

---

# STACK-NUMBER IS NOT BOUNDED BY QUEUE-NUMBER

Vida Dujmović<sup>‡</sup>, David Eppstein<sup>§</sup>, Robert Hickingbotham<sup>¶</sup>, Pat Morin<sup>‡</sup>, David R. Wood<sup>‡</sup>

November 7, 2020

---

**ABSTRACT.** We describe a family of graphs with queue-number at most 4 but unbounded stack-number. This resolves open problems of Heath, Leighton and Rosenberg (1992) and Blankenship and Oporowski (1999).

---

## 1 Introduction

Stacks and queues are fundamental data structures in computer science, but which is more powerful? In 1992, Heath, Leighton and Rosenberg [28, 29] introduced an approach for answering this question by defining the graph parameters *stack-number* and *queue-number* (defined below), which respectively measure the power of stacks and queues for representing graphs. The following fundamental problems, implicit in [28, 29], were made explicit by Dujmović and Wood [21]<sup>1</sup>:

- Is stack-number bounded by queue-number?
- Is queue-number bounded by stack-number?

If stack-number is bounded by queue-number but queue-number is not bounded by stack-number, then stacks would be considered to be more powerful than queues. Similarly, if the converse holds, then queues would be considered to be more powerful than stacks. Despite extensive research on stack- and queue-numbers, these fundamental questions have remained unsolved.

We now formally define stack- and queue-number. Let  $G$  be a graph and let  $<$  be a total order on  $V(G)$ . Two disjoint edges  $vw, xy \in E(G)$  with  $v < w$  and  $x < y$  *cross* with respect to  $<$  if  $v < x < w < y$  or  $x < v < y < w$ , and *nest* with respect to  $<$  if  $v < x < y < w$  or  $x < v < w < y$ . Let  $\varphi : E(G) \rightarrow \{1, \dots, k\}$  for some integer  $k \geq 1$ . Then  $(<, \varphi)$  is a  $k$ -*stack layout* of  $G$  if  $vw$  and  $xy$  do not cross for all edges  $vw, xy \in E(G)$  with  $\varphi(vw) = \varphi(xy)$ . Similarly,

---

<sup>‡</sup>School of Computer Science and Electrical Engineering, University of Ottawa, Ottawa, Canada (vida.dujmovic@uottawa.ca). Research supported by NSERC.

<sup>§</sup>Department of Computer Science, University of California, Irvine, California, USA (eppstein@uci.edu).

<sup>¶</sup>School of Mathematics, Monash University, Melbourne, Australia (robert.hickingbotham@monash.edu).

<sup>‡</sup>School of Computer Science, Carleton University, Ottawa, Canada (morin@scs.carleton.ca). Research supported by NSERC.

<sup>‡</sup>School of Mathematics, Monash University, Melbourne, Australia (david.wood@monash.edu). Research supported by the Australian Research Council.

<sup>1</sup>A *graph parameter* is a function  $\alpha$  such that  $\alpha(G) \in \mathbb{R}$  for every graph  $G$  and such that  $\alpha(G_1) = \alpha(G_2)$  for all isomorphic graphs  $G_1$  and  $G_2$ . A graph parameter  $\alpha$  is *bounded* by a graph parameter  $\beta$  if there exists a function  $f$  such that  $\alpha(G) \leq f(\beta(G))$  for every graph  $G$ .

$(\prec, \varphi)$  is a  $k$ -queue layout of  $G$  if  $vw$  and  $xy$  do not nest for all edges  $vw, xy \in E(G)$  with  $\varphi(vw) = \varphi(xy)$ . See Figure 1 for examples. The smallest integer  $s$  for which  $G$  has an  $s$ -stack layout is called the *stack-number* of  $G$ , denoted  $\text{sn}(G)$ . The smallest integer  $q$  for which  $G$  has a  $q$ -queue layout is called the *queue-number* of  $G$ , denoted  $\text{qn}(G)$ .



Figure 1: A 2-queue layout and a 2-stack layout of the triangulated grid graph  $H_4$  defined below. Edges drawn above the vertices are assigned to the first queue/stack and edges drawn below the vertices are assigned to the second queue/stack.

Given a  $k$ -stack layout  $(\prec, \varphi)$  of a graph  $G$ , for each  $i \in \{1, \dots, k\}$ , the set  $\varphi^{-1}(i)$  behaves like a stack, in the sense that each edge  $vw \in \varphi^{-1}(i)$  with  $v \prec w$  corresponds to an element in a sequence of stack operations, such that if we traverse the vertices in the order of  $\prec$ , then  $vw$  is pushed onto the stack at  $v$  and popped off the stack at  $w$ . Similarly, each set  $\varphi^{-1}(i)$  in a queue layout behaves like a queue. In this way, the stack-number and queue-number respectively measure the power of stacks and queues to represent graphs.

Note that stack layouts are equivalent to book embeddings (first defined by Ollmann [34] in 1973), and stack-number is also known as *page-number*, *book-thickness* or *fixed outer-thickness*. Stack and queue layouts have other applications including computational complexity [10, 11, 19, 26], RNA folding [27], graph drawing in two [1, 2, 39] and three dimensions [15, 16, 18, 40], and fault-tolerant multiprocessing [12, 36–38]. See [3–5, 13, 14, 20, 22, 30, 43, 44] for bounds on the stack- and queue-number for various graph classes.

### Is Stack-Number Bounded by Queue-Number?

This paper considers the first of the above questions. In a positive direction, Heath et al. [28] showed that every 1-queue graph has a 2-stack layout. On the other hand, they described graphs that need exponentially more stacks than queues. In particular,  $n$ -vertex ternary hypercubes have queue-number  $O(\log n)$  and stack-number  $\Omega(n^{1/9-\epsilon})$  for any  $\epsilon > 0$ .

Our key contribution is the following theorem, which shows that stack-number is not bounded by queue-number.

