REFEREE REPORT ON "ASYMPTOTICALLY OPTIMAL VERTEX RANKING OF PLANAR GRAPHS"

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General Comments

In its present form, this paper is full of typos and inaccuracies. I stopped in the proof of Lemma 29. It seems that the parameter β present in the statement has been added at some point, but that the proof has not been properly updated. I would like this proof to be fixed before further reading.

Section 1

Although the authors dedicate paragraph 1.1.1 to vertex ranking, they refrain to precise that vertex ranking is equal to tree-depth (see [29]) and (consequently) to mention quite a few relevant works. Also, the rank number of a graph G is the minimum clique number of a trivially perfect super-graph of G (which is an analog of similar characterizations of pathwidth and treewidth). Moreover, I do not understand why you refer to [33] for the notion of ranking, and not to [A.V. Iyer, H.D. Ratliff, and G. Vijayan. Optimal node ranking of trees. *Information Processing Letters*, 28(5), 225-229 (1988)], to which [3] explicitly refers to.

The notion of (vertex) t-ranking was introduced in [7] (without dedicated notation) and it would be fair to quote this reference when you recall its definition. The choice of $\chi_\ell(G)$(instead of $\chi_{\ell\text{-vr}}$ used in [19]) to denote ℓ-vertex ranking is questionable, particularly as you stress the essential difference with $(\ell+1)$-centered coloring, which is commonly denoted this way (since [29]).

p2, l59	… $K_3 \boxtimes P \cancel{K_3 \times P}$ …
p3, l100	There exists n-vertex planar…: maybe give the trivial example of grids.
p3, l105 − 109	Why not the more precise bounds $\chi_\infty(P_n) = \lceil \log_2(n+1) \rceil$ and $\chi_\infty(G) \le (t+1)\log_2 n$ (see e.g. [29])?
p3, l116	More generally…: the notation $O(\ell \log n)$ suggests that you mean that for any fixed proper minor-closed family $\mathcal{G}$ of graphs, $\chi_\ell(G) \in O(\ell \log n)$ for every positive integer ℓ and every n-vertex graph $G \in \mathcal{G}$.

p4,	Table 1	In [19], the bound $O(\ell \log n)$ is proved for every fixed proper minor closed class (Theorem 5.4). Why do you mention here that this bound holds only for planar graphs (thus contradicting what you wrote on the previous page, line 116)?
p4,	l142 – 143	You should include a proper definition of $(\ell+1)$-centered coloring: An $(\ell+1)$-centered coloring is a coloring such that every connected subgraph with at most ℓ colors has some uniquely colored vertex. The connection with ℓ-ranking does not seem right to me: there is a coloration with ℓ colors of the path on $2^{\ell-1}+1$ vertices that is not $(\ell+1)$-centered, but such that every subgraph of diameter at most $2^{\ell-1}$ has exactly one vertex of maximum color. For instance, consider 5 1 2 1 3 1 2 1 4 1 2 1 3 1 2 1 5 and $\ell = 5$. However, 2^{ℓ}-vertex ranking is a stronger notion than ℓ-centered coloring.

Section 2

It might be useful to precise that if a graph G is edge-maximal for a path-decomposition (resp. a tree-decomposition) of width t then it is an interval graph (resp. a chordal graph) with clique number at most $t+1$.

p7,	Lemma 11	Please give some reference (or at least refer to [N. Robertson, P.D. Seymour, Graph minors. II. Algorithmic aspects of tree-width, Journal of Algorithms, 7(3) 1986, pp. 309-322] for the unweighted version)
p7,	l241	~~the thesis of~~ Wulf ~~[37]~~ studies it extensively in his thesis [37].
p7,	l251	A proof . . . is due to Wood: if the lemma is due to Wood, then write "The following lemma, whose proof uses minor-monotony, is due to Wood"; otherwise, give a proper reference to Lemma 14.
p7,	l258	whose vertex set $V(G_1) \boxtimes V(G_2)$ is the Cartesian product $\cancel{V(G_1) \boxtimes V(G_2) :=} V(G_1) \times V(G_2)$
p8,	l276	implies that $U_{0,1} \neq u_{p,1}$, for otherwise . . .
p8,	l278	$\rho(u_{0,1}) < \max\{ \rho(u_{0,1}) \cancel{\rho_{u_{0,1}}}, \dots, \rho(u_{p,1}) \cancel{u_{p,1}} \}$
p8,	l281	any graph ~~is~~ G is
p8,	l282	A priori, the upper bound should be $8 \cdot 7^{\ell} + 1$. (Arguing that the product is not a complete graph to decrease the upper bound by 1 would obviously be of no interest here.)
p8,	l281 – 282	The following observation~~lemma~~. . . : either this is an observation, or you should replace the comments on lines 282-285 by a formal proof after the statement of the lemma.

p8, l1285 – 288

…in P) and contains cliques of order $3(\ell+1)$.
Observation 17. For any path P, $\overline{\chi}_\ell(K_3 \boxtimes P)=3(\ell+1)$.
~~We remark … $3(l+1)$.~~

p8, l1291 – 292

$\log x \le x-1$ for $x>0$ is a basic property of the natural logarithm. Hence (1) could definitely be reduced to

$$\log(x+a) = \log x + \log(1+a/x) \le \log x + \frac{a}{x}$$

p9, l1298

Either precise $i>0$ or define $\tau(-1)$.

