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The Structure of k-Planar Graphs

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THE STRUCTURE OF K-PLANAR GRAPHS

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July 13, 2019

Abstract. Dujmović et al. (FOCS 2019) recently proved that every planar graph is a subgraph of the strong product of a graph of bounded treewidth and a path. This tool has been used to solve longstanding problems on queue layouts, non-repetitive colouring, p-centered colouring, and implicit graph encoding. We generalise this result for k-planar graphs, where a graph is *k-planar* if it has a drawing in the plane in which each edge is involved in at most k crossings. In particular, we prove that every k-planar graph is a subgraph of the strong product of a graph of treewidth $O(k^5)$ and a path. This is the first result of this type for a non-minor-closed class of graphs. It implies, amongst other results, that k-planar graphs have non-repetitive chromatic number upper-bounded by a function of k. All these results generalise for drawings of graphs on arbitrary surfaces. In fact, we work in a much more general setting based on so-called shortcut systems that are of independent interest.

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1 Introduction

A graph is *k-planar* if it has a drawing in the plane in which each edge is involved in at most k crossings. Such graphs provide a natural generalisation of planar graphs, and are important in graph drawing research; see the recent bibliography on 1-planar graphs and the 140 references therein [26]. The present paper studies the structure of k-planar graphs and other more general classes of graphs.

Our main results generalise the following recent theorem of Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood [15] to a setting that includes k-planar graphs.

Theorem 1 ([15])**.** *Every planar graph is a subgraph of $H \boxtimes P$, for some graph H of treewidth at most 8 and for some path P.*

Here $\boxtimes$ is the strong product, and treewidth is an invariant that measures how 'tree-like' a given graph is; see Section 2 for formal definitions and see Figure 1 for an example. Loosely speaking, Theorem 1 says that every planar graph is contained in the product of a tree-like graph and a path. This enables combinatorial results for graphs of bounded treewidth to be generalised for planar graphs (with different constants). Theorem 1 has been the key tool in solving several well-known open problems. In particular, Dujmović et al. [15] use it to prove that planar graphs have bounded queue-number (resolving a conjecture of Heath, Leighton, and Rosenberg [22] from 1992); Dujmović, Esperet, Joret, Walczak, and Wood [14] use it to prove that planar graphs have bounded non-repetitive chromatic number (resolving a conjecture of Alon, Grytczuk, Hałuszczak, and Riordan [4] from 2002); and Bonamy, Gavoille, and Pilipczuk [6] use it to find shorter implicit representations of planar graphs (making progress on a sequence of results going back to 1988 [24, 25]).

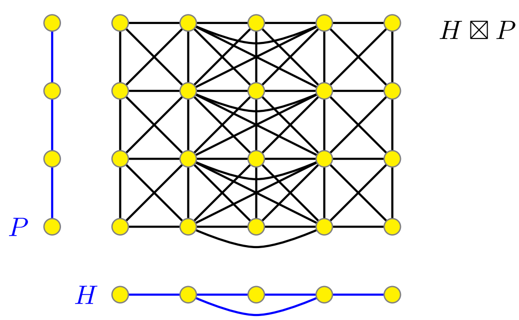

Figure 1: Example of a strong product.

We generalise Theorem 1 as follows.

Theorem 2. *Every k-planar graph is a subgraph of $H \boxtimes P$, for some graph H of treewidth $(18k^2 + 30k)\binom{k+4}{3} - 1$ and for some path P.*

Although k-planar graphs are the most high-profile target for a generalization of Theorem 1, we actually prove a substantially stronger result than Theorem 2 using the following definition. A collection $\mathcal{P}$ of paths in a graph G is a (k,d)-*shortcut system* (for G) if:

- every path in $\mathcal{P}$ has length at most k,[1] and
- for every $v \in V(G)$, the number of paths in $\mathcal{P}$ that use v as an internal vertex is at most d.

Each path $P \in \mathcal{P}$ is called a *shortcut*; if P has endpoints v and w then it is a vw-*shortcut*. Given a graph G and a (k,d)-shortcut system $\mathcal{P}$ for G, let $G^{\mathcal{P}}$ denote the supergraph of G obtained by adding the edge vw for each vw-shortcut in $\mathcal{P}$.

This definition is related to k-planarity because of the following observation:

Observation 1. *Every k-planar graph is a subgraph of $G^{\mathcal{P}}$ for some planar graph G and some $(k+1,2)$-shortcut system $\mathcal{P}$ for G.*

The proof of Observation 1 is trivial: Given a k-plane embedding of a graph G', create a planar graph G by adding a dummy vertex at each crossing point. For each edge $vw \in E(G')$ there is a path P in G between v and w of length at most $k+1$. Let $\mathcal{P}$ be the set of such paths P. For each vertex v of G, at most two paths in $\mathcal{P}$ use v as an internal vertex (since no original vertex of G' is an internal vertex of a path in $\mathcal{P}$). Thus $\mathcal{P}$ is a $(k+1,2)$-shortcut system for G, such that $G' \subseteq G^{\mathcal{P}}$.

We prove the following theorem that shows if a graph G is a subgraph of $H \boxtimes P$ and $\mathcal{P}$ is a shortcut system for G, then $G^{\mathcal{P}}$ is a subgraph of $J \boxtimes P$, where the treewidth of J is bounded by a function of the treewidth of H.

Theorem 3. *Let G be a subgraph of $H \boxtimes P$, for some graph H of treewidth at most t and for some path P. Let $\mathcal{P}$ be a (k,d)-shortcut system for G. Then $G^{\mathcal{P}}$ is a subgraph of $J \boxtimes P$ for some graph J of treewidth at most $d(k^3+3k)\binom{k+t}{t}-1$ and some path P.*

Theorems 1 and 3 and Observation 1 immediately imply Theorem 2 with $O(k^{11})$ instead of $O(k^5)$. Some further observations reduce the bound to $O(k^5)$; see Section 2.

Theorem 3 leads to several other results of interest. Here is one example. The *k-th power* of a graph G is the graph G^k with vertex set $V(G^k) := V(G)$, where $vw \in E(G^k)$ if and only if $\mathrm{dist}_G(v,w) \leqslant k$.[2] If G has maximum degree Δ, then $G^k = G^{\mathcal{P}}$ for some $(k, 2k\Delta^k)$-shortcut system $\mathcal{P}$; see Lemma 14. Theorems 2 and 3 then imply:

Theorem 4. *For every planar graph G with maximum degree Δ and for every integer $k \geqslant 1$, G^k is a subgraph of $H \boxtimes P$, for some graph H of treewidth at most $\Delta^k(2k^4+6k^2)\binom{k+8}{8}-1$.*

These theorems have applications in diverse areas, including queue layouts [15], non-repetitive colouring [14], and p-centered colouring [10], which we explore in Section 6. For example, we prove that k-planar graphs have bounded non-repetitive chromatic number (for fixed k). Prior to the recent work of Dujmović et al. [14], it was even open whether planar graphs have bounded non-repetitive chromatic number.

[1] A path of length k consists of k edges and $k+1$ vertices.

[2] For a graph G and two vertices $v,w \in V(G)$, $\mathrm{dist}_G(v,w)$ denotes the length of a shortest path, in G, with endpoints v and w. We define $\mathrm{dist}_G(v,w) := \infty$ if v and w are in different connected components of G.

Section 8 presents several examples of graph classes that can be obtained from a shortcut system applied to a planar graph, including graph powers, map graphs, string graphs, and nearest neighbour graphs. All of these results also apply, where instead of planar graphs, we consider graphs of bounded Euler genus. All of the applications discussed in Section 6 work on these graph classes.

2 Layerings, Decompositions and Partitions

This section defines concepts and results from the literature that will be important for our work.

In this paper, all graphs are finite and undirected. Unless specifically mentioned otherwise, all graphs are also simple. For any graph G and any set S (typically $S \subseteq V(G)$), let $G[S]$ denote the graph with vertex set $V(G) \cap S$ and edge set $\{uv \in E(G) : u, v \in S\}$. We use $G - S$ as a shorthand for $G[V(G) \setminus S]$. We use $G' \subseteq G$ to denote subgraph containment; that is, $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$.

We now formally define k-planar graphs. An *embedded graph* G is a graph with $V(G) \subset \mathbb{R}^2$ in which each edge $vw \in E(G)$ is a closed curve[3] in $\mathbb{R}^2$ with endpoints v and w and not containing any vertex of G in its interior. A *crossing* in an embedded graph G is a triple (p, vw, xy) with $p \in \mathbb{R}^2$, $vw, xy \in E(G)$ and such that $p \in (vw \cap xy) \setminus \{v, w, x, y\}$. An embedded graph G is *k-plane* if each edge of G takes part in at most k crossings. A (not necessarily embedded) graph G' is *k-planar* if there exists a k-plane graph G isomorphic to G'. Under these definitions, 0-planar graphs are exactly planar graphs and 0-plane graphs are exactly plane graphs.

We now define two concepts used in the theorems in Section 1: strong products and treewidth. The *strong product* of graphs A and B, denoted by $A \boxtimes B$, is the graph with vertex set $V(A) \times V(B)$, where distinct vertices $(v, x), (w, y) \in V(A) \times V(B)$ are adjacent if:

- $v = w$ and $xy \in E(B)$, or
- $x = y$ and $vw \in E(A)$, or
- $vw \in E(A)$ and $xy \in E(B)$.

A *tree-decomposition* $\mathcal{T}$ of a graph G consists of a tree T and a collection $\mathcal{T} = (B_x : x \in V(T))$ of subsets of $V(G)$ indexed by the nodes of T such that:

(i) for every $vw \in E(G)$, there exists some node $x \in V(T)$ with $v, w \in B_x$; and
(ii) for every $v \in V(G)$, the induced subgraph $T[v] := T[\{x : v \in B_x\}]$ is connected.

The *width* of the tree-decomposition $\mathcal{T}$ is $\max\{|B_x| : x \in V(T)\} - 1$. The *treewidth* $\operatorname{tw}(G)$ of a graph G is the minimum width of a tree-decomposition of G. Treewidth is the standard measure of how similar a graph is to a tree. Indeed, a connected graph has treewidth 1 if and only if it is a tree. Treewidth is of fundamental importance in structural and algorithmic graph theory; see [5, 21, 33] for surveys.

While strong products enable concise statements of the theorems in Section 1, to prove such results it is helpful to work with layerings and partitions, which we now introduce.

[3] A closed curve in a surface Σ is a continuous function $f : [0,1] \to \Sigma$. The points $f(0)$ and $f(1)$ are called the *endpoints* of the curve. When there is no danger of misunderstanding we treat a curve f as the point set $\{f(t) : 0 \leqslant t \leqslant 1\}$.