**Theorem 1.** *For every  $s \in \mathbb{N}$  there exists a graph  $G$  with  $\text{qn}(G) \leq 4$  and  $\text{sn}(G) > s$ .*

This demonstrates that stacks are not more powerful than queues for representing graphs.

## Cartesian Products

As illustrated in Figure 2, the graph  $G$  in Theorem 1 is the cartesian product<sup>2</sup>  $S_b \square H_n$ , where  $S_b$  is the star graph with root  $r$  and  $b$  leaves, and  $H_n$  is the dual of the hexagonal grid, defined by

$$\begin{aligned} V(H_n) &:= \{1, \dots, n\}^2 \quad \text{and} \quad E(H_n) := \{(x, y)(x+1, y) : x \in \{1, \dots, n-1\}, y \in \{1, \dots, n\}\} \\ &\quad \cup \{(x, y)(x, y+1) : x \in \{1, \dots, n\}, y \in \{1, \dots, n-1\}\} \\ &\quad \cup \{(x, y)(x+1, y+1) : x, y \in \{1, \dots, n-1\}\} . \end{aligned}$$



Figure 2:  $S_9 \square H_4$ .

We prove the following:

**Theorem 2.** *For every  $s \in \mathbb{N}$ , if  $b$  and  $n$  are sufficiently large compared to  $s$ , then*

$$\text{sn}(S_b \square H_n) > s.$$

We now show that  $\text{qn}(S_b \square H_n) \leq 4$ , which with Theorem 2 implies Theorem 1. We need the following definition due to Wood [41]. A queue layout  $(\varphi, <)$  is *strict* if for every vertex  $u \in V(G)$  and for all neighbours  $v, w \in N_G(u)$ , if  $u < v, w$  or  $v, w < u$ , then  $\varphi(uv) \neq \varphi(uw)$ . Let  $\text{sqn}(G)$  be the minimum integer  $k$  such that  $G$  has a strict  $k$ -queue layout. To see that  $\text{sqn}(H_n) \leq 3$ , order the vertices row-by-row and then left-to-right within a row, with vertical edges in one queue, horizontal edges in one queue, and diagonal edges in another queue (this construction puts the edges below the vertices in Figure 1 into two queues). Wood [41] proved that for all graphs  $G_1$  and  $G_2$ ,

$$\text{qn}(G_1 \square G_2) \leq \text{qn}(G_1) + \text{sqn}(G_2). \quad (1)$$

Of course,  $S_b$  has a 1-queue layout (since no two edges are nested for any vertex-ordering). Thus  $\text{qn}(S_b \square H_n) \leq 4$ .

<sup>2</sup>For graphs  $G_1$  and  $G_2$ , the *cartesian product*  $G_1 \square G_2$  is the graph with vertex set  $\{(v_1, v_2) : v_1 \in V(G_1), v_2 \in V(G_2)\}$ , where  $(v_1, v_2)(w_1, w_2) \in E(G_1 \square G_2)$  if  $v_1 = w_1$  and  $v_2 w_2 \in E(G_2)$ , or  $v_1 w_1 \in E(G_1)$  and  $v_2 = w_2$ . The *strong product*  $G_1 \boxtimes G_2$  is the graph obtained from  $G_1 \square G_2$  by adding the edge  $(v_1, v_2)(w_1, w_2)$  whenever  $v_1 w_1 \in E(G_1)$  and  $v_2 w_2 \in E(G_2)$ . Note that Pupyrev [35] independently suggested using graph products to show that stack-number is not bounded by queue-number.

---

Bernhart and Kainen [4] implicitly proved a result similar to (1) for stack layouts. Let  $\text{dsn}(G)$  be the minimum integer  $k$  such that  $G$  has a  $k$ -stack layout  $(\prec, \varphi)$  where  $\varphi$  is a proper edge-colouring of  $G$ ; that is,  $\varphi(vx) \neq \varphi(vy)$  for any two edges  $vx, vy$  with a common endpoint. Then for every graph  $G_1$  and every bipartite graph  $G_2$ ,

$$\text{sn}(G_1 \square G_2) \leq \text{sn}(G_1) + \text{dsn}(G_2). \quad (2)$$

The key difference between (1) and (2) is that  $G_2$  is assumed to be bipartite in (2). Theorem 2 says that this assumption is essential, since it is easily seen that  $\text{dsn}(H_n)$  is bounded, but the stack number of  $(S_b \square H_n)_{b,n \in \mathbb{N}}$  is unbounded by Theorem 2.

### Subdivisions

A noteworthy consequence of Theorem 1 is that it resolves a conjecture of Blankenship and Oporowski [6]. A graph  $G'$  is a *subdivision* of a graph  $G$  if  $G'$  can be obtained from  $G$  by replacing the edges  $vw$  of  $G$  by internally disjoint paths  $P_{vw}$  with endpoints  $v$  and  $w$ . If each  $P_{vw}$  has exactly  $k$  internal vertices, then  $G'$  is the  $k$ -subdivision of  $G$ . If each  $P_{vw}$  has at most  $k$  internal vertices, then  $G'$  is a  $(\leq k)$ -subdivision of  $G$ . Blankenship and Oporowski [6] conjectured that the stack-number of  $(\leq k)$ -subdivisions ( $k$  fixed) is not much less than the stack-number of the original graph. More precisely:

**Conjecture 1** ([6]). *There exists a function  $f$  such that for every graph  $G$  and integer  $k$ , if  $G'$  is any  $(\leq k)$ -subdivision of  $G$ , then  $\text{sn}(G) \leq f(\text{sn}(G'), k)$ .*

Dujmović and Wood [21] established a connection between this conjecture and the question of whether stack-number is bounded by queue-number. In particular, they showed that if Conjecture 1 was true, then stack-number would be bounded by queue-number. Since Theorem 1 shows that stack-number is not bounded by queue-number, Conjecture 1 is false. The proof of Dujmović and Wood [21] is based on the following key lemma: every graph  $G$  has a 3-stack subdivision with  $1 + 2\lceil \log_2 \text{qn}(G) \rceil$  division vertices per edge. Applying this result to the graph  $G = S_b \square H_n$  in Theorem 1, the 5-subdivision of  $S_b \square H_n$  has a 3-stack layout. If Conjecture 1 was true, then  $\text{sn}(S_b \square H_n)$  would be at most  $f(3, 5)$ , contradicting Theorem 1.

