1. STACK-NUMBER IS NOT BOUNDED BY QUEUE-NUMBER
or
2. STACK-NUMBER IS NOT QUEUE-NUMBER BOUNDED
or
3. STACKS ARE NOT MORE POWERFUL THAN QUEUES
or
4. STACKS ARE NO MORE POWERFUL THAN QUEUES

Votes Vida: 1 Robert: 3 Pat: 1 or 2 David: 3 or 4

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Abstract. We describe a family of graphs with queue-number at most 4 but unbounded stack-number. This resolves open problems of Heath, Leighton and Rosenberg (1992) and Blankenship and Oporwoski (1999).

1 Introduction

Stacks and queues are fundamental data structures in computer science, but which is more powerful? In 1992, Heath et al. [10] introduced an approach for answering this question by defining the graph parameters *stack-number* and *queue-number*, which respectively measure the power of stacks and queues for representing graphs.

Let G be a graph and let $\prec$ be a total order on $V(G)$. Two disjoint edges $vw, xy \in E(G)$ with $v \prec w$ and $x \prec y$ *cross* with respect to $\prec$ if $v \prec x \prec w \prec y$ or $x \prec v \prec y \prec w$, and *nest* with respect to $\prec$ if $v \prec x \prec y \prec w$ or $x \prec v \prec w \prec y$. Let $\varphi : E(G) \to \{1,\dots,k\}$ for some integer $k \geqslant 1$. Then $(\prec, \varphi)$ is a *k-stack layout* of G if, for every pair of edges $vw, xy \in E(G)$, if $\varphi(vw) = \varphi(xy)$ then vw and xy do not cross. Similarly, the pair $(\prec, \varphi)$ is a *k-queue layout* of G if, for every pair of edges $vw, xy \in E(G)$, if $\varphi(vw) = \varphi(xy)$ then vw and xy do not nest. The smallest integer s for which G has an s-stack layout is called the *stack-number* of G, denoted $\mathrm{sn}(G)$. The smallest integer q for which G has a q-queue layout is called the *queue-number* of G, denoted $\mathrm{qn}(G)$. Note that stack layouts are equivalent to book embeddings, and stack-number is also known as *page-number*, *book-thickness* or *fixed outer-thickness*.

PM: Suggestion: Replace second if then with $\varphi(vw) \neq \varphi(xy)$ or vw and xy do not cross.

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Given a k-stack layout $(\prec, \varphi)$ of a graph G, for each $i \in \{1, \dots, k\}$, the set $E_i := \{e \in E(G) : \varphi(e) = i\}$ behaves like a stack, in the sense that each edge $e = vw \in E_i$ with $v \prec w$ corresponds to an element in a sequence of stack opertions, such that if we traverse the vertices in the order of $\prec$, then e is pushed onto the stack at v and popped off the stack at w. Similarly, given a k-queue layout $(\prec, \varphi)$, each set E_i behaves like a queue. In this way, the stack-number and queue-number respectively measure the power of stacks and queues to represent graphs.

Should the following two paragraphs be moved to prior to the definitions PM: Yes

The following key problems are implcit in the work of Heath et al. [10], and made explicit by Dujmović and Wood [6] [1]:

- Is stack-number bounded by queue-number?
- Is queue-number bounded by stack-number?

If stack-number is bounded by queue-number but queue-number is not bounded by stack-number, then we would consider stacks to be more powerful than queues. Similarly, if the converse holds, then we would consider queues to be more powerful than stacks. Despite extensive research on stack and queue layouts of graphs (see [4, 5, 7] and the references therein), these fundamental questions have remained unsolved.

Is Stack-Number Bounded by Queue-number?

This paper considers the first of the above questions. In a positive direction, Heath et al. [10] showed that every 1-queue graph has a 2-stack layout. On the other hand, they described graphs that need exponentially more stacks than queues. In particular, n-vertex ternary hypercubes have queue-number $O(\log n)$ and stack-number $\Omega(n^{1/9-\epsilon})$ for any $\epsilon > 0$.

Our key contribution is the following theorem, which shows that stack-number is not bounded by queue-number. This demonstrates that stacks are not more powerful than queues in the sense discussed above.

