

---

# STACK-NUMBER IS NOT BOUNDED BY QUEUE-NUMBER

Vida Dujmović<sup>§</sup>, Robert Hickingbotham<sup>δ</sup>, Pat Morin<sup>σ</sup>, David R. Wood<sup>2</sup>

October 29, 2020

---

**ABSTRACT.** We describe a family of graphs with queue-number at most 4 but unbounded stack-number. This resolves open problems of Heath, Leighton and Rosenberg (1992) and Blankenship and Oporowski (1999).

---

## 1 Introduction

Stacks and queues are fundamental data structures in computer science, but which is more powerful? In 1992, Heath, Leighton and Rosenberg [12, 13] introduced an approach for answering this question by defining the graph parameters *stack-number* and *queue-number* (defined below), which respectively measure the power of stacks and queues for representing graphs. The following fundamental problems, implicit in [12, 13], were made explicit by Dujmović and Wood [8]<sup>1</sup>:

- Is stack-number bounded by queue-number?
- Is queue-number bounded by stack-number?

If stack-number is bounded by queue-number but queue-number is not bounded by stack-number, then stacks would be considered to be more powerful than queues. Similarly, if the converse holds, then queues would be considered to be more powerful than stacks. Despite extensive research on stack- and queue-numbers, these fundamental questions have remained unsolved.

We now formally define stack- and queue-number. Let  $G$  be a graph and let  $<$  be a total order on  $V(G)$ . Two disjoint edges  $vw, xy \in E(G)$  with  $v < w$  and  $x < y$  *cross* with respect to  $<$  if  $v < x < w < y$  or  $x < v < y < w$ , and *nest* with respect to  $<$  if  $v < x < y < w$  or  $x < v < w < y$ . Let  $\varphi : E(G) \rightarrow \{1, \dots, k\}$  for some integer  $k \geq 1$ . Then  $(<, \varphi)$  is a  $k$ -*stack layout* of  $G$  if, for every pair of edges  $vw, xy \in E(G)$ , if  $\varphi(vw) = \varphi(xy)$  then  $vw$  and  $xy$  do not cross. Similarly, the pair  $(<, \varphi)$  is a  $k$ -*queue layout* of  $G$  if, for every pair of edges

---

<sup>§</sup>School of Computer Science and Electrical Engineering, University of Ottawa, Ottawa, Canada (vida.dujmovic@uottawa.ca). Research supported by NSERC and the Ontario Ministry of Research and Innovation.

<sup>δ</sup>School of Mathematics, Monash University, Melbourne, Australia (robert.hickingbotham@monash.edu).

<sup>σ</sup>School of Computer Science, Carleton University, Ottawa, Canada (morin@scs.carleton.ca). Research supported by NSERC and the Ontario Ministry of Research and Innovation.

<sup>2</sup>School of Mathematics, Monash University, Melbourne, Australia (david.wood@monash.edu). Research supported by the Australian Research Council.

<sup>1</sup>A *graph parameter* is a function  $\alpha$  such that  $\alpha(G) \in \mathbb{R}$  for every graph  $G$ , such that  $\alpha(G_1) = \alpha(G_2)$  for all isomorphic graphs  $G_1$  and  $G_2$ . A graph parameter  $\alpha$  is *bounded* by a graph parameter  $\beta$  if there exists a function  $f$  such that  $\alpha(G) \leq f(\beta(G))$  for every graph  $G$ .

PM: Suggestion: Replace second if then with  $\varphi(vw) \neq \varphi(xy)$  or  $vw$  and  $xy$  do not cross.

---

$vw, xy \in E(G)$ , if  $\varphi(vw) = \varphi(xy)$  then  $vw$  and  $xy$  do not nest. The smallest integer  $s$  for which  $G$  has an  $s$ -stack layout is called the *stack-number* of  $G$ , denoted  $\text{sn}(G)$ . The smallest integer  $q$  for which  $G$  has a  $q$ -queue layout is called the *queue-number* of  $G$ , denoted  $\text{qn}(G)$ . Note that stack layouts are equivalent to book embeddings (first defined by Ollmann [15]), and stack-number is also known as *page-number*, *book-thickness* or *fixed outer-thickness*. See [1, 2, 5–7, 9, 14, 18, 19] and the references therein for work on stack- and queue-layouts.

Given a  $k$ -stack layout  $(\prec, \varphi)$  of a graph  $G$ , for each  $i \in \{1, \dots, k\}$ , the set  $\varphi^{-1}(i)$  behaves like a stack, in the sense that each edge  $vw \in \varphi^{-1}(i)$  with  $v \prec w$  corresponds to an element in a sequence of stack operations, such that if we traverse the vertices in the order of  $\prec$ , then  $vw$  is pushed onto the stack at  $v$  and popped off the stack at  $w$ . Similarly, each set  $\varphi^{-1}(i)$  in a queue layout behaves like a queue. In this way, the stack-number and queue-number respectively measure the power of stacks and queues to represent graphs.