### Is Queue-Number Bounded by Stack-Number?

It remains open whether queues are more powerful than stacks; that is, whether queue-number is bounded by stack-number. Several results are known about this problem. Heath et al. [28] showed that every 1-stack graph has a 2-queue layout. Dujmović et al. [14] showed that planar graphs have bounded queue-number. (Note that graph products also feature heavily in this proof.) Since 2-stack graphs are planar, this implies that 2-stack graphs have bounded queue-number. It is open whether 3-stack graphs have bounded queue-number. In fact, the case of three stacks is as hard as the general question. Dujmović and Wood [21] proved that queue-number is bounded by stack-number if and only if 3-stack graphs have bounded queue-number. Moreover, if this is true then queue-number is bounded by a polynomial function of stack-number.

## 2 Proof of Theorem 2

We now turn to the proof of our main result, the lower bound on  $\text{sn}(G)$ , where  $G := S_b \square H_n$ . Consider a hypothetical  $s$ -stack layout  $(\varphi, <)$  of  $G$  where  $n$  and  $b$  are chosen sufficiently large compared to  $s$  as detailed below. We begin with three lemmata that, for sufficiently large  $b$ , provide a large subgraph  $S_d$  of  $S_b$  for which the induced stack layout of  $S_d \square H_n$  is highly structured.

For each node  $v$  of  $S_b$ , define  $\pi_v$  as the permutation of  $\{1, \dots, n\}^2$  in which  $(x_1, y_1)$  appears before  $(x_2, y_2)$  if and only if  $(v, (x_1, y_1)) < (v, (x_2, y_2))$ . The following lemma is an immediate consequence of the Pigeonhole Principle:

**Lemma 1.** *There exists a permutation  $\pi$  of  $\{1, \dots, n\}^2$  and a set  $L_1$  of leaves of  $S_b$  of size  $a \geq b/(n^2)!$  such that  $\pi_v = \pi$  for each  $v \in L_1$ .*

For each leaf  $v$  in  $L_1$ , let  $\varphi_v$  be the edge colouring of  $H_n$  defined by  $\varphi_v(xy) := \varphi((v, x)(v, y))$  for each  $xy \in E(H_n)$ . Since  $H_n$  has maximum degree 6 and is not 6-regular, it has fewer than  $3n^2$  edges. Therefore there are fewer than  $s^{3n^2}$  edge colourings of  $H_n$  using  $s$  colours. Another application of the Pigeonhole Principle proves the following:

**Lemma 2.** *There exists a subset  $L_2 \subseteq L_1$  of size  $c \geq a/s^{3n^2}$  and an edge colouring  $\phi : E(H_n) \rightarrow \{1, \dots, s\}$  such that  $\varphi_v = \phi$  for each  $v \in L_2$ .*

Let  $S_c$  be the subgraph of  $S_b$  induced by  $L_2 \cup \{r\}$ . The preceding two lemmata ensure that, for distinct leaves  $v$  and  $w$  of  $S_c$ , the stack layouts of the isomorphic graphs  $G[\{(v, p) : p \in V(H_n)\}]$  and  $G[\{(w, p) : p \in V(H_n)\}]$  are identical. The next lemma is a statement about the relationships between the stack layouts of  $G[\{(v, p) : v \in V(S_c)\}]$  and  $G[\{(v, q) : v \in V(S_c)\}]$  for distinct  $p, q \in V(H_n)$ . It does not assert that these two layouts are identical but it does state that they fall into one of two categories.

**Lemma 3.** *There exists a sequence  $u_1, \dots, u_d \in L_2$  of length  $d \geq c^{1/2^{n^2-1}}$  such that, for each  $p \in V(H_n)$ , either  $(u_1, p) < (u_2, p) < \dots < (u_d, p)$  or  $(u_1, p) > (u_2, p) > \dots > (u_d, p)$ .*

*Proof.* Let  $p_1, \dots, p_{n^2}$  denote the vertices of  $H_n$  in any order. Begin with the sequence  $S_1 := v_{1,1}, \dots, v_{1,c}$  that contains all  $c$  elements of  $L_2$  ordered so that  $(v_{1,1}, p_1) < \dots < (v_{1,c}, p_1)$ . For each  $i \in \{2, \dots, n^2\}$ , the Erdős-Szekeres Theorem [24] implies that  $S_{i-1}$  contains a subsequence  $S_i := v_{i,1}, \dots, v_{i,|S_i|}$  of length  $|S_i| \geq \sqrt{|S_{i-1}|}$  such that  $(v_{i,1}, p_i) < \dots < (v_{i,|S_i|}, p_i)$  or  $(v_{i,1}, p_i) > \dots > (v_{i,|S_i|}, p_i)$ . It is straightforward to verify by induction on  $i$  that  $|S_i| \geq c^{1/2^{i-1}}$  resulting in a final sequence  $S_{n^2} =: L_3$  of length at least  $c^{1/2^{n^2-1}}$ .  $\square$

For the rest of the proof we work with the star  $S_d$  whose leaves are  $u_1, \dots, u_d$  described in Lemma 3. Consider the (improper) colouring of  $H_n$  obtained by colouring each vertex  $p \in V(H_n)$  *red* if  $(u_1, p) < \dots < (u_d, p)$  and colouring  $p$  *blue* if  $(u_1, p) > \dots > (u_d, p)$ . We need the following famous Hex Lemma [25].

**Lemma 4** ([25]). *Every vertex 2-colouring of  $H_n$  contains a monochromatic path on  $n$  vertices.*

Apply Lemma 4 with the above-defined colouring of  $H_n$ . We obtain a path  $R := p_1, \dots, p_n$  in  $H_n$  that, without loss of generality, consists entirely of red vertices; thus  $(u_1, p_j) < \dots < (u_d, p_j)$  for each  $j \in \{1, \dots, n\}$ . Let  $X$  be the subgraph  $S_d \square R$  of  $G$ .

