# STACK NUMBER IS NOT BOUNDED BY QUEUE-NUMBER
# or
# STACK NUMBER IS NOT QUEUE-NUMBER BOUNDED
# or
# STACKS ARE NOT MORE POWERFUL THAN QUEUES
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Abstract. We describe a family of graphs in which every member has queue number at most 4, but for every integer $s$, there is a member of the family whose stack number is greater than $s$. This resolves open problems of ??? and Blankenship and Oporwoski (???).

## 1 Introduction

Stacks and queues are fundamental structures in computer science. But which one is more powerful? In 1992, this question was explored by Heath et al. [9] where they compared the graph parameters *stack-number* and *queue-number* which respectively measure the power of stacks and queues for laying out graphs. They showed that every graph which admits a 1-queue layout has a 2-stack layout and that the ternary hypercubes requires exponentially more stacks than queues [9]. In particular, $n$-vertex ternary hypercubes have queue-number at most $2\log_3 n$, but stack-number at least $\Omega(n^{1/9-\epsilon})$ for any $\epsilon > 0$.

We say that a graph parameter $\alpha$ is *bounded* by $\beta$ if there exists a function $f$ such that for every graph $G$ we have $\alpha(G) \leqslant f(\beta(G))$. In 2005, Dujmović et al. [6] extended the study on the comparison of stacks and queues by asking whether stack-number is bounded by queue-number or queue-number bounded by stack-number. If stack-number is bounded by queue-number but queue-number is not bounded by stack-number, then we would consider stacks to be more powerful than queues. Similarly, if the converse holds, then we would consider queues to be more powerful than stacks. Despite extensive research on stack and queue layouts of graphs [references???], it has remained unclear which one is more powerful for laying out graphs.

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**UPDATE THE FOLLOWING PARAGRAPH WITH PAT'S NOTATION**

PM: We should probably replace $\prec$ and $\succ$ with $<$ and $>$, but I do think the edge colouring $\varphi$ is more convenient for what we're doing.

We now formally define stack and queue layouts of graphs. Let $G = (V, E)$ be a graph with disjoint edges $vw, xy$ and a linear ordering $\preccurlyeq$ of the vertices. Without loss of generality, we may assume that $v < w$, $x < y$ and $v < x$. We say that $vw$ and $xy$ *cross* if $v < x < w < y$, *nest* if $v < x < y < w$, and are *disjoint* if $v < w < x < y$. A *stack* is a set of pairwise non-crossing edges and a *queue* is a set of pairwise non-nesting edges. A $k$-queue layout of $G$ consist of a linear ordering $\preccurlyeq$ of its vertices and a partition $E_1, E_2, \dots, E_k$, of its edges into queues with respect to $\preccurlyeq$. The stack-number of a graph $G$, $\operatorname{sn}(G)$, is the minimum integer $k$ such that $G$ has a $k$-stack layout. Similarly, the queue-number of a graph $G$, $\operatorname{qn}(G)$, is the minimum integer $k$ such that $G$ has a $k$-queue layout. We note that stack layouts of graphs are related to book embeddings of graph and that stack-number is also known as *page-number*, *book-thickness*, or *fixed outer-thickness*.

Note that stacks and queues are closely related to breadth-first search (BFS) and depth-first search (DFS) layouts of graphs. It can be easily shown that a BFS vertex ordering of a tree admits a 1-queue layout and a DFS vertex ordering of a tree admits a 1-stack layout.

Our key contribution is the following theorem, which shows that stack-number is not bounded by queue-number.

**Theorem 1.** *For every $s \in \mathbb{N}$ there exists a graph $G$ with $\operatorname{qn}(G) \leqslant 4$ and $\operatorname{sn}(G) > s$.*

This demonstrates that stacks are not more powerful than queues for laying out graphs. It remains open whether queues are more powerful than stacks; that is, whether queue-number is bounded by stack-number. Before we specify the graph $G$ in Theorem 1, we must first introduce graph products.

