Analog Communication Theory:
A Text for EE501

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Note to Students. This text is an evolving entity. Please help make an OSU education more valuable by providing me feedback on this work. Small things like catching typos or big things like highlighting sections that are not clear are both important.

My goal in teaching communications (and in authoring this text) is to provide students with

1. the required theory,
2. an insight into the required tradeoffs between spectral efficiency, performance, and complexity that are required for a communication system design,
3. demonstration of the utility and applicability of the theory in the homework problems and projects,
4. a logical progression in thinking about communication theory.

Consequently this textbook will be more mathematical than most and does not discuss a host of examples of communication systems. Matlab is used extensively to illustrate the concepts of communication theory as it is a great visualization tool. To me the beauty of communication theory is the logical flow of ideas. I have tried to capture this progression in this text.

This book is written for the modern communications curriculum. Most modern communications curriculum at the undergraduate level have a networking course hence no coverage is given for networking. For communications majors it is expected that this course will be followed by a course in digital communications (EE702). The course objectives for EE501 that can be taught from this text are (along with their ABET criteria)

1. Students learn the bandpass representation for carrier modulated signals. (Criterion 3(a))
2. Students engage in engineering design of communications system components. (Criteria 3(c),(k))
3. Students learn to analyze the performance, spectral efficiency and complexity of the various options for transmitting analog message signals. (Criteria 3(e),(k))
4. Students learn to characterize noise in communication systems. (Criterion 3(a))

Prerequisites to this course are random variables (Math530 or STAT427) and a signal and systems course (EE351-2).

Many of my professional colleagues have made the suggestion that analog modulation concepts should be removed from the modern undergraduate curriculum. Comments such as "We do not teach about vacuum tubes so why should we teach about analog modulations?" are frequently heard. I heartily disagree with this opinion but not because I have a fondness for analog modulation but because analog modulation concepts are so important in modern communication systems. The theory and notation for signals and noise learned in this class will be a solid foundation for further explorations into modern communication systems. For example in the testing of modern communication systems and subsystems analog modulation and demodulation concepts are used extensively. In fact most of my good problems for the analog communication chapters have come as a result of my work in experimental wireless communications even though my research work has always been focused on digital communication systems! Another example of the utility of analog communications is that I am unaware of a synthesized signal generator that does not have an option to produce amplitude modulated (AM) and frequency modulated (FM) test signals. While modern communication engineers do not often design analog communication systems, the theory is still a useful tool. Consequently EE501 focuses on analog communications and noise but using a modern perspective that will provide students the tools to flourish in their careers.

Finally thanks go to the many people who commented on previous versions of these notes. Peter Doerschuk (PD) and Urbashi Mitra (UM) have contributed homework problems.

Contents

Chapter 1

Signals and Systems Review

This chapter provides a brief review of signals and systems theory usually taught in an undergraduate curriculum. The purpose of this review is to introduce the notation that will be used in this text. Most results will be given without proof as these can be found in most undergraduate texts in signals and systems [May84, ZTF89, OW97, KH97].

1.1 Signal Classification

A signal, $x(t)$, is defined to be a function of time ($t \in \mathcal{R}$). Signals in engineering systems are typically described with five different mathematical classifications:

1. Deterministic or Random,
2. Energy or Power,
3. Periodic or Aperiodic,
4. Complex or Real,
5. Continuous Time or Discrete Time.

We will only consider deterministic signals at this point as random signals are a subject of Chapter 8.

1.1.1 Energy versus Power Signals

Definition 1.1 *The energy, E_x, of a signal $x(t)$ is*

$$E_x = \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 \, dt. \tag{1.1}$$

$x(t)$ is called an energy signal when $E_x < \infty$. Energy signals are normally associated with pulsed or finite duration waveforms (e.g., speech or a finite length information transmission). In contrast, a signal is called a power signal if it does not have finite energy. In reality, all signals are energy signals (infinity is hard to produce in a physical system) but it is often mathematically convenient to model certain signals as power signals. For example, when considering a voice signal it is usually appropriate to consider the signal as an energy signal for voice recognition applications but in radio broadcast applications we often model voice as a power signal.

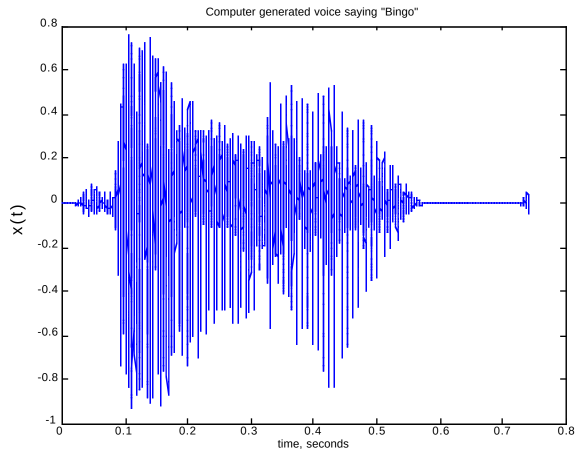


Figure 1.1: The time waveform for a computer generated voice saying "Bingo.".

Example 1.1: A pulse is an energy signal:

$$\begin{aligned} x(t) &= \frac{1}{\sqrt{T}} \quad 0 \leq t \leq T \\ &= 0 \quad \text{elsewhere} \end{aligned} \qquad E_x = 1 \tag{1.2}$$

Example 1.2: Not all energy signals have finite duration:

$$x(t) = 2W\frac{\sin(2\pi Wt)}{2\pi Wt} = 2W\mathrm{sinc}\,(2Wt) \qquad E_x = 2W \tag{1.3}$$

Example 1.3: A voice signal. Fig. 1.1 shows the time waveform for a computer generated voice saying "Bingo." This signal is an obvious energy signal due to it's finite time duration.

Definition 1.2 *The signal power, P_x, is*

$$P_x = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2\, dt \tag{1.4}$$

Note that if $E_x < \infty$ then $P_x = 0$ and if $P_x > 0$ then $E_x = \infty$.

Example 1.4:

$$x(t) = \cos(2\pi f_c t)$$

$$P_x = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \cos^2(2\pi f_c t)dt = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left(\frac{1}{2} + \frac{1}{2}\cos(4\pi f_c t)\right) dt = \frac{1}{2} \tag{1.5}$$

Throughout this text, signals will be considered and analyzed independent of physical units but the concept of units is worth a couple comments at this point. Energy is typically measured in Joules and power is typically measured in Watts. Most signals we consider as electrical engineers are voltages or currents, so to obtain energy and power in the appropriate units we need to specify a resistance (e.g., Watts=Volts^2/Ohms). To simplify notation we will just define energy and power as above which is equivalent to having the signal $x(t)$ measured in volts or amperes and the resistance being unity ($R = 1\Omega$).

1.1.2 Periodic versus Aperiodic

A periodic signal is one that repeats itself in time.

Definition 1.3 *$x(t)$ is a periodic signal when*

$$x(t) = x(t + T_0) \qquad \forall t \quad \textit{and for some} \quad T_0 \neq 0. \tag{1.6}$$

Definition 1.4 *The signal period is*

$$T = \min\left(|T_0|\right). \tag{1.7}$$

The fundamental frequency is then

$$f_T = \frac{1}{T}. \tag{1.8}$$

Example 1.5: The simplest example of a periodic signal is

$$x(t) = \cos(2\pi f_c t) \qquad\qquad T_0 = \frac{n}{f_c} \qquad\qquad T = \frac{1}{f_c} \tag{1.9}$$

Most periodic signals are power signals (note if the energy in one period is nonzero then the periodic signal is a power signal) and again periodicity is a mathematical convenience that is not rigorously true for any real signal. We use the model of periodicity when the signal has approximately the property in (1.6) over the time range of interest. An aperiodic signal is defined to be a signal that is not periodic.

Example 1.6: Aperiodic signals

$$x(t) = e^{-\frac{t}{\tau}} \qquad\qquad x(t) = \frac{1}{t} \tag{1.10}$$

1.1.3 Real versus Complex Signals

Complex signals arise often in communication systems analysis and design. The most common example is in the representation of bandpass signals and Chapter 3 discusses this in more detail. For this review we will consider some simple characteristics of complex signals. Define a complex signal and a complex exponential to be

$$z(t) = x(t) + jy(t) \qquad\qquad e^{j\theta} = \cos(\theta) + j\sin(\theta) \tag{1.11}$$

where $x(t)$ and $y(t)$ are both real signals. Note this definition is often known as Euler's rule. A magnitude and phase representation of a complex signal is also commonly used, i.e.,

$$z(t) = A(t)e^{j\theta(t)} \qquad \text{and} \qquad A(t) = |z(t)| \qquad \theta(t) = \arg(z(t)). \tag{1.12}$$

The complex conjugate operation is defined as

$$z^*(t) = x(t) - jy(t) = A(t)e^{-j\theta(t)}. \tag{1.13}$$

Some important formulas for analyzing complex signals are

$$\begin{aligned} |z(t)|^2 = A(t)^2 = z(t)z^*(t) = x(t)^2 + y(t)^2, &\qquad \cos(\theta)^2 + \sin(\theta)^2 = 1, \\ \Re\left[z(t)\right] = x(t) = A(t)\cos(\theta) = \tfrac{1}{2}\left[z(t) + z^*(t)\right], &\quad \cos(\theta) = \tfrac{1}{2}\left[e^{j\theta} + e^{-j\theta}\right], \\ \Im\left[z(t)\right] = y(t) = A(t)\sin(\theta) = \tfrac{1}{2j}\left[z(t) - z^*(t)\right], &\quad \sin(\theta) = \tfrac{1}{2j}\left[e^{j\theta} - e^{-j\theta}\right]. \end{aligned} \tag{1.14}$$

Example 1.7: The most common complex signal in communication engineering is the complex exponential, i.e.,

$$\exp\left[j2\pi f_m t\right] = \cos\left(2\pi f_m t\right) + j\sin\left(2\pi f_m t\right).$$

This signal will be the basis of the frequency domain understanding of signals.

1.1.4 Continuous Time Signals versus Discrete Time Signals

A signal, $x(t)$, is defined to be a continuous time signal if the domain of the function defining the signal contains intervals of the real line. A signal, $x(t)$, is defined to be a discrete time signal if the domain of the signal is a countable subset of the real line. Often a discrete signal is denoted x_k where k is an integer and a discrete signal often arises from (uniform) sampling of a continuous time signal, e.g., $x(k) = x(kT_s)$. Discrete signals and systems are of increasing importance because of the widespread use of computers and digital signal processors, but in communication systems the vast majority of the transmitted and received signals are continuous time signals. Consequently since this is a course in transmitter and receiver design (physical layer communications) we will be primarily concerned with continuous time signals and systems. Alternately, the digital computer is a great tool for visualization and discrete valued signal are processed in the computer. Section 1.4 will discuss in more detail how the computer and specifically the software package *Matlab* can be used to examine continuous time signal models.

1.2 Frequency Domain Characterization of Signals

Signal analysis can be completed in either the time or frequency domains. This section briefly overviews some simple results for frequency domain analysis. We first review the Fourier series representation for periodic signal, then discuss the Fourier transform for energy signals and finally relate the two concepts.

1.2.1 Fourier Series

If $x(t)$ is periodic with period T then $x(t)$ can be represented as

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{j\frac{2\pi nt}{T}} \tag{1.15}$$

where

$$x_n = \frac{1}{T}\int_0^T x(t) e^{-j\frac{2\pi nt}{T}} dt. \tag{1.16}$$

This is known as the complex exponential Fourier series. Note a sine-cosine Fourier series is also possible with equivalence due to Euler's rule. Note in general the x_n are complex numbers. In words: *A periodic signal, $x(t)$, with period T can be decomposed into a weighted sum of complex sinusoids with frequencies that are an integer multiple of the fundamental frequency.*

Figure 1.2: A periodic pulse train.

Example 1.8:

$$
\begin{aligned}
x(t) &= \cos\left(\frac{2\pi t}{T}\right) = \frac{1}{2}e^{j\frac{2\pi t}{T}} + \frac{1}{2}e^{-j\frac{2\pi t}{T}} \\
x_1 &= x_{-1} = \frac{1}{2} \qquad x_n = 0 \quad \text{for all other} \quad n
\end{aligned} \tag{1.17}
$$

Example 1.9: The waveform in Fig. 1.2 is an analytical representation of the transmitted signal in a simple radar or sonar system. A signal pulse is transmitted for τ seconds and the receiver then listens T seconds for returns from airplanes, ships, stormfronts, or other targets. After T seconds another pulse is transmitted. The Fourier series for this example is

$$
\begin{aligned}
x_n &= \frac{1}{T}\int_0^{\tau} \exp\left(-j\frac{2\pi nt}{T}\right) dt \\
&= \frac{1}{T} \left. \frac{\exp\left(\frac{-j2\pi nt}{T}\right)}{\frac{-j2\pi n}{T}} \right|_0^{\tau} \\
&= \frac{1}{T} \frac{1-\exp\left(\frac{-j2\pi n\tau}{T}\right)}{\frac{j2\pi n}{T}}
\end{aligned} \tag{1.18}
$$

Using $\sin(\theta) = \frac{e^{j\theta}-e^{-j\theta}}{j2}$ gives

$$
x_n = \frac{\tau}{T}\exp\left[-j\frac{\pi n\tau}{T}\right]\frac{\sin\left(\frac{\pi n\tau}{T}\right)}{\frac{\pi n\tau}{T}} = \frac{\tau}{T}\exp\left[-j\frac{\pi n\tau}{T}\right]\operatorname{sinc}\left(\frac{n\tau}{T}\right) \tag{1.19}
$$

A number of things should be noted about this example

1. τ and bandwidth of the waveform are inversely proportional, i.e., a smaller τ produces a larger bandwidth signal,
2. τ and the signal power are directly proportional.
3. If $T/\tau =$ integer, some terms will vanish (i.e., $\sin(m\pi) = 0$),
4. To produce a rectangular pulse requires an infinite number of harmonics.

Properties of the Fourier Series

Property 1.1 *If $x(t)$ is real then $x_n = x_{-n}^*$. This property is known as Hermitian symmetry. Consequently the Fourier series of a real signal is a Hermitian symmetric function of frequency. This implies that the magnitude of the Fourier series is an even function of frequency, i.e.,*

$$|x_n| = |x_{-n}| \tag{1.20}$$

and the phase of the Fourier series is an odd function of frequency, i.e.,

$$\arg(x_n) = -\arg(x_{-n}) \tag{1.21}$$

Property 1.2 *If $x(t)$ is real and an even function of time, i.e., $x(t) = x(-t)$, then all the coefficients of the Fourier series are real numbers.*

Property 1.3 *If $x(t)$ is real and odd, i.e., $x(t) = -x(-t)$, then all the coefficients of the Fourier series are imaginary numbers.*

Theorem 1.1 (Parseval)

$$P_x = \frac{1}{T}\int_0^T |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |x_n|^2 \tag{1.22}$$

Parseval's Theorem states that the power of a signal can be calculated using either the time or the frequency domain representation of the signal and the two results are identical.

