STACK-NUMBER IS NOT BOUNDED BY QUEUE-NUMBER

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November 3, 2020

ABSTRACT. We describe a family of graphs with queue-number at most 4 but unbounded stack-number. This resolves open problems of Heath, Leighton and Rosenberg (1992) and Blankenship and Oporowski (1999).

1 Introduction

Stacks and queues are fundamental data structures in computer science, but which is more powerful? In 1992, Heath, Leighton and Rosenberg [**? ?**] introduced an approach for answering this question by defining the graph parameters *stack-number* and *queue-number* (defined below), which respectively measure the power of stacks and queues for representing graphs. The following fundamental problems, implicit in [**? ?**], were made explicit by **?**][1]:

- Is stack-number bounded by queue-number?
- Is queue-number bounded by stack-number?

If stack-number is bounded by queue-number but queue-number is not bounded by stack-number, then stacks would be considered to be more powerful than queues. Similarly, if the converse holds, then queues would be considered to be more powerful than stacks. Despite extensive research on stack- and queue-numbers, these fundamental questions have remained unsolved.

We now formally define stack- and queue-number. Let G be a graph and let $\prec$ be a total order on $V(G)$. Two disjoint edges $vw, xy \in E(G)$ with $v \prec w$ and $x \prec y$ *cross* with respect to $\prec$ if $v \prec x \prec w \prec y$ or $x \prec v \prec y \prec w$, and *nest* with respect to $\prec$ if $v \prec x \prec y \prec w$ or $x \prec v \prec w \prec y$. Let $\varphi : E(G) \to \{1, \dots, k\}$ for some integer $k \geqslant 1$. Then $(\prec, \varphi)$ is a *k-stack layout* of G if vw and xy do not cross for all edges $vw, xy \in E(G)$ with $\varphi(vw) = \varphi(xy)$. Similarly,

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[1]A *graph parameter* is a function α such that $\alpha(G) \in \mathbb{R}$ for every graph G and such that $\alpha(G_1) = \alpha(G_2)$ for all isomorphic graphs G_1 and G_2. A graph parameter α is *bounded* by a graph parameter β if there exists a function f such that $\alpha(G) \leqslant f(\beta(G))$ for every graph G.

$(\prec, \varphi)$ is a *k-queue layout* of G if vw and xy do not nest for all edges $vw, xy \in E(G)$ with $\varphi(vw) = \varphi(xy)$. See Figure 1 for examples. The smallest integer s for which G has an s-stack layout is called the *stack-number* of G, denoted $\mathrm{sn}(G)$. The smallest integer q for which G has a q-queue layout is called the *queue-number* of G, denoted $\mathrm{qn}(G)$.

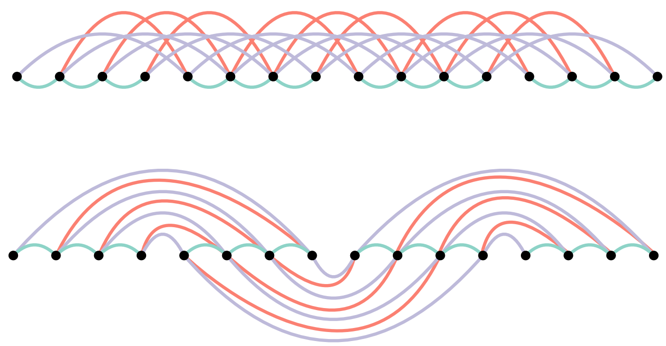

Figure 1: A 2-queue layout and a 2-stack layout of the triangulated grid graph H_4 defined below. Edges drawn above the vertices are assigned to the first queue/stack and edges drawn below the vertices are assigned to the second queue/stack.

Given a k-stack layout $(\prec, \varphi)$ of a graph G, for each $i \in \{1, \ldots, k\}$, the set $\varphi^{-1}(i)$ behaves like a stack, in the sense that each edge $vw \in \varphi^{-1}(i)$ with $v \prec w$ corresponds to an element in a sequence of stack operations, such that if we traverse the vertices in the order of $\prec$, then vw is pushed onto the stack at v and popped off the stack at w. Similarly, each set $\varphi^{-1}(i)$ in a queue layout behaves like a queue. In this way, the stack-number and queue-number respectively measure the power of stacks and queues to represent graphs.

