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Graph Product Structure for Non-Minor-Closed Classes

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GRAPH PRODUCT STRUCTURE FOR NON-MINOR-CLOSED CLASSES

Vida Dujmović[☿], *Pat Morin*[♁], *and David R. Wood*[♂]

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Abstract. Dujmović et al. (FOCS 2019) recently proved that every planar graph is a subgraph of the strong product of a graph of bounded treewidth and a path. Analogous results were obtained for graphs of bounded Euler genus or apex-minor-free graphs. These tools have been used to solve longstanding problems on queue layouts, non-repetitive colouring, p-centered colouring, and adjacency labelling. This paper proves analogous product structure theorems for various non-minor-closed classes. One noteable example is k-planar graphs (those with a drawing in the plane in which each edge is involved in at most k crossings). We prove that every k-planar graph is a subgraph of the strong product of a graph of treewidth $O(k^5)$ and a path. This is the first result of this type for a non-minor-closed class of graphs. It implies, amongst other results, that k-planar graphs have non-repetitive chromatic number upper-bounded by a function of k. All these results generalise for drawings of graphs on arbitrary surfaces. In fact, we work in a much more general setting based on so-called shortcut systems that are of independent interest. This leads to analogous results for map graphs, string graphs, graph powers, and nearest neighbour graphs.

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1 Introduction

The starting point for this work is the following 'product structure theorem' for planar graphs[1] by Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood [14].

Theorem 1 ([14]). *Every planar graph is a subgraph of:*

(a) $H \boxtimes P$ *for some graph* H *of treewidth at most* 8 *and for some path* P,

(b) $H \boxtimes P \boxtimes K_3$ *for some graph* H *of treewidth at most* 3 *and for some path* P.

Here $\boxtimes$ is the strong product[2], and treewidth[3] is an invariant that measures how 'tree-like' a given graph is; see Figure 1 for an example. Loosely speaking, Theorem 1 says that every planar graph is contained in the product of a tree-like graph and a path. This enables combinatorial results for graphs of bounded treewidth to be generalised for planar graphs (with different constants).

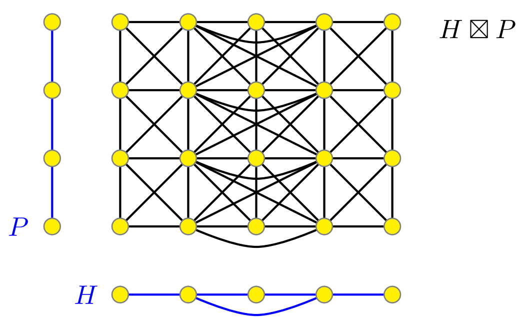

Figure 1: Example of a strong product.

Theorem 1 has been the key tool in solving the following well-known open problems:

- Dujmović et al. [14] use it to prove that planar graphs have bounded queue-number (resolving a conjecture of Heath, Leighton, and Rosenberg [25] from 1992).
- Dujmović, Esperet, Joret, Walczak, and Wood [15] use it to prove that planar graphs have bounded non-repetitive chromatic number (resolving a conjecture of Alon, Grytczuk, Hałuszczak, and Riordan [4] from 2002).

[1]In this paper, all graphs are finite and undirected. Unless specifically mentioned otherwise, all graphs are also simple. For any graph G and any set S (typically $S \subseteq V(G)$), let $G[S]$ denote the graph with vertex set $V(G) \cap S$ and edge set $\{uv \in E(G) : u, v \in S\}$. We use $G - S$ as a shorthand for $G[V(G) \setminus S]$. We use $G' \subseteq G$ to denote subgraph containment; that is, $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$.

[2]The *strong product* of graphs A and B, denoted by $A \boxtimes B$, is the graph with vertex set $V(A) \times V(B)$, where distinct vertices $(v,x),(w,y) \in V(A) \times V(B)$ are adjacent if $v = w$ and $xy \in E(B)$, or $x = y$ and $vw \in E(A)$, or $vw \in E(A)$ and $xy \in E(B)$.

[3]A *tree decomposition* $\mathcal{T}$ of a graph G consists of a tree T and a collection $\mathcal{T} = (B_x : x \in V(T))$ of subsets of $V(G)$ indexed by the nodes of T such that (i) for every $vw \in E(G)$, there exists some node $x \in V(T)$ with $v, w \in B_x$; and (ii) for every $v \in V(G)$, the induced subgraph $T[v] := T[\{x : v \in B_x\}]$ is connected. The *width* of the tree decomposition $\mathcal{T}$ is $\max\{|B_x| : x \in V(T)\} - 1$. The *treewidth* $\text{tw}(G)$ of a graph G is the minimum width of a tree decomposition of G. Treewidth is the standard measure of how similar a graph is to a tree. Indeed, a connected graph has treewidth 1 if and only if it is a tree. Treewidth is of fundamental importance in structural and algorithmic graph theory; see [5, 24, 36] for surveys.

- Dębski, Felsner, Micek, and Schröder [10] use it to make dramatic improvements to the best known bounds for p-centered colourings of planar graphs.
- Bonamy, Gavoille, and Pilipczuk [6] use it to find shorter adjacency labellings of planar graphs (improving on a sequence of results going back to 1988 [26, 27]), and Dujmović, Esperet, Gavoille, Joret, Micek, and Morin [16] have since used it to find asymptotically optimal adjacency labellings of planar graphs.
- The result of Dujmović et al. [16] implies that, for every integer $n > 0$, there is a *universal* graph U_n with $n^{1+o(1)}$ vertices such that every n-vertex planar graph is an induced subgraph of U_n.

All of these results hold for any graph class that has a product structure theorem analogous to Theorem 1; that is, for any graph class $\mathcal{G}$ where every graph in $\mathcal{G}$ is a subgraph of $H \boxtimes P \boxtimes K_\ell$ where H has bounded treewidth, P is a path, and ℓ is bounded[4]. These applications motivate finding product structure theorems for other graph classes. Dujmović et al. [14] prove product structure theorems for graphs of bounded Euler genus[5] and for apex-minor-free graphs[6], and Dujmović, Esperet, Walczak, and Wood [17] do so for graphs in any minor-closed class and with bounded maximum degree. See Section 1.3 for more precise statements and see [18] for a survey on this topic.

The purpose of this paper is to prove product structure theorems for several non-minor-closed classes of interest. Our results are the first of this type for non-minor-closed classes.

1.1 k-Planar Graphs

We start with the example of k-planar graphs. A graph is *k-planar* if it has a drawing in the plane in which each edge is involved in at most k crossings. Such graphs provide a natural generalisation of planar graphs, and are important in graph drawing research; see the recent bibliography on 1-planar graphs and the 140 references therein [28]. It is well-known that the family of k-planar graphs is not minor-closed. Indeed, 1-planar graphs may contain arbitrarily large complete graph minors [12]. Hence the above results are not applicable for k-planar graphs. We extend Theorem 1 as follows.

Theorem 2. *Every k-planar graph is a subgraph of $H \boxtimes P \boxtimes K_{18k^2+48k+30}$, for some graph H of treewidth $\binom{k+4}{3} - 1$ and for some path P.*

This theorem has applications in diverse areas, including queue layouts [14], non-repetitive colouring [15], p-centered colouring [10], and adjacency labelling [16], which

[4] It is easily seen that $\mathrm{tw}(H \boxtimes K_\ell) \leqslant (\mathrm{tw}(H)+1)\ell - 1$, so we may assume that $\ell = 1$ in this definition.

[5] The *Euler genus* of the orientable surface with h handles is $2h$. The *Euler genus* of the non-orientable surface with c cross-caps is c. The *Euler genus* of a graph G is the minimum integer g such that G embeds in a surface of Euler genus g. Of course, a graph is planar if and only if it has Euler genus 0; see [30] for more about graph embeddings in surfaces.

[6] A graph M is a *minor* of a graph G if a graph isomorphic to M can be obtained from a subgraph of G by contracting edges. A class $\mathcal{G}$ of graphs is *minor-closed* if for every graph $G \in \mathcal{G}$, every minor of G is in $\mathcal{G}$. A minor-closed class is *proper* if it is not the class of all graphs. For example, for fixed $g \geqslant 0$, the class of graphs with Euler genus at most g is a proper minor-closed class. A graph G is t-apex if it contains a set A of at most t vertices such that $G - A$ is planar. A 1-apex graph is *apex*. A minor-closed class $\mathcal{G}$ is apex-minor-free if some apex graph is not in $\mathcal{G}$.

we explore in Section 4. For example, we prove that k-planar graphs have bounded non-repetitive chromatic number (for fixed k). Prior to the recent work of Dujmović et al. [15], it was even open whether planar graphs have bounded non-repetitive chromatic number.

1.2 Shortcut Systems

Although k-planar graphs are the most high-profile target for a generalization of Theorem 1, we actually prove a substantially stronger result than Theorem 2 using the following definition. A non-empty set $\mathcal{P}$ of non-trivial paths in a graph G is a (k,d)-*shortcut system* (for G) if:

- every path in $\mathcal{P}$ has length at most k,[7] and
- for every $v \in V(G)$, the number of paths in $\mathcal{P}$ that use v as an internal vertex is at most d.

Each path $P \in \mathcal{P}$ is called a *shortcut*; if P has endpoints v and w then it is a vw-*shortcut*. Given a graph G and a (k,d)-shortcut system $\mathcal{P}$ for G, let $G^{\mathcal{P}}$ denote the supergraph of G obtained by adding the edge vw for each vw-shortcut in $\mathcal{P}$.

This definition is related to k-planarity because of the following observation:

Observation 1. *Every k-planar graph is a subgraph of $G^{\mathcal{P}}$ for some planar graph G and some $(k+1,2)$-shortcut system $\mathcal{P}$ for G.*

The proof of Observation 1 is trivial: Given a k-plane embedding of a graph G', create a planar graph G by adding a dummy vertex at each crossing point. For each edge $vw \in E(G')$ there is a path P in G between v and w of length at most $k+1$. Let $\mathcal{P}$ be the set of such paths P. For each vertex v of G, at most two paths in $\mathcal{P}$ use v as an internal vertex (since no original vertex of G' is an internal vertex of a path in $\mathcal{P}$). Thus $\mathcal{P}$ is a $(k+1,2)$-shortcut system for G, such that $G' \subseteq G^{\mathcal{P}}$. This idea can be pushed further to obtain a rough characterisation of k-planar graphs, which is interesting in its own right, and is useful for showing that various classes of graphs are k-planar (see Section 3.4).

The following theorem is the main contribution of the paper. It says that if a graph class $\mathcal{G}$ has a product structure theorem, then the class of graphs obtained by applying a shortcut system to graphs in $\mathcal{G}$ also has a product structure theorem.

Theorem 3. *Let G be a subgraph of $H \boxtimes P \boxtimes K_\ell$, for some graph H of treewidth at most t and for some path P. Let $\mathcal{P}$ be a (k,d)-shortcut system for G. Then $G^{\mathcal{P}}$ is a subgraph of $J \boxtimes P \boxtimes K_{d\ell(k^3+3k)}$ for some graph J of treewidth at most $\binom{k+t}{t}-1$ and some path P.*

Theorems 1(b) and 3 and Observation 1 imply Theorem 2 with $K_{6(k^3+3k)}$ instead of $K_{18k^2+48k+30}$. Some further observations presented in Section 3 lead to the improved result.

Theorem 3 is applicable for many graph classes in addition to k-planar graphs. Here is one example. The *k-th power* of a graph G is the graph G^k with vertex set $V(G^k) := V(G)$, where $vw \in E(G^k)$ if and only if $\text{dist}_G(v,w) \leqslant k$.[8] If G has maximum degree Δ, then $G^k = G^{\mathcal{P}}$

[7] A path of length k consists of k edges and $k+1$ vertices. A path is *trivial* if it has length 0 and *non-trivial* otherwise.