1.2.2 Fourier Transform

If $x(t)$ is an energy signal, then the Fourier transform is defined as

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt = \mathcal{F}\left\{x(t)\right\}. \tag{1.23}$$

$X(f)$ is in general complex and gives the frequency domain representation of $x(t)$. The inverse Fourier transform is

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df = \mathcal{F}^{-1}\left\{X(f)\right\}.$$

Example 1.10: Example 1.1(cont.). The Fourier transform of

$$x(t) = 2W\frac{\sin(2\pi Wt)}{2\pi Wt} = 2W\operatorname{sinc}(2Wt)$$

is given as

$$\begin{aligned} X(f) &= 1 \qquad |f| \leq W \\ &= 0 \qquad \text{elsewhere} \end{aligned} \tag{1.24}$$

Properties of the Fourier Transform

Property 1.4 *If $x(t)$ is real then the Fourier transform is Hermitian symmetric, i.e., $X(f) = X^*(-f)$. This implies*

$$|X(f)| = |X^*(-f)| \qquad \arg(X(f)) = -\arg(X^*(-f)) \tag{1.25}$$

Property 1.5 *If $x(t)$ is real and an even function of time, i.e., $x(t) = x(-t)$, then $X(f)$ is a real valued and even function of frequency.*

Property 1.6 *If $x(t)$ is real and odd, i.e., $x(t) = -x(-t)$, then $X(f)$ is an imaginary valued and odd function of frequency.*

Theorem 1.2 (Rayleigh's Energy)

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df \tag{1.26}$$

Theorem 1.3 (Convolution) *The convolution of two time functions, $x(t)$ and $h(t)$, is defined*

$$y(t) = x(t) \otimes h(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda. \tag{1.27}$$

The Fourier transform of $y(t)$ is given as

$$Y(f) = \mathcal{F}\left\{y(t)\right\} = H(f)X(f). \tag{1.28}$$

Theorem 1.4 (Duality) *If $X(f) = \mathcal{F}\left\{x(t)\right\}$ then*

$$x(f) = \mathcal{F}\left\{X(-t)\right\} \qquad x(-f) = \mathcal{F}\left\{X(t)\right\} \tag{1.29}$$

Theorem 1.5 Translation and Dilation *If $y(t) = x(at+b)$ then*

$$Y(f) = \frac{1}{|a|} X\left(\frac{f}{a}\right) \exp\left[j2\pi \frac{b}{a} f\right] \tag{1.30}$$

Theorem 1.6 Modulation *Multiplying any signal by a sinusoidal signal results in a frequency translation of the Fourier transforms, i.e.,*

$$x_c(t) = x(t)\cos(2\pi f_c t) \qquad \Rightarrow \qquad X_c(f) = \frac{1}{2}X(f-f_c) + \frac{1}{2}X(f+f_c) \tag{1.31}$$

$$x_c(t) = x(t)\sin(2\pi f_c t) \qquad \Rightarrow \qquad X_c(f) = \frac{1}{j2}X(f-f_c) - \frac{1}{j2}X(f+f_c) \tag{1.32}$$

Definition 1.5 *The correlation function of a signal $x(t)$ is*

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)dt \tag{1.33}$$

Definition 1.6 *The energy spectrum of a signal $x(t)$ is*

$$G_x(f) = X(f)X^*(f) = |X(f)|^2 . \tag{1.34}$$

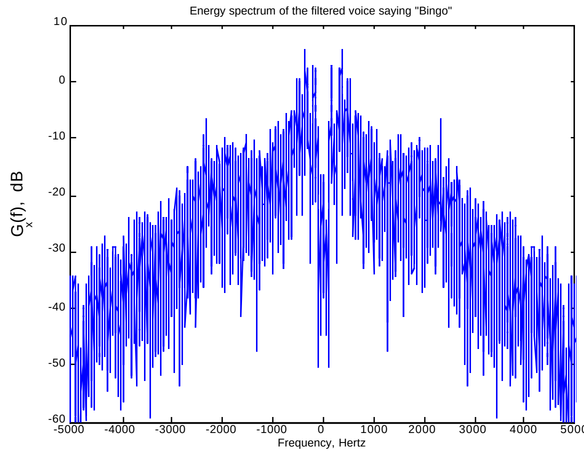


Figure 1.3: The energy spectrum of the computer generated voice signal from Example 1.2.

The energy spectral density is the Fourier transform of the correlation function, i.e.,

$$G_x(f) = \mathcal{F}\left\{R_x(\tau)\right\} \tag{1.35}$$

The energy spectrum is a functional description of how the energy in the signal $x(t)$ is distributed as a function of frequency. The units on an energy spectrum are Joules/Hz. Because of this characteristic two important characteristics of the energy spectral density are

$$\begin{aligned} G_x(f) &\geq 0 \qquad \forall f \qquad \text{(Energy in a signal cannot be negative valued)} \\ E_x &= \int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} G_x(f) df \end{aligned} \tag{1.36}$$

This last result is known as Rayleigh's energy theorem and the analogy to Parseval's theorem should be noted.

Example 1.11: Example 1.2(cont.). The energy spectrum of the computer generated voice signal is shown in Fig. 1.3. The two characteristics that stand out in examining this spectrum is that the energy in the signal starts to significantly drop off after about 2.5KHz and that the DC content of this voice signal is small.

Theory of Operation: Signal Analyzers

Electronic test equipment (power meters, spectrum analyzers, etc.) are tools which attempt to provide the important characteristics of electronic signals to the practicing engineer. The main difference between test equipment and the theoretical equations like those discussed in this chapter is that test equipment only observes the signal over a finite interval. For instance a power meter outputs a reading which is essentially a finite time average power.

Definition 1.7 *The average power over a time length of T is*

$$P_x(T) = \frac{1}{T}\int_0^T |x(t)|^2\, dt \quad \textit{Watts.} \tag{1.37}$$

Definition 1.8 *The Fourier transform of a signal truncated to a time length of T is*

$$X_T(f) = \int_0^T x(t)e^{-j2\pi f t}dt \tag{1.38}$$

An ideal spectrum analyzer produces the following measurement for a span of frequencies

$$S_x(f,T) = \frac{1}{T}\left|X_T(f)\right|^2 \quad \text{Watts/Hz.} \tag{1.39}$$

The function in (1.39) is often termed the sampled power spectral density and it is a direct analog to the energy spectral density of (1.34). The sampled power spectrum is a functional description of how the power in the truncated signal $x(t)$ is distributed as a function of frequency. The units on a power spectrum are Watts/Hz. Because of this characteristic two important characteristics of the power spectral density are

$$\begin{aligned} S_x(f,T) &\geq 0 \qquad \forall f \qquad \text{(Power in a signal cannot be negative valued)} \\ P_x(T) &= \int_0^T \left|x(t)\right|^2 dt = \int_{-\infty}^{\infty} S_x(f,T)df \end{aligned} \tag{1.40}$$

In practice the actual values output by a spectrum analyzer test equipment are closer to

$$\tilde{S}_x(f,T) = \frac{1}{T}\int_{-B_r/2}^{B_r/2} \left|X_T(f)\right|^2 \quad \text{Watts} \tag{1.41}$$

where B_r is denoted the resolution bandwidth (see [Pet89] for a good discussion of spectrum analyzers). Signal analyzers are designed to provide the same information for an engineer as signal and systems theory within the constraints of electronic measurement techniques.

Also when measuring electronic signals practicing engineers commonly use the decibel terminology. Decibel notation can express the ratio of two quantities, i.e.,

$$P = 10\log\left[\frac{P_1}{P_2}\right] \text{dB.} \tag{1.42}$$

In communication systems we often use this notation to compare the power of two signals. The decibel notation is used because it compresses the measurement scale ($10^6 = 60$dB). Also multiplies turn into adds which is often convenient in the discussion of electronic systems. The decibel notation can also be used for absolute measurements if P_2 becomes a reference level. The most commonly used absolute decibel scales are dBm (P_2=1 milliwatt) and dBW (P_2=1 Watt). In communication systems the ratio between the signal power and the noise power is an important quantity and this ratio is typically discussed with decibel notation.

1.2.3 Bandwidth of Signals

Engineers often like to quantify complex entities with simple parameterizations. This often helps in developing intuition and simplifies discussion among different members of an engineering team. Since the frequency domain description of signals can be a complex function engineers often like to describe signals in terms of their bandwidth. Unfortunately the bandwidth of a signal does not have a common definition across all engineering disciplines. The common definitions of bandwidth can be summarized as

1. X dB relative bandwidth, B_X Hz.
2. $P\%$ energy (power) integral bandwidth, B_P Hz.

For lowpass energy signals we have the following two definitions

Definition 1.9 *If a signal $x(t)$ has an energy spectrum $G_x(f)$ then B_X is determined as*

$$10\log\left(\max_f G_x(f)\right) = X + 10\log\left(G_x(B_X)\right) \tag{1.43}$$

where $G_x(B_X) > G_x(f)$ for $|f| > B_X$.

In words a signal has a relative bandwidth B_X if the energy spectrum is at least XdB down from the peak at all frequency at or above B_X Hz. Often used values for X in engineering practice are the 3dB bandwidth and the 40dB bandwidth.

Definition 1.10 *If a signal $x(t)$ has an energy spectrum $G_x(f)$ then B_P is determined as*

$$P = \frac{\int_{-B_P}^{B_P} G_x(f)df}{E_x} \tag{1.44}$$

In words a signal has a integral bandwidth B_P if the total energy in the interval $[-B_P, B_P]$ is equal to P%. Often used values for P in engineering practice are 98% and 99%.

For lowpass power signals similar ideas hold with $G_x(f)$ being replaced with $S_x(f,T)$., i.e.,

Definition 1.11 *If a signal $x(t)$ has a sampled power spectral density $S_x(f,T)$ then B_X is determined as*

$$10\log\left(\max_f S_x(f,T)\right) = X + 10\log\left(S_x(B_X,T)\right) \tag{1.45}$$

where $S_x(B_X,T) > S_x(f,T)$ for $|f| > B_X$.

Definition 1.12 *If a signal $x(t)$ has a sampled power spectral density $S_x(f,T)$ then B_P is determined as*

$$P = \frac{\int_{-B_P}^{B_P} S_x(f,T)df}{P_x(T)} \tag{1.46}$$

1.2.4 Fourier Transform Representation of Periodic Signals

The Fourier transform for power signals is not rigorously defined and yet we often want to use frequency representations of power signals. Typically the power signals we will be using in this text are periodic signals which have a Fourier series representation. A Fourier series can be represented in the frequency domain with the help of the following result

$$\delta(f - f_1) = \int_{-\infty}^{\infty} \exp\left[j2\pi f_1 t\right] \exp\left[-j2\pi f t\right] dt. \tag{1.47}$$

In other words a complex exponential of frequency f_1 can be represented in the frequency domain with an impulse at f_1. Consequently the Fourier transform of a periodic signal is represented as

$$X(f) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x_n \exp\left[j\frac{2\pi nt}{T}\right] \exp\left[-j2\pi f t\right] dt = \sum_{n=-\infty}^{\infty} x_n \delta\left(f - \frac{n}{T}\right) \tag{1.48}$$

Example 1.12:

$$\mathcal{F}\left(\cos(2\pi f_m t)\right) = \frac{1}{2}\delta(f - f_m) + \frac{1}{2}\delta(f + f_m)$$
$$\mathcal{F}\left(\sin(2\pi f_m t)\right) = \frac{1}{j2}\delta(f - f_m) - \frac{1}{j2}\delta(f + f_m)$$

1.2.5 Laplace Transforms

This text will use the one-sided Laplace transform for the analysis of transient signals in communications systems. The one–sided Laplace transform of a signal $x(t)$ is

$$X(s) = \mathcal{L}\left\{x(t)\right\} = \int_0^\infty x(t)\exp\left[-st\right] dt. \tag{1.49}$$

The use of the one-sided Laplace transform implies that the signal is zero for negative time. The inverse Laplace transform is given as

$$x(t) = \frac{1}{2\pi j}\oint X(s)\exp\left[st\right] ds. \tag{1.50}$$

The evaluation of the general inverse Laplace transform requires the evaluation of contour integrals in the complex plane. For most transforms of interest the results are available in tables.

Example 1.13:

$$x(t) = \sin(2\pi f_m t) \qquad X(s) = \frac{2\pi f_m}{s^2 + (2\pi f_m)^2}$$
$$x(t) = \cos(2\pi f_m t) \qquad X(s) = \frac{s}{s^2 + (2\pi f_m)^2}$$

1.3 Linear Time-Invariant Systems

Definition: A linear system is one in which superposition holds.
Definition: A time-invariant system is one in which a time shift in the input only changes the output by a time shift.
A linear time-invariant (LTI) system is described completely by an impulse response, $h(t)$. The output of the linear system is the convolution of the input signal with the impulse response, i.e.,

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda. \tag{1.51}$$

The Fourier transform (Laplace transform) of the impulse response is denoted the transfer function, $H(f) = \mathcal{F}\left\{h(t)\right\}$ $(H(s) = \mathcal{L}\left\{h(t)\right\})$, and by the convolution theorem for energy signals we have

$$Y(f) = H(f)X(f) \qquad (Y(s) = H(s)X(s)). \tag{1.52}$$

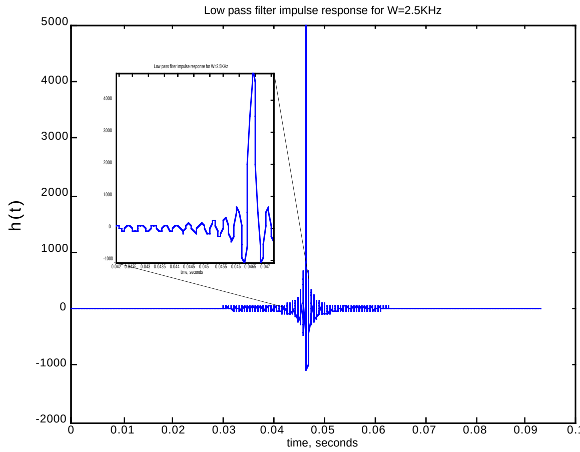


Figure 1.4: The impulse response, $h(t)$, of a filter obtained by truncating (93ms) and time-shifting (46.5ms) an ideal lowpass (W=2.5KHz) filter impulse response.

Example 1.14: If a linear time invariant filter had the impulse given in Example 1.1, i.e.,

$$h(t) = 2W\frac{\sin(2\pi Wt)}{2\pi Wt} = 2W\mathrm{sinc}\,(2Wt)$$

it would result in an ideal lowpass filter. A filter of this form has two problems for an implementation: the filter is anti-causal and the filter has an infinite impulse response. The obvious solution to the having an infinite duration impulse is to truncate the filter impulse response to a finite time duration. The way to make an anti-causal filter causal is simply to time shift the filter response. Fig. 1.4 shows the resulting impulse response when the ideal lowpass filter had a bandwidth of W=2.5KHz and the truncation of the impulse response is 93ms and the time shift of 46.5ms. The truncation will change the filter transfer function slightly while the delay will only add a phase shift. The resulting magnitude for the filter transfer function is shown in Fig. 1.5.

For a periodic input signal the output of the LTI system will be a periodic signal of the same period and have a Fourier series representation of

$$y(t) = \sum_{n=-\infty}^{\infty} H\left(\frac{n}{T}\right) x_n \exp\left[\frac{j2\pi nt}{T}\right] = \sum_{n=-\infty}^{\infty} y_n \exp\left[\frac{j2\pi nt}{T}\right] \tag{1.53}$$

where $y_n = H\left(\frac{n}{T}\right) x_n$. In other words the output of an LTI system with a periodic input will also be periodic. The Fourier series coefficients are the product of input signal's coefficients and the transfer function evaluated at the harmonic frequencies. Likewise the linear system output energy spectrum has a simple form

$$G_y(f) = H(f)H^*(f)G_x(f). \tag{1.54}$$

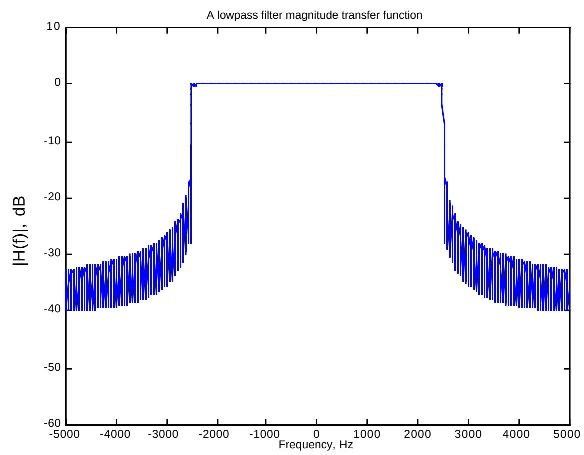


Figure 1.5: The resulting magnitude transfer function, $|H(f)|$.

