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All Solutions of a Class of Difference Equations are Truncated Periodic

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Abstract—We propose the difference equation $x_{n+1} = x_n - f(x_{n-k})$ as a model for a single neuron with no internal decay, where f satisfies the McCulloch-Pitts nonlinearity. It is shown that every solution is truncated periodic with the minimal period $2(2l+1)$ for some $l \geq 0$ such that $(k-l)/(2l+1)$ is a nonnegative even integer. The potential application of our results to neural networks is obvious. © 2002 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

In recent decades, the study of neural networks has attracted the attention of many researchers. Since large scale networks are very hard to analyze, we use networks of one or two neurons as prototypes to understand the dynamics of large scale networks. In the literature, the delay differential equation

$$\dot{x}(t) = -f(x(t-\tau)) \tag{1}$$

is used as the model for a single neuron with no internal decay, where $f : \mathbb{R} \to \mathbb{R}$ is either the sigmoid or a piecewise linear signal function and $\tau \geq 0$ is the synaptic transmission delay (see, for example, [1–3]).

As we know, differential equations with piecewise constant arguments have wide applications in certain biomedical models (see [4], for instance). So, we are inspired to model a single neuron with no internal decay by

$$\dot{x}(t) = -f(x([t])), \tag{2}$$

where $[\cdot]$ denotes the greatest integer function. In contrast to the assumptions on f for equation (1) in the literature, we assume that $f : \mathbb{R} \to \mathbb{R}$ is the function given by the piecewise constant McCulloch-Pitts nonlinearity,

$$f(u) = \begin{cases} 1, & \text{if } u > \sigma, \\ -1, & \text{if } u \leq \sigma. \end{cases} \tag{3}$$

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Here, $\sigma \in \mathbb{R}$ is referred to as the threshold.

Since the pioneering work of Cooke and Wiener [5] and Shah and Wiener [6], much progress has been made in the study of differential equations with piecewise constant arguments. A good survey is done by Cooke and Wiener [7]. However, most of the study deals with linear and scalar equations. To the best of our knowledge, there exist no results for the dynamics of model (2) with f satisfying (3). Surprisingly, as we will see, each solution of (2) will be periodic eventually.

To study (2), it is better to consider the following difference equation:

$$x_{n+1} = x_n - f(x_{n-k}), \qquad n = 0, 1, \dots, \tag{4}$$

where k is a nonnegative integer and f satisfies (3). In fact, denote $x_n = x(n)$. Then integrate (2) from n to $t \in [n, n+1)$ to get

$$x(t) = x(n) - \int_n^t f(x([s]))\,\mathrm{d}s = x(n) - f(x(n))(t-n).$$

Letting $t \to n+1$, we get

$$x_{n+1} = x_n - f(x_n),$$

which is a special case of (4) with $k = 0$. Hence, equation (4) can be used to model a single neuron with transmission delay.

For the sake of simplicity, let $\mathbb{Z}$ denote the set of all integers. For a, $b \in \mathbb{Z}$, define $\mathbb{Z}(a) = \{a, a+1, \dots\}$, and $\mathbb{Z}(a,b) = \{a, a+1, \dots, b\}$ whenever $a \le b$. By a solution of (4), we mean a sequence $\{x_n\}_{n=-k}^{\infty}$ that satisfies (4) for $n \in \mathbb{Z}(0)$. Clearly, for any $(\bar{x}_{-k}, \bar{x}_{-k+1}, \dots, \bar{x}_0) \in \mathbb{R}^{k+1}$, equation (4) has a unique solution $\{x_n\}_{n=-k}^{\infty}$ satisfying $x_n = \bar{x}_n$ for $n \in \mathbb{Z}(-k, 0)$. Because of (3), for a sequence $\{x_n\}_{n=-k}^{\infty}$, we say that $\{x_l, x_{l+1}, \dots, x_m\}$ with $l \le m$ is a positive semicycle (with respect to σ) if $x_n > \sigma$ for $n \in \mathbb{Z}(l, m)$ and l and m satisfy

$$\text{either } l = -k \qquad \text{or} \qquad x_{l-1} \le \sigma$$

and

$$\text{either } m = \infty \qquad \text{or} \qquad x_{m+1} \le \sigma.$$

We call $m - l + 1$ the length of the positive semicycle. A negative semicycle is defined similarly (replace $>$ by $\le$ and vice versa). In the sequel, a semicycle means either a positive semicycle or a negative semicycle. To state our main results in Section 2, we introduce a terminology. A sequence $\{x_n\}_{n=-k}^{\infty}$ is truncated periodic if there exist $n_0 \in \mathbb{Z}(-k)$ and $\omega \in \mathbb{Z}(1)$ such that

$$x_{n+n_0+\omega} = x_{n+n_0}, \qquad \text{for } n \in \mathbb{Z}(0),$$

and ω is called a period.