**Lemma 5.**  *$X$  contains a set of at least  $\min\{\lfloor d/2^n \rfloor, \lceil n/2 \rceil\}$  pairwise crossing edges with respect to  $<$ .*

*Proof.* Extend the total order  $<$  to a partial order over subsets of  $V(G)$ , where for all  $V, W \subseteq V(G)$ , we have  $V < W$  if and only if  $v < w$  for each  $v \in V$  and each  $w \in W$ . We abuse notation slightly by using  $<$  to compare elements of  $V(G)$  and subsets of  $V(G)$  so that, for  $v \in V(G)$  and  $V \subseteq V(G)$ ,  $v < V$  denotes  $\{v\} < V$ . We will define sets  $A_1 \supseteq \dots \supseteq A_n$  of leaves of  $S_d$  so that each  $A_i$  satisfies the following conditions:

- (C1)  $A_i$  contains  $d_i \geq d/2^{i-1}$  leaves of  $S_d$ .
- (C2) Each leaf  $v \in A_i$  defines an  $i$ -element vertex set  $Z_{i,v} := \{(v, p_j) : j \in \{1, \dots, i\}\}$ . For any distinct  $v, w \in A_i$ , the sets  $Z_{i,v}$  and  $Z_{i,w}$  are *separated* with respect to  $<$ ; that is,  $Z_{i,v} < Z_{i,w}$  or  $Z_{i,v} > Z_{i,w}$ .

Before defining  $A_1, \dots, A_n$  we first show how the existence of the set  $A_n$  implies the lemma. To avoid triple-subscripts, let  $d' := d_n \geq d/2^{n-1}$ . The set  $A_n$  defines vertex sets  $Z_{n,v_1} < \dots < Z_{n,v_{d'}}$  (see Figure 3). Recall that  $r$  is the root of  $S_b$  so it is adjacent to each of  $v_1, \dots, v_{d'}$  in  $S_d$ . Therefore, for each  $j \in \{1, \dots, n\}$  and each  $i \in \{1, \dots, d'\}$ , the edge  $(r, p_j)(v_i, p_j)$  is in  $X$ . Therefore,  $(r, p_j)$  is adjacent to an element of each of  $Z_{n,v_1}, \dots, Z_{n,v_{d'}}$ .

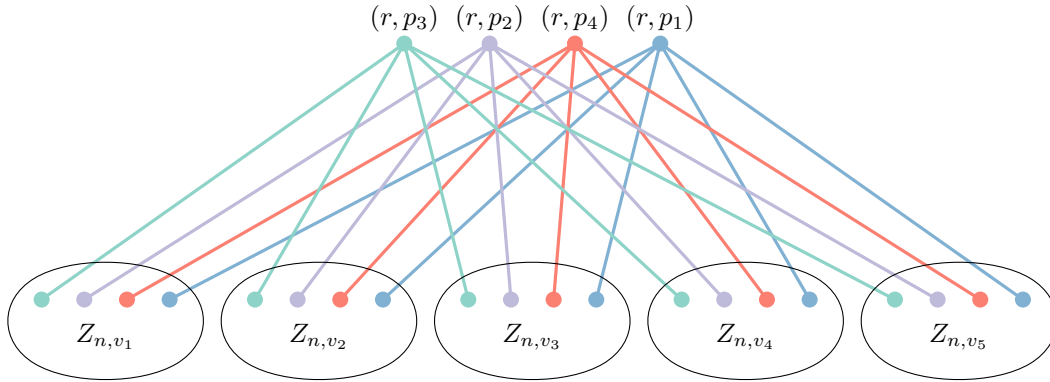


Figure 3: The sets  $Z_{n,v_1}, \dots, Z_{n,v_{d'}}$  where  $n = 4$  and  $d' = 5$ .

Since  $Z_{n,v_1}, \dots, Z_{n,v_{d'}}$  are separated with respect to  $<$ , when viewed from afar, this situation looks like a complete bipartite graph  $K_{n,d'}$  with the root vertices  $L := \{(r, p_j) : j \in \{1, \dots, n\}\}$  in one part and the groups  $R := Z_{n,v_1} \cup \dots \cup Z_{n,v_{d'}}$  in the other part. Any linear ordering of  $K_{n,d'}$  has a large set of pairwise crossing edges so, intuitively, the induced subgraph  $X[L \cup R]$  should also have a large set of pairwise crossing edges. We can formalize this as follows: Label the vertices in  $L$  as  $r_1, \dots, r_n$  so that  $r_1 < \dots < r_n$ . Then at least one of the following two cases applies (see Figure 4):

1.  $Z_{n, \lfloor d'/2 \rfloor} < r_{\lceil n/2 \rceil}$  in which case the graph between  $r_{\lceil n/2 \rceil}, \dots, r_n$  and  $Z_{n,1}, \dots, Z_{n, \lfloor d'/2 \rfloor}$  has a set of at least  $\min\{\lfloor d'/2 \rfloor, \lceil n/2 \rceil\}$  pairwise-crossing edges.

2.  $r_{\lceil n/2 \rceil} < Z_{\lceil d'/2 \rceil + 1}$  in which case the graph between  $r_1, \dots, r_{\lceil n/2 \rceil}$  and  $Z_{\lceil d'/2 \rceil + 1}, \dots, Z_{d'}$  has a set of  $\min\{\lfloor d'/2 \rfloor, \lceil n/2 \rceil\}$  pairwise-crossing edges.