Example 1.15: If the voice signal of Example 1.2 is passed through the filter of Example 1.14 then due to (1.54) the output energy spectrum should filtered heavily outside of 2.5KHz and roughly the same within 2.5KHz. Figure 1.6 shows the measured output energy spectrum and the this measured value matches exactly that predicted by theory. This is an example of how a communications engineers might use linear system theory to predict system performance. This signal will be used as a message signal throughout the remainder of this text to illustrate the ideas of analog communication.

An important linear system for the study of frequency modulated (FM) signals is the differentiator. The differentiator is described with the following transfer function

$$y(t) = \frac{d^n x(t)}{dt^n} \qquad \Leftrightarrow \qquad Y(f) = (j2\pi f)^n X(f). \tag{1.55}$$

1.4 Utilizing Matlab

The signals that are discussed in a course on communications are typically defined over a continuous time variable, e.g., $x(t)$. Matlab is an excellent package for visualization and learning in communications engineering and will be used liberally throughout this text. Unfortunately Matlab uses signals which are defined over a discrete time variable, $x(k)$. Discrete time signal processing basics is covered in EE352 and more comprehensive treatments are given in [Mit98, OS99, PM88, Por97]. This section provides a brief discussion of how to transition between the continuous time functions (the analog communication theory) and the discrete functions (Matlab). The examples considered in Matlab will reflect the types of signals you might measure when testing signals in the lab.

1.4.1 Sampling

The simplest way to convert a continuous time signal, $x(t)$, into a discrete time signal, $x(k)$ is to sample the continuous time signal, i.e.

$$x(k) = x(kT_s + \epsilon)$$

where T_s is the time between samples. The sample rate is denoted $f_s = \frac{1}{T_s}$. This conversion is often termed analog–to–digital conversion (ADC) and is a common operation in practical communication

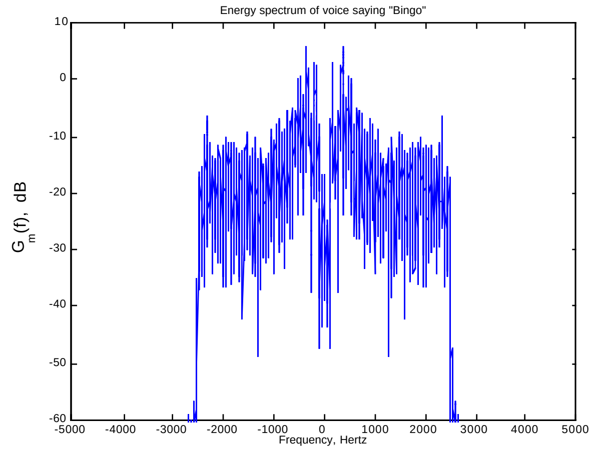


Figure 1.6: The output energy spectrum of the voice signal after filtering by a lowpass filter of bandwidth 2.5KHz.

system implementations. The discrete time version of the signal is a faithful representation of the continuous time signal if the sampling rate is high enough.

To see that this is true it is useful to introduce some definitions for discrete time signals.

Definition 1.13 *For a discrete time signal $x(k)$, the discrete time Fourier transform (DTFT) is*

$$X\left(e^{j2\pi f}\right) = \sum_{k=-\infty}^{\infty} x(k) e^{j2\pi fk} \tag{1.56}$$

For clarity when the frequency domain representation of a continuous time signal is discussed it will be denoted as a function of f, e.g., $X(f)$ and when the frequency domain representation of a discrete time signal is discussed it will be denoted as a function of $e^{j2\pi f}$, e.g., $X\left(e^{j2\pi f}\right)$. This DTFT is a continuous function of frequency and since no time index is associated with $x(k)$ the range of where the function can potentially take unique values is $f \in [-0.5, 0.5]$. Matlab has built in functions to compute the discrete Fourier transform (DFT) which is simply the DTFT evaluated at uniformly spaced points.

For a sampled signal, the DTFT is related to the Fourier transform of the continuous time signal via [Mit98, OS99, PM88, Por97]

$$X\left(e^{j2\pi f}\right) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X\left(\frac{f-n}{T_s}\right) \tag{1.57}$$

Examining (1.57) shows that if the sampling rate is higher than twice the highest significant frequency component of the continuous time signal then

$$X\left(e^{j2\pi f}\right) = \frac{1}{T_s} X\left(\frac{f}{T_s}\right) \tag{1.58}$$

and

$$X(f) = T_s X\left(e^{j2\pi (fT_s)}\right) \tag{1.59}$$

Note the highest significant frequency component is often quantified through the definition of the bandwidth (see Section 1.2.3). Consequently the rule of thumb for sampling is that the sampling rate should be at least twice the bandwidth of the signal. A sampling rate of exactly twice the bandwidth of the signal is known as Nyquist's sampling rate.

1.4.2 Integration

Many characteristics of continuous time signals are defined by an integral. For example the energy of a signal is given in (1.1) as an integral. The Fourier transform, the correlation function, convolution, and the power are other examples of signal characteristics defined through integrals. Matlab does not have the ability to evaluate integrals but the values of the integral can be approximated to any level of accuracy desired. The simplest method of computing an approximation solution to an integral is given by the Riemann sum first introduced in calculus.

Definition 1.14 *A Riemann sum approximation to an integral is*

$$\int_a^b x(t)dt \approx \frac{b-a}{N}\sum_{k=1}^{N} x\left(a-\epsilon+\frac{k(b-a)}{N}\right) = h\sum_{k=1}^{N} x\left(k\right) \tag{1.60}$$

where $\epsilon \in [-(b-a)/N, 0]$ *and* h *is the step size for the sum.*

A Riemann sum will converge to the true value of the integral as the number of points in the sum goes to infinity. Note that the DTFT for sampled signals can actually be viewed as a Riemann sum approximation to the Fourier transform.

1.4.3 Commonly Used Functions

This section details some Matlab functions that can be used to implement the signals and systems that are discussed in this chapter. Help with the details of these functions is available in Matlab.

Standard Stuff

- `cos` - cosine function
- `sin` - sine function
- `sinc` - sinc function
- `sqrt` - square root function
- `log10` - log base ten, this is useful for working in dB
- `max` - maximum of a vector
- `sum` - sum of the elements of a vector

Complex signals

- `abs` - $|\bullet|$
- `angle` - $\arg(\bullet)$
- `real` - $\Re\left[\bullet\right]$
- `imag` - $\Im\left[\bullet\right]$
- `conj` - $(\bullet)^*$

Input and Output

- `plot` - 2D plotting
- `xlabel, ylabel, axis` - formatting plots
- `sound,soundsc` - play vector as audio
- `load` - load data
- `save` - save data

Frequency Domain and Linear Systems

- `fft` - computes a discrete Fourier transform (DTF)
- `fftshift` - shifts an DFT output so when it is plotted it looks more like a Fourier transform
- `conv` - convolves two vectors

1.5 Homework Problems

Problem 1.1. Let two complex numbers be given as

$$z_1 = x_1 + jy_1 = a_1 \exp(j\theta_1) \qquad\qquad z_2 = x_2 + jy_2 = a_2 \exp(j\theta_2) \tag{1.61}$$

Find

a) $\Re[z_1 + z_2]$.

b) $|z_1 + z_2|$.

c) $\Im[z_1 z_2]$.

d) $\arg[z_1 z_2]$.

e) $|z_1 z_2|$.

Problem 1.2. Plot, find the period, and find the Fourier series representation of the following periodic signals

a) $x(t) = 2\cos(200\pi t) + 5\sin(400\pi t)$

b) $x(t) = 2\cos(200\pi t) + 5\sin(300\pi t)$

c) $x(t) = 2\cos(150\pi t) + 5\sin(250\pi t)$

Problem 1.3. Consider the two signals

$$x_1(t) = m(t)\cos(2\pi f_c t) \qquad x_2(t) = m(t)\sin(2\pi f_c t)$$

where the bandwidth of $m(t)$ is much less than f_c. Compute the simplest form for the following four signals

a) $y_1(t) = x_1(t)\cos(2\pi f_c t)$

b) $y_2(t) = x_1(t)\sin(2\pi f_c t)$

c) $y_3(t) = x_2(t)\cos(2\pi f_c t)$

d) $y_4(t) = x_2(t)\sin(2\pi f_c t)$

Postulate how a communications engineer might use these results to recover a signal, $m(t)$, from $x_1(t)$ or $x_2(t)$.

Problem 1.4. (Design Problem) This problem gives you a little feel for microwave signal processing and the importance of the Fourier series. You have at your disposal

1) a signal generator which produces $\pm$ 1V amplitude square wave in a 1 Ω system where the fundamental frequency, f_1, is tunable from 1KHz to 50MHz

2) an ideal bandpass filter with a center frequency of 175MHz and a bandwidth of 30MHz.

The design problem is

a) Select an f_1 such that when the signal generator is cascaded with the filter that the output will be a single tone at 180MHz. There might be more than one correct answer (that often happens in real life engineering).

b) Calculate the amplitude of the resulting sinusoid.

Problem 1.5. This problems exercises the signal and system tools. Compute the Fourier transform of

a)

$$\begin{aligned} u(t) &= A \qquad 0 \le t \le T \\ &= 0 \qquad \text{elsewhere} \end{aligned} \tag{1.62}$$

b)

$$\begin{aligned} u(t) &= A\sin\left(\frac{\pi t}{T}\right) \qquad 0 \le t \le T \\ &= 0 \qquad \text{elsewhere} \end{aligned} \tag{1.63}$$

and give the value of A such that E_u=1.

Problem 1.6. This problem is an example of a problem which is best solved with the help of a computer. The signal $u(t)$ is passed through an ideal lowpass filter of bandwidth B/T Hz. For the signals given in Problem 1.5 with unit energy make a plot of the output energy vs. B. *Hint: Recall the trapezoidal rule from calculus to approximately compute the this energy.*

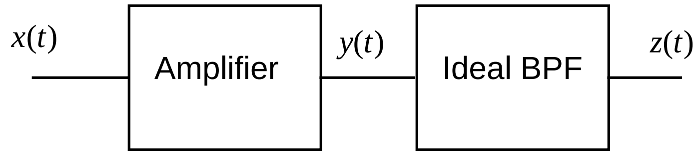


Figure 1.7: The system diagram for Problem 1.7.

Problem 1.7. This problem uses signal and system theory to compute the output of a simple memoryless nonlinearity. An amplifier is an often used device in communication systems and is simply modeled as an ideal memoryless system, i.e.,

$$y(t) = a_1 x(t).$$

This model is an excellent model until the signal levels get large then nonlinear things start to happen which can produce unexpected changes in the output signals. These changes often have a significant impact in a communication system design. As an example of this characteristic consider the system in Figure 1.7 with following signal model

$$x(t) = b_1 \cos(200000\pi t) + b_2 \cos(202000\pi t),$$

the ideal bandpass filter has a bandwidth of 10 KHz centered at 100KHz, and the amplifier has the following memoryless model

$$y(t) = a_1 x(t) + a_3 x^3(t).$$

Give the system output, $z(t)$, as a function of a_1, a_3, b_1, b_3.

Problem 1.8. (PD) A nonlinear device that is often used in communication systems is a quadratic memoryless nonlinearity. For such a device if $x(t)$ is the input the output is given as

$$y(t) = ax(t) + bx^2(t)$$

a) If $x(t) = A\cos(2\pi f_m t)$ what is $y(t)$ and $Y(f)$?

b) If

$$\begin{aligned} X(f) &= A \qquad ||f| - f_c| \leq f_m \\ &= 0 \qquad \text{otherwise} \end{aligned} \tag{1.64}$$

what is $y(t)$ and $Y(f)$?

c) A quadratic nonlinearity is often used in a *frequency doubler*. What component would you need to add in series with this quadratic memoryless nonlinearity such that you could put a sine wave in and get a sine wave out of twice the input frequency?

Problem 1.9. Consider the following signal

$$\begin{aligned} x(t) &= \cos(2\pi f_1 t) + a\sin(2\pi f_1 t) \\ &= X_A(a)\cos(2\pi f_1 t + X_p(a)) \end{aligned} \tag{1.65}$$

a) Find $X_A(a)$.

b) Find $X_p(a)$.

c) What is the power of $x(t)$, P_x?

d) Is $x(t)$ periodic? If so what is the period and the Fourier series representation of $x(t)$?

Problem 1.10. Consider a signal and a linear system as depicted in Fig. 1.8 where

$$x(t) = A + \cos(2\pi f_1 t)$$

and

$$\begin{aligned} h(t) &= \frac{1}{\sqrt{T}} \qquad 0 \leq\leq T \\ &= 0 \qquad \text{elsewhere.} \end{aligned} \tag{1.66}$$

Compute the output $y(t)$.

Problem 1.11. For the signal

$$x(t) = 23\frac{\sin(2\pi 147 t)}{2\pi 147 t} \tag{1.67}$$

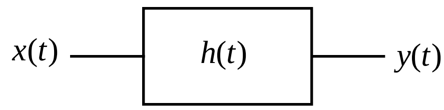


Figure 1.8: The system for Problem 1.10.

a) Compute $X(f)$.

b) Compute E_x.

c) Compute $y(t) = \frac{dx(t)}{dt}$.

d) Compute $Y(f)$

Hint: Some of the computations have an easy way and a hard way so think before turning the crank!
Problem 1.12. The three signals seen in Figure 1.9 are going to be used to exercise your signals and systems theory.

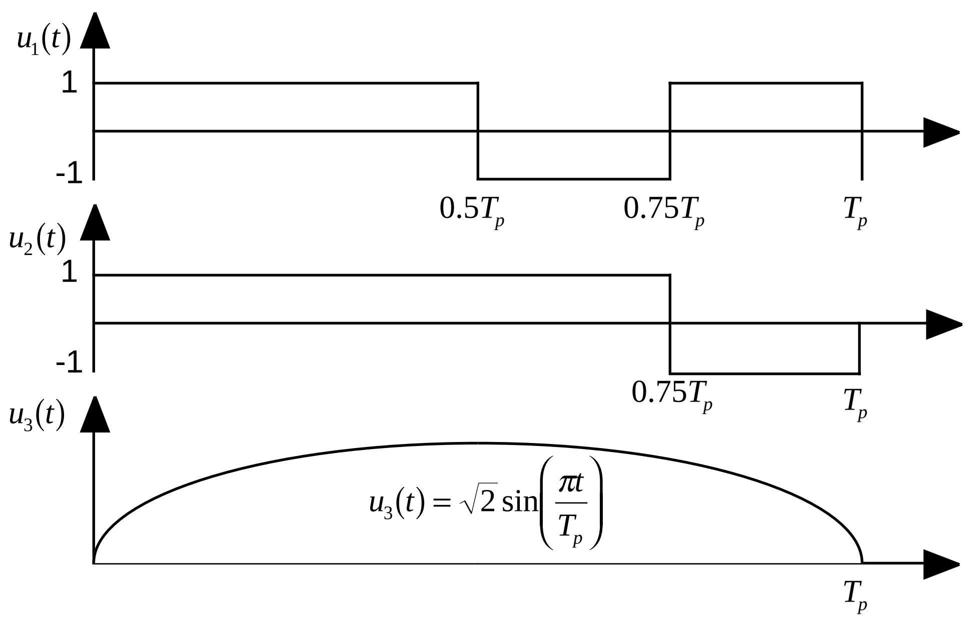


Figure 1.9: Pulse shapes considered for Problem 1.12.

Compute for each signal

a) The Fourier transform, $U(f)$,

b) The energy spectral density, $G_u(f)$,

c) The correlation function, $R_u(\tau)$. What is the energy, E_u.

Hint: Some of the computations have an easy way and a hard way so think before turning the crank!

