STACK NUMBER IS NOT QUEUE-NUMBER BOUNDED

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Abstract. We describe a family of graphs in which every member has queue number at most X, but for every integer s, there is a member of the family whose stack number is greater than s.

1 Introduction

We will prove the following theorem.

Theorem 1. *There exists a family $\mathcal{F}$ of graphs for which* $\mathrm{qn}(G) \le X$ *for every* $G \in \mathcal{F}$ *and, for every* $s \in \mathbb{N}$*, there exists* $G \in \mathcal{F}$ *for which* $\mathrm{sn}(G) > s$.

Specifically,

$$\mathcal{F} := \{S_B \square Q_n : B, Q \in \mathbb{N}\}$$

where S_B denotes the star with B leaves and Q_n is the triangulated $n \times n$ grid.

2 The Proof

Let S_B denote the star graph with root r and B leaves. Let Q be the $n \times n$ triangulated grid, defined as follows: $V(Q) := \{1, \dots, n\}^2$ and

$$\begin{aligned} E(Q) := &\{(x,y)(x+1,y) : x \in \{1,\dots,n-1\}, y \in \{1,\dots,n\}\} \\ &\cup \{(x,y)(x,y+1) : x \in \{1,\dots,n\}, y \in \{1,\dots,n-1\}\} \\ &\cup \{(x,y)(x+1,y+1) : x,y \in \{1,\dots,n-1\}\} \ . \end{aligned}$$

We will consider the graph $G := S_B \square Q$.

Lemma 1. $\mathrm{qn}(G) \le 4$.

Proof. From David's email:

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Q is planar with a Hamiltonian cycle (assuming n is even), so sn(Q) <=
2.

Also sqn(Q) <= 3: Order the vertices row-by-row and then left-to-right
within a row, vertical edges in one queue, horizontal edges in one
queue and diagonal edges in another queue.

My old paper shows qn(G \box H) <= qn(G) + sqn(H), so qn( S \box Q ) <= 4.
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□

Now, consider a hypothetical s-stack layout $(\varphi, \prec)$ of G.

For each node v of S_b, we define π_v as the permutation of $\{1,\dots,n\}^2$ in which (x_1,y_1) appears before (x_2,y_2) if and only $(v,x_1,y_1) \prec (v,x_2,y_2)$. The following lemma is an immediate consequence of the Pigeonhole Principle:

Lemma 2. *There exists a permutation π of $\{1,\dots,n\}^2$ and a set L_1 of leaves of S_B of size $B_1 \geq \lceil B/(n^2)! \rceil$ such that $\pi_v = \pi$ for each $v \in L_1$.*

For each leaf v in L, consider the subgraph Q_v of G induced by the vertex set $\{(v,x,y) : x,y \in \{1,\dots,n\}$. The edge colouring φ used in the stack layout gives an edge colouring of Q_v using s colours. The graph Q_v is isomorphic to Q, so the edge colouring of Q_v defines an edge colouring of Q. We call this colouring of Q $\varphi_v : Q \to \{1,\dots,s\}$. The graph Q has less than $7n^2$ edges, so there are fewer than s^{7n^2} edge colourings of Q. Another application of the Pigeonhole Principle proves the following:

Lemma 3. *There exists a subset $L_2 \subseteq L_1$ of size $B_2 \geq B_1/s^{7n^2}$ and an edge colouring $\varphi_0 : Q \to \{1,\dots,s\}$ such that $\varphi_v = \varphi_0$ for each $v \in L_2$.*

Lemma 4. *There exists a sequence $L_3 := u_1,\dots,u_{B_3}$ with $\{u_1,\dots,u_{B_3}\} \subseteq L_2$ of length $B_3 \geq (B_2)^{1/2^{n^2-1}}$ such that, for each $p \in V(Q)$, $(u_1,p) \prec (u_2,p) \prec \cdots \prec (u_{B_3},p)$ or $(u_1,p) \succ (u_2,p) \succ \cdots \succ (u_{B_3},p)$.*

Proof. Let $p_1,\dots,p_{n^2}$ denote the vertices of Q, in any order. Begin with the sequence $S_1 := v_{1,1},\dots,v_{1,b}$ that contains all $b_1 := B_2$ elements of L_2 ordered so that $(v_{1,1},p_1) \prec \cdots (v_{1,b},p_1)$. For each $i \in \{2,\dots,n^2\}$, the Erdős-Szekeres Theorem implies that, S_{i-1} contains a subsequence $S_i := v_{i,1},\dots,v_{i,b_i}$ of length $b_i \geq \sqrt{|S_{i-1}|}$ such that $(v_{i,1},p_i) \prec \cdots \prec (v_{i,b_i},p_i)$ or $(v_{i,1},p_i) \succ \cdots \succ (v_{i,b_i},p_i)$. It is straightforward to verify by induction that $b_i \geq B_3^{1/2^{i-1}}$ resulting in a final sequence S_{n^2} of length at least $B_2^{1/2^{n^2-1}}$. □