[8] For a graph G and two vertices $v,w \in V(G)$, $\text{dist}_G(v,w)$ denotes the length of a shortest path, in G, with endpoints v and w. We define $\text{dist}_G(v,w) := \infty$ if v and w are in different connected components of G.

for some $(k, 2k\Delta^k)$-shortcut system $\mathcal{P}$; see Lemma 12. Theorems 1(b) and 3 then imply:

Theorem 4. *For every planar graph G with maximum degree Δ and for every integer $k \geqslant 1$, G^k is a subgraph of $H \boxtimes P \boxtimes K_{6k^2(k^2+3)\Delta^k}$ for some graph H of treewidth at most $\binom{k+3}{3} - 1$ and some path P.*

Section 5 presents further examples of graph classes that can be constructed using shortcut systems, including map graphs, string graphs, and k-nearest neighbour graphs. Theorem 3 implies product structure theorems for each of these classes. All of the above-mentioned applications also hold for these examples.

1.3 Generalisations

As mentioned above, product structure theorems have been established for several minor-closed classes in addition to planar graphs. The first generalises Theorem 1 for graphs of bounded Euler genus.

Theorem 5 ([14])**.** *Every graph of Euler genus g is a subgraph of:*

(a) $H \boxtimes P$ for some graph H of treewidth at most $2g + 8$ and some path P.
(b) $H \boxtimes P \boxtimes K_{\max\{2g,3\}}$ for some graph H of treewidth at most 4 and for some path P.

Dujmović et al. [14] generalised Theorem 5 for apex-minor-free graphs as follows.

Theorem 6 ([14])**.** *For every apex graph X, there exists $c \in \mathbb{N}$ such that every X-minor-free graph is a subgraph of $H \boxtimes P$ for some graph H with $\operatorname{tw}(H) \leqslant c$ and some path P.*

The assumption that X is apex is needed in Theorem 6, since if the class of X-minor-free graphs has a product structure theorem analogous to Theorem 1, then X is apex [14]. On the other hand, Dujmović et al. [17] proved a product structure theorem for bounded degree graphs in any minor-closed class.

Theorem 7 ([17])**.** *For every graph X there exists $c \in \mathbb{N}$ such that for every $\Delta \in \mathbb{N}$, every X-minor-free graph G with maximum degree at most Δ is a subgraph of $H \boxtimes P$ for some graph H with $\operatorname{tw}(H) \leqslant c\Delta$ and for some path P.*

1.4 Layered Partitions

While strong products enable concise statements of the theorems in Section 1, to prove such results it is helpful to work with layerings and partitions, which we now introduce.

A *layering* of a graph G is a sequence $\mathcal{L} = \langle L_0, L_1, \dots\rangle$ such that $\{L_0, L_1, \dots\}$ is a partition of $V(G)$ and for every edge $vw \in E(G)$, if $v \in L_i$ and $w \in L_j$ then $|j - i| \leqslant 1$. For any partition $\mathcal{P} = \{S_1, \dots, S_p\}$ of $V(G)$, a *quotient graph* $H = G/\mathcal{P}$ has a p-element vertex set $V(H) = \{x_1, \dots, x_p\}$ and $x_i x_j \in E(H)$ if and only if there exists an edge $vw \in E(G)$ such that $v \in S_i$ and $w \in S_j$. To highlight the importance of the quotient graph H, we call $\mathcal{P}$ an *H-partition* and write this concisely as $\mathcal{P} = \{S_x : x \in V(H)\}$ so that each element of $\mathcal{P}$ is indexed by the vertex it creates in H.

For any partition $\mathcal{P}$ of $V(G)$ and any layering $\mathcal{L}$ of G we define the *layered width* of $\mathcal{P}$ with respect to $\mathcal{L}$ as $\max\{|L \cap P| : L \in \mathcal{L}, P \in \mathcal{P}\}$. For any partition $\mathcal{P}$ of $V(G)$, we define the *layered width* of $\mathcal{P}$ as the minimum, over all layerings $\mathcal{L}$ of G, of the layered width of $\mathcal{P}$ with respect to $\mathcal{L}$.

Dujmović et al. [14] introduced the study of partitions with bounded layered width such that the quotient has some additional desirable property, like small treewidth. Dujmović et al. define a class $\mathcal{G}$ of graphs to *admit bounded layered partitions* if there exist $t, \ell \in \mathbb{N}$ such that every graph $G \in \mathcal{G}$ has an H-partition of layered width at most ℓ for some graph $H = H(G)$ of treewidth at most t.

These definitions relate to strong products as follows.

Lemma 1 ([14])**.** *For every graph H, a graph G has an H-partition of layered width at most ℓ if and only if G is a subgraph of $H \boxtimes P \boxtimes K_\ell$ for some path P.*

As an example of the use of layered partitions, to prove Theorem 1(a), Dujmović et al. [14] showed that every planar graph has an H-partition of layered width 1 for some planar graph H of treewidth at most 8. The proof is constructive and gives a simple quadratic-time algorithm for finding the corresponding partition and layering. At the core of their work is the elegant proof by Pilipczuk and Siebertz [34] of the following result:

Theorem 8 ([34])**.** *Every planar triangulation G has an H-partition $\mathcal{P}$ such that* $\operatorname{tw}(H) \leqslant 8$ *and $G[P]$ is a shortest path in G for each $P \in \mathcal{P}$.*

Indeed, the above-mentioned result of Dujmović et al. [14] is a slight strengthening of Theorem 8, where for each $P \in \mathcal{P}$ no two vertices of P have the same distance to some fixed root vertex r.

The following result is the main technical contribution of the paper. Loosely speaking, it shows that if a graph G admits bounded layered partitions, then so too does $G^{\mathcal{P}}$ for every shortcut system $\mathcal{P}$ of G.

Theorem 9. *Let G be a graph having an H-partition of layered width ℓ in which H has treewidth at most t and let $\mathcal{P}$ be a (k,d)-shortcut system for G. Then $G^{\mathcal{P}}$ has a J-partition of layered width at most $d\ell(k^3+3k)$ for some graph J of treewidth at most $\binom{k+t}{t}-1$.*

Note that Theorem 9 is equivalent to Theorem 3 by Lemma 1.

2 Shortcut Systems

The purpose of this section is to prove our main technical result, Theorem 9. This theorem shows how, given a (k,d)-shortcut system $\mathcal{P}$ of a graph G, a H-partition of G can be used to obtain a J-partition of $G^{\mathcal{P}}$ where the layered width does not increase dramatically and the treewidth of J is not much more than the treewidth of H.

For convenience, it will be helpful to assume that $\mathcal{P}$ contains a length-1 vw-shortcut for every edge $vw \in E(G)$. Since $G^{\mathcal{P}}$ is defined to be a supergraph of G, this assumption has no effect on $G^{\mathcal{P}}$ but eliminates special cases in some of our proofs.

For a tree T rooted at some node $x_0 \in V(T)$, we say that a node $a \in V(T)$ is a *T-ancestor* of $x \in V(T)$ (and x is a *T-descendant* of a) if a is a vertex of the path, in T, from x_0 to x. Note that each node $x \in V(T)$ is a T-ancestor and T-descendant of itself. We say that a

T-ancestor $a \in V(T)$ of $x \in V(T)$ is a *strict* T-ancestor of x if $a \neq x$. The T-*depth* of a node $x \in V(T)$ is the length of the path, in T, from x_0 to x. For each node $x \in V(T)$, define

$$T_x := T[\{y \in V(T) : x \text{ is a } T\text{-ancestor of } y\}]$$

to be the maximal subtree of T rooted at x.

We begin with a standard technique that allows us to work with a normalised tree decomposition:

Lemma 2. *For every graph H of treewidth t, there is a rooted tree T with $V(T) = V(H)$ and a width-t T-decomposition $(B_x : x \in V(T))$ of H that has following additional properties:*

(T1) for each node $x \in V(H)$, the subtree $T[x] := T[\{y \in V(T) : x \in B_y\}]$ is rooted at x; and consequently

(T2) for each edge $xy \in E(H)$, one of x or y is a T-ancestor of the other.

Proof. That (T1) implies (T2) is a standard observation: If two subtrees intersect, then one contains the root of the other. Thus, it suffices to construct width-t tree decomposition that satisfies (T2).

Begin with any width-t tree decomposition $(B_x : x \in V(T_0))$ of H that uses some tree T_0. Select any node $x \in V(T_0)$, add a leaf x_0, with $B_{x_0} = \emptyset$, adjacent to x and root T_0 at x_0. (The purpose of x_0 is to ensure that every node x for which B_x is non-empty has a parent.) Let $f : V(H) \to V(T)$ be the function that maps each $x \in V(H)$ onto the root of the subtree $T_0[x] := T_0[\{y \in V(T_0) : x \in B_y\}]$. If f is not one-to-one, then select some distinct pair $x, y \in V(H)$ with $a := f(x) = f(y)$. Subdivide the edge between a and its parent in T by introducing a new node a' with $B_{a'} = B_a \setminus \{x\}$. This modification reduces the number of distinct pairs $x, y \in V(H)$ with $f(x) = f(y)$, so repeatedly performing this modification will eventually produce a tree decomposition $(B_x : x \in V(T_0))$ of H in which f is one-to-one.

Next, consider any node $a \in V(T_0)$ such that there is no vertex $x \in V(H)$ with $f(x) = a$. In this case, $B_a \subseteq B_{a'}$ where a' is the parent of a since any $x \in B_a \setminus B_{a'}$ would have $f(x) = a$. In this case, contract the edge aa' in T_0, eliminating the node a. Repeating this operation will eventually produce a width-t tree decomposition of $(B_x : x \in V(T_0))$ where f is a bijection between $V(H)$ and $V(T_0)$. Renaming each node $a \in V(T_0)$ as $f^{-1}(a)$ gives a tree decomposition $(B_x : x \in V(T))$ with $V(T) = V(H)$. By the definition of f, the tree decomposition $(B_x : x \in V(T))$ satisfies (T1). □

The following lemma shows how to interpret an H-partition of G and a tree decomposition of H as a hierarchical decomposition of G; refer to Figure 2.

Lemma 3. *Let G be a graph; let $\mathcal{L} := \langle L_1, \dots, L_h \rangle$ be a layering of G; let $\mathcal{Y} := (Y_x : x \in V(H))$ be an H-partition of G of layered width at most ℓ with respect to $\mathcal{L}$ where H has treewidth at most t; and let $\mathcal{T} := (B_x : x \in V(T))$ be a tree decomposition of H satisfying the conditions of Lemma 2. For each $x \in V(T)$, let $V_x := \bigcup_{y \in V(T_x)} Y_y$, $F_x := \{w \in V(G) : vw \in E(G), v \in V_x, w \notin V_x\}$, and $N_x := V_x \cup F_x$. Then,*

(Y1) For each $x \in V(T)$, there is no edge $vw \in E(G)$ with $v \in V_x$ and $w \in V(G) \setminus N_x$.

(Y2) For each $x \in V(T)$, there is a set $\{a_1, \dots, a_{t'}\}$ of $t' \leqslant t$ strict T-ancestors of x such that $F_x \subseteq \bigcup_{i=1}^{t'} Y_{a_i}$.

