

# Minor-Closed Graph Classes with Bounded Layered Pathwidth

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## Abstract

We prove that a minor-closed class of graphs has bounded layered pathwidth if and only if some apex-forest is not in the class.

## 1 Introduction

Pathwidth and treewidth are graph parameters that respectively measure how similar a given graph is to a path or a tree. These parameters are of fundamental importance in structural graph theory, especially in Roberston and Seymour’s graph minors series. They also have numerous applications in algorithmic graph theory. Indeed, many NP-complete problems are solvable in polynomial time on graphs of bounded treewidth.

Recently, Dujmović, Morin, and Wood [18] introduced the notion of layered treewidth. Loosely speaking, a graph has bounded layered treewidth if it has a tree decomposition and a layering such that each bag of the tree decomposition contains a bounded number of vertices in each layer (defined formally below). This definition is interesting since several natural graph classes, such as planar graphs, that have unbounded treewidth have bounded layered treewidth. Bannister, Devanny, Dujmović, Eppstein, and Wood [1] introduced layered pathwidth, which is analogous to layered treewidth where the tree decomposition is required to be a path decomposition.

The purpose of this paper is to characterise the minor-closed graph classes with bounded layered pathwidth.

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## 1.1 Definitions

Before continuing, we define the above notions. A *tree decomposition* of a graph  $G$  is a collection  $(B_x \subseteq V(G) : x \in V(T))$  of subsets of  $V(G)$  (called *bags*) indexed by the nodes of a tree  $T$ , such that:

- (i) for every edge  $vw$  of  $G$ , some bag  $B_x$  contains both  $v$  and  $w$ , and
- (ii) for every vertex  $v$  of  $G$ , the set  $\{x \in V(T) : v \in B_x\}$  induces a non-empty connected subtree of  $T$ .

The *width* of a tree decomposition is the size of the largest bag minus 1. The *treewidth* of a graph  $G$ , denoted by  $\text{tw}(G)$ , is the minimum width of a tree decomposition of  $G$ . Tree decompositions were first introduced by Halin [23] and independently by Robertson and Seymour [30].

A *path decomposition* of a graph  $G$  is a tree decomposition in which the underlying tree is a path. The *pathwidth* of  $G$ , denoted by  $\text{pw}(G)$ , is the minimum width of a path decomposition of  $G$ .

A *layering* of a graph  $G$  is a partition  $(V_0, V_1, \dots, V_t)$  of  $V(G)$  such that for every edge  $vw \in E(G)$ , if  $v \in V_i$  and  $w \in V_j$  then  $|i - j| \leq 1$ . Each set  $V_i$  is called a *layer*. For example, for a vertex  $r$  of a connected graph  $G$ , if  $V_i$  is the set of vertices at distance  $i$  from  $r$ , then  $(V_0, V_1, \dots)$  is a layering of  $G$ , called the *bfs layering* of  $G$  starting from  $r$ .

Dujmović et al. [18] introduced the following definition. The *layered width* of a tree decomposition  $(B_x : x \in V(T))$  of a graph  $G$  is the minimum integer  $\ell$  such that, for some layering  $(V_0, V_1, \dots, V_t)$  of  $G$ , each bag  $B_x$  contains at most  $\ell$  vertices in each layer  $V_i$ . The *layered treewidth* of a graph  $G$  is the minimum layered width of a tree decomposition of  $G$ . Bannister et al. [1] defined the *layered pathwidth* of a graph  $G$  to be the minimum layered width of a path decomposition of  $G$ .

## 1.2 Examples and Applications

Several interesting graph classes have bounded layered treewidth (despite having unbounded treewidth). For example, Dujmović et al. [18] proved that every planar graph has layered treewidth at most 3, and more generally that every graph with Euler genus  $g$  has layered treewidth at most  $2g + 3$ . Note that layered treewidth and layered pathwidth are not minor-closed parameters (unlike treewidth and pathwidth). In fact, several graph classes that contain arbitrarily large clique minors have bounded layered treewidth or bounded layered pathwidth. For example, every graph that can be drawn on a surface of Euler genus  $g$  with at most  $k$  crossings per edge has layered treewidth at most  $2(2g + 3)(k + 1)$ . Even with  $g = 0$  and  $k = 1$ , this family includes graphs with arbitrarily large clique minors. Map graphs have similar behaviour [16].

Bannister et al. [1] identified the following natural graph classes that have bounded layered pathwidth (despite having unbounded pathwidth): every squaregraph has layered pathwidth 1; every bipartite outerplanar graph has layered pathwidth 1; every outerplanar graph has layered pathwidth at most 2; every Halin graph has layered pathwidth at most 2; and every unit disc graph with clique number  $k$  has layered pathwidth at most  $4k$ .

Part of the motivation for studying graphs with bounded layered treewidth or pathwidth is that such graphs have several desirable properties. For example, Norin proved that every  $n$ -vertex graph with layered treewidth  $k$  has treewidth less than  $2\sqrt{kn}$  (see [18]). This leads to a very simple proof of the Lipton-Tarjan separator theorem. A standard trick leads to an upper bound of  $11\sqrt{kn}$  on the pathwidth (see [16]).

Another application is to track- and queue-number: Dujmović et al. [18] proved that every  $n$ -vertex graph with layered treewidth  $k$  has track- and queue-number  $O(k \log n)$ . This leads to the best known bounds on the track- and queue-number of several natural graph classes. For graphs with bounded layered pathwidth, the dependence on  $n$  can be eliminated: Bannister et al. [1] proved that every graph with layered pathwidth  $k$  has track- and queue-number at most  $3k$ .

Graph colouring is another application area for layered treewidth. Esperet and Joret [20] proved that every graph with maximum degree  $\Delta$  and Euler genus  $g$  is (improperly) 3-colourable with bounded clustering, which means that each monochromatic component has size bounded by some function of  $\Delta$  and  $g$ . This resolved an old open problem even in the planar case ( $g = 0$ ). The clustering function proved by Esperet and Joret [20] is roughly  $O(\Delta^{32\Delta^{2g}})$ . While Esperet and Joret [20] made no effort to reduce this function, their method will not lead to a sub-exponential clustering bound. On the other hand, Liu and Wood [27] recently proved that every graph with layered treewidth  $k$  and maximum degree  $\Delta$  is 3-colourable with clustering  $O(k^{19}\Delta^{37})$ . In particular, every graph with Euler genus  $g$  and maximum degree  $\Delta$  is 3-colorable with clustering  $O(g^{19}\Delta^{37})$ . This greatly improves upon the clustering bound of Esperet and Joret [20]. Moreover, the proof by Liu and Wood [27] is relatively simple, avoiding many technicalities that arise when dealing with graph embeddings. This result highlights the utility of layered treewidth as a general tool.

### 1.3 Characterisations

We now turn to the question of characterising those minor-closed classes that have bounded treewidth. The key example is the  $n \times n$  grid graph, which has treewidth  $n$ . Indeed, Robertson and Seymour [31] proved that every graph with sufficiently large treewidth contains the  $n \times n$  grid as a minor. The next theorem follows since every planar graph is a minor of some grid graph. Several subsequent works have improved the bounds [7, 8, 15, 26, 32].

**Theorem 1** (Robertson and Seymour [31]). *A minor-closed class has bounded treewidth if and only if some planar graph is not in the class.*

An analogous result for pathwidth holds, where the complete binary tree is the key example (the analogue of grid graphs for treewidth). Let  $T_h$  be the complete binary tree of height  $h$ . It is well known and easily proved that  $\text{pw}(T_h) = \lceil h/2 \rceil$ , and every forest is a minor of some complete binary tree. Robertson and Seymour [29] proved the converse.