1.6 Example Solutions

Problem 1.8. The output of the quadratic nonlinearity is modeled as

$$y(t) = ax(t) + bx^2(t). \tag{1.68}$$

It is useful to recall that multiplication in the time domain results in convolution in the frequency domain, i.e.,

$$z(t) = x(t) \times x(t) \qquad Z(f) = X(f) \otimes X(f). \tag{1.69}$$

a) $x(t) = A\cos(2\pi f_m t)$ which results in

$$\begin{aligned} y(t) &= aA\cos(2\pi f_m t) + bA^2\cos^2(2\pi f_m t) \\ &= aA\cos(2\pi f_m t) + bA^2\left(\frac{1}{2} + \frac{1}{2}\cos(4\pi f_m t)\right) \\ &= \frac{aA}{2}\exp[j2\pi f_m t] + \frac{aA}{2}\exp[-j2\pi f_m t] + \frac{bA^2}{2} \\ &\quad + \frac{bA^2}{4}\exp[j4\pi f_m t] + \frac{bA^2}{4}\exp[-j4\pi f_m t] \end{aligned} \tag{1.70}$$

The frequency domain representation for the output signal is given as

$$Y(f) = \frac{aA}{2}\delta(f - f_m) + \frac{aA}{2}\delta(f + f_m) + \frac{bA^2}{2}\delta(f) + \frac{bA^2}{4}\delta(f - 2f_m) + \frac{bA^2}{4}\delta(f + 2f_m) \tag{1.71}$$

The first two terms in (1.71) are due to the linear term in (1.68) while the last three terms are due to the square law term. It is interesting to note that these last three terms can be viewed as being obtained by convolving the two "delta" function frequency domain representation of a cosine wave, i.e., $2\cos(2\pi f_m t) = \exp[j2\pi f_m t] + \exp[-j2\pi f_m t]$, with itself.

b) The input signal is

$$\begin{aligned} X(f) &= A \qquad ||f| - f_c| \leq f_m \\ &= 0 \qquad \text{otherwise.} \end{aligned} \tag{1.72}$$

Taking the inverse Fourier transform gives

$$x(t) = 4Af_m\frac{\sin(\pi f_m t)}{\pi f_m t}\cos(2\pi f_c t) \tag{1.73}$$

The output time signal of this quadratic nonlinearity is

$$\begin{aligned} x(t) &= 4aAf_m\frac{\sin(\pi f_m t)}{\pi f_m t}\cos(2\pi f_c t) + b\left(4Af_m\frac{\sin(\pi f_m t)}{\pi f_m t}\cos(2\pi f_c t)\right)^2 \\ &= 4aAf_m\frac{\sin(\pi f_m t)}{\pi f_m t}\cos(2\pi f_c t) + b\left(4Af_m\frac{\sin(\pi f_m t)}{\pi f_m t}\right)^2\left(\frac{1}{2} + \frac{1}{2}\cos(4\pi f_m t)\right) \end{aligned} \tag{1.74}$$

The frequency domain representation of $y(t)$ can be computed by taking the Fourier transform or by using $Y(f) = aX(f) + b(X(f) \otimes X(f))$. The resulting Fourier transform is plotted in Fig. 1.10.

c) Looking at part a) one can see that a tone in will produce a tone out if all frequencies except those at $2f_m$ are eliminated. Consequently a quadratic nonlinearity followed by a BPF will produce a frequency doubler. The block diagram for this frequency doubler is shown in Fig. 1.11.

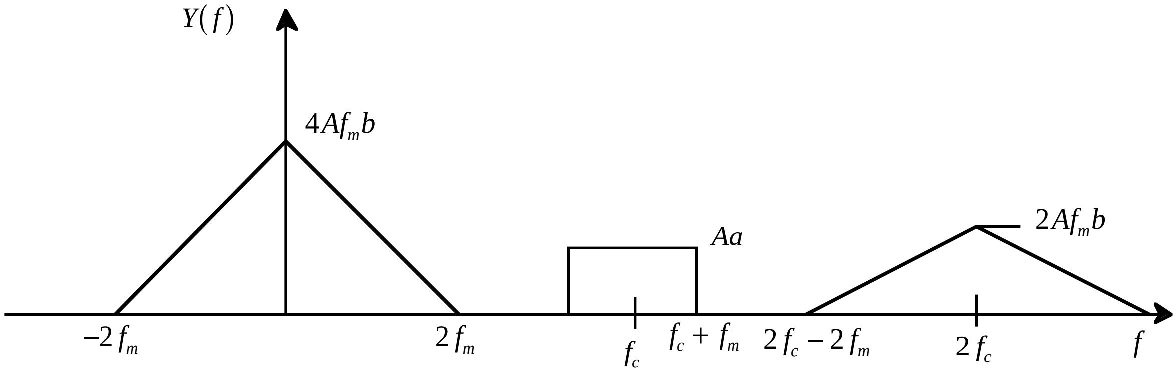


Figure 1.10: Output spectrum of the quadratic nonlinearity for part b).

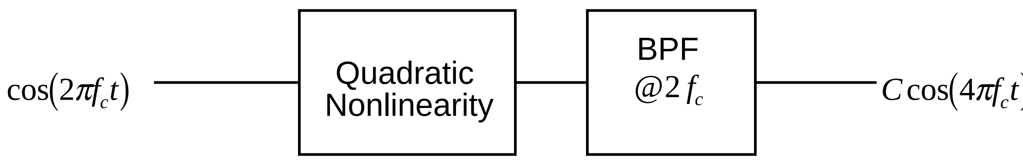


Figure 1.11: A frequency doubler.

1.7 Mini-Projects

Goal: To give exposure

1. to a small scope engineering design problem in communications
2. to the dynamics of working with a team
3. to the importance of engineering communication skills (in this case oral presentations).

Presentation: The forum will be similar to a design review at a company (only much shorter) The presentation will be of 5 minutes in length with an overview of the given problem and solution. The presentation will be followed by questions from the audience (your classmates and the professor). Each team member should be prepared to make the presentation on the due date.

Project 1.1. In designing a communication system, bandwidth efficiency is often a top priority. In analog communications one way to get bandwidth efficiency is to limit the bandwidth of the message signal. The big engineering tradeoff is how narrow to make the bandwidth of the message signal. If the bandwidth is made too narrow the message will be distorted. If the bandwidth is made to wide spectrum will be wasted.

Get the Matlab file `chap1ex2.m` from the class web page along with the computer generated voice signal. Use this file to make an estimate how small the bandwidth of a voice signal can be and still enable high fidelity communications. There is no single or right answer to the problem but engineering judgment must be used. How would your answer be different if you were only concerned with word recognition or if you wanted to maintain speaker recognition as well. Please detail your reasons for your solution.

Chapter 2

Review of Probability and Random Variables

This chapter is intended to review introductory material on random variables typically covered in a undergraduate curriculum. The most important use of this section will be the introduction of the notation used in this text. More proofs will be provided in this chapter than the last because typically this subject matter is not as synthesized at the start of a senior level communications course. Texts that give a more detailed treatment of the subject of probability and random variables are [DR87, LG89, Hel91, CM86, Dev00, Sti99].

2.1 Axiomatic Definitions of Probability

Characterization of random events or experiments is critical for communication system design and analysis. A majority of the analyses of random events or experiments are extensions of the simple axioms or definitions presented in this section. A random experiment is characterized by a probability space consisting of a sample space Ω, a field $\mathcal{F}$, and a probability measure $P(\bullet)$.

Example 2.1: Consider the random experiment of rolling a fair dice

$$\Omega = \{1, 2, 3, 4, 5, 6\} \tag{2.1}$$

The field, $\mathcal{F}$, is the set of all possible combinations of outputs, i.e., consider the following outcomes $A_1 = \{\text{the die shows a 1}\}$, $A_2 = \{\text{the die shows an even number}\}$, $A_3 = \{\text{the die shows a number less than 4}\}$, and $A_4 = \{\text{the die shows an odd number}\}$ which implies

$$A_1, A_2, A_3, A_4 \in \mathcal{F} \tag{2.2}$$

Definition 2.1 *Events A and B are mutually exclusive if $A \cap B = \emptyset$*

Axioms of Probability. For $A, B \in \mathcal{F}$, a field defined over a sample space, Ω, the probability measure, $P(\bullet)$, for this probability space must satisfy the following axioms

1. For any event A, $P(A) \geq 0$
2. $P(\Omega) = 1$
3. If A and B are mutually exclusive events then $P(A \cup B) = P(A) + P(B)$.

These three axioms are the building blocks for probability theory and an understanding of random events that characterize communication system performance.

Example 2.2: Example 2.1(continued). The rolling of a fair die

$$\begin{array}{cccc} A_1 = \{1\} & A_2 = \{2,4,6\} & A_3 = \{1,2,3\} & A_4 = \{1,3,5\} \\ & P[\{1\}] = P[\{2\}] = & \cdots = P[\{6\}] & \\ P[A_1] = \frac{1}{6} & P[A_2] = \frac{1}{2} & P[A_3] = \frac{1}{2} & P[A_4] = \frac{1}{2} \end{array} \tag{2.3}$$

Definition 2.2 (Complement) *The complement of a set A, denoted A^C, is the set of all elements of Ω that are not an element of A.*

Theorem 2.1 (Poincare) *For N events $A_1, A_2, ...A_N$*

$$P[A_1 \cup A_2 \cup .... \cup A_N] = S_1 - S_2 + ... + (-1)^{N-1} S_N$$

where

$$S_k = \sum_{i_1 < i_2 < .... < i_k} P[A_{i_1} \cap A_{i_2} \cap .... \cap A_{i_k}] \qquad i_1 \geq 1,\, i_k \leq N.$$

Proof: The results are given for N=2 and the proof for other cases is similar (if not more tedious). Note that events A and $B \cap A^C$ are mutually exclusive events. Consequently we have

$$\begin{aligned} P[A \cup B] &= P[(A \cap B^C) \cup B] = P[A \cap B^C] + P[B] && (2.4) \\ P[A \cup B] &= P[(B \cap A^C) \cup A] = P[B \cap A^C] + P[A] && (2.5) \\ P[A \cup B] &= P[(A \cap B^C) \cup (A \cap B) \cup (B \cap A^C)] = P[A \cap B^C] + P[A \cup B] + P\left[B \cup A^C\right]. && (2.6) \end{aligned}$$

Using (2.4)+(2.5)-(2.6) gives the desired result. □

Example 2.3:

$$\begin{aligned} P[A \cup B] &= P[A] + P[B] - P[A \cap B] \\ P[A \cup B \cup C] &= P[A] + P[B] + P[C] - P[A \cap B] - P[B \cap C] - P[A \cap C] + P[A \cap B \cap C] \end{aligned} \tag{2.7}$$

Example 2.4: Example 2.1(continued).

$$P[A_2 \cup A_3] = P[A_2] + P[A_3] - P[A_2 \cap A_3] \tag{2.8}$$

$$A_2 \cap A_3 = 2 \qquad P[A_2 \cap A_3] = \frac{1}{6} \tag{2.9}$$

$$P[A_2 \cup A_3] = P[1,2,3,4,6] = \frac{1}{2} + \frac{1}{2} - \frac{1}{6} = \frac{5}{6} \tag{2.10}$$

Definition 2.3 (Conditional Probability) *Let $(\Omega, \mathcal{F}, P)$ be a probability space with sets $A, B \in \mathcal{F}$ and $P[B] \neq 0$. The conditional or a posteriori probability of event A given an event B, denoted $P[A|B]$, is defined as*

$$P[A|B] = \frac{P[A \cap B]}{P[B]}.$$

$P[A\,|B]$ is interpreted as event A's probability after the experiment has been performed and event B is observed.

Definition 2.4 (Independence) *Two events $A, B \in \mathcal{F}$ are independent if and only if*

$$P[A \cap B] = P[A]\,P[B]\,.$$

Independence is equivalent to

$$P[A|B] = P[A]\,. \tag{2.11}$$

For independent events A and B, the a posteriori or conditional probability $P[A|B]$ is equal to the a prioir probability $P[A]$. Consequently if A and B are independent, then observing B reveals nothing about the relative probability of the occurrence of event A.

Example 2.5: Example 2.1(continued). If the die is rolled and you are told the outcome is even, A_2, how does that change the probability of event A_3?

$$P[A_3\,|A_2] = \frac{P(A_2 \cap A_3)}{P(A_2)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

Since $P[A_3\,|A_2] \neq P[A_3]$, event A_3 is not independent of event A_2.

Definition 2.5 *A collectively exhaustive set of events is one for which*

$$P[A_1 \cup A_2 \cup .... \cup A_N] = 1 \qquad \textit{or equivalently} \qquad A_1 \cup A_2 \cup .... \cup A_N = \Omega$$

Theorem 2.2 (Total Probability) *For N mutually exclusive, collectively events $(A_1, A_2, ...A_N)$ and $B \in \Omega$, then*

$$P[B] = \sum_{i=1}^{N} P[B|A_i]P[A_i]\,.$$

Proof: The probability of event B can be written as

$$P[B] = P[B \cap \Omega] = P\left[B \cap \bigcup_{i=1}^{N} A_i\right] = P\left[\bigcup_{i=1}^{N} B \cap A_i\right].$$

The events $B \cap A_i$ are mutually exclusive, so

$$P[B] = P\left[\bigcup_{i=1}^{N} B \cap A_i\right] = \sum_{i=1}^{N} P[B \cap A_i] = \sum_{i=1}^{N} P[B\,|A_i]P[A_i]\,. \qquad \square$$

Example 2.6: Example 2.1(continued). Note A_2 and A_4 are a set of mutually exclusive collectively exhaustive events with

$$P[A_3|A_4] = \frac{P[A_3 \bigcap A_4]}{P[A_4]} = \frac{P[\{1,3\}]}{P[A_4]} = \frac{\frac{2}{6}}{\frac{1}{2}} = \frac{2}{3}.$$

This produces

$$P[A_3] = P[A_3|A_2]\,P(A_2) + P[A_3|A_4]\,P(A_3) = \frac{1}{3}\left[\frac{1}{2}\right] + \left[\frac{2}{3}\right]\left[\frac{1}{2}\right] = \frac{3}{6} = \frac{1}{2}$$

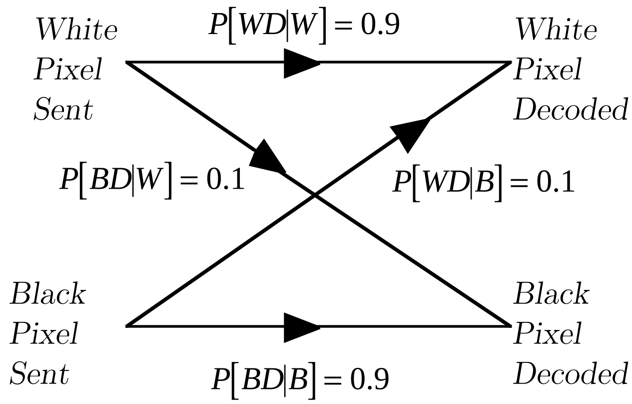


Figure 2.1: The binary symmetric channel.

