STACK NUMBER IS NOT QUEUE-NUMBER BOUNDED

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Abstract. We describe a family of graphs in which every member has queue number at most 4, but for every integer s, there is a member of the family whose stack number is greater than s. This resolves open problems of ??? and Blankenship and Oporwoski (???).

1 Introduction

STACKS vs QUEUES

LINEAR LAYOUTS, CROSSINGS AND NESTINGS

STACKS AND QUEUES

STACK-NUMBER AND QUEUE-NUMBER

BOUNDED PARAMETERS

IMPORTANT PAPERS [3, 5, 7–9]

IS STACK-NUMBER BOUNDED BY QUEUE-NUMBER?

Heath et al. [7] showed that every 1-queue graph has a 2-stack layout. Heath et al. [7] showed that the ternary hypercubes requires exponentially more stacks than queues. In particular, n-vertex ternary hypercubes have queue-number at most $2\log_3 n$, but stack-number at least $\Omega(n^{1/9-\epsilon})$ for any $\epsilon > 0$. We prove the following theorem, which shows that stack-number is not bounded by queue-number.

Theorem 1. *For every $s \in \mathbb{N}$ there exists a graph G with $\operatorname{qn}(G) \leqslant 4$ and $\operatorname{sn}(G) > s$.*

IS QUEUE-NUMBER BOUNDED BY STACK-NUMBER?

Heath et al. [7] showed that every 1-stack graph has a 2-queue layout. Dujmović et al. [4] showed that planar graphs have bounded queue-number. In particular, 2-stack graphs have bounded queue-number. It is open whether 3-stack graphs have bounded queue-number. In fact, the case of three stacks is as hard as the general question. Dujmović

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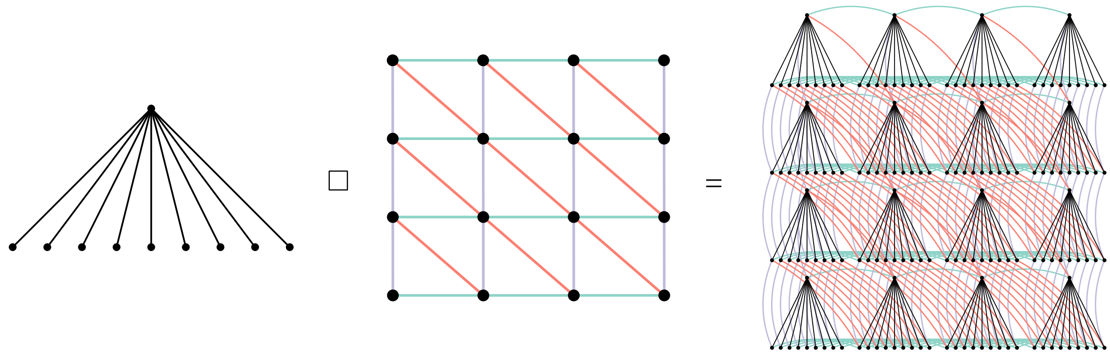

Figure 1: $S_9 \square Q_4$.

and Wood [5] proved that queue-number is bounded by stack-number if and only if 3-stack graphs have bounded queue-number. Moreover, if this is true then stack-number is bounded by a polynomial function of queue-number.

BLANKENSHIP AND OPOROWSKI CONJECTURE [1, 2]

The graph G in Theorem 1 is obtained as follow. Let S_b denote the star graph with root r and b leaves. For an even positive integer n, let Q_n be the $n \times n$ triangulated grid, defined by $V(Q_n) := \{1, \dots, n\}^2$ and

$$\begin{aligned} E(Q_n) := \{&(x,y)(x+1,y) : x \in \{1,\dots,n-1\}, y \in \{1,\dots,n\}\} \\ &\cup \{(x,y)(x,y+1) : x \in \{1,\dots,n\}, y \in \{1,\dots,n-1\}\} \\ &\cup \{(x,y+1)(x+1,y) : x,y \in \{1,\dots,n-1\}\} \ . \end{aligned}$$

We consider the graph $G := S_b \square Q_n$. See Figure 1. That is, $V(G) = V(S_b) \times V(Q_n)$ where vertices $(v_1, w_1), (v_2, w_2) \in V(G)$ are adjacent whenever $v_1 = w_1$ and $v_2 w_2 \in E(Q_n)$, or $v_1 w_1 \in E(S_b)$ and $v_2 = w_2$.