**Theorem 2** (Robertson and Seymour [29]). *A minor-closed class has bounded pathwidth if and only if some forest is not in the class.*

Note that Bienstock, Robertson, Seymour, and Thomas [2] proved the following quantitatively stronger result: for every forest  $T$  (with at least one edge) every graph containing no  $T$  minor has pathwidth at most  $|V(T)| - 2$ .

Now consider layered analogues of Theorems 1 and 2. A graph  $G$  is *apex* if  $G - v$  is planar for some vertex  $v$ . If  $G$  is the apex graph obtained from the  $n \times n$  grid by adding one dominant

vertex  $v$ , then  $G$  has layered treewidth at least  $\frac{n+1}{3}$  since every layering of  $G$  uses at most three layers. Dujmović et al. [18] proved the following converse.

**Theorem 3** (Dujmović et al. [18]). *A minor-closed class has bounded layered treewidth if and only if some apex graph is not in the class.*

Note that the proof of Theorem 3 uses the graph minor structure theorem and thus relies on Theorem 1.

A graph  $G$  is an *apex-forest* if  $G - v$  is a forest for some vertex  $v$ . The following theorem is the main result of this paper.

**Theorem 4.** *A minor-closed class has bounded layered pathwidth if and only if some apex-forest is not in the class.*

Theorem 4 is analogous to Theorem 3 for layered treewidth. However, unlike the proof of Theorem 3 which depends on Theorem 1, our proof of Theorem 4 does not depend on Theorem 2. In fact, Theorem 4 implies Theorem 2, as we now explain. Let  $T$  be a forest, and let  $G$  be a graph with no  $T$  minor. Let  $T^+$  be the apex-forest obtained from  $T$  by adding a dominant vertex  $v$ . Let  $G^+$  be the graph obtained from  $G$  by adding a dominant vertex  $x$ . Suppose for the sake of contradiction that  $G^+$  contains  $T^+$  as a minor. Since  $x$  is dominant in  $G^+$ , we may assume that  $\{x\}$  is the image in the  $T^+$ -model of some vertex  $w$  of  $T^+$ . Thus  $G = G^+ - x$  contains  $T^+ - w$  as a minor. Since  $v$  is dominant in  $T^+$ , the graph  $T^+ - w$  contains a subgraph isomorphic to  $T$  (even if  $w \neq v$ ). Thus  $G$  contains  $T$  as a minor, which is a contradiction. Hence  $G^+$  is  $H^+$ -minor-free. By Theorem 4,  $G^+$  has layered pathwidth at most  $c = c(T^+)$ . Since  $G^+$  has radius 1, at most three layers are used. Thus  $G^+$  and  $G$  have pathwidth less than  $3c$ .

DW: Do we need to define model? Do we need to define minor?

DW: We need to check carefully that the result of Dang [9] does not use Theorem 2.

Layered treewidth is closely related to the notion of ‘local treewidth’, which was first introduced by Eppstein [19] under the guise of the ‘treewidth-diameter’ property. A graph class  $\mathcal{G}$  has *bounded local treewidth* if there is a function  $f$  such that for every graph  $G$  in  $\mathcal{G}$ , for every vertex  $v$  of  $G$  and for every integer  $r \geq 0$ , the subgraph of  $G$  induced by the vertices at distance at most  $r$  from  $v$  has treewidth at most  $f(r)$ ; see [10, 11, 19, 21]. If  $f(r)$  is a linear function, then  $\mathcal{G}$  has *linear local treewidth*. Dujmović et al. [18] made the following easy observation: If every graph in some class  $\mathcal{G}$  has layered treewidth at most  $\ell$ , then  $\mathcal{G}$  has linear local treewidth with  $f(r) = \ell(2r + 1) - 1$ . We define local pathwidth similarly. A graph class  $\mathcal{G}$  has *bounded local pathwidth* if there is a function  $f$  such that for every graph  $G$  in  $\mathcal{G}$ , for every vertex  $v$  of  $G$  and for every integer  $r \geq 0$ , the subgraph of  $G$  induced by the vertices at distance at most  $r$  from  $v$  has pathwidth at most  $f(r)$ . The observation of Dujmović et al. [18] extends to the setting of local pathwidth: If every graph in some class  $\mathcal{G}$  has layered pathwidth at most  $\ell$ , then  $\mathcal{G}$  has linear local pathwidth with  $f(r) = \ell(2r + 1) - 1$ .

Theorem 4 is extended to capture local pathwidth by the following theorem, which also provides a structural description in terms of a tree decomposition with certain properties. If  $T$  is a tree indexing a tree decomposition of a graph  $G$ , then for each vertex  $v$  of  $G$ , let  $T[v]$

denote the subtree of  $T$  induced by those nodes corresponding to bags that contain  $v$ . Thus  $T[v]$  is non-empty and connected.

**Theorem 5.** *The following are equivalent for a minor-closed class  $\mathcal{G}$ :*

- (1) *some apex-forest graph is not in  $\mathcal{G}$ ,*
- (2)  *$\mathcal{G}$  has bounded local pathwidth,*
- (3)  *$\mathcal{G}$  has linear local pathwidth.*
- (4)  *$\mathcal{G}$  has bounded layered pathwidth,*
- (5) *each graph  $G \in \mathcal{G}$  has a tree decomposition  $T$  of bounded width, such that  $T[v]$  has bounded pathwidth for each vertex  $v$  of  $G$ ,*

Here is some intuition about (5). Suppose that  $\mathcal{G}$  excludes some apex-forest graph as a minor. Since every apex-forest graph is planar, by Theorem 1, the graphs in  $\mathcal{G}$  have bounded treewidth. Thus we should expect that the tree decompositions in (5) have bounded width. Moreover, if  $\mathcal{G}$  has bounded layered pathwidth, then  $G[N(v)]$  has bounded pathwidth for each vertex  $v$  in each graph  $G \in \mathcal{G}$ . Property (5) takes this idea further, and says that each graph  $G \in \mathcal{G}$  has a tree decomposition indexed by  $T$  such that for each vertex  $v$  in  $G$ , the subtree  $T[v]$  has bounded pathwidth, which implies that  $G[N(v)]$  has bounded pathwidth (since the width of the tree decomposition is bounded).

In Section 2 we prove  $(5) \implies (4) \implies (3) \implies (2) \implies (1)$ . In Section 3 we close the loop by proving  $(1) \implies (5)$ . Throughout the proof we use the following ‘universal’ apex-forest graph. Let  $Q_k$  be the graph obtained from the complete binary tree  $T_k$  by adding one dominant vertex. Note that  $\text{pw}(Q_k) = \lceil k/2 \rceil + 1$  and the layered pathwidth of  $Q_k$  is  $\Omega(k)$ , since every layering of  $Q_k$  uses at most three layers. Since every forest is a minor of some complete binary tree, every apex-forest graph is a minor of some  $Q_k$ .

## 2 Downward Implications

We start with a few simple but useful lemmas.