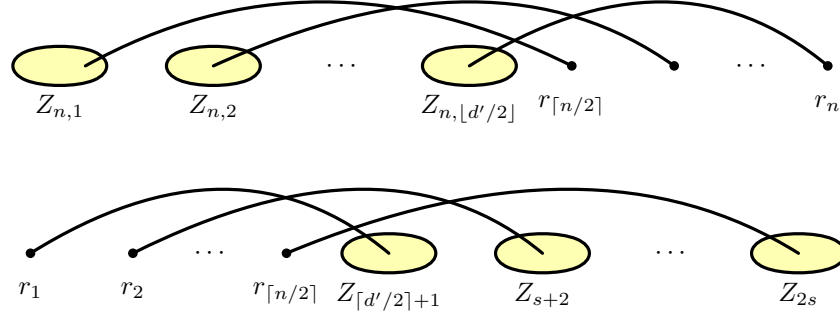


Figure 4: The two cases in the proof of Lemma 5.

Since, by (C1),  $d' \geq d/2^{n-1}$ , either case results in a set of pairwise-crossing edges of size at least  $\min\{\lfloor d/2^n \rfloor, \lceil n/2 \rceil\}$ , as claimed.

All that remains is to define the sets  $A_1 \supseteq \dots \supseteq A_n$  that satisfy (C1) and (C2). Let  $A_1$  be the set of all the leaves of  $S_d$ . For each  $i \in \{2, \dots, n\}$ , the set  $A_i$  is defined as follows: Let  $Z_1, \dots, Z_{|A_{i-1}|}$  denote the sets  $Z_{i-1,v}$  for each  $v \in A_{i-1}$  ordered so that  $Z_1 < \dots < Z_{|A_{i-1}|}$ . By Property (C2), this is always possible. Label the vertices of  $A_{i-1}$  as  $v_1, \dots, v_{|A_{i-1}|}$  so that  $(v_1, p_{i-1}) < \dots < (v_r, p_{i-1})$ . (This is equivalent to naming them so that  $(v_j, p_{i-1}) \in Z_j$  for each  $j \in \{1, \dots, |A_{i-1}|\}$ .) Define the set  $A_i := \{v_{2k+1} : k \in \{0, \dots, \lfloor (|A_{i-1}| - 1)/2 \rfloor\}\} = \{v_j \in A_{i-1} : j \text{ is odd}\}$ . This completes the definition of  $A_1, \dots, A_n$ .

All that remains is to verify that  $A_i$  satisfies (C1) and (C2) for each  $i \in \{1, \dots, n\}$ . We do this by induction on  $i$ . The base case  $i = 1$  is trivial so we assume from this point on that  $i \in \{2, \dots, n\}$ . To see that  $A_i$  satisfies (C1) just observe that  $|A_i| = \lceil |A_{i-1}|/2 \rceil \geq |A_{i-1}|/2 \geq d/2^{i-1}$ , where the final inequality follows by applying the inductive hypothesis  $|A_{i-1}| \geq d/2^{i-2}$ . Now all that remains is to show that  $A_i$  satisfies (C2).

Recall that, for each  $v \in A_{i-1}$ , the edge  $e_v := (v, p_{i-1})(v, p_i)$  is in  $X$ . We have the following properties:

- (P1) By Lemma 2,  $\phi(e_v) = \phi(p_{i-1}p_i)$  for each  $v \in A_{i-1}$ .
- (P2) Since  $p_{i-1}$  and  $p_i$  are both red,  $(v, p_{i-1}) < (w, p_{i-1})$  if and only if  $(v, p_i) < (w, p_i)$  for each  $v, w \in A_{i-1}$ .
- (P3) By Lemma 1,  $(v, p_{i-1}) < (v, p_i)$  for every  $v \in A_{i-1}$  or  $(v, p_{i-1}) > (v, p_i)$  for every  $v \in A_{i-1}$ .

We claim that these three conditions imply that the vertex sets  $\{(v, p_{i-1}) : v \in A_{i-1}\}$  and  $\{(v, p_i) : v \in A_{i-1}\}$  interleave perfectly with respect to  $<$ . More precisely:

**Claim 1.**  $(v_1, p_{i-1+t}) < (v_1, p_{i-t}) < (v_2, p_{i-1+t}) < (v_2, p_{i-t}) \dots < (v_r, p_{i-1+t}) < (v_r, p_{i-t})$  for some  $t \in \{0, 1\}$ .

*Proof of Claim 1.* By (P3) we may assume, without loss of generality, that  $(v, p_{i-1}) < (v, p_i)$  for each  $v \in A_{i-1}$ , in which case we are trying to prove the claim for  $t = 0$ . Therefore, it is sufficient to show that  $(v_j, p_i) < (v_{j+1}, p_{i-1})$  for each  $j \in \{1, \dots, r-1\}$ . For the sake of



contradiction, suppose  $(v_j, p_i) > (v_{j+1}, p_{i-1})$  for some  $j \in \{1, \dots, r-1\}$ . By the labelling of  $A_{i-1}$ ,  $(v_j, p_{i-1}) < (v_{j+1}, p_{i-1})$  so, by (P2),  $(v_j, p_i) < (v_{j+1}, p_i)$ . Therefore

$$(v_j, p_{i-1}) < (v_{j+1}, p_{i-1}) < (v_j, p_i) < (v_{j+1}, p_i) .$$

Therefore the edges  $e_{v_j} = (v_j, p_{i-1})(v_j, p_i)$  and  $e_{v_{j+1}} = (v_{j+1}, p_{i-1})(v_{j+1}, p_i)$  cross with respect to  $<$ . But this is a contradiction since, by (P1),  $\varphi(e_{v_j}) = \varphi(e_{v_{j+1}}) = \phi(p_{i-1}p_i)$ . This contradiction completes the proof of Claim 1.  $\square$

We now complete the proof that  $A_i$  satisfies (C2). Apply Claim 1 and assume without loss of generality that  $t = 0$ , so that

$$(v_1, p_{i-1}) < (v_1, p_i) < (v_2, p_{i-1}) < (v_2, p_i) \cdots < (v_r, p_{i-1}) < (v_r, p_i) .$$

