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Graph Product Structure for Non-Minor-Closed Classes

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Abstract:	Dujmovi\'c~et~al.~[\emph{J.~ACM}~'20] recently proved that every planar graph is isomorphic to a subgraph of the strong product of a bounded treewidth graph and a path. Analogous results were obtained for graphs of bounded Euler genus or apex-minor-free graphs. These tools have been used to solve longstanding problems on queue layouts, non-repetitive colouring, \$p\$-centered colouring, and adjacency labelling. This paper proves analogous product structure theorems for various non-minor-closed classes. One noteable example is \$k\$-planar graphs (those with a drawing in the plane in which each edge is involved in at most \$k\$ crossings). We prove that every \$k\$-planar graph is isomorphic to a subgraph of the strong product of a graph of treewidth \$O(k^5)\$ and a path. This is the first result of this type for a non-minor-closed class of graphs. It implies, amongst other results, that \$k\$-planar graphs have non-repetitive chromatic number upper-bounded by a function of \$k\$. All these results generalise for drawings of graphs on arbitrary surfaces. In fact, we work in a more general setting based on so-called shortcut systems, which are of independent interest. This leads to analogous results for certain types of map graphs, string graphs, graph powers, and nearest neighbour graphs.

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Graph Product Structure for Non-Minor-Closed Classes
Vida Dujmović, Pat Morin, David R. Wood
Journal of Combinatorial Theory Series B (JCTB9758)
Response to Referee Reports

Both referees strongly recommend acceptance. We have revised the paper according to the referee reports as detailed below.

Referee #1

1. Lemma 3: This lemma is really about "normalised" tree decompositions, and says that for each $x \in V(T)$, the set $V(T_x)$ has at most t neighbours in H. Call this property (T3) and remove (Y1) through (Y5), which should now be obvious.

This lemma has been eliminated completely. (See the next response.)

2. Proof of Theorem 9: There is too much notation throughout. As one example, it is hard to remember which of T_x, Y_x, V_x, F_x, N_x, S_x, X_v, B_x, C_x is which. If $\phi(S)$ is written for the subset of $V(G)$ corresponding to a set $S \subseteq V(H)$ and appropriate shorthand and standard notation for neighbourhoods are used, then Y_x, V_x, F_x, and N_x can be replaced by $\phi(x)$, $\phi(T_x)$, $N(\phi(T_x))$, and $V(G) \setminus N[\phi(T_x)]$. The set X_v is only used in the statement of Claim 1, so it does not need a name; say what these vertices are explicitly. It may be more indicative to write $\phi'(x)$ for S_x and B'_x for C_x.

We agree that the notation was excessive (an unfortunate consequence of the general result on (k,d)-shortcut systems having evolved from a specific result about k-planar graphs). We have eliminated the notations V_x, N_x, F_x, X_x, and T_x.

The notations Y_x and B_x are needed since we need names for the parts of the H-decomposition of G and the bags in the tree-decomposition of H. The notation C_x is limited to the proof of (newly-numbered) Claim 3, where it is used to denote the bags in the tree-decomposition that is the subject of Claim 3.

We have avoided the introduction of any new notation, including ϕ. The new presentation is much shorter and clearer. We thank the referee for making us revisit this proof carefully. The final paper is better for it.

3. Theorem 10: Unfortunately I have to suggest removing it. I don't see the motivation for making H planar or for improving the constants. Are the improved constants at all close to tight? Is there another reason for including it?

There is a lot of motivation for proving this result with H of treewidth 3 and planar. First, the treewidth bound is best possible (even for planar graphs). Having H planar means that H is in fact a planar 3-tree (also called stack polytope or Appolonian network). It follows that we get better bounds for queue-number for example.

Since the initial submission, there is now even more justification for including this material. A result of Bose et al on vertex ranking relies critically on the simple treewidth of H, as does the result of Debski et al on p-centered colouring. In both case, if H does not have simple treewidth 3 then one does not get an asymptotically tight bound.

Also, while adding details requested by Referee #2 we discovered that a small change in the proof (removing both edges of a crossing pair instead of just one) results in a better constant and leads to a more general result that applies to d-framed graphs. This, in turn, leads to better results for d-map graphs.

The former Theorem 10 (on 1-planar graphs) is now a consequence of Theorem 23 on (d-framed graphs).

4. Sections 4 and 5: I think as they are currently organized they distract from the purpose of the paper, as stated in the introduction, to "prove product structure theorems for several non-minor-closed classes of interest." Which of Corollaries 1–4 and Theorems 12–17 are most important? I would remove any mention of the applications from the examples section, put that section first, and state in that section a single main theorem with all of the best bounds on the product structure. The applications section is mostly a survey and I think should be focused on the most exciting new corollaries. Theorems 18 and 19 should be in the applications section.

We have reorganised the paper, addressing this comment.

1. abstract: "This leads to analogous results for map graphs, string graphs, graph powers, and nearest neighbour graphs." $\longrightarrow$ Add "certain types of" or be more specific. Same comment for immediately after Theorem 4.

Done.

2. page 1: "A tree-decomposition T consists of a tree T and a collection T " $\longrightarrow$ Delete the first T and define a T-decomposition.

Done.

3. pages 2 – 4: The paragraph following Theorem 2 and Section 1.3 (except for the fact that the apex assumption is necessary) is repetition of things discussed earlier on in the introduction.

We have shortened the paragraph, avoiding the repetition.

4. page 6, line 14: "Thus, it suffices to construct width-t tree decomposition that satisfies (T2)." $\longrightarrow$ should be (T1)

Done.

5. page 6, line 16: "Select any node $x \in V(T_0)$" $\longrightarrow$ "Select any nonempty node $x \in V(T_0)$"

All the bags at this point are non-empty. We have revised to make this clear.

6. page 6, last paragraph of Lemma 2: This isn't quite right since you have to be careful with x_0.

Both of these have been fixed by first insisting that we start with a tree decomposition having no empty bags and then adding the (empty) root x_0 which is used only in the first stage to ensure that every problematic node has a an edge to its parent that we can subdivide. Then the node x_0 is removed before the second stage where, by definition, every problematic/redundant node has a parent.

7. page 8, item (i): "S has small layered width with respect to the layering L" $\longrightarrow$ "S has small layered width with respect to the layering L of G". Perhaps mention something about how it is easy to find the new layering.

We have added "Once we have established (i) and (ii), the result follows easily since a layering of $G^{\mathcal{S}}$ is easily obtained from $\mathcal{L}$ by 'compressing' groups of k consecutive layers."

8. general comment: Claims 1, 2, and 4 are especially belaboured by notation as they are quite straightforward.

Claims 1 and 2 are now gone, leaving only Claims 3, 4, and 5 whose proofs have now been simplified.

9. page 8, line 10: "We say that a vertex $w \in Y_x$ contributes a vertex $v \in S_x$ if v participates in x." This needs to be fixed.

Replaced with "We say that a vertex $w \in Y_x$ *contributes* a vertex $v \in S_x$ if $v = w$ or if some path in $\mathcal{S}$ that contains v has w as an internal vertex."

10. page 10, Claim 5: Indices could mostly be removed by continuing to talk about T-ancestors instead. Move the definition of $H+$ and its directed version up, along with the explanation of what will be proven, so that it isn't necessary to define the s_i.

This is greatly simplified in the new version.

11. page 10, Claim 5: The name $P_{ww'}$ is strange since in case 2. there is no w' defined.

This has disappeared in the new version

12. page 11, first paragraph of Section 3: "This section applies our main results for shortcut systems to prove graph product structure theorem..." $\longrightarrow$ "to prove a graph product structure theorem..."

Done.

Referee #2

Page 3. The sentence 'For each $vw \in E(G')$ there is a path P in G between v and we of length at most $k+1$" is correct but should be made more precise. For example, there could be a short path between v and w which does not involve any new dummy vertices. You mean to say that the 'subdivided path' between v and w has length at most $k+1$.

Done.

Page 3. You should add the assumption that no three edges cross at the same point (which you can assume by perturbing the embedding). Otherwise, you will not get a $(k+1, 2)$-shortcut system.

Done.

P is used twice in the statement of Theorem 3. It is probably safer (and necessary?) to use a different variable for the second occurrence.

We now use $\mathcal{S}$ for shortcut systems.

Page 6. Regarding (T1), technically a subtree of a rooted tree can be rooted at any vertex, so it is a bit imprecise to say that $T[x]$ is rooted at x. I suggest adding that a subtree of a tree rooted at r, is always considered to be rooted at the vertex closest to r.

Fixed, though in a different way. Since we have to define lowest common T-ancestors anyway, we define the subtree T' of T to be rooted at at the lowest common T-ancestor of $V(T')$.

Page 6. Replace 'that satisfies (T2)' by 'that satisfies (T1)'.

Done.

Page 6. Replace 'parent in T' by 'parent in T_0'.

Done.

Page 6. The term 'hierarchical decomposition' is not defined.

The lemma that this sentence referred to is now gone, and so is the sentence.

Page 7. Although it is clear for me, 'separation' has not been defined. Perhaps a blanket note saying all undefined terms are in Diestel's textbook should be added.

This sentence is now gone.

Page 7. In the very last sentence, I think B_{x_i} should be B_{z_i} (twice). Moreover, the way the proof is written, it seems to suggest that we only get that $a \in B_{z_i}$ for each $i \in \{1, \dots, r\}$. The conclusion that $a \in B_{z_i}$ for each $i \in \{0, \dots, r\}$ is true though. In a normalized tree decomposition, if xy is an edge and x is a T-ancestor of y, then x must be in B_y (since the subtrees T_x and T_y intersect).

Revised accordingly.

Page 8. Can you add a brief explanation why $\mathcal{S}$ is a partition of $V(G)$? It is clear that the sets in $\mathcal{S}$ are disjoint, but why do they cover $V(G)$?

We changed this sentence to "Since $a : V(G) \to V(T)$ is a function with domain $V(G)$ and range $V(T)$, $\mathcal{P} := (S_x : x \in V(T))$ is a sequence of pairwise disjoint sets that cover $V(G)$."

Page 11. Replace 'belong to F_δ' by 'belongs to F_δ'.

This is gone in the new version.

Page 11. In footnote 11, what the authors call a 'closed curve' I would call just a 'curve'. To me, a closed curve satisfies the additional condition that $f(0) = f(1)$.

Revised accordingly.

Page 13. Could it be that the subgraph induced by the vertices of a kite is a K_4 with some parallel edges? This seems relevant later in the proof.

The referee is correct, the induced graph was the wrong thing to use here. Rather, it is the set of edges and vertices incident to the four faces that appear at a crossing. This issue no longer appears in the new version that works with d-framed graphs. The relationship between 4-framed multigraphs and 1-planar graphs is now stated as a lemma.

Page 13. Perhaps I am misunderstanding something, but I do not see why 'none of the edges vx, xw, wy, or yv are crossed by any other edges of G.' For example, see the attached PDF for a picture where vw, xw, wy, and yv are all crossed by other edges of G.

It appears that the referee forgot that we are dealing with an edge-maximal multigraph at this point. We have revised the proof so that this misunderstanding cannot happen.

Page 13. The attached PDF also shows that the definition of 'kite face' may not be well-defined. For example, I do not see why there cannot be some edges of G 'inside' a kite face. Even if this is not the case, I think it is more precise to say that a kite face has 'two and a half edges' and two vertices of G on its boundary' rather than 'three edges and two vertices of G on its boundary'.

Our definition of kite was not exactly what we meant. This section has been completely revised.

Page 13. Regarding the proof of Lemma 4, I think it is better to include all the details from reference [14]. As far as I know, JCTB does not have a page limit, so I do not see an issue with just reproducing the entire proof and telling the reader that they can skip all the details if they wish. This is just my personal opinion though, so the authors can ignore this request if they choose.

Upon reflection we agree with the referee here. Indeed, while cleaning up this proof we have also improved the constant from 30 to 7. The proof is similar to the previous proof except that graph G' is obtained by removing both crossing edges involved in a crossing (the spars of a kite), rather than just one. We now use the language of framed multigraph, for consistency with the literature. Thanks to the referee for forcing us to write this proof more carefully so that we could discover this improvement.

Page 15. Regarding the proof of Theorem 11, if it really is the same proof as the proof of Theorem 2, why not just prove Theorem 11 (from which Theorem 2 follows as a special case)?

We have written the paper starting with simpler cases before moving onto the more general. This approach may use a few more words, but we think the paper becomes more accessible for more readers this way.

Page 16. At the end of Page 16, I do not see why 'every pair of edges cross at most once.' Without further assumptions, it seems as if two paths in the path system could intersect several times. Therefore, by drawing 'each edge vw of G alongside $P_v w$ in G_0', there may be two edges which cross several times. It seems this can be fixed by choosing the path system carefully by 'uncrossing' paths which intersect at more than two internal vertices.

The referee is right here: two edges may cross more than once. We do not want to 'uncross' since this does not maintain the bound on the number of crossings per edge. We have revised the proof avoiding the claim that every pair of edges cross at most once. The lemma itself has not changed.

Page 18. In point 1 in the proof of Lemma 9, I think '$\phi(w)$' should be '$\alpha(w)$'.

Revised accordingly.

Page 18. Say H' is connected since X is connected.

This proof is now removed since the authors of [10] included this result in the journal version of their paper.

Page 18. Replace '$|i' - i''| < p$' by '$0 < |i' - i''| < p$'.

