# Intro to Linear Algebra

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# A Short Overview of Linear Algebra

- Consider the following linear regression model in scalar notation:
- $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_n x_n + \epsilon$
- And the same model in matrix notation:
- $\mathbf{y} = \mathbf{X}\beta + \epsilon$;
- where $\mathbf{y}$ is a vector of values for the dependent variable, $\mathbf{X}$ is a matrix of values for the independent variables, $\beta$ is vector of coefficients $\epsilon$ is a vector of disturbance terms

# Vectors

- A vector is a series of numbers in a particular sequence
- They are defined by a bold faced, lower case letter
- And can take the form of row vectors:

$$\mathbf{v} = [1, 2, 3, 4]$$

- or column vectors:

$$\mathbf{v}' = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

# Mathematical Operations With Vectors

- Given the following two vectors $\mathbf{u}$ and $\mathbf{v}$:

$$\mathbf{u} = [1, 2, 3, 4], \text{ and } \mathbf{v} = [4, 3, 2, 1]$$

- We can add perform vector addition $(\mathbf{u} + \mathbf{v})$ or vector subtraction $(\mathbf{u} - \mathbf{v})$
- Additionally, we can multiple or divide a vector by a scalar

# Mathematical Operations With Vectors

- Vector addition (subtraction) involves adding (subtracting) the corresponding elements of two vectors
- To perform either operation the vectors *must be either both be column or both row vectors and they must have the same number of elements*
- Or, general terms, the vectors must be conformable, meaning the first vector is a size that conforms with the second
- If two vectors are nonconformable, then we cannot complete the operation

# Vector Addition and Subtraction

- General Rules:

$$\mathbf{u} + \mathbf{v} = [u_1, u_2, u_3, u_4] + [v_1, v_2, v_3, v_4]$$

$$= [u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4]$$

$$\mathbf{u} - \mathbf{v} = [u_1, u_2, u_3, u_4] - [v_1, v_2, v_3, v_4]$$

$$= [u_1 - v_1, u_2 - v_2, u_3 - v_3, u_4 - v_4]$$

# Vector Addition and Subtraction

- Examples:

$$\mathbf{u} + \mathbf{v} = [1, 2, 3, 4] + [4, 3, 2, 1]$$

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# Vector Addition and Subtraction

- Examples:

$$\mathbf{u} + \mathbf{v} = [1, 2, 3, 4] + [4, 3, 2, 1]$$

$$= [1 + 4, 2 + 3, 3 + 2, 4 + 1]$$

$$= [5, 5, 5, 5]$$

# Scalar Multiplication and Division

- Scalar multiplication and division simple involves multiplying the scalar by each element of the vector:

$$3 * \mathbf{u} = [3 * u_1, 3 * u_2, 3 * u_3, 3 * u_4]$$

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$$= [3 * 1, 3 * 2, 3 * 3, 3 * 4]$$

$$= [3, 6, 9, 12]$$

# Elementary Formal Properties of Vector Algebra

| | |
|---|---|
| Commutative Property | $\mathbf{u} + \mathbf{v} = (\mathbf{v} + \mathbf{u})$ |
| Additive Associative Property | $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\ \mathbf{v} + \mathbf{w})$ |
| Vector Distributive Property | $s(\mathbf{u} + \mathbf{v}) = s\mathbf{u} + s\mathbf{v}$ |
| Scalar Distributive Property | $(s + t)\mathbf{u} = s\mathbf{u} + t\mathbf{u}$ |
| Zero Property | $\mathbf{u} + 0 = \mathbf{u} \longleftrightarrow \mathbf{u}$ - $\mathbf{u} = 0$ |
| Zero Multiplicative Property | $0\mathbf{u} = 0$ |

# Basics of Matrices

- A matrix is simply a rectangular array of numbers (e.g. any dataset)
- The individual elements of a matrix are ordered by row and column
- And the matrix itself is defined by these two dimensions, such that a matrix with $i$ rows and $j$ columns is of dimension $i$ x $j$
- Example:

$$\underset{2\times 2}{\mathbf{X}} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

# Basics of Matrices

- More generally, we can define the elements on any $n \times p$ matrix by subscripting
- Example:

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} & \cdots & \cdots & \cdots & x_{1(p-1)} & x_{1p} \\ x_{21} & x_{22} & \cdots & \cdots & \cdots & x_{2(p-1)} & x_{2p} \\ \vdots & \vdots & \ddots & & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & & \ddots & \vdots & \vdots \\ x_{(n-1)1} & x_{(n-1)2} & \cdots & \cdots & \cdots & x_{(n-1)(p-1)} & x_{(n-1)p} \\ x_{n1} & x_{n2} & \cdots & \cdots & \cdots & x_{n(p-1)} & x_{np} \end{bmatrix}$$

# Special Matrices

- Square Matrix — A matrix with the same number of rows and columns

$$\mathbf{X} = \begin{bmatrix} 5 & -3 & 5 \\ 1 & 8 & 7 \\ -56 & 3 & 21 \end{bmatrix}$$

- Symmetric Matrix — All elements above the diagonal (upper-left to lower-right) are equal (special case of the square matrix)

$$\mathbf{X} = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 6 \\ 2 & 5 & 6 & 1 \end{bmatrix}$$

# Special Matrices

- Diagonal Matrix – All off-diagonal elements are 0 (special case of symmetric, and thus of the square matrix)

$$\mathbf{X} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

- Identity Matrix – Special case of the diagonal matrix, where all diagonal elements are 1. This is the matrix equivalent of the scalar 1

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Matrix Addition and Subtraction

- Addition and subtraction of matrices is quite simple and simply involves adding (or subtracting) the corresponding elements
- This requires that the matrices have the same number or rows *and* columns to be conformable
- Given $\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$ and $\mathbf{Y} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}$, then

$$\mathbf{X} + \mathbf{Y} = \begin{bmatrix} x_{11} + y_{11} & x_{12} + y_{12} \\ x_{21} + y_{21} & x_{22} + y_{22} \end{bmatrix}$$

- Or

$$\mathbf{X} - \mathbf{Y} = \begin{bmatrix} x_{11} - y_{11} & x_{12} - y_{12} \\ x_{21} - y_{21} & x_{22} - y_{22} \end{bmatrix}$$

# Properties of (Conformable) Matrix Manipulation

| | |
|---|---|
| Commutative Property | $\mathbf{X} + \mathbf{Y} = (\mathbf{Y} + \mathbf{X})$ |
| Additive Associative Property | $(\mathbf{X} + \mathbf{Y}) + \mathbf{Z} = \mathbf{X} + (\mathbf{Y} + \mathbf{Z})$ |
| Matrix Distributive Property | $s(\mathbf{X} + \mathbf{Y}) = s\mathbf{X} + s\mathbf{Y}$ |
| Scalar Distributive Property | $(s + t)\mathbf{X} = s\mathbf{X} + t\mathbf{X}$ |
| Zero Property | $\mathbf{X} + 0 = \mathbf{X}$ and $\mathbf{X}$ - $\mathbf{X} = 0$ |

# Matrix Multiplication

- Matrix multiplication is slightly more complicated than addition or subtraction
- The principle issue is one of conformability; for two matrices to be conformable for purposes of multiplication the number of *columns* in the first matrix must be the same as the number of *rows* in the second matrix
- This generates the second issue: the order matters. $\mathbf{XY} \neq \mathbf{YX}$
- For example: $\underset{(k \times n)}{\mathbf{X}} \underset{(n \times p)}{\mathbf{Y}}$ would be valid, but $\underset{(n \times p)}{\mathbf{Y}} \underset{(k \times n)}{\mathbf{X}}$ would not be (unless $k = p$)

# Matrix Multiplication

- In the **XY** example, the resulting matrix would be of size $k$ x $p$
- The language we use to describe matrix multiplication reflects the importance of order.
- For this example, we would say either:
  - **X** pre-multiplies **Y**, or
  - **Y** post-multiplies **X**
- Formally, if we define a matrix **Z** as the product of **XY**, then
  - We can define each element of **Z** as $\mathbf{z}_{kp} = \sum_n x_{kn} y_{np}$
  - In words: The element in the $k$th row and the $p$th column of **Z** is obtained by multiplying the elements of the $k$th row of **X** by the corresponding elements of the $p$th column of **Y** and summing over all terms