Theorem 2.3 (Bayes) *For N mutually exclusive, collectively exhaustive events $\{A_1, A_2, \cdots A_N\}$ and $B \in \Omega$, then the conditional probability of the event A_j given that event B is observed is*

$$P[A_j|B] = \frac{P[B|A_j]\,P[A_j]}{P[B]} = \frac{P[B|A_j]\,P[A_j]}{\sum\limits_{i=1}^{N} P[B \cap A_i]}.$$

Proof: The definition of conditional probability gives

$$P[A_j \cap B] = P[A_j|B]\,P[B] = P[B|A_j]\,P[A_j].$$

Rearrangement and total probability complete the proof. □

Example 2.7: (Binary symmetric channel) A facsimile machine divides a document up into small regions (i.e., pixels) and decides whether each pixel is black or white. Reasonable a priori statistics for facsimile transmission is

$$P[\text{A pixel is white}] = P[W] = 0.8 \qquad P[\text{A pixel is black}] = P[B] = 0.2$$

This pixel value is transmitted across a telephone line and the receiving fax machine makes a decision about whether a black or white pixel was sent. Fig. 2.1 is a simplified representation of this operation in an extremely noisy situation. If a black pixel is decoded (BD) what is the probability a white pixel was sent, $P(W|BD)$? Bayes rule gives a straightforward solution to this problem, i.e.,

$$\begin{aligned} P(W|BD) &= \frac{P[BD|W]\,P[W]}{P[BD]} && (2.12) \\ &= \frac{P[BD|W]\,P[W]}{P[BD|W]\,P[W] + P[BD|B]\,P[B]} \\ &= \frac{(0.1)0.8}{(0.1)(0.8)+(0.9)(0.2)} = 0.3077 && (2.13) \end{aligned}$$

2.2 Random Variables

In communications, the information transmission is often corrupted by random noise. This random noise manifests itself as a random voltage (a real number) at the output of the electrical circuits in the receiver. The concept of a random variable (RV) links the axiomatic definition of probability with these observed real random numbers.

Definition 2.6 *Let $(\Omega, \mathcal{F}, P)$ be a probability space. A real random variable $X(\omega)$ is a single-valued function or mapping from Ω to the real line $(\mathcal{R})$.*

There are three types of random variables: discrete, continuous, and mixed. A discrete random variable has a finite (or countably infinite) number of possible values. A continuous random variable take values in some interval of the real line of nonzero length. A mixed random variable is a convex combination of a discrete and a continuous random variable. To simplify the notation when no ambiguity exists, X represents the random variable $X(\omega)$ (the experimental outcome index is dropped) and $x = X(\omega)$ represents a particular realization of this random variable.

Example 2.8: Matlab has a built in random number generator. In essence this function when executed performs an experiment and produces a real number output. Each time this function is run a different real value is returned. Go to Matlab and type `rand(1)` and see what happens. Each time you run this function it returns a number that is unable to be predicted. We can characterize in many ways the outputs but not completely predict them. In this text we will call the observed real number resulting from the random experiment a sample from the random variable.

Random variables are completely characterized by either of two related functions: the cumulative distribution function (CDF) or the probability density function (PDF)[1]. These functions are the subject of the next two sections.

2.2.1 Cumulative Distribution Function

Definition 2.7 *For a random variable $X(\omega)$, the cumulative distribution function (CDF) is a function $F_X(x)$ defined as*

$$F_X(x) = P\left(X(\omega) \leq x\right) \qquad \forall \quad x \in \mathcal{R}.$$

Again to simplify the notation when no ambiguity exists, the CDF of the random variable X will be written as $F(x)$. Fig.2.2 shows example plots of the CDF for discrete, continuous and mixed random variables. The discrete random variable has a CDF with a stairstep form, the steps occur at the points of the possible values of the random variable. The continuous random variable has a CDF which is continuous and the mixed random variable has both continuous intervals and jumps.

Properties of a CDF

1. $F_X(x)$ is a monotonically increasing function[2], i.e.,

$$x_1 < x_2 \qquad \Rightarrow \qquad F_X(x_1) \leq F_X(x_2).$$

2. $0 \leq F(x) \leq 1$.

3. $F(-\infty) = P(x \leq -\infty) = 0$ and $F(\infty) = P(x \leq \infty) = 1$.

[1] Continuous RVs have PDFs but discrete random variables have probability mass functions (PMF).
[2] This is sometimes termed a nondecreasing function.

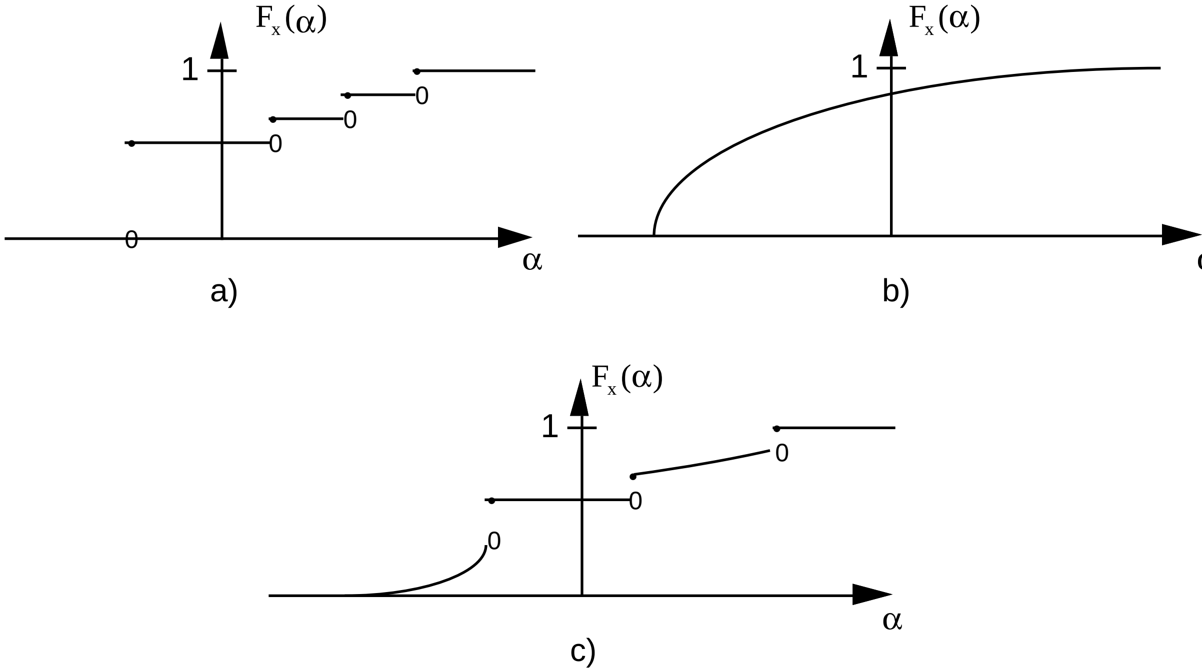


Figure 2.2: Example CDF for a) discrete RV, b) continuous RV, and c) mixed RV.

4. A CDF is right continuous.
5. $P(X > x) = 1 - F(x)$ and $P(x_1 < X \leq x_2) = F(x_2) - F(x_1)$.
6. The probability of the random variable taking a particular value, a, is given as

$$P\left(X = a\right) = F_X(a) - F_X(a^-).$$

Continuous random variables take any value with zero probability since the CDF is continuous, while discrete and mixed random variables take values with nonzero probabilities since the CDF has jumps. The discrete random variable has a CDF with the form

$$F_X\left(x\right) = \sum_{k=1}^{N} P\left(X = a_k\right) U\left(x - a_k\right) \tag{2.14}$$

where $U()$ is the unit step function, N is the number of jumps in the CDF, and a_k are the locations of the jumps.

2.2.2 Probability Density Function

Definition 2.8 *For a continuous random variable $X(\omega)$, the probability density function (PDF) is a function $f_X(x)$ defined as*

$$f_X(x) = \frac{dP\left(X(\omega) \leq x\right)}{dx} = \frac{dF_X\left(x\right)}{dx} \qquad \forall \qquad x \in \mathcal{R}. \tag{2.15}$$

Since the derivative in (2.15) can be rearranged to give

$$\lim_{\Delta \to 0} f_X\left(x\right) \Delta = \lim_{\Delta \to 0} P\left(x - \frac{\Delta}{2} < X(\omega) \leq x + \frac{\Delta}{2}\right),$$

the PDF can be thought of as the probability "density" in a very small interval around $X = x$. Discrete random variables do not have a probability "density" spread over an interval but do have probability mass concentrated at points. So the idea of a density function for a discrete random variable is not consistent. The analogous quantity to a PDF for a continuous RV for a discrete RV is the probability mass function (PMF),

$$p_X(x) = P(X = x).$$

The idea of the probability density function can be extended to discrete and mixed random variables by utilizing the notion of the Dirac delta function [CM86].

Properties of a PDF

1. $F_X(x) = \int_{-\infty}^{x} f_X(\beta)\, d\beta$.
2. $f_X(x) \geq 0$.
3. $F_X(\infty) = 1 = \int_{-\infty}^{\infty} f_X(\beta)\, d\beta$.
4. $P(x_1 < X \leq x_2) = \int_{x_1}^{x_2} f_X(\beta)\, d\beta$.

Again, if there is no ambiguity in the expression, the PDF of the random variable X is written as $f(x)$ and if there is no ambiguity about whether a RV is continuous or discrete, the PDF is written $p(x)$. Knowing the PDF (or equivalently the CDF) allows you to completely describe any random event associated with a random variable.

Example 2.9: The `rand(•)` function in Matlab produces a sample from what is commonly termed a uniformly distributed random variable. The PDF for a uniformly distributed random variable is given as

$$\begin{aligned} f_X(x) &= \frac{1}{b-a} \qquad a \leq x \leq b \\ &= 0 \qquad \text{otherwise.} \end{aligned} \tag{2.16}$$

The function in Matlab has $a = 0$ and $b = 1$. Likewise the CDF is

$$\begin{aligned} F_X(x) &= 0 \qquad x \leq a \\ &= \frac{x-a}{b-a} \qquad a \leq x \leq b \\ &= 1 \qquad x \geq b. \end{aligned} \tag{2.17}$$

2.2.3 Moments and Statistical Averages

A communication engineer often calculates the statistical average of a function of a random variable. The average value or expected value of a function $g(X)$ with respect to a random variable X is

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) p_X(x) dx.$$

Average or expected values are numbers which provides some information about the random variable. Average values are one number characterizations of random variables but are not a complete description in themselves like a PDF or CDF. A good example of a statistical average and one often used to characterize RVs is given by the mean value. The mean value is defined as

$$E(X) = m_X = \int_{-\infty}^{\infty} x p_X(x) dx.$$

The mean is the average value of the random variable. The n^{th} moment of a random variable is a generalization of the mean and is defined as

$$E\left(X^n\right) = m_{x,n} = \int_{-\infty}^{\infty} x^n p_X(x)dx.$$

The mean square value, $E(x^2)$, is frequently used in the analysis of a communication system (e.g., average power). Another function of interest is a central moment (a moment around the mean value) of a random variable. The n^{th} central moment is defined as

$$E\left((X - m_X)^n\right) = \sigma_{x,n} = \int_{-\infty}^{\infty} (x - m_X)^n p_X(x)dx.$$

The most commonly used second central moment is the variance, $\sigma_{x,2}$, which provides a measure of the spread of a random variable around the mean. The variance has the shorthand notation $\sigma_{x,2} = \sigma_X^2 = \text{var}(x)$. The relation between the variance and the mean square value is given by

$$E\left(X^2\right) = m_X^2 + \sigma_X^2.$$

The expectation operator is linear, i.e.,

$$E\left(g_1(x) + g_2(x)\right) = E\left(g_1(x)\right) + E\left(g_2(x)\right).$$

2.2.4 The Gaussian Random Variable

The most common random variable (RV) encountered in communications system analysis is the Gaussian random variable. This RV is a very good approximation to many of the physical phenomena that occur in electronic communications (e.g., the noise voltage generated by thermal motion of electrons in conductors). The reason for the common usage of the Gaussian RV is the Central Limit Theorem (see Section 2.3.5) and that analysis with Gaussian RVs is often tractable. This section will review the characteristics of the Gaussian RV. A Gaussian or normal RV, X, of mean m_X and variance σ_X^2 is denoted $N(m_X, \sigma_X^2)$. The Gaussian random variable is a continuous random variable taking values over the whole real line and a PDF of

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left[-\frac{(x - m_X)^2}{2\sigma_X^2}\right].$$

The Gaussian RV is characterized completely by it's mean and it variance. The Gaussian PDF is often referred to as the bell shaped curve. The mean shifts the centroid of the bell shaped curve as shown in Fig. 2.3. The variance is a measure of the spread in the values the random variable realizes. A large variance implies that the random variable is likely to take values far from the mean, whereas a small variance implies that a large majority of the values a random variable takes is near the mean. As an example Fig. 2.4 plots the density functions for three zero mean Gaussian RVs with variances of 0.25, 1, and 4.

A closed form expression for the CDF of the Gaussian RV does not exist. The CDF is expressed as

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left[-\frac{(x_1 - m_X)^2}{2\sigma_X^2}\right] dx_1.$$

The CDF can be expressed in terms of the erf function [Ae72], which is given as

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt.$$

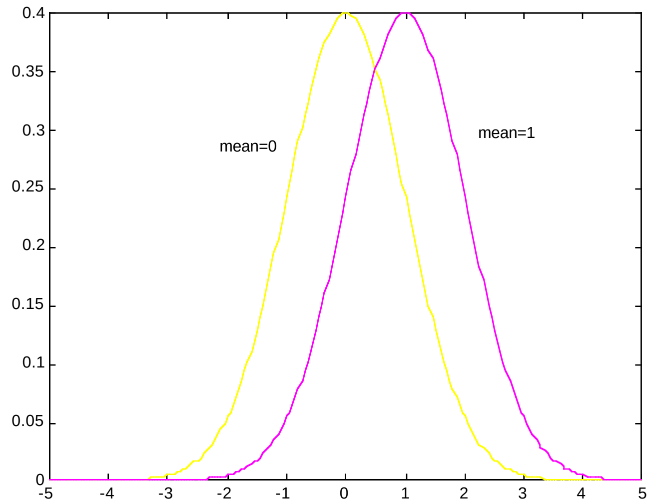


Figure 2.3: Plots of the PDF of a unit variance Gaussian random variable with m_X=0,1.

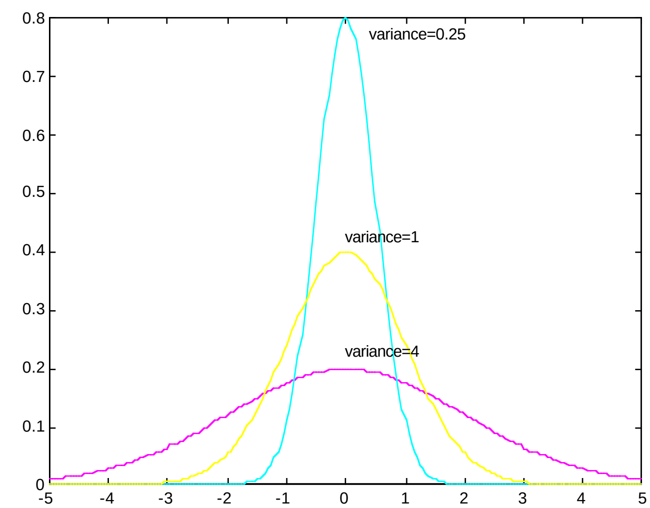


Figure 2.4: Plots of the PDF of a zero mean Gaussian random variable with σ_X^2=0.25,1,4.

The CDF of a Gaussian RV is then given as

$$F_X(x) = \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{x - m_X}{\sqrt{2}\sigma_X}\right).$$

While the CDF of the Gaussian random variables is defined using different functions by various authors, this function is used in this text because it is commonly used in math software packages (e.g., Matlab). Three properties of the erf function important for finding probabilities associated with Gaussian random variables are given as

$$\begin{aligned} \mathrm{erf}(\infty) &= 1 \\ \mathrm{erfc}(z) &= 1 - \mathrm{erf}(z) \\ \mathrm{erf}(-z) &= -\mathrm{erf}(z). \end{aligned}$$

Example 2.10: The `randn(•)` function in Matlab produces a sample from a Gaussian distributed random variable with $m_X = 0$ and $\sigma_X = 1$.