A *layering* of a graph G is a sequence $\mathcal{L} = \langle V_0, V_1, \ldots\rangle$ such that $\{V_0, V_1, \ldots\}$ is a partition of $V(G)$ and for every edge $vw \in E(G)$, if $v \in V_i$ and $w \in V_j$ then $|j-i| \leqslant 1$. For any partition $\mathcal{P} = \{S_1, \ldots, S_p\}$ of $V(G)$, a *quotient graph* $H = G/\mathcal{P}$ has a p-element vertex set $V(H) = \{x_1, \ldots, x_p\}$ and $x_i x_j \in E(H)$ if and only if there exists an edge $vw \in E(G)$ such that $v \in S_i$ and $w \in S_j$. To highlight the importance of the quotient graph H, we call $\mathcal{P}$ an *H-partition* and write this concisely as $\mathcal{P} = \{S_x : x \in V(H)\}$ so that each element of $\mathcal{P}$ is indexed by the vertex it creates in H.

For any partition $\mathcal{P}$ of $V(G)$ and any layering $\mathcal{L}$ of G we define the *layered width* of $\mathcal{P}$ with respect to $\mathcal{L}$ as $\max\{|L \cap P| : L \in \mathcal{L}, P \in \mathcal{P}\}$. For any partition $\mathcal{P}$ of $V(G)$, we define the *layered width* of $\mathcal{P}$ as the minimum, over all layerings $\mathcal{L}$ of G, of the layered width of $\mathcal{P}$ with respect to $\mathcal{L}$.

2.1 Previous Work on Partitions of Minor-Closed Classes

Dujmović et al. [15] introduced the study of partitions with bounded layered width such that the quotient has some additional desirable property, like small treewidth. They defined a class $\mathcal{G}$ of graphs to *admit bounded layered partitions* if there exist $t, \ell \in \mathbb{N}$ such that every graph $G \in \mathcal{G}$ has an H-partition of layered width at most ℓ for some graph $H = H(G)$ of treewidth at most t.

These definitions relate to strong products as follows.

Lemma 1 ([15])**.** *For every graph H, a graph G has an H-partition of layered width at most ℓ if and only if G is a subgraph of $H \boxtimes P \boxtimes K_\ell$ for some path P.*

Dujmović et al. [15] also showed it suffices to consider partitions of layered width 1.

Lemma 2 ([15])**.** *If a graph G has an H-partition of layered width ℓ for some graph H of treewidth at most t, then G has an H'-partition of layered width 1 for some graph H' of treewidth at most $(t+1)\ell - 1$. That is, if $G \subseteq H \boxtimes P \boxtimes K_\ell$ for some graph H of treewidth at most t and for some path P, then $G \subseteq H' \boxtimes P$ for some graph H' of treewidth at most $(t+1)\ell - 1$.*

Dujmović et al. [15] proved the following result, which with Lemma 2, implies Theorem 1.

Theorem 5 ([15])**.** *Every planar graph has:*

(a) an H-partition of layered width 1 for some planar graph H of treewidth at most 8, and
(b) an H-partition of layered width 3 for some planar graph H of treewidth at most 3.

Their proof is constructive and gives a simple quadratic-time algorithm for finding these partitions and corresponding layerings. The same authors proved the following generalisation of Theorems 1 and 5 for graphs embeddable on other surfaces.[4]

Theorem 6 ([15])**.** *Every graph of Euler genus g is a subgraph of:*

[4]The *Euler genus* of the orientable surface with h handles is $2h$. The *Euler genus* of the non-orientable surface with c cross-caps is c. The *Euler genus* of a graph G is the minimum integer g such that G embeds in a surface of Euler genus g. Of course, a graph is planar if and only if it has Euler genus 0; see [28] for more about graph embeddings in surfaces.

(a) $H \boxtimes P \boxtimes K_{\max\{2g,1\}}$ *for some graph* H *of treewidth at most* 9 *and for some path* P*;*
(b) $H \boxtimes P \boxtimes K_{\max\{2g,3\}}$ *for some graph* H *of treewidth at most* 4 *and for some path* P*.*

Equivalently, every graph of Euler genus g *has:*

(a) *an* H*-partition with layered width at most* $\max\{2g, 1\}$ *such that* H *has treewidth at most* 9*;*
(b) *an* H*-partition with layered width at most* $\max\{2g, 3\}$ *such that* H *has treewidth at most* 4*.*

Dujmović et al. [15] generalised Theorem 6 further as follows.[5]

Theorem 7 ([15])**.** *For every apex graph* X*, there exists* $c \in \mathbb{N}$ *such that every* X*-minor-free graph* G *has an* H*-partition with layered width 1 such that* H *has treewidth at most* c*. Equivalently,* $G \subseteq H \boxtimes P$ *for some path* P*.*

3 New Results

Apex-minor-free graphs, addressed by Theorem 7, are the largest minor-closed class that admit bounded layered partitions [15]. However, the family of k-planar graphs is not minor-closed. A graph G' obtained from a k-planar graph G by edge deletions and edge contractions may or may not be k-planar. Indeed, Dujmović, Eppstein, and Wood [12] construct 1-planar graphs that contain arbitrarily large complete graph minors. Our results for k-planar graphs are the first of this type for a non-minor-closed class.

The following result is the main theorem of the paper. Loosely speaking, it shows that if a graph G admits bounded layered partitions, then so to does $G^{\mathcal{P}}$ for every (k,d)-shortcut system $\mathcal{P}$ of G.

Theorem 8. *Let* G *be a graph having an* H*-partition of layered width* ℓ *in which* H *has treewidth at most* t *and let* $\mathcal{P}$ *be a* (k,d)*-shortcut system for* G*. Then* $G^{\mathcal{P}}$ *has a* J*-partition of layered width at most* $d\ell(k^3 + 3k)$ *for some graph* J *of treewidth at most* $\binom{k+t}{t} - 1$*.*

Theorem 8 and Lemma 2 immediately imply Theorem 3 in the introduction.

Using the relationship between k-planar graphs and $(k+1,2)$-shortcut systems along with a direct application of Theorem 8 we can obtain a slightly weaker version of Theorem 9, below. The minor modifications needed to obtain the stronger bound are described in Section 5.

Theorem 9. *Every* k*-planar graph has an* H*-partition of layered width at most* $18k^2 + 30k$ *in which* H *has treewidth at most* $\binom{k+4}{3} - 1$*.*

In the important special case of $k = 1$ we obtain better constants and an additional property (planarity) of H (see Section 5.1 for the proof):

[5]A graph M is a *minor* of a graph G if a graph isomorphic to M can be obtained from a subgraph of G by contracting edges. A class $\mathcal{G}$ of graphs is *minor-closed* if for every graph $G \in \mathcal{G}$, every minor of G is in $\mathcal{G}$. A minor-closed class is *proper* if it is not the class of all graphs. For example, for fixed $g \geqslant 0$, the class of graphs with Euler genus at most g is a proper minor-closed class. A graph G is t-apex if it contains a set A of at most t vertices such that $G - A$ is planar. A 1-apex graph is *apex*. A minor-closed class $\mathcal{G}$ is apex-minor-free if some apex graph is not in $\mathcal{G}$.

Theorem 10. *Every 1-planar graph has an H-partition of layered width at most 30 where H is planar and has treewidth at most 3.*

The definition of k-planar graphs naturally generalises for other surfaces. A graph G drawn on a surface Σ is (Σ, k)-plane if every edge of G is involved in at most k crossings. A graph G is (g, k)-planar if it is isomorphic to some (Σ, k)-plane graph, for some surface Σ with Euler genus at most g. Observation 1 immediately generalises as follows:

Observation 2. *Every (g, k)-planar graph is a subgraph of $G^{\mathcal{P}}$ for some graph G of Euler genus at most g and some $(k+1, 2)$-shortcut system $\mathcal{P}$ for G.*

We prove that (g, k)-planar graphs admit bounded layered partitions.

Theorem 11. *Every (g, k)-planar graph has an H-partition of layered width at most* $\max\{2g, 3\} \cdot (6k^2 + 10k)$ *in which H has treewidth at most $\binom{k+5}{4} - 1$.*

Again, a direct application of Theorem 8 using Theorem 6(b) implies Theorem 11 with a weaker bound on the layered width. We prove the stronger bound in Section 5.

Finally, we state the following corollary of Theorems 7 and 8 and Lemma 2. This is the most general result that follows from Theorem 8 and the work of Dujmović et al. [15].

Theorem 12. *For every apex graph X and for all integers $k, d \geqslant 1$, there is an integer c such that for every X-minor-free graph G and for every (k, d)-shortcut system $\mathcal{P}$ for G, $G^{\mathcal{P}}$ has an H-partition of layered width 1 such that H has treewidth at most c; that is $G^{\mathcal{P}} \subseteq H \boxtimes P$ for some path P.*

3.1 Previous Work on the Structure of (g, k)-Planar Graphs

Prior to this work, the strongest structural description of k-planar or (g, k)-planar graphs was in terms of layered treewidth, which we now define. A *layered tree-decomposition* $(\mathcal{L}, \mathcal{T})$ consists of a layering $\mathcal{L}$ and a tree-decomposition $\mathcal{T}$ of G. The layered width of $(\mathcal{L}, \mathcal{T})$ is $\max\{|L \cap B| : L \in \mathcal{L}, B \in \mathcal{T}\}$. The *layered treewidth* of G is the minimum layered width of any layered tree-decomposition of G. Dujmović, Morin, and Wood [13] proved that planar graphs have layered treewidth at most 3, that graphs of Euler genus g have layered treewidth at most $2g + 3$, and more generally that a minor-closed class has bounded layered treewidth if and only if it excludes some apex graph. Dujmović et al. [12] show that every k-planar graph has layered treewidth at most $6(k+1)$, and more generally that every (g, k)-planar graph has layered treewidth at most $(4g+6)(k+1)$. It follows from this result that (g, k)-planar graphs have treewidth $O(\sqrt{(g+1)(k+1)n})$ and thus have balanced separators of the same order, which can also be concluded from the work of Fox and Pach [17]. In related work, Grigoriev and Bodlaender [20] used structural results to obtain approximation algorithms for (g, k)-planar graphs, and Pach and Tóth [30] determined the maximum number of edges in a k-planar graph (up to a constant factor).

If a graph class admits bounded layered partitions, then it also has bounded layered treewidth. In particular, Dujmović et al. [15] proved that if a graph G has an H-partition with layered width at most ℓ such that H has treewidth at most k, then G has layered treewidth at most $(k+1)\ell$. So any property that holds for graphs of bounded layered treewidth also holds for G. What sets layered partitions apart from layered treewidth is that they lead to constant upper bounds on the queue-number and non-repetitive chromatic number, whereas for both these parameters, the best known upper bound obtainable via layered treewidth is $O(\log n)$; see Section 6.

4 Shortcut Systems

The purpose of this section is to prove our main result, Theorem 8. This theorem shows how, given a (k,d)-shortcut system $\mathcal{P}$ of a graph G, a H-partition of G can be used to obtain a J-partition of $G^{\mathcal{P}}$ where the layered width does not increase dramatically and the treewidth of J is not much more than the treewidth of H. Our main results for k-planar graphs (Theorem 9) and (g,k)-planar graphs (Theorem 11) follow; see Section 5.

For convenience, it will be helpful to assume that $\mathcal{P}$ contains a length-1 vw-shortcut for every edge $vw \in E(G)$. Since $G^{\mathcal{P}}$ is defined to be a supergraph of G, this assumption has no effect on $G^{\mathcal{P}}$ but eliminates special cases in some of our proofs.