# Matrix Multiplication

- In general:

$$\mathbf{XY} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}$$

$$= \begin{bmatrix} x_{11}y_{11} + x_{12}y_{21} & x_{11}y_{12} + x_{12}y_{22} \\ x_{21}y_{11} + x_{22}y_{21} & x_{21}y_{12} + x_{22}y_{22} \end{bmatrix}$$

# Matrix Multiplication

- Numerical Example:

$$\mathbf{XY} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 2 \\ 0 & 1 \end{bmatrix}$$

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$$\mathbf{XY} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (1)(-2) + (2)(0) & (1)(2) + (2)(1) \\ (3)(-2) + (4)(0) & (3)(2) + (4)(1) \end{bmatrix}$$

# Matrix Multiplication

- Numerical Example:

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$$= \begin{bmatrix} (1)(-2)+(2)(0) & (1)(2)+(2)(1) \\ (3)(-2)+(4)(0) & (3)(2)+(4)(1) \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 4 \\ -6 & 10 \end{bmatrix}$$

# Properties of (Conformable) Matrix Multiplication

| | |
|---|---|
| Associative Property | $(\mathbf{XY})\mathbf{Z} = \mathbf{X}(\mathbf{YZ})$ |
| Additive Distributive Property | $(\mathbf{X} + \mathbf{Y})\mathbf{Z} = \mathbf{XZ} + \mathbf{YZ}$ |
| Scalar Distributive Property | $s\mathbf{XY} = (\mathbf{X}s)\mathbf{Y} = \mathbf{X}(s\mathbf{Y}) = \mathbf{XY}s$ |
| Zero Multiplicative Property | $\mathbf{X}0 = 0$ |

# Matrix Transposition

- Matrix transposition simply involves switching the rows and columns of a matrix
- We denote the transpose of a matrix $\mathbf{X}$ by $\mathbf{X}'$ or $\mathbf{X}^T$
- And we say "$\mathbf{X}$ prime" or "$\mathbf{X}$ transpose"
- Example:

$$\mathbf{X} = \begin{bmatrix} 2 & 3 & 5 \\ 6 & 1 & 3 \end{bmatrix} \qquad \mathbf{X}' = \begin{bmatrix} 2 & 6 \\ 3 & 1 \\ 5 & 3 \end{bmatrix}$$

# Properties of Matrix Transposition

| | |
|---|---|
| Invertability | $(\mathbf{X}')' = \mathbf{X}$ |
| Additive Property | $(\mathbf{X}+\mathbf{Y})' = \mathbf{X}' + \mathbf{Y}'$ |
| Multiplicative Property | $(\mathbf{XY})' = \mathbf{Y}'\mathbf{X}'$ |
| General Multiplicative Property | $(\mathbf{X}_1\mathbf{X}_2 \ldots \mathbf{X}_{n-1}\mathbf{X}_n)' = \mathbf{X}'_n\mathbf{X}'_{n-1} \ldots \mathbf{X}'_2\mathbf{X}'_1$ |

# Matrix Inversion

- The inverse of a matrix is similar conceptually to the inverse of a scalar
- Not all matrices have a inverse. Specifically, only some, *square* matrices have an inverse
- We denote the inverse of a matrix $\mathbf{X}$ as $\mathbf{X}^{-1}$ and define it with the following property:
- $\mathbf{X}\mathbf{X}^{-1} = \mathbf{X}^{-1}\mathbf{X} = \mathbf{I}$
- Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

# Determinants

- A useful, but difficult to calculate, summary measure of a matrix is known as the determinant
- The determinant utilizes all the values and provide a summary of the structure of the matrix
- A determinant exists for all *square* matrices and in denoted as $\det(\mathbf{X})$ or $|\mathbf{X}|$

# Determinants

- For the case of a 2 x 2, calculating the determinant is relatively simple:

$$\det(\mathbf{X}) = |\mathbf{X}| = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = x_{11}x_{22} - x_{12}x_{21}$$

- Moving to larger matrices, the process of calculating a determinant become significantly more complex
- And involves the defining of submatrices created by deleting specific rows and columns.
- Thankfully, modern technology has largely elemented the need to hand-calculate determinants.