2.2.5 A Transformation of a Random Variable

In the analysis of communication system performance it is often necessary to characterize a random variable which is a transformation of another random variable. This transformation is expressed as

$$Y = g(X).$$

This section is concerned with finding the PDF or the CDF of the random variable, Y. The general technique is a two step process.
Step 1: Find the CDF of Y as a sum of integrals over the random variable X such that $g(X) < y$. This is expressed mathematically as

$$F_Y(y) = \sum_i \int_{R_i(y)} p_X(\beta)d\beta$$

where the $R_i(y)$ are intervals on the real line where X is such that $g(X) < y$.
Step 2: Find the PDF by differentiating the CDF found in Step 1 using Liebnitz Rule from calculus.

$$p_Y(y) = \frac{dF_Y(y)}{dy} = \sum_i \frac{d}{dy} \int_{R_i(y)} p_X(\beta)d\beta$$

Liebnitz Rule:

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)dx = f(b(t),t)\frac{db(t)}{dt} - f(a(t),t)\frac{da(t)}{dt} + \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t}dx$$

To illustrate this technique, three examples are given which are common to communication engineering.

Example 2.11: $Y = aX$ (the output of a linear amplifier with gain a having an input random voltage X).

Step 1:

$$F_Y(y) = \begin{cases} \int_{-\infty}^{\frac{y}{a}} p_X(\beta)d\beta & if \quad a > 0 \\ U(y) & if \quad a = 0 \\ \int_{\frac{y}{a}}^{\infty} p_X(\beta)d\beta & if \quad a < 0 \end{cases}$$

where $U(X)$ is the unit step function. Note for $a > 0$ that only one interval of the real line exists where $g(x) \leq y$ and that is $x \in [-\infty, y/a)$.

Step 2:

$$p_Y(y) = \begin{cases} \frac{1}{a} p_X\left(\frac{y}{a}\right) & if \quad a > 0 \\ \delta(y) & if \quad a = 0 \\ -\frac{1}{a} p_X\left(\frac{y}{a}\right) & if \quad a < 0. \end{cases}$$

Example 2.12: $Y = X^2$ (the output of a square law or power detector with an input random voltage X).

Step 1:

$$F_Y(y) = \begin{cases} \int_{-\sqrt{y}}^{\sqrt{y}} p_X(\beta)d\beta & if \quad , y \geq 0 \\ 0 & if \quad y < 0 \end{cases}$$

Step 2:

$$p_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} p_X\left(\sqrt{y}\right) + \frac{1}{2\sqrt{y}} p_X\left(-\sqrt{y}\right) & if \quad y \geq 0 \\ 0 & if \quad y < 0 \end{cases}$$

Example 2.13: $Y = XU(X)$. This example corresponds to the output of an ideal diode with an input random voltage X.

Step 1:

$$\begin{aligned} F_Y(y) &= \begin{cases} \int_{-\infty}^{y} p_X(\beta)d\beta & if \quad y > 0 \\ \int_{-\infty}^{0} p_X(\beta)d\beta & if \quad y = 0 \\ 0 & if \quad y < 0 \end{cases} \\ &= U(y) \int_{-\infty}^{y} p_X(\beta)d\beta \\ &= U(y) \int_{0}^{y} p_X(\beta)d\beta + U(y)F_x(0) \end{aligned} \tag{2.18}$$

Step 2:

$$p_Y(y) = U(y)\, p_X(y) + \delta(y) F_X(0)$$

2.3 Multiple Random Variables

In communication engineering, performance is often determined by more than one random variable. To analytically characterize the performance, a joint description of the random variables is necessary. All of the descriptions of a single random variable (PDF, CDF, moments, etc.) can be extended to the sets of random variables. This section highlights these extensions.

2.3.1 Joint Density and Distribution Functions

Again the random variables are completely characterized by either the joint CDF or the joint PDF. Note similar simplifications in notation will be used with joint CDFs and PDFs as was introduced in Section 2.2 when no ambiguity exists.

Definition 2.9 *The joint CDF of two random variables is*

$$F_{XY}(x,y) = P\left(X \leq x \cap Y \leq y\right) = P\left(X \leq x, Y \leq y\right).$$

Properties of the Joint CDF

1. $F_{XY}(x,y)$ is a monotonically nondecreasing function, i.e.,
$$x_1 < x_2 \text{ and } y_1 < y_2 \qquad \Rightarrow \qquad F_{XY}(x_1,y_1) \leq F_{XY}(x_2,y_2).$$
2. $0 \leq F_{XY}(x,y) \leq 1$.
3. $F_{XY}(-\infty,-\infty) = P(X \leq -\infty, Y \leq -\infty) = 0$ and $F_{XY}(\infty,\infty) = P(X \leq \infty, Y \leq \infty) = 1$.
4. $F_{XY}(x,-\infty) = 0$ and $F_{XY}(-\infty,y) = 0$.
5. $F_X(x) = F_{XY}(x,\infty)$ and $F_Y(y) = F_{XY}(\infty,y)$.
6. $P(x_1 < X \leq x_2, y_1 < Y \leq y_2) = F_{XY}(x_2,y_2) - F_{XY}(x_1,y_2) - F_{XY}(x_2,y_1) + F_{XY}(x_1,y_1)$.

Definition 2.10 *For two continuous random variables X and Y, the joint PDF, $f_{XY}(x,y)$, is*

$$f_{XY}(x,y) = \frac{\partial^2 P\left(X \leq x, Y \leq y\right)}{\partial x \partial y} = \frac{\partial^2 F_{XY}\left(x,y\right)}{\partial x \partial y} \qquad \forall \qquad x,y \in \mathcal{R}.$$

Note jointly distributed discrete random variables have a joint PMF, $p_{XY}(x,y)$, and as the joint PMF and PDF have similar characteristics this text often uses $p_{XY}(x,y)$ for both functions.

Properties of the Joint PDF or PMF

1. $F_{XY}\left(x,y\right) = \int_{-\infty}^{y}\int_{-\infty}^{x} p_{XY}\left(\alpha,\beta\right) d\alpha d\beta$.
2. $p_{XY}(x,y) \geq 0$.
3. $p_X\left(x\right) = \int_{-\infty}^{\infty} p_{XY}\left(x,y\right) dy$ and $p_Y\left(y\right) = \int_{-\infty}^{\infty} p_{XY}\left(x,y\right) dx$.
4. $P\left(x_1 < X \leq x_2, y_1 < Y \leq y_2\right) = \int_{x_1}^{x_2}\int_{y_1}^{y_2} p_{XY}\left(\alpha,\beta\right) d\alpha d\beta$.

Definition 2.11 *Let X and Y be random variables defined on the probability space $(\Omega, \mathcal{F}, P)$. The conditional or a posteriori PDF of Y given $X = x$, is*

$$p_{Y|X}\left(y \left| X = x\right.\right) = \frac{p_{XY}\left(x,y\right)}{p_X\left(x\right)}. \tag{2.19}$$

The conditional PDF, $p_{Y|X}\left(y \left| X = x\right.\right)$, is the PDF of the random variable Y after the random variable X is observed to take the value x.

Definition 2.12 *Two random variables x and y are independent if and only if*

$$p_{XY}\left(x,y\right) = p_X\left(x\right) p_Y\left(y\right).$$

Independence is equivalent to

$$p_{Y|X}\left(y\,|X=x\right)=p_{Y}\left(y\right),$$

i.e., Y is independent of X if no information is in the RV X about Y in the sense that the conditional PDF is not different than the unconditional PDF.

> *Example 2.14:* In Matlab each time `rand` or `randn` is executed it returns an independent sample from the corresponding uniform or Gaussian distribution.

Theorem 2.4 (Total Probability) *For two random variables X and Y the marginal density of Y is given by*

$$p_{Y}\left(y\right)=\int_{-\infty}^{\infty}p_{Y|X}\left(y\,|X=x\right)p_{X}\left(x\right)dx.$$

Proof: The marginal density is given as (Property 3 of PDFs)

$$p_{Y}\left(y\right)=\int_{-\infty}^{\infty}p_{XY}\left(x,y\right)dx.$$

Rearranging (2.19) and substituting completes the proof. □

Theorem 2.5 (Bayes) *For two random variables X and Y the conditional density is given by*

$$p_{Y|X}\left(y\,|X=x\right)=\frac{p_{X|Y}\left(x\,|Y=y\right)p_{Y}\left(y\right)}{p_{X}\left(x\right)}=\frac{p_{X|Y}\left(x\,|Y=y\right)p_{Y}\left(y\right)}{\int_{-\infty}^{\infty}p_{XY}\left(x,y\right)dy}.$$

Proof: The definition of conditional probability gives

$$p_{XY}\left(x,y\right)=p_{Y|X}\left(y\,|X=x\right)p_{X}\left(x\right)=p_{Y|X}\left(x\,|Y=y\right)p_{Y}\left(y\right).$$

Rearrangement and total probability completes the proof. □

A majority of the results presented in this section correspond to two random variables. The extension of these concepts to three or more random variables is straightforward.

2.3.2 Joint Moments and Statistical Averages

Joint moments and statistical averages are also of interest in communication system engineering. The general statistical average of a function $g(X,Y)$ of two random variables X and Y is given as

$$E\left[g(X,Y)\right]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(x,y)p_{XY}\left(x,y\right)dxdy.$$

A commonly used joint moment or statistical average is the correlation between two random variables X and Y, defined as

$$E\left[XY\right]=\mathrm{corr}\left(X,Y\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}xyp_{XY}\left(x,y\right)dxdy.$$

A frequently used joint central moment is the covariance between two random variables x and y, defined as

$$E\left[\left(X-m_X\right)\left(Y-m_Y\right)\right]=\mathrm{cov}\left(X,Y\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(x-m_X\right)\left(y-m_Y\right)p_{XY}\left(x,y\right)dxdy.$$

Definition 2.13 *The correlation coefficient is*

$$\rho_{XY}=\frac{cov\left(X,Y\right)}{\sqrt{var\left(X\right)var\left(Y\right)}}=\frac{E\left[\left(X-m_X\right)\left(Y-m_Y\right)\right]}{\sigma_X\sigma_Y}.$$

The correlation coefficient is a measure the statistical similarity of two random variables. If $|\rho_{XY}|=1$ for random variables X and Y, then X is a scalar multiple of Y. If $\rho_{XY}=0$ then the random variables are uncorrelated. Values of $|\rho_{XY}|$ between these two extremes provide a measure of the similarity of the two random variables (larger $|\rho_{XY}|$ being more similar).

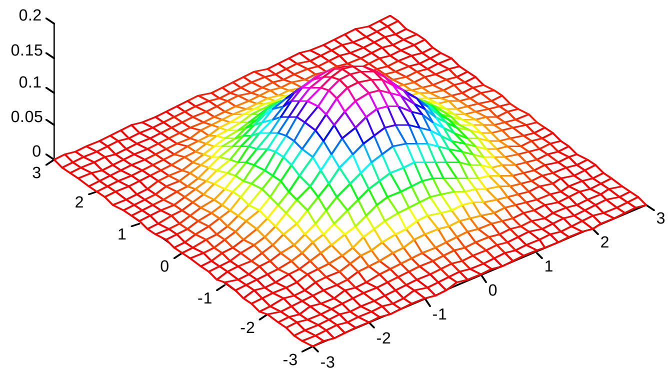


Figure 2.5: Plots of the joint PDF of two zero mean, unit variance, uncorrelated Gaussian random variables.

2.3.3 Two Gaussian Random Variables

Two jointly Gaussian random variables, X and Y, have a joint density function of

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{XY}^2}}\exp\left(-\frac{1}{2(1-\rho_{XY}^2)}\left(\frac{(x-m_X)^2}{\sigma_X^2} - \frac{2\rho_{XY}(x-m_X)(y-m_Y)}{\sigma_X\sigma_Y} + \frac{(y-m_Y)^2}{\sigma_Y^2}\right)\right).$$

where $m_X = E(X)$, $m_Y = E(Y)$, $\sigma_X^2 = \mathrm{var}(X)$, $\sigma_Y^2 = \mathrm{var}(Y)$, and ρ_{XY} is the correlation coefficient between the two random variables X and Y. This density function results in a three dimensional bell shaped curve which is stretched and shaped by the means, variances and the correlation coefficient. As an example Fig. 2.5 is a plot of the joint density function of two zero mean, unit variance, uncorrelated (ρ_{XY}=0) Gaussian random variables. The uncorrelatedness of the two random variables is what gives this density function its circular symmetry. In fact for Gaussian random variables uncorrelatedness implies that the two random variables are independent, i.e.,

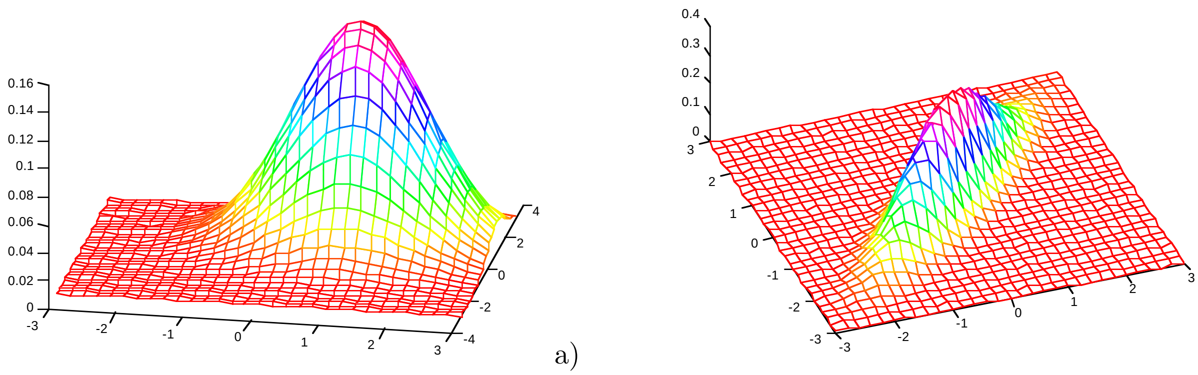
$$f_{XY}(x,y) = f_X(x)\, f_Y(y).$$

Changing the mean in the bivariate Gaussian density function again simply changes the center of the bell shaped curve. Fig. **??** is a plot of a bivariate Gaussian density function with $m_X = 1$, $m_Y = 1$, $\sigma_X = 1$, $\sigma_Y = 1$ and ρ_{XY}=0. The only difference between Fig. 2.6-a) and Fig. 2.5 (other than the perspective) is a translation of the bell shape to the new mean mean value (1,1). Changing the variances of joint Gaussian random variables changes the relative shape of the joint density function much like that shown in Fig. 2.4.

The effect of the correlation coefficient on the shape of the bivariate Gaussian density function is a more interesting concept. Recall that the correlation coefficient is

$$\rho_{XY} = \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{var}(X)\,\mathrm{var}(Y)}} = \frac{E[(X-m_X)(Y-m_Y)]}{\sigma_X\sigma_Y}.$$

if two random variables have a correlation coefficient greater than zero then these two random variables tend probabilistically to take values having the same sign. Also if these two random variables have a



a) Nonzero means, $m_X = 1$, $m_Y = 1$. b) Nonzero correlation coefficient, $\rho_{XY} = 0.9$

Figure 2.6: Plots of the joint PDF of two unit variance Gaussian random variables.

correlation coefficient close to unity then they probabilistically behave in a very similar fashion. A good example of this characteristic is seen in Fig. 2.6-b) which shows a plot of a bivariate Gaussian with $m_X = 0$, $m_Y = 0$, $\sigma_X = 1$, $\sigma_Y = 1$ and ρ_{XY}=0.9.

