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# STACK-NUMBER IS NOT BOUNDED BY QUEUE-NUMBER

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**ABSTRACT.** We describe a family of graphs with queue-number at most 4 but unbounded stack-number. This resolves open problems of Heath, Leighton and Rosenberg (1992) and Blankenship and Oporowski (1999).

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## 1 Introduction

Stacks and queues are fundamental data structures in computer science, but which is more powerful? In 1992, Heath, Leighton and Rosenberg [28, 29] introduced an approach for answering this question by defining the graph parameters *stack-number* and *queue-number* (defined below), which respectively measure the power of stacks and queues for representing graphs. The following fundamental problems, implicit in [28, 29], were made explicit by Dujmović and Wood [21]<sup>1</sup>:

- Is stack-number bounded by queue-number?
- Is queue-number bounded by stack-number?

If stack-number is bounded by queue-number but queue-number is not bounded by stack-number, then stacks would be considered to be more powerful than queues. Similarly, if the converse holds, then queues would be considered to be more powerful than stacks. Despite extensive research on stack- and queue-numbers, these fundamental questions have remained unsolved.

We now formally define stack- and queue-number. Let  $G$  be a graph and let  $<$  be a total order on  $V(G)$ . Two disjoint edges  $vw, xy \in E(G)$  with  $v < w$  and  $x < y$  *cross* with respect to  $<$  if  $v < x < w < y$  or  $x < v < y < w$ , and *nest* with respect to  $<$  if  $v < x < y < w$  or  $x < v < w < y$ . Let  $\varphi : E(G) \rightarrow \{1, \dots, k\}$  for some integer  $k \geq 1$ . Then  $(<, \varphi)$  is a  $k$ -*stack layout* of  $G$  if  $vw$  and  $xy$  do not cross for all edges  $vw, xy \in E(G)$  with  $\varphi(vw) = \varphi(xy)$ . Similarly,

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<sup>1</sup>A *graph parameter* is a function  $\alpha$  such that  $\alpha(G) \in \mathbb{R}$  for every graph  $G$  and such that  $\alpha(G_1) = \alpha(G_2)$  for all isomorphic graphs  $G_1$  and  $G_2$ . A graph parameter  $\alpha$  is *bounded* by a graph parameter  $\beta$  if there exists a function  $f$  such that  $\alpha(G) \leq f(\beta(G))$  for every graph  $G$ .

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$(\prec, \varphi)$  is a  $k$ -queue layout of  $G$  if  $vw$  and  $xy$  do not nest for all edges  $vw, xy \in E(G)$  with  $\varphi(vw) = \varphi(xy)$ . See Figure 1 for examples. The smallest integer  $s$  for which  $G$  has an  $s$ -stack layout is called the *stack-number* of  $G$ , denoted  $\text{sn}(G)$ . The smallest integer  $q$  for which  $G$  has a  $q$ -queue layout is called the *queue-number* of  $G$ , denoted  $\text{qn}(G)$ .



Figure 1: A 2-queue layout and a 2-stack layout of the triangulated grid graph  $H_4$  defined below. Edges drawn above the vertices are assigned to the first queue/stack and edges drawn below the vertices are assigned to the second queue/stack.

Given a  $k$ -stack layout  $(\prec, \varphi)$  of a graph  $G$ , for each  $i \in \{1, \dots, k\}$ , the set  $\varphi^{-1}(i)$  behaves like a stack, in the sense that each edge  $vw \in \varphi^{-1}(i)$  with  $v \prec w$  corresponds to an element in a sequence of stack operations, such that if we traverse the vertices in the order of  $\prec$ , then  $vw$  is pushed onto the stack at  $v$  and popped off the stack at  $w$ . Similarly, each set  $\varphi^{-1}(i)$  in a queue layout behaves like a queue. In this way, the stack-number and queue-number respectively measure the power of stacks and queues to represent graphs.

Note that stack layouts are equivalent to book embeddings (first defined by Ollmann [34] in 1973), and stack-number is also known as *page-number*, *book-thickness* or *fixed outer-thickness*. Stack and queue layouts have other applications including computational complexity [10, 11, 19, 26], RNA folding [27], graph drawing in two [1, 2, 39] and three dimensions [15, 16, 18, 40], and fault-tolerant multiprocessing [12, 36–38]. See [3–5, 13, 14, 20, 22, 30, 43, 44] for bounds on the stack- and queue-number for various graph classes.

### Is Stack-Number Bounded by Queue-Number?

This paper considers the first of the above questions. In a positive direction, Heath et al. [28] showed that every 1-queue graph has a 2-stack layout. On the other hand, they described graphs that need exponentially more stacks than queues. In particular,  $n$ -vertex ternary hypercubes have queue-number  $O(\log n)$  and stack-number  $\Omega(n^{1/9-\epsilon})$  for any  $\epsilon > 0$ . Our key contribution is the following theorem, which shows that stack-number is not bounded by queue-number.