Theorem 1. *For every $s \in \mathbb{N}$ there exists a graph G with $\operatorname{qn}(G) \leqslant 4$ and $\operatorname{sn}(G) > s$.*

The graph G in Theorem 1 is a cartesian product. For graphs G_1 and G_2, the *cartesian product* $G_1 \square G_2$ is the graph with vertex set $\{(v_1, v_2) : v_1 \in V(G_1), v_2 \in V(G_2)\}$, where $(v_1, v_2)(w_1, w_2) \in E(G_1 \square G_2)$ if $v_1 = w_1$ and $v_2 w_2 \in E(G_2)$, or $v_1 w_1 \in E(G_1)$ and $v_2 = w_2$.

Let S_b be the star graph with root r and b leaves. For $n \in \mathbb{N}$, let H_n be the dual of the hexagonal grid, defined by $V(H_n) := \{1, \dots, n\}^2$ and

$$\begin{aligned} E(H_n) := & \{(x,y)(x+1,y) : x \in \{1,\dots,n-1\}, y \in \{1,\dots,n\}\} \\ & \cup \{(x,y)(x,y+1) : x \in \{1,\dots,n\}, y \in \{1,\dots,n-1\}\} \\ & \cup \{(x,y)(x+1,y+1) : x,y \in \{1,\dots,n-1\}\} \ . \end{aligned}$$

The graph G in Theorem 1 is $S_b \square H_n$ where b and n are chosen to be sufficiently large compared to s, as illustrated in Figure 1. Note that Pupyrev [11] independently suggested using graph products to show that stack-number is not bounded by queue-number.

[1]A *graph parameter* is a function α such that $\alpha(G) \in \mathbb{R}$ for every graph G, such that $\alpha(G_1) = \alpha(G_2)$ for all isomorphic graphs G_1 and G_2. A graph parameter α is *bounded* by a graph parameter β if there exists a function f such that for every graph G we have $\alpha(G) \leqslant f(\beta(G))$.

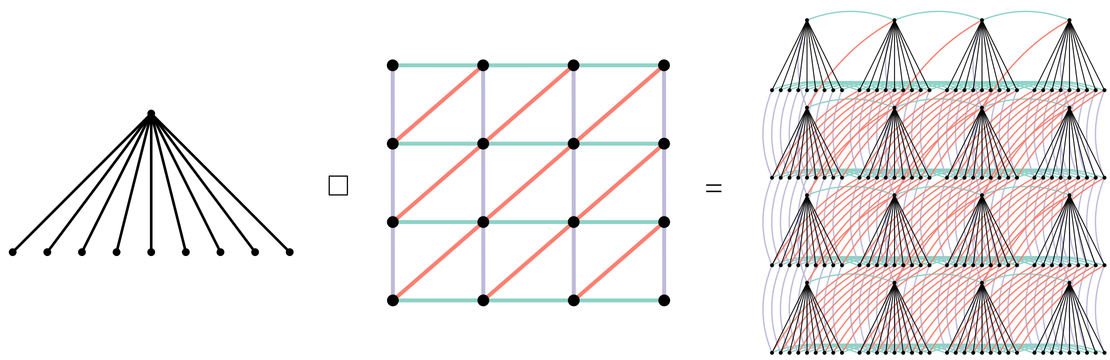

Figure 1: $S_9 \square H_4$.

Subdivisions

A noteworthy consequence of Theorem 1 is that it resolves a conjecture of Blankenship and Oporowski [2, 3]. A graph G' is a *subdivision* of a graph G if G' can be obtained from G by replacing the edges vw of G by internally disjoint paths P_{vw} with endpoints v and w. If each P_{vw} has eaxctly k internal vertices, then G' is the *k-subdivision*. If each P_{vw} has at most k internal vertices, then G' is a *$(\leqslant k)$-subdivision*. Blankenship and Oporowski [2] conjectured that the stack-number of $(\leqslant k)$-subdivisions (k fixed) is not much less than that of the original graph. More precisely:

Conjecture 1 ([2]). *There exists a function f such that for every graph G and integer k, if G' is any $(\leqslant k)$-subdivision of G, then $\operatorname{sn}(G) \leqslant f(\operatorname{sn}(G'), k)$.*

