Some Properties of Graph Products*

Robert Hickingbotham David R. Wood

November 8, 2020

Abstract

1 Introduction

Numerous graph parameters and properties have been determined for graph products:

chromatic number of Cartesian products [1] and of direct products [**?**],

connectivity of Cartesian products [**?**]

etc

2 Degeneracy

A graph G is *d-degenerate* if every subgraph of G has minimum degree at most d. The *degeneracy* of G is the minimum integer d such that G is d-degenerate. Equivalently, the *degeneracy* of G is the maximum, taken over all subgraphs H of G, of the minimum degree of H.

Theorem 1. *For all graphs G_1 and G_2,*

$$\operatorname{degen}(G_1 \square G_2) = \operatorname{degen}(G_1) + \operatorname{degen}(G_2).$$

Proof. Let $d_i := \operatorname{degen}(G_i)$. First we prove the lower bound. By definition, each G_i has a subgraph H_i with minimum degree d_i. Thus $H_1 \square H_2$ is a subgraph of $G_1 \square G_2$ with minimum degree $d_1 + d_2$. Hence $\operatorname{degen}(G_1 \square G_2) \geqslant d_1 + d_2$.

*School of Mathematics, Monash University, Melbourne, Australia (robert.hickingbotham,david.wood@monash.edu). Research supported by the Australian Research Council.

Now we prove the upper bound. Let Z be a subgraph of $G_1 \square G_2$. Our goal is to show that Z has minimum degree at most $d_1 + d_2$. Let X be the projection of Z in G_1. That is, X is the subgraph of G_1 induced by the set of vertices $v_1 \in V(G_1)$ such that $(v_1, v_2) \in V(Z)$ for some $v_2 \in V(G_2)$. Since G_1 is d_1-degenerate, there is a vertex v_1 in X with $\deg_X(v_1) \leqslant d_1$. Let Y be the subgraph of G_2 induced by the set of vertices v_2 in G_2 such that $(v_1, v_2) \in V(X)$. By construction, Y is not empty. Since G_2 is d_2-degenerate, there is a vertex v_2 in Y with $\deg_Y(v_2) \leqslant d_2$. By construction, the vertex (v_1, v_2) of $G_1 \square G_2$ is in Z. The neighbourhood of (v_1, v_2) in Z is a subset of $\{(x, v_2) : v_1x \in E(X)\} \cup \{(v_1, y) : v_2y \in E(Y)\}$. Now $|\{(x, v_2) : v_1x \in E(X)\}| = \deg_X(v_1) \leqslant d_1$ and $|\{(v_1, y) : v_2y \in E(Y)\}| = \deg_Y(v_2) \leqslant d_2$. Thus (v_1, v_2) has degree at most $d_1 + d_2$ in Z. Hence $G_1 \square G_2$ is $(d_1 + d_2)$-degenerate, and $\operatorname{degen}(G_1 \square G_2) \leqslant d_1 + d_2$. □

Theorem 2. *Let G_i be a d_i-degenerate graph with maximum degree Δ_i for $i \in \{1, 2\}$. Then:*
(2) $G_1 \times G_2$ is $(\min\{d_1\Delta_2, d_2\Delta_1\})$-degenerate.
(3) $G_1 \boxtimes G_2$ is $(d_1 + d_2 + \min\{d_1\Delta_2, d_2\Delta_1\})$-degenerate.

These bounds are tight: For (2), the product of two complete bipartite graphs realise the bound. For (3), the product of two complete graphs realizes the bound.

Is $\operatorname{degen}(G_1 \times G_2) = \min\{\operatorname{degen}(G_1)\Delta(G_2), \operatorname{degen}(G_2)\Delta(G_1)\}$?

Question: Does there exist graphs G_1 and G_2 where $G_1 \boxtimes G_2$ is $(d_1+d_2+\min\{d_1\Delta_2, d_2\Delta_1\})$-degenerate $d_i \neq \Delta_i$ for $i \in \{1, 2\}$?

