1. Let $m$ and $n$ be positive integers. Let $a_1, a_2, \ldots, a_m$ be distinct elements of $\{1, 2, \ldots, n\}$ such that whenever $a_i + a_j \leq n$ for some $i, j, 1 \leq i \leq j \leq m$, there exists $k, 1 \leq k \leq m$, with $a_i + a_j = a_k$. Prove that
$\frac{a_1+a_2+\cdots+a_m}{m} \geq \frac{n+1}{2}$.

2. $ABC$ is an isosceles triangle with $AB = AC$. Suppose that

   1. $M$ is the midpoint of $BC$ and $O$ is the point on the line $AM$ such that $OB$ is perpendicular to $AB$;

   2. $Q$ is an arbitrary point on the segment $BC$ different from $B$ and $C$;

   3. $E$ lies on the line $AB$ and $F$ lies on the line $AC$ such that $E$, $Q$, $F$ are distinct and collinear.

   Prove that $OQ$ is perpendicular to $EF$ if and only if $QE = QF$.

3. For any positive integer $k$, let $f(k)$ be the number of elements in the set $\{k+1, k+2, \ldots, 2k\}$ whose base 2 representation has precisely three $1s$.

   • $(a)$ Prove that, for each positive integer $m$, there exists at least one positive integer $k$ such that $f(k) = m$.

   • $(b)$ Determine all positive integers $m$ for which there exists exactly one $k$ with $f(k) = m$.

4. Determine all ordered pairs $(m, n)$ of positive integers such that
$\frac{n^3+1}{mn-1}$
is an integer.

5. Let $S$ be the set of real numbers strictly greater than $-1$. Find all functions $f : S \to S$ satisfying the two conditions:

   1. $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$ for all $x$ and $y$ in $S$;

   2. $\frac{f(x)}{x}$ is strictly increasing on each of the intervals $-1 < x < 0$

   and $0 < x$.

6. Show that there exists a set $A$ of positive integers with the following property: For any infinite set S of primes there exist two positive integers $m \in A$ and $n \notin A$ each of which is a product of $k$ distinct elements of $S$ for some $k \geq 2$.