Dujmović and Wood [6] established a connection between this conjecture and the question of whether stack-number is bounded by queue-number. In particular, they showed that if Conjecture 1 is true then stack-number is bounded by queue-number. Since Theorem 1 shows that stack-number is not bounded by queue-number, Conjecture 1 is false. The proof of Dujmović and Wood [6] is based on the folllowing key lemma: every graph G has a 3-stack subdivision with $1 + 2\lceil \log_2 \operatorname{qn}(G) \rceil$ division vertices per edge. Applying this result to the graph $G = S_b \square H_n$ in Theorem 1, the 5-subdivision of $S_b \square H_n$ has a 3-stack layout. If Conjecture 1 was true, then $\operatorname{sn}(S_b \square H_n) \leqslant f(3, 5)$, contradicting Theorem 1.

Is Queue-number Bounded by Stack-Number?

It remains open whether queues are more powerful than stacks; that is, whether queue-number is bounded by stack-number. Several reults are known about this problem. Heath et al. [10] showed that every 1-stack graph has a 2-queue layout. Dujmović et al. [4] showed that planar graphs have bounded queue-number. (Note that graph products also feature heavily in this proof.) Since 2-stack graphs are planar, this implies that 2-stack graphs have bounded queue-number. It is open whether 3-stack graphs have bounded queue-number. In fact, the case of three stacks is as hard as the general question. Dujmović and Wood [6] proved that queue-number is bounded by stack-number if and only if 3-stack graphs have bounded queue-number. Moreover, if this is true then stack-number is bounded by a polynomial function of queue-number.

2 The Proof

First we prove that $\mathrm{qn}(S_b \square H_n) \leqslant 4$, as claimed in Theorem 1. We need the following definition due to Wood [12]. A queue layout $(\varphi, \prec)$ is *strict* if for every vertex $u \in V(G)$ and for all neighbours $v, w \in N_G(u)$, if $u \prec v, w$ or $v, w \prec u$, then $\varphi(uv) \neq \varphi(uw)$. Let $\mathrm{sqn}(G)$ be the minimum integer k such that G has a strict k-queue layout. To see that $\mathrm{sqn}(H_n) \leqslant 3$, order the vertices row-by-row and then left-to-right within a row, with vertical edges in one queue, horizontal edges in one queue, and diagonal edges in another queue. Wood [12] proved that $\mathrm{qn}(G \square H) \leqslant \mathrm{qn}(G) + \mathrm{sqn}(H)$ for all graphs G and H. Of course, S_b has a 1-queue layout (since no two edges are nested for any vertex-ordering). Thus $\mathrm{qn}(S_b \square H_n) \leqslant 4$.

We now turn to the proof of our main result, the lower bound on $\mathrm{sn}(G)$, where $G := S_b \square H_n$. Consider a hypothetical s-stack layout $(\varphi, \prec)$ of G where n and b are chosen sufficiently large compared to s as detailed below. We begin with three lemmata that, for sufficiently large b, provide a large subgraph S_d of S_b for which the induced stack layout of $S_d \square H_n$ is highly structured.

For each node v of S_b, define π_v as the permutation of $\{1, \dots, n\}^2$ in which (x_1, y_1) appears before (x_2, y_2) if and only $(v, x_1, y_1) \prec (v, x_2, y_2)$. The following lemma is an immediate consequence of the Pigeonhole Principle:

Lemma 1. *There exists a permutation π of $\{1, \dots, n\}^2$ and a set L_1 of leaves of S_b of size $b_1 \geqslant \lceil b/(n^2)! \rceil$ such that $\pi_v = \pi$ for each $v \in L_1$.*

For each leaf v in L, let φ_v be the edge colouring of H_n defined by $\varphi_v(x, y) := \varphi(v, (x, y))$. Since H_n has maximum degree 6 and is not 6-regular, it has less than $3n^2$ edges. Therefore there are fewer than s^{3n^2} edge colourings of H_n using s colours. Another application of the Pigeonhole Principle proves the following:

Lemma 2. *There exists a subset $L_2 \subseteq L_1$ of size $b_2 \geqslant b_1 / s^{3n^2}$ and an edge colouring $\varphi_0 : H_n \to \{1, \dots, s\}$ such that $\varphi_v = \varphi_0$ for each $v \in L_2$.*