**Lemma 6.** *Let  $T_1$  and  $T_2$  be subtrees of a tree  $T$ , such that  $T = T_1 \cup T_2$ . Then*

$$\text{pw}(T) + 1 \leq (\text{pw}(T_1) + 1) + (\text{pw}(T_2) + 1) .$$

*Proof.* Let  $B_1, \dots, B_s$  be a path decomposition of  $T_1$  with bag size at most  $\text{pw}(T_1) + 1$ . Each component of  $T - V(T_1)$  is contained in  $T_2$  and therefore has a path decomposition with bag size at most  $\text{pw}(T_2) + 1$ . For each such component  $J$  of  $T - V(T_1)$ , there is exactly one vertex  $v$  in  $T_1$  adjacent to some vertex in  $J$  (otherwise  $T$  would contain a cycle consisting of two edges between a path in  $T_1$  and a path in  $J$ ). Say  $v$  is in bag  $B_i$ . We say  $J$  *attaches* at  $v$  and at  $B_i$ . By doubling bags in the path decomposition of  $T_1$ , we may assume that distinct components of  $T - V(T_1)$  attach at distinct  $B_i$ . For each component  $J$  of  $T - V(T_1)$ , if  $D_1, \dots, D_t$  is a path decomposition of  $J$  with bag size at most  $\text{pw}(T_2) + 1$ , then replace  $B_i$  by  $B_i \cup D_1, \dots, B_i \cup D_t$ . We obtain a path decomposition of  $T$  with bag size at most  $(\text{pw}(T_1) + 1) + (\text{pw}(T_2) + 1)$ . The result follows.  $\square$

**Corollary 7.** *Let  $T_1, \dots, T_k$  be subtrees of a tree  $T$ , such that  $T = T_1 \cup \dots \cup T_k$ . Then*

$$\text{pw}(T) + 1 \leq \sum_{i=1}^k (\text{pw}(T_i) + 1) .$$

*Proof that (2) implies (1).* Assume that  $\mathcal{G}$  has bounded local pathwidth. Thus, for some constant  $k$ , for every graph  $G \in \mathcal{G}$  and every vertex  $v$  of  $G$ , we have  $\text{pw}(G[N[v]]) \leq k$ . Suppose on the contrary that  $G := Q_{2k}$  is in  $\mathcal{G}$ . Let  $v$  be the dominant vertex in  $Q_{2k}$ . Thus

$$k + 1 = \text{pw}(Q_{2k}) = \text{pw}(G[N[v]]) \leq k .$$

This contradiction proves that  $Q_{2k}$  is not in  $\mathcal{G}$ . □

It is immediate that (3) implies (2).

*Proof that (4) implies (3).* Assume that every graph  $G \in \mathcal{G}$  has layered pathwidth at most  $k$ . Let  $B_1, \dots, B_s$  be a path decomposition of  $G$  with layered width  $k$ , with respect to layering  $V_0, V_1, \dots, V_t$ . Let  $v$  be a vertex in  $V_i$ . Let  $r$  be a positive integer. Let  $H$  be the subgraph of  $G$  induced by the vertices at distance at most  $r$  from  $v$ . Thus  $V(H) \subseteq V_{i-r} \cup V_{i-r+1} \cup \dots \cup V_{i+r}$ . Each bag  $B_j$  contains at most  $k$  vertices in each layer. Hence  $B_1 \cap V(H), \dots, B_s \cap V(H)$  is a path decomposition of  $H$  with at most  $(2r + 1)k$  vertices in each bag. Therefore  $G$  has linear local pathwidth with binding function  $f(r) = (2r + 1)k - 1$ . □

*Proof that (5) implies (4).* Consider a graph  $G$  with a tree decomposition  $T$  of width  $w$ , such that  $\text{pw}(T[v]) \leq p$  for each vertex  $v$  of  $G$ . Since adding edges does not decrease the layered pathwidth, we may add edges to  $G$  between two non-adjacent vertices in the same bag of  $T$ . Now each bag is a clique, and  $G$  is chordal with maximum clique size  $w + 1$ . Let  $V_0, V_1, \dots, V_t$  be a bfs layering in  $G$ . That is,  $V_i$  is the set of vertices in  $G$  at distance  $i$  from some fixed vertex  $x$  of  $G$ . In particular,  $V_0 = \{x\}$ .

Consider a component  $H$  of  $G[V_i]$  for some  $i \geq 1$ . Let  $C_H$  be the set of vertices in  $V_{i-1}$  adjacent to some vertex in  $H$ . Then  $C_H$  is a clique of size at most  $w$  (see [17, 25]), called the *parent clique* of  $H$ . A bag containing  $w \in C_H$  is said to *belong* to  $w$ . Define  $X_H := \cup_{w \in C_H} T[w]$ . Since  $C_H$  is a clique, which is contained in a single bag of  $T$ , the subtrees whose union is  $X_H$  have a node in common. Thus  $X_H$  is a (connected) subtree of  $T$ . Moreover,  $X_H$  is the union of at most  $w$  subtrees, each with pathwidth at most  $p$ . Thus  $\text{pw}(X_H) + 1 \leq w(p + 1)$  by Corollary 7.

We now prove that  $X_H$  indexes a tree decomposition of  $G[V(H) \cup C_H]$  (using the same bags as  $T$  restricted to  $V(H) \cup C_H$ ). We first prove condition (ii). For a vertex  $v$  of  $C_H$ , the set of bags in  $X_H$  that contain  $v$  is precisely  $T[v]$ , which is non-empty and connected, by assumption. Now, consider a vertex  $v$  in  $H$ . Let  $w$  be the neighbour of  $v$  on a shortest  $vx$ -path in  $G$ . Thus  $w$  is in  $C_H$ . Since  $vw$  is an edge,  $v$  and  $w$  appear in a common bag of  $T$ , which is also in  $X_H$  (since that bag contains  $w$ ). Hence  $X_H[v]$  is non-empty. We now prove that  $X_H[v]$  is connected. Let  $B_1$  and  $B_2$  be distinct bags of  $X_H$  containing  $v$ . Let  $P$  be the  $B_1 B_2$ -path in  $T$ . Since  $T[v]$  is connected,  $v$  is in each bag in  $P$ . To conclude that  $X_H[v]$  is connected, it remains to prove that  $P \subseteq X_H$ . By construction, some vertex  $w_1 \in B_1 \cap C_H$  and some vertex  $w_2 \in B_2 \cap C_H$ . Since  $w_1$  and  $w_2$  are adjacent, each bag in  $P$  contains  $w_1$  or  $w_2$ . Hence  $P \subseteq X_H$  and  $X_H[v]$  is connected. This proves condition (ii). Now we prove condition (i). Since  $C_H$  is contained in some bag of  $X_H$ , condition (i) holds

for each edge with endpoints in  $C_H$ . For each edge  $vw$  with  $v \in V(H)$  and  $w \in C_H$ ,  $v$  and  $w$  are in a common bag of  $T$ , which is also in  $X_H$  (since that bag contains  $w$ ), as desired. Finally, consider an edge  $uv$  with  $u, v \in V(H)$ . Suppose on the contrary that  $u$  and  $v$  have no common neighbour in  $C_H$ . By construction,  $u$  has a neighbour  $w_1$  in  $C_H$ , and  $v$  has a neighbour  $w_2$  in  $C_H$ . Thus  $w_1 \neq w_2$ . Since  $C_H$  is a clique,  $w_1$  and  $w_2$  are adjacent. Since  $uw_2 \notin E(G)$  and  $vw_1 \notin E(G)$ , the 4-cycle  $(u, w_1, w_2, v)$  is chordless, and  $G$  is not chordal, which is a contradiction. Hence  $u$  and  $v$  have a common neighbour  $w$  in  $C_H$ . Thus  $\{u, v, w\}$  is a triangle in  $G$ , which is in a common bag of  $T$ , and therefore in a common bag of  $X_H$ , implying that  $u$  and  $v$  are in a common bag of  $X_H$ . This proves condition (i) in the definition of tree decomposition. Therefore  $X_H$  indexes a tree decomposition of  $G[V(H) \cup C_H]$ . By construction, it has width at most  $w$ .