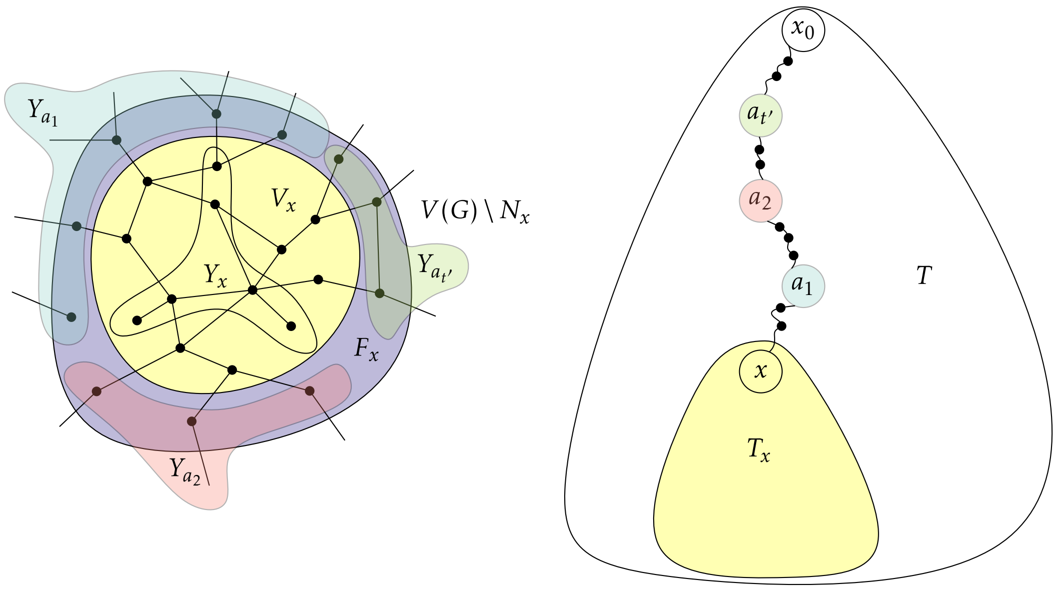

Figure 2: The sets Y_x, F_x, and V_x associated with $x \in V(T)$ and the ancestors $a_1, \dots, a_{t'}$ of X such that $F_x \subseteq \bigcup_{i=1}^{t'} Y_{a_i}$.

Before proving Lemma 3 we point out more properties that are immediately implied by it:

(Y3) $Y_x \subseteq V_x$ for every $x \in V(T)$.
(Y4) $V_x \subseteq V_a$ for every T-ancestor a of x.
(Y5) $N_x \subseteq N_a$ for every T-ancestor a of x.

Property (Y3) follows from the fact that V_x is the union of several sets, one of which is Y_x. Property (Y4) follows from the definition of V_x and the fact that $V(T_x) \subseteq V(T_a)$. To show Property (Y5) first note that, by (Y4) it suffices to consider vertices $w \in F_x = N_x \setminus V_x$. By definition, every vertex $w \in F_x$ is adjacent, in G, to a vertex $v \in V_x$. By (Y4), $v \in V_a$, so w is either in V_a or w satisfies the condition $vw \in E(G)$, $v \in V_a$, and $w \notin V_a$, so $w \in F_a$. In either case $w \in N_a = V_a \cup F_a$. Note that none of (Y3)–(Y5) depends on (Y2) (which is important, since (Y4) is used to establish (Y2) in the following proof).

Proof of Lemma 3. Property (Y1) is immediate from the definitions of F_x and N_x. In particular, $(N_x, V(G) \setminus V_x)$ is a separation of G with $F_x = N_x \cap (V(G) \setminus V_x)$.

To establish Property (Y2), consider some vertex $w \in F_x$. Since $w \in F_x$, there exists an edge $vw \in E(G)$ with $v \in V_x$ and $w \notin V_x$. Since $v \in V_x$, $v \in Y_{x'}$ for some T-descendant x' of x (possibly $x = x'$). Since $\mathcal{Y}$ is a partition, $w \in Y_a$ for some $a \notin V(T_x)$. Since $vw \in E(G)$, we have $x'a \in E(H)$. By (T2), one of a or x' is a T-ancestor of the other. Since $w \in Y_a \subseteq V_a$ and $w \notin V_x \supseteq V_{x'}$, (Y4) rules out the possibility that x' is a T-ancestor of a. Therefore, a is a T-ancestor of x which is a T-ancestor of x'. Let $z_0, \dots, z_r$ be the path in T from $z_0 := x'$ to $z_r := a$. For each $i \in \{0, \dots, r\}$, at least one of a or x' is in B_{z_i}, since $x'a \in E(H)$. However, by (T1) x' is not contained in B_{x_i} for any $i \in \{1, \dots, r\}$. Therefore $a \in B_{x_i}$ for each $i \in \{0, \dots, r\}$. In particular, a is contained in B_x. Property (Y2) now follows from the fact that $|B_x| \leqslant t+1$ and B_x contains x. □

We are now ready to prove our main result, which we restate here for convenience:

Theorem 9. *Let G be a graph having an H-partition of layered width ℓ in which H has treewidth at most t and let $\mathcal{P}$ be a (k,d)-shortcut system for G. Then $G^{\mathcal{P}}$ has a J-partition of layered width at most $d\ell(k^3+3k)$ for some graph J of treewidth at most $\binom{k+t}{t}-1$.*

Proof. Apply Lemma 3 to G and let $\mathcal{L}, \mathcal{Y}, \mathcal{T}, T, Y_x, V_x, F_x$, and N_x be defined as in Lemma 3, where the partition $\mathcal{Y}$ has width ℓ with respect to the layering $\mathcal{L}$.

For a node $x \in V(T)$, we say that a shortcut $P \in \mathcal{P}$ *crosses* x if Y_x contains an internal vertex of P, that is, $P = (v_0, \dots, v_r)$ and $\{v_1, \dots, v_{r-1}\} \cap Y_x \neq \emptyset$. We say that a vertex $v \in V(G)$ *participates* in x if $v \in Y_x$, or $\mathcal{P}$ contains a shortcut P with $v \in V(P)$ and P crosses x. We let X_v denote the set of nodes $x \in V(T)$ such that v participates in x.

Claim 1. *For any $v \in V(G)$ there exists a (unique) node $a(v) \in X_v$ such that $a(v)$ is a T-ancestor of every node in X_v.*

Proof. Let $Z := \{v\} \cup \{\{v_1, \dots, v_{r-1}\} : (v_0, \dots, v_r) \in \mathcal{P}, v \in \{v_0, \dots, v_r\}\}$. Then $G[Z]$ is connected because Z is the union of (vertex sets of) paths in G, each of which contains v.

We claim that v participates in a node $x \in V(T)$ if and only if $Z \cap Y_x \neq \emptyset$. If v participates in x then either $v \in Y_x$, so $Z \cap Y_x \supseteq \{v\}$; or $v \in \{v_0, \dots, v_r\}$ for some shortcut $(v_0, \dots, v_r) \in \mathcal{P}$ that crosses x, so $Z \cap Y_x \supseteq \{v_i\}$ for some $i \in \{1, \dots, r-1\}$. In the other direction, if $Z \cap Y_x \neq \emptyset$, then either $Z \cap Y_x \supseteq \{v\}$, so $v \in Y_x$; or $Z \cap Y_x \supseteq \{v_i\}$ where $i \in \{1, \dots, r\}$, $(v_0, \dots, v_r) \in \mathcal{P}$ and $v \in \{v_0, \dots, v_r\}$, so $v \in V(P)$ for a path $P = (v_0, \dots, v_r) \in \mathcal{P}$ that crosses x.

Let $X_H := \{x \in V(H) : Z \cap Y_x \neq \emptyset\}$. The connectivity of $G[Z]$ implies that $H[X_H]$ is connected. Choose $a(v) \in X_H$ to be the member of X_H that does not have a strict T-ancestor in X_H. Transitivity of the T-ancestor relationship, (T2), and connectivity of $H[X_H]$ implies that such an $a(v)$ exists and is a T-ancestor of every node $x \in X_H$, as required. □

For each $x \in V(T)$, define $S_x := \{v \in V(G) : a(v) = x\}$. Observe that $\mathcal{S} := (S_x : x \in V(T))$ is a partition of $V(G)$. We let $J := G^{\mathcal{P}}/\mathcal{S}$ denote the resulting quotient graph and we let $V(J) \subseteq V(T)$ in the obvious way, so that each $x \in V(J)$ is the vertex obtained by contracting S_x in $G^{\mathcal{P}}$. (Nodes $x \in V(T)$ with $S_x = \emptyset$ do not contribute a vertex to J.)

From this point onward, the plan is to show that:

(i) $\mathcal{S}$ has small layered width with respect to the layering $\mathcal{L}$ and that
(ii) J has small treewidth.

First we need to understand the relationship between sets $S_x \in \mathcal{S}$ and the related sets Y_x, V_x, F_x, and N_x.

Claim 2. *For every $x \in V(T)$, $S_x \subseteq V_x$.*

Proof. For the sake of contradiction, assume otherwise, so there exists some $v \in S_x \setminus V_x$. By (Y3), $Y_x \subseteq V_x$, so $v \notin Y_x$. Therefore, $\mathcal{P}$ contains a path P, with $v \in V(P)$, that crosses x. The path P contains a proper subpath $v_0, v_1, \ldots, v_r$ such that $v = v_0$ and $v_r \in Y_x$. Since $v \notin V_x$ and $v_r \in Y_x \subseteq V_x$, (Y1), implies that $v_i \in F_x$ for some $i \in \{0, \ldots, r-1\}$. Now (Y2) implies $v_i \in Y_a$ for some strict T-ancestor a of x. Therefore, either $v \in Y_a$ or P crosses a. But this implies that $a(v)$ is a T-ancestor of a, which is a strict T-ancestor of x, contradicting the assumption that $v \in S_x$. □

Next we complete Step (i) and show that $\mathcal{S}$ has small layered width with respect to the layering $\mathcal{L} = \langle L_1, \ldots, L_h \rangle$:

Claim 3. *For each $i \in \{1, \ldots, h\}$ and each $x \in V(J)$, $|S_x \cap L_i| \leqslant d\ell(k^2 + 3)$.*

Proof. Recall that S_x is defined by vertices that participate in x, and these are vertices that are either in Y_x or in shortcuts that cross Y_x. We say that a vertex $w \in Y_x$ *contributes* a vertex $v \in S_x$ if v participates in x. We upper bound the number of vertices in $S_x \cap L_i$ by upper-bounding the number of vertices contributed to $S_x \cap L_i$ by each $w \in Y_x$. Refer to Figure 3. If $w \in Y_x \cap L_i$ and no path in $\mathcal{P}$ includes w as an internal vertex then w contributes at most one vertex, itself, to $S_x \cap L_i$.

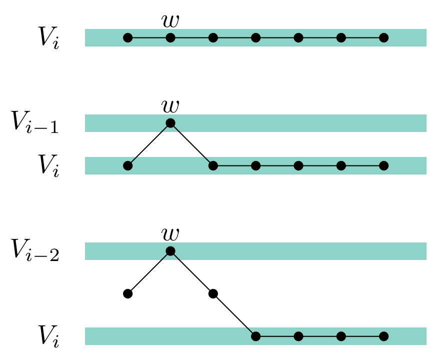

Figure 3: A path P containing an internal vertex $w \in Y_x \cap L_{i-j}$.

Otherwise, consider some path $P \in \mathcal{P}$ that contains w as an internal vertex. If $w \in L_i$, then P contributes at most $k+1$ vertices to $S_x \cap L_i$. If $w \in L_{i-1} \cup L_{i+1}$, then P contributes at most k vertices to $S_x \cap L_i$. If $w \in L_{i-j} \cup L_{i+j}$ for $j \geqslant 2$, then P contributes at most $k - j$ vertices to $S_x \cap L_i$.