The preceding two lemmata ensure that, for distinct leaves v and w of S_{b_2}, the stack layouts of the isomorphic graphs $H^v := G[\{(v, p) : p \in V(H_n)]$ and $H^w := G[\{(w, p) : p \in V(H_n)]$ are identical. The next lemma is a statement about the relationships between the stack layouts of $S^p := G[\{(v, p) : v \in V(S_{b_2})]$ and $S^q := G[\{(v, q) : v \in V(S_{b_2})]$ for distinct $p, q \in V(H_n)$. It cannot assert that these two layouts are identical but it does state that they fall into one of two categories.

Lemma 3. *There exists a sequence $L_3 := u_1, \dots, u_{b_3}$ with $\{u_1, \dots, u_{b_3}\} \subseteq L_2$ of length $b_3 \geqslant b_2^{1/2^{n^2-1}}$ such that, for each $p \in V(H_n)$, either $(u_1, p) \prec (u_2, p) \prec \cdots \prec (u_{b_3}, p)$ or $(u_1, p) \succ (u_2, p) \succ \cdots \succ (u_{b_3}, p)$.*

Proof. Let $p_1, \dots, p_{n^2}$ denote the vertices of H_n, in any order. Begin with the sequence $S_1 := v_{1,1}, \dots, v_{1,d_1}$ that contains all $d_1 := b_2$ elements of L_2 ordered so that $(v_{1,1}, p_1) \prec$

$\cdots \prec (v_{1,d_1}, p_1)$. For each $i \in \{2,\ldots,n^2\}$, the Erdős-Szekeres Theorem [8] implies that, S_{i-1} contains a subsequence $S_i := v_{i,1},\ldots,v_{i,d_i}$ of length $d_i \geqslant \sqrt{|S_{i-1}|}$ such that $(v_{i,1}, p_i) \prec \cdots \prec (v_{i,d_i}, p_i)$ or $(v_{i,1}, p_i) \succ \cdots \succ (v_{i,d_i}, p_i)$. It is straightforward to verify by induction on i that $d_i \geqslant b_2^{1/2^{i-1}}$ resulting in a final sequence $S_{n^2} =: L_3$ of length at least $b_2^{1/2^{n^2-1}}$. $\square$

Let $d := b_3$ and let S_d be the the subgraph of S_b induced by $\{r\} \cup \{u_1,\ldots,u_d\}$ where $u_1,\ldots,u_d$ is the sequence of leaves defined in Lemma 3. Consider the (improper) vertex colouring of H_n obtained by colouring each vertex $p \in V(H_n)$ *red* if $(u_1, p) \prec \cdots \prec (u_d, p)$ and colouring p *blue* if $(u_1, p) \succ \cdots \succ (u_d, p)$. We need the following famous Hex Lemma [9].

Lemma 4. *Every red–blue vertex colouring of the graph H_n contains an n-vertex path R consisting entirely of red vertices or entirely of blue vertices.*

Without loss of generality, assume that the path $R := p_1,\ldots,p_n$ defined by Lemma 4 consists entirely of red vertices, so that $(u_1, p_j) \prec \cdots \prec (u_d, p_j)$ for each $j \in \{1,\ldots,n\}$. Recall that $(\varphi, \prec)$ is a hypothetical s-stack layout of G and therefore it is also an s-stack layout of the subgraph $X := S_d \square R$. The following result finishes the proof by showing that this is not possible when $n > 2s$ and $d > s2^n$.

Lemma 5. *The graph X contains a set of edges of size at least* $\min\{d/2^n, n/2\}$ *that are pairwise crossing with respect to $\prec$.*

Proof. We will define two sequences of nested sets $A_1 \supseteq \cdots \supseteq A_n$ of leaves of S_d so that each A_i satisifies the following conditions:

(C1) A_i contain $d_i \geqslant d/2^{i-1}$ leaves of S_d.

(C2) Each leaf $v \in A_i$ defines an i-element vertex set $Z_{i,v} := \{(v, p_j) : j \in \{1,\ldots,i\}\}$. For any distinct $v, w \in A_i$, $Z_{i,v}$ and $Z_{i,w}$ are *separated* with respect to $\prec$, i.e., $Z_{i,v} \prec Z_{i,w}$ or $Z_{i,v} \succ Z_{i,w}$.