Since  $\text{pw}(X_H) + 1 \leq w(p+1)$  and  $X_H$  indexes a tree decomposition of  $G[V(H) \cup C_H]$  with width at most  $w$ , it follows that  $\text{pw}(V(H) \cup C_H) \leq (w(p+1) + 1)(w+1) - 1$ .

GJ: This implicitly uses a lemma along the lines "if treewidth is  $k$  and pathwidth of tree indexing tree decomposition is  $p$  then pathwidth is at most  $pk$ ", this should be explained

We now construct a path decomposition of  $G$  with layered width at most  $w(p+1)(w+1)$  with respect to layering  $V_0, V_1, \dots, V_t$ . Let  $G_i := G[V_0 \cup V_1 \cup \dots \cup V_i]$ . We now prove, by induction on  $i$ , that  $G_i$  has a path decomposition with layered width at most  $w(p+1)(w+1)$  with respect to layering  $V_0, V_1, \dots, V_i$ . This claim is trivial for  $i = 0$ . Now assume that  $B_1, \dots, B_q$  is a path decomposition of  $G_{i-1}$  with layered width at most  $w(p+1)(w+1)$  with respect to layering  $V_0, V_1, \dots, V_{i-1}$ . For each component  $H$  of  $G[V_i]$ , there is a bag  $B_j$  that contains  $C_H$ ; pick one such bag and call it the *parent bag* of  $H$ . By doubling the bag, we may assume that distinct components of  $G[V_i]$  have distinct parents bags. Now, for each component  $H$  of  $G[V_i]$  with parent bag  $B_j$ , if  $D_1, \dots, D_r$  is a path decomposition of  $G[V(H) \cup C_H]$  with width  $w(p+1)(w+1) - 1$ , then replace  $B_j$  by  $B_j \cup D_1, \dots, B_j \cup D_r$ . Doing this for each component of  $G[V_i]$  produces a path decomposition of  $G_i$  with layered width at most  $w(p+1)(w+1)$  with respect to layering  $V_0, V_1, \dots, V_i$ . In particular, we obtain a path decomposition of  $G$  with layered width at most  $w(p+1)(w+1)$  with respect to layering  $V_0, V_1, \dots, V_t$ .  $\square$

### 3 Proof that (1) implies (5)

Say that a tree decomposition of a graph  $G$  is  $(w, p)$ -good if its width is at most  $w$  and, for every  $v \in V(G)$ , the subtree  $T[v]$  has pathwidth at most  $p$ . The goal of this section is to show that if a graph  $G$  excludes some apex-forest graph  $H$  as a minor, then  $G$  has  $(w, p)$ -good tree decomposition for some  $w = w(H)$  and  $p = p(H)$ . Since every apex-forest graph is a minor of some  $Q_k$ , it suffices to prove this result for  $H = Q_k$ , in which case we denote  $w = w(k)$  and  $p = p(k)$ .

We will be working with two related trees  $S$  and  $T$  and one graph  $G$ . To help the reader keep track of things we will use variables  $a, b$ , and  $c$  as names for *nodes* of  $S$  and  $T$  and variables  $v, x, y$ , and  $z$  to refer to *vertices* of  $G$ .

We now give an outline of the proof. We make use of a recent result of Dang which implies that any 3-connected graph  $G$  with no  $Q_k$  minor has pathwidth at most  $w = w(k)$ . Thus, in this case  $G$  has a  $(w, 1)$ -good tree decomposition. Next we deal with cut vertices by showing

that, if each 2-connected component of a graph  $G$  has a  $(w, p)$ -good tree decomposition, then  $G$  has a  $(w, p + 1)$ -good tree decomposition.

Therefore, the main difficulty is to show that every 2-connected graph  $G$  with no  $Q_k$  minor has a  $(w, p)$ -good tree decomposition  $(B_a : a \in V(T))$ . By the results of Bienstock et al. [2] described in the introduction, if  $\text{pw}(T[v]) > 2^{h+1} - 3$  for some  $v \in V(G)$  then  $T[v]$  contains a  $T_h$  minor. We then show that this  $T_h$  minor in  $T[v]$ , for sufficiently large  $h$ , implies that  $G$  has a  $Q_k$  minor in which  $v$  plays the role of the apex vertex.

To construct the tree decomposition  $(B_a : a \in V(T))$  we use two tools: An SPQR-tree,  $S$ , represents  $G$  as a collection of subgraphs (S- and R-nodes) that are joined at 2-vertex cutsets (P-nodes). These subgraphs consist of cycles (S-nodes) and 3-connected graphs (R-nodes). Cycles have pathwidth 2 and, by the result of Dang discussed above, the 3-connected graphs have pathwidth at most  $w = w(k)$ . Replacing the S- and R-nodes of the SPQR-tree with these path decompositions produces the tree  $T$  in our tree decomposition.

To show that this tree decomposition is  $(w, p)$ -good, we first show that if  $T[v]$  contains a subdivision of a sufficiently large complete binary tree then the SPQR-tree  $S$  also contains a subdivision of a large complete binary tree all of whose nodes have subgraphs that contain  $v$ . Using this large binary tree in  $S$  we then piece together a subgraph of  $G$  that has a  $Q_k$  minor in which  $v$  is the apex vertex.

### 3.1 Dang's Result

First we show how a result of Dang [9, Theorem 1.1.5] implies that any 3-connected graph with no  $Q_k$  minor has pathwidth at most  $w = w(k)$ .

**Theorem 8** (Dang 2018). *Let  $P'$  be a graph with two distinct vertices  $u_1$  and  $u_2$  such that  $P' - \{u_1, u_2\}$  is a forest,  $Q'$  be a graph with a vertex  $v$  such that  $Q' \setminus \{v\}$  is an outerplanar graph, and  $R'$  be a tree with a cycle going through its leaves in order from the leftmost leaf to the rightmost leaf so that  $R'$  is planar. Then there exists a number  $w = w(P', Q', R')$  such that every 3-connected graph of pathwidth at least  $w$  has a  $P'$ ,  $Q'$ , or  $R'$  minor.*

To use Theorem 8 we need a small helper lemma. For every  $k \geq 0$ , let  $T_k^+$  be the complete binary tree  $T_k$  of height  $k$  plus a vertex adjacent to the leaves.

**Lemma 9.** *For every integer  $k \geq 0$ ,  $T_{2k}^+$  contains  $Q_k$  as a minor.*

*Proof.* The statement is immediate for  $k = 0$ . For  $k \geq 1$ , partition the edges of  $T_{2k}^+$  into the tree  $T_{2k}$  and the remaining edges, which form a star centered at some vertex  $v$ . Let  $a_1, a_2, a_3, a_4$  be the grandchildren of the root of  $T_{2k}$  ordered from left to right. Contract the entire subtree comprised of the subtree rooted at  $a_2$ , the subtree rooted at  $a_3$  and the path from  $a_2$  to  $a_3$ . Applying the same procedure recursively on  $T_{2(k-1)}^+$  rooted at  $a_1$  and  $T_{2(k-1)}^+$  rooted at  $a_4$  produces  $Q_k$ , as can be easily verified by induction.  $\square$

**Corollary 10.** *There exists a number  $w = w(k)$  such that every 3-connected graph of pathwidth at least  $w$  has a  $Q_k$  minor.*

*Proof.* Choose  $P'$  to be  $T_k$  plus two universal vertices  $u_1$  and  $u_2$  adjacent to every vertex of  $T_k$ . Choose  $Q'$  to be the “universal” outerplanar graph  $\nabla_k$ , whose weak dual is a complete



binary tree of height  $k$ , plus one additional apex vertex  $v$ . Choose  $R'$  to be  $T_{2k+1}$  plus a cycle on its leaves.