Note that stack layouts are equivalent to book embeddings (first defined by **?**] in 1973), and stack-number is also known as *page-number*, *book-thickness* or *fixed outer-thickness*. Stack and queue layouts have other applications including computational complexity [**? ? ? ?**], RNA folding [**?**], graph drawing in two [**? ? ?**] and three dimensions [**? ? ? ?**], and fault-tolerant multiprocessing [**? ? ? ?**]. See [**? ? ? ? ? ? ? ? ? ? ?**] for bounds on the stack- and queue-number for various graph classes.

Is Stack-Number Bounded by Queue-Number?

This paper considers the first of the above questions. In a positive direction, **?**] showed that every 1-queue graph has a 2-stack layout. On the other hand, they described graphs that need exponentially more stacks than queues. In particular, n-vertex ternary hypercubes have queue-number $O(\log n)$ and stack-number $\Omega(n^{1/9-\epsilon})$ for any $\epsilon > 0$.

Our key contribution is the following theorem, which shows that stack-number is not bounded by queue-number.

Theorem 1. *For every $s \in \mathbb{N}$ there exists a graph G with $\mathrm{qn}(G) \leqslant 4$ and $\mathrm{sn}(G) > s$.*

This demonstrates that stacks are not more powerful than queues for representing graphs.

Cartesian Products

As illustrated in Figure 2, the graph G in Theorem 1 is the cartesian product[2] $S_b \,\square\, H_n$, where S_b is the star graph with root r and b leaves, and H_n is the dual of the hexagonal grid, defined by

$$V(H_n) := \{1,\dots,n\}^2 \quad \text{and} \quad \begin{aligned} E(H_n) := &\{(x,y)(x+1,y) : x \in \{1,\dots,n-1\}, y \in \{1,\dots,n\}\} \\ &\cup \{(x,y)(x,y+1) : x \in \{1,\dots,n\}, y \in \{1,\dots,n-1\}\} \\ &\cup \{(x,y)(x+1,y+1) : x,y \in \{1,\dots,n-1\}\} \ . \end{aligned}$$

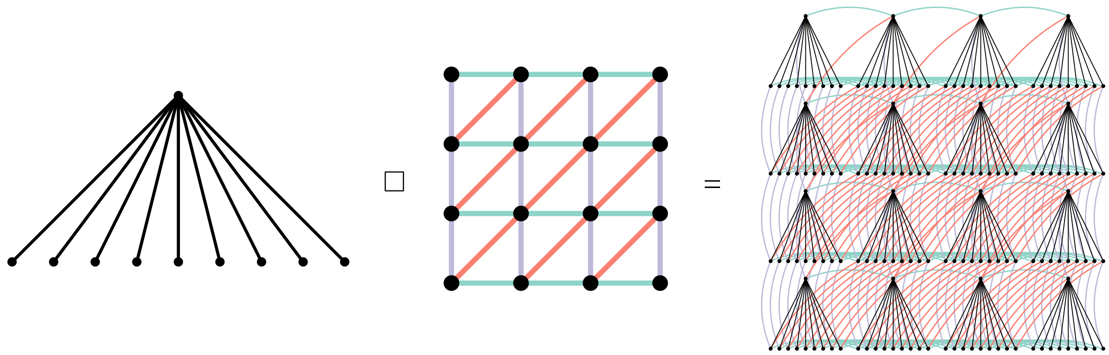

Figure 2: $S_9 \,\square\, H_4$.

We prove the following:

Theorem 2. *For every $s \in \mathbb{N}$, if b and n are sufficiently large compared to s, then*

$$\operatorname{sn}(S_b \,\square\, H_n) > s.$$

We now show that $\operatorname{qn}(S_b \square H_n) \leqslant 4$, which with Theorem 2 implies Theorem 1. We need the following definition due to **?**]. A queue layout $(\varphi, \prec)$ is *strict* if for every vertex $u \in V(G)$ and for all neighbours $v, w \in N_G(u)$, if $u \prec v, w$ or $v, w \prec u$, then $\varphi(uv) \neq \varphi(uw)$. Let $\operatorname{sqn}(G)$ be the minimum integer k such that G has a strict k-queue layout. To see that $\operatorname{sqn}(H_n) \leqslant 3$, order the vertices row-by-row and then left-to-right within a row, with vertical edges in one queue, horizontal edges in one queue, and diagonal edges in another queue (this construction puts the edges below the vertices in Figure 1 into two queues). **?**] proved that for all graphs G_1 and G_2,

$$\operatorname{qn}(G_1 \,\square\, G_2) \leqslant \operatorname{qn}(G_1) + \operatorname{sqn}(G_2). \tag{1}$$

Of course, S_b has a 1-queue layout (since no two edges are nested for any vertex-ordering). Thus $\operatorname{qn}(S_b \,\square\, H_n) \leqslant 4$.