2. THE MAIN RESULTS

We start with a result on the limiting behavior of solutions to (4).

LEMMA 1. *Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of (4). Then for any $n \in \mathbb{Z}(-k)$, there exist $n_1 \ge n$ and $n_2 \ge n$ such that $x_{n_1} > \sigma$ and $x_{n_2} \le \sigma$.*

PROOF. By way of contradiction, without loss of generality, we assume that there exists an $n_0 \in \mathbb{Z}(-k)$ such that

$$x_n > \sigma, \qquad \text{for } n \in \mathbb{Z}(n_0). \tag{5}$$

Then, by induction, we know that

$$x_{n_0+k+n} = x_{n_0+k} - n, \qquad \text{for } n \in \mathbb{Z}(0).$$

Thus, $x_n \to -\infty$ as $n \to \infty$, a contradiction to (5). This completes the proof. ∎

It follows from Lemma 1 that all semicycles of solutions to (4) are of finite length. Moreover, it is easy to observe that, for a solution $\{x_n\}_{n=-k}^{\infty}$ of (4), the length of the semicycle $\{x_{n_0}, \dots, x_{m_0}\}$ with $n_0 > 0$ is odd.

The following two results will be very essential to prove our main results.

PROPOSITION 2. *Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of (4). If $\{x_n\}_{n=-k}^{\infty}$ has a semicycle of length larger than k, then $\{x_n\}_{n=-k}^{\infty}$ is truncated periodic with the minimal period $2(2k+1)$.*

PROOF. Without loss of generality, we assume that $\{x_n\}_{n=-k}^{\infty}$ has a positive semicycle $\{x_{n_0}, x_{n_0+1}, \dots, x_{n_0+l}\}$ of length $l+1$, where $l \geq k$. Then, from (3) and (4), we can get

$$\begin{aligned} x_{n_0+l+m} &= x_{n_0+l} - m, && \text{for } m \in \mathbb{Z}(1, k+1), \\ x_{n_0+l+k+m+1} &= x_{n_0+l} - k + m - 1, && \text{for } m \in \mathbb{Z}(1, 2k+1), \\ x_{n_0+l+3k+m+2} &= x_{n_0+l} + k - m, && \text{for } m \in \mathbb{Z}(1, 2k+1). \end{aligned}$$

Note that $\sigma \geq x_{n_0+l+1} = x_{n_0+l} - f(x_{n_0+l-k}) = x_{n_0+l} - 1$. Thus, $x_{n_0+l+4k+3} = x_{n_0+l} - 1 \leq \sigma$, and hence, $\{x_{n_0+l+2k+2}, \dots, x_{n_0+l+4k+2}\}$ is a positive semicycle with length $2k+1$. Moreover, $x_{n_0+l+4k+2} = x_{n_0+l}$. In the above discussion, we only used the fact that $x_{n_0+l-k}, \dots, x_{n_0+l}$ are larger than σ. So the same arguments will give us

$$x_{n_0+l+n+2(2k+1)} = x_{n_0+l+n}, \qquad \text{for } n \in \mathbb{Z}(0).$$

That is, $\{x_n\}_{n=-k}^{\infty}$ is truncated periodic and $2(2k+1)$ is the minimal period. ∎

LEMMA 3. *Let $k \geq 3$ and $\{x_n\}_{n=-k}^{\infty}$ be a solution of (4). Assume that $\{x_n\}_{n=-k}^{\infty}$ has a semicycle, $\{x_{n_0}, \dots, x_{n_0+2l}\}$ with $n_0 > 0$, of length $2l+1 \in \mathbb{Z}(3, k)$. If no semicycle $\{x_{n_1}, \dots, x_{m_1}\}$ with $n_1 > n_0 + 2l$ has length larger than $2l+1$, then $(k-l)/(2l+1)$ is an even integer and $\{x_n\}_{n=-k}^{\infty}$ is truncated periodic with the minimal period $2(2l+1)$.*