Suggestion: Replace Q_n with H (for *Hex*)

2 Queue-Number Upper Bound

To prove that $\text{qn}(G) \leqslant 4$ in Theorem 1 we need the following definition due to Wood [10]. A queue layout $(\varphi, \prec)$ is *strict* if for every vertex $u \in V(G)$ and for all neighbours $v, w \in N_G(u)$ with $u \prec v, w$ or $v, w \prec u$, we have $\phi(uv) \neq \phi(uw)$. Let $\text{sqn}(G)$ be the minimum integer k such that G has a strict k-queue layout. Note that $\text{sqn}(Q_n) \leqslant 3$: Order the vertices row-by-row and then left-to-right within a row, with vertical edges in one queue, horizontal edges in one queue, and diagonal edges in another queue. Wood [10] proved that $\text{qn}(G \square H) \leqslant \text{qn}(G) + \text{sqn}(H)$ for all graphs G and H. Of course, S_b has a 1-queue layout (since no two edges are nested for any vertex-ordering). Thus

$$\text{qn}(S_b \square Q_n) \leqslant 4.$$

ADD TO DISCUSSION LATER: Q_n is planar with a Hamiltonian cycle (assuming n is even), so $\text{sn}(Q_n) \leqslant 2$

3 Stack-Number Lower Bound

Consider a hypothetical s-stack layout $(\varphi, \prec)$ of G where n and b are chosen sufficiently large compared to s as detailed below. We begin with three lemmata that, for sufficiently large b, allow us to find a large subgraph S_d of S_b for which the stack layout $(\varphi, \prec)$ of $S_d \square Q_n$ is highly structured.

Does anyone know if there is a standard box operator that is typeset like this $S \boxtimes Q$ or $S \boxdot Q$ instead of like this $S\square Q$ or like this $S\square Q$? I tried square and Box.

For each node v of S_b, we define π_v as the permutation of $\{1,\dots,n\}^2$ in which (x_1, y_1) appears before (x_2, y_2) if and only $(v, x_1, y_1) \prec (v, x_2, y_2)$. The following lemma is an immediate consequence of the Pigeonhole Principle:

Lemma 1. *There exists a permutation π of $\{1,\dots,n\}^2$ and a set L_1 of leaves of S_b of size $b_1 \geqslant \lceil b/(n^2)!\rceil$ such that $\pi_v = \pi$ for each $v \in L_1$.*

For each leaf v in L, let φ_v be the edge colouring of Q_n defined by $\varphi_v(x,y) := \varphi(v,x,y)$. The edges of Q_n are defined as the union of three sets, each of which has size less than n^2, so Q_n has less than $3n^2$ edges. Therefore are fewer than s^{3n^2} edge colourings of Q_n using s colours. Another application of the Pigeonhole Principle proves the following:

Lemma 2. *There exists a subset $L_2 \subseteq L_1$ of size $b_2 \geqslant b_1/s^{3n^2}$ and an edge colouring $\varphi_0 : Q_n \to \{1,\dots,s\}$ such that $\varphi_v = \varphi_0$ for each $v \in L_2$.*

The preceding two lemmas ensure that, for distinct leaves v and w of S_{b_2}, the stack layout of the isomorphic graphs $Q_v := G[\{(v,p) : p \in V(Q)]$ and $Q_w := G[\{(w,p) : p \in V(Q)]$ is identical. The next lemma is a statement about the relationships between the stack layouts of $S_p := G[\{(v,p) : v \in V(S_{b_2})]$ and $S_q := G[\{(v,q) : v \in V(S_{b_2})]$ for two distinct $p, q \in V(Q)$. It cannot assert that these two layouts are identical but it does state that they fall into one of two categories.