This proof is now removed since the authors of [10] included this result in the journal version of their paper.

Page 20. Where does the bound $d(d-3)/2$ come from? The naive bound I compute is $d(d-1)/2$.

By construction, if the path vxw is in $\mathcal{P}$, then xv and xw are non-consecutive edges incident to x. The bound of $d(d-1)/2$ arises if one ignores that xv and xw are non-consecutive.

Page 21. Replace the comma at the end of the statement of Theorem 14 with a period.

Done.

Page 23. Replace 'oberve' by 'observe'.

Done.

Page 24. Petr Hliněný (together with his student) has announced that the answer to the open question is yes, with $C = 3$ and $\ell = O(k^2)$.

We have been told that there is an error in their proof.

Revised manuscript Click here to view linked References

GRAPH PRODUCT STRUCTURE FOR NON-MINOR-CLOSED CLASSES

Vida Dujmović[☿], *Pat Morin*[♁], and *David R. Wood*[♂]

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Abstract. Dujmović et al. [*J. ACM* '20] recently proved that every planar graph is isomorphic to a subgraph of the strong product of a bounded treewidth graph and a path. Analogous results were obtained for graphs of bounded Euler genus or apex-minor-free graphs. These tools have been used to solve longstanding problems on queue layouts, non-repetitive colouring, p-centered colouring, and adjacency labelling. This paper proves analogous product structure theorems for various non-minor-closed classes. One noteable example is k-planar graphs (those with a drawing in the plane in which each edge is involved in at most k crossings). We prove that every k-planar graph is isomorphic to a subgraph of the strong product of a graph of treewidth $O(k^5)$ and a path. This is the first result of this type for a non-minor-closed class of graphs. It implies, amongst other results, that k-planar graphs have non-repetitive chromatic number upper-bounded by a function of k. All these results generalise for drawings of graphs on arbitrary surfaces. In fact, we work in a more general setting based on so-called shortcut systems, which are of independent interest. This leads to analogous results for certain types of map graphs, string graphs, graph powers, and nearest neighbour graphs.

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Contents

1 Introduction

The starting point for this work is the following 'product structure theorem' for planar graphs[1] by Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood [26] (with an improvement by Ueckerdt, Wood, and Yi [48]). A graph G is *contained* in a graph X if G is isomorphic to a subgraph of X.

Theorem 1 ([26, 48])**.** *Every planar graph is contained in:*

(a) $H \boxtimes P$ *for some graph* H *of treewidth at most* 6 *and for some path* P*,*

(b) $H \boxtimes P \boxtimes K_3$ *for some graph* H *of treewidth at most* 3 *and for some path* P*.*

Here $\boxtimes$ is the strong product,[2] and treewidth[3] is an invariant that measures how 'tree-like' a given graph is; see Figure 1 for an example. Loosely speaking, Theorem 1 says that every planar graph is contained in the product of a tree-like graph and a path. This enables combinatorial results for graphs of bounded treewidth to be generalised for planar graphs (with different constants).

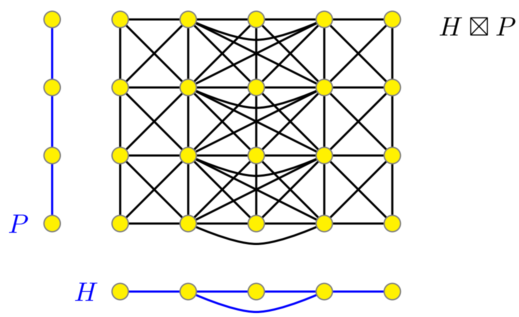


Figure 1: Example of a strong product.

Theorem 1 has been the key tool in solving the following well-known open problems:

- Dujmović et al. [26] use it to prove that planar graphs have bounded queue-number (resolving a conjecture of Heath, Leighton, and Rosenberg [36]).
- Dujmović, Esperet, Joret, Walczak, and Wood [23] use it to prove that planar graphs have bounded non-repetitive chromatic number (resolving a conjecture of Alon, Grytczuk, Hałuszczak, and Riordan [5]).

[1] In this paper, all graphs are finite and undirected. Unless mentioned otherwise, all graphs are also simple. For any graph G and any set S (typically $S \subseteq V(G)$), let $G[S]$ denote the graph with vertex set $V(G) \cap S$ and edge set $\{uv \in E(G) : u, v \in S\}$. We use $G - S$ as a shorthand for $G[V(G) \setminus S]$. Undefined terms are in [18].

[2] The *strong product* of graphs A and B, denoted by $A \boxtimes B$, is the graph with vertex set $V(A) \times V(B)$, where distinct vertices $(v,x),(w,y) \in V(A) \times V(B)$ are adjacent if $v = w$ and $xy \in E(B)$, or $x = y$ and $vw \in E(A)$, or $vw \in E(A)$ and $xy \in E(B)$.

[3] For a tree T, a *T-decomposition* of a graph G is a collection $\mathcal{T} = (B_x : x \in V(T))$ of subsets of $V(G)$ indexed by the nodes of T such that (i) for every $vw \in E(G)$, there exists some node $x \in V(T)$ with $v, w \in B_x$; and (ii) for every $v \in V(G)$, the induced subgraph $T[v] := T[\{x : v \in B_x\}]$ is connected. The *width* of $\mathcal{T}$ is $\max\{|B_x| : x \in V(T)\} - 1$. A *tree-decomposition* is a T-decomposition for any tree T. The *treewidth* $\text{tw}(G)$ of a graph G is the minimum width of a tree-decomposition of G. Treewidth is the standard measure of how similar a graph is to a tree. Indeed, a connected graph has treewidth 1 if and only if it is a tree. Treewidth is of fundamental importance in structural and algorithmic graph theory; see [8, 35, 47] for surveys.

- Dębski, Felsner, Micek, and Schröder [20] use it to prove that planar graphs have p-centered chromatic number $O(p^2 \log p)$ and give a matching lower bound.
- Dujmović, Esperet, Gavoille, Joret, Micek, and Morin [22] use it to find asymptotically optimal adjacency labellings of planar graphs (resolving a problem of Kannan, Naor, and Rudich [38]).
- Esperet, Joret, and Morin [31] use it to show the existence of a 'universal graph' with $n^{1+o(1)}$ vertices and edges that contains every n-vertex planar graph as an induced subgraph (resolving a problem of Babai, Chung, Erdős, Graham, and Spencer [6]).

In addition, Theorem 1 has been used to resolve or make substantial progress on a number of other problems on planar graphs, including ℓ-vertex ranking [11], twin-width [9], and odd colourings [27].

All of these results hold for any graph class that has a product structure theorem analogous to Theorem 1; that is, for any graph class $\mathcal{G}$ where every graph in $\mathcal{G}$ is contained in $H \boxtimes P \boxtimes K_\ell$ where H has bounded treewidth, P is a path, and ℓ is bounded.[4] These applications motivate finding product structure theorems for other graph classes. Dujmović et al. [26] prove product structure theorems for graphs of bounded Euler genus[5] and for apex-minor-free graphs,[6] and Dujmović, Esperet, Morin, Walczak, and Wood [24] do so for bounded-degree graphs in any minor-closed class. See [29] for a survey on this topic.

The main purpose of this paper is to prove product structure theorems for several non-minor-closed classes of interest. Our results are the first of this type for non-minor-closed classes.

1.1 k-Planar Graphs

We start with the example of k-planar graphs. A graph is *k-planar* if it has a drawing in the plane in which each edge is involved in at most k crossings, where no three edges cross at a single point (see Section 2.3 for a formal definition). Such graphs provide a natural generalisation of planar graphs, and are important in graph drawing research; see the recent bibliography on 1-planar graphs and the 140 references therein [40]. It is well-known that the family of k-planar graphs is not minor-closed. Indeed, 1-planar graphs may contain arbitrarily large complete graph minors [21]. Hence the above results are not applicable for k-planar graphs. We extend Theorem 1 as follows.

Theorem 2. *Every k-planar graph is contained in $H \boxtimes P \boxtimes K_{18k^2+48k+30}$, for some graph H of treewidth $\binom{k+4}{3} - 1$ and for some path P.*

For the important special case of 1-planar graphs, we obtain the following result with

[4]It is easily seen that $\text{tw}(H \boxtimes K_\ell) \leqslant (\text{tw}(H)+1)\ell - 1$, so we may assume that $\ell = 1$ in this statement.

[5]The *Euler genus* of a surface with h handles and c crosscaps is $2h + c$. The *Euler genus* of a graph G is the minimum integer g such that G embeds in a surface of Euler genus g. Of course, a graph is planar if and only if it has Euler genus 0; see [43] for more about graph embeddings in surfaces.

[6]A graph M is a *minor* of a graph G if a graph isomorphic to M can be obtained from a subgraph of G by contracting edges. A class $\mathcal{G}$ of graphs is *minor-closed* if for every graph $G \in \mathcal{G}$, every minor of G is in $\mathcal{G}$. A minor-closed class is *proper* if it is not the class of all graphs. For example, for fixed $g \geqslant 0$, the class of graphs with Euler genus at most g is a proper minor-closed class. A graph G is t-apex if it contains a set A of at most t vertices such that $G - A$ is planar. A 1-apex graph is *apex*. A minor-closed class $\mathcal{G}$ is apex-minor-free if some apex graph is not in $\mathcal{G}$.

H planar and a best possible bound on the treewidth of H.

Theorem 3. *Every* 1*-planar graph is contained in* $H \boxtimes P \boxtimes K_7$*, for some planar graph* H *of treewidth* 3 *and for some path* P.

For at least two applications, the planarity of H in Theorem 3 is critical in obtaining asymptotically tight bounds [11, 20]. Theorem 3 is proven in the more general setting of d-framed multigraphs, which also implies a product structure theorem for d-map graphs in which H is a planar with treewidth 3.

1.2 Shortcut Systems

The above result for k-planar graphs (Theorem 2) is in fact a special case of a more general result that relies on the following definition. A non-empty set $\mathcal{S}$ of non-trivial paths in a graph G is a *(k,d)-shortcut system* (for G) if:

- every path in $\mathcal{S}$ has length at most k,[7] and
- for every $v \in V(G)$, the number of paths in $\mathcal{S}$ that include v as an internal vertex is at most d.

Each path $P \in \mathcal{S}$ is called a *shortcut*; if P has endpoints v and w then it is a *vw-shortcut*. Given a graph G and a (k,d)-shortcut system $\mathcal{S}$ for G, let $G^{\mathcal{S}}$ denote the supergraph of G obtained by adding the edge vw for each vw-shortcut in $\mathcal{S}$.

This definition is related to k-planarity by the following observation:

Observation 4. *Every* k*-planar graph is contained in* $G^{\mathcal{S}}$ *for some planar graph* G *and some* $(k+1,2)$*-shortcut system* $\mathcal{S}$ *for* G.

The proof of Observation 4 is trivial: Given a k-plane embedding of a graph G', create a planar graph G by adding a dummy vertex at each crossing point. For each edge $vw \in E(G')$ there is a path P in G between v and w of length at most $k+1$ in which every internal vertex is a dummy vertex. Let $\mathcal{S}$ be the set of such paths P. For each vertex v of G, at most two paths in $\mathcal{S}$ use v as an internal vertex (since no original vertex of G' is an internal vertex of a path in $\mathcal{S}$). Thus $\mathcal{S}$ is a $(k+1,2)$-shortcut system for G, such that $G' \subseteq G^{\mathcal{S}}$. This idea can be pushed further to obtain a rough characterisation of k-planar graphs, which is interesting in its own right, and is useful for showing that various classes of graphs are k-planar (see Section 2.5).

The following theorem is one of the main contributions of the paper. It says that if a graph class $\mathcal{G}$ has a product structure theorem like Theorem 1(b), then so too does the class of graphs obtained by applying shortcut systems to graphs in $\mathcal{G}$.

Theorem 5. *Let* G *be a graph contained in* $H \boxtimes P \boxtimes K_\ell$*, for some graph* H *of treewidth at most* t *and for some path* P*. Let* $\mathcal{S}$ *be a* (k,d)*-shortcut system for* G*. Then* $G^{\mathcal{S}}$ *is contained in* $J \boxtimes P \boxtimes K_{d\ell(k^3+3k)}$ *for some graph* J *of treewidth at most* $\binom{k+t}{t}-1$ *and some path* P.

Theorem 1(b), Theorem 5, and Observation 4 imply Theorem 2 with $K_{6(k^3+3k)}$ instead of $K_{18k^2+48k+30}$. Some further observations presented in Section 2.3 lead to the improved

[7] A path of length k consists of k edges and $k+1$ vertices. A path is *trivial* if it has length 0 and *non-trivial* otherwise.

result. Theorem 5 is applicable for many graph classes in addition to k-planar graphs. Some examples are explored in Section 3.

1.3 Overview and Outline

Table 1 summarizes existing results on product structure theorems for minor-closed graph classes and new results for non-minor-closed graph classes.