For each  $j \in \{1, \dots, r-2\}$ , we have  $(v_{j+1}, p_{i-1}) \in Z_{j+1} < Z_{j+2}$ , so  $(v_j, p_i) < (v_{j+1}, p_{i-1}) < Z_{j+2}$ . Therefore  $Z_j \cup \{(v_j, p_i)\} < Z_{j+2}$ . By a symmetric argument,  $Z_j \cup \{(v_j, p_i)\} > Z_{j-2}$  for each  $j \in \{3, \dots, r\}$ . Finally, since  $(v_j, p_i) < (v_{j+2}, p_i)$  for each odd  $i \in \{1, \dots, r\}$ , we have  $Z_j \cup \{(v_j, p_i)\} < Z_{j+2} \cup \{(v_{j+2}, p_i)\}$  for each odd  $j \in \{1, \dots, r-2\}$ . Thus  $A_i$  satisfies (C2) since the sets  $Z_1 \cup \{(v_1, p_i)\}, Z_3 \cup \{(v_3, p_i)\}, \dots, Z_{2\lfloor (r-1)/2 \rfloor + 1} \cup \{(v_{2\lfloor (r-1)/2 \rfloor + 1}, p_i)\}$  are precisely the sets  $Z_{i,1}, \dots, Z_{i,d_i}$  determined by our choice of  $A_i$ .  $\square$

*Proof of Theorem 2.* Let  $G := S_b \square H_n$ , where  $n := 2s+1$  and  $b := (n^2)! s^{3n^2} ((s+1)2^n)^{2^{n^2-1}}$ . Suppose that  $G$  has an  $s$ -stack layout  $(\varphi, <)$ . In particular, there are no  $s+1$  pairwise crossing edges in  $G$  with respect to  $<$ . By Lemmas 1 to 3, we have  $a \geq b/(n^2)! = s^{3n^2} ((s+1)2^n)^{2^{n^2-1}}$  and  $c \geq a/s^{3n^2} \geq ((s+1)2^n)^{2^{n^2-1}}$  and  $d \geq c^{1/2^{n^2-1}} \geq (s+1)2^n$ . By Lemma 5, the graph  $X$ , which is a subgraph of  $G$ , contains  $\min\{\lfloor d/2^n \rfloor, \lceil n/2 \rceil\} = s+1$  pairwise crossing edges with respect to  $<$ . This contradiction shows that  $\text{sn}(G) > s$ .  $\square$

### 3 Reflections

We now mention some further consequences of our main result.

Nešetřil, Ossona de Mendez, and Wood [33] proved that graph classes with bounded stack-number or bounded queue-number have bounded expansion; see [32] for background on bounded expansion classes. The converse is not true, since cubic graphs (for example) have bounded expansion, unbounded stack-number [31] and unbounded queue-number [42]. However, prior to the present work it was open whether graph classes with polynomial expansion have bounded stack-number or bounded queue-number. It follows from the work of Dvořák, Huynh, Joret, Liu, and Wood [23, Theorem 19] that  $(S_b \square H_n)_{b,n \in \mathbb{N}}$  has polynomial expansion. So Theorem 2 implies there is a class of graphs with polynomial expansion and with unbounded stack-number. It remains open whether graph classes with polynomial expansion have bounded queue-number. See [14, 17] for several examples of graph classes with polynomial expansion and bounded queue-number.

Our main result also resolves a question of Bonnet, Geniet, Kim, Thomassé, and Watrigant [7] concerning *sparse twin-width*; see [7–9] for the definition and background on (sparse) twin-width. Bonnet et al. [7] proved that graphs with bounded stack-number have



bounded sparse twin-width, and they write that they “believe that the inclusion is strict”; that is, there exists a class of graphs with bounded sparse twin-width and unbounded stack-number. Theorem 2 confirms this intuition, since it follows from [7, Theorem 10] that the class of all subgraphs of  $(S_b \square H_n)_{b,n \in \mathbb{N}}$  has bounded sparse twin-width<sup>3</sup>. It remains open whether bounded sparse twin-width coincides with bounded queue-number.

Finally, we mention some more open problems:

- Recall that every 1-queue graph has a 2-stack layout [28] and we proved that there are 4-queue graphs with unbounded stack-number. The following questions remain open: Do 2-queue graphs have bounded stack-number? Do 3-queue graphs have bounded stack-number?
- Given the role of cartesian products in our proof, it is natural to ask when is  $\text{sn}(G_1 \square G_2)$  bounded? As illustrated in Figure 1,  $\text{sn}(H_n) \leq 2$ . So  $\text{sn}(G_1 \square G_2)$  can be unbounded even when  $G_1$  is a star and  $\text{sn}(G_2) \leq 2$ . Since  $\text{sn}(G_2) \leq 1$  if and only if  $G_2$  is outerplanar, the following questions naturally arise: Is  $\text{sn}(S \square H)$  bounded for every star  $S$  and outerplanar graph  $H$  with bounded degree? Is  $\text{sn}(T \square H)$  bounded for every tree  $T$  and outerplanar graph  $H$  with bounded degree? The assumption that  $H$  has bounded degree is needed since  $S_n \square S_n$  contains the 1-subdivision of  $K_{n,n}$ , which has unbounded stack-number [5].
- Since  $H_n \subseteq P \boxtimes P$  where  $P$  is the  $n$ -vertex path, Theorem 1 implies that  $\text{sn}(S \boxtimes P \boxtimes P)$  is unbounded for all stars  $S$  and paths  $P$ . It is easily seen that  $\text{sn}(S \boxtimes P)$  is bounded [35]. The following question naturally arises (independently asked by Pupyrev [35]): Is  $\text{sn}(T \boxtimes P)$  bounded for all trees  $T$  and paths  $P$ ? We conjecture the answer is “no”.