Lemma 3. *There exists a sequence $L_3 := u_1,\dots,u_{b_3}$ with $\{u_1,\dots,u_{b_3}\} \subseteq L_2$ of length $b_3 \geqslant (b_2)^{1/2^{n^2-1}}$ such that, for each $p \in V(Q_n)$, $(u_1,p) \prec (u_2,p) \prec \cdots \prec (u_{b_3},p)$ or $(u_1,p) \succ (u_2,p) \succ \cdots \succ (u_{b_3},p)$.*

Proof. Let $p_1,\dots,p_{n^2}$ denote the vertices of Q_n, in any order. Begin with the sequence $S_1 := v_{1,1},\dots,v_{1,d_1}$ that contains all $d_1 := b_2$ elements of L_2 ordered so that $(v_{1,1},p_1) \prec \cdots \prec (v_{1,d_1},p_1)$. For each $i \in \{2,\dots,n^2\}$, the Erdős-Szekeres Theorem implies that, S_{i-1} contains a subsequence $S_i := v_{i,1},\dots,v_{i,d_i}$ of length $d_i \geqslant \sqrt{|S_{i-1}|}$ such that $(v_{i,1},p_i) \prec \cdots \prec (v_{i,d_i},p_i)$ or $(v_{i,1},p_i) \succ \cdots \succ (v_{i,d_i},p_i)$. It is straightforward to verify by induction that $d_i \geqslant b_2^{1/2^{i-1}}$ resulting in a final sequence $S_{n^2} =: L_3$ of length at least $b_2^{1/2^{n^2-1}}$. □

Let $d := b_3$ and let S_d be the the subgraph of S_b induced by $\{r\} \cup \{u_1,\dots,u_d\}$ where $u_1,\dots,u_d$ is the sequence of leaves defined in Lemma 3. Consider the (improper) vertex colouring of Q_n obtained by colouring each vertex $p \in V(Q_n)$ *red* if $(u_1,p) \prec \cdots \prec (u_d,p)$ and colouring p *blue* if $(u_1,p) \succ \cdots \succ (u_d,p)$.

Lemma 4. *The graph Q_n contains an n-vertex path R consisting entirely of red vertices or entirely of blue vertices.*

Proof. The dual of Q_n is the board on which the game Hex is played. The well-known *Hex Lemma* states that any colouring of the vertices of Q_n with colours red and blue contains exactly one of the following [6]:

1. a path with endpoints $(x,1)$ and (x',n) consisting entirely of red vertices, for some $x,x' \in \{1,\dots,n\}$; or
2. a path with endpoints $(1,y)$ and (n,y') consisting entirely of blue vertices, for some $y,y' \in \{1,\dots,n\}$.

In either case, the path contains at least n vertices and therefore has a n-vertex subpath consisting entirely of red vertices or entirely of blue vertices. $\square$

Without loss of generality, assume that the path $R := p_1,\dots,p_n$ defined by Lemma 4 consists entirely of red vertices, so that $(u_1,p_j) \prec \dots \prec (u_d,p_j)$ for each $j \in \{1,\dots,n\}$. Recall that $(\varphi,\prec)$ is a hypothetical s-stack layout of G and therefore it is also an s-stack layout of the subgraph $X := S_d \square R$. The following result finishes the proof by showing that this is not possible when $n > 2s$ and $d > s2^{2s+1}$.

Lemma 5. *The graph X contains a set of edges of size at least* $\min\{d/2^n, n\}/2$ *that are pairwise crossing with respect to $\prec$.*

Proof. We will define two sequences of nested sets $A_1 \supseteq A_1 \supseteq A_n$ of leaves of S_d so that each A_i satisifies the following conditions:

(C1) A_i contain $d_i \geqslant d/2^i$ leaves of S_d.

(C2) Each leaf $v \in A_i$ defines an i-element vertex set $Z_{i,v} := \{(v,p_j) : j \in \{1,\dots,i\}\}$. For any distinct $v,w \in A_i$, $Z_{i,v}$ and $Z_{i,w}$ are *separated* with respect to $\prec$, i.e., $Z_{i,v} \prec Z_{i,w}$ or $Z_{i,v} \succ Z_{i,w}$.