3 Complete Bipartite Subgraphs

Theorem 3. *For all graphs G_1 and G_2 and for all integers $s, t \in \mathbb{N}$, we have $K_{s,t} \subseteq G_1 \times G_2$ if and only if $K_{a,b} \subseteq G_1$ and $K_{p,q} \subseteq G_2$ for some integers $a, b, p, q \geqslant 1$ with $s = ap$ and $t = bq$.*

Proof. finish this □

Theorem 4. *For all graphs G_1 and G_2 with $E(G_1) \neq \emptyset$ and $E(G_2) \neq \emptyset$ and for all integers $s, t \in \mathbb{N}$, we have $K_{s,t} \subseteq G_1 \square G_2$ if and only if $K_{s,t} \subseteq G_1$ or $K_{s,t} \subseteq G_2$, or $s, t \leqslant 2$.*

Proof. WRITE THIS If $K_{2,3} \subseteq G_2 \square G_2$ then $K_{2,3} \subseteq G_1$ or $K_{2,3} \subseteq G_2$. □

The next two lemmas characterise complete bipartite subgraphs in strong products. Let $K_{a,b,\overline{c}}$ be the graph obtained from the complete 3-partite graph $K_{a,b,c}$ by adding an edge between each pair of vertices in the part of size c. More formally, $V(K_{a,b,\overline{c}}) = A \cup B \cup C$, where A, B, C are pairwise disjoint sets with $|A| = a$, $|B| = b$ and $|C| = c$, such that $uv, vw, wu \in E(K_{a,b,\overline{c}})$ for all $u \in A$, $v \in B$, $w \in C$, and $w_1w_2 \in E(K_{a,b,\overline{c}})$ for all distinct $w_1, w_2 \in C$.

Lemma 5. *If $K_{a,b,\overline{c}} \subseteq G_1$ and $K_{p,q,\overline{r}} \subseteq G_2$, then $K_{ap+ar+cp,bq+br+cq,\overline{cr}} \subseteq G_1 \boxtimes G_2$.*

Proof. Let A, B, C be the subsets of $V(G_1)$ defining a $K_{a,b,\overline{c}}$ subgraph of G_1. Let P, Q, R be the subsets of $V(G_2)$ defining a $K_{p,q,\overline{r}}$ subgraph of G_2. Then $(A \times P) \cup (A \times R) \cup (C \times P), (B \times Q) \cup (B \times R) \cup (C \times Q), (C \times R)$ define a $K_{ap+ar+cp,bq+br+cq,\overline{cr}}$ subgraph in $G_1 \boxtimes G_2$. $\square$

Lemma 6. *If $K_{s,t} \subseteq G_1 \boxtimes G_2$ then $K_{a,b,\overline{c}} \subseteq G_1$ and $K_{p,q,\overline{r}} \subseteq G_2$ for some integers $a, b, c, p, q, r, x, y \geqslant 0$ with $s \leqslant ap + ar + cp + x$ and $t \leqslant bq + br + cq + y$ and $x + y \leqslant cr$.*

Proof. Let $\{S, T\}$ be the bipartition of a $K_{s,t}$ subgraph in $G_1 \boxtimes G_2$. Let S_i be the projection of S in G_i, and let T_i be the projection of T in G_i. Then $\{S_i \setminus T_i, T_i \setminus S_i, S_i \cap T_i\}$ are the colour classes of a complete 3-partite subgraph of G_i. Let $a := |S_1 \setminus T_1|$ and $b := |T_1 \setminus S_1|$ and $c := |S_1 \cap T_1|$. Let $p := |S_2 \setminus T_2|$ and $q := |T_2 \setminus S_2|$ and $r := |S_2 \cap T_2|$. Thus $K_{a,b,c} \subseteq G_1$, and $K_{p,q,r} \subseteq G_2$. Consider distinct vertices $u_1, v_1 \in S_1 \cap T_1$. Thus there are vertices $(u_1, u_2) \in S$ and $(v_1, v_2) \in T$, implying that $u_1v_1 \in E(G_1)$. Similarly, distinct vertices in $S_2 \cap T_2$ are adjacent in G_2. Hence $K_{a,b,\overline{c}} \subseteq G_1$, and $K_{p,q,\overline{r}} \subseteq G_2$. Let $Z := (S_1 \cap T_1) \times (S_2 \cap T_2)$ and $x := |Z \cap S|$ and $y := |Z \cap T|$. Thus $x + y \leqslant |Z| = cr$. Since $S \subseteq ((S_1 \setminus T_1) \times S_2) \cup ((S_1 \cap T_1) \times (S_2 \setminus T_2)) \cup (Z \cap S)$, we have $s \leqslant a(p+r) + cp + x$. Since $T \subseteq ((T_1 \setminus S_1) \times T_2) \cup ((S_2 \cap T_2) \times (T_2 \setminus S_2)) \cup (Z \cap T)$, we have $t \leqslant b(q + r) + cq + y$ $\square$