For a tree T rooted at some node $x_0 \in V(T)$, we we say that a node $a \in V(T)$ is a T-*ancestor* of $x \in V(T)$ (and x is a T-*descendant* of a) if a is a vertex of the path, in T, from x_0 to x. Note that each node $x \in v(T)$ is a T-ancestor and T-descendant of itself. We say that a T-ancestor $a \in V(T)$ of $x \in V(T)$ is a *strict* T-ancestor of x if $a \neq x$. The T-*depth* of a node $x \in V(T)$ is the length of the path, in T, from x_0 to x. For each node $x \in V(T)$, define

$$T_x := T[\{y \in V(T) : x \text{ is a } T\text{-ancestor of } y\}]$$

to be the maximal subtree of T rooted at x.

We begin with a fairly standard lemma about normalised tree decompositions.

Lemma 3. *For every graph H of treewidth t, there is a rooted tree T with $V(T) = V(H)$ and a T-decomposition $(B_x : x \in V(T))$ of H with width t that has following additional properties:*

(T1) For each node $x \in V(H)$, the subtree $T[x] := T[\{y \in V(T) : x \in B_y\}]$ is rooted at x.

(T2) For each edge $xy \in E(H)$, one of x or y is a T-ancestor of the other.

Proof. Begin with any width-t tree decomposition $(B_x : x \in V(T_0))$ of H that uses some tree T_0. Select any node $x \in V(T_0)$, add a leaf x_0, with $B_{x_0} = \emptyset$, adjacent to x and root T_0 at x_0. Let $f : V(H) \to V(T)$ be the function that maps each $x \in V(H)$ onto the root of the subtree $T_0[x] := T_0[\{y \in V(T_0) : x \in B_y]$. If f is not one-to-one, then select some distinct pair $x, y \in V(H)$ with $a := f(x) = f(y)$. Subdivide the edge between a and its parent in T by introducing a new node a' with $B_{a'} = B_a \setminus \{x\}$. This modification reduces the number of distinct pairs $x, y \in V(H)$ with $f(x) = f(y)$, so repeatedly performing this modification will eventually produce a tree-decomposition $(B_x : x \in V(T_0))$ of H in which f is one-to-one.

Next, consider any node $a \in V(T_0)$ such that there is no vertex $x \in V(H)$ with $f(x) = a$. In this case, $B_a \subseteq B_{a'}$ where a' is the parent of a since any $x \in B_a \setminus B_{a'}$ would have $f(x) = a$. In this case, contract the edge aa' in T_0, eliminating the node a. Repeating this operation will eventually produce a width-t tree-decomposition of $(B_x : x \in V(T_0))$ where f is a bijection between $V(H)$ and $V(T_0)$. Renaming each node $a \in V(T_0)$ as $f^{-1}(a)$ gives a tree-decomposition $(B_x : x \in V(T))$ with $V(T) = V(H)$.

By the definition of f, the tree-decomposition $(B_x : x \in V(T))$ satisfies (T1). To see that $(B_x : x \in V(T))$ satisifies (T2), observe that, if $xy \in E(H)$, then at least one of x or y is contained in B_z for every node z on the path from x to y in T. If neither x nor y is an ancestor of the other, then some node z on this path has T-depth less than that of x and y. If $x \in B_z$ this contradicts the fact that x is the root of $T[x]$. If $y \in B_z$ this contradicts the fact that y is the root of $T[y]$. □

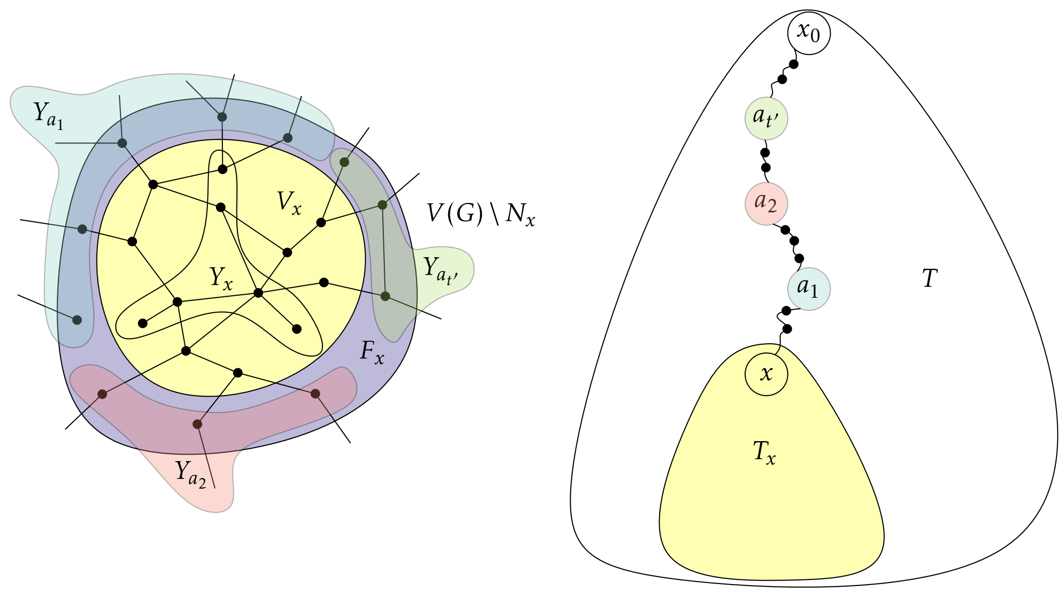

Figure 2: The sets Y_x, F_x, and V_x associated with $x \in V(T)$ and the ancestors $a_1, \dots, a_{t'}$ of X such that $F_x \subseteq \bigcup_{i=1}^{t'} Y_{a_i}$.

The following lemma shows how to interpret an H-partition of G and a tree-decomposition of H as a hierarchical decomposition of G; refer to Figure 2.

Lemma 4. *Let G be a graph; let $\mathcal{L} := \langle V_1, \dots, V_h \rangle$ be a layering of G; let $\mathcal{Y} := (Y_x : x \in V(H))$ be an H-partition of G of layered width at most ℓ with respect to $\mathcal{L}$ where H has treewidth at most t; and let $\mathcal{T} := (B_x : x \in V(T))$ be a tree-decomposition of H satisfying the conditions of Lemma 3. For each $x \in V(T)$, let $V_x := \bigcup_{y \in V(T_x)} Y_y$, $F_x := \{w \in V(G) : vw \in E(G), v \in V_x, w \notin V_x\}$, and $N_x := V_x \cup F_x$. Then,*

(Y1) $\mathcal{Y} := (Y_x : x \in V(T))$ is a partition of $V(G)$ of layered width at most ℓ with respect to $\mathcal{L}$.

(Y2) For each $x \in V(T)$, there is no edge $vw \in E(G)$ with $v \in V_x$ and $w \in V(G) \setminus N_x$.

(Y3) For each $x \in V(T)$, there is a set $\{a_1, \dots, a_{t'}\}$ of $t' \leqslant t$ strict T-ancestors of x such that $F_x \subseteq \bigcup_{i=1}^{t'} Y_{a_i}$.

Before proving Lemma 4 we point out more properties that are immediately implied by it:

(Y4) $Y_x \subseteq V_x$ for every $x \in V(T)$.

(Y5) $V_x \subseteq V_a$ for every T-ancestor a of x.

(Y6) $N_x \subseteq N_a$ for every T-ancestor a of x.

Property (Y4) follows from the fact that V_x is the union of several sets, one of which is Y_x. Property (Y5) follows from the definition of V_x and the fact that $V(T_x) \subseteq V(T_a)$. To show Property (Y6) first note that, by (Y5) it suffices to consider vertices $w \in F_x = N_x \setminus V_x$. By definition, every vertex $w \in F_x$ is adjacent, in G, to a vertex $v \in V_x$. By (Y5), $v \in V_a$, so w is either in V_a or w satisfies the condition $vw \in E(G)$, $v \in V_a$, and $w \notin V_a$, so $w \in F_a$. In either case $w \in N_a = V_a \cup F_a$. Note that none of (Y4)–(Y6) depends on (Y3) (which is important, since (Y5) is used to establish (Y3) in the following proof).

Proof of Lemma 4. Property (Y1) follows immediately from the fact that $V(T) = V(H)$ and the fact that $\mathcal{Y}$ has layered width at most ℓ with respect to $\mathcal{L}$. Property (Y2) is immediate from the definitions of F_x and N_x. In particular, $(N_x, V(G) \setminus V_x)$ is a separation of G with $F_x = N_x \cap (V(G) \setminus V_x)$.

To establish Property (Y3), consider some vertex $w \in F_x$. Since $w \in F_x$, there exists an edge $vw \in E(G)$ with $v \in V_x$ and $w \notin V_x$. Since $v \in V_x$, $v \in Y_{x'}$ for some T-descendant x' of x (possibly $x = x'$). Since $\mathcal{Y}$ is a partition, $w \in Y_a$ for some $a \notin V(T_x)$. Since $vw \in E(G)$, we have $x'a \in E(H)$. By (T2), one of a or x' is a T-ancestor of the other. Since $w \in Y_a \subseteq V_a$ and $w \notin V_x \supseteq V_{x'}$, (Y5) rules out the possibility that x' is a T-ancestor of a. Therefore, a is a T-ancestor of x which is a T-ancestor of x'. Let $z_0, \dots, z_r$ be the path in T from $z_0 := x'$ to $z_r := a$. For each $i \in \{0, \dots, r\}$, at least one of a or x' is in B_{z_i}. However, by (T1) x' is not contained in B_{x_i} for any $i \in \{1, \dots, r\}$. Therefore $a \in B_{x_i}$ for each $i \in \{0, \dots, r\}$. In particular, a is contained in B_x. Property (Y3) now follows from the fact that $|B_x| \leqslant t+1$ and B_x contains x. □

We are now ready to prove our main result, which we restate here for convenience:

Theorem 8. *Let G be a graph having an H-partition of layered width ℓ in which H has treewidth at most t and let $\mathcal{P}$ be a (k,d)-shortcut system for G. Then $G^{\mathcal{P}}$ has a J-partition of layered width at most $d\ell(k^3 + 3k)$ for some graph J of treewidth at most $\binom{k+t}{t} - 1$.*

Proof. Apply Lemma 4 to G and let $\mathcal{L}$, $\mathcal{Y}$, $\mathcal{T}$, T, Y_x, V_x, F_x, and N_x be defined as in Lemma 4.

For a node $x \in V(T)$, we say that a shortcut $P \in \mathcal{P}$ *crosses* x if Y_x contains an internal vertex of P, that is, $P = (v_0, \dots, v_r)$ and $\{v_1, \dots, v_{r-1}\} \cap Y_x \neq \emptyset$. We say that a vertex $v \in V(G)$ *participates* in x if $v \in Y_x$ or $\mathcal{P}$ contains a shortcut P with $v \in V(P)$ and P crosses x. We let X_v denote the set of nodes $x \in V(T)$ such that v participates in x.