Before defining $A_1,\ldots,A_n$ we first show how the existence of the set A_n implies the lemma. To avoid triple-subscripts, let $d' := d_n \geqslant d/2^n$. The set A_n defines vertex sets $Z_{n,v_1} \prec \cdots \prec Z_{n,v_{d'}}$. Refer to Figure 2. Recall that r is the root of S_b so it is adjacent to each of $v_1,\ldots,v_{d'}$ in S_b. Therefore, for each $j \in \{1,\ldots,n\}$ and each $i \in \{1,\ldots,d'\}$, the edge $(r, p_j)(v_i, p_j)$ is in X. Therefore, (r, p_j) is adjacent to an element of each of $Z_{n,v_1},\ldots,Z_{n,v_{d'}}$.

Since $Z_{n,v_1},\ldots,Z_{n,v_{d'}}$ are separated with respect to $\prec$, when viewed from afar, this situation looks like a complete bipartite graph $K_{n,d'}$ with the root vertices $L := \{(r, p_j) : j \in \{1,\ldots,n\}\}$ in one part and the groups $R := Z_{n,v_1} \cup \cdots \cup Z_{n,v_{d'}}$ in the other part. Any linear ordering of $K_{n,d'}$ has a large set of pairwise crossing edges so, intuitively, the graph induced by $L \cup R$ should also have a large set of pairwise crossing edges. Lemma 6, below, formalizes this and shows that this graph has a set of at least $\min\{d', n\}/2$ pairwise crossing edges.

All that remains is to define the sets $A_1 \supseteq \cdots \supseteq A_n$ that satisfy (C1) and (C2). The set A_1 contains all the leaves of S_d. For each $i \in \{2,\ldots,n\}$, the set A_i is defined as follows: Let $Z_1,\ldots,Z_r$ denote the sets $\{(v, p_j) : j \in \{1,\ldots,i-1\}\}$ for $v \in A_{i-1}$ ordered so that $Z_1 \prec \cdots \prec Z_r$. By Property (C2), this is always possible. Label the vertices of A_{i-1} as $v_1,\ldots,v_r$ so that

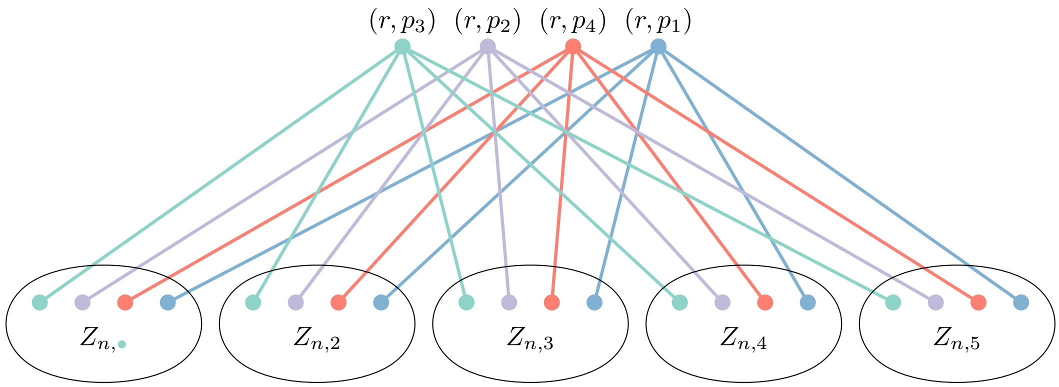

Figure 2: The sets $Z_{n,1},\dots,Z_{n,d'}$ ($n=4$, $d'=5$).

$(v_1,p_{i-1}) \prec \cdots \prec (v_r,p_{i-1})$. (This is equivalent to naming them so that $(v_j,p_{i-1}) \in Z_j$ for each $j \in \{1,\dots,r\}$.) We define the set $A_i := \{v_{2k+1} : k \in \{0,\dots,\lfloor (r-1)/2 \rfloor\}\} = \{v_j \in A_{i-1} : j \text{ is odd}\}$. This completes the definition of $A_1,\dots,A_n$.