### Is Stack-Number Bounded by Queue-number?

This paper considers the first of the above questions. In a positive direction, Heath et al. [12] showed that every 1-queue graph has a 2-stack layout. On the other hand, they described graphs that need exponentially more stacks than queues. In particular,  $n$ -vertex ternary hypercubes have queue-number  $O(\log n)$  and stack-number  $\Omega(n^{1/9-\epsilon})$  for any  $\epsilon > 0$ .

Our key contribution is the following theorem, which shows that stack-number is not bounded by queue-number. This demonstrates that stacks are not more powerful than queues for representing graphs.

**Theorem 1.** *For every  $s \in \mathbb{N}$  there exists a graph  $G$  with  $\text{qn}(G) \leq 4$  and  $\text{sn}(G) > s$ .*

As illustrated in Figure 1, the graph  $G$  in Theorem 1 is the cartesian product<sup>2</sup>  $S_b \square H_n$ , where  $S_b$  is the star graph with root  $r$  and  $b$  leaves, and  $H_n$  is the dual of the hexagonal grid, defined by

$$\begin{aligned} V(H_n) := \{1, \dots, n\}^2 \quad \text{and} \quad E(H_n) := & \{(x, y)(x+1, y) : x \in \{1, \dots, n-1\}, y \in \{1, \dots, n\}\} \\ & \cup \{(x, y)(x, y+1) : x \in \{1, \dots, n\}, y \in \{1, \dots, n-1\}\} \\ & \cup \{(x, y)(x+1, y+1) : x, y \in \{1, \dots, n-1\}\} . \end{aligned}$$

In Theorem 1,  $b$  and  $n$  are chosen to be sufficiently large compared to  $s$ . Note that Pupyrev [16] independently suggested using graph products to show that stack-number is not bounded by queue-number.

### Subdivisions

A noteworthy consequence of Theorem 1 is that it resolves a conjecture of Blankenship and Oporowski [4]. A graph  $G'$  is a *subdivision* of a graph  $G$  if  $G'$  can be obtained from  $G$  by replacing the edges  $vw$  of  $G$  by internally disjoint paths  $P_{vw}$  with endpoints  $v$  and  $w$ . If each  $P_{vw}$  has exactly  $k$  internal vertices, then  $G'$  is the  $k$ -*subdivision* of  $G$ . If each  $P_{vw}$  has at most  $k$  internal vertices, then  $G'$  is a  $(\leq k)$ -*subdivision* of  $G$ . Blankenship and Oporowski

---

<sup>2</sup>For graphs  $G_1$  and  $G_2$ , the *cartesian product*  $G_1 \square G_2$  is the graph with vertex set  $\{(v_1, v_2) : v_1 \in V(G_1), v_2 \in V(G_2)\}$ , where  $(v_1, v_2)(w_1, w_2) \in E(G_1 \square G_2)$  if  $v_1 = w_1$  and  $v_2 w_2 \in E(G_2)$ , or  $v_1 w_1 \in E(G_1)$  and  $v_2 = w_2$ .

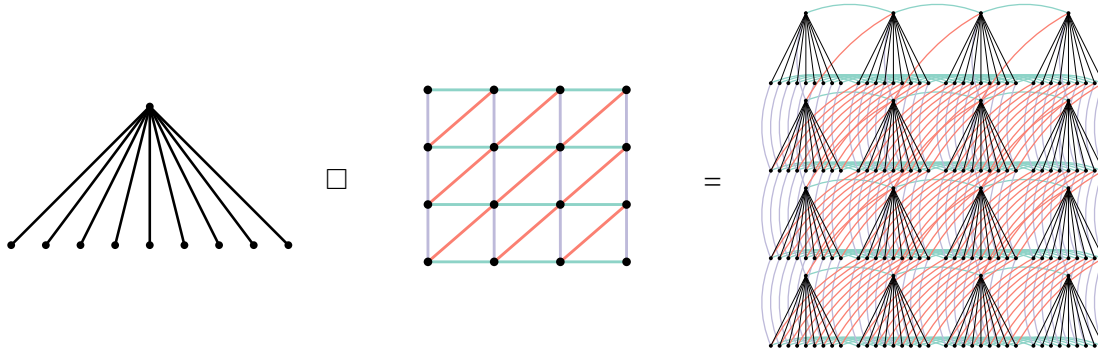


Figure 1:  $S_9 \square H_4$ .