Claim 1. *For any $v \in V(G)$ there exists a (unique) node $a(v) \in X_v$ such that $a(v)$ is a T-ancestor of every node in X_v.*

Proof. Let

$$Z := \{v\} \cup \{\{v_1, \dots, v_{r-1}\} : (v_0, \dots, v_r) \in \mathcal{P},\ v \in \{v_0, \dots, v_r\}\}$$

Clearly $G[Z]$ is connected because Z is the union of (vertex sets of) paths in G, each of which contains v.

We claim that v participates in a node $x \in V(T)$ if and only if $Z \cap Y_x \neq \emptyset$. If v participates in x then either $v \in Y_x$, so $Z \cap Y_x \supseteq \{v\}$; or $v \in \{v_0, \dots, v_r\}$ for some shortcut $(v_0, \dots, v_r) \in \mathcal{P}$

that crosses x, so $Z \cap Y_x \supseteq \{v_i\}$ for some $i \in \{1,\dots,r-1\}$. In the other direction, if $Z \cap Y_x \neq \emptyset$, then either $Z \cap Y_x \supseteq \{v\}$, so $v \in Y_x$; or $Z \cap Y_x \supseteq \{v_i\}$ where $i \in \{1,\dots,r\}$, $(v_0,\dots,v_r) \in \mathcal{P}$ and $v \in \{v_0,\dots,v_r\}$, so $v \in V(P)$ for a path $P = (v_0,\dots,v_r) \in \mathcal{P}$ that crosses x.

Let $Z_H := \{x \in V(H) : Z \cap Y_x \neq \emptyset\}$. The connectivity of $G[Z]$ implies that $H[X_H]$ is connected. Choose $a(v) \in X_H$ to be any member of X_H that does not have a strict T-ancestor in X_H. Transitivity of the T-ancestor relationship, (T2), and connectivity of $H[X_H]$ implies that $a(v)$ is a T-ancestor of every node $x \in X_H$, as required. □

For each $x \in V(T)$, we define the *separator*

$$S_x := \{v \in V(G) : a(v) = x\} \ .$$

Observe that $\mathcal{S} := (S_x : x \in V(T))$ is a partition of $V(G)$. We let $J := G^{\mathcal{P}}/\mathcal{S}$ denote the resulting quotient graph and we let $V(J) \subseteq V(T)$ in the obvious way, so that each $x \in V(J)$ is the vertex obtained by contracting S_x in $G^{\mathcal{P}}$. (Nodes $x \in V(T)$ with $S_x = \emptyset$ do not contribute a vertex to J.)

Claim 2. *For every $x \in V(T)$, $S_x \subseteq V_x$.*

Proof. For the sake of contradiction, assume otherwise, so there exists some $v \in S_x \setminus V_x$. By (Y4), $Y_x \subseteq V_x$, so $v \notin Y_x$. Therefore, $\mathcal{P}$ contains a path P, with $v \in V(P)$, that crosses x. The path P contains a subpath $v_0, v_1, \dots, v_r$ such that $v = v_0$ and $v_r \in Y_x$. Since $v \notin V_x$ and $v_r \in Y_x \subseteq V_x$, (Y2), implies that $v_i \in F_x$ for some $i \in \{0,\dots,r-1\}$. Now (Y3) implies $v_i \in Y_a$ for some strict T-ancestor a of x. Therefore, either $v \in Y_a$ or P crosses a. But this implies that $a(v)$ is a T-ancestor of a, which is a strict T-ancestor of x, contradicting the assumption that $v \in S_x$. □

Next we show that $\mathcal{S}$ has small layered width with respect to $\mathcal{L}$:

Claim 3. *For each $i \in \{1,\dots,h\}$ and each $x \in V(J)$, $|S_x \cap V_i| \leqslant d\ell(k^2+3)$.*

Proof. We count the number of vertices in $S_x \cap V_i$ by upper-bounding the number of vertices contributed to $S_x \cap V_i$ by each vertex $w \in Y_x$. Refer to Figure 3. If $w \in Y_x \cap V_i$ and no path in $\mathcal{P}$ includes w as an internal vertex then w contributes only one vertex, itself, to $S_x \cap V_i$.

Otherwise, consider some path $P \in \mathcal{P}$ that contains w as an internal vertex. If $w \in V_i$, then P contributes at most $k+1$ vertices to $S_x \cap V_i$. If $w \in V_{i-1} \cup V_{i+1}$, then P contributes at most k vertices to $S_x \cap V_i$. If $w \in V_{i-j} \cup V_{i+j}$ for $j \geqslant 2$, then P contributes at most $k-j$ vertices to $S_x \cap V_i$.

For any j, the number of vertices $w \in V_{i+j} \cap Y_x$ is at most ℓ. Each such vertex w is an internal vertex of at most d paths in $\mathcal{P}$. Therefore,

$$|S_x \cap V_i| \leqslant d\ell \cdot \left(k+1+2k+\sum_{j=2}^{k} 2(k-j)\right) = d\ell(k^2+3) \ . \qquad \square$$

Claim 4. *For each edge $xy \in E(J)$, one of x or y is a T-ancestor of the other.*

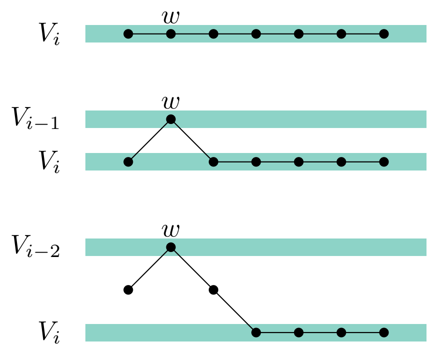

Figure 3: A path P containing an internal vertex $w \in Y_x \cap V_{i-j}$.

Proof. Suppose, for the sake of contradiction, that neither x nor y is a T-ancestor of the other. Since $xy \in E(J)$, $G^{\mathcal{P}}$ contains an edge vw with $v \in S_x$ and $w \in S_y$. Since $vw \in E(G^{\mathcal{P}})$, $\mathcal{P}$ contains a vw-shortcut P. By Claim 2, $v \in V_x$ and $w \in V_y$. By (Y5), if neither x nor y is a T-ancestor of the other, then V_x and V_y are disjoint. By (Y3) N_x and V_y are also disjoint. By (Y2) P contains an internal vertex $v' \in F_x$. By (Y3), $v' \in Y_a$ for some strict T-ancestor a of x. But this implies that $a(v) = a'$ so $v \in S_{a'}$ for some T-ancestor a' of a, contradicting the assumption that $v \in S_x$. □

Claim 5. *The graph J has a tree-decomposition in which every bag has size at most $\binom{k+t}{t}$.*

Proof. For the tree-decomposition $(C_x : x \in V(T))$ of J we use the same tree T used in the tree-decomposition $(B_x : x \in V(T))$ of H. For each node x of T, C_x contains x as well as every T-ancestor a of x such that J contains an edge ax' where x is a T-ancestor of x' (including the possibility that $x = x'$). Claim 4 ensures that, for every edge $ax' \in E(J)$, $a, x' \in C_{x'}$ and the connectivity of $T[a] := T[\{x \in V(T) : a \in C_x]$ is obvious. Therefore $(C_x : x \in V(T))$ is indeed a tree-decomposition of J. It remains to bound the size of each bag C_x.

Consider an arbitrary node $x \in V(T)$ where $x_0, \dots, x_r$ is the path from the root x_0 of T to $x_r := x$. To avoid triple-subscripts in what follows, we abuse notation slightly by using V_i, F_i, and N_i, as shorthands for V_{x_i}, F_{x_i} and N_{x_i}, respectively.

If $x_\delta \in C_x$, it is because $x_\delta x' \in E(J)$ for some T-descendant x' of x. This implies $G^{\mathcal{P}}$ contains an edge vw with $v \in S_{x'}$ and $w \in S_{x_\delta} = S_\delta$. This implies that $\mathcal{P}$ contains a vw-shortcut P_{vw}. Let v' be the second-last vertex of P_{vw} (so $v'w \in E(G)$).

Since $w \in S_\delta$, at least one of the following is true:

1. $\mathcal{P}$ contains a ww'-shortcut $P_{ww'}$ that has an internal vertex in Y_δ; or
2. $w \in Y_\delta$. In this case, we define $P_{ww'}$ to be the path of length 0 that contains only $w = w'$.

Let w'' denote the first vertex of $P_{ww'}$ contained in Y_δ.

Let $w_0, w_1, \dots, w_p$ be the path that begins $w_0 := v'$ and then follows the subpath of $P_{ww'}$ that begins at $w_1 := w$ and ends at $w_p := w''$. For each $i \in \{0, \dots, p\}$, let $s_i = \max\{j \in$

$\{1,\ldots,r\}:\{w_0,\ldots,w_i\}\subseteq V_j\}\}$, and let $a_i = x_{s_i}$. Note that $s_0,\ldots,s_p$ is a non-increasing sequence and $a_0,\ldots,a_p$ is a sequence of nodes of T whose distance from the root, x_0, of T is non-increasing.

By definition, $a_0 = x_r$. We claim that $a_p = x_\delta$, i.e., $s_p = \delta$. To see this, first observe that, for each $i \in \{1,\ldots,p\}$, $w_i \in V_\delta$ since, otherwise, an internal vertex of $P_{ww'}$ belong to F_δ, which would imply (by (Y3)) that $w \in S_{\delta'}$ for some $\delta' < \delta$, contradicting the assumption that $w \in S_\delta$. Therefore $s_p \geqslant \delta$. To see that $s_p < \delta - 1$, observe that either $w = w'' \in Y_\delta$ or $P_{ww'}$ contains an internal vertex w'' in Y_δ. By (Y1) and the defnition of V_x, $V_{\delta-1}$ does not contain w'', so $s_p < \delta - 1$.

Let H^+ denote the supergraph of H with vertex set $V(T)$ and in which $xy \in E(H^+)$ if and only there exists some $z \in V(T)$ such that $x, y \in B_z$. We claim that $a_0,\ldots,a_p$ is a lazy walk[6] in H^+. Indeed, if $a_i \neq a_{i+1}$ for some $i \in \{0,\ldots,p-1\}$ then this is precisely because $w_i \in V_{a_i}$ but $w_{i+1} \notin V_{a_i}$. By definition, $w_i \in Y_{a'_i}$ for some T-descendant a'_i of a_i. By (Y2), $w_{i+1} \in F_{a_i}$ so by (Y3) $w_{i+1} \in Y_{a''_i}$ for some strict T-ancestor a''_i of a_i. Since $w_i w_{i+1} \in E(G)$, $a'_i a''_i \in E(H)$. By (T1), $a''_i \in B_{a''_i}$ and $a''_i \in B_{a'_i}$. Since a_i is on the path from a'_i to a''_i in T this implies that $a''_i \in B_{a_i}$. Therefore $a_i a''_i \in E(H^+)$ as claimed.