Clearly,  $P'$  contains  $Q_k$  as a minor since  $P' - \{u_2\}$  is isomorphic to  $Q_k$ .  $Q'$  also contains  $Q_k$  as a minor because  $\nabla_k$  contains a complete binary tree of height  $k$  as a subgraph. Finally,  $R'$  also contains a  $Q_k$  minor: Contract the cycle, then we have a complete binary tree of height  $2k$  plus an apex vertex linked to its leaves, which contains  $Q_k$  as a minor by Lemma 9. Theorem 8 implies that there exists  $w = w(k)$  such that every 3-connected graph with pathwidth at least  $w$  contains at least one of  $P'$ ,  $Q'$ , or  $R'$  as a minor and therefore contains a  $Q_k$  minor.  $\square$

### 3.2 Dealing with Cut Vertices

**Lemma 11.** *Let  $G$  be a graph, each of whose 2-connected components has a  $(w, p)$ -good tree decomposition. Then  $G$  has a  $(w, p + 1)$ -good tree decomposition.*

*Proof.* Let  $C_1, \dots, C_r$  be  $G$ 's 2-connected components and let  $T_1, \dots, T_r$  be the trees used in  $(w, p)$ -good tree decompositions of these components.

We create a tree decomposition of  $G$  as follows: For each cut vertex  $v$  in  $G$ , we introduce a new tree node  $a_v$  with  $B_{a_v} = \{v\}$ . In each 2-connected component  $C_i$  that contains  $v$ , the tree decomposition  $(B_a : a \in V(T_i))$  has at least one node  $a$  such that  $v \in B_a$ , so we make  $a_v$  adjacent to exactly one such node for each  $C_i$ .

It is straightforward to verify that this defines a tree decomposition of  $G$  and we now argue this decomposition is  $(w, p + 1)$ -good. The resulting tree decomposition of  $G$  has width at most  $w$ . For each cut vertex  $v \in V(G)$ , the subtree  $T[v]$  is composed of some number of subtrees, each adjacent to  $a_v$  and each having a path decomposition of width at most  $p$ . We obtain a path decomposition of  $T[v]$  by concatenating the path decompositions of each subtree and adding  $v$  to every bag of the resulting path decomposition. The resulting path decomposition of  $T[v]$  has width at most  $p + 1$ .  $\square$

### 3.3 SPQR-Trees

In this section we quickly review SPQR-trees, a graph decomposition tool formalized by Di Battista and Tamassia [12, 13, 14].

Let  $G$  be a 2-connected graph. A *SPQR-tree*  $S$  of  $G$  is a tree in which each node  $a \in V(S)$  is associated with a minor  $H_a$  of  $G$ . For each node  $a$  of  $S$  each edge  $xy \in E(H_a)$  is classified either as a *virtual edge* or a *real edge*. The SPQR-tree  $S$  is defined recursively as follows (see Fig. 1):<sup>1</sup>

1. If  $G$  is a cycle, then  $S$  consists of a single node  $a$  (an S-node) in which  $H_a = G$  and all edges of  $H_a$  are real.
2. If  $G$  is 3-connected, then  $S$  consists of a single node  $a$  (an R-node) in which  $H_a = G$  and all edges of  $H_a$  are real.

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<sup>1</sup>This definition includes P-nodes consisting of only two virtual edges, which some works exclude because they are unnecessary. However, their inclusion in our definition makes some of our analysis simpler.

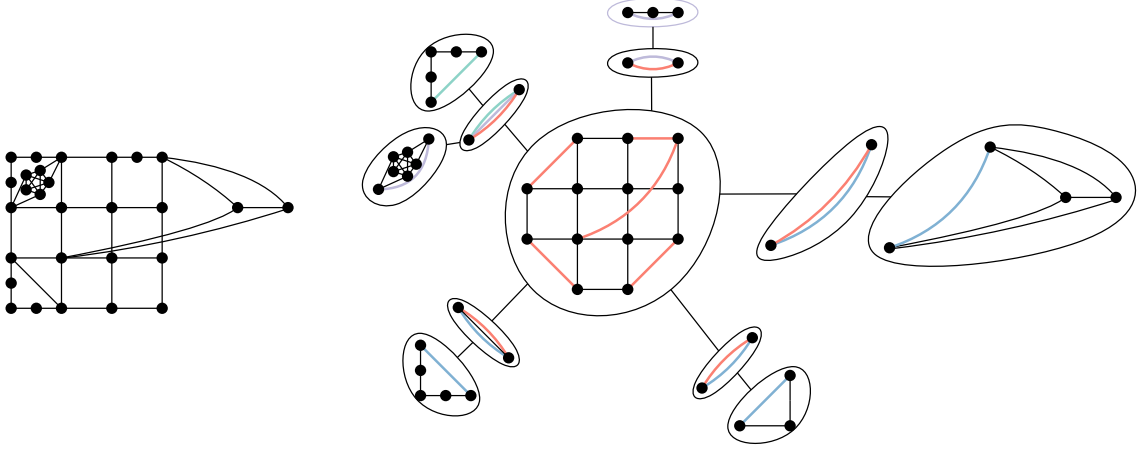


Figure 1: A graph and its SPQR-tree.

3. Otherwise  $G$  has a cutset  $\{x, y\}$  such that  $x$  and  $y$  each have degree at least 3. Then let  $C_1, \dots, C_r$ ,  $r \geq 2$ , be the connected components of  $G - \{x, y\}$ . For each  $i \in \{1, \dots, r\}$ , let  $\tilde{G}_i$  be  $G[V(C_i) \cup \{x, y\}]$  along with the additional edge  $xy$ . Note that (because of the inclusion of  $xy$ ) each  $\tilde{G}_i$  is 2-connected, so each has an SPQR-tree  $S_i$ . Then the SPQR-tree for  $G$  is obtained by creating a node  $a$  (a P-node) with  $H_a$  being a dipole graph with vertices  $x$  and  $y$  and having  $k$  virtual edges joining  $x$  and  $y$ . In addition to these virtual edges,  $H_a$  contains the real edge  $xy$  if  $xy \in E(G)$ . The construction and the fact that  $xy$  is an edge in each  $\tilde{G}_i$  imply that there exists exactly one node  $a_i$  in  $S_i$  such that  $xy$  is a real edge in  $H_{a_i}$ . To complete  $S$  we make  $a$  adjacent to each of  $a_1, \dots, a_k$  and we make the edge  $xy$  a virtual edge in each of  $H_{a_1}, \dots, H_{a_k}$ .