For any j, the number of vertices $w \in L_{i+j} \cap Y_x$ is at most ℓ. Each such vertex w is an internal vertex of at most d paths in $\mathcal{P}$. Therefore,

$$|S_x \cap L_i| \leqslant d\ell\left(k + 1 + 2k + \sum_{j=2}^{k} 2(k-j)\right) = d\ell(k^2 + 3) \ .$$ □

We now proceed with Step (ii), showing that J has small treewidth. To accomplish this, we construct a small width tree-decomposition $\mathcal{C} := (C_x : x \in V(T))$ of J using the same tree T used in the tree decomposition $\mathcal{T} := (B_x : x \in V(T))$ of H. The following claim will be useful in showing that the resulting decomposition has small width.

Claim 4. *For each edge $xy \in E(J)$, one of x or y is a T-ancestor of the other.*

Proof. Suppose, for the sake of contradiction, that neither x nor y is a T-ancestor of the other. Since $xy \in E(J)$, $G^{\mathcal{P}}$ contains an edge vw with $v \in S_x$ and $w \in S_y$. Since $vw \in E(G^{\mathcal{P}})$, $\mathcal{P}$ contains a vw-shortcut P.[9] By Claim 2, $v \in V_x$ and $w \in V_y$. By (Y4), if neither x nor y is a T-ancestor of the other, then V_x and V_y are disjoint. By (Y2), N_x and V_y are also disjoint. By (Y1) P contains an internal vertex $v' \in F_x$. By (Y2), $v' \in Y_a$ for some strict T-ancestor a of x. But this implies that $a(v) = a'$ so $v \in S_{a'}$ for some T-ancestor a' of a, contradicting the assumption that $v \in S_x$. □

Claim 5. *The graph J has a tree decomposition in which every bag has size at most $\binom{k+t}{t}$.*

Proof. For the tree decomposition $(C_x : x \in V(T))$ of J we use the same tree T used in the tree decomposition $(B_x : x \in V(T))$ of H. For each node x of T, we define C_x as follows: C_x contains x as well as every T-ancestor a of x such that J contains an edge ax' where x is a T-ancestor of x' (including the possibility that $x = x'$). Claim 4 ensures that, for every edge $ax' \in E(J)$, $a, x' \in C_{x'}$. The connectivity of $T[a] := T[\{x \in V(T) : a \in C_x\}]$ follows from the fact that, for every node $x' \in T[a]$, every node x on the path in T from x' to a is also a node of $T[a]$. Therefore $(C_x : x \in V(T))$ is indeed a tree decomposition of J. It remains to bound the size of each bag C_x.

Consider an arbitrary node $x \in V(T)$ where $x_0, \dots, x_r$ is the path from the root x_0 of T to $x_r := x$. To avoid triple-subscripts in what follows, we abuse notation slightly by using V_i, F_i, and N_i, as shorthands for V_{x_i}, F_{x_i} and N_{x_i}, respectively.

If $x_\delta \in C_x$, it is because $x_\delta x' \in E(J)$ for some T-descendant x' of x. This implies $G^{\mathcal{P}}$ contains an edge vw with $v \in S_{x'}$ and $w \in S_{x_\delta} = S_\delta$. This implies that $\mathcal{P}$ contains a vw-shortcut P_{vw}. Let v' be the second-last vertex of P_{vw} (so $v'w \in E(G)$).

Since $w \in S_\delta$, $a(w) = x_\delta$, so w participates in x_δ, so at least one of the following is true:

1. There exists $w' \in V(G)$ such that $\mathcal{P}$ contains a ww'-shortcut $P_{ww'}$ that has an internal vertex in Y_δ; or
2. $w \in Y_\delta$. In this case, we define $P_{ww'}$ to be the path of length 0 that contains only $w = w'$.

Let w'' denote the first vertex of $P_{ww'}$ contained in Y_δ.

Let $w_0, w_1, \dots, w_p$ be the path that begins $w_0 := v'$ and then follows the subpath of $P_{ww'}$ that begins at $w_1 := w$ and ends at $w_p := w''$. For each $i \in \{0, \dots, p\}$, let $s_i = \max\{j \in \{0, \dots, r\} : \{w_0, \dots, w_i\} \subseteq V_j\}\}$, and let $a_i = x_{s_i}$. Note that $s_0, \dots, s_p$ is a non-increasing sequence and $a_0, \dots, a_p$ is a sequence of nodes of T whose distance from the root, x_0, of T is non-increasing.

We claim that $a_0 = x_r$. Since $v \in S_{x'}$, $a(v) = x'$, so $V(P_{vw}) \subseteq V_{x'}$, otherwise v participates in a for some node a not in the subtree $T_{x'}$ rooted at x', but this contradicts Claim 4 since $a(v) = x'$ is a T-ancestor of every node in which v participates. Since $v' = w_0 \in V(P_{vw})$, $\{w_0\} \subseteq V_{x'} \subseteq V_r$, so $s_0 = r$ and $a_0 = x_r$.

[9] Recall that we have made the assumption that $\mathcal{P}$ contains a length-1 vw-shortcut for each edge $vw \in E(G)$.

We claim that $a_p = x_\delta$—that is, $s_p = \delta$. To see this, first observe that, for each $i \in \{1,\dots,p\}$, $w_i \in V_\delta$ since, otherwise, an internal vertex of $P_{ww'}$ belong to F_δ, which would imply (by (Y2)) that $w \in S_{\delta'}$ for some $\delta' < \delta$, contradicting the assumption that $w \in S_\delta$. Therefore $s_p \geqslant \delta$. To see that $s_p < \delta + 1$, observe that either $w = w'' \in Y_\delta$ or $P_{ww'}$ contains an internal vertex w'' in Y_δ. By the definition of V_x, $V_{\delta-1}$ does not contain w'', so $s_p < \delta + 1$.

Let H^+ denote the supergraph of H with vertex set $V(T)$ and in which $xy \in E(H^+)$ if and only there exists some $z \in V(T)$ such that $x, y \in B_z$. We claim that $a_0, \dots, a_p$ is a lazy walk[10] in H^+. Indeed, if $a_i \neq a_{i+1}$ for some $i \in \{0,\dots,p-1\}$ then this is precisely because $w_i \in V_{a_i}$ but $w_{i+1} \notin V_{a_i}$. By definition, $w_i \in Y_{a_i'}$ for some T-descendant a_i' of a_i. By (Y1), $w_{i+1} \in F_{a_i}$ so by (Y2) $w_{i+1} \in Y_{a_i''}$ for some strict T-ancestor a_i'' of a_i. Since $w_i w_{i+1} \in E(G)$, $a_i' a_i'' \in E(H)$. By (T1), $a_i'' \in B_{a_i''}$ and $a_i'' \in B_{a_i'}$. Since a_i is on the path from a_i' to a_i'' in T this implies that $a_i'' \in B_{a_i}$. Therefore $a_i a_i'' \in E(H^+)$ as claimed.

Thus, $a_0, \dots, a_p$ is a lazy walk in H^+ of length $p \leqslant k$ where the distance s_i between a_i and the root x_0 of T is non-decreasing. By removing repeated vertices this gives a path in the directed graph $\overrightarrow{H}^+$ obtained by directing each edge $xy \in E(H^+)$ from its T-descendant x towards its T-ancestor y. Finally, we are in a position to appeal to [34, Lemma 24] which states that the number of nodes in $\overrightarrow{H}^+$ that can be reached from any node x by a directed path of length at most k is at most $\binom{k+t}{t}$.

At this point, the proof is complete, but let us summarize. For each node $x \in V(T)$, C_x contains only T-ancestors of x (including x itself). For each ancestor x_δ of x contained in C_x, there is path from x to x_δ in $\overrightarrow{H}^+$ of length at most k. The number of ancestors of x that can be reached by paths of length at most k in $\overrightarrow{H}^+$ is at most $\binom{k+t}{t}$. Therefore $|C_x| \leqslant \binom{k+t}{t}$, as required. □

At this point, the proof of Theorem 9 is almost immediate from Claims 3 and 5, except that the layering $\mathcal{L}$ of G may not be a valid layering of $G^{\mathcal{P}}$. In particular, $G^{\mathcal{P}}$ may contain edges vw with $v \in L_i$ and $w \in L_{i+j}$ for any $j \in \{0,\dots,k\}$. To resolve this, we use a new layering $\mathcal{L}' := \langle L_0', \dots, L_h' \rangle$ in which $L_i' = \bigcup_{j=ki}^{ki+k-1} L_i$. This increases the layered width given by Claim 3 from $d\ell(k^2+3)$ to $d\ell(k^3+3k)$. Therefore G has an H-partition of layered width at most $d\ell(k^3+3k)$ in which H has treewidth at most $\binom{k+t}{t}$, completing the proof of Theorem 9. □

3 Allowing Crossings

This section applies our main results for shortcut systems to prove graph product structure theorem for graphs drawn with a bounded number of crossings per edge. Then we show how to improve the bounds in this case.

3.1 k-Planar Graphs

We first formally define k-planar graphs. An *embedded graph* G is a graph with $V(G) \subset \mathbb{R}^2$ in which each edge $vw \in E(G)$ is a closed curve[11] in $\mathbb{R}^2$ with endpoints v and w and not

[10] A *lazy walk* in a graph H is a walk in the pseudograph H' obtained by adding a loop at each vertex of H.

[11] A closed curve in a surface Σ is a continuous function $f : [0,1] \to \Sigma$. The points $f(0)$ and $f(1)$ are called the *endpoints* of the curve. When there is no danger of misunderstanding we treat a curve f as the point set

containing any vertex of G in its interior. A *crossing* in an embedded graph G is a triple (p, vw, xy) with $p \in \mathbb{R}^2$, $vw, xy \in E(G)$ and such that $p \in (vw \cap xy) \setminus \{v, w, x, y\}$. An embedded graph G is *k-plane* if each edge of G takes part in at most k crossings. A (not necessarily embedded) graph G' is *k-planar* if there exists a k-plane graph G isomorphic to G'. Under these definitions, 0-planar graphs are exactly planar graphs and 0-plane graphs are exactly plane graphs.

As mentioned in Section 1, Theorems 1 and 3 imply a product structure theorem for k-planar graphs. We get improved bounds as follows.

Proof of Theorem 2. Let G be a k-plane graph. We will assume, for ease of exposition, that any point $p \in \mathbb{R}^2$ is involved in at most one crossing (p, vw, xy) of G. This assumption is justified since it can be enforced by a slight deformation of the edges of G and the resulting (deformed) graph is also k-plane.

As in the proof of Observation 1, let G_0 be the plane graph obtained by adding a dummy vertex at each crossing in G. In this way, each edge $vw \in E(G)$ corresponds naturally to a path P_{vw} of length at most $k+1$ in G_0. Let $\mathcal{P} := \{P_{vw} : vw \in E(G)\}$. Observe that $\mathcal{P}$ is a $(k+1, 2)$-shortcut system for G_0 and that $G_0^{\mathcal{P}} \supseteq G$. Specifically, $G_0^{\mathcal{P}}$ contains every edge and vertex of G as well as the dummy vertices in $V(G_0) \setminus V(G)$ and their incident edges.

Since G_0 is planar, Theorem 1(b) and Lemma 1 implies that G_0 has an H-partition of layered width 3 for some planar graph H of treewidth at most 3. Applying Theorem 9 to G_0 and $\mathcal{P}$ immediately implies that G (an arbitrary k-planar graph) has an H-partition of layered width $6((k+1)^3 + 3(k+1))$ for some graph H of treewidth at most $\binom{k+4}{3} - 1$.