PROOF. Let $m_0 \in \mathbb{Z}(1)$ be such that $k \in \mathbb{Z}(m_0(2l+1), m_0(2l+1)+2l)$. Without loss of generality, we assume that $\{x_{n_0}, \dots, x_{n_0+2l}\}$ is a positive semicycle. First, we claim that

$$\sigma - l - 1 < x_n \leq \sigma + l + 1, \qquad \text{for } n \in \mathbb{Z}(n_0 + 2l + 1). \tag{6}$$

Let us show that $x_n \leq \sigma + l + 1$ for $n \in \mathbb{Z}(n_0 + 2l + 1)$ by contradiction (the proof for the other part is similar). Assume that there exists an $n_1 \in \mathbb{Z}(n_0 + 2l + 1)$ such that $x_{n_1} > \sigma + l + 1$. Let $n_2 \in \mathbb{Z}(-k, n_1)$ and $n_3 \in \mathbb{Z}(n_1)$ be the largest integer and smallest integer such that $x_{n_2} \leq \sigma$ and $x_{n_3} \leq \sigma$, respectively. Then $n_2 < n_1 - l - 1$. Otherwise, $n_2 \geq n_1 - l - 1$, which is positive. But, we have

$$\begin{aligned} x_{n_1} &= x_{n_2} + \sum_{m=n_2-k}^{n_1-k-1} (-f(x_m)) \\ &\leq x_{n_2} + \sum_{m=n_2-k}^{n_1-k-1} 1 \\ &\leq \sigma + (n_1 - n_2) \\ &\leq \sigma + l + 1, \end{aligned}$$

a contradiction to $x_{n_1} > \sigma + l + 1$. This proves $n_2 < n_1 - l - 1$. Similarly, we can show that $n_3 > n_1 + l + 1$. Thus, $\{x_{n_2+1}, \dots, x_{n_3-1}\}$ is a positive semicycle of length $n_3 - n_2 - 1$, which is larger than $2l+1$, a contradiction to the assumption of the lemma. This proves the claim. Using a similar argument, we can show that $x_{n_0+k} \in (\sigma + l, \sigma + l + 1]$. To simplify the following argument, for $\xi \in (\sigma + l, \sigma + l + 1]$, we denote A_ξ^+ be the positive semicycle $\{x_p, \dots, x_{p+2l}\}$

with $x_{p+l\pm m} = \xi - m$ for $m \in \mathbb{Z}(0,l)$ and A_ξ^- be the negative semicycle $\{x_q, \dots, x_{q+2l}\}$ with $x_{q+l\pm m} = \xi - (2l+1) + m$ for $m \in \mathbb{Z}(0,l)$. It follows from (3) and (4) that