Graph class	tw(H)	ℓ	Reference
planar	3^*	3	[26]
planar	4^*	2	[12]
planar	6^*	1	[48]
genus g	3^*	$\max\{2g, 3\}$	[19]
genus g	$2g+6$	1	[48]
apex-minor-free	$O(1)$	$O(1)$	[26]
k-planar	$\binom{k+4}{3}-1$	$18k^2+48k+30$	Theorem 2
(g,k)-planar	$\binom{k+4}{3}-1$	$\max\{2g,3\}(6k^2+16k+10)$	Theorem 11
(g,δ)-string	$\binom{\delta+4}{3}-1$	$\max\{2g,3\}(\delta^4+4\delta^3+9\delta^2+10\delta+4)$	Theorem 14
k-nearest-neighbour	$O(k^6)$	$O(k^4)$	Theorem 22
d-framed	3^*	$d+3\lfloor d/2\rfloor-3$	Theorem 23
1-planar	3^*	7	Theorem 3
d-map	3^*	$d+3\lfloor d/2\rfloor-3$	Theorem 26
(g,d)-map	9	$\max\{2g,3\}(7d^2-21d)$	Theorem 18

* these bounds also apply to the simple treewidth of H.

Table 1: Product structure theorems (of the form $G \subseteq H \boxtimes P \boxtimes K_\ell$).

The remainder of this paper is organized as follows:

- In Section 2 we prove our main result for shortcut systems (Theorem 5), and its optimisations for k-planar graphs and their genus-g generalisation (Theorems 2 and 11).
- Section 3 uses Theorems 2, 5 and 11 to derive product structure theorems for string graphs, powers of bounded-degree graphs, map graphs, and k-nearest neighbour graphs.
- Section 4 proves a product structure theorem for d-framed multigraphs in which the graph H has treewidth 3 and is planar. Using this result, we derive improved product structure theorems for 1-planar graphs and planar map graphs, again with a planar graph H.
- Section 5 quickly surveys applications of product structure theorems and the consequences of the current work.

2 Shortcut Systems and k-Planar Graphs

This section proves Theorem 5 and its specialization to k-planar graphs, Theorem 2. While strong products enable concise statements of the theorems in Section 1, to prove such results it is helpful to work with layerings and partitions, which we now introduce.

2.1 Layered Partitions

A *layering* of a graph G is an ordered partition $\mathcal{L} = \langle L_0, L_1, \ldots \rangle$ such that for every edge $vw \in E(G)$, if $v \in L_i$ and $w \in L_j$ then $|j - i| \leqslant 1$. For any partition $\mathcal{P} = \{S_1, \ldots, S_p\}$ of $V(G)$, a *quotient graph* $H = G/\mathcal{P}$ has a p-element vertex set $V(H) = \{x_1, \ldots, x_p\}$ and $x_i x_j \in E(H)$ if and only if there exists an edge $vw \in E(G)$ such that $v \in S_i$ and $w \in S_j$. To highlight the importance of the quotient graph H, we call $\mathcal{P}$ an *H-partition* and write this concisely as $\mathcal{P} = \{S_x : x \in V(H)\}$ so that each element of $\mathcal{P}$ is indexed by the corresponding vertex in H.

For any partition $\mathcal{P}$ of $V(G)$ and any layering $\mathcal{L}$ of G we define the *layered width* of $\mathcal{P}$ with respect to $\mathcal{L}$ as $\max\{|L \cap P| : L \in \mathcal{L}, P \in \mathcal{P}\}$. For any partition $\mathcal{P}$ of $V(G)$, we define the *layered width* of $\mathcal{P}$ as the minimum, over all layerings $\mathcal{L}$ of G, of the layered width of $\mathcal{P}$ with respect to $\mathcal{L}$.

These definitions relate to strong products as follows.

Lemma 6 ([26])**.** *For every graph H, a graph G has an H-partition of layered width at most ℓ if and only if G is contained in $H \boxtimes P \boxtimes K_\ell$ for some path P.*

As an example of the use of layered partitions, to prove Theorem 1(a), Dujmović et al. [26] build on an earlier result of Pilipczuk and Siebertz [46] to show that every planar graph has an H-partition of layered width 1 for some planar graph H of treewidth at most 8 (improved to 6 in [48]). The proof is constructive and gives a simple quadratic-time algorithm for finding the corresponding partition and layering.[8]

By Lemma 6, Theorem 5 is equivalent to the following result, whose proof is the subject of the next section.

Theorem 7. *Let G be a graph having an H-partition of layered width ℓ in which H has treewidth at most t and let $\mathcal{S}$ be a (k,d)-shortcut system for G. Then $G^{\mathcal{S}}$ has a J-partition of layered width at most $d\ell(k^3 + 3k)$ for some graph J of treewidth at most $\binom{k+t}{t} - 1$.*

2.2 Shortcut Systems

We now prove Theorem 7. For convenience, it will be helpful to assume that $\mathcal{S}$ contains a length-1 vw-shortcut for every edge $vw \in E(G)$. Since $G^{\mathcal{S}}$ is defined to be a supergraph of G, this assumption has no effect on $G^{\mathcal{S}}$ but eliminates special cases in some of our proofs.

Let T be a tree rooted at some node $x_0 \in V(T)$. A node $a \in V(T)$ is a *T-ancestor* of $x \in V(T)$ (and x is a *T-descendant* of a) if a is a vertex of the path, in T, from x_0 to x. Note that each node $x \in V(T)$ is a T-ancestor and T-descendant of itself. We say that a T-ancestor $a \in V(T)$ of $x \in V(T)$ is a *strict* T-ancestor of x if $a \neq x$. The *T-depth* of a node $x \in V(T)$ is the length of the path, in T, from x_0 to x. The *lowest common T-ancestor* of a non-empty set $S \subseteq V(T)$ is the T-ancestor of every node in S with maximum T-depth. In the following, we treat any subtree T' of T as a rooted tree whose root is the lowest common T-ancestor of $V(T')$.

We begin with a standard technique that allows us to work with a normalised tree-decomposition whose tree has the same vertex set as the graph it decomposes:

[8] Bose et al. [12] have recently given linear time algorithms for computing layered H-partitions of planar graphs.

Lemma 8. *For every graph H of treewidth t, there is a rooted tree T with $V(T) = V(H)$ and a width-t T-decomposition $(B_x : x \in V(T))$ of H that has following additional properties:*

(T1) for each node $x \in V(H)$, the subtree $T[x] := T[\{y \in V(T) : x \in B_y\}]$ is rooted at x; and consequently

(T2) for each edge $xy \in E(H)$, one of x or y is a T-ancestor of the other.

Proof. That (T1) implies (T2) is a standard observation: If two subtrees intersect, then one contains the root of the other. Thus, it suffices to construct a width-t tree-decomposition that satisfies (T1).

Begin with any width-t tree-decomposition $(B_x : x \in V(T_0))$ of H that uses some tree T_0 and has no empty bags. Select any node $x_1 \in V(T_0)$, add a leaf x_0, with $B_{x_0} = \emptyset$, adjacent to x_1 and root T_0 at x_0. (The purpose of x_0 is to ensure that every node x for which B_x is non-empty has a parent.) Let $f : V(H) \to V(T)$ be the function that maps each $x \in V(H)$ onto the root of the subtree $T_0[x] := T_0[\{y \in V(T_0) : x \in B_y\}]$. If f is not one-to-one, then select some distinct pair $x, y \in V(H)$ with $a := f(x) = f(y)$. Subdivide the edge between a and its parent in T_0 by introducing a new node a' with $B_{a'} = B_a \setminus \{x\}$. Now $f(y) = a'$ and $f(x) = a$, so this modification reduces the number of distinct pairs $x, y \in V(H)$ with $f(x) = f(y)$. Repeatedly performing this modification will eventually produce a tree-decomposition $(B_x : x \in V(T_0))$ of H in which f is one-to-one.

Next, remove the node x_0 from T_0 (so that x_1 becomes the new root of T_0). Consider any node $a \in V(T_0)$ such that there is no vertex $x \in V(H)$ with $f(x) = a$. In this case, $B_a \subseteq B_{a'}$ where a' is the parent of a since any $x \in B_a \setminus B_{a'}$ would have $f(x) = a$. In this case, contract the edge aa' in T_0, eliminating the node a. Repeating this operation will eventually produce a width-t tree-decomposition of $(B_x : x \in V(T_0))$ where f is a bijection between $V(H)$ and $V(T_0)$. Renaming each node $a \in V(T_0)$ as $f^{-1}(a)$ gives a tree-decomposition $(B_x : x \in V(T))$ with $V(T) = V(H)$. By the definition of f, the tree-decomposition $(B_x : x \in V(T))$ satisfies (T1). □

Proof of Theorem 7. Let $\mathcal{L} := \langle L_1, \dots, L_h \rangle$ be a layering of G; let $\mathcal{Y} := (Y_x : x \in V(H))$ be an H-partition of G of layered width at most ℓ with respect to $\mathcal{L}$; and let $\mathcal{T} := (B_x : x \in V(T))$ be a tree-decomposition of H satisfying the conditions of Lemma 8.

For a node $x \in V(T)$, we say that a shortcut $P \in \mathcal{S}$ *crosses* x if Y_x contains an internal vertex of P. In other words, if $P = v_0, \dots, v_r$ and $\{v_1, \dots, v_{r-1}\} \cap Y_x \neq \emptyset$. We say that a vertex $v \in V(G)$ *participates* in x if $v \in Y_x$ or if $\mathcal{S}$ contains a shortcut P with $v \in V(P)$ and P crosses x. For each $v \in V(G)$, let $a(v)$ denote the lowest common T-ancestor of all the nodes $x \in V(T)$ in which v participates.

For each $x \in V(T)$, define $S_x := \{v \in V(G) : a(v) = x\}$. Since $a : V(G) \to V(T)$ is a function with domain $V(G)$ and range $V(T)$, $\mathcal{P} := (S_x : x \in V(T))$ is a sequence of pairwise disjoint sets that cover $V(G)$. Let $J := G^{\mathcal{S}}/\mathcal{P}$ denote the resulting quotient graph. We consider $V(J) \subseteq V(T)$, where each $x \in V(J)$ is the vertex obtained by contracting S_x in $G^{\mathcal{S}}$. (Any node $x \in V(T)$ with $S_x = \emptyset$ does not contribute a vertex to J.)

From this point onward, the plan is to show that:

(i) $\mathcal{P}$ has small layered width with respect to the layering $\mathcal{L}$ of G, and

(ii) J has small treewidth.

Once we have established (i) and (ii), the result follows easily since a layering of $G^{\mathcal{S}}$ is easily obtained from $\mathcal{L}$ by 'compressing' groups of k consecutive layers. We begin with Step (i):

Claim 1. *For each $i \in \{1,\dots,h\}$ and each $x \in V(J)$, $|S_x \cap L_i| \leqslant d\ell(k^2+3)$.*

Proof. Recall that S_x is defined by vertices that participate in x, and these are vertices that are either in Y_x or in shortcuts that cross Y_x. We say that a vertex $w \in Y_x$ *contributes* a vertex $v \in S_x$ if $v = w$ or if some path in $\mathcal{S}$ that contains v has w as an internal vertex. We upper bound the number of vertices in $S_x \cap L_i$ by upper-bounding the number of vertices contributed to $S_x \cap L_i$ by each $w \in Y_x$.

Refer to Figure 2. If $w \in Y_x \cap L_i$ and no path in $\mathcal{S}$ includes w as an internal vertex then w contributes at most one vertex, itself, to $S_x \cap L_i$. Otherwise, consider some path $P \in \mathcal{S}$ that contains w as an internal vertex. If $w \in L_i$, then P contributes at most $k+1$ vertices to $S_x \cap L_i$. If $w \in L_{i-1} \cup L_{i+1}$, then P contributes at most k vertices to $S_x \cap L_i$. If $w \in L_{i-j} \cup L_{i+j}$ for $j \geqslant 2$, then P contributes at most $k-j$ vertices to $S_x \cap L_i$.

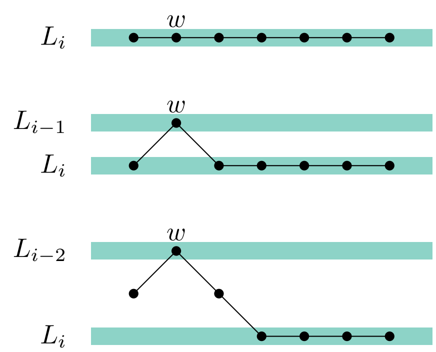


Figure 2: A path P containing an internal vertex $w \in Y_x \cap L_{i-j}$.

For any j, the number of vertices $w \in L_{i+j} \cap Y_x$ is at most ℓ. Each such vertex w is an internal vertex of at most d paths in $\mathcal{S}$. Therefore,

$$|S_x \cap L_i| \leqslant d\ell\left(k+1+2k+\sum_{j=2}^{k} 2(k-j)\right) = d\ell(k^2+3) \ . \qquad \square$$

We now proceed with Step (ii), showing that J has small treewidth. To accomplish this, we construct a small width tree-decomposition $\mathcal{C} := (C_x : x \in V(T))$ of J using the same tree T used in the tree-decomposition $\mathcal{T}$ of H. The following claim will be useful in showing that the resulting decomposition has small width.

Claim 2. *For each edge $xy \in E(J)$, one of x or y is a T-ancestor of the other.*

Proof. If $xy \in E(J)$ then there exists a shortcut $P := v_0, \dots, v_r \in \mathcal{S}$ with endpoints $v_0 \in S_x$ and $v_r \in S_y$.[9] For each $i \in \{0, \dots, r\}$, let Y_{x_i} be the part in $\mathcal{Y}$ that contains v_i. Since P is a connected subgraph of G, $H[\{x_0, \dots, x_r\}]$ is a connected subgraph of H. Since $\mathcal{T}$ is a normalized tree-decomposition of H, (T2) implies that some node x_i is a T-ancestor of all nodes in $x_0, \dots, x_r$.