We can reduce the layered width of the H-partition of G from $O(k^3)$ to $O(k^2)$ by observing that the dummy vertices in $V(G_0) \setminus V(G)$ do not contribute to the layered width of this partition. In this setting, the proof of Claim 3 is simpler since each vertex $w \in Y_x$ contributes at most two vertices to $L_i \cap Y_x$. More precisely, each path $P \in \mathcal{P}$ containing an internal (dummy) vertex $w \in Y_x \cap (L_{i-j} \cup L_{i+j})$ contributes: (i) at most two vertices to $S_x \cap L_i$ for $j \in \{0, \dots, \lfloor (k+1)/2 \rfloor\}$; (ii) at most one vertex to $S_x \cap L_j$ for $j \in \{\lfloor (k+1)/2 \rfloor + 1, \dots, k+1\}$; or (iii) no vertices to $S_x \cap L_j$ for $j > k+1$. Redoing the calculation at the end of the proof of Claim 3 then yields

$$|S_x \cap Y_i| \leqslant d\ell\Big(2 + 4\Big\lfloor \tfrac{k+1}{2} \Big\rfloor + 2\Big\lceil \tfrac{k+1}{2} \Big\rceil\Big) = d\ell\Big(2 + 2(k+1) + 2\Big\lfloor \tfrac{k+1}{2} \Big\rfloor\Big) \leqslant d\ell(3k+5) = 18k + 30 \ .$$

With this change, the layered width of the partition given by Theorem 9 becomes $(18k + 30)(k+1) = 18k^2 + 48k + 30$. The result follows from Lemma 1. □

3.2 1-Planar Graphs

In the important special case of 1-planar graphs we obtain better constants and an additional property (planarity) of H.

Theorem 10. *Every 1-planar graph is a subgraph of $H \boxtimes P \boxtimes K_{30}$ for some planar graph H with treewidth at most 3 and for some path P.*

$\{f(t) : 0 \leqslant t \leqslant 1\}$.

Let G be an edge-maximal 1-plane multigraph with no two parallel edges on the boundary of a single face. Here, edge-maximal should be taken to mean that, if any two vertices v and w appear on a common face[12] F, then there is an edge $vw \in E(G)$ that is contained in the boundary of F. We assume that no two edges incident to a common vertex cross each other since, in a 1-plane graph, such a crossing can always be removed by a local modification to obtain an isomorphic 1-plane graph in which the two edges do not cross.[13]

A *kite* in G is the subgraph $K = G[\{v,w,x,y\}]$ induced by the endpoints of a pair of crossing edges $vw, xy \in E(G)$. It follows from edge-maximality that every kite is isomorphic to the complete graph K_4. The edges vw and xy are called *spars* of K. The cycle $vxwy$ is called the *sail* of K. It follows from edge-maximality that none of the edges vx, xw, wy, or yv are crossed by any other edges of G. Thus any edge that is a spar of a kite K is not part of a sail of any kite K'. Observe that any spar of K is incident on exactly four *kite faces* of G, each of which has three edges and two vertices of G on its boundary.

The 1-plane graph G has a plane triangulation G' as a subgraph that can be obtained by removing one spar from each kite in G. Observe that, for any spar $xy \in E(G) \setminus E(G')$ that crosses $vw \in E(G')$, G' contains the path vxw (and vyw). It follows that $\mathrm{dist}_{G'}(v,w) \leqslant 2$.

Our proof of Theorem 10 follows quickly from the following technical lemma, which is an extension of the analagous result for plane graphs [14].

Lemma 4. *The setup:*

1. *Let G and G' be defined as above.*
2. *Let T be a BFS spanning tree of G' rooted at some vertex r.*
3. *For each integer $j \geqslant 0$, let $L_j = \{v \in V(G) : \mathrm{dist}_T(r,v) = j\}$.*
4. *Let F be a cycle in G' with r in the exterior of F and such that*
   (a) *No edge of F is crossed by any edge of G; and*
   (b) *$V(F)$ can be partitioned into $P_1,\dots,P_k$, for some $k \in \{1,2,3\}$ such that for each $i \in \{1,\dots,k\}$,*
      i. *$F[P_i]$ is a path; and*
      ii. *$|V(P_i) \cap L_j| \leqslant 15$ for all integers $j \geqslant 0$.*
5. *Let N and N' be the subgraphs of G and G' consisting only of those edges and vertices contained in F or the interior of F.*

Then N has an H-partition $\mathcal{P} = \{S_x : x \in V(H)\}$ such that:

1. *H is planar;*
2. *for all integers $j \geqslant 0$, and all $x \in V(H)$, $|S_x \cap L_j| \leqslant 15$;*
3. *for each $i \in \{1,\dots,k\}$, there exists some $x_i \in V(H)$ such that $P_i = S_{x_i}$; and*
4. *H has a tree decomposition in which every bag has size at most 4 and such that some bag contains $x_1,\dots,x_k$.*

Proof. This proof is very similar to the proof of Lemma 14 by Dujmović et al. [14]. Rather than duplicate every detail of that proof here, we focus on the differences and refer the

[12] The *faces* of an embedded graph G are the connected components of $\mathbb{R}^2 \setminus \bigcup_{vw \in E(G)} vw$. We say that a vertex $v \in V(G)$ appears on a face F if v is contained in the closure of F.

[13] While this is true for 1-plane graphs it is not true for k-plane graphs with $k \geqslant 3$; the uncrossing operation can increase the number of crossings on a particular edge from k to $2(k-1)$.

reader to the original proof for the remaining details.

The proof is by induction on the number of vertices of N. First note that N' is a near-triangulation. If $k = 3$, set $R_i := P_i$ for each $i \in \{1,2,3\}$. Otherwise, as in [14], split $P_1,\dots,P_k$ to partition $V(F)$ into three sets R_1, R_2, and R_3 such that each $F[R_i]$ is a non-empty path and each R_i contains vertices from exactly one of $P_1,\dots,P_k$.

Next, as in [14], use Sperner's Lemma to find an inner face $\tau = v_1v_2v_3$ of N' such that, T contains disjoint vertical paths Q_1,Q_2,Q_3 such that each Q_i begins at v_i, ends at some vertex in R_i, and whose internal vertices (if any) are contained in $N' - V(F)$.

Let $\overline{Y}$ denote the subgraph of N' consisting of vertices and edges of Q_1, Q_2, Q_3, and τ. Let $\overline{Y}^+$ denote the subgraph of N consisting of the vertices and edges of $\overline{Y}$ plus the vertices and edges of every kite formed by a crossing between an edge of G and an edge of $\overline{Y}$.

We claim that, for each integer $i \geqslant 0$, $|V(\overline{Y}^+) \cap L_i| \leqslant 15$. First observe that, since Q_1,Q_2,Q_3 are each vertical paths in T, $\overline{Y}$ contains at most three vertices of L_i, each incident on at most two edges of $\overline{Y}$. Since $\mathrm{dist}_{G'}(v,w) \leqslant 2$ for each $vw \in E(G)$, any vertex $x \in V(\overline{Y}^+) \setminus V(\overline{Y}) \cap L_i$, is incident to an edge $xy \in E(G)$ that crosses one of the at most six edges in $\overline{Y}$ having an endpoint in L_i. These at most six edges have at most 12 endpoints. Therefore $|V(\overline{Y}^+) \setminus V(\overline{Y}) \cap L_i| \leqslant 6 \times 2 = 12$, so $|V(\overline{Y}^+) \cap L_i| \leqslant 12 + 3 = 15$.

Let M and M^+ denote the subgraph of G containing the edges and vertices of $\overline{Y}$, respectively $\overline{Y}^+$, and the edges and vertices of F. The graph M^+ has some number of bounded faces, all contained in the interior of F. Some of the bounded faces of M^+ are kite faces. Let $F_1,\dots,F_m$ be the non-kite bounded faces of M^+.

We claim that, for each $i \in \{1,\dots,m\}$, the boundary of F_i is a cycle in G' that contains no spars. Otherwise, some edge vw contributes to the boundary of F_i but is crossed by an edge $xy \in E(G)$. Then, $vw \notin E(F)$ since no edge of F is crossed by any edge of G. Therefore $vw \in E(\overline{Y}^+)$ so $xy \in E(Y^+)$. But then the only faces of M^+ incident to vw are kite faces. In particular vw cannot be incident to the non-kite face F_i.

Observe that each of the faces $F_1,\dots,F_m$ is contained in a single internal face of M. Let $Y^+ := \overline{Y}^+ - F$. Therefore, $V(F_i)$ can be partitioned into at most three sets P_1', P_2', and P_3' where $P_1' \subset V(Y^+)$, $P_2' \subseteq P_a$, $P_3' \subseteq P_b$ for some $a,b \in \{1,2,3\}$, and $F_i[P_j']$ is a path, for each $j \in \{1,2,3\}$.

Finally, the subgraph N_i of G consisting of the edges and vertices of G contained in F_i or its interior does not contain one of the three vertices of τ. Therefore, we can apply induction using the cycle F_i and the partition P_1',P_2',P_3' of $V(C_i)$ to obtain the desired H-partition and tree decomposition of N_i.

The proof finishes in the same way as the proof in [14]. The paths $P_1,\dots,P_k$, and $S = V(Y^+)$ become elements of the H-partition. Elements in each of the H-partitions of $N_1,\dots,N_3$ that intersect $P_1,\dots,P_k$, or $V(\overline{Y}^+ - F)$ are discarded and all the resulting sets are combined to obtain an H-partition of G. The desired tree decomposition of G is obtained in exactly the same way as in the proof of Lemma 14 in [14], except that now each node x has a child for each face F_i of M_x^+ that contains a vertex of G in its interior.

The planarity of H comes from two properties:

1. $G/\mathcal{P}$ and $G^+/\mathcal{P}^+$ are isomorphic, where G^+ is the triangulation obtained by adding dummy vertices at each crossing in G and $\mathcal{P}^+$ is the partition we obtain by adding a dummy vertex z to $\overline{Y}^+$ if $\overline{Y}^+$ contains an edge vw that contains z in its interior.
2. $G^+[\overline{Y}^+ - F]$ is connected. To see this, first observe that $\overline{Y} - F$ is connected, and then observe that every vertex of $\overline{Y}^+$ is either a vertex of $\overline{Y}$ or adjacent to a vertex of $\overline{Y}$.

Since G^+ is planar, the second point implies that $H = G^+/\mathcal{P}$ is planar. □

Using Lemma 4, the proof of Theorem 10 is now straightforward.

Proof of Theorem 10. Given a 1-plane graph G, add edges to make it edge-maximal so that it has outer face $F = v_1v_2v_3$. Next, add a vertex r adjacent to v_1, v_2, and v_3 to obtain an edge-maximal 1-plane graph $\overline{G}$ with one vertex r of degree 3 on its outer face.

Let G' be the plane graph obtained by removing one spar from each kite of $\overline{G}$ and let T be a BFS tree of G' rooted at r. Now apply Lemma 4 with $G = \overline{G}$, G', F, and $P_i = \{v_i\}$ for each $i \in \{1,2,3\}$. This gives an H-partition $\{S_x : x \in V(H)\}$ of $\overline{G} - \{r\} \supseteq G$ in which H is planar and has treewidth at most 3.