[4] conjectured that the stack-number of  $(\leq k)$ -subdivisions ( $k$  fixed) is not much less than the stack-number of the original graph. More precisely:

**Conjecture 1** ([4]). *There exists a function  $f$  such that for every graph  $G$  and integer  $k$ , if  $G'$  is any  $(\leq k)$ -subdivision of  $G$ , then  $\text{sn}(G) \leq f(\text{sn}(G'), k)$ .*

Dujmović and Wood [8] established a connection between this conjecture and the question of whether stack-number is bounded by queue-number. In particular, they showed that if Conjecture 1 is true, then stack-number is bounded by queue-number. Since Theorem 1 shows that stack-number is not bounded by queue-number, Conjecture 1 is false. The proof of Dujmović and Wood [8] is based on the following key lemma: every graph  $G$  has a 3-stack subdivision with  $1 + 2\lceil \log_2 \text{qn}(G) \rceil$  division vertices per edge. Applying this result to the graph  $G = S_b \square H_n$  in Theorem 1, the 5-subdivision of  $S_b \square H_n$  has a 3-stack layout. If Conjecture 1 was true, then  $\text{sn}(S_b \square H_n) \leq f(3, 5)$ , contradicting Theorem 1.

### Is Queue-number Bounded by Stack-Number?

It remains open whether queues are more powerful than stacks; that is, whether queue-number is bounded by stack-number. Several results are known about this problem. Heath et al. [12] showed that every 1-stack graph has a 2-queue layout. Dujmović et al. [6] showed that planar graphs have bounded queue-number. (Note that graph products also feature heavily in this proof.) Since 2-stack graphs are planar, this implies that 2-stack graphs have bounded queue-number. It is open whether 3-stack graphs have bounded queue-number. In fact, the case of three stacks is as hard as the general question. Dujmović and Wood [8] proved that queue-number is bounded by stack-number if and only if 3-stack graphs have bounded queue-number. Moreover, if this is true then stack-number is bounded by a polynomial function of queue-number.

## 2 Stack and Queue Layouts of Cartesian Products

Add discussion of result of Bernhart and Kainen [2]:  $\text{sn}(G \square H) \leq \text{sn}(G) + \text{dsn}(H)$  for bipartite  $H$ . Highlight the key difference between stack and queue layouts is that we need  $H$  to be bipartite here.

mention results of Pupyrev [16] about bipartite graphs?

First we prove that  $\text{qn}(S_b \square H_n) \leq 4$ , as claimed in Theorem 1. We need the following definition due to Wood [17]. A queue layout  $(\varphi, <)$  is *strict* if for every vertex  $u \in V(G)$  and for all neighbours  $v, w \in N_G(u)$ , if  $u < v, w$  or  $v, w < u$ , then  $\varphi(uv) \neq \varphi(uw)$ . Let  $\text{sqn}(G)$  be the minimum integer  $k$  such that  $G$  has a strict  $k$ -queue layout. To see that  $\text{sqn}(H_n) \leq 3$ , order the vertices row-by-row and then left-to-right within a row, with vertical edges in one queue, horizontal edges in one queue, and diagonal edges in another queue. Wood [17] proved that  $\text{qn}(G \square H) \leq \text{qn}(G) + \text{sqn}(H)$  for all graphs  $G$  and  $H$ . Of course,  $S_b$  has a 1-queue layout (since no two edges are nested for any vertex-ordering). Thus  $\text{qn}(S_b \square H_n) \leq 4$ .

### 3 The Main Proof

We now turn to the proof of our main result, the lower bound on  $\text{sn}(G)$ , where  $G := S_b \square H_n$ . Consider a hypothetical  $s$ -stack layout  $(\varphi, <)$  of  $G$  where  $n$  and  $b$  are chosen sufficiently large compared to  $s$  as detailed below. We begin with three lemmata that, for sufficiently large  $b$ , provide a large subgraph  $S_d$  of  $S_b$  for which the induced stack layout of  $S_d \square H_n$  is highly structured.

For each node  $v$  of  $S_b$ , define  $\pi_v$  as the permutation of  $\{1, \dots, n\}^2$  in which  $(x_1, y_1)$  appears before  $(x_2, y_2)$  if and only if  $(v, (x_1, y_1)) < (v, (x_2, y_2))$ . The following lemma is an immediate consequence of the Pigeonhole Principle:

**Lemma 1.** *There exists a permutation  $\pi$  of  $\{1, \dots, n\}^2$  and a set  $L_1$  of leaves of  $S_b$  of size  $b_1 \geq \lceil b/(n^2)! \rceil$  such that  $\pi_v = \pi$  for each  $v \in L_1$ .*

For each leaf  $v$  in  $L_1$ , let  $\varphi_v$  be the edge colouring of  $H_n$  defined by  $\varphi_v(x, y) := \varphi(v, (x, y))$ . Since  $H_n$  has maximum degree 6 and is not 6-regular, it has fewer than  $3n^2$  edges. Therefore there are fewer than  $s^{3n^2}$  edge colourings of  $H_n$  using  $s$  colours. Another application of the Pigeonhole Principle proves the following:

**Lemma 2.** *There exists a subset  $L_2 \subseteq L_1$  of size  $b_2 \geq b_1/s^{3n^2}$  and an edge colouring  $\varphi_0 : E(H_n) \rightarrow \{1, \dots, s\}$  such that  $\varphi_v = \varphi_0$  for each  $v \in L_2$ .*

Let  $S_{b_2}$  be the subgraph of  $S_b$  induced by  $L_2 \cup \{r\}$ . The preceding two lemmata ensure that, for distinct leaves  $v$  and  $w$  of  $S_{b_2}$ , the stack layouts of the isomorphic graphs  $H^v := G[\{(v, p) : p \in V(H_n)\}]$  and  $H^w := G[\{(w, p) : p \in V(H_n)\}]$  are identical. The next lemma is a statement about the relationships between the stack layouts of  $S^p := G[\{(v, p) : v \in V(S_{b_2})\}]$  and  $S^q := G[\{(v, q) : v \in V(S_{b_2})\}]$  for distinct  $p, q \in V(H_n)$ . It does not assert that these two layouts are identical but it does state that they fall into one of two categories.

**Lemma 3.** *There exists a sequence  $L_3 := u_1, \dots, u_{b_3}$  with  $\{u_1, \dots, u_{b_3}\} \subseteq L_2$  of length  $b_3 \geq b_2^{1/2n^2-1}$  such that, for each  $p \in V(H_n)$ , either  $(u_1, p) < (u_2, p) < \dots < (u_{b_3}, p)$  or  $(u_1, p) > (u_2, p) > \dots > (u_{b_3}, p)$ .*

*Proof.* Let  $p_1, \dots, p_{n^2}$  denote the vertices of  $H_n$  in any order. Begin with the sequence  $S_1 := v_{1,1}, \dots, v_{1,d_1}$  that contains all  $d_1 := b_2$  elements of  $L_2$  ordered so that  $(v_{1,1}, p_1) < \dots < (v_{1,d_1}, p_1)$ . For each  $i \in \{2, \dots, n^2\}$ , the Erdős-Szekeres Theorem [10] implies that  $S_{i-1}$  contains a subsequence  $S_i := v_{i,1}, \dots, v_{i,d_i}$  of length  $d_i \geq \sqrt{|S_{i-1}|}$  such that  $(v_{i,1}, p_i) < \dots <$

delete  $\lceil$   
and  $\rceil$ ?

There is an annoying notation overlap of  $\varphi_v$  and  $\varphi_0$ . Replace  $\varphi_0$  by  $\phi$ ?

Do we need  $H^v$  and  $H^w$ ?

To we need  $S^p$  and  $S^q$ ?

$(v_{i,d_i}, p_i)$  or  $(v_{i,1}, p_i) > \dots > (v_{i,d_i}, p_i)$ . It is straightforward to verify by induction on  $i$  that  $d_i \geq b_2^{1/2^{i-1}}$  resulting in a final sequence  $S_{n^2} =: L_3$  of length at least  $b_2^{1/2^{n^2-1}}$ .  $\square$

Let  $d := b_3$  and let  $S_d$  be the subgraph of  $S_b$  induced by  $\{r\} \cup \{u_1, \dots, u_d\}$  where  $u_1, \dots, u_d$  is the sequence of leaves defined in Lemma 3.

Do we need “ where  $u_1, \dots, u_d$  is the sequence of leaves defined in Lemma 3”?

Consider the (improper) vertex colouring of  $H_n$  obtained by colouring each vertex  $p \in V(H_n)$  red if  $(u_1, p) < \dots < (u_d, p)$  and colouring  $p$  blue if  $(u_1, p) > \dots > (u_d, p)$ . We need the following famous Hex Lemma [11].

**Lemma 4** ([11]). *Every red–blue vertex colouring of the graph  $H_n$  contains an  $n$ -vertex path  $R$  consisting entirely of red vertices or entirely of blue vertices.*

Without loss of generality, assume that the path  $R := p_1, \dots, p_n$  defined by Lemma 4 (with the above-defined colouring) consists entirely of red vertices, so that  $(u_1, p_j) < \dots < (u_d, p_j)$  for each  $j \in \{1, \dots, n\}$ . Recall that  $(\varphi, <)$  is a hypothetical  $s$ -stack layout of  $G$  and therefore it is also an  $s$ -stack layout of the subgraph  $X := S_d \square R$ . In particular, there is no set of greater than  $s$  pairwise crossing edges in  $X$ . The following result finishes the proof by showing that such a set exists when  $n > 2s$  and  $d > s2^n$  (which is implied if  $b \geq (n^2)! s^{3n^2} d^{2^{n^2-1}}$ ).