Before defining $A_1,\dots,A_n$ we first show how the existence of the set A_n implies the lemma. To avoid triple-subscripts, let $d' := d_n \geqslant d/2^n$. The set A_n defines vertex sets $Z_{n,v_1} \prec \dots \prec Z_{n,v_{d'}}$. Refer to Figure 2. Recall that r is the root of S_b so it is adjacent to each of $v_1,\dots,v_{d'}$ in S_b. Therefore, for each $j \in \{1,\dots,n\}$ and each $i \in \{1,\dots,d'\}$, the edge $(r,p_j)(v_i,p_j)$ is in X. Therefore, (r,p_j) is adjacent to an element of each of $Z_{n,v_1},\dots,Z_{n,v_{d'}}$.

Since $Z_{n,v_1},\dots,Z_{n,v_{d'}}$ are separated with respect to $\prec$, when viewed from afar, this situation looks like a complete bipartite graph $K_{n,d'}$ with the root vertices $L := \{(r,p_j) : j \in \{1,\dots,n\}\}$ in left part and the groups $R := Z_{n,v_1} \cup \dots \cup Z_{n,v_{d'}}$ in the right part. Any linear ordering of $K_{n,d'}$ has a large set of pairwise crossing edges so, intuitively, the graph induced by $L \cup R$ should also have a large set of pairwise crossing edges. Lemma 6, below, formalizes this and shows that this graph has a set of at least $\min\{d',n\}/2$ pairwise crossing edges.

All that remains is to define the sets $A_1 \supseteq \dots \supseteq A_n$. The set A_1 contains all the leaves of S_d. For each $i \in \{2,\dots,n\}$, the set A_i is defined as follows: Let $Z_1,\dots,Z_r$ denote the sets $\{\{(v,p_j) : j \in \{1,\dots,i-1\}\} : v \in A_{i-1}\}$ ordered so that $Z_1 \prec \dots \prec Z_r$. By Property (C2), this

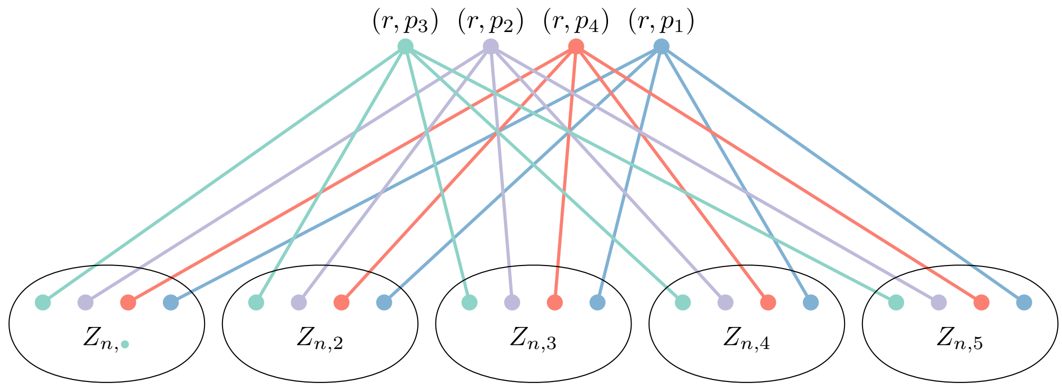

Figure 2: The sets $Z_{n,1},\ldots,Z_{n,d'}$ ($n=4$, $d'=5$).

is always possible. Label the vertices of A_{i-1} as $v_1,\ldots,v_r$ so that $(v_1,p_{i-1}) \prec \cdots \prec (v_r,p_{i-1})$. (This is equivalent to naming them so that $(v_j,p_j)\in Z_j$ for each $j\in\{1,\ldots,r\}$.) We define the set $A_i := \{v_{2k+1} : k\in\{0,\ldots,\lfloor(r-1)/2\rfloor\}$. This completes the definition of $A_1,\ldots,A_n$.