Use the layering $\mathcal{L} = \langle L'_0, L'_1 \dots \rangle$ where $L'_i = L_{2i} \cup L_{2i+1}$ for each integer $i \geqslant 0$. That this is a layering of G follows from the fact that $\mathrm{dist}_{G'}(v,w) \leqslant 2$ for every edge $vw \in E(G)$. Since $|L_j \cap S_x| \leqslant 15$ for every integer $j \geqslant 0$, $|L'_i \cap S_x| \leqslant 30$ for every integer $i \geqslant 0$ and every $x \in V(H)$. The result follows from Lemma 1. □

3.3 (g,k)-Planar Graphs

The definition of k-planar graphs naturally generalises for other surfaces. A graph G drawn on a surface Σ is (Σ,k)-plane if every edge of G is involved in at most k crossings. A graph G is (g,k)-planar if it is isomorphic to some (Σ,k)-plane graph, for some surface Σ with Euler genus at most g. Observation 1 immediately generalises as follows:

Observation 2. *Every (g,k)-planar graph G is a subgraph of $G_0^{\mathcal{P}}$ for some graph G_0 of Euler genus at most g and some $(k+1,2)$-shortcut system $\mathcal{P}$ for G_0. Moreover, $V(G) \subseteq V(G_0)$ and for every edge $vw \in E(G)$ there is a vw-path P in G_0 of length at most $k+1$, such that every internal vertex in P has degree at most 4 in G_0.*

Theorems 3 and 5(b) imply a product structure theorem for (g,k)-planar graphs. The obtained bounds are improved by the following result, which is proved using exactly the same approach used in the proof of Theorem 2 (applying Theorem 5(b) instead of Theorem 1(b)). We omit repeating the details.

Theorem 11. *Every (g,k)-planar graph is a subgraph of $H \boxtimes P \boxtimes K_\ell$ for some graph H with $\mathrm{tw}(H) \leqslant \binom{k+5}{4} - 1$, where $\ell := \max\{2g,3\} \cdot (6k^2 + 16k + 10)$.*

Prior to this work, the strongest structural description of k-planar or (g,k)-planar graphs (or any of the other classes presented in Section 5) was in terms of layered treewidth, which we now define. A *layered tree decomposition* $(\mathcal{L},\mathcal{T})$ consists of a layering $\mathcal{L}$ and a tree decomposition $\mathcal{T}$ of G. The layered width of $(\mathcal{L},\mathcal{T})$ is $\max\{|L \cap B| : L \in \mathcal{L}, B \in \mathcal{T}\}$. The *layered* *treewidth* of G is the minimum layered width of any layered tree decomposition of G. Dujmović, Morin, and Wood [13] proved that planar graphs have layered treewidth at most 3, that graphs of Euler genus g have layered treewidth at most $2g+3$, and more generally that a minor-closed class has bounded layered treewidth if and only if it excludes some apex graph. Dujmović et al. [12] show that every k-planar graph has layered treewidth at most $6(k+1)$, and more generally that every (g,k)-planar graph has layered treewidth at most $(4g+6)(k+1)$. It follows from this result that (g,k)-planar graphs have treewidth $O(\sqrt{(g+1)(k+1)n})$ and thus have balanced separators of the same order, which can also be concluded from the work of Fox and Pach [20]. In related work, Grigoriev and Bodlaender [23] used structural results to obtain approximation algorithms for (g,k)-planar graphs, and Pach and Tóth [32] determined the maximum number of edges in a k-planar graph (up to a constant factor).

If a graph class admits bounded layered partitions, then it also has bounded layered treewidth. In particular, if $\mathcal{P} = (P_x : x \in V(H))$ is an H-partition of G of layered width ℓ with respect to some layering $\mathcal{L}$ of G and $(B_x : x \in V(T))$ is a width-t tree decomposition of H, then setting $C_x = \bigcup_{y \in B_x} P_y$ for each $x \in V(T)$ gives a tree decomposition $(C_x : x \in V(T))$ of G that has layered treewidth $(t+1)\ell$ [14]. Therefore, any property that holds for graphs of bounded layered treewidth also holds for G. What sets layered partitions apart from layered treewidth is that they lead to constant upper bounds on the queue-number and non-repetitive chromatic number, whereas for both these parameters, the best known upper bound obtainable via layered treewidth is $O(\log n)$; see Section 4.

3.4 Rough Characterisation

Observation 2 shows that (g,k)-planar graphs can be obtained by a shortcut system applied to a graph of Euler genus g, where internal vertices on the paths have bounded degree. This observation and the following converse result together provide a rough characterisation of (g,k)-planar graphs, which is interesting in its own right, and is useful for showing that various classes of graphs are (g,k)-planar.

Lemma 5. *Fix integers $g \geqslant 0$ and $k, \Delta \geqslant 2$. Let G_0 be a graph of Euler genus at most g. Let G be a graph with $V(G) \subseteq V(G_0)$ such that for every edge $vw \in E(G)$ there is a vw-path P_{vw} in G_0 of length at most k, such that every internal vertex on P_{vw} has degree at most Δ in G_0. Then G is $(g, 2k(k+1)\Delta^k)$-planar.*

Proof. For a vertex x of G_0 with degree at most Δ, and for $i \in \{1,\dots,k-1\}$, say a vertex v is *i-close* to x if there is a vx-path P in G_0 of length at most i such that every internal vertex in P has degree at most Δ in G_0. For each edge vw of G, say that vw *passes through* each internal vertex on P_{vw}. Say vw passes through x. Then v is i-close to x and w is j-close to x for some $i,j \in \{1,\dots,k-1\}$ with $i+j \leqslant k$. At most Δ^i vertices are i-close to x. Thus, the number of edges of G that pass through x is at most

$$\sum_{i=1}^{k-1}\sum_{j=1}^{k-i}\Delta^i\Delta^j = \sum_{i=1}^{k-1}\Delta^i\sum_{j=1}^{k-i}\Delta^j < \sum_{i=1}^{k-1}\Delta^i 2\Delta^{k-i} = \sum_{i=1}^{k-1} 2\Delta^k < 2k\Delta^k \;.$$

Draw each edge vw of G alongside P_{vw} in G_0, so that every pair of edges cross at most once. Every edge of G that crosses vw passes through a vertex on P_{vw} (including v and/or w if they too have degree at most Δ). Since P_{vw} has at most $k+1$ vertices, and less than $2k\Delta^k$ edges of G pass through each vertex on P_{vw}, the edge vw is crossed by less than $2k(k+1)\Delta^k$ edges in G. Hence G is $(g, 2k(k+1)\Delta^k)$-planar. $\square$

4 Applications

Here we discuss some of the consequences of the above theorems for k-planar and (g,k)-planar graphs.

4.1 Queue Layouts

For an integer $k \geqslant 0$, a *k-queue layout* of a graph G consists of a linear ordering $\preceq$ of $V(G)$ and a partition $\{E_1, E_2, \dots, E_k\}$ of $E(G)$, such that for $i \in \{1, 2, \dots, k\}$, no two edges in E_i are nested with respect to $\preceq$. That is, it is not the case that $v \prec x \prec y \prec w$ for edges $vw, xy \in E_i$. The *queue-number* of a graph G, denoted by $\mathrm{qn}(G)$, is the minimum integer k such that G has a k-queue layout. Queue-number was introduced by Heath et al. [25], who famously conjectured that planar graphs have bounded queue-number. Dujmović et al. [14] recently proved this conjecture using Theorem 1 and the following lemma. Indeed, resolving this question was the motivation for the development of Theorem 1.

Lemma 6 ([14])**.** *If $G \subseteq H \boxtimes P \boxtimes K_\ell$ then $\mathrm{qn}(G) \leqslant 3\ell\,\mathrm{qn}(H) + \lfloor \frac{3}{2}\ell \rfloor \leqslant 3\ell\, 2^{\mathrm{tw}(H)} - \lceil \frac{3}{2}\ell \rceil$.*

Lemma 6 and Theorem 11 imply that (g,k)-planar graphs have queue-number at most $g2^{O(k^4)}$. Note that Dujmović et al. [14] previously proved the bound of $O(g^{k+2})$ using Theorem 5 and an ad-hoc method. Our result provides a better bound when $g > 2^{k^3}$. In the case of 1-planar graphs we can improve further. Alam, Bekos, Gronemann, Kaufmann, and Pupyrev [3] proved that every planar graph with treewidth at most 3 has queue-number at most 5. Thus the graph H in Theorem 10 has queue-number at most 5. Lemma 6 and Theorem 10 then imply:

Corollary 1. *Every 1-planar graph has queue-number at most* $3 \times 30 \times 5 + \lfloor \frac{3}{2} \times 30 \rfloor = 495$.

4.2 Non-Repetitive Colouring

The next two applications are in the field of graph colouring. For our purposes, a *c-colouring* of a graph G is any function $\phi\colon V(G) \to C$, where C is a set of size at most c. A c-colouring ϕ of G is *non-repetitive* if, for every path $v_1, \dots, v_{2h}$ in G, there exists $i \in \{1, \dots, h\}$ such that $\phi(v_i) \neq \phi(v_{i+h})$. The *non-repetitive chromatic number* $\pi(G)$ of G is the minimum integer c such that G has a non-repetitive c-colouring. This concept, introduced by Alon et al. [4], has since been widely studied; see [15] for more than 40 references. Up until recently the main open problem in the field has been whether planar graphs have bounded non-repetitive chromatic number, first asked by Alon et al. [4]. Dujmović et al. [15] recently solved this question using Theorem 1 and the following lemma.

Lemma 7 ([15])**.** *If $G \subseteq H \boxtimes P \boxtimes K_\ell$ then $\pi(G) \leqslant \ell\, 4^{\mathrm{tw}(H)+1}$.*

Lemma 7 and Theorems 2, 10 and 11 imply the following result:

Corollary 2.

- *For every* 1*-planar graph* G, $\pi(G) \leqslant 30 \times 4^4 = 7680$.
- *For every* k*-planar graph* G, $\pi(G) \leqslant (18k^2 + 48k + 30)4^{\binom{k+4}{3}}$.
- *For every* (g,k)*-planar graph* G, $\pi(G) \leqslant \max\{2g, 3\} \cdot (6k^2 + 16k + 10)4^{\binom{k+5}{4}}$.

Prior to the current work, the strongest upper bound on the non-repetitive chromatic number of n-vertex k-planar graphs was $O(k \log n)$ [13].

4.3 Centered Colourings

A c-colouring ϕ of G is *p-centered* if, for every connected subgraph $X \subseteq G$, $|\{\phi(v) : v \in V(X)\}| > p$ or there exists some $v \in V(X)$ such that $\phi(v) \neq \phi(w)$ for every $w \in V(X) \setminus \{v\}$. In words, either X receives more than p colours or some vertex in X receives a unique colour. Let $\chi_p(G)$ be the minimum integer c such that G has a p-centered c-colouring. Centered colourings are important since they characterise classes of bounded expansion, which is a key concept in the sparsity theory of Nešetřil and Ossona de Mendez [31].

Pilipczuk and Siebertz [34] and Dębski et al. [10] use Theorem 8 and Theorem 1(b), respectively, to show that $\chi_p(G)$ is polynomial in p when G is planar or of bounded Euler genus. We use the following lemma due to Pilipczuk and Siebertz [35, Lemma 15].

Lemma 8 ([35]). *Every graph* H *of treewidth at most* t *has* $\chi_p(H) \leqslant \binom{p+t}{t}$.

The following lemma is implicitly in the work of Dębski et al. [10, Proof of Theorem 2.1]. We include the proof for completeness.

Lemma 9 ([10]). *If* $G \subseteq H \boxtimes P \boxtimes K_\ell$ *then* $\chi_p(G) \leqslant \ell(p+1)\chi_p(H)$.