We claim that at least one of v_0 or v_r participates in x_i. If $i = 0$ then $v_0 \in Y_{x_0} = Y_{x_i}$ so v_0 participates in x_i. Similarly, if $i = r$ then v_r participates in x_r. Otherwise $i \in \{1, \dots, r-1\}$ so v_i is an internal vertex of the shortcut P, so P crosses x_i. Since $v_0, v_r \in V(P)$, this implies that v_0 and v_r both participate in x_i.

Suppose, without loss of generality, that v_0 participates in x_i. Then $a(v_0) = x$ is a T-ancestor of x_i which a T-ancestor of x_r. Finally $a(v_r) = y$ is a T-ancestor of x_r. Therefore, both x and y are contained in the path from the root of T to x_r, so at least one of x or y is a T-ancestor of the other. This completes the proof of Claim 2 □

Claim 3. *The graph J has a tree-decomposition in which every bag has size at most $\binom{k+t}{t}$.*

Proof. For the tree-decomposition $(C_x : x \in V(T))$ of J we use the same tree T used in the tree-decomposition $(B_x : x \in V(T))$ of H. For each node x of T, we define C_x as follows: C_x contains x as well as every T-ancestor a of x such that J contains an edge ax' where x is a T-ancestor of x' (including the possibility that $x = x'$). Claim 2 ensures that, for every edge $ax' \in E(J)$, both a and x are contained in $C_{x'}$. The connectivity of $T[a] := T[\{x \in V(T) : a \in C_x\}]$ follows from the fact that, for every node $x' \in V(T[a])$, every node x on the path in T from a to x' is also a node of $T[a]$. Therefore $(C_x : x \in V(T))$ is indeed a tree-decomposition of J.

It remains to show that each bag in this tree-decomposition has size at most $\binom{k+t}{t}$. We will do this by appealing to an elegant result of Pilipczuk and Siebertz [46] on k-reachability in directed skeletons, which we now explain.

Let H^+ denote the supergraph of H that contains the edge xy if and only if some bag $B_z \in \mathcal{T}$ contains both x and y (that is, H^+ is the chordal closure of H with respect to $\mathcal{T}$). Let $\overrightarrow{H}^+$ be the directed graph obtained by directing each edge xy of H^+ in the direction $\overrightarrow{xy}$ so that y is a T-ancestor of x. Observe that any directed path in $\overrightarrow{H}^+$ that begins at x leads to a T-ancestor of x. Pilipczuk and Siebertz [46, Lemma 13] show that, for any $x \in V(\overrightarrow{H}^+)$, the number of strict T-ancestors of x that can be reached by a directed path of length at most k that begins at x is at most $\binom{k+t}{t} - 1$.

Consider an arbitrary node $x \in V(T)$ and suppose that some strict T-ancestor a of x is contained in C_x. We will show that $\overrightarrow{H}^+$ contains a directed path from x to a of length at most k. By [46, Lemma 13], this implies that the number of strict T-ancestors of x contained in C_x is at most $\binom{k+t}{t} - 1$ and therefore $|C_x| \leqslant \binom{k+t}{t}$.

Since $a \in C_x$, there exists a T-descendant x' of x such that $ax' \in E(J)$. Therefore, there exists a shortcut $P := v_0, \dots, v_r$ in $\mathcal{S}$ with $v_0 \in S_{x'}$ and $v_r \in S_a$. For each $i \in \{0, \dots, r\}$, let Y_{x_i} be the part of $\mathcal{Y}$ that contains v_i. For each $i \in \{1, \dots, r-1\}$, $v_i \in Y_{x_i}$ is an internal vertex of P, so

[9] Recall that we have made the assumption that $\mathcal{S}$ contains a length-1 vw-shortcut for each edge $vw \in E(G)$.

v_r participates in x_i. Therefore $a = a(v_r)$ is a T-ancestor of x_i for each $i \in \{1, \dots, r-1\}$.

Since $v_0, \dots, v_r$ is a path in G, $x_0, \dots, x_r$ is a lazy walk in H.[10] Let j be the minimum integer such that x_j is a strict T-ancestor of x. Since $x' = x_0$ is a T-descendant of x and $a = x_r$ is a strict T-ancestor of x, $j \in \{1, \dots, r\}$ and x_{j-1} is a T-descendant of x (possibly $x_{j-1} = x$). Since $x_{j-1}x_j \in E(H)$, $x_j \in B_{x_{j-1}}$ and therefore $x_j \in B_x$. Therefore $xx_j \in E(H^+)$.

Therefore $W := y_0, \dots, y_s := x, x_j, x_{j+1}, \dots, x_r$ is a lazy walk in H^+ of length $r \leqslant k$. For each $j \in \{0, \dots, s\}$, let z_j be the lowest common T-ancestor of $y_0, \dots, y_j$. The same reasoning used to conclude that $xx_j \in E(H^+)$ can be used to conclude that $z_{i-1} = z_i$ or $z_{i-1}z_i \in E(H^+)$ for each $i \in \{1, \dots, s\}$. Therefore $z_0, \dots, z_s$ is a directed lazy walk in $\overrightarrow{H}^+$. Finally, let $\overrightarrow{W}$ be the path in $\overrightarrow{H}^+$ obtained by removing removing duplicates from $z_0, \dots, z_s$. Then $\overrightarrow{W}$ is a directed path in $\overrightarrow{H}^+$ from x to a of length at most $s \leqslant r \leqslant k$. This completes the proof of Claim 3. □

At this point, the proof of Theorem 7 is almost immediate from Claims 1 and 3, except that the layering $\mathcal{L}$ of G may not be a valid layering of $G^{\mathcal{S}}$. In particular, $G^{\mathcal{S}}$ may contain edges vw with $v \in L_i$ and $w \in L_{i+j}$ for any $j \in \{0, \dots, k\}$. To resolve this, we use a new layering $\mathcal{L}' := \langle L'_0, \dots, L'_h \rangle$ in which $L'_i = \bigcup_{j=ki}^{ki+k-1} L_i$. This increases the layered width given by Claim 1 from $d\ell(k^2+3)$ to $d\ell(k^3+3k)$. Therefore G has an H-partition of layered width at most $d\ell(k^3+3k)$ in which H has treewidth at most $\binom{k+t}{t} - 1$, completing the proof of Theorem 7. □

2.3 k-Planar Graphs

We first formally define k-planar graphs. For a surface Σ, a *Σ-embedded graph* G is a graph with $V(G) \subset \Sigma$ in which each edge $vw \in E(G)$ is a curve[11] in Σ with endpoints v and w and not containing any vertex of G in its interior. The *faces* of G are the connected components of $\mathbb{R}^2 \setminus \bigcup_{vw \in E(G)} vw$. A *crossing* in a Σ-embedded (multi)graph G is an unordered pair of distinct edges $\{vw, xy\} \subseteq E(G)$ whose interiors have a point in common.

If an embedded graph G has no crossings then it is *non-crossing* graph. When G is non-crossing and F is a face of G, then let *$V(F)$* be the set of vertices in G on the boundary of F. When F is homeomorphic to a disk, the *facial walk* of F is the cycle consisting of the edges and vertices of G on the boundary of F. If $\Sigma = \mathbb{R}^2$ is the Euclidean plane and each edge of G is involved in at most k crossings, then G is a k-plane graph. A graph G is *k-planar* if is isomorphic to some k-plane graph. A 0-plane graph is a *plane* graph and a 0-planar graph is a *planar* graph.

Without any loss of generality we may assume that the interiors of any two edges of a Σ-embedded graph G have at most one point in common and that no point in $\mathbb{R}^2$ is contained in the interior of three or more edges of G. This assumption can be enforced by local changes that do not change the number of crossings each edge is involved in. If p is

[10] A *lazy walk* is a graph G is a walk in the pseudograph obtained by adding a self-loop at each vertex of G.

[11] A *curve* in a surface Σ is a continuous function $f : [0,1] \to \Sigma$. The points $f(0)$ and $f(1)$ are called the *endpoints* of the curve. When there is no danger of misunderstanding we treat a curve f as the point set $\{f(t) : 0 \leqslant t \leqslant 1\}$. The *interior* of f is the point set $\{f(t) : 0 < t < 1\}$.

a common point in the interior of two crossing edges vw and xy, then we say that vw and xy *cross at* p.

As mentioned in Section 1, Theorems 1 and 5 imply a product structure theorem for k-planar graphs. We get improved bounds as follows.

Proof of Theorem 2. Let G be a k-plane graph. As in the proof of Observation 4, let G_0 be the plane graph obtained by adding a dummy vertex at each crossing in G. In this way, each edge $vw \in E(G)$ corresponds naturally to a path P_{vw} of length at most $k+1$ in G_0. Let $\mathcal{S} := \{P_{vw} : vw \in E(G)\}$. Observe that $\mathcal{S}$ is a $(k+1,2)$-shortcut system for G_0 and that $G_0^{\mathcal{S}} \supseteq G$. Specifically, $G_0^{\mathcal{S}}$ contains every edge and vertex of G as well as the dummy vertices in $V(G_0) \setminus V(G)$ and their incident edges.

Since G_0 is planar, Theorem 1(b) and Lemma 6 implies that G_0 has an H-partition of layered width 3 for some planar graph H of treewidth at most 3. Applying Theorem 7 to G_0 and $\mathcal{S}$ immediately implies that G (an arbitrary k-planar graph) has an H-partition of layered width $6((k+1)^3 + 3(k+1))$ for some graph H of treewidth at most $\binom{k+4}{3} - 1$.

We can reduce the layered width of the H-partition of G from $O(k^3)$ to $O(k^2)$ by observing that the dummy vertices in $V(G_0) \setminus V(G)$ do not contribute to the layered width of this partition. In this setting, the proof of Claim 1 is simpler since each vertex $w \in Y_x$ contributes at most two vertices to $L_i \cap Y_x$. More precisely, each path $P \in \mathcal{S}$ containing an internal (dummy) vertex $w \in Y_x \cap (L_{i-j} \cup L_{i+j})$ contributes: (i) at most two vertices to $S_x \cap L_i$ for $j \in \{0,\dots,\lfloor (k+1)/2 \rfloor\}$; (ii) at most one vertex to $S_x \cap L_j$ for $j \in \{\lfloor (k+1)/2 \rfloor + 1,\dots,k+1\}$; or (iii) no vertices to $S_x \cap L_j$ for $j > k+1$. Redoing the calculation at the end of the proof of Claim 1 then yields

$$|S_x \cap Y_i| \leqslant d\ell\left(2 + 4\left\lfloor \tfrac{k+1}{2} \right\rfloor + 2\left\lceil \tfrac{k+1}{2} \right\rceil\right) = d\ell\left(2 + 2(k+1) + 2\left\lfloor \tfrac{k+1}{2} \right\rfloor\right) \leqslant d\ell(3k+5) = 18k + 30 \ .$$

With this change, the layered width of the partition given by Theorem 7 becomes $(18k + 30)(k+1) = 18k^2 + 48k + 30$. The result follows from Lemma 6. $\square$

Theorem 2 shows that every k-planar graph is contained in $H \boxtimes P \boxtimes K_\ell$ for some graph H with treewidth $O(k^3)$ where $\ell \leqslant O(k^2)$. In some applications, the treewidth of H is much more significant than the value of ℓ, which leads to the following question:

Open Problem: Does there exist a function $\ell : \mathbb{N} \to \mathbb{N}$ and a universal constant C such that every k-planar graph is contained in $H \boxtimes P \boxtimes K_{\ell(k)}$ for some graph H with treewidth at most C? Perhaps $C = 3$, which is the case for planar graphs (Theorem 1(b)) and for 1-planar graphs (Theorem 3). Note that $C \geqslant 3$ even for planar graphs [26].

2.4 (g,k)-Planar Graphs

As mentioned in the introduction, product structure theorems have been established for several minor-closed classes in addition to planar graphs, including the following version of Theorem 1 for graphs of bounded Euler genus.

Theorem 9 ([19, 26, 48])**.** *Every graph of Euler genus g is contained in:*

(a) $H \boxtimes P$ for some graph H of treewidth at most $2g+6$ and some path P.

(b) $H \boxtimes P \boxtimes K_{\max\{2g,3\}}$ for some planar graph H of treewidth at most 3 and for some path P.

The definition of k-planar graphs naturally generalises to genus-g surfaces. A Σ-embedded graph G is *(Σ,k)-plane* if every edge of G is involved in at most k crossings. A graph G is *(g,k)-planar* if it is isomorphic to some (Σ,k)-plane graph, for some surface Σ with Euler genus at most g. Observation 4 immediately generalises as follows:

Observation 10. *Every (g,k)-planar graph G is contained in $G_0^{\mathcal{S}}$ for some graph G_0 of Euler genus at most g and some $(k+1,2)$-shortcut system $\mathcal{S}$ for G_0. Moreover, $V(G) \subseteq V(G_0)$ and for every edge $vw \in E(G)$ there is a vw-path P in G_0 of length at most $k+1$, such that every internal vertex in P has degree at most 4 in G_0.*

Theorems 5 and 9(b) imply a product structure theorem for (g,k)-planar graphs. The resulting bounds are improved by the following theorem, which is proved using exactly the same approach used in the proof of Theorem 2 (applying Theorem 9(b) instead of Theorem 1(b)). We omit repeating the details.

