta_0)$. By the Brouwer fixed-point theorem, H has a fixed point $(y_1^*, y_2^*, y_3^*) \in N(\bar{y}, \delta_0)$. The solution of (3.4) with initial condition $y_{-2} = y_1^*$, $y_{-1} = y_2^*$, $y_0 = y_3^*$ is a periodic solution of period 3. The theorem is now complete.

REMARK 3.1. It is obvious that $(1/\beta^2)(1 + 1/\beta^2 + 1/\beta^4) < 1$ for $\beta \in ((1 + \sqrt{5})/2, \infty)$, and thus, (3.2) can be satisfied if we choose small ϵ.

EXAMPLE. Let $c > 0$ and $g_0(x) = (1 - e^{-cx})/(1 + e^{-cx})$. Then

$$|g_0(x) - 1| = \frac{2e^{-cx}}{1 + e^{-cx}} \le 2e^{-cr}, \qquad \text{if } x \ge r,$$

$$|g_0(x) + 1| = \frac{2}{1 + e^{-cx}} \le 2e^{-cr}, \qquad \text{if } x \le -r.$$

Assuming $\beta \in ((1 + \sqrt{5})/2, \infty)$, $r \in (0, (\beta^2 - \beta - 1)/(\beta^3 - 1))$, (3.2) holds, and $c \ge (1/r)\ln(2/\epsilon)$ ($\epsilon < 2$), then (3.1) holds for $g = g_0$. According to Theorem 3.1,

$$x_{n+1} = \beta x_n - g_0(x_n), \qquad n = 0, 1, \dots,$$

has periodic orbits of arbitrary periods.

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