Let $Q$ and $G$ be graph. We define a graph product $G \times Q$ with the vertex set:

$$V(G \times Q) = \{(a, v) : a \in V(G), v \in V(Q)\}.$$

A potential edge $(a, v)(b, u)$ in $G \times Q$ can be classified as:

- $G$-edge: $ab \in E(G)$ and $v = u$;
- $Q$-edge: $uv \in E(Q)$ and $a = b$; or
- direct-edge: $ab \in E(G)$ and $uv \in E(Q)$.

The *Cartesian product* $G \square Q$ consists of the $G$-edges and $Q$-edges. The *direct product* $G \times Q$ consist of the direct edges. The *strong product* $G \boxtimes Q$ consist of the $G$-edges, $Q$-edges and the direct edges.

Graph products are a powerful tool in structural graph theory and have been used to solve many open problems concerning queue layouts of graphs [4, 5]. The graph $G$ in Theorem 1 is obtained as follows (See Figure 1): Let $S_b$ denote the star graph with root $r$ and $b$ leaves. For an even positive integer $n$, let $H_n$ be the Hex $n \times n$ grid, defined by $V(H_n) := \{1, \dots, n\}^2$ and

$$
\begin{aligned}
E(H_n) := &\{(x,y)(x+1,y) : x \in \{1,\dots,n-1\}, y \in \{1,\dots,n\}\} \\
&\cup \{(x,y)(x,y+1) : x \in \{1,\dots,n\}, y \in \{1,\dots,n-1\}\} \\
&\cup \{(x,y+1)(x+1,y) : x,y \in \{1,\dots,n-1\}\} \ .
\end{aligned}
$$

Note that the Hex graph corresponds to the board used in the game *Hex* which was invented by the mathematician Peit Hein in 1942. In this paper, we prove Theorem 1 with the graph $G := S_b \square H_n$ where $b$ and $n$ are large compared to $s$. We note that Pupyrev first suggested using graph products to show that stack-number is not bounded by queue-number [11].

Another noteworthy consequence of Theorem 1 is that it resolves a conjecture of Blankenship and Oporowski [1, 2]. A graph $G'$ is a *subdivision* of a graph $G$ if it can be obtained by replacing every edge by a path. The vertices in $G' - G$ are called the *division vertices*. A subdivision of $G$ with exactly $t$ division vertices for each edge of $G$ is called the *$t$-subdivision* of $G$. In 1999, Blankeship and Oporowski conjectured that the subdivision of any graph is not too much smaller than that of the original graph. More precisely:

**Conjecture 1.** *[1, 2] There exists a function $f$ such that for every graph $G$ and the 2-subdivision $G'$ of $G$ we have $\mathrm{sn}(G) \leqslant f(\mathrm{sn}(G'))$.*

Dujmović et al. [6], in their extensive study of layout of graph subdivisions, proved the following.

**Theorem 2.** *[6] For every integer $s \geqslant 3$, every graph $G$ has an $s$-stack subdivision with $1 + 2\lceil \log_{s-1} \mathrm{qn}(G) \rceil$ division vertices per edge.*

As a consequence of this result, Dujvonic et al. showed that if the Blankeship and Oporowski Conjecture is true, then it follows that stack-number is bounded by queue-number. But as Theorem 1 proves that this is not the case, it follows that Conjecture 1 is indeed false. We note that for our graph $G$ in Theorem 1, since $\mathrm{qn}(G) \leqslant 4$ as a consequence of Theorem 2 we have the following corollary.

**Corollary 1.** *For the graph $G$ specified in Theorem 1, there exists a 3-stack layout of the 5-subdivision of $G$.*

This demonstrates that the stack-number of a graph can be unboundedly larger than that of a subdivision of it.