[2]For graphs G_1 and G_2, the *cartesian product* $G_1 \,\square\, G_2$ is the graph with vertex set $\{(v_1, v_2) : v_1 \in V(G_1), v_2 \in V(G_2)\}$, where $(v_1, v_2)(w_1, w_2) \in E(G_1 \,\square\, G_2)$ if $v_1 = w_1$ and $v_2 w_2 \in E(G_2)$, or $v_1 w_1 \in E(G_1)$ and $v_2 = w_2$. The *strong product* $G_1 \boxtimes G_2$ is the graph obtained from $G_1 \,\square\, G_2$ by adding the edge $(v_1, v_2)(w_1, w_2)$ whenever $v_1 w_1 \in E(G_1)$ and $v_2 w_2 \in E(G_2)$. Note that **?**] independently suggested using graph products to show that stack-number is not bounded by queue-number.

?] implicitly proved a result similar to (1) for stack layouts. Let $\mathrm{dsn}(G)$ be the minimum integer k such that G has a k-stack layout $(\prec, \varphi)$ where φ is a proper edge-colouring of G; that is, $\varphi(vx) \neq \varphi(vy)$ for any two edges vx, vy with a common endpoint. Then for every graph G_1 and every bipartite graph G_2,

$$\mathrm{sn}(G_1 \square G_2) \leqslant \mathrm{sn}(G_1) + \mathrm{dsn}(G_2). \tag{2}$$

The key difference between (1) and (2) is that G_2 is assumed to be bipartite in (2). Theorem 2 says that this assumption is essential, since it is easily seen that $\mathrm{dsn}(H_n)$ is bounded, but $\mathrm{sn}(S_b \square H_n)$ is unbounded by Theorem 2.

Subdivisions

A noteworthy consequence of Theorem 1 is that it resolves a conjecture of **?**]. A graph G' is a *subdivision* of a graph G if G' can be obtained from G by replacing the edges vw of G by internally disjoint paths P_{vw} with endpoints v and w. If each P_{vw} has exactly k internal vertices, then G' is the *k-subdivision* of G. If each P_{vw} has at most k internal vertices, then G' is a *$(\leqslant k)$-subdivision* of G. **?**] conjectured that the stack-number of $(\leqslant k)$-subdivisions (k fixed) is not much less than the stack-number of the original graph. More precisely:

Conjecture 1 ([**?**]). *There exists a function f such that for every graph G and integer k, if G' is any $(\leqslant k)$-subdivision of G, then* $\mathrm{sn}(G) \leqslant f(\mathrm{sn}(G'), k)$.

?] established a connection between this conjecture and the question of whether stack-number is bounded by queue-number. In particular, they showed that if Conjecture 1 were true, then stack-number would be bounded by queue-number. Since Theorem 1 shows that stack-number is not bounded by queue-number, Conjecture 1 is false. The proof of **?**] is based on the following key lemma: every graph G has a 3-stack subdivision with $1 + 2\lceil \log_2 \mathrm{qn}(G) \rceil$ division vertices per edge. Applying this result to the graph $G = S_b \square H_n$ in Theorem 1, the 5-subdivision of $S_b \square H_n$ has a 3-stack layout. If Conjecture 1 were true, then $\mathrm{sn}(S_b \square H_n)$ would be at most $f(3,5)$, contradicting Theorem 1.

Is Queue-Number Bounded by Stack-Number?

It remains open whether queues are more powerful than stacks; that is, whether queue-number is bounded by stack-number. Several reults are known about this problem. **?**] showed that every 1-stack graph has a 2-queue layout. **?**] showed that planar graphs have bounded queue-number. (Note that graph products also feature heavily in this proof.) Since 2-stack graphs are planar, this implies that 2-stack graphs have bounded queue-number. It is open whether 3-stack graphs have bounded queue-number. In fact, the case of three stacks is as hard as the general question. **?**] proved that queue-number is bounded by stack-number if and only if 3-stack graphs have bounded queue-number. Moreover, if this is true then queue-number is bounded by a polynomial function of stack-number.