Theorem 7. *For all graphs G_1 and G_2, we have $K_{s,t} \subseteq G_1 \boxtimes G_2$ if and only if $K_{a,b,\overline{c}} \subseteq G_1$ and $K_{p,q,\overline{r}} \subseteq G_2$ for some integers $a, b, c, p, q, r, x, y \geqslant 0$ with $s \leqslant ap + ar + cp + x$ and $t \leqslant bq + br + cq + y$ and $x + y \leqslant cr$.*

Proof. The $(\Longrightarrow)$ direction is Lemma 6.

$(\Longleftarrow)$ Suppose that $K_{a,b,\overline{c}} \subseteq G_1$ and $K_{p,q,\overline{r}} \subseteq G_2$ for some integers $a, b, c, p, q, r \geqslant 0$ with $s \leqslant ap + ar + cp + x$ and $t \leqslant bq + br + cq + y$ and $x + y \leqslant cr$. By Lemma 5, we have $K_{ap+ar+cp,bq+br+cq,\overline{cr}} \subseteq G_1 \boxtimes G_2$. Splitting the colour class of size cr into two sets of size x and y, and combining these sets with the first and second colour classes, we obtain a $K_{ap+ar+cp+x,bq+br+cq+y}$ subgraph in $G_1 \boxtimes G_2$. Since $ap + ar + cp + x \geqslant s$ and $bq + br + cq + y \geqslant y$, we have $K_{s,t} \subseteq G_1 \boxtimes G_2$. $\square$

Corollary 8. *$S_\ell \boxtimes H_n$ contains no $K_{8,8}$.*

Proof. Suppose that $S_\ell \boxtimes H_n$ contains $K_{8,8}$. By Lemma 6, $K_{a,b,\overline{c}} \subseteq S$ and $K_{p,q,\overline{r}} \subseteq H_n$, where $8 \leqslant ap + ar + cp + x$ and $8 \leqslant bq + br + cq + y$ and $x + y \leqslant cr$. If $p + q + r \geqslant 8$, then H_n would contain a complete bipartite subgraph on 8 vertices, which is not possible since H_n contains no $K_{1,7}$, no $K_{2,6}$ and no $K_{3,3}$. Thus $p + q + r \leqslant 7$. Without loss of generality, $a \leqslant b$. Since S_ℓ contains no triangle, $c \leqslant 2$.

Case. $c = 0$: Thus $x = y = 0$. Since S_ℓ contains no 4-cycle, $a \leqslant 1$ and $8 \leqslant p + r$, which is a contradiction.

Case. $c = 1$: Since S_ℓ contains no triangle, $a = 0$. Thus $8 \leqslant p + x \leqslant p + x + y \leqslant p + r$, which is a contradiction.

Case. $c = 2$: Since S_ℓ contains no triangle, $a = b = 0$. Thus $8 \leqslant 2p + x$ and $8 \leqslant 2q + y$, implying $16 \leqslant 2p + 2q + x + y \leqslant 2p + 2q + 2r$ and $8 \leqslant p + q + r$, which is a contradiction. $\square$

4 Complete Multipartite Subgraphs

References

[1] Gert Sabidussi. Graphs with given group and given graph-theoretical properties. *Canad. J. Math.*, 9:515–525, 1957.