All that remains is to verify that A_i satisfies (C1) and (C2). We do this by induction on i. The base case $i=1$ is trivial so we assume from this point on that $i\in\{2,\ldots,n\}$. To see that A_i satisfies (C1) just observe that $|A_i| = \lceil r/2\rceil \geqslant r/2 = |A_{i-1}|/2 \geqslant d/2^i$. All that remains is to show that A_i satisfies (C2).

For each $j\in\{i-1,i\}$, let $Q_j := Q_n[\{(v,p_j) : v\in A_{i-1}\}]$. Recall that, for each $v\in A_{i-1}$, the edge $e_v := (v,p_{i-1})(v,p_i)$ is in X. We have the following properties:

Tiny notation conflict here: Q_n, Q_i, Q_{i-1}.

(P1) By Lemma 2, $\varphi(e_v) = \varphi_0(p_{i-1},p_i)$ does not depend on v. In particular for distinct $v,w\in A_{i-1}$ the edges e_v and e_w do not cross.

(P2) By the application of Lemma 4 the order of vertices in Q_{i-1} by $\prec$ is identical to the order of vertices in Q_i by $\prec$. That is $(v,p_{i-1})\prec(w,p_{i-1})$ if and only if $(v,p_i)\prec(w,p_i)$ for each $v,w\in A_{i-1}$.

(P3) By Lemma 1, $(v,p_{i-1})\prec(v,p_i)$ for every $v\in A_{i-1}$ or $(v,p_{i-1})\succ(v,p_i)$ for every $v\in A_{i-1}$.

We claim that these three conditions imply that the vertex sets Q_{i-1} and Q_i interleave perfectly with respect to $\prec$. More precisely:

Claim 1. *$(v_1,p_{i-1+t})\prec(v_1,p_{i-t})\prec(v_2,p_{i-1+t})\prec(v_2,p_{i-t})\cdots\prec(v_r,p_{i-1+t})\prec(v_r,p_{i-t})$ for some $t\in\{0,1\}$.*

Proof of Claim 1. By (P3) we may assume, without loss of generality, that $(v,p_{i-1})\prec(v,p_i)$ for each $v\in A_{i-1}$, in which case we are trying to prove the claim for $t=0$. It is sufficient, therefore to show that $(v_j,p_i)\prec(v_{j+1},p_{i-1})$ for each $j\in\{1,\ldots,r-1\}$. For the sake of contradiction, suppose $(v_j,p_i)\succ(v_{j+1},p_{i-1})$ for some $j\in\{1,\ldots,r-1\}$. By definition $(v_j,p_{i-1})\prec(v_{j+1},p_{i-1})$ so, by (P2) $(v_j,p_i)\prec(v_{j+1},p_i)$. Therefore

$$(v_j,p_{i-1})\prec(v_{j+1},p_{i-1})\prec(v_j,p_i)\prec(v_{j+1},p_i) \ .$$

Therefore the edges $(v_j,p_{i-1})(v_j,p_i)$ and $(v_{j+1},p_{i-1})(v_{j+1},p_i)$ cross. But this contradicts (P1) and completes the proof of Claim 1. $\square$

Now, apply Claim 1 and assume, without loss of generality that $t = 0$, so that

$$(v_1, p_{i-1}) \prec (v_1, p_i) \prec (v_2, p_{i-1}) \prec (v_2, p_i) \cdots \prec (v_r, p_{i-1}) \prec (v_r, p_i) \ .$$

For each $j \in \{1, \ldots, r-2\}$, $(v_{j+1}, p_{i-1}) \in Z_{j+1} \prec Z_{j+2}$, so $(v_j, p_i) \prec (v_{j+1}, p_{i-1}) \prec Z_{j+2}$. Therefore $Z_j \cup \{(v_j, p_i)\} \prec Z_{j+2}$. By a symmetric argument, $Z_j \cup \{(v_j, p_i)\} \succ Z_{j-2}$ for each $j \in \{2, \ldots, r\}$. Finally, since $(v_j, p_i) \prec (v_{j+2}, p_i)$ for each odd $i \in \{1, \ldots, r\}$, we have $Z_j \cup \{(v_j, p_i)\} \prec Z_{j+2} \cup \{(v_{j+2}, p_i)\}$ for each odd $j \in \{1, \ldots, r-2\}$. Thus A_i satisifies (C2) since the sets $Z_1 \cup \{(v_1, p_i)\}, Z_3 \cup \{(v_3, p_i)\}, \ldots, Z_{2\lfloor(\lfloor r-1)/2\rfloor+1} \cup (v_{2\lfloor(\lfloor r-1)/2\rfloor+1}, p_i)$ are precisely the sets $Z_{i,1}, \ldots, Z_{i,d_i}$ determined by our choice of A_i. □