Other Connections

To conclude this introduction, we mentions some other properties of the graph $S_b \square H_n$. **?**] proved that graph classes with bounded stack-number or bounded queue-number have bounded expansion (see [?] for background on bounded expansion classes). The converse is not true, since cubic graphs (for example) have bounded expansion and unbounded stack-number [?] and unbounded queue-number [?]. However, prior to the present work it was open whether graph classes with polynomial expansion have bounded stack-number or bounded queue-number. It follows from the work of ? , Theorem 19] that $\{S_b \square H_n : b, n \in \mathbb{N}\}$ has polynomial expansion. So Theorem 2 implies there is a class of graphs with polynomial expansion and with unbounded stack-number. It remains open whether graph classes with polynomial expansion have bounded queue-number.

Another consequence of our result is that it resolves a question of twinwidthIIpaper concerning the novel graph parameter *twin-width* (see twinwidthIpaper for definition and background on twin-width). A graph class $\mathcal{C}$ is called *hereditary* if for every graph G in $\mathcal{C}$, every induced subgraph of G is also contained in $\mathcal{C}$. A hereditary class of graphs $\mathcal{C}$ with bounded twin-width has *bounded sparse twin-width* if there exists an integer t such that no graphs in $\mathcal{C}$ contains $K_{t,t}$ as a subgraph. twinwidthIIpaper proved that the class of graphs with bounded stack-number has bounded sparse twin-width. They believe that the inclusion is strict; that is, there exists a hereditary class of graphs with bounded sparse twin-width and unbounded stack-number. Theorem 2 implies this is so.

To see this, we first note that every tree has bounded twin-width (twinwidthIpaper). Let G and H be graphs with bounded twin-width where H also has bounded maximum degree. Then the strong product of G and H also has bounded twin-width (twinwidthIIpaper, Theorem 9). In particular, for all positive integers b, n, the graph $G^{(b,n)} := S_b \boxtimes P_n \boxtimes P_n$ has bounded twin-width. Note also that $G^{(b,n)}$ does not contain $K_{10,10}$ as a subgraph. Let $\mathcal{C}$ be the class of graphs that consists of $G^{(b,n)}$ and every induced subgraph of $G^{(b,n)}$ for all integers b, n. Then $\mathcal{C}$ is a hereditary class of graphs with bounded sparse twin-width (twinwidthIIpaper, Theorem 10). Since $G^{(b,n)}$ contains $S_b \square H_n$ as a subgraph it follows by Theorem 2 that $G^{(b,n)}$ has unbounded stack-number. This confirms that the class of graphs with bounded sparse twin-width does not coincide with the class of graphs with bounded stack-number as suspected. It remains open whether bounded sparse twin-width coincides with bounded queue-number.

Should the above two subsections go to Section 3?

2 Proof of Theorem 2

We now turn to the proof of our main result, the lower bound on $\text{sn}(G)$, where $G := S_b \square H_n$. Consider a hypothetical s-stack layout $(\varphi, \prec)$ of G where n and b are chosen sufficiently large compared to s as detailed below. We begin with three lemmata that, for sufficiently large b, provide a large subgraph S_d of S_b for which the induced stack layout of $S_d \square H_n$ is highly structured.

For each node v of S_b, define π_v as the permutation of $\{1, \dots, n\}^2$ in which (x_1, y_1) appears before (x_2, y_2) if and only if $(v, (x_1, y_1)) \prec (v, (x_2, y_2))$. The following lemma is an immediate consequence of the Pigeonhole Principle:

Lemma 1. *There exists a permutation π of $\{1, \dots, n\}^2$ and a set L_1 of leaves of S_b of size $a \geqslant b/(n^2)!$ such that $\pi_v = \pi$ for each $v \in L_1$.*

For each leaf v in L_1, let φ_v be the edge colouring of H_n defined by $\varphi_v(xy) := \varphi((v, x)(v, y))$

for each $xy \in E(H_n)$. Since H_n has maximum degree 6 and is not 6-regular, it has fewer than $3n^2$ edges. Therefore there are fewer than s^{3n^2} edge colourings of H_n using s colours. Another application of the Pigeonhole Principle proves the following:

Lemma 2. *There exists a subset $L_2 \subseteq L_1$ of size $c \geqslant a/s^{3n^2}$ and an edge colouring $\phi : E(H_n) \to \{1,\dots,s\}$ such that $\varphi_v = \phi$ for each $v \in L_2$.*

Let S_c be the subgraph of S_b induced by $L_2 \cup \{r\}$. The preceding two lemmata ensure that, for distinct leaves v and w of S_c, the stack layouts of the isomorphic graphs $G[\{(v,p) : p \in V(H_n)\}]$ and $G[\{(w,p) : p \in V(H_n)\}]$ are identical. The next lemma is a statement about the relationships between the stack layouts of $G[\{(v,p) : v \in V(S_c)\}]$ and $G[\{(v,q) : v \in V(S_c)\}]$ for distinct $p,q \in V(H_n)$. It does not assert that these two layouts are identical but it does state that they fall into one of two categories.