Theorem 11. *Every (g,k)-planar graph is contained in $H \boxtimes P \boxtimes K_\ell$ for some graph H with $\operatorname{tw}(H) \leqslant \binom{k+4}{3} - 1$, where $\ell := \max\{2g,3\} \cdot (6k^2+16k+10)$.*

2.5 Rough Characterisation

Observation 10 shows that (g,k)-planar graphs can be obtained by a shortcut system applied to a graph of Euler genus g, where internal vertices on the paths have bounded degree. This observation and the following converse result together provide a rough characterisation of (g,k)-planar graphs, which is interesting in its own right, and is useful for showing that various classes of graphs are (g,k)-planar.

Lemma 12. *Fix integers $g \geqslant 0$ and $k, \Delta \geqslant 2$. Let G_0 be a graph of Euler genus at most g. Let G be a graph with $V(G) \subseteq V(G_0)$ such that for every edge $vw \in E(G)$ there is a vw-path P_{vw} in G_0 of length at most k, such that every internal vertex on P_{vw} has degree at most Δ in G_0. Then G is $(g, 2k(k+1)\Delta^k)$-planar.*

Proof. For a vertex x of G_0 with degree at most Δ, and for $i \in \{1,\dots,k-1\}$, say a vertex v is *i-close* to x if there is a vx-path P in G_0 of length at most i such that every internal vertex in P has degree at most Δ in G_0. For each edge vw of G, say that vw *passes through* each internal vertex on P_{vw}. Say vw passes through x. Then v is i-close to x and w is j-close to x for some $i,j \in \{1,\dots,k-1\}$ with $i+j \leqslant k$. At most Δ^i vertices are i-close to x. Thus, the number of edges of G that pass through x is at most

$$\sum_{i=1}^{k-1}\sum_{j=1}^{k-i}\Delta^i\Delta^j = \sum_{i=1}^{k-1}\Delta^i\sum_{j=1}^{k-i}\Delta^j < \sum_{i=1}^{k-1}\Delta^i 2\Delta^{k-i} = \sum_{i=1}^{k-1} 2\Delta^k < 2k\Delta^k \ .$$

Fix a (Σ,k)-plane drawing of G_0 in a surface Σ of Euler genus at most g. Choose $\epsilon > \epsilon' > 0$. For each vertex v of G_0, let B_v be the *ball* $\{x \in \mathbb{R}^2 : \operatorname{dist}(x,v) \leqslant \epsilon\}$. For each edge vw of G_0, let C_{vw} be the *channel* $\{x \in \mathbb{R}^2 : \operatorname{dist}(x,vw) \leqslant \epsilon'\} \backslash (B_v \cup B_w)$. We may choose $\epsilon > \epsilon' > 0$ sufficiently small so that (i) $B_v \cap B_w = \emptyset$ for distinct $v,w \in V(G_0)$, (ii) $C_{vw} \neq \emptyset$ for $vw \in E(G_0)$, (iii)

$B_v \cap C_{xy} = \emptyset$ for $v \in V(G_0)$ and $xy \in E(G_0)$, and (iv) $C_{vw} \cap C_{xy} = \emptyset$ for distinct $vw, xy \in E(G_0)$. Draw each edge vw of G following the sequence of balls and channels defined by the vertices and edges in P_{vw}. This can be done so that whenever edges vw and xy of G cross, the crossing point is in B_z for some vertex z of G_0 that is internal in P_{vw} or in P_{xy}, and vw and xy cross at most once in each such B_z.

Thus, for each edge vw of G, every edge of G that crosses vw passes through a vertex on P_{vw} (including v and/or w if they too have degree at most Δ). Since P_{vw} has at most $k+1$ vertices, and less than $2k\Delta^k$ edges of G pass through each vertex on P_{vw}, the edge vw is crossed by less than $2k(k+1)\Delta^k$ edges in G. Hence G is $(g, 2k(k+1)\Delta^k)$-planar. □

3 Graphs Derived from Shortcut Systems

In this section we give four examples of well-known graph families that can be expressed in terms of shortcut systems and which therefore have product structure theorems.

3.1 String Graphs

A *string graph* is the intersection graph of a set of curves in the plane with no three curves meeting at a single point; see [33, 34, 45] for example. For an integer $\delta \geqslant 2$, if each curve is in at most δ intersections with other curves, then the corresponding string graph is called a *δ-string graph*. A *(g,δ)-string* graph is defined analogously for curves on a surface of Euler genus at most g.

Lemma 13. *Every (g,δ)-string graph G is contained in $G_0^{\mathcal{S}}$ for some graph G_0 with Euler genus at most g and some $(\delta+1,\delta+1)$-shortcut system $\mathcal{S}$ for G_0.*

Proof. Let $\mathcal{C} = \{C_v : v \in V(G)\}$ be a set of curves in a surface of Euler genus at most g whose intersection graph is G. Let G_0 be the graph obtained by adding a vertex at the intersection point of every pair of curves in $\mathcal{C}$ that intersect, where two such consecutive vertices on a curve C_v are adjacent in G_0. For each vertex $v \in V(G)$, if C_v intersects $k \leqslant \delta$ other curves, then introduce a new vertex called v on C_v between the $\lfloor \frac{k}{2} \rfloor$-th vertex already on C_v and the $\lfloor \frac{k}{2} + 1 \rfloor$-th such vertex. For each edge vw of G, there is a path P_{vw} of length at most $2\lceil \frac{\delta}{2} \rceil \leqslant \delta + 1$ in G_0 between v and w. Let $\mathcal{S}$ be the set of all such paths P_{vw}. Consider a vertex x in G_0 that is an internal vertex on some path in $\mathcal{S}$. Then x is at the intersection of C_v and C_w for some edge $vw \in E(G)$. If some path $P \in \mathcal{S}$ passes through x, then $P = P_{vu}$ for some edge vu incident to v, or $P = P_{wu}$ for some edge wu incident to w. At most $\lceil \frac{\delta}{2} \rceil$ paths in $\mathcal{S}$ corresponding to edges incident to v pass through x, and similarly for edges incident to w. Thus at most $2\lceil \frac{\delta}{2} \rceil \leqslant \delta + 1$ paths in $\mathcal{S}$ use x as an internal vertex. Thus $\mathcal{S}$ is a $(\delta+1,\delta+1)$-shortcut system for G_0, and by construction, $G \subseteq G_0^{\mathcal{S}}$. □

Lemma 13 and Theorems 5 and 9 now imply:

Theorem 14. *For integers $g \geqslant 0$ and $\delta \geqslant 2$, let $\ell := \max\{2g,3\}(\delta^4 + 4\delta^3 + 9\delta^2 + 10\delta + 4)$ and $t := \binom{\delta+4}{3} - 1$. Then every (g,δ)-string graph is contained in $H \boxtimes P \boxtimes K_\ell$ for some path P and for some graph H with treewidth t,*

3.2 Powers of Bounded Degree Graphs

Recall that the *k-th power* of a graph G is the graph G^k with vertex set $V(G^k) := V(G)$, where $vw \in E(G^k)$ if and only if $\mathrm{dist}_G(v,w) \leqslant k$. If G is planar with maximum degree Δ, then G^k is $2k(k+1)\Delta^k$-planar by Lemma 12. Thus we can immediately conclude that bounded powers of planar graphs of bounded degree have product structure. However, the bounds we obtain are improved by the following lemma that constructs a shortcut system directly.

Lemma 15. *If a graph G has maximum degree Δ, then $G^k = G^{\mathcal{S}}$ for some $(k, 2k\Delta^k)$-shortcut system $\mathcal{S}$.*

Proof. For each pair of vertices x and y in G with $\mathrm{dist}_G(x,y) \in \{1,\dots,k\}$, fix an xy-path P_{xy} of length $\mathrm{dist}_G(x,y)$ in G. Let $\mathcal{S} := \{P_{xy} : \mathrm{dist}_G(x,y) \in \{1,\dots,k\}\}$. Say P_{xy} uses some vertex v as an internal vertex. If $\mathrm{dist}_G(v,x) = i$ and $\mathrm{dist}_G(v,y) = j$, then $i,j \in \{1,\dots,k-1\}$ and $i+j \leqslant k$. The number of vertices at distance i from v is at most Δ^i. Thus the number of paths in $\mathcal{S}$ that use v as an internal vertex is at most

$$\sum_{i=1}^{k-1}\sum_{j=1}^{k-i}\Delta^i\Delta^j = \sum_{i=1}^{k-1}\Delta^i\sum_{j=1}^{k-i}\Delta^j < \sum_{i=1}^{k-1}\Delta^i(2\Delta^{k-i}) < 2k\Delta^k \ .$$

Hence $\mathcal{S}$ is a $(k, 2k\Delta^k)$-shortcut system. $\square$

Theorem 5 and Lemma 15 imply:

Theorem 16. *Let G be contained in $H \boxtimes P \boxtimes K_\ell$ with maximum degree Δ, for some graph H of treewidth at most t and for some path P. Then for every integer $k \geqslant 1$, the k-th power G^k is contained in $J \boxtimes P \boxtimes K_{2k\ell\Delta^k(k^3+3k)}$ for some graph J of treewidth at most $\binom{k+t}{t} - 1$ and some path P.*

3.3 Map Graphs

Map graphs are defined as follows. Start with a graph G_0 embedded in a surface of Euler genus g, with each face labelled a 'nation' or a 'lake', where each vertex of G_0 is incident with at most d nations. Let G be the graph whose vertices are the nations of G_0, where two vertices are adjacent in G if the corresponding faces in G_0 share a vertex. Then G is called a *(g,d)-map graph*. A $(0,d)$-map graph is called a (plane) *d-map graph*; see [16, 32] for example. The $(g,3)$-map graphs are precisely the graphs of Euler genus at most g; see [21].

There is a natural drawing of a map graph obtained by positioning each vertex of G inside the corresponding nation and each edge of G as a curve passing through the corresponding vertex of G_0. It is easily seen that each edge is in at most $\lfloor\frac{d-2}{2}\rfloor\lceil\frac{d-2}{2}\rceil$ crossings; see [21]. Thus G is $(g, \lfloor\frac{d-2}{2}\rfloor\lceil\frac{d-2}{2}\rceil)$-planar. Also note that Lemma 12 with $k=2$ implies that G is $(g, O(d^2))$-planar. Theorem 11 then establishes a product structure theorem for map graphs, but we get much better bounds by constructing a shortcut system directly. The following lemma is reminiscent of the characterisation of (g,d)-map graphs in terms of the half-square of bipartite graphs [16, 21].

Lemma 17. *Every (g,d)-map graph G is contained in $G_1^{\mathcal{S}}$ for some graph G_1 with Euler genus at most g and some $(2, \frac{1}{2}d(d-3))$-shortcut system $\mathcal{S}$ for G_1.*

Proof. Let G be a (g,d)-map graph. So there is a graph G_0 embedded in a surface of Euler genus g, with each face labelled a 'nation' or a 'lake', where each vertex of G_0 is incident with at most d nations. Let N be the set of nations. Then $V(G) = N$ where two vertices are adjacent in G if the corresponding nation faces of G_0 share a vertex. Let G_1 be the graph with $V(G_1) := V(G_0) \cup N$, where distinct vertices $v, w \in N$ are adjacent in G_1 if the boundaries of the corresponding nations have an edge of G_0 in common, and $v \in V(G_0)$ and $w \in N$ are adjacent in G_1 if v is on the boundary of the nation corresponding to w. Observe that G_1 embeds in the same surface as G_0 with no crossings, and that each vertex in $V(G_0)$ has degree at most d in G_1. Consider an edge $vw \in E(G)$. If the nations corresponding to v and w share an edge of G_0, then vw is an edge of G_1. Otherwise, v and w have a common neighbour x in $V(G_0)$. In the latter case, let P_{vw} be the path (v,x,w). Let $\mathcal{S}$ be the set of all such paths P_{vw}. Each vertex $x \in V(G_0)$ is the middle vertex on at most $\frac{1}{2}d(d-3)$ paths in $\mathcal{S}$. Thus $\mathcal{S}$ is a $(2, \frac{1}{2}d(d-3))$-shortcut system for G_1, and by construction, $G \subseteq G_1^{\mathcal{S}}$. □

Theorems 5 and 9 and Lemma 17 imply:

Theorem 18. *For integers $g \geqslant 0$ and $d \geqslant 3$, let $\ell := \max\{2g, 3\}(7d^2 - 21d)$. Then every (g,d)-map graph is contained in $H \boxtimes P \boxtimes K_\ell$ for some path P and for some graph H with treewidth 9,*

3.4 k-Nearest-Neighbour Graphs

In this section, we show that k-nearest neighbour graphs of point sets in the plane are $O(k^2)$-planar. For two points $x, y \in \mathbb{R}^2$, let $d_2(x,y)$ denote the Euclidean distance between x and y. The k-nearest-neighbour graph of a point set $P \subset \mathbb{R}^2$ is the geometric graph G with vertex set $V(G) = P$, where the edge set is defined as follows. For each point $v \in P$, let $N_k(v)$ be the set of k points in P closest to v. Then $vw \in E(G)$ if and only if $w \in N_k(v)$ or $v \in N_k(w)$. (The edges of G are straight-line segments joining their endpoints.) See [10] for a survey of results on k-nearest neighbour graphs and other related proximity graphs.