## References

- [1] PATRIZIO ANGELINI, GIUSEPPE DI BATTISTA, FABRIZIO FRATI, MAURIZIO PATRIGNANI, AND IGNAZ RUTTER. [Testing the simultaneous embeddability of two graphs whose intersection is a biconnected or a connected graph](#). *J. Discrete Algorithms*, 14:150–172, 2012.
- [2] MICHAEL BAUR AND ULRIC BRANDES. [Crossing reduction in circular layouts](#). In *Proc. 30th International Workshop on Graph-Theoretic Concepts in Computer Science (WG'04)*, vol. 3353 of *Lecture Notes in Computer Science*, pp. 332–343. Springer, 2004.
- [3] MICHAEL A. BEKOS, HENRY FÖRSTER, MARTIN GRONEMANN, TAMARA MCHEDLIDZE, FABRIZIO MONTECCHIANI, CHRYSANTHI N. RAFTOPOULOU, AND TORSTEN UECKERDT. [Planar graphs of bounded degree have bounded queue number](#). *SIAM J. Comput.*, 48(5):1487–1502, 2019.
- [4] FRANK R. BERNHART AND PAUL C. KAINEN. [The book thickness of a graph](#). *J. Combin. Theory Ser. B*, 27(3):320–331, 1979.
- [5] ROBIN BLANKENSHIP. [Book embeddings of graphs](#). Ph.D. thesis, Department of Mathematics, Louisiana State University, U.S.A., 2003.

<sup>3</sup>For graph classes  $\mathcal{G}$  and  $\mathcal{H}$ , let  $\mathcal{G} \boxtimes \mathcal{H} := \{G \boxtimes H : G \in \mathcal{G}, H \in \mathcal{H}\}$ . If  $\mathcal{G}$  and  $\mathcal{H}$  both have bounded twin-width,  $\mathcal{H}$  has bounded maximum degree, and no graph in  $\mathcal{G} \boxtimes \mathcal{H}$  contains a  $K_{t,t}$  subgraph for some constant  $t$ , then Bonnet et al. [7, Theorem 10] proved that the class of all subgraphs of graphs in  $\mathcal{G} \boxtimes \mathcal{H}$  has bounded sparse twin-width. Apply this result with  $\mathcal{G} := (S_b)_{b \in \mathbb{N}}$  and  $\mathcal{H} := (H_n)_{n \in \mathbb{N}}$ . Every star graph has twin-width 0 and planar graphs have bounded twin-width [9]. Since  $S_b \boxtimes H_n$  is 7-degenerate, it contains no  $K_{8,8}$  subgraph. Thus the class of all subgraphs of graphs in  $\mathcal{G} \boxtimes \mathcal{H}$  has bounded sparse twin-width. In particular, the class of all subgraphs of  $(S_b \square H_n)_{b,n \in \mathbb{N}}$  has bounded sparse twin-width.

- 
- [6] ROBIN BLANKENSHIP AND BOGDAN OPOROWSKI. [Drawing subdivisions of complete and complete bipartite graphs on books](#). Tech. Rep. 1999-4, Department of Mathematics, Louisiana State University, U.S.A., 1999.
  - [7] ÉDOUARD BONNET, COLIN GENIET, EUN JUNG KIM, STÉPHAN THOMASSÉ, AND RÉMI WATRIGANT. [Twin-width II: small classes](#). In *Proc. Annual ACM-SIAM Symp. on Discrete Algorithms (SODA '21)*. 2020. arXiv:2006.09877.
  - [8] ÉDOUARD BONNET, COLIN GENIET, EUN JUNG KIM, STÉPHAN THOMASSÉ, AND RÉMI WATRIGANT. [Twin-width III: Max independent set and coloring](#). 2020, arXiv:2007.14161.
  - [9] ÉDOUARD BONNET, EUN JUNG KIM, STÉPHAN THOMASSÉ, AND RÉMI WATRIGANT. [Twin-width I: tractable FO model checking](#). In *Proc. 61st IEEE Symp. on Foundations of Comput. Sci. (FOCS '20)*. 2020. arXiv:2004.14789.
  - [10] JEAN BOURGAIN. [Expanders and dimensional expansion](#). *C. R. Math. Acad. Sci. Paris*, 347(7-8):357–362, 2009.
  - [11] JEAN BOURGAIN AND AMIR YEHUDAYOFF. [Expansion in  \$SL\_2\(\mathbb{R}\)\$  and monotone expansion](#). *Geometric and Functional Analysis*, 23(1):1–41, 2013.
  - [12] FAN R. K. CHUNG, F. THOMSON LEIGHTON, AND ARNOLD L. ROSENBERG. [Embedding graphs in books: a layout problem with applications to VLSI design](#). *SIAM J. Algebraic Discrete Methods*, 8(1):33–58, 1987.
  - [13] GIUSEPPE DI BATTISTA, FABRIZIO FRATI, AND JÁNOS PACH. [On the queue number of planar graphs](#). *SIAM J. Comput.*, 42(6):2243–2285, 2013.
  - [14] VIDA DUJMOVIĆ, GWENAËL JORET, PIOTR MICEK, PAT MORIN, TORSTEN UECKERDT, AND DAVID R. WOOD. [Planar graphs have bounded queue-number](#). *J. ACM*, 67(4):22, 2020.
  - [15] VIDA DUJMOVIĆ, PAT MORIN, AND DAVID R. WOOD. [Layout of graphs with bounded tree-width](#). *SIAM J. Comput.*, 34(3):553–579, 2005.
  - [16] VIDA DUJMOVIĆ, PAT MORIN, AND DAVID R. WOOD. [Layered separators in minor-closed graph classes with applications](#). *J. Combin. Theory Ser. B*, 127:111–147, 2017.
  - [17] VIDA DUJMOVIĆ, PAT MORIN, AND DAVID R. WOOD. [Graph product structure for non-minor-closed classes](#). 2019, arXiv:1907.05168.
  - [18] VIDA DUJMOVIĆ, ATTILA PÓR, AND DAVID R. WOOD. [Track layouts of graphs](#). *Discrete Math. Theor. Comput. Sci.*, 6(2):497–522, 2004.
  - [19] VIDA DUJMOVIĆ, ANASTASIOS SIDIROPOULOS, AND DAVID R. WOOD. [Layouts of expander graphs](#). *Chicago J. Theoret. Comput. Sci.*, 2016(1), 2016.
  - [20] VIDA DUJMOVIĆ AND DAVID R. WOOD. [On linear layouts of graphs](#). *Discrete Math. Theor. Comput. Sci.*, 6(2):339–358, 2004.
  - [21] VIDA DUJMOVIĆ AND DAVID R. WOOD. [Stacks, queues and tracks: Layouts of graph subdivisions](#). *Discrete Math. Theor. Comput. Sci.*, 7:155–202, 2005.
  - [22] VIDA DUJMOVIĆ AND DAVID R. WOOD. [Graph treewidth and geometric thickness parameters](#). *Discrete Comput. Geom.*, 37(4):641–670, 2007.
  - [23] ZDENĚK DVOŘÁK, TONY HUYNH, GWENAËL JORET, CHUN-HUNG LIU, AND DAVID R. WOOD. [Notes on graph product structure theory](#). In *2019-20 MATRIX Annals*. Springer, to appear. arXiv:2001.08860.
  - [24] PAUL ERDŐS AND GEORGE SZEKERES. [A combinatorial problem in geometry](#). *Compositio Math.*, 2:463–470, 1935.
  - [25] DAVID GALE. [The game of Hex and the Brouwer fixed-point theorem](#). *Amer. Math. Monthly*, 86(10):818–827, 1979.
  - [26] ZVI GALIL, RAVI KANNAN, AND ENDRE SZEMERÉDI. [On 3-pushdown graphs with large separators](#). *Combinatorica*, 9(1):9–19, 1989.
-