Heath et al. [9] showed that every 1-stack graph has a 2-queue layout. Dujmović et al. [4] showed that planar graphs have bounded queue-number. In particular, 2-stack graphs have bounded queue-number. It is open whether 3-stack graphs have bounded queue-number. In fact, the case of three stacks is as hard as the general question. Dujmović and Wood [6] proved that queue-number is bounded by stack-number if and only if 3-stack graphs have bounded queue-number. Moreover, if this is true then stack-number is bounded by a polynomial function of queue-number.

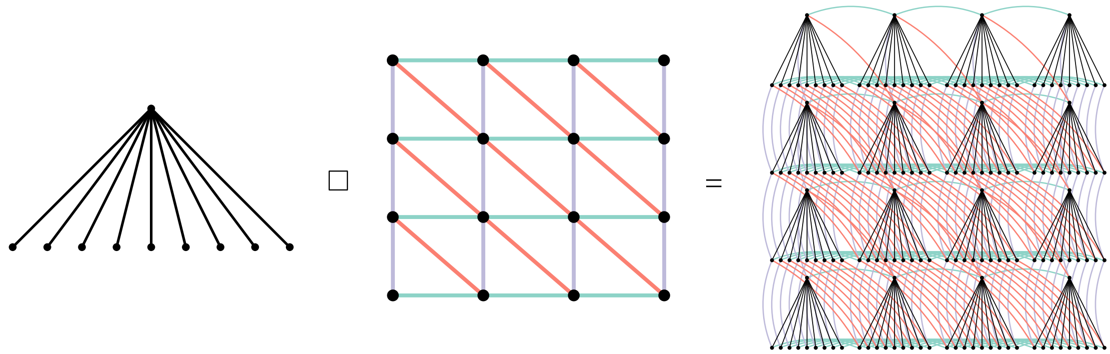

Figure 1: $S_9 \square Q_4$.

PM: Does anyone know if there is a standard box operator that is typeset like this $S \boxtimes Q$ or $S \boxdot Q$ instead of like this $S\square Q$ or like this $S\square Q$? I tried square and Box. DW: I defined `CartProd` which typesets okay $A \square B$ .

The graph $G$ in Theorem 1 is obtained as follows (See Figure 1): Let $S_b$ denote the star graph with root $r$ and $b$ leaves. For an even positive integer $n$, let $Q_n$ be the $n \times n$ triangulated grid, defined by $V(Q_n) := \{1,\dots,n\}^2$ and

$$\begin{aligned} E(Q_n) := &\{(x,y)(x+1,y) : x \in \{1,\dots,n-1\}, y \in \{1,\dots,n\}\} \\ &\cup \{(x,y)(x,y+1) : x \in \{1,\dots,n\}, y \in \{1,\dots,n-1\}\} \\ &\cup \{(x,y+1)(x+1,y) : x,y \in \{1,\dots,n-1\}\} \ . \end{aligned}$$

We consider the graph $G := S_b \square Q_n$. That is, $V(G) = V(S_b) \times V(Q_n)$ where vertices $(v_1,w_1),(v_2,w_2) \in V(G)$ are adjacent whenever $v_1 = w_1$ and $v_2w_2 \in E(Q_n)$, or $v_1w_1 \in E(S_b)$ and $v_2 = w_2$.

SAY SOMETHING ABOUT THE RESULTS OF **?** ]. I EXPECT WE SOLVE SOME OPEN PROBLEM HERE.

PM: Suggestion: Replace $Q_n$ with $H_n$ (for Hex). DW: I like this change.

PM:Not really, his open problem is about $H \boxtimes P$ where $H$ has bounded treewidth.