The following result, which is immediate from Ábrego, Monroy, Fernández-Merchant, Flores-Peñaloza, Hurtado, Sacristán, and Saumell [1, Corollary 4.2.6] states that k-nearest-neighbour graphs have bounded maximum degree:

Lemma 19. *The degree of every vertex in a k-nearest-neighbour graph is at most $6k$.*

We make use of the following well-known observation (see for example, Bose, Morin, Stojmenović, and Urrutia [13, Lemma 2]):

Observation 20. *If $v_0, \dots, v_3$ are the vertices of a convex quadrilateral in counterclockwise order then there exists at least one $i \in \{0, \dots, 3\}$ such that $\max\{d_2(v_i, v_{i-1}), d_2(v_i, v_{i+1})\} < d_2(v_{i-1}, v_{i+1})$, where subscripts are taken modulo 4.*

Lemma 21. *Every k-nearest-neighbour graph is $O(k^2)$-planar.*

Proof. Let G be a k-nearest-neighbour graph and consider any edge $vw \in E(G)$. Let $xy \in E(G)$ be an edge that crosses vw. Note that $vxwy$ are the vertices of a convex quadrilateral in (without loss of generality) counter-clockwise order. Then we say that

1. xy is of Type v if $\max\{d_2(v,x), d_2(v,y)\} < d_2(x,y)$;
2. xy is of Type w if $\max\{d_2(w,x), d_2(w,y)\} < d_2(x,y)$; or
3. xy is of Type C otherwise.

If xy is of Type C, then Observation 20 implies that $\max\{d_2(x,v), d_2(x,w)\} < d_2(v,w)$ without loss of generality. In this case, we call x a Type C vertex. We claim that $V(G)$ contains at most $k-1$ Type C vertices. Indeed, more than $k-1$ Type C vertices would contradict the fact that $vw \in E(G)$ since every Type C vertex is closer to both v and w than $d_2(v,w)$.

Next observe that, if xy is of Type v, then at least one of xv or yv is in $E(G)$ in which case we call x (respectively y) a Type v vertex. By Lemma 19, there are at most $6k$ Type v vertices. Similarly, there are at most $6k$ Type w vertices.

Thus, in total, there are at most $13k-1$ Type v, Type w, and Type C vertices. By Lemma 19, each of these vertices is incident with at most $6k$ edges that cross vw. Therefore, there are at most $78k^2 - 6k$ edges of G that cross vw. Since this is true for every edge $vw \in E(G)$, G is $(78k^2 - 6k)$-planar. □

Note that Lemma 21 is tight up to the leading constant: Every k-nearest neighbour graph on $n \geqslant k+1$ vertices has at least $kn/2$ edges and at most kn edges. For $k \geqslant 7$, the Crossing Lemma [3, 42] implies that the total number of crossings is therefore $\Omega(k^3 n)$ so that the average number of crossings per edge is $\Omega(k^2)$.

Lemma 21 and Theorem 2 immediately imply

Theorem 22. *Every k-nearest neighbour graph is a subgraph of $H \boxtimes P \boxtimes K_\ell$ for some graph H of treewidth $O(k^6)$ and for some $\ell \in O(k^4)$.*

4 Framed Graphs

For any integer $d \geqslant 3$, an embedded (multi)graph G is *d-framed* if it has a biconnected plane spanning (multi)subgraph G_0, whose facial cycles each have length at most d, and such that the interior of each edge in $E(G) \setminus E(G_0)$ is contained in the interior of some face of G_0. The embedded (multi)graph G_0 is called the *frame* of G. This definition (for simple graphs) was introduced by Bekos, Lozzo, Griesbach, Gronemann, Montecchiani, and Raftopoulou [7].

Every d-framed multigraph can be described by a $(2, d(d-3)/2)$-shortcut system applied to a plane multigraph by adding a vertex inside each face F of G_0 adjacent to the vertices of F and creating a shortcut between each non-adjacent pair of vertices in $V(F)$. Thus Theorem 5 is applicable. However, for framed graphs we prove the following stronger result, where the treewidth bound on H is best possible.[12]

Theorem 23. *For any integer $d \geqslant 3$, every d-framed multigraph is contained in $H \boxtimes P \boxtimes K_{d+3\lfloor d/2 \rfloor - 3}$ for some planar graph H with treewidth at most 3 and for some path P.*

[12] A multigraph G is *contained* in a graph X if the simple graph underlying G is contained in X.

Theorem 23 is used below to obtain product structure theorems for 1-planar multigraphs and for map graphs. Note that even for simple 1-planar graphs it is essential that we allow frames with parallel edges.

Theorem 23 is a consequence of the following technical lemma, which is an extension of the analogous result for plane graphs [26]. The following definition is a convenient way to establish the planarity of H in Theorem 23: A T-decomposition of a graph G is *t-simple* if each of its bags has size at most $t+1$ and, for each t-element subset $\{v_1,\dots,v_t\} \subseteq V(G)$, at most two bags contain $v_1,\dots,v_t$. Finally, a *BFS spanning forest* T of G rooted at a set $V_0 \subseteq V(G)$ is a spanning forest of G with the property that for each $v \in V(G)$,

$$\min\{\mathrm{dist}_G(v,w) : w \in V_0\} = \min\{\mathrm{dist}_T(v,w) : w \in V_0\}.$$

Lemma 24. *The setup:*

1. *Let G be a d-framed multigraph and let G_0 be a frame of G.*
2. *Let F_0 be the outer face of G_0 and let T be a BFS spanning forest of G_0 rooted at $V(F_0)$.*
3. *For every integer $j \geqslant 0$, let $L_j = \{v \in V(G) : \mathrm{dist}_T(v, V(F_0)) = j\}$.*
4. *Let F be a cycle in G_0 whose vertices are partitioned into $k \leqslant 3$ sets $P_1,\dots,P_k$ such that for each $i \in \{1,\dots,k\}$,*
   (a) $F[P_i]$ is connected; and
   (b) P_i has a partition $\{X_i, Y_i\}$ where $|X_i| \leqslant d-3$ and $|Y_i \cap L_j| \leqslant 3$ for each $j \geqslant 0$.
5. *Let N and N_0 be the subgraphs of G and G_0 consisting only of those edges and vertices contained in the interior or boundary of F.*

Then N has an H-partition $\mathcal{P} = \{S_x : x \in V(H)\}$ such that:

(i) for each $i \in \{1,\dots,k\}$, there exists $x_i \in V(H)$ such that $P_i = S_{x_i}$;
(ii) For each $x \in V(H)$, S_x has a partition $\{X_x, Y_x\}$ where $|X_x| \leqslant d-3$ and $|Y_x \cap L_j| \leqslant 3$ for each $j \geqslant 0$;
(iii) H has a 3-simple tree-decomposition $\mathcal{T}$, and if $k=3$ then $\{x_1,x_2,x_3\}$ is contained in exactly one bag of $\mathcal{T}$.

Proof. This proof is similar to the proof of Lemma 13 by Dujmović et al. [26], but is complicated by several factors. In particular, the reader should keep in mind that G and G_0 are multigraphs and that the cycle F may consist of two vertices and two (parallel) edges.

We proceed by induction on (x,y) where x is the number of inner vertices of N_0 and y is the number of inner faces of N_0. The induction base case occurs when N_0 has no inner vertices (that is, $V(N_0) = V(F)$). In this case, simply take $\mathcal{P} := \{P_1,\dots,P_k\}$ and verify that the preconditions of the lemma (the 'setup') ensure that $\mathcal{P}$ satisfies the requirements (i)–(iii):

(i) By definition, each of $P_1,\dots,P_k$ is a part of $\mathcal{P}$.
(ii) Assumption 4(b) ensures, for each $x \in V(H)$, the existence of X_x and Y_x such that $|X_x| \leqslant d-3$ and $|Y_x \cap L_j| \leqslant 3$ for each integer $j \geqslant 0$.
(iii) The trivial tree-decomposition of H that has a single bag containing $\{x_1,\dots,x_k\}$ is 3-simple and, if $k=3$, has exactly one bag that contains $\{x_1,\dots,x_3\}$.

We now move onto the case in which N_0 contains at least one inner vertex. Our H-partition $\mathcal{P}$ will include P_1, P_2 , P_3 and a (possibly empty) part S that will be determined by three vertices $v_1v_2v_3$ that belong to a common inner face τ of N_0. We first explain how

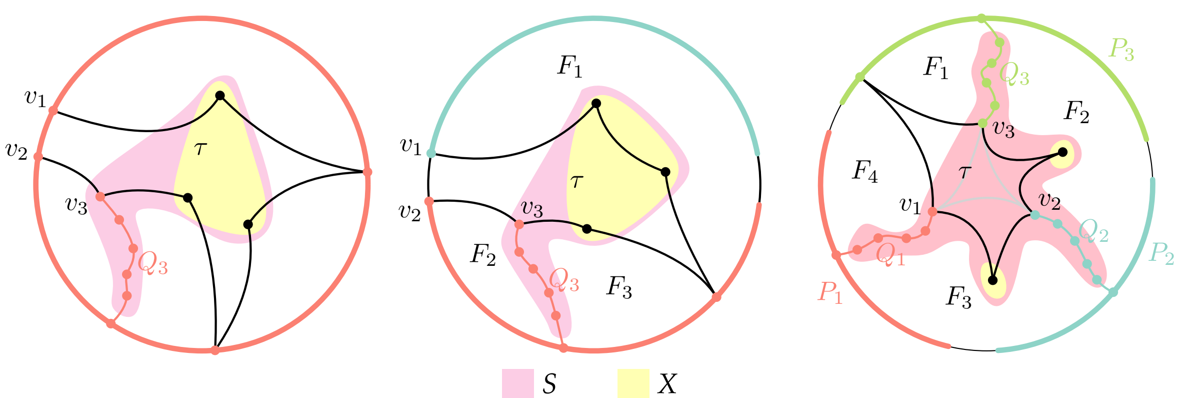
v_1, v_2 and v_3 are chosen and then explain how these are used to define S. There is one case to consider for each possible value of k (see Figure 3):

1. If $k = 1$ then let $\tau := v_1, \ldots, v_p$ be an inner face of N_0 that contains at least one edge $v_1 v_2$ of F on its boundary.
2. If $k = 2$ then let $\tau := v_1, \ldots, v_p$ be an inner face of N_0 that contains at least one edge $v_1 v_2$ of F on its boundary and such that $v_1 \in P_1$ and $v_2 \in P_2$.
3. If $k = 3$ then assign each vertex v of N_0 a colour $\alpha(v) \in \{1, \ldots, 3\}$ as follows: Let w be the first vertex of $V(F)$ encountered on the path in T from v to the appropriate vertex of F_0. (This implies that $w = v$ when v is a vertex of F.) Since $w \in V(F)$, $w \in P_i$ for some $i \in \{1, \ldots, k\}$ and we define $\alpha(v) := i$. Now, for each $d' \in \{4, \ldots, d\}$ and each d'-sided inner face F of N_0, add $d' - 3$ edges inside of F to split it into 3-sided faces. This yields a plane supergraph N_0' of N_0 in which each inner face has exactly three edges on its boundary. By Sperner's Lemma (see [2]) there exists an inner face $\tau' := v_1, v_2, v_3$ of N_0' such that $\alpha(v_i) = i$ for each $i \in \{1, 2, 3\}$. Let $\tau := v_1, \ldots, v_p$ be the facial cycle of N_0 that contains τ'.

We now define the vertices in S. For each $i \in \{1, 2, 3\}$, let Q_i be the shortest path, in T, from v_i to a vertex in $V(F)$. Let $\overline{S}$ denote the subgraph of N_0 consisting of vertices and edges of Q_1, Q_2, Q_3, and τ. Let $S := V(\overline{S}) \setminus V(F)$, let $Y := (V(Q_1) \cup V(Q_2) \cup V(Q_3)) \setminus V(F)$, and let $X := S \setminus Y$. Observe that $|V(Q_i) \cap L_j| \leqslant 1$ for each $i \in \{1, 2, 3\}$ and each $j \geqslant 0$. Therefore $|Y \cap L_j| \leqslant 3$ for each $j \geqslant 0$. Furthermore, $X \subseteq V(\tau) \setminus \{x_1, x_2, x_3\}$, so $|X| \leqslant p - 3 \leqslant d - 3$. Therefore, using $S := Y \cup X$ as a part in the partition $\mathcal{P}$ satisfies condition (ii).

Let M denote the subgraph of N_0 containing the edges and vertices of $\overline{S}$ and the edges and vertices of F. By our choice of τ, the graph M is 2-connected. Let $F_1, \ldots, F_m, \tau$ be the inner faces of M. By our choice of $v_1 v_2 v_3$, the vertices of each F_i can be partitioned into at most three sets $P_{i,1}$, $P_{i,2}$, and $P_{i,3}$ where $P_{i,1} \subset Y$, $P_{i,2} \subseteq P_b$, and $P_{i,3} \subseteq P_c$ for some $b, c \in \{1, \ldots, k\}$.

In the following, in order to avoid introducing even more notation or constantly adding the modifier "when it exists", we assume that $V(F_i) \cap S \neq \emptyset$ and that there are exactly two



$k = 1$ $k = 2$ $k = 3$

Figure 3: Three cases when choosing the face τ of N_0.

distinct $b, c \in \{1, \dots, k\}$ such that $V(F_i) \cap P_b \neq \emptyset$ and $V(F_i) \cap P_c \neq \emptyset$. When this assumption does not hold, exactly the same reasoning can be used, while omitting one or more of $P_{i,1}$, $P_{i,2}$ or $P_{i,3}$.