**Lemma 5.** *The graph  $X$  contains a set of edges of size at least  $\min\{d/2^n, n/2\}$  that are pairwise crossing with respect to  $<$ .*

*Proof.* We will define sets  $A_1 \supseteq \dots \supseteq A_n$  of leaves of  $S_d$  so that each  $A_i$  satisfies the following conditions:

- (C1)  $A_i$  contains  $d_i \geq d/2^{i-1}$  leaves of  $S_d$ .
- (C2) Each leaf  $v \in A_i$  defines an  $i$ -element vertex set  $Z_{i,v} := \{(v, p_j) : j \in \{1, \dots, i\}\}$ . For any distinct  $v, w \in A_i$ , the sets  $Z_{i,v}$  and  $Z_{i,w}$  are separated with respect to  $<$ ; that is,  $Z_{i,v} < Z_{i,w}$  or  $Z_{i,v} > Z_{i,w}$ .

Before defining  $A_1, \dots, A_n$  we first show how the existence of the set  $A_n$  implies the lemma. To avoid triple-subscripts, let  $d' := d_n \geq d/2^{n-1}$ . The set  $A_n$  defines vertex sets  $Z_{n,v_1} < \dots < Z_{n,v_{d'}}$ . Refer to Figure 2. Recall that  $r$  is the root of  $S_b$  so it is adjacent to each of  $v_1, \dots, v_{d'}$  in  $S_d$ . Therefore, for each  $j \in \{1, \dots, n\}$  and each  $i \in \{1, \dots, d'\}$ , the edge  $(r, p_j)(v_i, p_j)$  is in  $X$ . Therefore,  $(r, p_j)$  is adjacent to an element of each of  $Z_{n,v_1}, \dots, Z_{n,v_{d'}}$ .

Since  $Z_{n,v_1}, \dots, Z_{n,v_{d'}}$  are separated with respect to  $<$ , when viewed from afar, this situation looks like a complete bipartite graph  $K_{n,d'}$  with the root vertices  $L := \{(r, p_j) : j \in \{1, \dots, n\}\}$  in one part and the groups  $R := Z_{n,v_1} \cup \dots \cup Z_{n,v_{d'}}$  in the other part. Any linear ordering of  $K_{n,d'}$  has a large set of pairwise crossing edges so, intuitively, the induced subgraph  $X[L \cup R]$  should also have a large set of pairwise crossing edges. Lemma 6, below, formalizes this and shows that this graph has a set of at least  $\min\{d', n\}/2$  pairwise crossing edges, which implies the lemma.

All that remains is to define the sets  $A_1 \supseteq \dots \supseteq A_n$  that satisfy (C1) and (C2). Let  $A_1$  be the set of all the leaves of  $S_d$ . For each  $i \in \{2, \dots, n\}$ , the set  $A_i$  is defined as follows: Let

This paragraph is notation-heavy. Replace  $d_i$  by  $|S_i|$ ?

Replace  $b, b_1, b_2, b_3$  by  $a, b, c, d$  to avoid double-subscripts?

In figure 1, replace  $Z_{n,i}$  by  $Z_{n,v_i}$  and replace green dot by  $v_1$  in  $Z_{n,v_1}$

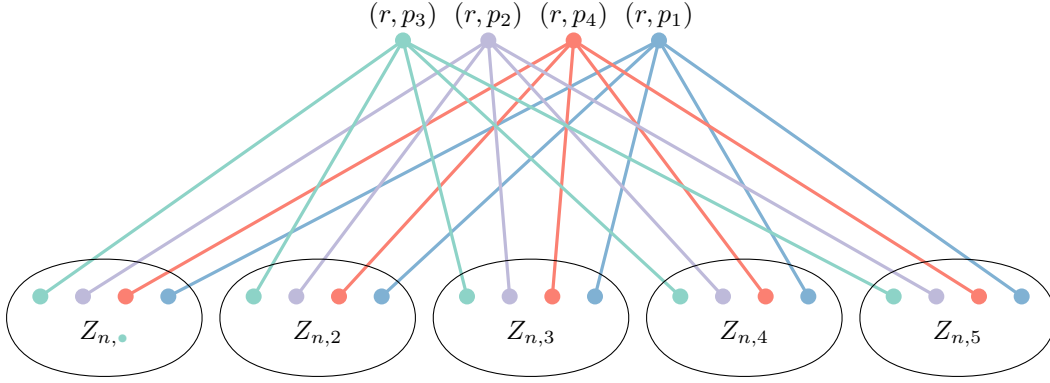


Figure 2: The sets  $Z_{n,1}, \dots, Z_{n,d'}$  ( $n = 4$ ,  $d' = 5$ ).

$Z_1, \dots, Z_r$  denote the sets  $\{(v, p_j) : j \in \{1, \dots, i-1\}\}$  for  $v \in A_{i-1}$  ordered so that  $Z_1 < \dots < Z_r$ .

Let's not use  $r$  here, since  $r$  is the root of the star. Replace  $r$  by  $|A_{i-1}|$ ?

replace  $\{(v, p_j) : j \in \{1, \dots, i-1\}\}$  by  $Z_{i-1,v}$ ?