Lemma 3. *There exists a sequence $u_1,\dots,u_d \in L_2$ of length $d \geqslant c^{1/2^{n^2-1}}$ such that, for each $p \in V(H_n)$, either $(u_1,p) \prec (u_2,p) \prec \cdots \prec (u_d,p)$ or $(u_1,p) \succ (u_2,p) \succ \cdots \succ (u_d,p)$.*

Proof. Let $p_1,\dots,p_{n^2}$ denote the vertices of H_n in any order. Begin with the sequence $S_1 := v_{1,1},\dots,v_{1,c}$ that contains all c elements of L_2 ordered so that $(v_{1,1},p_1) \prec \cdots \prec (v_{1,c},p_1)$. For each $i \in \{2,\dots,n^2\}$, the Erdős-Szekeres Theorem [**?**] implies that S_{i-1} contains a subsequence $S_i := v_{i,1},\dots,v_{i,|S_i|}$ of length $|S_i| \geqslant \sqrt{|S_{i-1}|}$ such that $(v_{i,1},p_i) \prec \cdots \prec (v_{i,|S_i|},p_i)$ or $(v_{i,1},p_i) \succ \cdots \succ (v_{i,|S_i|},p_i)$. It is straightforward to verify by induction on i that $|S_i| \geqslant c^{1/2^{i-1}}$ resulting in a final sequence $S_{n^2} =: L_3$ of length at least $c^{1/2^{n^2-1}}$. □

For the rest of the proof we work with the star S_d whose leaves are $u_1,\dots,u_d$ described in Lemma 3. Consider the (improper) colouring of H_n obtained by colouring each vertex $p \in V(H_n)$ *red* if $(u_1,p) \prec \cdots \prec (u_d,p)$ and colouring p *blue* if $(u_1,p) \succ \cdots \succ (u_d,p)$. We need the following famous Hex Lemma [**?**].

Lemma 4 ([**?**])**.** *Every vertex 2-colouring of H_n contains a monochromatic path on n vertices.*

Apply Lemma 4 with the above-defined colouring of H_n. We obtain a path $R := p_1,\dots,p_n$ in H_n that, without loss of generality, consists entirely of red vertices; thus $(u_1,p_j) \prec \cdots \prec (u_d,p_j)$ for each $j \in \{1,\dots,n\}$. Let X be the subgraph $S_d \square R$ of G.

Lemma 5. *X contains a set of at least $\min\{\lfloor d/2^n \rfloor, \lceil n/2 \rceil\}$ pairwise crossing edges with respect to $\prec$.*

Proof. Extend the total order $\prec$ to a partial order over subsets of $V(G)$, where for all $V,W \subseteq V(G)$, we have $V \prec W$ if and only if $v \prec w$ for each $v \in V$ and each $w \in W$. We abuse notation slightly by using $\prec$ to compare elements of $V(G)$ and subsets of $V(G)$ so that, for $v \in V(G)$ and $V \subseteq V(G)$, $v \prec V$ denotes $\{v\} \prec V$. We will define sets $A_1 \supseteq \cdots \supseteq A_n$ of leaves of S_d so that each A_i satisifies the following conditions:

(C1) A_i contains $d_i \geqslant d/2^{i-1}$ leaves of S_d.

(C2) Each leaf $v \in A_i$ defines an i-element vertex set $Z_{i,v} := \{(v, p_j) : j \in \{1, \dots, i\}\}$. For any distinct $v, w \in A_i$, the sets $Z_{i,v}$ and $Z_{i,w}$ are *separated* with respect to $\prec$; that is, $Z_{i,v} \prec Z_{i,w}$ or $Z_{i,v} \succ Z_{i,w}$.

Before defining $A_1, \dots, A_n$ we first show how the existence of the set A_n implies the lemma. To avoid triple-subscripts, let $d' := d_n \geqslant d/2^{n-1}$. The set A_n defines vertex sets $Z_{n,v_1} \prec \dots \prec Z_{n,v_{d'}}$ (see Figure 3). Recall that r is the root of S_b so it is adjacent to each of $v_1, \dots, v_{d'}$ in S_d. Therefore, for each $j \in \{1, \dots, n\}$ and each $i \in \{1, \dots, d'\}$, the edge $(r, p_j)(v_i, p_j)$ is in X. Therefore, (r, p_j) is adjacent to an element of each of $Z_{n,v_1}, \dots, Z_{n,v_{d'}}$.