- 
- [27] CHRISTIAN HASLINGER AND PETER F. STADLER. [RNA structures with pseudo-knots: Graph-theoretical, combinatorial, and statistical properties](#). *Bull. Math. Biology*, 61(3):437–467, 1999.
- [28] LENWOOD S. HEATH, F. THOMSON LEIGHTON, AND ARNOLD L. ROSENBERG. [Comparing queues and stacks as mechanisms for laying out graphs](#). *SIAM J. Discrete Math.*, 5(3):398–412, 1992.
- [29] LENWOOD S. HEATH AND ARNOLD L. ROSENBERG. [Laying out graphs using queues](#). *SIAM J. Comput.*, 21(5):927–958, 1992.
- [30] MICHAEL KAUFMANN, MICHAEL A. BEKOS, FABIAN KLUTE, SERGEY PUPYREV, CHRYSANTHI N. RAFTOPOULOU, AND TORSTEN UECKERDT. [Four pages are indeed necessary for planar graphs](#). *J. Comput. Geom.*, 11(1):332–353, 2020.
- [31] SETH M. MALITZ. [Graphs with  \$E\$  edges have pagenumber  \$O\(\sqrt{E}\)\$](#) . *J. Algorithms*, 17(1):71–84, 1994.
- [32] JAROSLAV NEŠETŘIL AND PATRICE OSSONA DE MENDEZ. [Sparsity](#), vol. 28 of *Algorithms and Combinatorics*. Springer, 2012.
- [33] JAROSLAV NEŠETŘIL, PATRICE OSSONA DE MENDEZ, AND DAVID R. WOOD. [Characterisations and examples of graph classes with bounded expansion](#). *European J. Combin.*, 33(3):350–373, 2011.
- [34] L. TAYLOR OLLMANN. On the book thicknesses of various graphs. In FREDERICK HOFFMAN, ROY B. LEVOW, AND ROBERT S. D. THOMAS, eds., *Proc. 4th Southeastern Conference on Combinatorics, Graph Theory and Computing*, vol. VIII of *Congr. Numer.*, p. 459. Utilitas Math., 1973.
- [35] SERGEY PUPYREV. [Book embeddings of graph products](#). 2020, arXiv:2007.15102.
- [36] ARNOLD L. ROSENBERG. The DIOGENES approach to testable fault-tolerant arrays of processors. *IEEE Trans. Comput.*, C-32:902–910, 1983.
- [37] ARNOLD L. ROSENBERG. Book embeddings and wafer-scale integration. In *Proc. 17th Southeastern International Conf. on Combinatorics, Graph Theory, and Computing*, vol. 54 of *Congr. Numer.*, pp. 217–224. 1986.
- [38] ARNOLD L. ROSENBERG. [DIOGENES, circa 1986](#). In *Proc. VLSI Algorithms and Architectures*, vol. 227 of *Lecture Notes in Comput. Sci.*, pp. 96–107. Springer, 1986.
- [39] FARHAD SHAHROKHI, ONDREJ SÝKORA, LÁSZLÓ A. SZÉKELY, AND IMRICH VŘÍTO. [Book embeddings and crossing numbers](#). In ERNST W. MAYR, GUNTHER SCHMIDT, AND GOTTFRIED TINHOFFER, eds., *Proc. 20th International Workshop on Graph-Theoretic Concepts in Computer Science (WG '94)*, vol. 903 of *Lecture Notes in Comput. Sci.*, pp. 256–268. Springer, 1994.
- [40] DAVID R. WOOD. [Bounded degree book embeddings and three-dimensional orthogonal graph drawing](#). In PETRA MUTZEL, MICHAEL JÜNGER, AND SEBASTIAN LEIPERT, eds., *Proc. 9th International Symposium on Graph Drawing (GD '01)*, vol. 2265 of *Lecture Notes in Computer Science*, pp. 312–327. Springer, 2001.
- [41] DAVID R. WOOD. [Queue layouts of graph products and powers](#). *Discrete Math. Theor. Comput. Sci.*, 7(1):255–268, 2005.
- [42] DAVID R. WOOD. [Bounded-degree graphs have arbitrarily large queue-number](#). *Discrete Math. Theor. Comput. Sci.*, 10(1):27–34, 2008. arXiv:0601293.
- [43] MIHALIS YANNAKAKIS. [Embedding planar graphs in four pages](#). *J. Comput. System Sci.*, 38(1):36–67, 1989.
- [44] MIHALIS YANNAKAKIS. [Planar graphs that need four pages](#). *J. Combin. Theory Ser. B*, 145:241–263, 2020.
-