**Theorem 1.** *For every  $s \in \mathbb{N}$  there exists a graph  $G$  with  $\text{qn}(G) \leq 4$  and  $\text{sn}(G) > s$ .*

This demonstrates that stacks are not more powerful than queues for representing graphs.

## Cartesian Products

As illustrated in Figure 2, the graph  $G$  in Theorem 1 is the cartesian product<sup>2</sup>  $S_b \square H_n$ , where  $S_b$  is the star graph with root  $r$  and  $b$  leaves, and  $H_n$  is the dual of the hexagonal grid, defined by

$$\begin{aligned} V(H_n) &:= \{1, \dots, n\}^2 \quad \text{and} \quad E(H_n) := \{(x, y)(x+1, y) : x \in \{1, \dots, n-1\}, y \in \{1, \dots, n\}\} \\ &\quad \cup \{(x, y)(x, y+1) : x \in \{1, \dots, n\}, y \in \{1, \dots, n-1\}\} \\ &\quad \cup \{(x, y)(x+1, y+1) : x, y \in \{1, \dots, n-1\}\} . \end{aligned}$$



Figure 2:  $S_9 \square H_4$ .

We prove the following:

**Theorem 2.** *For every  $s \in \mathbb{N}$ , if  $b$  and  $n$  are sufficiently large compared to  $s$ , then*

$$\text{sn}(S_b \square H_n) > s.$$

We now show that  $\text{qn}(S_b \square H_n) \leq 4$ , which with Theorem 2 implies Theorem 1. We need the following definition due to Wood [41]. A queue layout  $(\varphi, <)$  is *strict* if for every vertex  $u \in V(G)$  and for all neighbours  $v, w \in N_G(u)$ , if  $u < v, w$  or  $v, w < u$ , then  $\varphi(uv) \neq \varphi(uw)$ . Let  $\text{sqn}(G)$  be the minimum integer  $k$  such that  $G$  has a strict  $k$ -queue layout. To see that  $\text{sqn}(H_n) \leq 3$ , order the vertices row-by-row and then left-to-right within a row, with vertical edges in one queue, horizontal edges in one queue, and diagonal edges in another queue (this construction puts the edges below the vertices in Figure 1 into two queues). Wood [41] proved that for all graphs  $G_1$  and  $G_2$ ,

$$\text{qn}(G_1 \square G_2) \leq \text{qn}(G_1) + \text{sqn}(G_2). \quad (1)$$

Of course,  $S_b$  has a 1-queue layout (since no two edges are nested for any vertex-ordering). Thus  $\text{qn}(S_b \square H_n) \leq 4$ .

<sup>2</sup>For graphs  $G_1$  and  $G_2$ , the *cartesian product*  $G_1 \square G_2$  is the graph with vertex set  $\{(v_1, v_2) : v_1 \in V(G_1), v_2 \in V(G_2)\}$ , where  $(v_1, v_2)(w_1, w_2) \in E(G_1 \square G_2)$  if  $v_1 = w_1$  and  $v_2 w_2 \in E(G_2)$ , or  $v_1 w_1 \in E(G_1)$  and  $v_2 = w_2$ . The *strong product*  $G_1 \boxtimes G_2$  is the graph obtained from  $G_1 \square G_2$  by adding the edge  $(v_1, v_2)(w_1, w_2)$  whenever  $v_1 w_1 \in E(G_1)$  and  $v_2 w_2 \in E(G_2)$ . Note that Pupyrev [35] independently suggested using graph products to show that stack-number is not bounded by queue-number.

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Bernhart and Kainen [4] implicitly proved a result similar to (1) for stack layouts. Let  $\text{dsn}(G)$  be the minimum integer  $k$  such that  $G$  has a  $k$ -stack layout  $(\prec, \varphi)$  where  $\varphi$  is a proper edge-colouring of  $G$ ; that is,  $\varphi(vx) \neq \varphi(vy)$  for any two edges  $vx, vy$  with a common endpoint. Then for every graph  $G_1$  and every bipartite graph  $G_2$ ,

$$\text{sn}(G_1 \square G_2) \leq \text{sn}(G_1) + \text{dsn}(G_2). \quad (2)$$

The key difference between (1) and (2) is that  $G_2$  is assumed to be bipartite in (2). Theorem 2 says that this assumption is essential, since it is easily seen that  $\text{dsn}(H_n)$  is bounded, but  $\text{sn}(S_b \square H_n)$  is unbounded by Theorem 2.