Proof. By Lemma 1, G has an H-partition $(\mathcal{L} = \langle L_0, L_1, \dots \rangle, \mathcal{P} = (B_x : x \in V(H))$ with layered width at most ℓ. Use a product colouring $\phi : V(G) \to \{1,\dots,\ell\} \times \{0,\dots,p\} \times \{1,\dots,\chi_p(H)\}$. For each integer $i \geqslant 0$ and each $x \in V(H)$, assign the colour $\phi(v) := (\alpha(v), \beta(v), \gamma(v))$ to each vertex $v \in L_i \cap B_x$ such that:

1. $\alpha(v)$ is unique among $\{\phi(w) : w \in L_i \cap B_x\}$, which is possible since $|L_i \cap B_x| \leqslant \ell$,
2. $\beta(v) = i \bmod (p+1)$, and
3. $\gamma(v) = \gamma(x)$ where $\gamma : V(H) \to \{1,\dots,\chi_p(H)\}$ is a p-centered colouring of H.

To show this is a p-centered colouring, consider some connected subgraph $X \subseteq G$.

First suppose that there exists $v, w \in V(X)$ with $v \in L_i$ and $w \in L_j$ with $j - i \geqslant p$. Since X is connected, X contains a path from v to w. By the definition of layering, this path contains at least one vertex from $L_{i'}$ for each $i' \in \{i, i+1, \dots, j\}$. Therefore, $|\{\beta(v') : v' \in V(X)\}| \geqslant j - i + 1 > p$, so X receives more than p distinct colours.

Otherwise, $V(X) \subseteq L_i \cup \dots \cup L_{i+s}$ for some $s < p$. Let $H' := H[\{x \in V(H) : B_x \cap V(X) \neq \emptyset]$. If $|\{\gamma(x) : x \in V(H')\}| > p$ then $|\{\gamma(v) : v \in V(X)\}| > p$ so $|\{\phi(v) : v \in V(X)\}| > p$ and we are done. Otherwise, since γ is a p-centered colouring of H, there must exist some $x \in V(H')$ such that $\gamma(x) \neq \gamma(y)$ for every $y \in V(H') \setminus \{x\}$. For any $v, w \in B_x$ with $v \neq w$, either $v, w \in L_{i'}$ for some $i' \in \{i, i+1, \dots, i+s\}$ in which case $\alpha(v) \neq \alpha(w)$; or $v \in L_{i'}$ and $w \in L_{i''}$ with $|i' - i''| < p$, in which case $\beta(v) \neq \beta(w)$. Therefore every vertex $v \in B_x$ receives a colour $\phi(v)$ distinct from every colour in $\{\phi(z) : z \in X \setminus \{x\}\}$. Therefore, every vertex in B_x receives a colour distinct from every other vertex in X. $\square$

Lemmas 8 and 9 and Theorems 2, 10 and 11 immediately imply the following results, for every $p \geqslant 2$:

Corollary 3.

- *For every* 1*-planar graph* G, $\chi_p(G) \leqslant 5(p+3)(p+2)(p+1)^2$.
- *For every* k*-planar graph* G, $\chi_p(G) \leqslant (18k^2+48k+30)(p+1)\binom{p+\binom{k+4}{3}-1}{\binom{k+4}{3}-1}$.
- *For every* (g,k)*-planar graph* G, $\chi_p(G) \leqslant \max\{2g,3\}\cdot(6k^2+16k+10)(p+1)\binom{p+\binom{k+5}{4}-1}{\binom{k+5}{4}-1}$.

Prior to the current work, the strongest known upper bounds on the p-centered chromatic number of (g,k)-planar graphs G were doubly-exponential in p, as we now explain. Dujmović et al. [12] proved that G has layered treewidth $(4g+6)(k+1)$. Van den Heuvel and Wood [37] showed that this implies that G has r-strong colouring number at most $(4g+6)(k+1)(2r+1)$. By a result of Zhu [38], G has r-weak colouring number at most $((4g+6)(k+1)(2r+1))^r$, which by another result of Zhu [38] implies that G has p-centered chromatic number at most $((4g+6)(k+1)(2^{p-1}+1))^{2^{p-2}}$. The above results are substantial improvements, providing bounds on $\chi_p(G)$ that are polynomial in p for fixed g and k.

5 Examples

This section describes several examples of graph classes that can be obtained from a shortcut system typically applied to graphs of bounded Euler genus.

5.1 Map Graphs

Map graphs are defined as follows. Start with a graph G_0 embedded in a surface of Euler genus g, with each face labelled a 'nation' or a 'lake', where each vertex of G_0 is incident with at most d nations. Let G be the graph whose vertices are the nations of G_0, where two vertices are adjacent in G if the corresponding faces in G_0 share a vertex. Then G is called a (g,d)*-map graph*. A $(0,d)$-map graph is called a (plane) d*-map graph*; see [9, 19] for example. The $(g,3)$-map graphs are precisely the graphs of Euler genus at most g; see [12]. So (g,d)-map graphs generalise graphs embedded in a surface, and we now assume that $d \geqslant 4$ for the remainder of this section.

There is a natural drawing of a map graph obtained by positioning each vertex of G inside the corresponding nation and each edge of G as a curve passing through the corresponding vertex of G_0. It is easily seen that each edge is in at most $\lfloor\frac{d-2}{2}\rfloor\lceil\frac{d-2}{2}\rceil$ crossings; see [12]. Thus G is $(g,\lfloor\frac{d-2}{2}\rfloor\lceil\frac{d-2}{2}\rceil)$-planar. Also note that Lemma 5 with $k=2$ implies that G is $(g,O(d^2))$-planar. Theorem 11 then establishes a product structure theorem for map graphs, but we get much better bounds by constructing a shortcut system directly. The following lemma is reminiscent of the characterisation of (g,d)-map graphs in terms of the half-square of bipartite graphs [9, 12].

Lemma 10. *Every (g,d)-map graph G is a subgraph of $G_1^{\mathcal{P}}$ for some graph G_1 with Euler genus at most g and some $(2, \frac{1}{2}d(d-3))$-shortcut system $\mathcal{P}$ for G_1.*

Proof. Let G be a (g,d)-map graph. So there is a graph G_0 embedded in a surface of Euler genus g, with each face labelled a 'nation' or a 'lake', where each vertex of G_0 is incident with at most d nations. Let N be the set of nations. Then $V(G) = N$ where two vertices are adjacent in G if the corresponding nation faces of G_0 share a vertex. Let G_1 be the graph with $V(G_1) := V(G_0) \cup N$, where distinct vertices $v, w \in N$ are adjacent in G_1 if the boundaries of the corresponding nations have an edge of G_0 in common, and $v \in V(G_0)$ and $w \in N$ are adjacent in G_1 if v is on the boundary of the nation corresponding to w. Observe that G_1 embeds in the same surface as G_0 with no crossings, and that each vertex in $V(G_0)$ has degree at most d in G_1. Consider an edge $vw \in E(G)$. If the nations corresponding to v and w share an edge of G_0, then vw is an edge of G_1. Otherwise, v and w have a common neighbour x in $V(G_0)$. In the latter case, let P_{vw} be the path (v,x,w). Let $\mathcal{P}$ be the set of all such paths P_{vw}. Each vertex $x \in V(G_0)$ is the middle vertex on at most $\frac{1}{2}d(d-3)$ paths in $\mathcal{P}$. Thus $\mathcal{P}$ is a $(2, \frac{1}{2}d(d-3))$-shortcut system for G_1, and by construction, $G \subseteq G_1^{\mathcal{P}}$. □

Theorems 1(b), 3 and 5(b) and Lemmas 6 to 10 imply the following results.

Theorem 12. *Every d-map graph G:*

- *is a subgraph of $H \boxtimes P \boxtimes K_{21d(d-3)}$ for some path P and for some graph H with* $\text{tw}(H) \leqslant 9$,
- *has queue-number* $\text{qn}(G) < 32225\, d(d-3)$.
- *has p-centered chromatic number* $\chi_p(G) \leqslant 21d(d-3)(p+1)\binom{p+9}{9}$,
- *has non-repetitive chromatic number* $\pi(G) \leqslant 21 \cdot 4^{10} d(d-3)$.

Theorem 13. *For integers $g \geqslant 0$ and $d \geqslant 4$, if $\ell := 7d(d-3)\max\{2g, 3\}$ then every (g,d)-map graph G:*

- *is a subgraph of $H \boxtimes P \boxtimes K_\ell$ for some path P and for some graph H with* $\text{tw}(H) \leqslant 14$,
- *has queue-number* $\text{qn}(G) < 49151\, \ell$.
- *has p-centered chromatic number* $\chi_p(G) \leqslant \ell(p+1)\binom{p+14}{14}$,
- *has non-repetitive chromatic number* $\pi(G) \leqslant 4^{15}\, \ell$.

These results give the first constant upper bound on the non-repetitive chromatic number of map graphs, the first polynomial bounds on the p-centered chromatic number of map graphs, and the best known bounds on the queue-number of map graphs.

5.2 String Graphs

A *string graph* is the intersection graph of a set of curves in the plane with no three curves meeting at a single point; see [21, 22, 33] for example. For an integer $\delta \geqslant 2$, if each curve is in at most δ intersections with other curves, then the corresponding string graph is called a *δ-string graph*. A *(g,δ)-string* graph is defined analogously for curves on a surface of Euler genus at most g.

Lemma 11. *Every (g,δ)-string graph G is a subgraph of $G_0^{\mathcal{P}}$ for some graph G_0 with Euler genus at most g and some $(\delta+1, \delta+1)$-shortcut system $\mathcal{P}$ for G_0.*

Proof. Let $\mathcal{C} = \{C_v : v \in V(G)\}$ be a set of curves in a surface of Euler genus at most g whose intersection graph is G. Let G_0 be the graph obtained by adding a vertex at the intersection point of every pair of curves in $\mathcal{C}$ that intersect, where two such consecutive vertices on a curve C_v are adjacent in G_0. For each vertex $v \in V(G)$, if C_v intersects $k \leqslant \delta$ other curves, then introduce a new vertex called v on C_v between the $\lfloor \frac{k}{2} \rfloor$-th vertex already on C_v and the $\lfloor \frac{k}{2} + 1 \rfloor$-th such vertex. For each edge vw of G, there is a path P_{vw} of length at most $2\lceil \frac{\delta}{2} \rceil \leqslant \delta + 1$ in G_0 between v and w. Let $\mathcal{P}$ be the set of all such paths P_{vw}. Consider a vertex x in G_0 that is an internal vertex on some path in $\mathcal{P}$. Then x is at the intersection of C_v and C_w for some edge $vw \in E(G)$. If some path $P \in \mathcal{P}$ passes through x, then $P = P_{vu}$ for some edge vu incident to v, or $P = P_{wu}$ for some edge wu incident to w. At most $\lceil \frac{\delta}{2} \rceil$ paths in $\mathcal{P}$ corresponding to edges incident to v pass through x, and similarly for edges incident to w. Thus at most $2\lceil \frac{\delta}{2} \rceil \leqslant \delta + 1$ paths in $\mathcal{P}$ use x as an internal vertex. Thus $\mathcal{P}$ is a $(\delta+1, \delta+1)$-shortcut system for G_0, and by construction, $G \subseteq G_0^{\mathcal{P}}$. □

A similar proof to that of Lemma 11 shows that every (g,δ)-string graph is $(g, 2\delta^2)$-planar. Theorems 1(b), 3 and 5(b) and Lemmas 6 and 11 imply:

Theorem 14. *For integers $g \geqslant 0$ and $\delta \geqslant 2$, let $\ell := \max\{2g, 3\}(\delta^4 + 4\delta^3 + 9\delta^2 + 10\delta + 4)$, and $t := \binom{\delta+5}{4} - 1$ if $g \geqslant 1$ and $t := \binom{\delta+4}{3} - 1$ if $g = 0$. Then every (g,δ)-string graph:*

- *is a subgraph of $H \boxtimes P \boxtimes K_\ell$ for some path P and for some graph H with treewidth t,*
- *has queue-number $\operatorname{qn}(G^k) \leqslant 3\ell\, 2^t - \lceil \frac{3}{2}\ell \rceil$,*

Our results also give bounds on the non-repetitive chromatic number and the p-centered chromatic number of (g,δ)-string graphs, but the bounds are weak, since such graphs G have maximum degree at most 2δ, implying that $\pi(G) \leqslant (4 + o(1))\delta^2$ and $\chi_p(G) \leqslant p(64\delta)^2$ by results of Dujmović, Joret, Kozik, and Wood [11] and Dębski et al. [10], respectively.