## 2 Queue-Number Upper Bound

To prove that $\mathrm{qn}(G) \leqslant 4$ in Theorem 1 we need the following definition due to Wood [12]. A queue layout $(\varphi, \prec)$ is *strict* if for every vertex $u \in V(G)$ and for all neighbours $v,w \in N_G(u)$ with $u \prec v,w$ or $v,w \prec u$, we have $\varphi(uv) \neq \varphi(uw)$. Let $\mathrm{sqn}(G)$ be the minimum integer $k$ such that $G$ has a strict $k$-queue layout. Note that $\mathrm{sqn}(Q_n) \leqslant 3$: Order the vertices row-by-row and then left-to-right within a row, with vertical edges in one queue, horizontal edges in one queue, and diagonal edges in another queue. Wood [12] proved that $\mathrm{qn}(G \square H) \leqslant \mathrm{qn}(G) + \mathrm{sqn}(H)$ for all graphs $G$ and $H$. Of course, $S_b$ has a 1-queue layout (since no two edges are nested for any vertex-ordering). Thus

$$\mathrm{qn}(S_b \square Q_n) \leqslant 4.$$

ADD TO DISCUSSION LATER: $Q_n$ is planar with a Hamiltonian cycle (assuming $n$ is even), so $\mathrm{sn}(Q_n) \leqslant 2$

## 3 Stack-Number Lower Bound

Consider a hypothetical $s$-stack layout $(\varphi,\prec)$ of $G$ where $n$ and $b$ are chosen sufficiently large compared to $s$ as detailed below. We begin with three lemmata that, for sufficiently large $b$, allow us to find a large subgraph $S_d$ of $S_b$ for which the stack layout $(\varphi,\prec)$ of $S_d \square Q_n$ is highly structured.

For each node $v$ of $S_b$, we define $\pi_v$ as the permutation of $\{1,\ldots,n\}^2$ in which $(x_1,y_1)$ appears before $(x_2,y_2)$ if and only $(v,x_1,y_1)\prec(v,x_2,y_2)$. The following lemma is an immediate consequence of the Pigeonhole Principle:

**Lemma 1.** *There exists a permutation $\pi$ of $\{1,\ldots,n\}^2$ and a set $L_1$ of leaves of $S_b$ of size $b_1 \geqslant \lceil b/(n^2)!\rceil$ such that $\pi_v=\pi$ for each $v\in L_1$.*

If we cared, we could improve this to $b/2^{cn^2}$ since we only use the weaker property (P3) in the final proof.

For each leaf $v$ in $L$, let $\varphi_v$ be the edge colouring of $Q_n$ defined by $\varphi_v(x,y) := \varphi(v,x,y)$. The edges of $Q_n$ are defined as the union of three sets, each of which has size less than $n^2$, so $Q_n$ has less than $3n^2$ edges. Therefore are fewer than $s^{3n^2}$ edge colourings of $Q_n$ using $s$ colours. Another application of the Pigeonhole Principle proves the following:

**Lemma 2.** *There exists a subset $L_2 \subseteq L_1$ of size $b_2 \geqslant b_1/s^{3n^2}$ and an edge colouring $\varphi_0 : Q_n \to \{1,\ldots,s\}$ such that $\varphi_v=\varphi_0$ for each $v\in L_2$.*

The preceding two lemmata ensure that, for distinct leaves $v$ and $w$ of $S_{b_2}$, the stack layout of the isomorphic graphs $Q_v := G[\{(v,p) : p\in V(Q)]$ and $Q_w := G[\{(w,p) : p\in V(Q)]$ is identical. The next lemma is a statement about the relationships between the stack layouts of $S_p := G[\{(v,p) : v\in V(S_{b_2})]$ and $S_q := G[\{(v,q) : v\in V(S_{b_2})]$ for two distinct $p,q\in V(Q)$. It cannot assert that these two layouts are identical but it does state that they fall into one of two categories.