### Subdivisions

A noteworthy consequence of Theorem 1 is that it resolves a conjecture of Blankenship and Oporowski [6]. A graph  $G'$  is a *subdivision* of a graph  $G$  if  $G'$  can be obtained from  $G$  by replacing the edges  $vw$  of  $G$  by internally disjoint paths  $P_{vw}$  with endpoints  $v$  and  $w$ . If each  $P_{vw}$  has exactly  $k$  internal vertices, then  $G'$  is the  $k$ -subdivision of  $G$ . If each  $P_{vw}$  has at most  $k$  internal vertices, then  $G'$  is a  $(\leq k)$ -subdivision of  $G$ . Blankenship and Oporowski [6] conjectured that the stack-number of  $(\leq k)$ -subdivisions ( $k$  fixed) is not much less than the stack-number of the original graph. More precisely:

**Conjecture 1** ([6]). *There exists a function  $f$  such that for every graph  $G$  and integer  $k$ , if  $G'$  is any  $(\leq k)$ -subdivision of  $G$ , then  $\text{sn}(G) \leq f(\text{sn}(G'), k)$ .*

Dujmović and Wood [21] established a connection between this conjecture and the question of whether stack-number is bounded by queue-number. In particular, they showed that if Conjecture 1 were true, then stack-number would be bounded by queue-number. Since Theorem 1 shows that stack-number is not bounded by queue-number, Conjecture 1 is false. The proof of Dujmović and Wood [21] is based on the following key lemma: every graph  $G$  has a 3-stack subdivision with  $1 + 2\lceil \log_2 \text{qn}(G) \rceil$  division vertices per edge. Applying this result to the graph  $G = S_b \square H_n$  in Theorem 1, the 5-subdivision of  $S_b \square H_n$  has a 3-stack layout. If Conjecture 1 were true, then  $\text{sn}(S_b \square H_n)$  would be at most  $f(3, 5)$ , contradicting Theorem 1.

### Is Queue-Number Bounded by Stack-Number?

It remains open whether queues are more powerful than stacks; that is, whether queue-number is bounded by stack-number. Several results are known about this problem. Heath et al. [28] showed that every 1-stack graph has a 2-queue layout. Dujmović et al. [14] showed that planar graphs have bounded queue-number. (Note that graph products also feature heavily in this proof.) Since 2-stack graphs are planar, this implies that 2-stack graphs have bounded queue-number. It is open whether 3-stack graphs have bounded queue-number. In fact, the case of three stacks is as hard as the general question. Dujmović and Wood [21] proved that queue-number is bounded by stack-number if and only if 3-stack graphs have bounded queue-number. Moreover, if this is true then queue-number is bounded by a polynomial function of stack-number.

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## Other Connections

To conclude this introduction, we mention some other properties of the graph  $S_b \square H_n$ . Nešetřil, Ossona de Mendez, and Wood [33] proved that graph classes with bounded stack-number or bounded queue-number have bounded expansion (see [32] for background on bounded expansion classes). The converse is not true, since cubic graphs (for example) have bounded expansion, unbounded stack-number [31] and unbounded queue-number [42]. However, prior to the present work it was open whether graph classes with polynomial expansion have bounded stack-number or bounded queue-number. It follows from the work of Dvořák, Huynh, Joret, Liu, and Wood [23, Theorem 19] that  $\{S_b \square H_n : b, n \in \mathbb{N}\}$  has polynomial expansion. So Theorem 2 implies there is a class of graphs with polynomial expansion and with unbounded stack-number. It remains open whether graph classes with polynomial expansion have bounded queue-number. See [14, 17] for several examples of graph classes with polynomial expansion and bounded queue-number.

Another consequence of our result is that it resolves a question of Bonnet, Geniet, Kim, Thomassé, and Watrigant [7] concerning the graph parameter *twin-width* (see [7–9] for the definition and background on twin-width). A graph class  $\mathcal{C}$  is called *hereditary* if for every graph  $G$  in  $\mathcal{C}$ , every induced subgraph of  $G$  is also in  $\mathcal{C}$ . A hereditary class of graphs  $\mathcal{C}$  with bounded twin-width has *bounded sparse twin-width* if there exists  $t \in \mathbb{N}$  such that no graph in  $\mathcal{C}$  contains  $K_{t,t}$  as a subgraph. Bonnet et al. [7] proved that any hereditary class of graphs with bounded stack-number has bounded sparse twin-width. Bonnet et al. [7] write that they “believe that the inclusion is strict”; that is, there exists a hereditary class of graphs with bounded sparse twin-width and unbounded stack-number. Theorem 2 confirms this intuition. To see this, first note that every star  $S_k$  has twin-width 0 and that every planar graph has bounded twin-width [9]. Note also that  $S_b \boxtimes H_n$  does not contain  $K_{8,8}$  as a subgraph as it is 7-degenerate. By [7, Theorem 10], it follows that  $S_b \square H_n$  has bounded twin-width. Let  $\mathcal{G}$  be the class of graphs that consists of  $S_b \square H_n$  and every induced subgraph of it. Since  $S_b \square H_n$  does not contain  $K_{5,5}$  as a subgraph since it is 4-degenerate, it follows that  $\mathcal{G}$  is a hereditary class of graphs with bounded sparse twin-width [7, Theorem 10] and unbounded stack-number. This confirms that the class of graphs with bounded sparse twin-width does not coincide with the class of graphs with bounded stack-number, as suspected by Bonnet et al. [7]. It remains open whether bounded sparse twin-width coincides with bounded queue-number.