$$x_{n_0+k+m} = x_{n_0+k} - m, \qquad \text{for } m \in \mathbb{Z}(0, 2l+1).$$

We claim that $\{x_{n_0+k-l}, \dots, x_{n_0+k+l}\}$ is a positive semicycle of type $A^+_{x_{n_0+k}}$. In fact, let $n_4 \in \mathbb{Z}(-k, n_0+k)$ be the largest integer such that $x_{n_4} \le \sigma$. Then $n_4 < n_0+k-l$. If $n_4 < n_0+k-l-1$, then $\{x_{n_4+1}, \dots, x_{n_0+k+l}\}$ is a positive semicycle of length larger than $2l+1$, a contradiction to the assumption of the lemma. So $n_4 = n_0 + k - l - 1$, and hence, $n_4 + 1 = n_0 + k - l$. Since the distance between each two consecutive elements in $\{x_{n_0+k-l}, \dots, x_{n_0+k}\}$ is just one, we conclude that $x_{n_0+k-m} = x_{n_0+k} - m$ for $m \in \mathbb{Z}(0,l)$. Thus, $\{x_{n_0+k-l}, \dots, x_{n_0+k+l}\}$ is a positive semicycle of type $A^+_{x_{n_0+k}}$. Since $x_{n_0+k+2l+1} \in (\sigma - l - 1, \sigma - l]$, similarly, we can show that $\{x_{n_0+k+l+1}, \dots, x_{n_0+k+3l+1}\}$ is a negative semicycle of type $A^-_{x_{n_0+k}}$. Continuing in this way, we can show that $\{x_{n_0+m_0k-m_0l}, \dots, x_{n_0+m_0k+(m_0+2)l+m_0}\}$ consists of A_ξ^+, A_ξ^-, …, consecutively, in total of m_0+1 semicycles, where $\xi = x_{n_0+m_0k-(m_0-1)l}$. Since $x_{n_0+(m_0+1)k-m_0l} \in (\sigma+l, \sigma+l+1]$ and $n_0 + (m_0+1)k - m_0l \in \mathbb{Z}(n_0 + m_0k + m_0l + m_0, n_0 + m_0k + (m_0+2)l + m_0)$, m_0 must be even and $n_0 + (m_0+1)k - m_0l = n_0 + m_0k + (m_0+1)l + m_0$. Thus, $k = (2m_0+1)l + m_0$. That is, $(k-l)/(2l+1)$ is an even integer. Moreover, we can easily see that $\{x_n\}_{n=-k}^\infty$ is truncated periodic with the minimal period $2(2l+1)$. This completes the proof. ∎

For the sake of simplicity, we introduce another notation. For $k \in \mathbb{Z}(0)$, denote $P(k) = \{l \in \mathbb{Z}(0);\ (k-l)/(2l+1) \in 2\mathbb{Z}(0)\}$. Obviously, $k \in P(k)$ for each $k \in \mathbb{Z}(0)$. Moreover, $0 \in P(k)$ if k is even and $0 \notin P(k)$ if k is odd.

Now, we are ready to prove the main results of this paper.

THEOREM 4. *Every solution $\{x_n\}_{n=-k}^\infty$ of (4) is truncated periodic with the minimal period $2(2l+1)$ for some $l \in P(k)$.*

PROOF. First, we assume that $\{x_n\}_{n=-k}^\infty$ has a semicycle of length larger than 1 eventually. Obviously, it has a semicycle of length larger than k for $k \in \mathbb{Z}(0,2)$. If $k \ge 3$, by Lemma 3, $\{x_n\}_{n=-k}^\infty$ has a semicycle of length larger than k eventually or it is truncated with the minimal period $2(2l+1)$ for some $l \in P(k)\setminus\{0,k\}$. Thus, by Proposition 2, $\{x_n\}_{n=-k}^\infty$ is truncated periodic with the minimal period $2(2l+1)$ for some $l \in P(k) \setminus \{0\}$. Now, assume that all semicycles of $\{x_n\}_{n=-k}^\infty$ are of length 1 eventually. Then there exists $n_0 \in \mathbb{Z}(-k)$ such that, for $n \in \mathbb{Z}(0)$,

$$x_{n_0+2n} \le \sigma \qquad \text{and} \qquad x_{n_0+2n+1} > \sigma.$$

If k is odd, then $\sigma \ge x_{n_0+k+1} = x_{n_0+k} - f(x_{n_0}) = x_{n_0+k} + 1 > \sigma$, a contradiction. Thus, not all semicycles of $\{x_n\}_{n=-k}^\infty$ have length 1 eventually. If k is even, then

$$\begin{aligned} x_{n_0+k+1} &= x_{n_0+k} - f(x_{n_0}) = x_{n_0+k} + 1, \\ x_{n_0+k+2} &= x_{n_0+k+1} - f(x_{n_0+1}) = x_{n_0+k+1} - 1 = x_{n_0+k}. \end{aligned}$$

By induction, for $n \in \mathbb{Z}(0)$, we have

$$x_{n_0+k+2n} = x_{n_0+k} \qquad \text{and} \qquad x_{n_0+k+2n+1} = x_{n_0+k} + 1.$$

Hence, $\{x_n\}_{n=-k}^\infty$ is truncated periodic with the minimal period 2. This completes the proof. ∎

Theorem 4 implies that all solutions of (4) are truncated periodic. This is very interesting. For example, when $k = 0$, it reflects the fact of Boolean property of computers. When there is delay, we can have more than one kind of periodic solutions. Hence, if many neurons are interconnected, we are expecting more complicated dynamical behavior and this can have potential applications in neural networks.

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