**Lemma 3.** *There exists a sequence $L_3 := u_1,\ldots,u_{b_3}$ with $\{u_1,\ldots,u_{b_3}\}\subseteq L_2$ of length $b_3 \geqslant b_2^{1/2^{n^2-1}}$ such that, for each $p\in V(Q_n)$, $(u_1,p)\prec(u_2,p)\prec\cdots\prec(u_{b_3},p)$ or $(u_1,p)\succ(u_2,p)\succ\cdots\succ(u_{b_3},p)$.*

*Proof.* Let $p_1,\ldots,p_{n^2}$ denote the vertices of $Q_n$, in any order. Begin with the sequence $S_1 := v_{1,1},\ldots,v_{1,d_1}$ that contains all $d_1 := b_2$ elements of $L_2$ ordered so that $(v_{1,1},p_1)\prec\cdots\prec(v_{1,d_1},p_1)$. For each $i\in\{2,\ldots,n^2\}$, the Erdős-Szekeres Theorem [7] implies that, $S_{i-1}$ contains a subsequence $S_i := v_{i,1},\ldots,v_{i,d_i}$ of length $d_i \geqslant \sqrt{|S_{i-1}|}$ such that $(v_{i,1},p_i)\prec\cdots\prec(v_{i,d_i},p_i)$ or $(v_{i,1},p_i)\succ\cdots\succ(v_{i,d_i},p_i)$. It is straightforward to verify by induction on $i$ that $d_i \geqslant b_2^{1/2^{i-1}}$ resulting in a final sequence $S_{n^2} =: L_3$ of length at least $b_2^{1/2^{n^2-1}}$. □

Let $d := b_3$ and let $S_d$ be the the subgraph of $S_b$ induced by $\{r\}\cup\{u_1,\ldots,u_d\}$ where $u_1,\ldots,u_d$ is the sequence of leaves defined in Lemma 3. Consider the (improper) vertex colouring of $Q_n$ obtained by colouring each vertex $p\in V(Q_n)$ *red* if $(u_1,p)\prec\cdots\prec(u_d,p)$ and colouring $p$ *blue* if $(u_1,p)\succ\cdots\succ(u_d,p)$.

**Lemma 4.** *The graph $Q_n$ contains an $n$-vertex path $R$ consisting entirely of red vertices or entirely of blue vertices.*

*Proof.* The dual of $Q_n$ is the board on which the game Hex is played. The well-known *Hex Lemma* states that any colouring of the vertices of $Q_n$ with colours red and blue contains exactly one of the following [8]:

1. a path with endpoints $(x,1)$ and $(x',n)$ consisting entirely of red vertices, for some $x,x' \in \{1,\dots,n\}$; or
2. a path with endpoints $(1,y)$ and $(n,y')$ consisting entirely of blue vertices, for some $y,y' \in \{1,\dots,n\}$.

In either case, the path contains at least $n$ vertices and therefore has a $n$-vertex subpath consisting entirely of red vertices or entirely of blue vertices. $\square$

Without loss of generality, assume that the path $R := p_1,\dots,p_n$ defined by Lemma 4 consists entirely of red vertices, so that $(u_1,p_j) \prec \cdots \prec (u_d,p_j)$ for each $j \in \{1,\dots,n\}$. Recall that $(\varphi,\prec)$ is a hypothetical $s$-stack layout of $G$ and therefore it is also an $s$-stack layout of the subgraph $X := S_d \square R$. The following result finishes the proof by showing that this is not possible when $n > 2s$ and $d > s2^n$.

**Lemma 5.** *The graph $X$ contains a set of edges of size at least* $\min\{d/2^n, n/2\}$ *that are pairwise crossing with respect to $\prec$.*

*Proof.* We will define two sequences of nested sets $A_1 \supseteq \cdots \supseteq A_n$ of leaves of $S_d$ so that each $A_i$ satisifies the following conditions:

(C1) $A_i$ contain $d_i \geqslant d/2^{i-1}$ leaves of $S_d$.

(C2) Each leaf $v \in A_i$ defines an $i$-element vertex set $Z_{i,v} := \{(v,p_j) : j \in \{1,\dots,i\}\}$. For any distinct $v,w \in A_i$, $Z_{i,v}$ and $Z_{i,w}$ are *separated* with respect to $\prec$, i.e., $Z_{i,v} \prec Z_{i,w}$ or $Z_{i,v} \succ Z_{i,w}$.