p9, l1304

which is ~~again~~ valid …

p9, l1306

For any positive integer i…

p9, l1307

the solution $x \in [\tau(i), k]$ to the equation …(You prove the existence of a unique solution in the interval $[\tau(i), k]$, but not that the solution is unique on the domain on which $(\log^{(i)} k)^k/(\log^{(i)} x)^x$ is defined.)

p9, l1310

You should add the remark that $\gamma_{i,k}$ is a decreasing function (used in proof of Lemma 29).

Section 3

There is a problem with the bound in Lemma 20 (see detailed comments below), whose fixing requires an update of the proof of Lemma 22.

p9, l1313

The idea ~~in this lower bound~~ is to …

p10, l1336 – 340

Let $k=\chi_2(U^{(h,m)})$ ~~Suppose for the sake … $hm+1$~~ and let $\phi : V(U^{(h,m)} \to \{1,\dots,k\}$ be a 2-ranking of $U^{(h,m)}$ ~~$U_{(h,m)}$~~. (…) This gives $\chi_2(U^{(h,m)}) \ge hm+1$ ~~the desired contradiction~~ since … $a_0 \in L_0$, $k \ge \phi(a_0) \ge mh+1$ ~~$> k$~~.

p10, l1349

There is something wrong here: as $m \to \infty$ we have

$$\begin{aligned}\sum_{i=0}^{m} (|U|(hm+1))^i &= (|U|(hm+1))^m \cdot (1+o(1)) \\ &= (|U|hm)^m \cdot \left(1+\frac{1}{hm}\right)^m \cdot (1+o(1)) \\ &= (|U|hm)^m \cdot (1+e^{1/h}+o(1))\end{aligned}$$

Thus if h and $|U|$ are fixed the asymptotic value is not equivalent to $(|U|hm)^m$.

p10, l353 – 354 | take $hm+1$ ~~$h+1$~~ disjoint copies … $x \in V(T_{hm})$ ~~$x \in V(T_h)$~~ … For each $i \in \{0,\ldots,hm\}$ ~~$i \in \{0,\ldots,h\}$~~, …

p11, l360 | ~~The base case … by~~ Karpas et al. [19] have shown ~~who show~~ … $\chi_2(T) \geq r$. As the … $(\log^{(0)} r)^r$, this establishes the base case $t=1$.

p11, l365 | ~~Now we take the graph~~ Let $G := U^{(h,m)}$.

p11, l368 | Update the upper bound (including the error term in the first line) and the consecutive computations

Section 4

p11, l391 | in ?? we discuss …: missing reference

p12, l423 | it is well known: please give a reference

p13, l447 | Consider ~~the~~ a greedy path …

p13, l449 | It is ~~well known~~ easily checked. (Note that this is related to the "well known" fact mentioned in the proof of the preceding lemma. Maybe you could add some basic properties of pathwidth to the Preliminaries section with a reference?)

p13, l453 | each $i \in \{1,\ldots,\cancel{p-1}\ p\}$. (or even $p+1$, as you probably wish to define $y_{p+1} = r(u_p) = m$ as well.)

p13, l454 | If you decide to define y_{p+1}, you can write $\{y_i,\ldots,y_{i+1}\}$ instead of $\{y_i,\ldots,m\}$.

p13, l465 | Why do you claim $B_{y_p} = B_m$? $y_p = r(u_{p-1})$ thus if $B_{y_p} = B_m$, then $u_0,\ldots,u_{p-1}$ is a shortest path from B_1 to B_m. I think you wanted to consider $y_0,\ldots,y_{p+1}$, $G'_1,\ldots,G'_{p+1}$, etc. so that $B_{y_{p+1}} = B_m$. If this is so, you should update the proof.

p14, l478 | $\{w_a,w_b\} \subseteq V(G'_j)$

p14, l484 | G'_{p+1} is not defined.

p14, l492 | The set $\Lambda(T)$ should also include the root (which could have a single children) and all the leaves of T. Otherwise, if T is a path then $\hat{H}$ is empty.

p14, l497 | For each pair of ~~branching~~ nodes …

p15, l517 | $\cancel{\chi_2(\hat{H})}$ $\chi_\ell(\hat{H})$

p15, l534	(...) the first block has more layers attached to it: this is not consistent with the second sentence of the paragraph ("The main idea ... each of which consists of $\ell + 1$ consecutive BFS layers").
p15, l535	I assume t, d, l, β are positive integers. Maybe you should precise somewhere that $\mathbb{N}$ does not include 0.
p16, l548	The β factor disappeared? I assume this is a typo. ~~implies that~~ rewrites as $n_H \leq \beta \cdot (\log^{(t-2)} k)^k / (\log^{(t-2)} c)^c$.
p16, l550	The inequality $\gamma_H \geq (\log c)^c$ has no reason to hold if $\beta > 1$.