Thus, $a_0,\ldots,a_p$ is a lazy walk in H^+ of length $p \leqslant k$ where the distance s_i between a_i and the root x_0 of T is non-decreasing. By removing repeated vertices this gives a path in the directed graph $\overrightarrow{H}^+$ obtained by directing each edge $xy \in E(H^+)$ from its T-descendant x towards its T-ancestor y. Finally, we are in a position to appeal to [32, Lemma 24] which states that the number of nodes in $\overrightarrow{H}^+$ that can be reached from any node x by a directed path of length at most k is at most $\binom{k+t}{t}$. □

At this point, the proof of Theorem 8 is almost immediate from Claim 3 and Claim 5, except that the layering $\mathcal{L}$ of G may not be a valid layering of $G^{\mathcal{P}}$. In particular, $G^{\mathcal{P}}$ may contain edges vw with $v \in V_i$ and $w \in V_{i+j}$ for any $j \in \{0,\ldots,k\}$. To resolve this, we use a new layering $\mathcal{L}' := \langle V'_0,\ldots,V'_h\rangle$ in which $V'_i = \bigcup_{j=ki}^{ki+k-1} V_i$. This increases the layered width given by Claim 3 from $d\ell(k^2+3)$ to $d\ell(k^3+3k)$. Therefore G has an H-partition of layered width at most $d\ell(k^3+3k)$ in which H has treewidth at most $\binom{k+t}{t}$, completing the proof of Theorem 8. □

5 k-Planar Graphs

This section shows that the constants in Theorem 8 can be improved in the case of k-planar graphs. Let G be a k-plane graph. We will assume, for ease of exposition, that any point $p \in \mathbb{R}^2$ is involved in at most one crossing (p, vw, xy) of G. This assumption is justified since it can be enforced by a slight deformation of the edges of G and the resulting (deformed) graph is also k-plane.

As in the proof of Observation 1, let G_0 be the plane graph obtained by adding a dummy vertex at each crossing in G. In this way, each edge $vw \in E(G)$ corresponds naturally to a path P_{vw} of length at most $k+1$ in G_0. Let $\mathcal{P} := \{P_{vw} : vw \in E(G)\}$. Observe that $\mathcal{P}$

[6] A *lazy walk* in a graph H is a walk in the pseudograph H' obtained by adding a self loop to each vertex of H.

is a $(k+1,2)$-shortcut system for G_0 and that $G_0^{\mathcal{P}} \supseteq G$. Specifically, $G_0^{\mathcal{P}}$ contains every edge and vertex of G as well as the dummy vertices in $V(G_0)\setminus V(G)$ and their incident edges.

Since G_0 is planar, Theorem 5(b) implies that G_0 has an H-partition of layered width 3 for some planar graph H of treewidth at most 3. Applying Theorem 8 to G_0 and $\mathcal{P}$ immediately implies that G (an arbitrary k-planar graph) has an H-partition of layered width $6((k+1)^3+3(k+1))$ for some graph H of treewidth at most $\binom{k+4}{3}-1$.

We can reduce the layered width of the H-partition of G from $O(k^3)$ to $O(k^2)$ by observing that the dummy vertices in $V(G_0)\setminus V(G)$ do not contribute to the layered width of this partition. In this setting, the proof of Claim 3 is simpler since each vertex $w \in Y_x$ contributes at most two vertices to $V_i \cap Y_x$. More precisely, each path $P \in \mathcal{P}$ containing an internal (dummy) vertex $w \in Y_x \cap (V_{i-j} \cup V_{i+j})$ contributes: (i) at most two vertices to $S_x \cap V_i$ for $j \in \{0,\dots,\lfloor (k+1)/2 \rfloor\}$; (ii) at most one vertex to $S_x \cap V_j$ for $j \in \{\lfloor (k+1)/2 \rfloor,\dots,k+1\}$; or (iii) no vertices to $S_x \cap V_j$ for $j > k+1$. Redoing the calculation at the end of the proof of Claim 3 then yields

$$\begin{aligned}|S_x \cap Y_i| \leqslant d\ell\left(2+4\left\lfloor\frac{k+1}{2}\right\rfloor+2\left(k+1-\left\lfloor\frac{k+1}{2}\right\rfloor\right)\right) &= d\ell\left(2k+2\left\lfloor\frac{k+1}{2}\right\rfloor+4\right)\\ &\leqslant d\ell(3k+5) = 18k+30\ .\end{aligned}$$

With this change, the layered width of the partition given by Theorem 8 becomes $18k^2+30k$. This establishes Theorem 9.

Exactly the same approach using Theorem 6(b) instead of Theorem 5(b) establishes Theorem 11 for (g,k)-planar graphs.

5.1 1-Planar Graphs

This section shows that the constants in Theorem 8 can be further improved in the important case of 1-planar graphs. Let G be an edge-maximal 1-plane multigraph with no two parallel edges on the boundary of a single face. Here, edge-maximal should be taken to mean that, if any two vertices v and w appear on a common face[7] F, then there is an edge $vw \in E(G)$ that is contained in the boundary of F. We assume that no two edges incident to a common vertex cross each other since, in a 1-plane graph, such a crossing can always be removed by a local modification to obtain an isomorphic 1-plane graph in which the two edges do not cross.[8]

A *kite* in G is the subgraph $K = G[\{v,w,x,y\}]$ induced by the endpoints of a pair of crossing edges $vw, xy \in E(G)$. It follows from edge-maximality that every kite is isomorphic to the complete graph K_4. The edges vw and xy are called *spars* of K. The cycle $vxwy$ is called the *sail* of K. It follows from edge-maximality that none of the edges vx, xw, wy, or yv are crossed by any other edges of G. Thus any edge that is a spar of a kite K is not part of a sail of any kite K'. Observe that any spar of K is incident on exactly four *kite faces* of G, each of which has three edges and two vertices of G on its boundary.

[7] The *faces* of an embedded graph G are the connected components of $\mathbb{R}^2 \setminus \bigcup_{vw\in E(G)} vw$. We say that a vertex $v \in V(G)$ appears on a face F if v is contained in the closure of F.

[8] While this is true for 1-plane graphs it is not true for k-plane graphs with $k \geqslant 3$; the uncrossing operation can increase the number of crossing on a particular edge from k to $2(k-1)$.

The 1-plane graph G has a plane triangulation G' as a subgraph that can be obtained by removing one spar from each kite in G. Observe that, for any spar $xy \in E(G) \setminus E(G')$ that crosses $vw \in E(G')$, G' contains the path vxw (and vyw). It follows that $\mathrm{dist}_{G'}(v,w) \leqslant 2$.

Our proof of Theorem 10 follows quickly from the following technical lemma, which is an extension of the analagous result for plane graphs [15].

Lemma 5. *The setup:*

1. *Let G and G' be defined as above.*
2. *Let T be a BFS spanning tree of G' rooted at some vertex r.*
3. *For each integer $j \geqslant 0$, let $V_j = \{v \in V(G) : \mathrm{dist}_T(r,v) = j\}$.*
4. *Let F be a cycle in G' with r in the exterior of F and such that*
   (a) *No edge of F is crossed by any edge of G; and*
   (b) *$V(F)$ can be partitioned into $P_1,\dots,P_k$, for some $k \in \{1,2,3\}$ such that for each $i \in \{1,\dots,k\}$,*
      i. *$F[P_i]$ is a path; and*
      ii. *$|V(P_i) \cap V_j| \leqslant 15$ for all integers $j \geqslant 0$.*
5. *Let N and N' be the subgraphs of G and G' consisting only of those edges and vertices contained in F or the interior of F.*

Then N has an H-partition $\mathcal{P} = \{S_x : x \in V(H)\}$ such that:

1. *H is planar;*
2. *for all integers $j \geqslant 0$, and all $x \in V(H)$, $|S_x \cap V_j| \leqslant 15$;*
3. *for each $i \in \{1,\dots,k\}$, there exists some $x_i \in V(H)$ such that $P_i = S_{x_i}$; and*
4. *H has a tree decomposition in which every bag has size at most 4 and such that some bag contains $x_1,\dots,x_k$.*

Proof. This proof is very similar to the proof of Lemma 14 by Dujmović et al. [15]. Rather than duplicate every detail of that proof here, we focus on the differences and refer the reader to the original proof for the remaining details.

The proof is by induction on the number of vertices of N. First note that N' is a near-triangulation. If $k = 3$, set $R_i := P_i$ for each $i \in \{1,2,3\}$. Otherwise, as in [15], split $P_1,\dots,P_k$ to partition $V(F)$ into three sets R_1, R_2, and R_3 such that each $F[R_i]$ is a non-empty path and each R_i contains vertices from exactly one of $P_1,\dots,P_k$.

Next, as in [15], use Sperner's Lemma to find an inner face $\tau = v_1v_2v_3$ of N' such that, T contains disjoint vertical paths Q_1, Q_2, Q_3 such that each Q_i begins at v_i, ends at some vertex in R_i, and whose internal vertices (if any) are contained in $N' - V(F)$.

Let $\overline{Y}$ denote the subgraph of N' consisting of vertices and edges of Q_1, Q_2, Q_3, and τ. Let $\overline{Y}^+$ denote the subgraph of N consisting of the vertices and edges of $\overline{Y}$ plus the vertices and edges of every kite formed by a crossing between an edge of G and an edge of $\overline{Y}$.

We claim that, for each integer $i \geqslant 0$, $|V(\overline{Y}^+) \cap V_i| \leqslant 15$. First observe that, since Q_1, Q_2, Q_3 are each vertical paths in T, $\overline{Y}$ contains at most three vertices of V_i, each incident on at most two edges of $\overline{Y}$. Since $\mathrm{dist}_{G'}(v,w) \leqslant 2$ for each $vw \in E(G)$, any vertex $x \in V(\overline{Y}^+) \setminus V(\overline{Y}) \cap V_i$, is incident to an edge $xy \in E(G)$ that crosses one of the at most six edges in $\overline{Y}$ having an endpoint in V_i. These at most six edges have at most 12 endpoints.

Therefore $|V(\overline{Y}^+) \setminus V(\overline{Y}) \cap V_i| \leqslant 6 \times 2 = 12$, so $|V(\overline{Y}^+) \cap V_i| \leqslant 12 + 3 = 15$.

Let M and M^+ denote the subgraph of G containing the edges and vertices of $\overline{Y}$, respectively $\overline{Y}^+$, and the edges and vertices of F. The graph M^+ has some number of bounded faces, all contained in the interior of F. Some of the bounded faces of M^+ are kite faces. Let $F_1, \ldots, F_m$ be the non-kite bounded faces of M^+.

We claim that, for each $i \in \{1, \ldots, m\}$, the boundary of F_i is a cycle in G' that contains no spars. Otherwise, some edge vw contributes to the boundary of F_i but is crossed by an edge $xy \in E(G)$. Then, $vw \notin E(F)$ since no edge of F is crossed by any edge of G. Therefore $vw \in E(\overline{Y}^+)$ so $xy \in E(Y^+)$. But then the only faces of M^+ incident to vw are kite faces. In particular vw cannot be incident to the non-kite face F_i.

Observe that each of the faces $F_1, \ldots, F_m$ is contained in a single internal face of M. Let $Y^+ := \overline{Y}^+ - F$. Therefore, $V(F_i)$ can be partitioned into at most three sets P_1', P_2', and P_3' where $P_1' \subset V(Y^+)$, $P_2' \subseteq P_a$, $P_3' \subseteq P_b$ for some $a, b \in \{1, 2, 3\}$, and $F_i[P_j']$ is a path, for each $j \in \{1, 2, 3\}$.