Since M is 2-connected and $\overline{S}$ is connected, $F_i[P_{i,j}]$ is connected, for each $j \in \{1, 2, 3\}$. Let N_i and $N_{0,i}$ be the subgraphs of N and N_0 contained in the interior or boundary of F_i. Every inner vertex of $N_{0,i}$ is an inner vertex of N_0. Every inner face of $N_{0,i}$ is an inner face of N_0 and τ is not an inner face of $N_{0,i}$. Therefore $N_{0,i}$ has no more inner vertices than N_0 and has fewer inner faces than N_0. Therefore, we can apply induction using the cycle F_i and the partition $P_{i,1}, P_{i,2}, P_{i,3}$ of $V(F_i)$. The result of the induction is a H_i-partition $\mathcal{P}_i$ of N_i satisfying (i)–(iii) above. We now define our partition

$$\mathcal{P} := \{Y, P_1, \dots, P_k\} \cup \bigcup_{i=1}^{m} \{P \in \mathcal{P}_i : P \cap V(M) = \emptyset\} \enspace .$$

It is straightforward to verify that $\mathcal{P}$ is indeed a partition of $V(N)$ and, by definition this is an H-partition of N for the graph $H := N/\mathcal{P}$. It remains to verify that $\mathcal{P}$ satisfies (i)–(iii).

By construction $\mathcal{P}$ contains parts $P_1, \dots, P_k$, so $\mathcal{P}$ satisfies (i). By Assumption 4(b) each of the parts P_1, P_2, P_3 satisfies condition (ii). We have already argued that the part S satisfies condition (ii). The inductive hypothesis ensures that each of the remaining parts satisfies condition (ii). Therefore $\mathcal{P}$ satisfies (ii).

It remains to construct a tree-decomposition $\mathcal{T} := (B_y : y \in V(T_0))$ of H that satisfies (iii). Let x_0 be the vertex of H obtained from contracting S and, for each $j \in \{1, \dots, k\}$, let x_j denote the vertex of H obtained from contracting P_j. The tree T_0 has a node y_0 with bag $B_{y_0} := \{x_0, \dots, x_k\}$. The inductive hypothesis ensures that, for each $i \in \{1, \dots, m\}$, $H_i := N_i/\mathcal{P}_i$ has a tree-decomposition $\mathcal{T}_i := (B_z : z \in V(T_i))$ that satisfies (iii). Let $a_i, b_i, c_i \in V(H_i)$ be the vertices obtained by contracting $P_{i,1}$, $P_{i,2}$, and $P_{i,3}$, respectively. Since a_i, b_i, c_i form a clique in H_i, there exists a bag B_{z_i} in $\mathcal{T}_i$ that contains $\{a_i, b_i, c_i\}$.

Recall that $P_{i,1} := V(N_i) \cap S$, $P_{i,2} := V(N_i) \cap P_b$ and $P_{i,3} := V(N_i) \cap P_c$ for some $b, c \in \{1, \dots, k\}$. For each bag B_z, $z \in V(T_i)$, replace each occurrence of a_i by x_0, replace each occurrence of b_i by x_b and replace each occurrence of c_i by x_c. These replacements do not increase the size of any bag and they result in a tree-decomposition of the graph obtained from H_i by renaming the vertices a_i, b_i, and c_i with the names x_0, x_a, and x_b. Now add an edge that joins y_0 to z_i so that T_i becomes a subtree of T_0 that is adjacent to y_0. We perform this operation for each $i \in \{1, \dots, m\}$ to obtain our final tree-decomposition $\mathcal{T} := (B_y : y \in V(T_0))$.

It is now straightforward to verify that $\mathcal{T}$ is a tree-decomposition of $H_0 := N_0/\mathcal{P}$ in which each bag has size at most 4. To see that $\mathcal{T}$ is also a tree-decomposition of $H := N/\mathcal{P}$, consider any edge $vw \in E(N) \setminus E(N_0)$ with $v \in P_\alpha$ and $w \in P_\beta$. We must verify that some bag of $\mathcal{T}$ contains α and β. If both v and w are in $V(M)$, then $\alpha = x_a$ and $\beta = x_b$ for some $a, b \in \{0, \dots, 3\}$, in which case $B_{y_0} = \{x_0, \dots, x_k\}$ contains α and β. If neither v nor w are in $V(M)$ then, since no edge of $N \subseteq G$ crosses an edge of $M \subseteq G_0$, both v and w are contained in the interior of the same face F_i of M. So, by the inductive hypothesis there exists a bag of $\mathcal{T}_i$ that contains α and β and this becomes a bag of $\mathcal{T}$ that contains α and β. Finally, if $v \in V(M)$ and $w \notin V(M)$, then $v \in P_{i,j}$ for some $j \in \{1, 2, 3\}$ and w is in the interior of some face F_i of M. By induction, some bag of $\mathcal{T}_i$ contains β and α', where α' is the vertex of H_i obtained by contracting $P_{i,j}$. However, before T_i is attached to T_0 each occurence of α' in $\mathcal{T}_i$ is replaced by α, so some bag of $\mathcal{T}$ contains both α and β. Therefore $\mathcal{T}$ is a tree-decomposition of H.

Next we verify that, when $k = 3$ exactly one bag of $\mathcal{T}$ contains $\{x_1, \dots, x_3\}$. The bag B_{y_0} contains $x_1, \dots, x_k$. When $k = 3$, each neighbour of z_i of y_0 has a bag that contains at most two of $\{x_1, x_2, x_3\}$. Therefore, when $k = 3$, B_y is the unique bag in $\mathcal{T}$ that contains $\{x_1, x_2, x_3\}$.

It remains to show that any triple of vertices of H is contained in at most two bags of $\mathcal{T}$. By the inductive hypothesis, the only triples we need to be concerned about are the 3-element subsets of $\{x_0, x_1, \dots, x_k\}$. The case $k = 1$ is trivial since no triples can be formed by $\{x_0, x_1\}$. When $k = 2$, $B_{y_0} = \{x_0, x_1, x_2\}$. In this case, the fact that τ contains an edge v_1v_2 of F with $v_1 \in P_{x_1}$ and $v_2 \in P_{x_2}$ ensures that there is at most one face F_i of M that has vertices of S, P_1, and P_2 on its boundary (see the middle part of Figure 3). By the inductive hypothesis (applied to H_i) the bag B_{z_i} is the only bag of $\mathcal{T}_i$ that contains $\{x_0, x_1, x_2\}$. Therefore there are at most two bags, B_y and B_{z_i} of $\mathcal{T}$ that contain $\{x_0, x_1, x_2\}$. This establishes (iii) for $k \in \{1, 2\}$ so, from this point on we assume that $k = 3$.

We claim that, for distinct $a, b \in \{1, 2, 3\}$ there is at most one face F_i such that $V(F_i) \cap S_{x_j} \neq \emptyset$ for each $j \in \{0, a, b\}$. Suppose to the contrary that there are two such faces F_{i_1} and F_{i_2}. Create a graph M^+ from M by placing a vertex v_{i_b} inside F_{i_b} and adjacent to each vertex in $V(F_{i_b})$ for each $b \in \{1, 2\}$. Since M is planar and F_{i_1} and F_{i_2} are distinct faces of M, M^+ is planar. However, contracting each of $S_{x_0}, \dots, S_{x_3}$ in M^+ produces a graph isomorphic to the complete bipartite graph $K_{3,3}$, a contradiction. (In this contracted graph, x_0, x_a, x_b are on one side of the bipartition and v_{i_a}, v_{i_b}, and the node in $\{x_1, x_2, x_3\} \setminus \{x_a, x_b\}$ is on the other side.)

We have already established that, when $k = 3$, the only bag that contains $\{x_1, x_2, x_3\}$ is B_{y_0}, so we need only consider triples $\{x_0, x_a, x_b\}$ for distinct $a, b \in \{1, 2, 3\}$. By the preceding claim, there is at most one face F_i with vertices of S, P_a, and P_b on its boundary. By the inductive hypothesis, the only other bag in $\mathcal{T}$ that contains $\{x_0, x_a, x_b\}$ is the bag B_{z_i}. This completes the proof. $\square$

Using Lemma 24, the proof of Theorem 23 is now straightforward.

Proof of Theorem 23. Let G be a d-framed multigraph with frame G_0 having outer face F_0 Let T be a BFS forest of G_0 rooted at $V(F_0)$ and let $\langle L_0, L_1, \dots \rangle$ be the resulting BFS layering. Let $P_1 := V(F_0)$, let Y_1 be any subset of $\min\{|P_1|, 3\}$ vertices in P_1 and let $X_1 := P_1 \setminus Y_1$. Now apply Lemma 24 on G, G_0, T, $F = F_0$, $k = 1$, and $P_1 = Y_1 \cup X_1$. This gives an H-partition $\mathcal{P} := \{S_x : x \in V(H)\}$ of G in which H has a 3-simple tree-decomposition. Since any graph having a 3-simple tree-decomposition is planar [30, 39, 41], H is planar.

Define $\mathcal{L} = \langle L'_0, L'_1 \dots \rangle$ where $L'_i = L_{\lfloor d/2 \rfloor i} \cup \dots \cup L_{\lfloor d/2 \rfloor (i+1)-1}$ for each integer $i \geqslant 0$. This is a layering of G since $\mathrm{dist}_{G_0}(v, w) \leqslant \lfloor \frac{d}{2} \rfloor$ for each edge $vw \in E(G)$. Since $|L_j \cap Y_x| \leqslant 3$ for each integer $j \geqslant 0$, $|L'_i \cap Y_x| \leqslant 3 \lfloor \frac{d}{2} \rfloor$ and therefore $|L'_i \cap S_x| = |L'_i \cap Y_x| + |L'_i \cap X_x| \leqslant \lfloor \frac{d}{2} \rfloor + d - 3$ for each integer $i \geqslant 0$ and each $x \in V(H)$. The result now follows from Lemma 6. $\square$

4.1 1-Planar Graphs

Theorem 3 is an immediate consequence of Theorem 23 and the next lemma. Similar results appear in earlier works [7, 14, 15, 17] but we state the precise lemma and provide a proof here for the sake of completeness.

Lemma 25. *A multigraph G is 1-planar if and only if it is contained in a 4-framed multigraph.*

Proof. Let G be a 1-plane multigraph. We may assume that no two edges incident to a common vertex of G cross, since such a crossing can be removed by a local modification to obtain an isomorphic 1-plane graph in which the two edges do not cross[13]. We make G into an *edge-maximal* 1-plane multigraph, by repeating the following operation: If two vertices v and w of G appear on a common face F and there is no edge $vw \in E(G)$ contained in the boundary of F, then add the edge vw, embedded in the face F. This may introduce parallel edges into G, but no two such edges appear on the boundary of a common face.

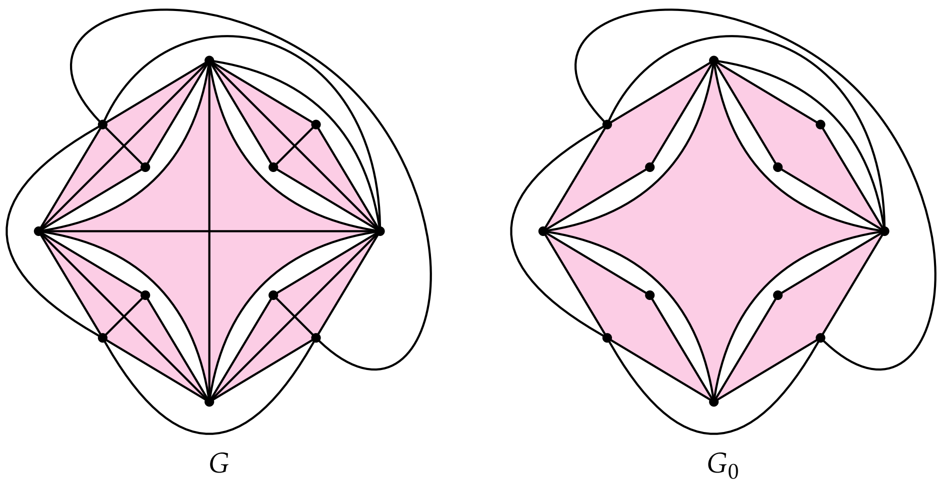


Figure 4: The graph G_0 obtained by removing pairs of crossing edges from G is a plane multigraph whose faces all have three or four sides.

Refer to Figure 4. To understand the structure of G, it is helpful to consider the plane graph G' obtained by replacing each pair of edges vw and xy that cross at p with four edges vp, wp, xp, and yp meeting at a newly added *dummy vertex* p. Let F be a face of G_0 and let $W := w_0, \dots, w_{r-1}$ be the facial walk around F. In the following paragraphs, subscripts on the vertices in W are implicitly taken modulo r. By edge-maximality, if W contains only non-dummy vertices, then it contains exactly three vertices and F is bounded by three edges of G.