By Property (C2), this is always possible. Label the vertices of  $A_{i-1}$  as  $v_1, \dots, v_r$  so that  $(v_1, p_{i-1}) < \dots < (v_r, p_{i-1})$ . (This is equivalent to naming them so that  $(v_j, p_{i-1}) \in Z_j$  for each  $j \in \{1, \dots, r\}$ .) Define the set  $A_i := \{v_{2k+1} : k \in \{0, \dots, \lfloor (r-1)/2 \rfloor\}\} = \{v_j \in A_{i-1} : j \text{ is odd}\}$ . This completes the definition of  $A_1, \dots, A_n$ .

All that remains is to verify that  $A_i$  satisfies (C1) and (C2). We do this by induction on  $i$ . The base case  $i = 1$  is trivial so we assume from this point on that  $i \in \{2, \dots, n\}$ . To see that  $A_i$  satisfies (C1) just observe that  $|A_i| = \lceil r/2 \rceil \geq r/2 = |A_{i-1}|/2 \geq d/2^{i-1}$ . All that remains is to show that  $A_i$  satisfies (C2).

For each  $j \in \{i-1, i\}$ , let  $H^j := H_n[\{(v, p_j) : v \in A_{i-1}\}]$ .

The graph  $H^j$  has no edges. All we care about is the vertex set, so let's not bother taking the induced subgraph.

Recall that, for each  $v \in A_{i-1}$ , the edge  $e_v := (v, p_{i-1})(v, p_i)$  is in  $X$ . We have the following properties:

(P1) By Lemma 2,  $\varphi(e_v) = \varphi_0(p_{i-1}p_i)$  for each  $v \in A_{i-1}$ .

(P2) By the application of Lemma 4,  $(v, p_{i-1}) < (w, p_{i-1})$  if and only if  $(v, p_i) < (w, p_i)$  for each  $v, w \in A_{i-1}$ .

Replace "By the application of Lemma 4" by "Since  $p_{i-1}$  and  $p_i$  are red". In fact, all we need here is that  $p_{i-1}$  and  $p_i$  are the same colour.

(P3) By Lemma 1,  $(v, p_{i-1}) < (v, p_i)$  for every  $v \in A_{i-1}$  or  $(v, p_{i-1}) > (v, p_i)$  for every  $v \in A_{i-1}$ .

We claim that these three conditions imply that the vertex sets of  $H^{i-1}$  and  $H^i$  interleave perfectly with respect to  $<$ . More precisely:

**Claim 1.**  $(v_1, p_{i-1+t}) < (v_1, p_{i-t}) < (v_2, p_{i-1+t}) < (v_2, p_{i-t}) \cdots < (v_r, p_{i-1+t}) < (v_r, p_{i-t})$  for some  $t \in \{0, 1\}$ .



*Proof of Claim 1.* By (P3) we may assume, without loss of generality, that  $(v, p_{i-1}) < (v, p_i)$  for each  $v \in A_{i-1}$ , in which case we are trying to prove the claim for  $t = 0$ . It is sufficient, therefore to show that  $(v_j, p_i) < (v_{j+1}, p_{i-1})$  for each  $j \in \{1, \dots, r-1\}$ . For the sake of contradiction, suppose  $(v_j, p_i) > (v_{j+1}, p_{i-1})$  for some  $j \in \{1, \dots, r-1\}$ . By definition  $(v_j, p_{i-1}) < (v_{j+1}, p_{i-1})$  so, by (P2),  $(v_j, p_i) < (v_{j+1}, p_i)$ . Therefore

$$(v_j, p_{i-1}) < (v_{j+1}, p_{i-1}) < (v_j, p_i) < (v_{j+1}, p_i) .$$

Therefore the edges  $e_{v_j} = (v_j, p_{i-1})(v_j, p_i)$  and  $e_{v_{j+1}} = (v_{j+1}, p_{i-1})(v_{j+1}, p_i)$  cross with respect to  $<$ . But this is a contradiction since, by (P1),  $\varphi(e_{v_j}) = \varphi(e_{v_{j+1}}) = \varphi_0(p_{i-1}p_i)$ . This contradiction completes the proof of Claim 1.  $\square$

Replace  
“By definition” by  
“By the labelling  
of  $A_{i-1}$ ”

We now complete the proof that  $A_i$  satisfies (C2). Apply Claim 1 and assume without loss of generality that  $t = 0$ , so that

$$(v_1, p_{i-1}) < (v_1, p_i) < (v_2, p_{i-1}) < (v_2, p_i) \cdots < (v_r, p_{i-1}) < (v_r, p_i) .$$