Finally, the subgraph N_i of G consisting of the edges and vertices of G contained in F_i or its interior does not contain one of the three vertices of τ. Therefore, we can apply induction using the cycle F_i and the partition P_1', P_2', P_3' of $V(C_i)$ to obtain the desired H-partition and tree decomposition of N_i.

The proof finishes in the same way as the proof in [15]. The paths $P_1, \ldots, P_k$, and $S = V(Y^+)$ become elements of the H-partition. Elements in each of the H-partitions of $N_1, \ldots, N_3$ that intersect $P_1, \ldots, P_k$, or $V(\overline{Y}^+ - F)$ are discarded and all the resulting sets are combined to obtain an H-partition of G. The desired tree decomposition of G is obtained in exactly the same way as in the proof of Lemma 14 in [15], except that now each node x has a child for each face F_i of M_x^+ that contains a vertex of G in its interior.

The planarity of H comes from two properties:

1. $G/\mathcal{P}$ and $G^+/\mathcal{P}^+$ are isomorphic, where G^+ is the triangulation obtained by adding dummy vertices at each crossing in G and $\mathcal{P}^+$ is the partition we obtain by adding a dummy vertex z to $\overline{Y}^+$ if $\overline{Y}^+$ contains an edge vw that contains z in its interior.

2. $G^+[\overline{Y}^+ - F]$ is connected. To see why this is so, first observe that $\overline{Y} - F$ is connected, and then observe that every vertex of $\overline{Y}^+$ is either a vertex of $\overline{Y}$ or adjacent to a vertex of $\overline{Y}$.

Since G^+ is planar, the second point implies that $H = G^+/\mathcal{P}$ is planar. □

Using Lemma 5, the proof of Theorem 10 is now straightforward.

Proof of Theorem 10. Given a 1-plane graph G, add edges to make it edge-maximal so that it has an outer face $F = v_1 v_2 v_3$. Next, add a vertex r adjacent to v_1, v_2, and v_3 to obtain an edge-maximal 1-plane graph $\overline{G}$ with one vertex r of degree 3 on its outer face.

Let G' be the plane graph obtained by removing one spar from each kite of $\overline{G}$ and let T be a BFS tree of G' rooted at r. Now apply Lemma 5 with $G = \overline{G}$, G', F, and $P_i = \{v_i\}$

for each $i \in \{1,2,3\}$. This gives an H-partition $\{S_x : x \in V(H)\}$ of $\overline{G} - \{r\} \supseteq G$ in which H is planar and has treewidth at most 3.

Use the layering $\mathcal{L} = \langle V_0', V_1' \dots \rangle$ where $V_i' = V_{2i} \cup V_{2i+1}$ for each integer $i \geqslant 0$. That this is a layering of G follows from the fact that $\text{dist}_{G'}(v,w) \leqslant 2$ for every edge $vw \in E(G)$. Since $|V_j \cap S_x| \leqslant 15$ for every integer $j \geqslant 0$, $|V_i' \cap S_x| \leqslant 30$ for every integer $i \geqslant 0$ and every $x \in V(H)$. □

6 Applications

Here we discuss some of the consequences of our main theorems for k-planar and (g,k)-planar graphs.

6.1 Queue Layouts

For an integer $k \geqslant 0$, a *k-queue layout* of a graph G consists of a linear ordering $\preceq$ of $V(G)$ and a partition $\{E_1, E_2, \dots, E_k\}$ of $E(G)$, such that for $i \in \{1,2,\dots,k\}$, no two edges in E_i are nested with respect to $\preceq$. That is, it is not the case that $v \prec x \prec y \prec w$ for edges $vw, xy \in E_i$. The *queue-number* of a graph G, denoted by $\text{qn}(G)$, is the minimum integer k such that G has a k-queue layout. Queue-number was introduced by Heath et al. [22], who famously conjectured that planar graphs have bounded queue-number. Dujmović et al. [15] recently proved this conjecture using Theorem 5 and the following lemma. Indeed, resolving this question was the motivation for the development of Theorems 1 and 5.

Lemma 6 ([15]). *If a graph G has an H-partition of layered width ℓ, then*

$$\text{qn}(G) \leqslant 3\ell\, \text{qn}(H) + \lfloor \tfrac{3}{2}\ell \rfloor \ .$$

Dujmović et al. [13] proved that queue-number is bounded by a function of treewidth. The best known bound is due to Wiechert [34]:

Theorem 13 ([34]). *For every graph G, $\text{qn}(G) \leqslant 2^{\text{tw}(G)} - 1$.*

Lemma 6 and Theorems 11 and 13 imply that (g,k)-planar graphs have queue-number at most $g2^{O(k^4)}$. Note that Dujmović et al. [15] previously proved the bound of $O(g^{k+2})$ using Theorem 6 and an ad-hoc method. Our result provides a better bound when $g > 2^{k^3}$. In the case of 1-planar graphs we can improve further. Alam, Bekos, Gronemann, Kaufmann, and Pupyrev [3] proved that every planar graph with treewidth at most 3 has queue-number at most 5. Thus the graph H in Theorem 10 has queue-number at most 5. Lemma 6 and Theorem 10 then imply:

Proposition 1. *Every 1-planar graph has queue-number at most* $3 \times 30 \times 5 + \lfloor \frac{3}{2} \times 30 \rfloor = 495$.

6.2 Non-Repetitive Colouring

The next two applications are in the field of graph colouring. For our purposes, a *c-colouring* of a graph G is any function $\phi \colon V(G) \to C$, where C is a set of size at most c.

A c-colouring ϕ of G is *non-repetitive* if, for every path $v_1,\dots,v_{2h}$ in G, there exists some $i \in \{1,\dots,h\}$ such that $\phi(v_i) \neq \phi(v_{i+h})$. The *non-repetitive chromatic number* $\pi(G)$ of G is the minimum integer c such that G has a non-repetitive c-colouring. This concept was introduced by Alon et al. [4] and has since been widely studied; see [14] for more than 40 references. Up until recently the main open problem in the field has been whether planar graphs have bounded non-repetitive chromatic number, first asked by Alon et al. [4]. Dujmović et al. [14] recently solved this question using Theorem 5 and the following lemma.

Lemma 7 ([14])**.** *If a graph G has an H-partition of layered width at most ℓ in which H has treewidth at most t, then $\pi(G) \leqslant \ell\, 4^{t+1}$.*

Lemma 7 and Theorems 9 to 11 immediately imply the following results:

Corollary 1. *For every k-planar graph G, $\pi(G) \leqslant (18k^2 + 30k)4^{\binom{k+4}{3}}$.*

Corollary 2. *For every 1-planar graph G, $\pi(G) \leqslant 30 \times 4^4 = 7680$.*

Corollary 3. *For every (g,k)-planar graph G,*

$$\pi(G) \leqslant \max\{2g,3\} \cdot (6k^2 + 10k)4^{\binom{k+5}{4}}.$$

Prior to the current work, the strongest upper bound on the non-repetitive chromatic number of n-vertex k-planar graphs was $O(k \log n)$ [13].

6.3 Centered Colourings

A c-colouring ϕ of G is *p-centered* if, for every connected subgraph $X \subseteq G$, $|\{\phi(v) : v \in V(X)\}| > p$ or there exists some $v \in V(X)$ such that $\phi(v) \neq \phi(w)$ for every $w \in V(X) \setminus \{v\}$. In words, either X receives more than p colours or some vertex in X receives a unique colour. Let $\chi_p(G)$ be the minimum integer c such that G has a p-centered c-colouring. Centered colourings are important since they characterise classes of bounded expansion, which is a key concept in the sparsity theory of Nešetřil and Ossona de Mendez [29].

We make use of the following lemma due to Pilipczuk and Siebertz [32].

Lemma 8 ([32])**.** *Every graph H of treewidth at most t has $\chi_p(H) \leqslant (t+1)\binom{p+t}{t}$.*

The following lemma is implicitly due to Dębski et al. [10]. We include the proof for completeness.

Lemma 9 ([10])**.** *Every graph G that has an H-partition of layered width at most ℓ has $\chi_p(G) \leqslant \ell(p+1)\,\chi_p(H)$.*

Proof. Let $(\mathcal{L} = \langle V_0, V_1, \dots\rangle, \mathcal{P} = (B_x : x \in V(H))$ be an H-partition of G having layered width at most ℓ. Use a product colouring $\phi : V(G) \to \{1,\dots,\ell\} \times \{0,\dots,p\} \times \{1,\dots,\chi_p(H)\}$. For each integer $i \geqslant 0$ and each $x \in V(H)$, assign the colour $\phi(v) := (\alpha(v), \beta(v), \gamma(v))$ to each vertex $v \in V_i \cap B_x$ such that:

1. $\alpha(v)$ is unique among $\{\phi(w) : w \in V_i \cap B_x\}$, which is possible since $|V_i \cap B_x| \leqslant \ell$,
2. $\beta(v) = i \bmod (p+1)$, and

3. $\gamma(v) = \gamma(x)$ where $\gamma : V(H) \to \{1,\dots,\chi_p(H)\}$ is a p-centered colouring of H.

To show this is a p-centered colouring, consider some connected subgraph $X \subseteq G$.

First suppose that there exists $v,w \in V(X)$ with $v \in V_i$ and $w \in V_j$ with $j - i \geqslant p$. Since $G[X]$ is connected, $G[X]$ contains a path from v to w. By the definition of layering, this path contains at least one vertex from $V_{i'}$ for each $i' \in \{i, i+1, \dots, j\}$. Therefore, $|\{\beta(v') : v' \in X\}| \geqslant j - i + 1 > p$, so X receives more than p distinct colours.

Otherwise, $X \subseteq V_i, \dots, V_{i+s}$ for some $s < p$. Let $H' := H[\{x \in V(H) : B_x \cap X \neq \emptyset]$. If $|\{\gamma(x) : x \in V(H')\}| > p$ then $|\{\gamma(v) : v \in X\}| > p$ so $|\{\phi(v) : v \in X\}| > p$ and we are done. Otherwise, since γ is a p-centered colouring of H, there must exist some $x \in V(H')$ such that $\gamma(x) \neq \gamma(y)$ for every $y \in V(H') \setminus \{x\}$. For any $v,w \in B_x$ with $v \neq w$, either $v,w \in V_{i'}$ for some $i' \in \{i, i+1, \dots, i+s\}$ in which case $\alpha(v) \neq \alpha(w)$; or $v \in V_{i'}$ and $w \in V_{i''}$ with $|i' - i''| < p$, in which case $\beta(v) \neq \beta(w)$. Therefore every vertex $v \in B_x$ receives a colour $\phi(v)$ distinct from every colour in $\{\phi(z) : z \in X \setminus \{x\}\}$. Therefore, every vertex in B_x receives a colour distinct from every other vertex in X. $\square$

Lemmas 8 and 9 and Theorems 9 to 11 immediately imply the following results, for every $p \geqslant 2$:

Corollary 4. *For every k-planar graph G,*

$$\chi_p(G) \leqslant (18k^2 + 30k)(p+1)\binom{k+4}{3}\binom{p + \binom{k+4}{3} - 1}{\binom{k+4}{3} - 1} .$$

Corollary 5. *For every 1-planar graph G,*

$$\chi_p(G) \leqslant 20(p+3)(p+2)(p+1)^2 .$$

Corollary 6. *For every (g,k)-planar graph G,*

$$\chi_p(G) \leqslant \max\{2g, 3\} \cdot (6k^2 + 10k)(p+1)\binom{k+5}{4}\binom{p + \binom{k+5}{4} - 1}{\binom{k+5}{4} - 1} .$$

Prior to the current work, the strongest known upper bounds on the p-centered chromatic number of (g,k)-planar graphs G were doubly-exponential in p, as we now explain. Dujmović et al. [12] proved that G has layered treewidth $(4g+6)(k+1)$. Van den Heuvel and Wood [23] showed that this implies that G has r-strong colouring number at most $(4g+6)(k+1)(2r+1)$. By a result of Zhu [35], G has r-weak colouring number at most $((4g+6)(k+1)(2r+1))^r$, which by another result of Zhu [35] implies that G has p-centered chromatic number at most $((4g+6)(k+1)(2^{p-1}+1))^{2^{p-2}}$. Corollaries 4 to 6 are substantial improvements over these results, providing bounds on $\chi_p(G)$ that are polynomial in p for fixed g and k.