All that remains is to verify that A_i satisfies (C1) and (C2). We do this by induction on i. The base case $i=1$ is trivial so we assume from this point on that $i \in \{2,\dots,n\}$. To see that A_i satisfies (C1) just observe that $|A_i| = \lceil r/2 \rceil \geqslant r/2 = |A_{i-1}|/2 \geqslant d/2^{i-1}$. All that remains is to show that A_i satisfies (C2).

For each $j \in \{i-1,i\}$, let $H^j := H_n[\{(v,p_j) : v \in A_{i-1}\}]$. Recall that, for each $v \in A_{i-1}$, the edge $e_v := (v,p_{i-1})(v,p_i)$ is in X. We have the following properties:

(P1) By Lemma 2, $\varphi(e_v) = \varphi_0(p_{i-1},p_i)$ for each $v \in A_{i-1}$.

(P2) By the application of Lemma 4, $(v,p_{i-1}) \prec (w,p_{i-1})$ if and only if $(v,p_i) \prec (w,p_i)$ for each $v,w \in A_{i-1}$.

(P3) By Lemma 1, $(v,p_{i-1}) \prec (v,p_i)$ for every $v \in A_{i-1}$ or $(v,p_{i-1}) \succ (v,p_i)$ for every $v \in A_{i-1}$.

We claim that these three conditions imply that the vertex sets of H^{i-1} and H^i interleave perfectly with respect to $\prec$. More precisely:

Claim 1. *$(v_1,p_{i-1+t}) \prec (v_1,p_{i-t}) \prec (v_2,p_{i-1+t}) \prec (v_2,p_{i-t}) \cdots \prec (v_r,p_{i-1+t}) \prec (v_r,p_{i-t})$ for some $t \in \{0,1\}$.*

Proof of Claim 1. By (P3) we may assume, without loss of generality, that $(v,p_{i-1}) \prec (v,p_i)$ for each $v \in A_{i-1}$, in which case we are trying to prove the claim for $t=0$. It is sufficient, therefore to show that $(v_j,p_i) \prec (v_{j+1},p_{i-1})$ for each $j \in \{1,\dots,r-1\}$. For the sake of contradiction, suppose $(v_j,p_i) \succ (v_{j+1},p_{i-1})$ for some $j \in \{1,\dots,r-1\}$. By definition $(v_j,p_{i-1}) \prec (v_{j+1},p_{i-1})$ so, by (P2) $(v_j,p_i) \prec (v_{j+1},p_i)$. Therefore

$$(v_j,p_{i-1}) \prec (v_{j+1},p_{i-1}) \prec (v_j,p_i) \prec (v_{j+1},p_i) \ .$$

Therefore the edges $(v_j,p_{i-1})(v_j,p_i)$ and $(v_{j+1},p_{i-1})(v_{j+1},p_i)$ cross with repect to $\prec$. But this is a contradiction since, by (P1), $\varphi((v_j,p_{i-1})(v_j,p_i)) = \varphi((v_{j+1},p_{i-1})(v_{j+1},p_i)) = \varphi_0(p_{i-1}p_i)$. This contradiction completes the proof of Claim 1. □

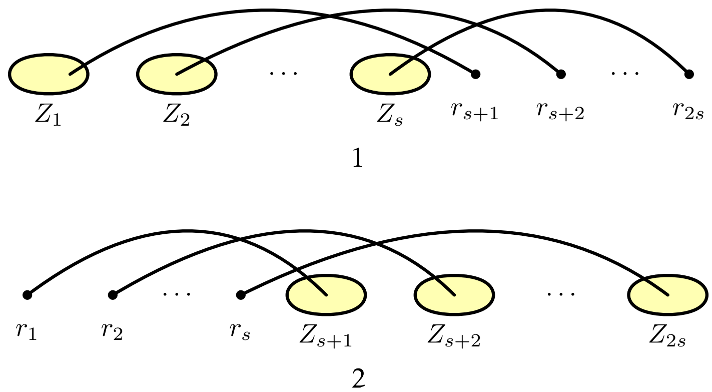

Figure 3: The two cases in the proof of Lemma 6.

