Prove: There exist disjoint $I, J \subseteq \{1 \dots n\}$ s.t.

(i) $|I| = |J| \geq n / q^{O(1)}$

(ii) for each $i, i' \in I$ $x_i < x_{i'}$ iff $i < i'$

(iii) for each $j, j' \in J$ $x_j < x_{j'}$ if $j > j'$

Prove: There exist $I' \subseteq I$ and $J' \subseteq J$ s.t.

(a) $I'$ contains $i_1 < i_2 < \dots < i_r$.

(b) $J'$ contains $j_1 > j_2 > \dots > j_r$.

(c) For each $k \in \{1 \dots r\}$, $G$ contains a path $P_k$ from $x_{i_k}$ to $x_{j_k}$ of length at most $d \in O(\log q)$.

(d) $P_1 \dots P_k$ are vertex disjoint.

(e) $r \geq n^{\alpha}$, for some fixed $\alpha > 0$.

By known lemma and (c)-(d), $\{1 \dots r\}$ can be partitioned into $s \leq q^{O(d)} = e^{O(d \log q)} = e^{O(\log^2 q)}$ sets $R_1 \dots R_s$ such that, for each $k, k' \in R_\ell$, $x_{i_k} < x_{i_{k'}} \iff y_{i_k} < y_{i_{k'}}$

Since $r \geq n^{\alpha}$,

$$\frac{r}{s} \geq \frac{n^{\alpha}}{e^{O(\log^c s)}} \geq \frac{n^{\alpha}}{e^{O((\log\log n)^c)}} \geq e^{\alpha \log n - O((\log\log n)^c)} > 1,$$

so by the pigeonhole principle, $|R_\ell| \geq \lceil \frac{r}{s} \rceil \geq 2$ for at least one $\ell \in \{1 \dots s\}$. But now we have $k < k' \in R_\ell$ s.t. $x_{i_k} < x_{i_{k'}}$ and $x_{j_k} < x_{j_{k'}}$. But this contradicts (iii.)

QED.