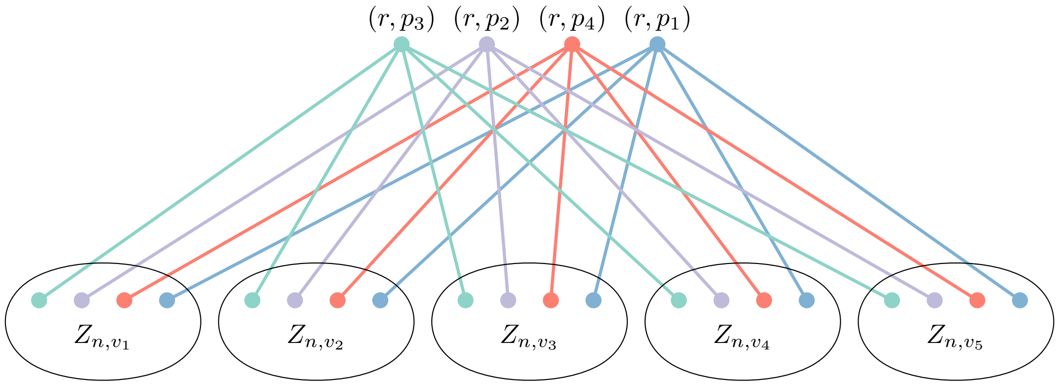

Figure 3: The sets $Z_{n,v_1}, \dots, Z_{n,v_{d'}}$ where $n = 4$ and $d' = 5$.

Since $Z_{n,v_1}, \dots, Z_{n,v_{d'}}$ are separated with respect to $\prec$, when viewed from afar, this situation looks like a complete bipartite graph $K_{n,d'}$ with the root vertices $L := \{(r, p_j) : j \in \{1, \dots, n\}\}$ in one part and the groups $R := Z_{n,v_1} \cup \dots \cup Z_{n,v_{d'}}$ in the other part. Any linear ordering of $K_{n,d'}$ has a large set of pairwise crossing edges so, intuitively, the induced subgraph $X[L \cup R]$ should also have a large set of pairwise crossing edges. We can formalize this as follows: Label the vertices in L as $r_1, \dots, r_n$ so that $r_1 \prec \dots \prec r_n$. Then at least one of the following two cases applies (see Figure 4):

1. $Z_{n,\lfloor d'/2 \rfloor} \prec r_{\lceil n/2 \rceil}$ in which case the graph between $r_{\lceil n/2 \rceil}, \dots, r_n$ and $Z_{n,1}, \dots, Z_{n,\lfloor d'/2 \rfloor}$ has a set of at least $\min\{\lfloor d'/2 \rfloor, \lceil n/2 \rceil\}$ pairwise-crossing edges.

2. $r_{\lceil n/2 \rceil} \prec Z_{\lceil d'/2 \rceil + 1}$ in which case the graph between $r_1, \dots, r_{\lceil n/2 \rceil}$ and $Z_{\lceil d'/2 \rceil + 1}, \dots, Z_{d'}$ has a set of $\min\{\lfloor d'/2 \rfloor, \lceil n/2 \rceil\}$ pairwise-crossing edges.

Since, by (C1), $d' \geqslant d/2^{n-1}$, either case results in a set of pairwise-crossing edges of size at least $\min\{\lfloor d/2^n \rfloor, \lceil n/2 \rceil\}$, as claimed.

All that remains is to define the sets $A_1 \supseteq \dots \supseteq A_n$ that satisfy (C1) and (C2). Let A_1 be the set of all the leaves of S_d. For each $i \in \{2, \dots, n\}$, the set A_i is defined as follows: Let $Z_1, \dots, Z_{|A_{i-1}|}$ denote the sets $Z_{i-1,v}$ for each $v \in A_{i-1}$ ordered so that $Z_1 \prec \dots \prec Z_{|A_{i-1}|}$. By Property (C2), this is always possible. Label the vertices of A_{i-1} as $v_1, \dots, v_{|A_{i-1}|}$ so that $(v_1, p_{i-1}) \prec \dots \prec (v_r, p_{i-1})$. (This is equivalent to naming them so that $(v_j, p_{i-1}) \in Z_j$ for each $j \in \{1, \dots, |A_{i-1}|\}$.) Define the set $A_i := \{v_{2k+1} : k \in \{0, \dots, \lfloor (|A_{i-1}| - 1)/2 \rfloor\}\} = \{v_j \in A_{i-1} : j \text{ is odd}\}$. This completes the definition of $A_1, \dots, A_n$.