If W contains a dummy vertex w_i, then neither w_{i-1} nor w_{i+1} is a dummy vertex since if w_{i-1} (respectively w_{i+1}) were a dummy vertex then the edge of G that contains $w_{i-1}w_i$ (respectively w_iw_{i+1}) would be involved in at least two crossings. Therefore each dummy

[13]While this is true for 1-plane graphs it is not true for k-plane graphs with $k \geqslant 3$; the uncrossing operation can increase the number of crossings on a particular edge from k to $2(k-1)$.

vertex w_i of W is flanked by two non-dummy vertices $w_{i-1}, w_{i+1} \in V(G)$. Since w_i has degree $4 > 1$ in G' and no two edges incident to a common vertex cross each other, $w_{i-1} \neq w_{i+1}$. Since G is edge-maximal the edge $w_{i-1}w_{i+1}$ is an edge of G on the boundary of F. Therefore, if F has a dummy vertex w_i in its facial walk W, then $W = w_{i-1}, w_i, w_{i+1}$ and F is bounded by three edges of G', each of which is contained in a different edge of G.

Consider some pair of edges vw and xy that cross at point p, which is a degree-4 vertex of G'. Since no pair of edges incident to a common vertex cross each other, v, w, x, and y are all distinct. There are four faces $F_1, \dots, F_4$ of G' with p on their boundary and, from the preceding discussion each of vx, xw, wy, and yv is an uncrossed edge of G. Therefore, by removing each pair of crossing edges from G we obtain a plane multigraph G_0 each of whose faces is bounded by three or four edges. Therefore G is a 4-framed graph with frame G_0. □

4.2 Plane Map Graphs

In addition to 1-planar graphs, framed graphs can be used to obtain the following improved product structure theorem for (plane) d-map graphs.

Theorem 26. *Every d-map graph is contained in $H \boxtimes P \boxtimes K_{d+3\lfloor d/2 \rfloor - 3}$ for some planar graph H with $\mathrm{tw}(H) \leqslant 3$ and for some path P.*

Theorem 26 is an immediate consequence of Theorem 23 and the next lemma. Similar results to Lemma 27 appear in earlier works [7, 14, 15, 17] but we state the precise lemma and provide a proof here for the sake of completeness.

Lemma 27. *For every integer $d \geqslant 3$, every d-map graph is a spanning subgraph of a d-framed multigraph.*

Proof. Let G_0 be a graph embedded in the plane, with each face labelled a nation or a face, and where each vertex of G_0 is incident with at most d nations and let G be the corresponding map graph. Let G_0^* be the dual graph of G_0. So the vertices of G_0^* correspond to faces of G_0, and each vertex of G_0^* is a nation vertex or a lake vertex.

Let x be a vertex of G_0, let F_x be the corresponding face of G_0^*, and let $v_1, \dots, v_s$ be the facial cycle of F_x. Let $C := w_1, \dots, w_r$ be the subsequence of $v_1, \dots, v_s$ consisting of only the nation vertices. Since x is incident to at most d nations, $r \leqslant d$. Call C_x the *nation cycle* of F_x. Note that if $r = 1$ then the "nation cycle" has no edges, and if $r = 2$ then the "nation cycle" has one edge.

Let G_1 be a plane supergraph of G_0^* obtained by adding the nation cycle of each face F of G_0, and then triangulating each face with more than d vertices on its boundary. Each nation cycle of length at least 3 is now a facial cycle of G_1, and no face of G_1 has more than d vertices on its boundary. Let $\widehat{G_1}$ be the d-framed graph whose frame is G_1.

By definition, $V(G) = V(\widehat{G_1})$. To prove the claim, it suffices to show that $E(G) \subseteq E(\widehat{G_1})$. Indeed, if $vw \in E(G)$ then the nation faces corresponding to v and w have a common vertex x on their boundary. The vertex x corresponds to a face F_x in G_0^* and the facial cycle of F_x contains v and w. Therefore, the nation cycle C_x of F_x contains v and w. If C_x has length

2 then $vw \in E(G_1) \subseteq E(\widehat{G_1})$. If C_x has length at least 3 then C_x bounds a face in G_1, so $vw \in E(\widehat{G_1})$. $\square$

5 Applications

As discussed in the introduction, product structure has been used to resolve a number of problems on planar graphs. In most cases, these results hold for any graph class with product structure. In this section, we survey some of the consequences of this for the graph classes considered in the previous two sections.

5.1 Queue Layouts

For an integer $k \geqslant 0$, a *k-queue layout* of a graph G consists of a linear ordering $\preceq$ of $V(G)$ and a partition $\{E_1, E_2, \dots, E_k\}$ of $E(G)$, such that for $i \in \{1, 2, \dots, k\}$, no two edges in E_i are nested with respect to $\preceq$. That is, it is not the case that $v \prec x \prec y \prec w$ for edges $vw, xy \in E_i$. The *queue-number* of a graph G, denoted by $\operatorname{qn}(G)$, is the minimum integer k such that G has a k-queue layout. Queue-number was introduced by Heath et al. [36], who famously conjectured that planar graphs have bounded queue-number. Dujmović et al. [26] recently proved this conjecture using Theorem 1 and the following lemma. Indeed, resolving this question was the motivation for the development of Theorem 1.

Lemma 28 ([26])**.** *If $G \subseteq H \boxtimes P \boxtimes K_\ell$ then $\operatorname{qn}(G) \leqslant 3\ell \operatorname{qn}(H) + \lfloor \frac{3}{2}\ell \rfloor \leqslant 3\ell\, 2^{\operatorname{tw}(H)} - \lceil \frac{3}{2}\ell \rceil$.*

Since all the product structure theorems presented thus far upper bound the treewidth of H, Lemma 28 immediately implies bounds on the queue number of all graphs in these classes. An improvement can be made for 1-planar and d-map graphs since then H is planar with treewidth at most 3. Alam, Bekos, Gronemann, Kaufmann, and Pupyrev [4] proved that every planar graph with treewidth at most 3 has queue-number at most 5. Thus the graph H in Theorems 3 and 26 has queue-number at most 5.

The following corollary summarizes all of these results.

Corollary 29. *The following bounds hold for the queue number of any graph from each of the following classes:*

- *k-planar:* $2^{O(k^3)}$
- *(g,k)-planar:* $\max\{1,g\} \cdot 2^{O(k^3)}$
- *(g,δ)-string:* $\max\{1,g\} \cdot 2^{O(\delta^3)}$
- *1-planar:* 115
- *d-map:* $\lfloor \frac{33}{2}(d + 3\lfloor \frac{d}{2} \rfloor - 3) \rfloor$
- *(g,d)-map:* $\max\{1,g\} \cdot 2^{O(d)}$
- *k-nearest-neighbour:* $2^{O(k^6)}$

Note that Dujmović et al. [26] previously proved the bound of $O(g^{k+2})$ for k-planar graphs using Theorem 9 and an ad-hoc method. Our result provides a better bound when $g > 2^{k^2}$.

5.2 Non-Repetitive Colouring

The next two applications are in the field of graph colouring. For our purposes, a *c-colouring* of a graph G is any function $\phi\colon V(G) \to C$, where C is a set of size at most c. A c-colouring ϕ of G is *non-repetitive* if, for every path $v_1,\dots,v_{2h}$ in G, there exists $i \in \{1,\dots,h\}$ such that $\phi(v_i) \neq \phi(v_{i+h})$. The *non-repetitive chromatic number* $\pi(G)$ of G is the minimum integer c such that G has a non-repetitive c-colouring. This concept, introduced by Alon et al. [5], has since been widely studied; see [23] for more than 40 references. Up until recently the main open problem in the field has been whether planar graphs have bounded non-repetitive chromatic number, first asked by Alon et al. [5]. Dujmović et al. [23] recently solved this question using Theorem 1 and the following lemma.

Lemma 30 ([23])**.** *If* $G \subseteq H \boxtimes P \boxtimes K_\ell$ *then* $\pi(G) \leqslant \ell\, 4^{\operatorname{tw}(H)+1}$.

Combining Lemma 30 with our results on product structure, we obtain the following corollary:

Corollary 31. *The following bounds hold for the non-repetitive chromatic number of any graph from each of the following classes:*

- *k-planar:* $(18k^2 + 48k + 30) \cdot 4^{\binom{k+4}{3}}$
- *(g,k)-planar:* $\max\{2g, 3\} \cdot (6k^2 + 16k + 10)4^{\binom{k+4}{3}}$
- *1-planar:* 1792
- *d-map:* $256(d + 3\lfloor \frac{d}{2} \rfloor - 3)$
- *(g,d)-map graph:* $(7d^2 - 21d) \cdot 4^{10}$

Prior to the current work, the strongest upper bound on the non-repetitive chromatic number of n-vertex k-planar graphs was $O(k \log n)$ [28]. Our results also give bounds on the non-repetitive chromatic number of (g,δ)-string graphs and k-nearest-neighbour graphs, but these bounds are weaker than existing results based on maximum degree. The bounds obtained from product structure are exponential in δ and k^2, respectively. However, each of these graph classes has degree bounded by $O(\delta)$ and $O(k^2)$, respectively, and therefore have non-repetitive chromatic number $O(\delta^2)$ and $O(k^4)$, respectively [25].

5.3 Centered Colourings

A c-colouring ϕ of G is *p-centered* if, for every connected subgraph $X \subseteq G$, $|\{\phi(v) : v \in V(X)\}| > p$ or there exists some $v \in V(X)$ such that $\phi(v) \neq \phi(w)$ for every $w \in V(X) \setminus \{v\}$. In words, either X receives more than p colours or some vertex in X receives a unique colour. Let $\chi_p(G)$ be the minimum integer c such that G has a p-centered c-colouring. Centered colourings are important since they characterise classes of bounded expansion, which is a key concept in the sparsity theory of Nešetřil and Ossona de Mendez [44].

Dębski et al. [20] use Theorem 1 and Theorem 1(b), respectively, to show that $\chi_p(G) \in O(p^2 \log p)$ when G is planar or of bounded Euler genus. Upper bounds on χ_p for graphs of given treewidth [46] and for graph products [20] imply the next lemma.

Lemma 32 ([20, 46])**.** *For every graph H of treewidth at most t and for every path P, if $G \subseteq H \boxtimes P \boxtimes K_\ell$ then*

$$\chi_p(G) \leqslant \ell(p+1)\,\chi_p(H) \leqslant \ell(p+1)\binom{p+t}{t}.$$

For planar graphs of treewidth at most 3 a $O(p^2 \log p)$ upper bound is known:

Lemma 33. *[20] For every planar graph H of treewidth at most 3,*

$$\chi_p(H) \leqslant (p+1)(p\lceil\log(p+1)\rceil + 2p + 1) \ .$$

Combining Lemmas 32 and 33 with our results on product structure, we obtain the following corollary:

Corollary 34. *The following bounds hold for the p-centered chromatic number of any graph from each of the following classes:*

- *k-planar:* $(18k^2 + 48k + 30) \cdot 4^{\binom{k+4}{3}}$
- *(g,k)-planar:* $\max\{2g, 3\} \cdot (6k^2 + 16k + 10)(p+1)\binom{p+\binom{k+4}{3}-1}{\binom{k+4}{3}-1}$
- *1-planar:* $7(p+1)^2(p\lceil\log(p+1)\rceil + 2p + 1)$
- *d-map:* $(d + 3\lfloor d/2 \rfloor - 3) \cdot (p+1)^2(p\lceil\log(p+1)\rceil + 2p + 1)$
- *(g,d)-map:* $\max\{2g, 3\} \cdot 7d(d-3)(p+1) \cdot \binom{p+9}{9}$

Prior to the current work, the strongest known upper bounds on the p-centered chromatic number of (g,k)-planar graphs G were doubly-exponential in p, as we now explain. Dujmović et al. [21] proved that G has layered treewidth $(4g+6)(k+1)$. Van den Heuvel and Wood [37] showed this implies that G has r-strong colouring number at most $(4g+6)(k+1)(2r+1)$. By a result of Zhu [49], G has r-weak colouring number at most $((4g+6)(k+1)(2r+1))^r$, which by another result of Zhu [49] implies that G has p-centered chromatic number at most $((4g+6)(k+1)(2^{p-1}+1))^{2^{p-2}}$. The above results are substantial improvements, providing bounds on $\chi_p(G)$ that are polynomial in p for fixed g and k.

As with non-repetitive chromatic number, our results also imply bounds on the p-centered chromatic numer of (g,δ)-string graphs, powers of bounded-degree graphs, and k-nearest-neighbour graphs, but these bounds are weaker than the bounds implied by the fact that these graphs have bounded maximum degree [20].

5.4 Universal Graphs

As mentioned in Section 1, Dujmović et al. [22] and Esperet et al. [31] prove results about adjacency labelling schemes for planar graphs. Stated in terms of universal graphs, their main theorem is interpreted as follows:

Theorem 35 ([22, 31])**.** *For every fixed integer t and every integer $n > 0$ there exists a graph U_n with $n^{1+o(1)}$ vertices and edges such that for every graph H of treewidth at most t and path P, every n-vertex subgraph of $H \boxtimes P$ is isomorphic to an induced subgraph of U_n.*

Combining Theorem 35 with our results on product structure yields the following:

Corollary 36. *For every fixed graph X and all fixed integers $d, \delta, \Delta, g, k > 0$, and every integer $n > 0$, there exists a graph U_n with $n^{1+o(1)}$ vertices and edges such that U_n contains the following graphs as induced subgraphs:*

- *every n-vertex (g,k)-planar graph;*

- *every n-vertex (g,d)-map graph;*
- *every n-vertex (g,δ)-string graph;*
- *every n-vertex graph G^k where G is X-minor-free and has maximum degree at most Δ;*
- *every k-nearest neighbour graph of n points in $\mathbb{R}^2$.*

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