DW: We definitely need some explanation about the excluded  $K_{t,t}$ .

RH: Need to include a further explanation that the graphs do not contain  $K_{t,t}$  as a subgraph due to degeneracy.

## 2 Proof of Theorem 2

We now turn to the proof of our main result, the lower bound on  $\text{sn}(G)$ , where  $G := S_b \square H_n$ . Consider a hypothetical  $s$ -stack layout  $(\varphi, <)$  of  $G$  where  $n$  and  $b$  are chosen sufficiently large compared to  $s$  as detailed below. We begin with three lemmata that, for sufficiently large  $b$ , provide a large subgraph  $S_d$  of  $S_b$  for which the induced stack layout of  $S_d \square H_n$  is

Should the above two sub-sections go to Section 3?

highly structured.

For each node  $v$  of  $S_b$ , define  $\pi_v$  as the permutation of  $\{1, \dots, n\}^2$  in which  $(x_1, y_1)$  appears before  $(x_2, y_2)$  if and only if  $(v, (x_1, y_1)) < (v, (x_2, y_2))$ . The following lemma is an immediate consequence of the Pigeonhole Principle:

**Lemma 1.** *There exists a permutation  $\pi$  of  $\{1, \dots, n\}^2$  and a set  $L_1$  of leaves of  $S_b$  of size  $a \geq b/(n^2)!$  such that  $\pi_v = \pi$  for each  $v \in L_1$ .*

For each leaf  $v$  in  $L_1$ , let  $\varphi_v$  be the edge colouring of  $H_n$  defined by  $\varphi_v(xy) := \varphi((v, x)(v, y))$  for each  $xy \in E(H_n)$ . Since  $H_n$  has maximum degree 6 and is not 6-regular, it has fewer than  $3n^2$  edges. Therefore there are fewer than  $s^{3n^2}$  edge colourings of  $H_n$  using  $s$  colours. Another application of the Pigeonhole Principle proves the following:

**Lemma 2.** *There exists a subset  $L_2 \subseteq L_1$  of size  $c \geq a/s^{3n^2}$  and an edge colouring  $\phi : E(H_n) \rightarrow \{1, \dots, s\}$  such that  $\varphi_v = \phi$  for each  $v \in L_2$ .*

Let  $S_c$  be the subgraph of  $S_b$  induced by  $L_2 \cup \{r\}$ . The preceding two lemmata ensure that, for distinct leaves  $v$  and  $w$  of  $S_c$ , the stack layouts of the isomorphic graphs  $G[\{(v, p) : p \in V(H_n)\}]$  and  $G[\{(w, p) : p \in V(H_n)\}]$  are identical. The next lemma is a statement about the relationships between the stack layouts of  $G[\{(v, p) : v \in V(S_c)\}]$  and  $G[\{(v, q) : v \in V(S_c)\}]$  for distinct  $p, q \in V(H_n)$ . It does not assert that these two layouts are identical but it does state that they fall into one of two categories.

**Lemma 3.** *There exists a sequence  $u_1, \dots, u_d \in L_2$  of length  $d \geq c^{1/2^{n^2-1}}$  such that, for each  $p \in V(H_n)$ , either  $(u_1, p) < (u_2, p) < \dots < (u_d, p)$  or  $(u_1, p) > (u_2, p) > \dots > (u_d, p)$ .*

*Proof.* Let  $p_1, \dots, p_{n^2}$  denote the vertices of  $H_n$  in any order. Begin with the sequence  $S_1 := v_{1,1}, \dots, v_{1,c}$  that contains all  $c$  elements of  $L_2$  ordered so that  $(v_{1,1}, p_1) < \dots < (v_{1,c}, p_1)$ . For each  $i \in \{2, \dots, n^2\}$ , the Erdős-Szekeres Theorem [24] implies that  $S_{i-1}$  contains a subsequence  $S_i := v_{i,1}, \dots, v_{i,|S_i|}$  of length  $|S_i| \geq \sqrt{|S_{i-1}|}$  such that  $(v_{i,1}, p_i) < \dots < (v_{i,|S_i|}, p_i)$  or  $(v_{i,1}, p_i) > \dots > (v_{i,|S_i|}, p_i)$ . It is straightforward to verify by induction on  $i$  that  $|S_i| \geq c^{1/2^{i-1}}$  resulting in a final sequence  $S_{n^2} =: L_3$  of length at least  $c^{1/2^{n^2-1}}$ .  $\square$

For the rest of the proof we work with the star  $S_d$  whose leaves are  $u_1, \dots, u_d$  described in Lemma 3. Consider the (improper) colouring of  $H_n$  obtained by colouring each vertex  $p \in V(H_n)$  *red* if  $(u_1, p) < \dots < (u_d, p)$  and colouring  $p$  *blue* if  $(u_1, p) > \dots > (u_d, p)$ . We need the following famous Hex Lemma [25].