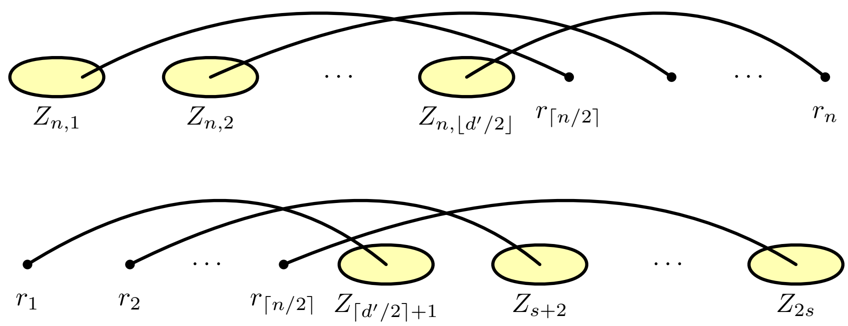

Figure 4: The two cases in the proof of Lemma 5.

All that remains is to verify that A_i satisfies (C1) and (C2) for each $i \in \{1,\ldots,n\}$. We do this by induction on i. The base case $i = 1$ is trivial so we assume from this point on that $i \in \{2,\ldots,n\}$. To see that A_i satisfies (C1) just observe that $|A_i| = \lceil |A_{i-1}|/2\rceil \geqslant |A_{i-1}|/2 \geqslant d/2^{i-1}$, where the final inequality follows by applying the inductive hypothesis $|A_{i-1}| \geqslant d/2^{i-2}$. Now all that remains is to show that A_i satisfies (C2).

Recall that, for each $v \in A_{i-1}$, the edge $e_v := (v,p_{i-1})(v,p_i)$ is in X. We have the following properties:

(P1) By Lemma 2, $\varphi(e_v) = \phi(p_{i-1}p_i)$ for each $v \in A_{i-1}$.
(P2) Since p_{i-1} and p_i are both red, $(v,p_{i-1}) \prec (w,p_{i-1})$ if and only if $(v,p_i) \prec (w,p_i)$ for each $v,w \in A_{i-1}$.
(P3) By Lemma 1, $(v,p_{i-1}) \prec (v,p_i)$ for every $v \in A_{i-1}$ or $(v,p_{i-1}) \succ (v,p_i)$ for every $v \in A_{i-1}$.

We claim that these three conditions imply that the vertex sets $\{(v,p_{i-1}) : v \in A_{i-1}\}$ and $\{(v,p_i) : v \in A_{i-1}\}$ interleave perfectly with respect to $\prec$. More precisely:

Claim 1. $(v_1,p_{i-1+t}) \prec (v_1,p_{i-t}) \prec (v_2,p_{i-1+t}) \prec (v_2,p_{i-t}) \cdots \prec (v_r,p_{i-1+t}) \prec (v_r,p_{i-t})$ *for some* $t \in \{0,1\}$.

Proof of Claim 1. By (P3) we may assume, without loss of generality, that $(v,p_{i-1}) \prec (v,p_i)$ for each $v \in A_{i-1}$, in which case we are trying to prove the claim for $t = 0$. Therefore, it is sufficient to show that $(v_j,p_i) \prec (v_{j+1},p_{i-1})$ for each $j \in \{1,\ldots,r-1\}$. For the sake of contradiction, suppose $(v_j,p_i) \succ (v_{j+1},p_{i-1})$ for some $j \in \{1,\ldots,r-1\}$. By the labelling of A_{i-1}, $(v_j,p_{i-1}) \prec (v_{j+1},p_{i-1})$ so, by (P2), $(v_j,p_i) \prec (v_{j+1},p_i)$. Therefore

$$(v_j,p_{i-1}) \prec (v_{j+1},p_{i-1}) \prec (v_j,p_i) \prec (v_{j+1},p_i) \ .$$

Therefore the edges $e_{v_j} = (v_j,p_{i-1})(v_j,p_i)$ and $e_{v_{j+1}} = (v_{j+1},p_{i-1})(v_{j+1},p_i)$ cross with respect to $\prec$. But this is a contradiction since, by (P1), $\varphi(e_{v_j}) = \varphi(e_{v_{j+1}}) = \phi(p_{i-1}p_i)$. This contradiction completes the proof of Claim 1. □