7 Rough Characterisation

This section presents a rough characterisation of (g,k)-planar graphs, which is interesting in its own right, and is useful for showing that various classes of graphs are (g,k)-planar. First note the following easy result (just add a degree-4 vertex at each crossing point of G, as in the proof of Observations 1 and 2).

Lemma 10. *For every (g,k)-planar graph G there is a graph G_0 of Euler genus at most g, such that $V(G) \subseteq V(G_0)$ and for every edge $vw \in E(G)$ there is a vw-path P in G_0 of length at most $k+1$, such that every internal vertex in P has degree at most 4 in G_0.*

The following converse of Lemma 10 provides a sufficient condition for a graph to be (g,k)-planar that is slightly more restrictive than being obtained from a (k,d)-shortcut system of a (g,k)-planar graph.

Lemma 11. *Fix integers $g \geqslant 0$ and $k,\Delta \geqslant 2$. Let G_0 be a graph of Euler genus at most g. Let G be a graph with $V(G) \subseteq V(G_0)$ such that for every edge $vw \in E(G)$ there is a vw-path P_{vw} in G_0 of length at most k, such that every internal vertex on P_{vw} has degree at most Δ in G_0. Then G is $(g, 2k(k+1)\Delta^k)$-planar.*

Proof. For a vertex x of G_0 with degree at most Δ, and for $i \in \{1,\dots,k-1\}$, say a vertex v is *i-close* to x if there is a vx-path P in G_0 of length at most i such that every internal vertex in P has degree at most Δ in G_0. For each edge vw of G, say that *vw passes through* each internal vertex on P_{vw}. Say vw passes through x. Then v is i-close to x and w is j-close to x for some $i,j \in \{1,\dots,k-1\}$ with $i+j \leqslant k$. At most Δ^i vertices are i-close to x. Thus, the number of edges of G that pass through x is at most

$$\sum_{i=1}^{k-1}\sum_{j=1}^{k-i}\Delta^i\Delta^j = \sum_{i=1}^{k-1}\Delta^i\sum_{j=1}^{k-i}\Delta^j < \sum_{i=1}^{k-1}\Delta^i 2\Delta^{k-i} = \sum_{i=1}^{k-1} 2\Delta^k < 2k\Delta^k \ .$$

Draw each edge vw of G alongside P_{vw} in G_0, so that every pair of edges cross at most once. Every edge of G that crosses vw passes through a vertex on P_{vw} (including v and/or w if they too have degree at most Δ). Since P_{vw} has at most $k+1$ vertices, and less than $2k\Delta^k$ edges of G pass through each vertex on P_{vw}, the edge vw is crossed by less than $2k(k+1)\Delta^k$ edges in G. Hence G is $(g, 2k(k+1)\Delta^k)$-planar. □

Together Lemmas 10 and 11 provide a rough characterisation of (g,k)-planar graphs.

8 Examples

This section describes several examples of graph classes that can be obtained from a shortcut system applied to graphs of bounded Euler genus.

8.1 Map Graphs

Map graphs are defined as follows. Start with a graph G_0 embedded in a surface of Euler genus g, with each face labelled a 'nation' or a 'lake', where each vertex of G_0 is incident with at most d nations. Let G be the graph whose vertices are the nations of G_0, where two vertices are adjacent in G if the corresponding faces in G_0 share a vertex. Then G is called a *(g,d)-map graph*. A $(0,d)$-map graph is called a (plane) *d-map graph*; see [9, 16] for example. The $(g,3)$-map graphs are precisely the graphs of Euler genus at most g; see [12]. So (g,d)-map graphs generalise graphs embedded in a surface, and we now assume that $d \geqslant 4$ for the remainder of this section.

There is a natural drawing of a map graph obtained by positioning each vertex of G inside the corresponding nation and each edge of G as a curve passing through the corresponding vertex of G_0. It is easily seen that each edge is in at most $\lfloor\frac{d-2}{2}\rfloor\lceil\frac{d-2}{2}\rceil$ crossings; see [12]. Thus G is $(g,\lfloor\frac{d-2}{2}\rfloor\lceil\frac{d-2}{2}\rceil)$-planar. Also note that Lemma 11 with $k=2$ implies that G is $(g,O(d^2))$-planar. Theorem 11 then shows that map graphs admit bounded layered partitions, but we get much better bounds by constructing a shortcut system directly. The following lemma is reminiscent of the characterisation of (g,d)-map graphs in terms of the half-square of bipartite graphs [9, 12].

Lemma 12. *Every (g,d)-map graph G is a subgraph of $G_1^{\mathcal{P}}$ for some graph G_1 with Euler genus at most g and some $(2,\frac{1}{2}d(d-3))$-shortcut system $\mathcal{P}$ for G_1.*

Proof. Let G be a (g,d)-map graph. So there is a graph G_0 embedded in a surface of Euler genus g, with each face labelled a 'nation' or a 'lake', where each vertex of G_0 is incident with at most d nations. Let N be the set of nations. Then $V(G)=N$ where two vertices are adjacent in G if the corresponding nation faces of G_0 share a vertex. Let G_1 be the graph with $V(G_1):=V(G_0)\cup N$, where distinct vertices $v,w\in N$ are adjacent in G_1 if the boundaries of the corresponding nations have an edge of G_0 in common, and $v\in V(G_0)$ and $w\in N$ are adjacent in G_1 if v is on the boundary of the nation corresponding to w. Observe that G_1 embeds in the same surface as G_0 with no crossings, and that each vertex in $V(G_0)$ has degree at most d in G_1. Consider an edge $vw\in E(G)$. If the nations corresponding to v and w share an edge of G_0, then vw is an edge of G_1. Otherwise, v and w have a common neighbour x in $V(G_0)$. In the latter case, let P_{vw} be the path (v,x,w). Let $\mathcal{P}$ be the set of all such paths P_{vw}. Each vertex $x\in V(G_0)$ is the middle vertex on at most $\frac{1}{2}d(d-3)$ paths in $\mathcal{P}$. Thus $\mathcal{P}$ is a $(2,\frac{1}{2}d(d-3))$-shortcut system for G_1, and by construction, $G\subseteq G_1^{\mathcal{P}}$. □

Theorem 5(b), Theorem 6(b) and Theorem 8 and Lemma 12 imply:

Theorem 14. *Every (g,d)-map graph has an H-partition with layered width at most $7d(d-3)\max\{2g,3\}$ for some graph H with treewidth at most $\binom{2+4}{4}-1=14$ or treewidth at most $\binom{2+3}{3}-1=9$ if $g=0$.*

Theorem 14 and Lemma 7 imply the first known constant upper bound on the non-repetitive chromatic number of map graphs:

Corollary 7. *Every (g,d)-map graph G has non-repetitive chromatic number $\pi(G)\leqslant 7\cdot 4^{15}\,d(d-3)\max\{2g,3\}$, and if $g=0$ then $\pi(G)\leqslant 21\cdot 4^{10}\,d(d-3)$.*

Lemmas 8 and 9 and Theorem 14 imply that the first known bounds on the p-centered chromatic number of map graphs that are polynomial in p.

Corollary 8. *Every (g,d)-map graph G has p-centered chromatic number*

$$\chi_p(G)\leqslant 105\,d(d-3)\max\{2g,3\}(p+1)\binom{p+14}{14}\ ,$$

and every plane d-map graph G has p-centered chromatic number

$$\chi_p(G)\leqslant 210\,d(d-3)(p+1)\binom{p+9}{9}\ .$$

Theorems 13 and 14 and Lemma 6 imply the best known bounds on the queue-number of map graphs:

Corollary 9. *Every (g,d)-map graph has queue-number at most $21 \cdot 2^{14} d(d-3)\max\{2g,3\}$, and every plane d-map graph has queue-number at most $63 \cdot 2^9 d(d-3)$.*

8.2 String Graphs

A *string graph* is the intersection graph of a set of curves in the plane with no three curves meeting at a single point; see [18, 19, 31] for example. For an integer $\delta \geqslant 2$, if each curve is in at most δ intersections with other curves, then the corresponding string graph is called a *δ-string graph*. A *(g,δ)-string* graph is defined analogously for curves on a surface of Euler genus at most g.

Lemma 13. *Every (g,δ)-string graph G is a subgraph of $G_0^{\mathcal{P}}$ for some graph G_0 with Euler genus at most g and some $(\delta+1,\delta+1)$-shortcut system $\mathcal{P}$ for G_0.*

Proof. Let $\mathcal{C} = \{C_v : v \in V(G)\}$ be a set of curves in a surface of Euler genus at most g whose intersection graph is G. Let G_0 be the graph obtained by adding a vertex at the intersection point of every pair of curves in $\mathcal{C}$ that intersect, where two such consecutive vertices on a curve C_v are adjacent in G_0. For each vertex $v \in V(G)$, if C_v intersects $k \leqslant \delta$ other curves, then introduce a new vertex called v on C_v between the $\lfloor \frac{k}{2} \rfloor$-th vertex already on C_v and the $\lfloor \frac{k}{2}+1 \rfloor$-th such vertex. For each edge vw of G, there is a path P_{vw} of length at most $2\lceil \frac{\delta}{2} \rceil \leqslant \delta+1$ in G_0 between v and w. Let $\mathcal{P}$ be the set of all such paths P_{vw}. Consider a vertex x in G_0 that is an internal vertex on some path in $\mathcal{P}$. Then x is at the intersection of C_v and C_w for some edge $vw \in E(G)$. If some path $P \in \mathcal{P}$ passes through x, then $P = P_{vu}$ for some edge vu incident to v, or $P = P_{wu}$ for some edge wu incident to w. At most $\lceil \frac{\delta}{2} \rceil$ paths in $\mathcal{P}$ corresponding to edges incident to v pass through x, and similarly for edges incident to w. Thus at most $2\lceil \frac{\delta}{2} \rceil \leqslant \delta+1$ paths in $\mathcal{P}$ use x as an internal vertex. Thus $\mathcal{P}$ is a $(\delta+1,\delta+1)$-shortcut system for G_0, and by construction, $G \subseteq G_0^{\mathcal{P}}$. □

A similar proof to that of Lemma 13 shows that every (g,δ)-string graph is $(g,2\delta^2)$-planar.