**Lemma 4** ([25]). *Every vertex 2-colouring of  $H_n$  contains a monochromatic path on  $n$  vertices.*

Apply Lemma 4 with the above-defined colouring of  $H_n$ . We obtain a path  $R := p_1, \dots, p_n$  in  $H_n$  that, without loss of generality, consists entirely of red vertices; thus  $(u_1, p_j) < \dots < (u_d, p_j)$  for each  $j \in \{1, \dots, n\}$ . Let  $X$  be the subgraph  $S_d \square R$  of  $G$ .

**Lemma 5.**  *$X$  contains a set of at least  $\min\{\lfloor d/2^n \rfloor, \lceil n/2 \rceil\}$  pairwise crossing edges with respect to  $<$ .*



*Proof.* Extend the total order  $<$  to a partial order over subsets of  $V(G)$ , where for all  $V, W \subseteq V(G)$ , we have  $V < W$  if and only if  $v < w$  for each  $v \in V$  and each  $w \in W$ . We abuse notation slightly by using  $<$  to compare elements of  $V(G)$  and subsets of  $V(G)$  so that, for  $v \in V(G)$  and  $V \subseteq V(G)$ ,  $v < V$  denotes  $\{v\} < V$ . We will define sets  $A_1 \supseteq \dots \supseteq A_n$  of leaves of  $S_d$  so that each  $A_i$  satisfies the following conditions:

- (C1)  $A_i$  contains  $d_i \geq d/2^{i-1}$  leaves of  $S_d$ .
- (C2) Each leaf  $v \in A_i$  defines an  $i$ -element vertex set  $Z_{i,v} := \{(v, p_j) : j \in \{1, \dots, i\}\}$ . For any distinct  $v, w \in A_i$ , the sets  $Z_{i,v}$  and  $Z_{i,w}$  are *separated* with respect to  $<$ ; that is,  $Z_{i,v} < Z_{i,w}$  or  $Z_{i,v} > Z_{i,w}$ .

Before defining  $A_1, \dots, A_n$  we first show how the existence of the set  $A_n$  implies the lemma. To avoid triple-subscripts, let  $d' := d_n \geq d/2^{n-1}$ . The set  $A_n$  defines vertex sets  $Z_{n,v_1} < \dots < Z_{n,v_{d'}}$  (see Figure 3). Recall that  $r$  is the root of  $S_b$  so it is adjacent to each of  $v_1, \dots, v_{d'}$  in  $S_d$ . Therefore, for each  $j \in \{1, \dots, n\}$  and each  $i \in \{1, \dots, d'\}$ , the edge  $(r, p_j)(v_i, p_j)$  is in  $X$ . Therefore,  $(r, p_j)$  is adjacent to an element of each of  $Z_{n,v_1}, \dots, Z_{n,v_{d'}}$ .

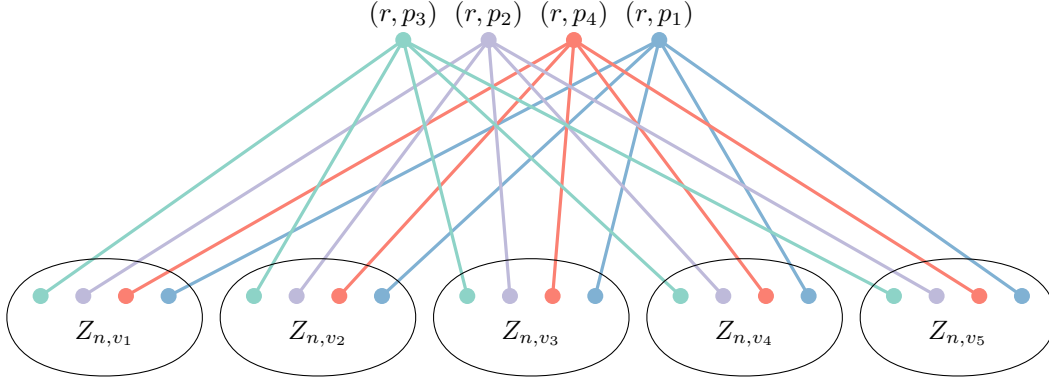


Figure 3: The sets  $Z_{n,v_1}, \dots, Z_{n,v_{d'}}$  where  $n = 4$  and  $d' = 5$ .