Before defining $A_1,\dots,A_n$ we first show how the existence of the set $A_n$ implies the lemma. To avoid triple-subscripts, let $d' := d_n \geqslant d/2^n$. The set $A_n$ defines vertex sets $Z_{n,v_1} \prec \cdots \prec Z_{n,v_{d'}}$. Refer to Figure 2. Recall that $r$ is the root of $S_b$ so it is adjacent to each of $v_1,\dots,v_{d'}$ in $S_b$. Therefore, for each $j \in \{1,\dots,n\}$ and each $i \in \{1,\dots,d'\}$, the edge $(r,p_j)(v_i,p_j)$ is in $X$. Therefore, $(r,p_j)$ is adjacent to an element of each of $Z_{n,v_1},\dots,Z_{n,v_{d'}}$.

Since $Z_{n,v_1},\dots,Z_{n,v_{d'}}$ are separated with respect to $\prec$, when viewed from afar, this situation looks like a complete bipartite graph $K_{n,d'}$ with the root vertices $L := \{(r,p_j) : j \in \{1,\dots,n\}\}$ in one part and the groups $R := Z_{n,v_1} \cup \cdots \cup Z_{n,v_{d'}}$ in the other part. Any linear ordering of $K_{n,d'}$ has a large set of pairwise crossing edges so, intuitively, the graph induced by $L \cup R$ should also have a large set of pairwise crossing edges. Lemma 6, below, formalizes this and shows that this graph has a set of at least $\min\{d',n\}/2$ pairwise crossing edges.

All that remains is to define the sets $A_1 \supseteq \cdots \supseteq A_n$ that satisfy (C1) and (C2). The set $A_1$ contains all the leaves of $S_d$. For each $i \in \{2,\dots,n\}$, the set $A_i$ is defined as follows: Let $Z_1,\dots,Z_r$ denote the sets $\{(v,p_j) : j \in \{1,\dots,i-1\}\}$ for $v \in A_{i-1}$ ordered so that $Z_1 \prec \cdots \prec Z_r$.

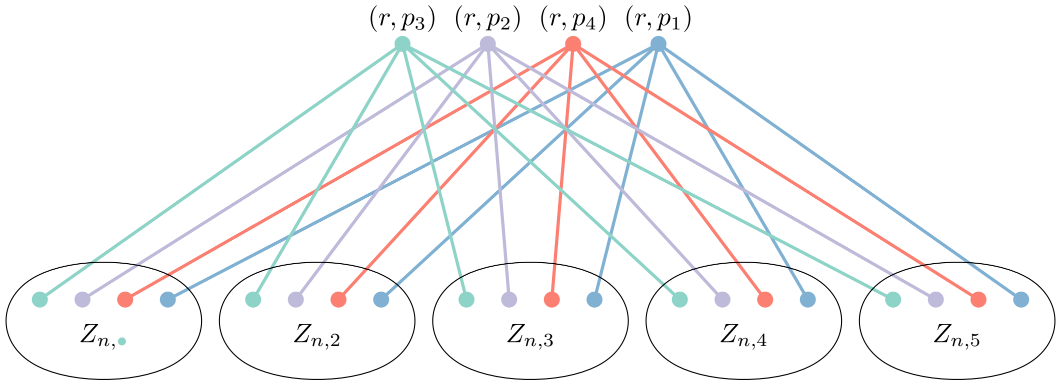


Figure 2: The sets $Z_{n,1},\dots,Z_{n,d'}$ ($n = 4$, $d' = 5$).

By Property (C2), this is always possible. Label the vertices of $A_{i-1}$ as $v_1,\dots,v_r$ so that $(v_1,p_{i-1}) \prec \cdots \prec (v_r,p_{i-1})$. (This is equivalent to naming them so that $(v_j,p_{i-1}) \in Z_j$ for each $j \in \{1,\dots,r\}$.) We define the set $A_i := \{v_{2k+1} : k \in \{0,\dots,\lfloor (r-1)/2 \rfloor\}\} = \{v_j \in A_{i-1} : j \text{ is odd}\}$. This completes the definition of $A_1,\dots,A_n$.