5.3 Powers of Bounded Degree Graphs

Recall that the *k-th power* of a graph G is the graph G^k with vertex set $V(G^k) := V(G)$, where $vw \in E(G^k)$ if and only if $\operatorname{dist}_G(v,w) \leqslant k$. If G is planar with maximum degree Δ, then G^k is $2k(k+1)\Delta^k$-planar by Lemma 5. Thus we can immediately conclude that bounded powers of planar graphs of bounded degree admit bounded layered partitions. However, the bounds we obtain are improved by the following lemma that constructs a shortcut system directly.

Lemma 12. *If a graph G has maximum degree Δ, then $G^k = G^{\mathcal{P}}$ for some $(k, 2k\Delta^k)$-shortcut system $\mathcal{P}$.*

Proof. For each pair of vertices x and y in G with $\operatorname{dist}_G(x,y) \in \{1,\dots,k\}$, fix an xy-path P_{xy} of length $\operatorname{dist}_G(x,y)$ in G. Let $\mathcal{P} := \{P_{xy} : \operatorname{dist}_G(x,y) \in \{1,\dots,k\}\}$. Say P_{xy} uses some vertex v as an internal vertex. If $\operatorname{dist}_G(v,x) = i$ and $\operatorname{dist}_G(v,y) = j$, then $i, j \in \{1,\dots,k-1\}$ and $i + j \leqslant k$. The number of vertices at distance i from v is at most Δ^i. Thus the number of paths in $\mathcal{P}$ that use v as an internal vertex is at most

$$\sum_{i=1}^{k-1}\sum_{j=1}^{k-i}\Delta^i\Delta^j = \sum_{i=1}^{k-1}\Delta^i\sum_{j=1}^{k-i}\Delta^j < \sum_{i=1}^{k-1}\Delta^i(2\Delta^{k-i}) < 2k\Delta^k \quad .$$

Hence $\mathcal{P}$ is a $(k, 2k\Delta^k)$-shortcut system. $\square$

Theorem 3 and Lemma 12 imply:

Theorem 15. *Let G be a subgraph of $H \boxtimes P \boxtimes K_\ell$ with maximum degree Δ, for some graph H of treewidth at most t and for some path P. Then for every integer $k \geqslant 1$, the k-th power G^k is a subgraph of $J \boxtimes P \boxtimes K_{2k\ell\Delta^k(k^3+3k)}$ for some graph J of treewidth at most $\binom{k+t}{t} - 1$ and some path P.*

Theorems 1(b), 5(b) and Theorem 3 and Lemmas 5 to 9 and 12 imply the following result, which with $g = 0$ implies Theorem 4 in the introduction.

Theorem 16. *For integers $g \geqslant 0$ and $k, \Delta \geqslant 1$, let $\ell := \max\{2g, 3\}(2k^4+6k^2)\Delta^k$, and $t := \binom{k+4}{4}-1$ if $g \geqslant 1$ and $t := \binom{k+3}{3}-1$ if $g = 0$. Then for every graph G of Euler genus g and maximum degree Δ,*

- *G^k is a subgraph of $H \boxtimes P \boxtimes K_\ell$ for some path P and for some graph H with treewidth t*
- *G^k is $(g, 2k(k+1)\Delta^k)$-planar,*
- *G^k has queue-number $\operatorname{qn}(G^k) \leqslant 3\ell \cdot 2^t - \lceil \frac{3}{2}\ell \rceil$.*
- *G^k has p-centered chromatic number $\chi_p(G^k) \leqslant \ell(p+1)\binom{p+t}{t}$,*
- *G^k has non-repetitive chromatic number $\pi(G^k) \leqslant \ell\, 4^{t+1}$.*

This result is the first constant upper bound on the queue-number of bounded powers of graphs with bounded degree and bounded Euler genus. For every graph G, since G^k has maximum degree at most Δ^k, a result of Dujmović et al. [11] implies that $\pi(G^k) \leqslant (1+o(1))\Delta^{2k}$. Theorem 16 improves upon this bound when $k, g \ll \Delta$. Similarly, a result of Dębski et al. [10] implies that $\chi_p(G^k) \leqslant 1024p\Delta^{2k}$ and Theorem 16 improves upon this bound when $p, k, g \ll \Delta$.

Lemmas 6 to 9 and Theorems 7 and 15 imply the following analogous result for powers of graphs in any minor-closed class with bounded maximum degree.

Theorem 17. *For every graph X there exists an integer c such that for all integers $k, \Delta \geqslant 1$, if $t := 2k\Delta^k(k^3+3k)\binom{k+c\Delta}{c\Delta} - 1$ and G is an X-minor-free graph with maximum degree Δ, then:*

- *G^k is a subgraph of $H \boxtimes P$ for some graph H with treewidth t and for some path P.*
- *G^k has queue-number at most $3 \cdot 2^t - 2$,*
- *G^k has p-centered chromatic number at most $(p+1)\binom{p+t}{t}$.*

5.4 k-Nearest-Neighbour Graphs

In this section, we show that k-nearest neighbour graphs of point sets in the plane are $O(k^2)$-planar. For two points $x, y \in \mathbb{R}^2$, let $d_2(x,y)$ denote the Euclidean distance between x and y. The k-nearest-neighbour graph of a point set $P \subset \mathbb{R}^2$ is the geometric graph G with vertex set $V(G) = P$, where the edge set is defined as follows. For each point $v \in P$, let $N_k(v)$ be the set of k points in P closest to v. Then $vw \in E(G)$ if and only if $w \in N_k(v)$ or $v \in N_k(w)$. (The edges of G are straight-line segments joining their endpoints.) See [8] for a survey of results on k-nearest neighbour graphs and other related proximity graphs.

The following result, which is immediate from Ábrego, Monroy, Fernández-Merchant, Flores-Peñaloza, Hurtado, Sacristán, and Saumell [1, Corollary 4.2.6] states that k-nearest-neighbour graphs have bounded maximum degree:

Lemma 13. *The degree of every vertex in a k-nearest-neighbour graph is at most $6k$.*

We make use of the following well-known observation (see for example, Bose, Morin, Stojmenović, and Urrutia [7, Lemma 2]):

Observation 3. *If $v_0, \ldots, v_3$ are the vertices of a convex quadrilateral in counterclockwise order then there exists at least one $i \in \{0, \ldots, 3\}$ such that $\max\{d_2(v_i, v_{i-1}), d_2(v_i, v_{i+1})\} < d_2(v_{i-1}, v_{i+1})$, where subscripts are taken modulo 4.*

Lemma 14. *Every k-nearest-neighbour graph is $O(k^2)$-planar.*

Proof. Let G be a k-nearest-neighbour graph and consider any edge $vw \in E(G)$. Let $xy \in E(G)$ be an edge that crosses vw. Note that $vxwy$ are the vertices of a convex quadrilateral in (without loss of generality) counterclockwise order. Then we say that

1. xy is of Type v if $\max\{d_2(v,x), d_2(v,y)\} < d_2(x,y)$;
2. xy is of Type w if $\max\{d_2(w,x), d_2(w,y)\} < d_2(x,y)$; or
3. xy is of Type C otherwise.

If xy is of Type C, then Observation 3 implies that $\max\{d_2(x,v), d_2(x,w)\} < d_2(v,w)$ without loss of generality. In this case, we call x a Type C vertex. We claim that $V(G)$ contains at most $k-1$ Type C vertices. Indeed, more than $k-1$ Type C vertices would contradict the fact that $vw \in E(G)$ since every Type C vertex is closer to both v and w than $d_2(v,w)$.

Next oberve that, if xy is of Type v, then at least one of xv or yv is in $E(G)$ in which case we call x (respectively y) a Type v vertex. By Lemma 13, there are at most $6k$ Type v vertices. Similarly, there are at most $6k$ Type w vertices.

Thus, in total, there are at most $13k-1$ Type v, Type w, and Type C vertices. By Lemma 13, each of these vertices is incident with at most $6k$ edges that cross vw. Therefore, there are at most $78k^2 - 6k$ edges of G that cross vw. Since this is true for every edge $vw \in E(G)$, G is $(78k^2 - 6k)$-planar. □

Note that Lemma 14 is tight up to the leading constant: Every k-nearest neighbour graph on $n \geqslant k+1$ vertices has at least $kn/2$ edges and at most kn edges. For $k \geqslant 7$, the Crossing Lemma [2, 29] implies that the total number of crossings is therefore $\Omega(k^3 n)$ so that the average number of crossings per edge is $\Omega(k^2)$.

Lemmas 6, 8, 9 and 14 and Theorem 2 imply:

Corollary 4. *For every integer $k \geqslant 1$ there exists integers $t \leqslant O(k^6)$ and $\ell \leqslant O(k^4)$ such that every k-nearest-neighbour graph:*

- *is a subgraph of $H \boxtimes P \boxtimes K_\ell$ for some graph H with treewidth t and some path P,*
- *has queue-number at most $2^{O(k^6)}$, and*
- *has p-centered chromatic number at most $\ell(p+1)\binom{p+t}{t}$.*

Lemma 7 and Corollary 4 also give bounds on the non-repetitive chromatic number of a k-nearest neighbour graph G. However, the bound is weak, since G has maximum degree at most $6k$, implying that $\pi(G) \leqslant (36+o(1))k^2$ by a result of Dujmović et al. [11].

6 Wrapping Up

As mentioned in Section 1, Dujmović et al. [16] prove results about adjacency labelling schemes for planar graphs. Stated in terms of universal graphs, their main theorem is interpreted as follows:

Theorem 18 ([16])**.** *For every fixed integer t and every integer $n > 0$ there exists a graph U_n with $n^{1+o(1)}$ vertices such that for every graph H of treewidth at most t and path P, every n-vertex subgraph of $H \boxtimes P$ is isomorphic to an induced subgraph of U_n.*

Combining this with Theorems 11, 13, 14 and 17 and Corollary 4 yields the following:

Theorem 19. *For every fixed graph X and all fixed integers $d, \delta, \Delta, g, k > 0$, and every integer $n > 0$, there exists a graph U_n with $n^{1+o(1)}$ vertices such that U_n contains all of the following graphs as induced subgraphs:*

- *every n-vertex (g,k)-planar graph;*
- *every n-vertex (g,d)-map graph;*
- *every n-vertex (g,δ)-string graph;*
- *every n-vertex graph G^k where G is X-minor-free and has maximum degree at most Δ;*
- *every k-nearest neighbour graph of n points in $\mathbb{R}^2$.*

We finish with an open problem. Theorem 2 shows that every k-planar graph is a subgraph of $H \boxtimes P \boxtimes K_\ell$ for some graph H with treewidth $O(k^3)$ where $\ell \leqslant O(k^2)$. Does there exist a function $\ell : \mathbb{N} \to \mathbb{N}$ and a universal constant C such that every k-planar graph is a subgraph of $H \boxtimes P \boxtimes K_{\ell(k)}$ for some graph H with treewidth at most C? Perhaps $C = 3$. Note that $C \geqslant 3$ even for planar graphs [14].

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