Since  $Z_{n,v_1}, \dots, Z_{n,v_{d'}}$  are separated with respect to  $<$ , when viewed from afar, this situation looks like a complete bipartite graph  $K_{n,d'}$  with the root vertices  $L := \{(r, p_j) : j \in \{1, \dots, n\}\}$  in one part and the groups  $R := Z_{n,v_1} \cup \dots \cup Z_{n,v_{d'}}$  in the other part. Any linear ordering of  $K_{n,d'}$  has a large set of pairwise crossing edges so, intuitively, the induced subgraph  $X[L \cup R]$  should also have a large set of pairwise crossing edges. We can formalize this as follows: Label the vertices in  $L$  as  $r_1, \dots, r_n$  so that  $r_1 < \dots < r_n$ . Then at least one of the following two cases applies (see Figure 4):

1.  $Z_{n,\lfloor d'/2 \rfloor} < r_{\lceil n/2 \rceil}$  in which case the graph between  $r_{\lceil n/2 \rceil}, \dots, r_n$  and  $Z_{n,1}, \dots, Z_{n,\lfloor d'/2 \rfloor}$  has a set of at least  $\min\{\lfloor d'/2 \rfloor, \lceil n/2 \rceil\}$  pairwise-crossing edges.
2.  $r_{\lceil n/2 \rceil} < Z_{\lceil d'/2 \rceil+1}$  in which case the graph between  $r_1, \dots, r_{\lceil n/2 \rceil}$  and  $Z_{\lceil d'/2 \rceil+1}, \dots, Z_{d'}$  has a set of  $\min\{\lfloor d'/2 \rfloor, \lceil n/2 \rceil\}$  pairwise-crossing edges.

Since, by (C1),  $d' \geq d/2^{n-1}$ , either case results in a set of pairwise-crossing edges of size at least  $\min\{\lfloor d/2^n \rfloor, \lceil n/2 \rceil\}$ , as claimed.

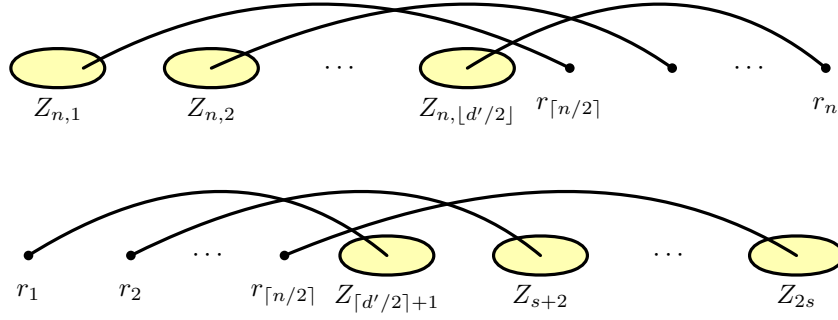


Figure 4: The two cases in the proof of Lemma 5.

All that remains is to define the sets  $A_1 \supseteq \dots \supseteq A_n$  that satisfy (C1) and (C2). Let  $A_1$  be the set of all the leaves of  $S_d$ . For each  $i \in \{2, \dots, n\}$ , the set  $A_i$  is defined as follows: Let  $Z_1, \dots, Z_{|A_{i-1}|}$  denote the sets  $Z_{i-1,v}$  for each  $v \in A_{i-1}$  ordered so that  $Z_1 < \dots < Z_{|A_{i-1}|}$ . By Property (C2), this is always possible. Label the vertices of  $A_{i-1}$  as  $v_1, \dots, v_{|A_{i-1}|}$  so that  $(v_1, p_{i-1}) < \dots < (v_r, p_{i-1})$ . (This is equivalent to naming them so that  $(v_j, p_{i-1}) \in Z_j$  for each  $j \in \{1, \dots, |A_{i-1}|\}$ .) Define the set  $A_i := \{v_{2k+1} : k \in \{0, \dots, \lfloor (|A_{i-1}| - 1)/2 \rfloor\}\} = \{v_j \in A_{i-1} : j \text{ is odd}\}$ . This completes the definition of  $A_1, \dots, A_n$ .

All that remains is to verify that  $A_i$  satisfies (C1) and (C2) for each  $i \in \{1, \dots, n\}$ . We do this by induction on  $i$ . The base case  $i = 1$  is trivial so we assume from this point on that  $i \in \{2, \dots, n\}$ . To see that  $A_i$  satisfies (C1) just observe that  $|A_i| = \lceil |A_{i-1}|/2 \rceil \geq |A_{i-1}|/2 \geq d/2^{i-1}$ , where the final inequality follows by applying the inductive hypothesis  $|A_{i-1}| \geq d/2^{i-2}$ . Now all that remains is to show that  $A_i$  satisfies (C2).

Recall that, for each  $v \in A_{i-1}$ , the edge  $e_v := (v, p_{i-1})(v, p_i)$  is in  $X$ . We have the following properties:

- (P1) By Lemma 2,  $\varphi(e_v) = \phi(p_{i-1}p_i)$  for each  $v \in A_{i-1}$ .
- (P2) Since  $p_{i-1}$  and  $p_i$  are both red,  $(v, p_{i-1}) < (w, p_{i-1})$  if and only if  $(v, p_i) < (w, p_i)$  for each  $v, w \in A_{i-1}$ .
- (P3) By Lemma 1,  $(v, p_{i-1}) < (v, p_i)$  for every  $v \in A_{i-1}$  or  $(v, p_{i-1}) > (v, p_i)$  for every  $v \in A_{i-1}$ .