Let  $S$  be the SPQR-tree of a 2-connected graph  $G$ . For each node  $a$  of  $S$ , we let  $E_r(H_a)$  denote the set of real edges in  $H_a$  and  $E_v(H_a)$  denote the multiset of virtual edges in  $H_a$ . For a connected subtree  $S'$  of  $S$  we define  $G[S']$  as the subgraph of  $G$  whose vertex set is  $V(G[S']) = \bigcup_{a \in V(S')} V(H_a)$  and whose edge set is  $E(G[S']) = \bigcup_{a \in V(S')} E_r(H_a)$ . For a vertex  $v \in V(G)$ , we denote the subtree of  $S$  induced by  $v$  as  $S[v] = S[\{a \in V(S) : v \in V(H_a)\}]$ . We make use of the following properties of  $S$ :

1. Every R-node and S-node is adjacent only to P-nodes and no two P-nodes are adjacent.
2. The degree of any node  $a$  is equal to the number of virtual edges in  $H_a$ .
3. For every vertex  $v \in V(G)$ ,  $S[v]$  is connected.
4. If  $a$  is an R-node or S-node, then  $H_a$  is a simple graph, i.e.,  $H_a$  does not contain any parallel edges.
5. If a P-node  $a$  has degree 2 and both its neighbors are S-nodes then  $H_a$  has a real edge.
6. For any node  $a$  of  $S$  and any component  $S'$  of  $S - \{a\}$ ,  $G[S']$  is connected.
7. For each  $xy \in E(G)$  there is exactly one node  $a$  of  $S$  for which  $xy$  is a real edge in  $H_a$ .

### 3.4 The Good Tree Decomposition

To obtain our good tree decomposition  $(B_a : a \in V(T))$  of a 2-connected graph  $G$  we start with the SPQR-tree  $S$  for  $G$ . For each R-node or S-node  $a$  of  $S$  we take a minimum-width path decomposition  $(B_c : c \in V(P_a))$  of  $H_a$ . We say that the node  $a$  *generates* the nodes in the path  $P_a$  and that each node in  $P_a$  is *generated by*  $a$ .

Each S- or R-node  $a$  is adjacent to some set of P-nodes in  $S$ . For each such P-node  $b$  whose dipole graph  $H_b$  has vertices  $x$  and  $y$ , the edge  $xy$  is a (virtual) edge in  $H_a$  and therefore  $x$  and  $y$  appear in some common bag  $B_c$  with  $c \in V(P_a)$ . We make  $c$  and  $b$  adjacent in  $T$ . This defines the tree  $T$  in the tree decomposition. The contents of  $T$ 's bags are obvious. Each P-node  $a$  of  $S$  becomes a node in  $T$  whose bag contains only the two vertices of  $H_a$ . Every node  $a$  in  $T$  that is generated by an S- or R-node  $a'$  of  $S$  is a node in some path decomposition of  $H_{a'}$  and already has an associated bag  $B_a$  that it inherits from this path decomposition.

It is straightforward to verify that  $(B_a : a \in V(T))$  is indeed a tree decomposition of  $G$ : For any particular vertex  $v \in V(G)$  the connectivity of the subtree  $T[v]$  follows from Property 3 of SPQR-trees and the equivalent property for the path decompositions that include  $v$ . Each edge  $xy$  of  $G$  appears as an edge in  $H_a$  for at least one node  $a$  of  $S$  and therefore  $x$  and  $y$  appear in a common bag in the path decomposition of  $H_a$ .

Each bag  $B_a$  of  $(B_a : a \in V(T))$  either has size in  $\{2, 3\}$  (when  $a$  is generated by a P-node or an S-node) or it has size at most  $w(k) + 1$  where  $w(k)$  is the function in Corollary 10 (when  $a$  is generated by an R-node). Thus,  $(B_a : a \in V(T))$  is a tree decomposition of  $G$  whose width is upper bounded by a function of  $k$ . What remains is to show that, for every  $v \in V(G)$ ,  $T[v]$  has pathwidth that is upper bounded by a function of  $k$ .

In the remainder of this section, we fix  $G$  to be a 2-connected graph,  $S$  to be the SPQR-tree of  $G$  and  $(B_a : a \in V(T))$  to be a tree decomposition obtained using the procedure described above.

**Lemma 12.** *For all integers  $h \geq 1$ , if  $T[v]$  has pathwidth greater than  $2^{2h+1} - 3$ , then  $S[v]$  contains a subdivision of  $T_h$ .*

*Proof.* In a rooted binary tree we call the root and every degree 3 node *branching nodes*. Every branching node and every leaf is a *distinctive node*. We will use the convention that all binary trees are ordered, possibly arbitrarily, so that we can distinguish between the left and right child of a branching node. For a node  $a$  in the rooted binary tree  $T$ , we denote by  $\hat{a}$  the subtree rooted at  $a$ , i.e., the subtree of  $T$  containing all nodes that have  $a$  as an ancestor, including  $a$  itself.

Recall, from the result of Bienstock et al. [2] discussed in the introduction, that if  $T[v]$  has pathwidth greater than  $2^{2h+1} - 3$  then  $T$  contains a subdivision  $T'$  of  $T_{2h}$ . Note that  $T'$  does not immediately imply the existence of  $T_h$  in  $S[v]$  since two or more distinctive nodes of  $T'$  may have been generated by the same node of  $S$ . Label each node of  $T'$  with the node of  $S$  that generated it. Recall that each node  $a$  in  $S$  generates a path in  $T$  so, if we consider a maximal subset of nodes of  $T'$  with a common label then this subset induces a path in  $T'$ .

We claim that  $T'$  contains a subdivision  $T''$  of  $T_h$  such that each of the distinctive nodes of  $T''$  has a unique label. We establish this claim by induction on  $h$ : If  $h = 0$  then the claim is trivial. Otherwise, let  $a$  be the root of  $T'$  and let  $a'$  and  $a''$  be the highest branching nodes in the left and right subtrees of  $\hat{a}$ , respectively. Let  $a_1$  and  $a_2$  be the highest distinctive nodes

in the left and right subtrees of  $\hat{a}'$ , respectively, and let  $a_3$  and  $a_4$  be the highest distinctive nodes in the left and right subtrees of  $\hat{a}''$ , respectively. Since each label induces a path in  $T'$ , at least one of  $\{a_1, a_2\}$ , say  $a_1$ , and at least one of  $\{a_3, a_4\}$ , say  $a_4$ , does not have the same label as  $a$ . Furthermore, since  $a_1$  and  $a_4$  are separated by  $a$ , the set of labels of nodes in  $\hat{a}_1$  is disjoint from the set of labels of nodes in  $\hat{a}_4$ . Applying induction on  $\hat{a}_1$  and  $\hat{a}_4$  yields two subdivisions of  $T_{h-1}$  in which each distinctive node has a unique label. Connecting these two subdivisions with the unique path from  $a_1$  to  $a_4$  yields the desired subdivision of  $T_h$  in which each distinctive node has a unique label.

Now, each distinctive node in  $T''$  has a unique label and therefore corresponds to a unique node of  $S$ . Thus, if we contract all nodes of  $T''$  sharing a common label, then we obtain a subtree  $T'''$  of  $S[v]$  that is a subdivision of  $T_h$ .  $\square$

Thus far we have established that if  $T[v]$  has sufficiently high pathwidth, then  $S[v]$  contains a subdivision of a large complete binary tree.

**Lemma 13.** *If  $S[v]$  contains a subdivision of  $T_{7(k+1)}$  then  $G$  contains a  $Q_k$  minor.*

*Proof.* First we note that if  $S[v]$  contains a subdivision of  $T_{7(k+1)}$  then  $S[v]$  contains a subdivision  $T'$  of  $T_{k+1}$  such that the path between any pair of distinctive nodes in  $T'$  has length at least 7.