All that remains is to verify that $A_i$ satisfies (C1) and (C2). We do this by induction on $i$. The base case $i = 1$ is trivial so we assume from this point on that $i \in \{2,\dots,n\}$. To see that $A_i$ satisfies (C1) just observe that $|A_i| = \lceil r/2 \rceil \geqslant r/2 = |A_{i-1}|/2 \geqslant d/2^{i-1}$. All that remains is to show that $A_i$ satisfies (C2).

For each $j \in \{i-1, i\}$, let $Q_j := Q_n[\{(v,p_j) : v \in A_{i-1}\}]$. Recall that, for each $v \in A_{i-1}$, the edge $e_v := (v,p_{i-1})(v,p_i)$ is in $X$. We have the following properties:

Tiny notation conflict here: $Q_n$, $Q_i$, $Q_{i-1}$.

(P1) By Lemma 2, $\varphi(e_v) = \varphi_0(p_{i-1},p_i)$ for each $v \in \{A_{i-1}$.

(P2) By the application of Lemma 4, $(v,p_{i-1}) \prec (w,p_{i-1})$ if and only if $(v,p_i) \prec (w,p_i)$ for each $v,w \in A_{i-1}$.

(P3) By Lemma 1, $(v,p_{i-1}) \prec (v,p_i)$ for every $v \in A_{i-1}$ or $(v,p_{i-1}) \succ (v,p_i)$ for every $v \in A_{i-1}$.

We claim that these three conditions imply that the vertex sets $Q_{i-1}$ and $Q_i$ interleave perfectly with respect to $\prec$. More precisely:

**Claim 1.** *$(v_1,p_{i-1+t}) \prec (v_1,p_{i-t}) \prec (v_2,p_{i-1+t}) \prec (v_2,p_{i-t}) \cdots \prec (v_r,p_{i-1+t}) \prec (v_r,p_{i-t})$ for some $t \in \{0,1\}$.*

*Proof of Claim 1.* By (P3) we may assume, without loss of generality, that $(v,p_{i-1}) \prec (v,p_i)$ for each $v \in A_{i-1}$, in which case we are trying to prove the claim for $t = 0$. It is sufficient, therefore to show that $(v_j,p_i) \prec (v_{j+1},p_{i-1})$ for each $j \in \{1,\dots,r-1\}$. For the sake of contradiction, suppose $(v_j,p_i) \succ (v_{j+1},p_{i-1})$ for some $j \in \{1,\dots,r-1\}$. By definition $(v_j,p_{i-1}) \prec (v_{j+1},p_{i-1})$ so, by (P2) $(v_j,p_i) \prec (v_{j+1},p_i)$. Therefore

$$(v_j,p_{i-1}) \prec (v_{j+1},p_{i-1}) \prec (v_j,p_i) \prec (v_{j+1},p_i) \ .$$

Therefore the edges $(v_j,p_{i-1})(v_j,p_i)$ and $(v_{j+1},p_{i-1})(v_{j+1},p_i)$ cross with repect to $\prec$. But this is a contradiction since, by (P1), $\varphi((v_j,p_{i-1})(v_j,p_i)) = \varphi((v_{j+1},p_{i-1})(v_{j+1},p_i)) = \varphi_0(p_{i-1}p_i)$. This contradiction completes the proof of Claim 1. □