While it is easy to see that the marginal densities, i.e.,

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dxdy$$

from Fig. 2.6 would be the same as the those obtained in Fig. 2.5, the conditional densities would be much different, e.g.,

$$f_{X|Y}\left(x\,|y\right) = \frac{1}{\sigma_X\sqrt{2\pi\left(1-\rho_{XY}^2\right)}} \exp\left(-\frac{1}{2\sigma_X^2(1-\rho_{XY}^2)}\left(x - m_X - \frac{\rho_{XY}\sigma_X}{\sigma_Y}\left(y-m_Y\right)\right)^2\right).$$

Since the random variables whose PDF is shown in Fig. 2.6-b) are highly correlated, the resulting values the random variables take are very similar. This is reflected in the knife edge characteristic that the density function takes along the line $Y = X$.

2.3.4 Transformations of Random Variables

In general, the problem of finding a probabilistic description (joint CDF or PDF) of a joint transformation of random variables is very difficult. To simplify the presentation two of the most common and practical transformations will be considered: the single function of n random variables and the one-to-one transformation.

Single Function of n Random Variables

A single function transformation of n random variables is expressed as

$$Y = g\left(X_1, X_2, \ldots, X_n\right).$$

where $X_1, \ldots, X_n$ are the n original random variables. This section is concerned with finding the PDF or the CDF of the random variable Y. The general technique is the identical two step process used for a single random variable (the regions of integration now become volumes instead of intervals).

Step 1: Find the CDF of Y as a sum of integrals over the random variables $X_1 \dots X_n$. This is expressed mathematically as

$$F_Y(y) = \sum_i \int \underset{R_i(y)}{\cdots} \int p_{X_1 \dots X_n}(x_1, x_2 \dots x_n) dx_1 \dots dx_n$$

where the $R_i(y)$ are n dimensional volumes where $X_1 \dots X_n$ are such that $g(X_1, \dots, X_n) < y$.
Step 2: Find the PDF by differentiating the CDF found in Step 1 using Liebnitz Rule, i.e.,

$$p_Y(y) = \frac{dF_Y(y)}{dy} = \sum_i \frac{d}{dy} \int \underset{R_i(y)}{\cdots} \int p_{X_1 \dots X_n}(x_1, x_2, \dots, x_n) dx_1 \dots dx_n$$

Example 2.15: A transformation which is very important for analyzing the performance of the standard demodulator used with amplitude modulation (AM) is $Y = \sqrt{X_1^2 + X_2^2}$.
Step 1:

$$F_Y(y) = P\left(\sqrt{X_1^2 + X_2^2} \leq y\right) = U(y) \int_{-y}^{y} dx_1 \int_{-\sqrt{y^2 - x_1^2}}^{\sqrt{y^2 - x_1^2}} p_{X_1 X_2}(x_1, x_2) dx_2$$

Step 2:

$$p_Y(y) = U(y) \int_{-y}^{y} \frac{y}{\sqrt{y^2 - x_1^2}} \left(p_{X_1 X_2}\left(x_1, \sqrt{y^2 - x_1^2}\right) + p_{X_1 X_2}\left(x_1, -\sqrt{y^2 - x_1^2}\right)\right) dx_1$$

If in this example X_1 and X_2 are independent ($\rho = 0$) and zero mean Gaussian random variables with equal variances, the joint density is given as

$$f_{X_1 X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_X^2} \exp\left(-\frac{x_1^2 + x_2^2}{2\sigma_X^2}\right).$$

and $f_Y(y)$ reduces to

$$f_Y(y) = \frac{y}{\sigma_X^2} \exp\left(-\frac{y^2}{2\sigma_X^2}\right) U(y). \tag{2.20}$$

This PDF is known as the Rayleigh density and appears quite frequently in communication system analysis.

One-to-One Transformations

Consider a one-to-one transformation of the form

$$\begin{aligned} Y_1 &= g_1(X_1, X_2, ..., X_n) \\ Y_2 &= g_2(X_1, X_2, ..., X_n) \\ &\vdots \\ Y_n &= g_n(X_1, X_2, ..., X_n). \end{aligned} \tag{2.21}$$

Since the transformation is one-to-one, the inverse functions exist and are given as

$$\begin{aligned} X_1 &= h_1(Y_1, Y_2, ..., Y_n) \\ X_2 &= h_2(Y_1, Y_2, ..., Y_n) \\ &\vdots \\ X_n &= = h_n(Y_1, Y_2, ..., Y_n). \end{aligned} \tag{2.22}$$

Since the probability mass in infinitesimal volumes in both the original X coordinate system and the Y coordinate system must be identical, the PDF of the Y's is given as

$$p_Y\left(y_1, y_2, ...y_n\right) = |J|\, p_X\left(h_1\left(y_1, y_2, ...y_n\right), ..., h_n\left(y_1, y_2, ...y_n\right)\right).$$

where $|J|$ is the Jacobian defined as

$$|J| = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_2}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_1} \\ \frac{\partial x_1}{\partial y_2} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_2} \\ \vdots & & \ddots & \\ \frac{\partial x_1}{\partial y_n} & \frac{\partial x_2}{\partial y_n} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix}.$$

Example 2.16: A one-to-one transformation which is very important in communications is

$$\begin{aligned} Y_1 &= \sqrt{X_1^2 + X_2^2} & \qquad X_1 &= Y_1 \sin\left(Y_2\right) \\ Y_2 &= \tan^{-1}\left(\frac{X_1}{X_2}\right) & \qquad X_2 &= Y_1 \cos\left(Y_2\right) \end{aligned}$$

The Jacobian of the transformation is given by $|J| = Y_1$ which gives

$$p_{Y_1 Y_2}\left(y_1, y_2\right) = y_1 p_{X_1 X_2}\left(y_1 \sin\left(y_2\right), y_1 \cos\left(y_2\right)\right).$$

2.3.5 Central Limit Theorem

Theorem 2.6 *If X_k is a sequence of independent, identically distributed (i.i.d.) random variables with mean m_x and variance σ^2 and Y_n is a random variable defined as*

$$Y_n = \frac{1}{\sqrt{n\sigma^2}} \sum_{k=1}^{n} \left(X_k - m_x\right)$$

then

$$\lim_{n\to\infty} p_{Y_n}(y) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{y^2}{2}\right] = N\left(0, 1\right).$$

Proof: The proof uses the characteristic function, a power series expansion and the uniqueness property of the characteristic function . Details are available in most introductory texts on probability [DR87, LG89, Hel91]. □

The Central Limit Theorem (CLT) implies that the sum of arbitrarily distributed random variables tends to a Gaussian random variable as the number of terms in the sum gets large. Because many physical phenomena in communications systems are due to interactions of large numbers of events (e.g., random electron motion in conductors or large numbers of scatters in wireless propagation), this theorem is one of the major reasons why the Gaussian random variable is so prevalent in communication system analysis.

2.4 Homework Problems

Problem 2.1. This problem exercises axiomatic definitions of probability. A communications system used in a home health care system needs to communicate four patient services, two digital and two analog. The two digital services are a 911 request and a doctor appointment request. The two analog services are the transmission of an electrocardiogram (EKG) and the transmission of audio transmissions from a stethoscope. The patient chooses each of the services randomly depending on the doctor's prior advice and the patient current symptoms.

a) For design purposes it is typical to model the average patient's requested services as a random experiment. Define a sample space for the requested services.

A market survey has shown that the probability that a patient will request an EKG is 0.4 and the probability that a patient will request 911 is 0.001. The probability that a digital service will be requested is 0.1.

b) Compute the probability of requesting a doctor's appointment.

c) Compute the probability of requesting a audio transmission from a stethoscope.

Problem 2.2. This problem exercises the idea of conditional probability. In the Problem 2.1 if you know the requested service is a digital service what is the probability that a 911 request is initiated?
Problem 2.3. A simple problem to exercise the axioms of probability. Two events A and B are defined by the same random experiment. $P(A) = 0.5$, $P(B) = 0.4$, and $P(A \cap B)$ =0.2

a) Compute $P(A \cup B)$.

b) Are events A and B independent?

Problem 2.4. Your roommate challenges you to a game of chance. He proposes the following game. A coin is flipped two times, if heads comes up twice she/he gets a dollar from you and if tails comes up twice you get a dollar from him/her. You know your roommate is a schemer so you know that there is some chance that this game is rigged. To this end you assign the following mathematical framework.

- $P(F)$ = probability that the game is fair= $\frac{5}{6}$
- $P(H|F)$ = probability that a head results if the game is fair = 0.5
- $P(H|UF)$ = probability that a head results if the game is unfair = 0.75

Assume conditioned on the fairness of the game that each flip of the coin is independent.

a) What is the probability that the game is unfair, $P(UF)$?

b) What is the probability that two heads appear given the coin is unfair?

c) What is the probability that two heads appear?

d) If two heads appear on the first trial of the game what is the probability that you are playing an unfair game?

Problem 2.5. Certain digital communication schemes use redundancy in the form of an error control code to improve the reliability of communication. The compact disc recording media is one such example. Assume a code can correct 2 or fewer errors in a block of N coded bits. If each bit detection is independent with a probability of error of $P(E) = 10^{-3}$

1. Plot the probability of correct block detection as a function of N.

2. How small must N be so that the probability the block is detected incorrectly is less 10^{-6}?

Problem 2.6. The probability of accessing the OSU computer system is characterized on a weekday with the following events

1. A={dialing is between 7:00PM-1:00AM},
2. B={dialing is between 1:00AM-7:00PM},
3. C={a connection is made},
4. D={a connection failed}.

The system is characterized with

$$P\left(\text{Connecting given dialing is between 7:00PM-1:00AM}\right) = P(C|A) = 0.1$$

$$P\left(\text{Connecting given dialing is between 1:00AM-7:00PM}\right) = P(C|B) = 0.75.$$

a) If a person randomly dials with uniform probability during the day what is $P(A)$?

b) If a person randomly dials with uniform probability during the day what is $P(C)$?

c) If a person randomly dials with uniform probability during the day what is the $P(A \text{ and } C)$?

d) Compute $P(A|C)$.

Problem 2.7. In a particular magnetic disk drive bits are written and read individually. The probability of a bit error is $P(E) = 10^{-8}$. Bit errors are independent from bit to bit. In a computer application the bits are grouped into 16 bit words. What is the probability that an application word will be in error (at least one of the 16 bits is in error) when the bits are read from this magnetic disk drive.

Problem 2.8. A simple yet practical problem to exercise the idea of Bayes rule and total probability. Air traffic control radars (ATCR) are used to direct traffic in and around airports. If a plane is detected where it is not suppose to be then the controllers must initiate some evasive actions. Consequently for the users of a radar system the two most important parameters of a radar system are

- $P(TA|DT)$ = probability of target being absent when a target is detected.
- $P(TP|ND)$ = probability of target being present when no detection is obtained.

Radar designers on the other hand like to quote performance in the following two parameters (because they can easily be measured)

- $P(DT|TA)$ = probability of detecting a target when a target is absent.
- $P(DT|TP)$ = probability of detecting a target when a target is present.

Imagine that you are a high priced consultant for the Federal Aviation Administration (FAA) and that the FAA has the following requirements for it's next generation ATCR

- $P(TA|DT) = 0.01$
- $P(TP|ND) = 0.0001$

A detailed study shows that planes are present only 1% of the time, $P(TP) = 0.01$. A contractor, Huge Aircrash Co., has offered a radar system to the government with the following specifications

- $P(DT|TA) = 0.00005$

- $P(DT|TP) = 0.9$

Would you recommend the government purchase this system and why?
Problem 2.9. The following are some random events

1. The sum of the roll of two dice.
2. The hexidecimal representation of an arbitrary 4 bits in an electronic memory.
3. The top card in a randomly shuffled deck.
4. The voltage at the output of a diode in a radio receiver.

Determine which events are well modeled with a random variable. For each random variable determine if the random variable is continuous, discrete, or mixed. Characterized the sample space and the mapping from the sample space if possible.
Problem 2.10. A random variable has a density function given as

$$\begin{aligned} f_X(x) &= 0 \qquad x < 0 \\ &= K_1 \qquad 0 \leq x < 1 \\ &= K_2 \qquad 1 \leq x < 2 \\ &= 0 \qquad x \geq 2 \end{aligned}$$

a) If the mean is 1.2 find K_1 and K_2.

b) Find the variance using the value of K_1 and K_2 computed in a).

c) Find the probability that $X \leq 1.5$ using the value of K_1 and K_2 computed in a).

Problem 2.11. In communications the phase shift induced by propagation between transmitter and receiver, ϕ_p, is often modeled as a random variable. A common model is to have

$$\begin{aligned} f_{\phi_p}(\phi) &= \frac{1}{2\pi} \qquad -\pi \leq \phi \leq \pi \\ &= 0 \qquad \text{elsewhere} \end{aligned}$$

This is commonly referred to as a uniformly distributed random variable.

a) Find the CDF, $F_{\Phi_p}(\phi)$.

b) Find the mean and variance of this random phase shift.

c) It turns out that a communication system will work reasonably well if the coherent phase reference at the receiver, $\hat{\phi}_p$, is within 30° of the true value of ϕ_p. If you implement a receiver with $\hat{\phi}_p = 0$, what is the probability the communication system will work.

d) Assuming you can physically move your system and change the propagation delay to obtain an independent phase shift. What is the probability that your system will work at least one out of two times?

e) How many independent locations would you have to try to ensure a 90% chance of getting your system to work?

Problem 2.12. You have just started work at Yeskia which is a company that manufactures FM transmitters for use in commercial broadcast. An FCC rule states that the carrier frequency of each station must be within 1 part per million of the assigned center frequency (i.e., if the station is assigned a 100MHz frequency then the oscillators deployed must be $|f_c - 100\text{MHz}| < 100\text{Hz}$). The oscillator designers within Yeskia have told you that the output frequency of their oscillators is well modeled as a Gaussian random variable with a mean of the desired channel frequency and a variance of σ_f^2. Assume the lowest assigned center frequency is 88.1MHz and the highest assigned center frequency is 107.9MHz.

a) Which assigned center frequency has the tightest absolute set accuracy constraint?

b) If σ_f=40Hz, what is the probability that a randomly chosen oscillator will meet the FCC specification?

c) What value of σ_f should you force the designers to achieve to guarantee that the probability that a randomly chosen oscillator does not meet FCC rules is less than 10^{-7}.

Problem 2.13. X is a Gaussian random variable with a mean of 2 and a variance of 4.

a) Plot $f_X(x)$.

b) Plot $F_X(x)$.

c) Plot the function $g(x) = P\left(|X-2| < x\right)$.

Problem 2.14. Consider two random samples, X_1 at the input to a communication system, and X_2, at the output of a communication system and model them as jointly Gaussian random variables. Plot the joint probability density function, $f_{X_1X_2}(x_1, x_2)$ for

a) $m_1 = m_2 = 0$, $\sigma_1 = \sigma_2 = 1$, and $\rho = 0$.

b) $m_1 = m_2 = 0$, $\sigma_1 = 2 \quad \sigma_2 = 1$, and $\rho = 0$.

c) $m_1 = m_2 = 0$, $\sigma_1 = \sigma_2 = 1$, and $\rho = 0.8$.

d) $m_1 = 1 \quad m_2 = 2$, $\sigma_1 = \sigma_2 = 1$, and $\rho = 0$.

e) $m_1 = 1 \quad m_2 = 2$, $\sigma_1 = 2 \quad \sigma_2 = 1$, and $\rho = 0.8$.

f) Rank order the five cases in terms of how much information is available in the output, X_2, about the input X_1.

Problem 2.15. Consider a uniform random variable with and PDF given in (2.16) with $a = 0$ and $b = 1$.

a) Compute m_X.

b) Compute σ_X.

c) Compute $P(X \leq 0.4)$.

d) Use `rand` in Matlab and produce 100 independent samples of a uniform random variable. How many of these 100 samples are less than or equal to 0.4? Does this coincide with the results you obtained in c)?