Lemma 13 and Theorem 8, Theorem 5(b) and Theorem 6(b) imply:

Theorem 15. *Every (g,δ)-string graph has an H-partition of layered width at most* $\max\{2g,3\}(\delta^4+4\delta^3+9\delta^2+10\delta+4)$ *for some graph H with treewidth at most $\binom{\delta+5}{4}-1$ or treewidth at most $\binom{\delta+4}{3}-1$ if $g=0$.*

Lemmas 8 and 9 and Theorem 15 imply the first known bounds on the p-centered chromatic number of map graphs that are polynomial in p.

Corollary 10. *Every (g,δ)-string graph has p-centered chromatic number*

$$\chi_p(G) \leqslant \max\{2g,3\}(\delta^4+4\delta^3+9\delta^2+10\delta+4)(p+1)\tbinom{\delta+5}{4}\binom{p+\binom{\delta+5}{4}-1}{\binom{\delta+5}{4}-1},$$

and every plane $(0,\delta)$-string graph has p-centered chromatic number

$$\chi_p(G) \leqslant \max\{2g,3\}(\delta^4+4\delta^3+9\delta^2+10\delta+4)(p+1)^{\binom{\delta+4}{3}}\binom{p+\binom{\delta+4}{3}-1}{\binom{\delta+4}{3}-1} .$$

Theorems 13 and 15 and Lemma 6 imply the following bounds on the queue-number of map graphs:

Corollary 11. *Every (g,δ)-string graph has queue-number at most*

$$3\max\{2g,3\}(\delta^4+4\delta^3+9\delta^2+10\delta+4)2^{\binom{\delta+5}{4}-1} .$$

and every plane $(0,\delta)$-map graph has queue-number at most

$$9(\delta^4+4\delta^3+9\delta^2+10\delta+4)2^{\binom{\delta+4}{3}-1} .$$

Lemma 7 and Theorem 15 also give bounds on the non-repetitive chromatic number of (g,δ)-string graphs, but the bound is weak, since such graphs G have maximum degree at most 2δ, implying that $\pi(G) \leqslant (4+o(1))\delta^2$ by a result of Dujmović, Joret, Kozik, and Wood [11].

8.3 Powers of Bounded Degree Graphs

Recall that the *k-th power* of a graph G is the graph G^k with vertex set $V(G^k) := V(G)$, where $vw \in E(G^k)$ if and only if $\mathrm{dist}_G(v,w) \leqslant k$. If G is planar with maximum degree Δ, then G^k is $2k(k+1)\Delta^k$-planar by Lemma 11. Thus we can immediately conclude that bounded powers of planar graphs of bounded degree admit bounded layered partitions. However, the bounds we obtain are improved by the following lemma that constructs a shortcut system directly.

Lemma 14. *If a graph G has maximum degree Δ, then $G^k = G^{\mathcal{P}}$ for some $(k, 2k\Delta^k)$-shortcut system $\mathcal{P}$.*

Proof. For each pair of vertices x and y in G with $\mathrm{dist}_G(x,y) \in \{1,\dots,k\}$, fix an xy-path P_{xy} of length $\mathrm{dist}_G(x,y)$ in G. Let $\mathcal{P} := \{P_{xy} : \mathrm{dist}_G(x,y) \in \{1,\dots,k\}\}$. Say P_{xy} uses some vertex v as an internal vertex. If $\mathrm{dist}_G(v,x) = i$ and $\mathrm{dist}_G(v,y) = j$, then $i,j \in \{1,\dots,k-1\}$ and $i+j \leqslant k$. The number of vertices at distance i from v is at most Δ^i. Thus the number of paths in $\mathcal{P}$ that use v as an internal vertex is at most

$$\sum_{i=1}^{k-1}\sum_{j=1}^{k-i}\Delta^i\Delta^j = \sum_{i=1}^{k-1}\Delta^i\sum_{j=1}^{k-i}\Delta^j < \sum_{i=1}^{k-1}\Delta^i(2\Delta^{k-i}) < 2k\Delta^k .$$

Hence $\mathcal{P}$ is a $(k, 2k\Delta^k)$-shortcut system. □

Theorem 5(b), Theorem 6(b), Theorem 8 and Lemma 14 imply the following result, which with Lemma 1 implies Theorem 4 in Section 1.

Theorem 16. *Fix $k \geqslant 1$. Let G be a graph of Euler genus g and maximum degree Δ. Then G^k has an H-partition of layered width at most $\max\{2g,3\}(2k^4+6k^2)\Delta^k$ for some graph H with treewidth at most $\binom{k+4}{4}-1$, and treewidth at most $\binom{k+3}{3}-1$ if $g=0$.*

Lemmas 6 to 9 and 11 and Theorem 16 immediately imply the following result:

Corollary 12. *For all integers $k \geqslant 1$ and $g \geqslant 0$, if G is a graph with Euler genus g and maximum degree Δ, then:*

- *G^k is $(g, 2k(k+1)\Delta^k)$-planar,*
- *G^k has queue-number*

$$\mathrm{qn}(G^k) \leqslant 3\max\{2g,3\}(2k^4+6k^2)\,\Delta^k\,2^{\binom{k+4}{4}-1},$$

- *G^k has p-centered chromatic number*

$$\chi_p(G^k) \leqslant \max\{2g,3\}(2k^4+6k^2)\Delta^k(p+1)\binom{k+4}{4}\binom{p+\binom{k+4}{4}-1}{\binom{k+4}{4}-1},$$

- *G^k has non-repetitive chromatic number*

$$\pi(G^k) \leqslant \max\{2g,3\}(2k^4+6k^2)\Delta^k\,4^{\binom{k+4}{4}}.$$

This result is the first constant upper bound on the queue-number and p-centered chromatic number of bounded powers of graphs with bounded degree and bounded Euler genus. For every graph G, since G^k has maximum degree at most Δ^k, a result of Dujmović et al. [11] implies that $\pi(G^k) \leqslant (1+o(1))\Delta^{2k}$. Corollary 12 improves upon this bound when $k \ll \Delta$ and G has Euler genus g.

8.4 k-Nearest-Neighbour Graphs

In this section, we show that k-nearest neighbour graphs of point sets in the plane are $O(k^2)$-planar. For two points $x,y \in \mathbb{R}^2$, let $d_2(x,y)$ denote the Euclidean distance between x and y. The k-nearest-neighbour graph of a point set $P \subset \mathbb{R}^2$ is the geometric graph G with vertex set $V(G)=P$, where the edge set is defined as follows. For each point $v \in P$, let $N_k(v)$ be the set of k points in P closest to v. Then $vw \in E(G)$ if and only if $w \in N_k(v)$ or $v \in N_k(w)$. (The edges of G are straight-line segments joining their endpoints.) See [8] for a survey of results on k-nearest neighbour graphs and other related proximity graphs.

The following result, which is immediate from Ábrego, Monroy, Fernández-Merchant, Flores-Peñaloza, Hurtado, Sacristán, and Saumell [1, Corollary 4.2.6] states that k-nearest-neighbour graphs have bounded maximum degree:

Lemma 15. *The degree of every vertex in a k-nearest-neighbour graph is at most $6k$.*

We make use of the following well-known observation (see for example, Bose, Morin, Stojmenović, and Urrutia [7, Lemma 2]):

Observation 3. *If $v_0,\dots,v_3$ are the vertices of a convex quadrilateral in counterclockwise order then there exists at least one $i \in \{0,\dots,3\}$ such that $\max\{d_2(v_i,v_{i-1}), d_2(v_i,v_{i+1})\} < d_2(v_{i-1},v_{i+1})$, where subscripts are taken modulo 4.*

Lemma 16. *Every k-nearest-neighbour graph is $O(k^2)$-planar.*

Proof. Let G be a k-nearest-neighbour graph and consider any edge $vw \in E(G)$. Let $xy \in E(G)$ be an edge that crosses vw. Note that $vxwy$ are the vertices of a convex quadrilateral in (without loss of generality) counterclockwise order. Then we say that

1. xy is of Type v if $\max\{d_2(v,x), d_2(v,y)\} < d_2(x,y)$;
2. xy is of Type w if $\max\{d_2(w,x), d_2(w,y)\} < d_2(x,y)$; or
3. xy is of Type C otherwise.

If xy is of Type C, then Observation 3 implies that (without loss of generality)

$$\max\{d_2(x,v), d_2(x,w)\} < d_2(v,w) \ .$$

In this case, we call x a Type C vertex. We claim that $V(G)$ contains at most $k-1$ Type C vertices. Indeed, more than $k-1$ Type C vertices would contradict the fact that $vw \in E(G)$ since every Type C vertex is closer to both v and w than $d_2(v,w)$.

Next oberve that, if xy is of Type v, then at least one of xv or yv is in $E(G)$ in which case we call x (respectively y) a Type v vertex. By Lemma 15, there are at most $6k$ Type v vertices. Similarly, there are at most $6k$ Type w vertices.

Thus, in total, there are at most $13k-1$ Type v, Type w, and Type C vertices. By Lemma 15, each of these vertices is incident with at most $6k$ edges that cross vw. Therefore, there are at most $78k^2 - 6k$ edges of G that cross vw. Since this is true for every edge $vw \in E(G)$, G is $(78k^2 - 6k)$-planar. □

We note that Lemma 16 is tight up to the leading constant: Every k-nearest neighbour graph on n vertices has at least $kn/2$ edges and at most kn edges. For $k \geqslant 7$, the Crossing Lemma [2, 27] implies that the total number of crossings is therefore $\Omega(k^3 n)$ so that the average number of crossings per edge is $\Omega(k^2)$.

Lemmas 6, 8, 9 and 16 and Theorem 9 imply:

Corollary 13. *Every k-nearest-neighbour graph has:*

- *an H-partition of layered width $O(k^4)$ in which H has treewidth $O(k^6)$,*
- *queue-number at most $2^{O(k^6)}$,*
- *p-centered chromatic number at most $O(k^4)(p+1)(t+1)\binom{p+t}{t}$ for some integer $t \leqslant O(k^6)$.*

Lemma 7 and Corollary 13 also give bounds on the non-repetitive chromatic number of a k-nearest neighbour graph G. However, the bound is weak, since G has maximum degree at most $6k$, implying that $\pi(G) \leqslant (36 + o(1))k^2$ by a result of Dujmović et al. [11].

9 An Open Problem

Theorem 9 shows that k-planar graphs have H-partitions of layered width $O(k^2)$ in which H has treewidth $O(k^3)$. We conclude with the following question: Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ and a universal constant C such that every k-planar graph has an H-partition of layered width at most $f(k)$ in which H has treewidth at most C? Perhaps $C = 3$. Note that $C \geqslant 3$ even for planar graphs [15].

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