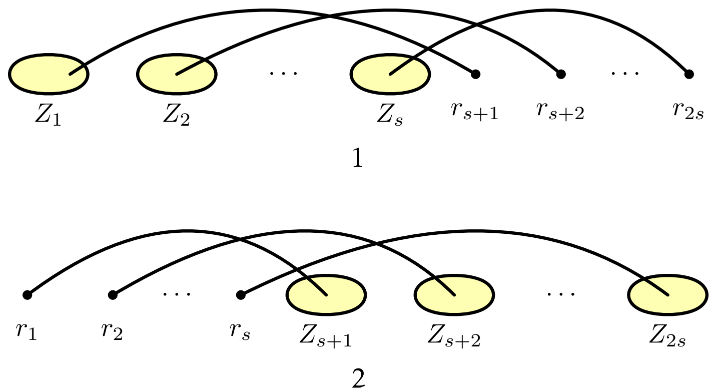


Figure 3: The two cases in the proof of Lemma 6.

Now, apply Claim 1 and assume without loss of generality that $t = 0$, so that

$$(v_1, p_{i-1}) \prec (v_1, p_i) \prec (v_2, p_{i-1}) \prec (v_2, p_i) \cdots \prec (v_r, p_{i-1}) \prec (v_r, p_i) \enspace .$$

DW: Why are these $\prec$'s red? PM: Just me keeping track of which one was the assumption, they don't need to be red.

For each $j \in \{1, \ldots, r-2\}$, $(v_{j+1}, p_{i-1}) \in Z_{j+1} \prec Z_{j+2}$, so $(v_j, p_i) \prec (v_{j+1}, p_{i-1}) \prec Z_{j+2}$. Therefore $Z_j \cup \{(v_j, p_i)\} \prec Z_{j+2}$. By a symmetric argument, $Z_j \cup \{(v_j, p_i)\} \succ Z_{j-2}$ for each $j \in \{2, \ldots, r\}$. Finally, since $(v_j, p_i) \prec (v_{j+2}, p_i)$ for each odd $i\S_4 in\{1, \ldots, r\}$, we have $Z_j \cup \{(v_j, p_i)\} \prec Z_{j+2} \cup \{(v_{j+2}, p_i)\}$ for each odd $j \in \{1, \ldots, r-2\}$. Thus $A_i$ satisifies (C2) since the sets $Z_1 \cup \{(v_1, p_i)\}, Z_3 \cup \{(v_3, p_i)\}, \ldots, Z_{2\lfloor(\rfloor r-1)/2\rfloor+1} \cup (v_{2\lfloor(\rfloor r-1)/2\rfloor+1}, p_i)$ are precisely the sets $Z_{i,1}, \ldots, Z_{i,d_i}$ determined by our choice of $A_i$. □

**Lemma 6.** *Let $G$ be any graph, let $\prec$ be any linear ordering of $V(G)$, let $Z_1 \prec \cdots \prec Z_{2s}$ be subsets of $V(G)$, and let $r_1 \prec \cdots \prec r_{2s}$ be vertices of $G$ such that, for each $i, j \in \{1, \ldots, 2s\}$, $G$ contains an edge $r_i z_j$ with $z_j \in Z_j$. Then $G$ contains a set of $s$ edges that are pairwise crossing with respect to $\prec$.*

*Proof.* At least one of the following two cases applies (see Figure 3):

1. $Z_s \prec r_{s+1}$ in which case the graph between $r_{s+1}, \ldots, r_{2s}$ and $Z_1, \ldots, Z_s$ has a set of $s$ pairwise-crossing edges.

2. $r_s \prec Z_{s+1}$ in which case the graph between $r_1, \ldots, r_s$ and $Z_{s+1}, \ldots, Z_{2s}$ has a set of $s$ pairwise-crossing edges. □

## 4 Open Problems

Recall that every 1-queue graph has a 2-stack layout [9] and we proved that there are 4-queue graphs with unbounded stack-number. The following questions remain open: Do 2-queue graphs have bounded stack-number? Do 3-queue graphs have bounded stack-number?

Is $\operatorname{sn}(T \square H)$ bounded for every tree $T$ and outerplanar graph $H$ with bounded degree?

Is $\operatorname{sn}(T \boxtimes P)$ bounded for every tree $T$ and path $P$?

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