Now, apply Claim 1 and assume without loss of generality that $t = 0$, so that

$$(v_1, p_{i-1}) \prec (v_1, p_i) \prec (v_2, p_{i-1}) \prec (v_2, p_i) \cdots \prec (v_r, p_{i-1}) \prec (v_r, p_i) \ .$$

For each $j \in \{1, \ldots, r-2\}$, $(v_{j+1}, p_{i-1}) \in Z_{j+1} \prec Z_{j+2}$, so $(v_j, p_i) \prec (v_{j+1}, p_{i-1}) \prec Z_{j+2}$. Therefore $Z_j \cup \{(v_j, p_i)\} \prec Z_{j+2}$. By a symmetric argument, $Z_j \cup \{(v_j, p_i)\} \succ Z_{j-2}$ for each $j \in \{2, \ldots, r\}$. Finally, since $(v_j, p_i) \prec (v_{j+2}, p_i)$ for each odd $i\S_4 in\{1, \ldots, r\}$, we have $Z_j \cup \{(v_j, p_i)\} \prec Z_{j+2} \cup \{(v_{j+2}, p_i)\}$ for each odd $j \in \{1, \ldots, r-2\}$. Thus A_i satisifies (C2) since the sets $Z_1 \cup \{(v_1, p_i)\}, Z_3 \cup \{(v_3, p_i)\}, \ldots, Z_{2\lfloor(\rfloor r-1)/2\rfloor+1} \cup (v_{2\lfloor(\rfloor r-1)/2\rfloor+1}, p_i)$ are precisely the sets $Z_{i,1}, \ldots, Z_{i,d_i}$ determined by our choice of A_i. □

Lemma 6. *Let G be any graph, let $\prec$ be any linear ordering of $V(G)$, let $Z_1 \prec \cdots \prec Z_{2s}$ be subsets of $V(G)$, and let $r_1 \prec \cdots \prec r_{2s}$ be vertices of G such that, for each $i, j \in \{1, \ldots, 2s\}$, G contains an edge $r_i z_j$ with $z_j \in Z_j$. Then G contains a set of s edges that are pairwise crossing with respect to $\prec$.*

Proof. At least one of the following two cases applies (see Figure 3):

1. $Z_s \prec r_{s+1}$ in which case the graph between $r_{s+1}, \ldots, r_{2s}$ and $Z_1, \ldots, Z_s$ has a set of s pairwise-crossing edges.

2. $r_s \prec Z_{s+1}$ in which case the graph between $r_1, \ldots, r_s$ and $Z_{s+1}, \ldots, Z_{2s}$ has a set of s pairwise-crossing edges. □

3 Open Problems

Recall that every 1-queue graph has a 2-stack layout [10] and we proved that there are 4-queue graphs with unbounded stack-number. The following questions remain open: Do 2-queue graphs have bounded stack-number? Do 3-queue graphs have bounded stack-number?

Given the role of cartesian products in our proof, it is natural to ask when is $\text{sn}(G_1 \square G_2)$ bounded? Note that H_n is a subgraph of a planar Hamiltonian graphs (namely, H_{2n}), so $\text{sn}(H_n) \leqslant 2$. So $\text{sn}(G_1 \square G_2)$ can be unbounded even when G_1 is a star and $\text{sn}(G_2) \leqslant 2$. Since $\text{sn}(G_2) \leqslant 1$ if and only G_2 is outerplanar, the following question naturally arises: Is $\text{sn}(S \square H)$ bounded for every star S and outerplanar graph H with bounded degree? Is $\text{sn}(T \square H)$ bounded for every tree T and outerplanar graph H with bounded degree? The assumption that H has bounded degree is needed since $S_n \square S_n$ contain the 1-subdivision of $K_{n,n}$, which has unbounded stack-number [1].

MENTION RESULTS OF Pupyrev [11] about bipartite graphs.

Since $H_n \subseteq P \boxtimes P$ where P is the n-vertex path, Theorem 1 implies that $\text{sn}(S \boxtimes P \boxtimes P)$ is unbounded for stars S and paths P. It is easily seen that $\text{sn}(S \boxtimes P)$ is bounded [11]. The following question naturally arises (independently asked by Pupyrev [11]): Is $\text{sn}(T \boxtimes P)$ bounded for every tree T and path P? We conjecture the answer is "no".

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