Let $d = B_3$ and let S_d be the the subgraph of S_b induced by $\{r\} \cup \{u_1,\dots,u_d\}$ where $u_1,\dots,u_d$ is the sequence of leaves defined in Lemma 4. Consider the vertex colouring of Q obtained by colouring each vertex $p \in V(Q)$ *red* if $(u_1,p) \prec \cdots \prec (u_d,p)$ and colouring p *blue* if $(u_1,p) \succ \cdots \succ (u_d,p)$.

Lemma 5. *The graph Q contains an n-vertex path R consisting entirely of red vertices or entirely of blue vertices.*

Proof. The dual of Q is the board on which the game Hex is played. The well-known *Hex Lemma* states that any colouring of the vertices of Q with colours red and blue contains exactly one of the following [**?**]:

1. a path with endpoints $(x,1)$ and (x',n) consisting entirely of red vertices, for some $x,x' \in \{1,\dots,n\}$; or

2. a path with endpoints $(1, y)$ and (n, y') consisting entirely of blue vertices, for some $y, y' \in \{1, \ldots, n\}$.

In either case, the path contains at least n vertices and therefore has a n-vertex subpath consisting entirely of red vertices or entirely of blue vertices. $\square$

Without loss of generality, assume that the path $R := p_1, \ldots, p_n$ defined by Lemma 5 consists entirely of red vertices, so that $(u_1, p_j) \prec \cdots \prec (u_d, p_j)$ for each $j \in \{1, \ldots, n\}$. Recall that $(\varphi, \prec)$ is a hypothetical s-stack layout of G and therefore it is also an s-stack layout of the subgraph $X := S_d \square R$. The following result finishes the proof by showing that this is not possible when $n > 2s$ and $d > s2^{2s+1}$.

Lemma 6. *The graph X contains a set of edges of size at least $\min\{d/2^n, n\}/2$ that are pairwise crossing with respect to $\prec$.*

Proof. We will define two sequences of nested sets $A_1 \supseteq A_1 \supseteq A_n$ of leaves of S_d so that each A_i satisifies the following conditions:

(C1) A_i contain $d_i \geq d/2^i$ leaves of S_d.

(C2) Each leaf $v \in A_i$ defines an i-element vertex set $Z_{i,v} := \{(v, p_j) : j \in \{1, \ldots, i\}\}$. For any distinct $v, w \in A_i$, $Z_{i,v}$ and $Z_{i,w}$ are separated with respect to $\prec$. In other words, $Z_{i,v} \prec Z_{i,w}$ or $Z_{i,v} \succ z_{i,w}$.

Before defining $A_1, \ldots, A_n$ we first show how the existence of the set A_n implies the lemma. To avoid triple-subscripts, let $d' := d_n \geq d/2^n$. The set A_n defines vertex sets $Z_{n,v_1} \prec \cdots \prec Z_{n,v_{d'}}$. Recall that r is the root of S_b so it is adjacent to each of $v_1, \ldots, v_{d'}$ in S_b. Therefore, for each $j \in \{1, \ldots, n\}$ and each $i \in \{1, \ldots, d'\}$, the edge $(r, p_j)(v_i, p_j)$ is in X. Therefore, (r, p_j) is adjacent to an element of each of $Z_{n,v_1}, \ldots, Z_{n,v_{d'}}$.

Since $Z_{n,v_1}, \ldots, Z_{n,v_{d'}}$ are separated with respect to $\prec$, when viewed from afar, this situation looks like a complete bipartite graph $K_{n,d'}$ with the root vertices $L := \{(r, p_j) : j \in \{1, \ldots, n\}\}$ in left part and the groups $R := Z_{n,v_1} \cup \cdots \cup Z_{n,v_{d'}}$ in the right part. Any linear ordering of $K_{n,d'}$ has a large set of pairwise crossing edges so, intuitively, the graph induced by $L \cup R$ should also have a large set of pairwise crossing edges. Lemma 7, below, formalizes this and shows that this graph has a set of at least $\min\{d', n\}/2$ pairwise crossing edges.