For each  $j \in \{1, \dots, r-2\}$ , we have  $(v_{j+1}, p_{i-1}) \in Z_{j+1} < Z_{j+2}$ , so  $(v_j, p_i) < (v_{j+1}, p_{i-1}) < Z_{j+2}$ . Therefore  $Z_j \cup \{(v_j, p_i)\} < Z_{j+2}$ . By a symmetric argument,  $Z_j \cup \{(v_j, p_i)\} > Z_{j-2}$  for each  $j \in \{3, \dots, r\}$ . Finally, since  $(v_j, p_i) < (v_{j+2}, p_i)$  for each odd  $i \in \{1, \dots, r\}$ , we have  $Z_j \cup \{(v_j, p_i)\} < Z_{j+2} \cup \{(v_{j+2}, p_i)\}$  for each odd  $j \in \{1, \dots, r-2\}$ . Thus  $A_i$  satisfies (C2) since the sets  $Z_1 \cup \{(v_1, p_i)\}, Z_3 \cup \{(v_3, p_i)\}, \dots, Z_{2\lfloor (r-1)/2 \rfloor + 1} \cup \{(v_{2\lfloor (r-1)/2 \rfloor + 1}, p_i)\}$  are precisely the sets  $Z_{i,1}, \dots, Z_{i,d_i}$  determined by our choice of  $A_i$ .  $\square$

**Lemma 6.** Let  $G$  be any graph, let  $<$  be any linear ordering of  $V(G)$ , let  $Z_1 < \dots < Z_{2s}$  be subsets of  $V(G)$ , and let  $r_1 < \dots < r_{2s}$  be vertices of  $G$  such that, for each  $i, j \in \{1, \dots, 2s\}$ ,  $G$  contains an edge  $r_i z_j$  with  $z_j \in Z_j$ . Then  $G$  contains a set of  $s$  edges that are pairwise crossing with respect to  $<$ .

*I think we should not re-use  $s$  in this lemma. More importantly, do we really need Lemma 6? It could be easily merged into the proof of Lemma 5 where Lemma 6 is used, and this would avoid having to translate notation. It took me a while to realise that  $r_1, \dots, r_{2s}$  corresponds to  $L$  in Lemma 5.*

*Proof.* At least one of the following two cases applies (see Figure 3):

1.  $Z_s < r_{s+1}$  in which case the graph between  $r_{s+1}, \dots, r_{2s}$  and  $Z_1, \dots, Z_s$  has a set of  $s$  pairwise-crossing edges.
2.  $r_s < Z_{s+1}$  in which case the graph between  $r_1, \dots, r_s$  and  $Z_{s+1}, \dots, Z_{2s}$  has a set of  $s$  pairwise-crossing edges.  $\square$

## 4 Open Problems

Recall that every 1-queue graph has a 2-stack layout [12] and we proved that there are 4-queue graphs with unbounded stack-number. The following questions remain open: Do

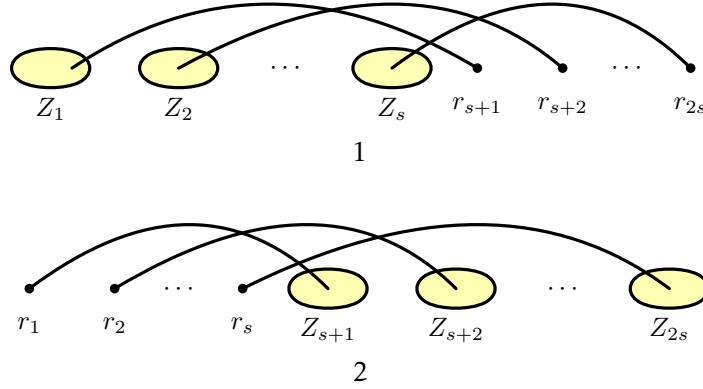


Figure 3: The two cases in the proof of Lemma 6.

2-queue graphs have bounded stack-number? Do 3-queue graphs have bounded stack-number?

Given the role of cartesian products in our proof, it is natural to ask when is  $\text{sn}(G_1 \square G_2)$  bounded? Note that  $H_n$  is a subgraph of a planar Hamiltonian graphs (namely,  $H_{2n}$ ), so  $\text{sn}(H_n) \leq 2$ . So  $\text{sn}(G_1 \square G_2)$  can be unbounded even when  $G_1$  is a star and  $\text{sn}(G_2) \leq 2$ . Since  $\text{sn}(G_2) \leq 1$  if and only if  $G_2$  is outerplanar, the following question naturally arises: Is  $\text{sn}(S \square H)$  bounded for every star  $S$  and outerplanar graph  $H$  with bounded degree? Is  $\text{sn}(T \square H)$  bounded for every tree  $T$  and outerplanar graph  $H$  with bounded degree? The assumption that  $H$  has bounded degree is needed since  $S_n \square S_n$  contain the 1-subdivision of  $K_{n,n}$ , which has unbounded stack-number [3].