Lemma 6. *Let G be any graph, let $\prec$ be any linear ordering of $V(G)$, let $Z_1 \prec \cdots \prec Z_{2s}$ be subsets of $V(G)$, and let $r_1 \prec \cdots \prec r_{2s}$ be vertices of G such that, for each $i, j \in \{1, \ldots, 2s\}$, G contains an edge $r_i z_j$ with $z_j \in Z_j$. Then G contains a set of s edges that are pairwise crossing with respect to $\prec$.*

Proof. At least one of the following two cases applies:

1. $Z_s \prec r_{s+1}$ in which case the graph between $r_{s+1}, \ldots, r_{2s}$ and $Z_1, \ldots, Z_s$ has a set of s pairwise-crossing edges.

2. $r_s \prec Z_{s+1}$ in which case the graph between $r_1, \ldots, r_s$ and $Z_{s+1}, \ldots, Z_{2s}$ has a set of s pairwise-crossing edges. □

4 Open Problems

Recall that every 1-queue graph has a 2-stack layout [7] and we proved that there are 4-queue graphs with unbounded stack-number. The following questions remain open: Do 2-queue graphs have bounded stack-number? Do 3-queue graphs have bounded stack-number?

Is $\text{sn}(T \square H)$ bounded for every tree T and outerplanar graph H with bounded degree?

Is $\text{sn}(T \boxtimes P)$ bounded for every tree T and path P?

References

[1] Robin Blankenship and Bogdan Oporowski. Drawing subdivisions of complete and complete bipartite graphs on books. Tech. Rep. 1999-4, Department of Mathematics, Louisiana State University, U.S.A., 1999.

[2] Robin Blankenship and Bogdan Oporowski. Book embeddings of graphs and minor-closed classes. In *Proc. 32nd Southeastern Int'l Conf. on Combinatorics, Graph Theory and Computing*. Department of Mathematics, Louisiana State University, 2001.

[3] Fan R. K. Chung, F. Thomson Leighton, and Arnold L. Rosenberg. Embedding graphs in books: a layout problem with applications to VLSI design. *SIAM J. Algebraic Discrete Methods*, 8(1):33–58, 1987.

[4] Vida Dujmović, Gwenaël Joret, Piotr Micek, Pat Morin, Torsten Ueckerdt, and David R. Wood. Planar graphs have bounded queue-number. *J. ACM*, 67(4):22, 2020.

[5] Vida Dujmović and David R. Wood. Stacks, queues and tracks: Layouts of graph subdivisions. *Discrete Math. Theor. Comput. Sci.*, 7:155–202, 2005. MR: 2164064.

[6] David Gale. The game of Hex and the Brouwer fixed-point theorem. *Amer. Math. Monthly*, 86(10):818–827, 1979. MR: 0551501.

[7] Lenwood S. Heath, F. Thomson Leighton, and Arnold L. Rosenberg. Comparing queues and stacks as mechanisms for laying out graphs. *SIAM J. Discrete Math.*, 5(3):398–412, 1992. MR: 1172748.

[8] Lenwood S. Heath and Arnold L. Rosenberg. Laying out graphs using queues. *SIAM J. Comput.*, 21(5):927–958, 1992. MR: 1181408.

[9] Sergey Pupyrev. Book embeddings of graph products. 2020, arXiv:2007.15102.

[10] David R. Wood. Queue layouts of graph products and powers. *Discrete Math. Theor. Comput. Sci.*, 7(1):255–268, 2005. MR: 2183176.