It is convenient to work with a simplified SPQR-tree  $S'$  and graph  $G'$  obtained by repeating the following operation exhaustively: Consider some edge  $ab$  of  $S$  with  $a \in V(T')$  and  $b \notin V(T')$ . The edge  $ab$  is associated with some virtual edge  $xy$  in  $H_a$ . In  $S'$  we replace the virtual edge  $xy$  in  $H_a$  with a real edge. At the same time we remove the maximal subtree  $\hat{b}$  of  $S$  that contains  $b$  and not  $a$ . By Property 6 of SPQR trees, in  $G'$  this operation is equivalent to contracting all the real edges in  $\bigcup_{c \in V(\hat{b})} E_r(H_c)$  and removing any resulting parallel edges. Since the resulting graph  $G'$  is a minor of  $G$ , this operation is safe in the sense that the existence of a  $Q_k$  minor in  $G'$  implies the existence of a  $Q_k$  minor in  $G$ .

With this simplification, the tree  $T'$  is the SPQR-tree for the graph  $G'$  and every virtual edge is incident to  $v$ . We now turn our efforts to finding the  $Q_k$  minor in  $G'$ . Recall that  $Q_k$  is the union of a complete binary tree  $T_k$  and an apex vertex. We begin by finding a subdivision  $T''$  of  $T_{k+1}$  in  $G'$ . In this subdivision, each edge of  $T_{k+1}$  that joins a node to its left child is represented by a path  $P_{\mu\nu}$  joining a branching node  $\mu$  to a distinctive node  $\nu$ . We show that  $G'$  contains a path from  $v$  to some *anchor node*  $\eta$  of  $P_{\mu\nu}$  with  $\eta \neq \nu$ , which is vertex disjoint from  $T''$  except for  $\eta$ . Furthermore, except for their common endpoint  $v$ , all of these paths are disjoint. The union of  $T''$  and these paths contains a  $Q_k$  minor since contracting the path from each anchor node to its closest ancestor branching node produces  $Q_k$ . See Fig. 2.

Let  $a$  be a branching node of  $T'$  and let  $b$  be the nearest distinctive node in one of  $a$ 's two subtrees. Consider the path  $a = c_1, c_2, \dots, c_r = b$  in  $T'$ . For each  $i \in \{1, \dots, r-1\}$ , the edge  $c_i c_{i+1}$  is associated with a cutset  $\{v, x_i\}$  in  $G'$  and  $vx_i$  is a virtual edge in  $H_{c_i}$  and  $H_{c_{i+1}}$ . Note that this implies that, for each  $i \in \{2, \dots, r\}$ ,  $H_{c_i}$  contains both vertices  $x_i$  and  $x_{i-1}$ .

Refer to Fig. 3. We claim that, for each  $i \in \{2, \dots, r-1\}$ ,  $H_{c_i}$  contains a path  $P_i$  from  $x_{i-1}$  to  $x_i$  that does not contain  $v$ . When  $c_i$  is a P-node, this claim is trivial since, in this case,  $x_{i-1} = x_i$ . The case in which  $c_i$  is an S-node or R-node is also easy: In these cases  $H_{c_i}$  is 2-connected, therefore there is a path from  $x_{i-1}$  to  $x_i$  that avoids  $v$ . Now note that the paths  $P_1, \dots, P_{r-1}$  are disjoint, except for each of the common endpoints  $x_i$  where  $P_i$  ends and  $P_{i+1}$

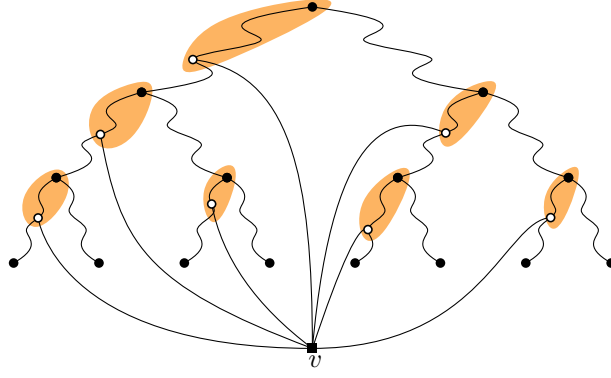


Figure 2: Finding a  $Q_k$  minor in the proof of Lemma 13. Distinctive nodes are indicated by black disks and anchor nodes by white circles.

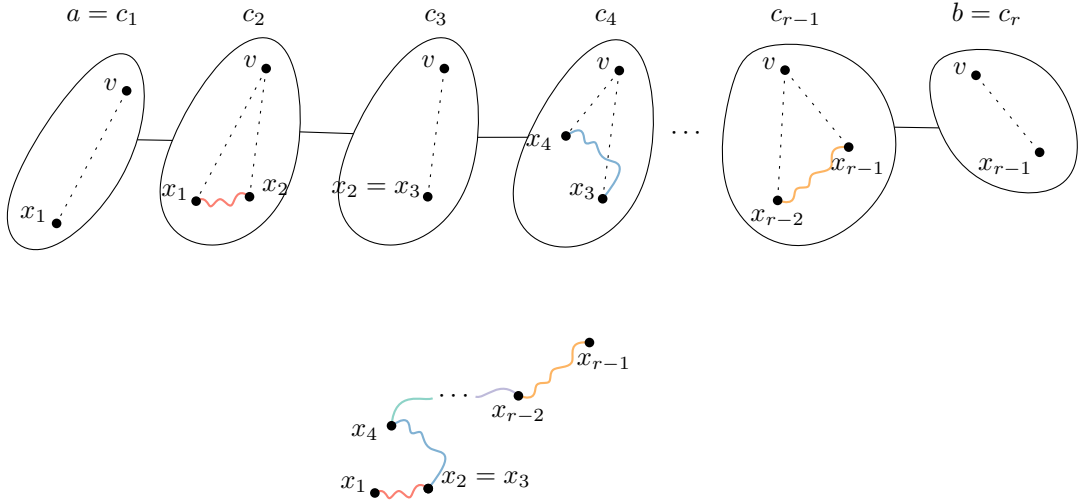


Figure 3: Finding a path connecting a vertex of  $H_a$  to a vertex of  $H_b$ .

begins. This is because each  $\{v, x_i\}$  is a cutset of  $G'$  that separates  $\bigcup_{j=1}^i V(H_{c_j}) \setminus \{v, x_i\}$  from  $\bigcup_{j=i+1}^r V(H_{c_j}) \setminus \{v, x_i\}$ . By concatenating  $P_2, \dots, P_{r-1}$  we obtain a path  $P_{ab}$  from  $x_1 \in V(H_a)$  to  $x_{r-1} \in V(H_b)$  that we call the *subdivision path* for nodes  $a$  and  $b$ .

By Properties 2 and 4 of SPQR-trees and the fact that all virtual edges are incident to  $v$ , every branching node  $a$  of  $T'$  is either an R-node or a P-node. In the case where  $a$  is a P-node, all the subdivision paths that begin or end at a vertex of  $H_a$  include the same vertex of  $H_a$ . In the case where  $a$  is an R-node, each subdivision path that begins or ends at a vertex of  $H_a$  includes a different vertex (for up to 3 different vertices  $x, y$ , and  $z$ ). Now, since  $H_a$  is 3-connected, these three vertices are in the same component of  $H_a - \{v\}$ . In particular,  $H_a - \{v\}$  contains an edge-minimal tree that includes  $x, y$ , and  $z$ . Adding each of these trees to the union of all subdivision paths produces the subdivision  $T''$  of  $T_{k+1}$ .