Since  $H_n \subseteq P \boxtimes P$  where  $P$  is the  $n$ -vertex path, Theorem 1 implies that  $\text{sn}(S \boxtimes P \boxtimes P)$  is unbounded for stars  $S$  and paths  $P$ . It is easily seen that  $\text{sn}(S \boxtimes P)$  is bounded [16]. The following question naturally arises (independently asked by Pupyrev [16]): Is  $\text{sn}(T \boxtimes P)$  bounded for every tree  $T$  and path  $P$ ? We conjecture the answer is “no”.

## References

- [1] MICHAEL A. BEKOS, HENRY FÖRSTER, MARTIN GRONEMANN, TAMARA MCHEDLIDZE, FABRIZIO MONTECCHIANI, CHRYSANTHI N. RAFTOPOULOU, AND TORSTEN UECKERDT. [Planar graphs of bounded degree have bounded queue number](#). *SIAM J. Comput.*, 48(5):1487–1502, 2019.
- [2] FRANK R. BERNHART AND PAUL C. KAINEN. [The book thickness of a graph](#). *J. Combin. Theory Ser. B*, 27(3):320–331, 1979.
- [3] ROBIN BLANKENSHIP. [Book embeddings of graphs](#). Ph.D. thesis, Department of Mathematics, Louisiana State University, U.S.A., 2003.
- [4] ROBIN BLANKENSHIP AND BOGDAN OPOROWSKI. [Drawing subdivisions of complete and complete bipartite graphs on books](#). Tech. Rep. 1999-4, Department of Mathematics, Louisiana State University, U.S.A., 1999.
- [5] GIUSEPPE DI BATTISTA, FABRIZIO FRATI, AND JÁNOS PACH. [On the queue number of planar graphs](#). *SIAM J. Comput.*, 42(6):2243–2285, 2013.



- 
- [6] VIDA DUJMOVIĆ, GWENAËL JORET, PIOTR MICEK, PAT MORIN, TORSTEN UECKERDT, AND DAVID R. WOOD. [Planar graphs have bounded queue-number](#). *J. ACM*, 67(4):22, 2020.
  - [7] VIDA DUJMOVIĆ AND DAVID R. WOOD. [On linear layouts of graphs](#). *Discrete Math. Theor. Comput. Sci.*, 6(2):339–358, 2004.
  - [8] VIDA DUJMOVIĆ AND DAVID R. WOOD. [Stacks, queues and tracks: Layouts of graph subdivisions](#). *Discrete Math. Theor. Comput. Sci.*, 7:155–202, 2005.
  - [9] VIDA DUJMOVIĆ AND DAVID R. WOOD. [Graph treewidth and geometric thickness parameters](#). *Discrete Comput. Geom.*, 37(4):641–670, 2007.
  - [10] PAUL ERDŐS AND GEORGE SZEKERES. [A combinatorial problem in geometry](#). *Compositio Math.*, 2:463–470, 1935.
  - [11] DAVID GALE. [The game of Hex and the Brouwer fixed-point theorem](#). *Amer. Math. Monthly*, 86(10):818–827, 1979.
  - [12] LENWOOD S. HEATH, F. THOMSON LEIGHTON, AND ARNOLD L. ROSENBERG. [Comparing queues and stacks as mechanisms for laying out graphs](#). *SIAM J. Discrete Math.*, 5(3):398–412, 1992.
  - [13] LENWOOD S. HEATH AND ARNOLD L. ROSENBERG. [Laying out graphs using queues](#). *SIAM J. Comput.*, 21(5):927–958, 1992.
  - [14] MICHAEL KAUFMANN, MICHAEL A. BEKOS, FABIAN KLUTE, SERGEY PUPYREV, CHRYSANTHI N. RAFTOPOULOU, AND TORSTEN UECKERDT. [Four pages are indeed necessary for planar graphs](#). *J. Comput. Geom.*, 11(1):332–353, 2020.
  - [15] L. TAYLOR OLLMANN. On the book thicknesses of various graphs. In FREDERICK HOFFMAN, ROY B. LEVOW, AND ROBERT S. D. THOMAS, eds., *Proc. 4th Southeastern Conference on Combinatorics, Graph Theory and Computing*, vol. VIII of *Congr. Numer.*, p. 459. Utilitas Math., 1973.
  - [16] SERGEY PUPYREV. [Book embeddings of graph products](#). 2020, arXiv:2007.15102.
  - [17] DAVID R. WOOD. [Queue layouts of graph products and powers](#). *Discrete Math. Theor. Comput. Sci.*, 7(1):255–268, 2005.
  - [18] MIHALIS YANNAKAKIS. [Embedding planar graphs in four pages](#). *J. Comput. System Sci.*, 38(1):36–67, 1989.
  - [19] MIHALIS YANNAKAKIS. [Planar graphs that need four pages](#). *J. Combin. Theory Ser. B*, 145:241–263, 2020.