All that remains is to define the sets $A_1 \supseteq \cdots \supseteq A_n$. The set A_1 contains all the leaves of S_d. For each $i \in \{2, \ldots, n\}$, the set A_i is defined as follows: Let $Z_1, \ldots, Z_r$ denote the sets $\{\{(v, p_j) : j \in \{1, \ldots, i-1\}\} : v \in A_{i-1}$ ordered so that $Z_1 \prec \cdots \prec Z_r$. Label the vertices of A_{i-1} $v_1, \ldots, v_r$ so that $(v_1, p_{i-1}) \prec \cdots \prec (v_r, p_{i-1})$. (This is equivalent to naming them so that $(v_j, p_j) \in Z_j$ for each $j \in \{1, \ldots, r\}$.)

Now we define the set $A_i := \{v_{2k+1} : k \in \{0, \ldots, \lfloor (r-1)/2 \rfloor\}$. All that remains is to verify that A_i satisfies (C1) and (C2). To see that A_i satisfies (c1) just observe that $|A_i| = \lceil r/2 \rceil \geq r/2 = |A_{i-1}|/2 \geq d/2^i$. All that remains is to show that A_i satisfies (C2).

For each $j \in \{i-1, i\}$, let $Q_j := \{(v, p_j) : v \in A_{i-1}\}$. Recall that, for each $v \in A_{i-1}$, the edge $e_v := (v, p_{i-1})(v, p_i)$ is in X. We have the following properties:

(P1) By Lemma 3, $\varphi(e_v) = \varphi_0(p_{i-1}, p_i)$ does not depend on v. In particular for distinct $v, w \in A_{i-1}$ the edges e_v and e_w do not cross.

(P2) By the application of Lemma 5 the order of vertices in Q_{i-1} by $\prec$ is identical to the order of vertices in Q_i by $\prec$. That is $(v, p_{i-1}) \prec (w, p_{i-1})$ if and only if $(v, p_i) \prec (w, p_i)$ for each $v, w \in A_{i-1}$.

(P3) By Lemma 2, $(v, p_{i-1}) \prec (v, p_i)$ for every $v \in A_{i-1}$ or $(v, p_{i-1}) \succ (v, p_i)$ for every $v \in A_{i-1}$.

These three conditions imply that the vertex sets Q_{i-1} and Q_i interleave perfectly with respect to $\prec$. More precisely,

$$(v_1, p_{i-1+b}) \prec (v_1, p_{i-b}) \prec (v_2, p_{i-1+b}) \prec (v_2, p_{i-b}) \cdots \prec (v_r, p_{i-1+b}) \prec (v_r, p_{i-b})$$

for some $b \in \{0, 1\}$. Suppose, without loss of generality that $b = 0$. [TODO: Explain why (P1)–(P3) imply a perfect interleave.]

For each odd $j \in \{1, \ldots, r-2\}$ we have $(v_j, p_i) \prec (v_{j+1}, p_{i-1}) \prec Z_{j+2}$. Therefore $Z_j \cup \{(v_j, p_i)\} \prec Z_{j+2}$. By a symmetric argument, $Z_j \cup \{(v_j, p_i)\} \succ Z_{j-2}$ for each odd $j \in \{3, \ldots, r\}$. Finally, since $(v_j, p_i) \prec (v_{j+2}, p_i)$ for each odd $i \in \{1, \ldots, r\}$, we have $Z_j \cup \{(v_j, p_i)\} \prec Z_{j+2} \cup \{(v_{j+2}, p_i)\}$ for each odd $j \in \{1, \ldots, r-2\}$. Thus A_i satisifies (C2) since the sets $Z_1 \cup \{(v_1, p_i)\}, Z_3 \cup \{(v_3, p_i)\}, \ldots, Z_{2\lfloor(\rfloor r-1)/2\rfloor+1} \cup (v_{2\lfloor(\rfloor r-1)/2\rfloor+1}, p_i)$ are precisely the sets $Z_{i,1}, \ldots, Z_{i,d_i}$ determined by our choice of A_i. □

Lemma 7. *Let G be any graph, let $\prec$ be any linear ordering of $V(G)$, let $Z_1 \prec \cdots \prec Z_{2s}$ be subsets of $V(G)$, and let $r_1 \prec \cdots \prec r_{2s}$ be vertices of G such that, for each $i, j \in \{1, \ldots, 2s\}$, G contains an edge $r_i z_j$ with $z_j \in Z_j$. Then G contains a set of s edges that are pairwise crossing with respect to $\prec$.*

Proof. At least one of the following two cases applies:

1. $Z_s \prec r_{s+1}$ in which case the graph between $r_{s+1}, \ldots, r_{2s}$ and $Z_1, \ldots, Z_s$ has a set of s pairwise-crossing edges.

2. $r_s \prec Z_{s+1}$ in which case the graph between $r_1, \ldots, r_s$ and $Z_{s+1}, \ldots, Z_{2s}$ has a set of s pairwise-crossing edges. □