Next we show how to construct paths from  $v$  to anchor nodes. Let  $a$  be a branching node of  $T'$ , let  $b$  be the highest distinctive node in  $a$ 's left subtree and let  $c_1 = a, \dots, c_r = b$  be the path in  $T'$  having endpoints  $a$  and  $b$ . Thus far we have established that  $G'[\{c_2, \dots, c_{r-1}\}]$  contains a simple path  $P_{ab}$  from  $x_1$  to  $x_{r-1}$  that does not include  $v$ . We now show that  $G'[\{c_2, c_3, c_4, c_5\}]$  contains an *apex path*  $P'$  from  $v$  to some *anchor node* of  $P_{ab}$  such that the

internal vertices of  $P'$  are disjoint from  $V(P_{ab}) \cup V(H_b)$ . We first describe the path  $P'$  in  $G'[\{c_2, c_3, c_4, c_5\}]$  and then show that  $P'$  does not contain any vertex of  $V(H_b) \setminus \{v\}$ . There are two cases to consider:

1.  $c_i$  is an R-node, for some  $i \in \{2, \dots, 5\}$ : Since  $H_{c_i}$  is 3-connected, there are three paths in  $H_{c_i}$  with endpoints  $v$  and  $x_i$  and no other vertices in common. Since  $H_{c_i}$  has only two virtual edges, at least one of these paths uses only real edges in  $H_{c_i}$ . This path therefore contains a subpath  $P'$  joining  $v$  to some vertex of  $P_{ab}$  (the anchor vertex) that is otherwise disjoint from  $P_{ab}$ .
2. Otherwise, none of  $c_2, \dots, c_5$  is an R-node. Property 1 of SPQR-trees implies that, for at least one  $i \in \{2, 3\}$ ,  $c_i$  is an S-node,  $c_{i+1}$  is a P-node and  $c_{i+2}$  is an S-node. But in this case, Property 5 of SPQR-trees ensures that  $H_{c_{i+1}}$  contains the real edge  $vx_{i+1}$ . Therefore, in this case there is a single edge path  $P'$  from  $v$  to  $P_{ab}$ .

It remains to show that  $P'$  does not contain any vertices of  $V(H_b) \setminus \{v\}$ . By Properties 1 and 6 of SPQR-trees, for any  $x \in V(G') \setminus \{v\}$ , the subtree  $T'[x]$  of  $T'$  consisting only of nodes  $a$  such that  $x \in V(H_a)$  is a star, i.e., the distance between any two nodes in  $T'[x]$  is at most 2. Now, since the distance between any two distinctive nodes of  $T'$  is at least 7,  $r \geq 8$  and therefore  $H_b = H_{c_r}$  has no vertex, except  $v$ , in common with any of  $H_{c_2}, \dots, H_{c_5}$ . Therefore,  $P'$  joins  $v$  to a vertex in  $P_{ab}$  that is not in  $H_b$ , as required.

Adding the set of all apex paths to  $T''$  then produces a subgraph of  $G'$  that contains a  $Q_k$ -minor.  $\square$

Let  $Q_k^*$  be the graph (shown in Fig. 2) obtained by subdividing each edge of  $T_{k+1}$  and adding a vertex  $v$  adjacent to all subdivision vertices. The proof above actually shows that, if  $S[v]$  contains a subdivision of  $T_{9(k+1)}$  then  $G$  contains a *subdivision* of  $Q_k^*$ . This is a little bit surprising, since I would have guessed that  $v$  would have to be a contracted subgraph of  $G$ , not a single high-degree vertex. Is it worth writing the proof to highlight this? GJ: Yes it is surprising at first sight. Note that in the R-nodes we do need to forbid  $Q_k$  as a minor, not enough to forbid a  $Q_k^*$ -subdivision. So there the minor relation is needed.

Finally, we have all the pieces in place to complete the proof.

*Proof that (1) implies (5).* Let  $G$  be a graph excluding some apex-forest graph  $H$  as a minor. As explained earlier,  $G$  contains no  $Q_k$  minor for some  $k = k(H)$ . We wish to show that there are  $w$  and  $p$  that depend only on  $k$  such that  $G$  has a  $(w, p)$ -good tree decomposition. By Lemma 11 we may assume that  $G$  is 2-connected.

Consider the good tree decomposition  $(B_a : a \in V(T))$  of  $G$  described in Section 3.4. This decomposition has width at most  $w$  where  $w = w(k)$  is the function that appears (implicitly) in Theorem 8 and Corollary 10. We claim that, for each  $v \in V(G)$ ,  $\text{pw}(T[v]) \leq 2^{14(k+1)} - 3$ , so that this tree decomposition is  $(w, 2^{14(k+1)} - 3)$ -good. Otherwise, by Lemma 12, there is a vertex  $v \in V(G)$  such that the SPQR-tree  $S$  has a subtree  $S[v]$  that contains a subdivision of  $T_{7(k+1)}$ . Therefore, by Lemma 13,  $G$  contains a  $Q_k$  minor, contradicting the supposition that  $G$  has no  $Q_k$  minor.  $\square$

Depending on where we decide to submit this paper, we may want to discuss algorithmic complexity, which the following theorem sketches.

DW: I vote to keep this.

**Theorem 14.** *There exists a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  and an algorithm that takes as input an  $n$ -vertex graph and outputs, in  $O(f(k)n)$  time, an  $(f(k), f(k))$ -good tree decomposition of  $G$  and a layered path decomposition of  $G$  of layered width at most  $f(k)$ , where  $k$  is largest integer such that  $G$  contains a  $Q_k$  minor.*

*Proof.* In the following, for each  $i \in \{0, \dots, 5\}$ ,  $f_i : \mathbb{N} \rightarrow \mathbb{N}$  is an unspecified function that is known to exist. The fact that  $G$  has no  $Q_k$ -minor implies that the number of edges of  $G$  is at most  $f_0(k)n$  (see [28]). Computing the SPQR-tree  $S$  of  $G$  can be done in  $O(f_0(k)n)$  time and the total size of the graphs  $\{H_a : a \in V(S)\}$  is at most  $f_0(k)n$  (see [22]). Computing a path decomposition of  $H_a$  with width at most 2 for each S-node or P-node  $a$  is easily done in time linear in the size of  $H_a$ . For each R-node  $a$ , the pathwidth of  $H_a$  is at most  $p = p(k) = O(1)$ . Therefore, computing a minimum-width path decomposition of  $H_a$  for an R-node  $a$  can be done in  $O(f_1(k)|V(H_a)|)$  time [3–6, 24]. These path decompositions are all that is needed to construct the tree  $T$  and an  $(f_2(k), f_3(k))$ -good tree decomposition  $\{B_a : a \in V(T)\}$  in  $O(f_1(k)n)$  time.

DW: I would delete reference [3], since [4] is the updated journal version.

The proof that (5) implies (4) in Section 2 is constructive and immediately gives an  $O(f_4(k)n)$  time algorithm to convert the  $(f_2(k), f_3(k))$ -good tree decomposition into a layered path decomposition of layered width at most  $f_5(k)$ . This establishes the theorem for  $f(k) = \max\{f_i(k) : i \in \{1, \dots, 5\}\}$ .  $\square$

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