We claim that these three conditions imply that the vertex sets  $\{(v, p_{i-1}) : v \in A_{i-1}\}$  and  $\{(v, p_i) : v \in A_{i-1}\}$  interleave perfectly with respect to  $<$ . More precisely:

**Claim 1.**  $(v_1, p_{i-1+t}) < (v_1, p_{i-t}) < (v_2, p_{i-1+t}) < (v_2, p_{i-t}) \dots < (v_r, p_{i-1+t}) < (v_r, p_{i-t})$  for some  $t \in \{0, 1\}$ .

*Proof of Claim 1.* By (P3) we may assume, without loss of generality, that  $(v, p_{i-1}) < (v, p_i)$  for each  $v \in A_{i-1}$ , in which case we are trying to prove the claim for  $t = 0$ . Therefore, it is sufficient to show that  $(v_j, p_i) < (v_{j+1}, p_{i-1})$  for each  $j \in \{1, \dots, r-1\}$ . For the sake of contradiction, suppose  $(v_j, p_i) > (v_{j+1}, p_{i-1})$  for some  $j \in \{1, \dots, r-1\}$ . By the labelling of  $A_{i-1}$ ,  $(v_j, p_{i-1}) < (v_{j+1}, p_{i-1})$  so, by (P2),  $(v_j, p_i) < (v_{j+1}, p_i)$ . Therefore

$$(v_j, p_{i-1}) < (v_{j+1}, p_{i-1}) < (v_j, p_i) < (v_{j+1}, p_i) .$$

Therefore the edges  $e_{v_j} = (v_j, p_{i-1})(v_j, p_i)$  and  $e_{v_{j+1}} = (v_{j+1}, p_{i-1})(v_{j+1}, p_i)$  cross with respect to



<. But this is a contradiction since, by (P1),  $\varphi(e_{v_j}) = \varphi(e_{v_{j+1}}) = \phi(p_{i-1}p_i)$ . This contradiction completes the proof of Claim 1.  $\square$

We now complete the proof that  $A_i$  satisfies (C2). Apply Claim 1 and assume without loss of generality that  $t = 0$ , so that

$$(v_1, p_{i-1}) < (v_1, p_i) < (v_2, p_{i-1}) < (v_2, p_i) \cdots < (v_r, p_{i-1}) < (v_r, p_i) .$$

For each  $j \in \{1, \dots, r-2\}$ , we have  $(v_{j+1}, p_{i-1}) \in Z_{j+1} < Z_{j+2}$ , so  $(v_j, p_i) < (v_{j+1}, p_{i-1}) < Z_{j+2}$ . Therefore  $Z_j \cup \{(v_j, p_i)\} < Z_{j+2}$ . By a symmetric argument,  $Z_j \cup \{(v_j, p_i)\} > Z_{j-2}$  for each  $j \in \{3, \dots, r\}$ . Finally, since  $(v_j, p_i) < (v_{j+2}, p_i)$  for each odd  $i \in \{1, \dots, r\}$ , we have  $Z_j \cup \{(v_j, p_i)\} < Z_{j+2} \cup \{(v_{j+2}, p_i)\}$  for each odd  $j \in \{1, \dots, r-2\}$ . Thus  $A_i$  satisfies (C2) since the sets  $Z_1 \cup \{(v_1, p_i)\}, Z_3 \cup \{(v_3, p_i)\}, \dots, Z_{2\lfloor (r-1)/2 \rfloor + 1} \cup \{(v_{2\lfloor (r-1)/2 \rfloor + 1}, p_i)\}$  are precisely the sets  $Z_{i,1}, \dots, Z_{i,d_i}$  determined by our choice of  $A_i$ .  $\square$

*Proof of Theorem 2.* Let  $G := S_b \square H_n$ , where  $n := 2s + 1$  and  $b := (n^2)! s^{3n^2} ((s+1)2^n)^{2^{n^2-1}}$ . Suppose that  $G$  has an  $s$ -stack layout  $(\varphi, <)$ . In particular, there are no  $s+1$  pairwise crossing edges in  $G$  with respect to  $<$ . By Lemmas 1 to 3, we have  $a \geq b/(n^2)! = s^{3n^2} ((s+1)2^n)^{2^{n^2-1}}$  and  $c \geq a/s^{3n^2} \geq ((s+1)2^n)^{2^{n^2-1}}$  and  $d \geq c^{1/2^{n^2-1}} \geq (s+1)2^n$ . By Lemma 5, the graph  $X$ , which is a subgraph of  $G$ , contains  $\min\{\lfloor d/2^n \rfloor, \lceil n/2 \rceil\} = s+1$  pairwise crossing edges with respect to  $<$ . This contradiction shows that  $\text{sn}(G) > s$ .  $\square$

### 3 Open Problems

Recall that every 1-queue graph has a 2-stack layout [28] and we proved that there are 4-queue graphs with unbounded stack-number. The following questions remain open: Do 2-queue graphs have bounded stack-number? Do 3-queue graphs have bounded stack-number?