We now complete the proof that A_i satisfies (C2). Apply Claim 1 and assume without loss of generality that $t = 0$, so that

$$(v_1,p_{i-1}) \prec (v_1,p_i) \prec (v_2,p_{i-1}) \prec (v_2,p_i) \cdots \prec (v_r,p_{i-1}) \prec (v_r,p_i) \ .$$

For each $j \in \{1, \dots, r-2\}$, we have $(v_{j+1}, p_{i-1}) \in Z_{j+1} \prec Z_{j+2}$, so $(v_j, p_i) \prec (v_{j+1}, p_{i-1}) \prec Z_{j+2}$. Therefore $Z_j \cup \{(v_j, p_i)\} \prec Z_{j+2}$. By a symmetric argument, $Z_j \cup \{(v_j, p_i)\} \succ Z_{j-2}$ for each $j \in \{3, \dots, r\}$. Finally, since $(v_j, p_i) \prec (v_{j+2}, p_i)$ for each odd $i \in \{1, \dots, r\}$, we have $Z_j \cup \{(v_j, p_i)\} \prec Z_{j+2} \cup \{(v_{j+2}, p_i)\}$ for each odd $j \in \{1, \dots, r-2\}$. Thus A_i satisfies (C2) since the sets $Z_1 \cup \{(v_1, p_i)\}, Z_3 \cup \{(v_3, p_i)\}, \dots, Z_{2\lfloor (r-1)/2 \rfloor + 1} \cup (v_{2\lfloor (r-1)/2 \rfloor + 1}, p_i)$ are precisely the sets $Z_{i,1}, \dots, Z_{i,d_i}$ determined by our choice of A_i. □

Proof of Theorem 2. Let $G := S_b \square H_n$, where $n := 2s+1$ and $b := (n^2)!\, s^{3n^2} ((s+1)2^n)^{2^{n^2-1}}$. Suppose that G has an s-stack layout $(\varphi, \prec)$. In particular, there are no $s+1$ pairwise crossing edges in G with respect to $\prec$. By Lemmas 1 to 3, we have $a \geqslant b/(n^2)! = s^{3n^2}((s+1)2^n)^{2^{n^2-1}}$ and $c \geqslant a/s^{3n^2} \geqslant ((s+1)2^n)^{2^{n^2-1}}$ and $d \geqslant c^{1/2^{n^2-1}} \geqslant (s+1)2^n$. By Lemma 5, the graph X, which is a subgraph of G, contains $\min\{\lfloor d/2^n \rfloor, \lceil n/2 \rceil\} = s+1$ pairwise crossing edges with respect to $\prec$. This contradictions shows that $\mathrm{sn}(G) > s$. □

3 Open Problems

Recall that every 1-queue graph has a 2-stack layout [**?**] and we proved that there are 4-queue graphs with unbounded stack-number. The following questions remain open: Do 2-queue graphs have bounded stack-number? Do 3-queue graphs have bounded stack-number?

Given the role of cartesian products in our proof, it is natural to ask when is $\mathrm{sn}(G_1 \square G_2)$ bounded? As illustrated in Figure 1, $\mathrm{sn}(H_n) \leqslant 2$. So $\mathrm{sn}(G_1 \square G_2)$ can be unbounded even when G_1 is a star and $\mathrm{sn}(G_2) \leqslant 2$. Since $\mathrm{sn}(G_2) \leqslant 1$ if and only if G_2 is outerplanar, the following questions naturally arise: Is $\mathrm{sn}(S \square H)$ bounded for every star S and outerplanar graph H with bounded degree? Is $\mathrm{sn}(T \square H)$ bounded for every tree T and outerplanar graph H with bounded degree? The assumption that H has bounded degree is needed since $S_n \square S_n$ contains the 1-subdivision of $K_{n,n}$, which has unbounded stack-number [**?**].

Since $H_n \subseteq P \boxtimes P$ where P is the n-vertex path, Theorem 1 implies that $\mathrm{sn}(S \boxtimes P \boxtimes P)$ is unbounded for stars S and paths P. It is easily seen that $\mathrm{sn}(S \boxtimes P)$ is bounded [**?**]. The following question naturally arises (independently asked by **?**]): Is $\mathrm{sn}(T \boxtimes P)$ bounded for every tree T and path P? We conjecture the answer is “no”.