Given the role of cartesian products in our proof, it is natural to ask when is  $\text{sn}(G_1 \square G_2)$  bounded? As illustrated in Figure 1,  $\text{sn}(H_n) \leq 2$ . So  $\text{sn}(G_1 \square G_2)$  can be unbounded even when  $G_1$  is a star and  $\text{sn}(G_2) \leq 2$ . Since  $\text{sn}(G_2) \leq 1$  if and only if  $G_2$  is outerplanar, the following questions naturally arise: Is  $\text{sn}(S \square H)$  bounded for every star  $S$  and outerplanar graph  $H$  with bounded degree? Is  $\text{sn}(T \square H)$  bounded for every tree  $T$  and outerplanar graph  $H$  with bounded degree? The assumption that  $H$  has bounded degree is needed since  $S_n \square S_n$  contains the 1-subdivision of  $K_{n,n}$ , which has unbounded stack-number [5].

Since  $H_n \subseteq P \boxtimes P$  where  $P$  is the  $n$ -vertex path, Theorem 1 implies that  $\text{sn}(S \boxtimes P \boxtimes P)$  is unbounded for stars  $S$  and paths  $P$ . It is easily seen that  $\text{sn}(S \boxtimes P)$  is bounded [35]. The following question naturally arises (independently asked by Pupyrev [35]): Is  $\text{sn}(T \boxtimes P)$  bounded for every tree  $T$  and path  $P$ ? We conjecture the answer is “no”.

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## A Properties of Graph Products – NOT TO KEEP

**Lemma 6.** *Let  $G_i$  be a  $d_i$ -degenerate graph for  $i \in \{1, 2\}$ . Then  $G_1 \square G_2$  is  $(d_1 + d_2)$ -degenerate.*

*Proof.* Let  $Z$  be a subgraph of  $G_1 \square G_2$ . Our goal is to show that  $Z$  has minimum degree at most  $d_1 + d_2$ . Let  $X$  be the projection of  $Z$  in  $G_1$ . That is,  $X$  is the subgraph of  $G_1$  induced by the set of vertices  $v_1 \in V(G_1)$  such that  $(v_1, v_2) \in V(Z)$  for some  $v_2 \in V(G_2)$ . Since  $G_1$  is  $d_1$ -degenerate, there is a vertex  $v_1$  in  $X$  with  $\deg_X(v_1) \leq d_1$ . Let  $Y$  be the subgraph of  $G_2$  induced by the set of vertices  $v_2$  in  $G_2$  such that  $(v_1, v_2) \in V(Z)$ . By construction,  $Y$  is not empty. Since  $G_2$  is  $d_2$ -degenerate, there is a vertex  $v_2$  in  $Y$  with  $\deg_Y(v_2) \leq d_2$ . By construction, the vertex  $(v_1, v_2)$  of  $G_1 \square G_2$  is in  $Z$ . The neighbourhood of  $(v_1, v_2)$  in  $Z$  is a subset of  $\{(x, v_2) : v_1 x \in E(X)\} \cup \{(v_1, y) : v_2 y \in E(Y)\}$ . Now  $|\{(x, v_2) : v_1 x \in E(X)\}| = \deg_X(v_1) \leq d_1$  and  $|\{(v_1, y) : v_2 y \in E(Y)\}| = \deg_Y(v_2) \leq d_2$ . Thus  $(v_1, v_2)$  has degree at most  $d_1 + d_2$  in  $Z$ . Hence  $G_1 \square G_2$  is  $(d_1 + d_2)$ -degenerate.  $\square$

**Corollary 1.** *Let  $G_i$  be a  $d_i$ -degenerate graph for  $i \in \{1, 2\}$ . Then  $G_1 \square G_2$  contains no  $K_{d_1+d_2+1, d_1+d_2+1}$ .*

Can we characterise when  $K_{s,t}$  is in  $G_1 \square G_2$ ?

If  $K_{s_1, t_1} \subseteq G_1$  and  $K_{s_2, t_2} \subseteq G_2$ , then  $K_{s_1 t_2, s_2 t_1} \subseteq G_1 \boxtimes G_2$ . Is it true that if  $K_{p, q} \subseteq G_1 \boxtimes G_2$ , then  $K_{s_1, t_1} \subseteq G_1$  and  $K_{s_2, t_2} \subseteq G_2$ , for some  $s_1, t_1, s_2, t_2$  with  $s_1 t_2 = p$  and  $s_2 t_1 = q$ ?