Problem 2.16. The received signal strength in a wireless radio environment is often well modeled as Rayleigh random variable (see (2.20)). For this problem assume the received signal strength at a particular location for a mobile phone is a random variable, X, with a Rayleigh distribution. A cellular phone using frequency modulation needs to have a signal level of at least $X = 20$mV to operate in an acceptable fashion. Field tests on the OSU campus have shown the average signal power is 1mW (in a 1 Ω system).

a) Using (2.20) compute the average signal power, $E\left[X^2\right]$, as a function of σ_X.

b) Compute the probability that a cellular phone will work at a randomly chosen location on the OSU campus.

c) Assume you physically move and obtain an independent received signal strength. What is the probability that your cellular phone will work at least one out of two times?

d) How many independent locations would you have to try to ensure a 90% chance of getting your cellular phone to work?

Problem 2.17. This problem both give a nice insight into the idea of a correlation coefficient and shows how to generate correlated random variables in simulation. X and W are two zero mean independent random variables where $E\left[X^2\right] = 1$ and $E\left[W^2\right] = \sigma_W^2$. A third random variable is defined as $Y = \rho X + W$ where ρ is a deterministic constant such that $-1 \leq \rho \leq 1$.

a) Prove $E[XW] = 0$. In other words prove that independence implies uncorrelatedness.

b) Choose σ_W^2 such that $\sigma_Y^2 = 1$.

c) Find ρ_{XY} when σ_W^2 is chosen as in part b).

Problem 2.18. You are designing a phase locked loop as an FM demodulator. The requirement for your design is that the loop bandwidth must be greater than 5KHz and less than 7KHz. You have computed the loop bandwidth and it is given as

$$B_L = 4R^3 + 2000$$

where R is a resistor in the circuit. It is obvious that choosing R=10 will solve your problem. Unfortunately resistors are random valued.

a) If the resistors used in manufacturing your FM demodulator are uniformly distributed between 9 and 11Ω, what is the probability that the design will meet the requirements?

b) If the resistors used in manufacturing your FM demodulator are Gaussian random variables with a mean of 10Ω and a standard deviation of 0.5Ω, what is the probability that the design will meet the requirements?

Problem 2.19. In general uncorrelatedness is a weaker condition than independence and this problem will demonstrate this characteristic.

a) If X and Y are independent random variables prove $E[XY] = 0$.

b) Consider the joint discrete random variable X and Y with a joint PMF of

$$\begin{aligned} p_{XY}(2,0) &= \frac{1}{3} \\ p_{XY}(-1,-1) &= \frac{1}{3} \\ p_{XY}(-1,1) &= \frac{1}{3} \\ p_{XY}(x,y) &= 0 \qquad \text{elsewhere.} \end{aligned} \tag{2.23}$$

What are the marginal PMFs, $p_X(x)$ and $p_Y(y)$? Are X and Y independent random variables? Are X and Y uncorrelated?

c) Show that two jointly Gaussian random variables which are uncorrelated (i.e., $\rho_{XY} = 0$) are also independent. This makes the Gaussian random variable an exception to the general rule.

Problem 2.20. The probability of having a cellular phone call dropped in the middle of a call while driving on the freeway in the big city is characterized by following events

1. A={call is during rush hour},
2. B={call is not during rush hour},
3. C={the call is dropped},
4. D={the call is normal }.

Rush hour is defined to be 7-9:00AM and 4-6:00PM. The system is characterized with

$$P\left(\text{Drop during rush hour}\right) = P(C|A) = 0.3$$

$$P\left(\text{Drop during nonrush hour}\right) = P(C|B) = 0.1.$$

a) A person's time to make a call, T, might be modeled as a random variable with

$$\begin{aligned} f_T(t) &= \frac{1}{24} \qquad 0 \leq t \leq 24 \\ &= 0 \qquad \text{otherwise} \end{aligned} \tag{2.24}$$

where time is measured using a 24 hour clock (i.e., 4:00PM=16). Find $P\left(A\right)$.

b) What is $P\left(C\right)$?

c) What is the $P\left(A \text{ and } C\right) = P\left(A \bigcap C\right)$?

d) Compute $P\left(A|C\right)$.

Problem 2.21. This problem examines the sum of two random variables works through the computation of the result PDF. Formally $Y = X_1 + X_2$.

a) Use `rand` in Matlab and produce 2 vectors, $\vec{X}_1$ and $\vec{X}_2$ of 1000 independent samples of a uniform random variable. The `rand` function produce random variable uniformly distributed on $[0, 1]$. Add these two vectors together and plot a histogram of the resulting vector.

b) Find the CDF of Y in the form

$$F_Y\left(y\right) = \sum_i \int \underset{R_i\left(y\right)}{\cdots} \int p_{X_1,X_2}(x_1, x_2)dx_1dx_2 \tag{2.25}$$

as given in Section 2.3.4. Identify all regions in the x_1x_2 plane, $R_i(y)$, where $x_1 + x_2 < y$ where y is a constant.

c) Find the PDF by taking the derivative of the CDF in b) and simplify as much as possible.

d) Show that if X_1 and X_2 are independent random variables that the resultant PDF of Y is given as the convolution of the PDF of X_1 and X_2.

e) Compute the PDF of Y if X_1 and X_2 are independent random variable uniformly distributed on $[0, 1]$. Looking back at the histogram produced in a), does the resulting answer make sense? Verify the result further by considering 10000 length vectors and repeating a).

2.5 Example Solutions

Problem 2.6. Using 24 hour time keeping to produce a mapping from time to a real number results in a uniform dialing model for the placement of calls having a PDF given as

$$\begin{aligned} f_T(t) &= \frac{1}{24} \qquad 0 \leq t \leq 24 \\ &= 0 \qquad \text{elsewhere.} \end{aligned} \tag{2.26}$$

a) $P(A) = P(0 \leq t \leq 1 \cup 19 \leq t \leq 24) = \int_0^1 f_T(t)dt = \int_{19}^{24} f_T(t)dt = 1/24 + 5/24 = 1/4.$

b) Here we use total probability over events A and B.

$$P(C) = P(C|A)P(A) + P(C|B)P(B) = 0.1 * 0.25 + 0.75 * 0.75 = 0.5875. \tag{2.27}$$

Note here use was made of $P(B) = 1 - P(A) = 0.75$.

c) Using the definition of conditional probability gives $P(A \cap C) = P(A, C) = P(C|A)P(A) = 0.25 * 0.1 = 0.025$.

d) Here we use the definition of conditional probability.

$$P(A|C) = \frac{P(A,C)}{P(C)} = \frac{0.025}{0.5875} = 0.04255 \tag{2.28}$$

Note if we had not completed part b) and c) then we could have used Bayes Rule

$$P(A|C) = \frac{P(C|A)P(A)}{P(C)} = \frac{P(C|A)P(A)}{P(C|A)P(A) + P(C|B)P(B)}. \tag{2.29}$$

Chapter 3

Complex Baseband Representation of Bandpass Signals

3.1 Introduction

Almost every communication system operates by modulating an information bearing waveform onto a sinusoidal carrier. As examples, Table 3.1 lists the carrier frequencies of various methods of electronic communication.

Type of Transmission	Center Frequency of Transmission
Telephone Modems	1600-1800 Hz
AM radio	530-1600 KHz
CB radio	27 MHz
FM radio	88-108 MHz
VHF TV	178-216 MHz
Cellular radio	850 MHz
Indoor Wireless Networks	1.8GHz
Commercial Satellite Downlink	3.7-4.2 GHz
Commercial Satellite Uplink	5.9-6.4 GHz
Fiber Optics	2×10^{14} Hz

Table 3.1: Carrier frequency assignments for different methods of information transmission.

One can see by examining Table 3.1 that the carrier frequency of the transmitted signal is not the component which contains the information. Instead it is the signal modulated on the carrier which contains the information. Hence a method of characterizing a communication signal which is independent of the carrier frequency is desired. This has led communication system engineers to use a **complex baseband representation** of communication signals to simplify their job. All of the communication systems mentioned in Table 3.1 can be and typically are analyzed with this complex baseband representation. This handout develops the complex baseband representation for deterministic signals. Other references which do a good job of developing these topics are [Pro89, PS94, Hay83, BB99]. One advantage of the complex baseband representation is simplicity. All signals are lowpass signals and the fundamental ideas behind modulation and communication signal processing are easily developed. Also any receiver that processes the received waveform digitally uses the complex baseband representation to develop the baseband processing algorithms.

3.2 Baseband Representation of Bandpass Signals

The first step in the development of a complex baseband representation is to define a bandpass signal.

Definition 3.1 *A bandpass signal, $x_c(t)$, is a signal whose one-sided energy spectrum is both: 1) centered at a non-zero frequency, f_C, and 2) does not extend in frequency to zero (DC).*

The two sided transmission bandwidth of a signal is typically denoted by B_T Hertz so that the one-sided spectrum of the bandpass signal is zero except in $[f_C - B_T/2, f_C + B_T/2]$. This implies that a bandpass signal satisfies the following constraint: $B_T/2 < f_C$. Fig. 3.1 shows a typical bandpass spectrum. Since a bandpass signal, $x_c(t)$, is a physically realizable signal it is real valued and consequently the energy spectrum will always be symmetric around $f = 0$. The relative sizes of B_T and f_C are not important, only that the spectrum takes negligible values around DC. In telephone modem communications this region of negligible spectral values is only about 300Hz while in satellite communications it can be many Gigahertz.

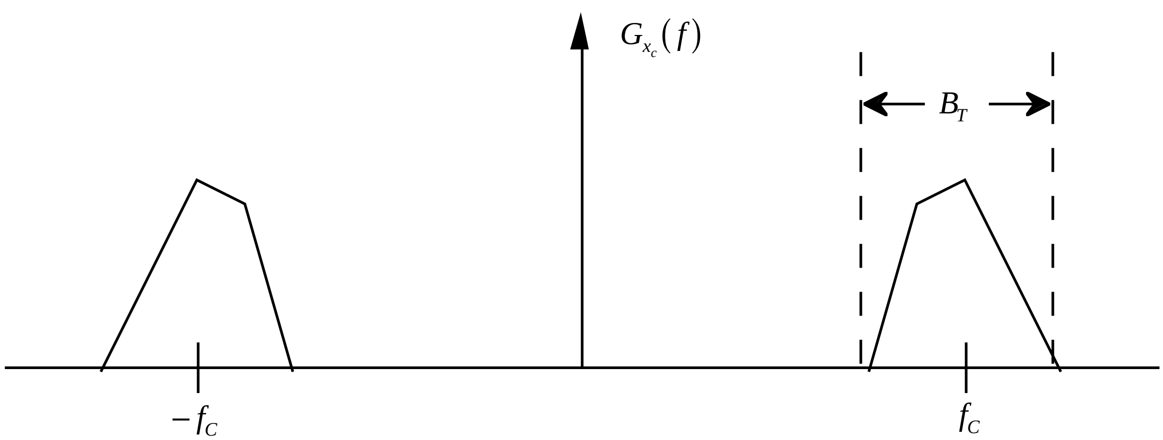


Figure 3.1: Energy spectrum of a bandpass signal.

A bandpass signal has a representation of

$$x_c(t) = x_I(t)\sqrt{2}\cos(2\pi f_c t) - x_Q(t)\sqrt{2}\sin(2\pi f_c t) \tag{3.1}$$

$$= x_A(t)\sqrt{2}\cos\left(2\pi f_c t + x_P(t)\right) \tag{3.2}$$

where f_c is denoted the carrier frequency with $f_C - B_T/2 \leq f_c \leq f_C + B_T/2$. The signal $x_I(t)$ in (3.1) is normally referred to as the **in-phase (I)** component of the signal and the signal $x_Q(t)$ is normally referred to as the **quadrature (Q)** component of the bandpass signal. $x_I(t)$ and $x_Q(t)$ are real valued lowpass signals with a one-sided non–negligible energy spectrum no larger than B_T Hertz. It should be noted that the center frequency of the bandpass signal, f_C, (see Fig. 3.1) and the carrier frequency, f_c are not always the same. While f_c can theoretically take a continuum of values in most applications an obvious value of f_c will give the simplest representation[1]. The carrier signal is normally thought of as the cosine term, hence the I component is in-phase with the carrier. Likewise the sine term is 90° out-of-phase (in quadrature) with the cosine or carrier term, hence the Q component is quadrature to the carrier. Equation (3.1) is known as the canonical form of a bandpass signal. Equation (3.2) is the amplitude and phase form of the bandpass signal, where $x_A(t)$ is the **amplitude** of the signal and $x_P(t)$

[1]This idea will become more obvious in Chapter 5.

is the **phase** of the signal. A bandpass signal has two degrees of freedom and the I/Q or the amplitude and phase representations are equivalent. The transformations between the two representations are given by

$$x_A(t) = \sqrt{x_I(t)^2 + x_Q(t)^2} \qquad x_P(t) = \tan^{-1}\left[x_Q(t), x_I(t)\right] \tag{3.3}$$

and

$$x_I(t) = x_A(t)\cos\left(x_P(t)\right) \qquad x_Q(t) = x_A(t)\sin\left(x_P(t)\right). \tag{3.4}$$

Note that the tangent function in (3.3) has a range of $[-\pi, \pi]$ (i.e., both the sign $x_I(t)$ and $x_Q(t)$ and the ratio of $x_I(t)$ and $x_Q(t)$ are needed to evaluate the function). The particulars of the communication design analysis determine which form for the bandpass signal is most applicable.

A complex valued signal, denoted the **complex envelope**, is defined as

$$x_z(t) = x_I(t) + jx_Q(t) = x_A(t)\exp\left[jx_P(t)\right].$$

The original bandpass signal can be obtained from the complex envelope by

$$x_c(t) = \sqrt{2}\Re\left[x_z(t)\exp\left[j2\pi f_c t\right]\right].$$

Since the complex exponential only determines the center frequency, the complex signal $x_z(t)$ contains all the information in $x_c(t)$. Using this complex baseband representation of bandpass signals greatly simplifies the notation for communication system analysis. As the quarter goes along hopefully the additional simplicity will become very evident.

Example 3.1: Consider the bandpass signal

$$x_c(t) = 2\cos(2\pi f_m t)\sqrt{2}\cos(2\pi f_c t) - \sin(2\pi f_m t)\sqrt{2}\sin(2\pi f_c t)$$

where $f_m < f_c$. A plot of this bandpass signal is seen in Fig. 3.2 with $f_c = 10 f_m$. Obviously we have

$$x_I(t) = 2\cos(2\pi f_m t) \qquad x_Q(t) = \sin(2\pi f_m t)$$

and

$$x_z(t) = 2\cos(2\pi f_m t) + j\sin(2\pi f_m t).$$

The amplitude and phase can be computed as

$$x_A(t) = \sqrt{1 + 3\cos^2(2\pi f_m t)} \qquad x_P(t) = \tan^{-1}\left[\sin(2\pi f_m t), 2\cos(2\pi f_m t)\right].$$

A plot of the amplitude and phase of this signal is seen in Fig. 3.3.

The next item to consider is methods to translate between a bandpass signal and a complex envelope signal. Basically a bandpass signal is generated from its I and Q components in a straightforward fashion corresponding to (3.1). Likewise a complex envelope signal is generated from the bandpass signal with a similar architecture. Using the results

$$\begin{aligned} x_c(t)\sqrt{2}\cos(2\pi f_c t) &= x_I(t) + x_I(t)\cos(4\pi f_c t) - x_Q(t)\sin(4\pi f_c t) \\ x_c(t)\sqrt{2}\sin(2\pi f_c t) &= -x_Q(t) + x_Q(t)\cos(4\pi f_c t) + x_I(t)\sin(4\pi f_c t) \end{aligned} \tag{3.5}$$

Fig. 3.4 shows these transformations where the lowpass filters remove the $2f_c$ terms in (3.5). Note in the Fig. 3.4 the boxes with $\pi/2$ are phase shifters (i.e., $\cos(\theta - \pi/2) = \sin(\theta)$) typically implemented with delay elements. The structure in Fig. 3.4 is fundamental to the study of all modulation techniques.

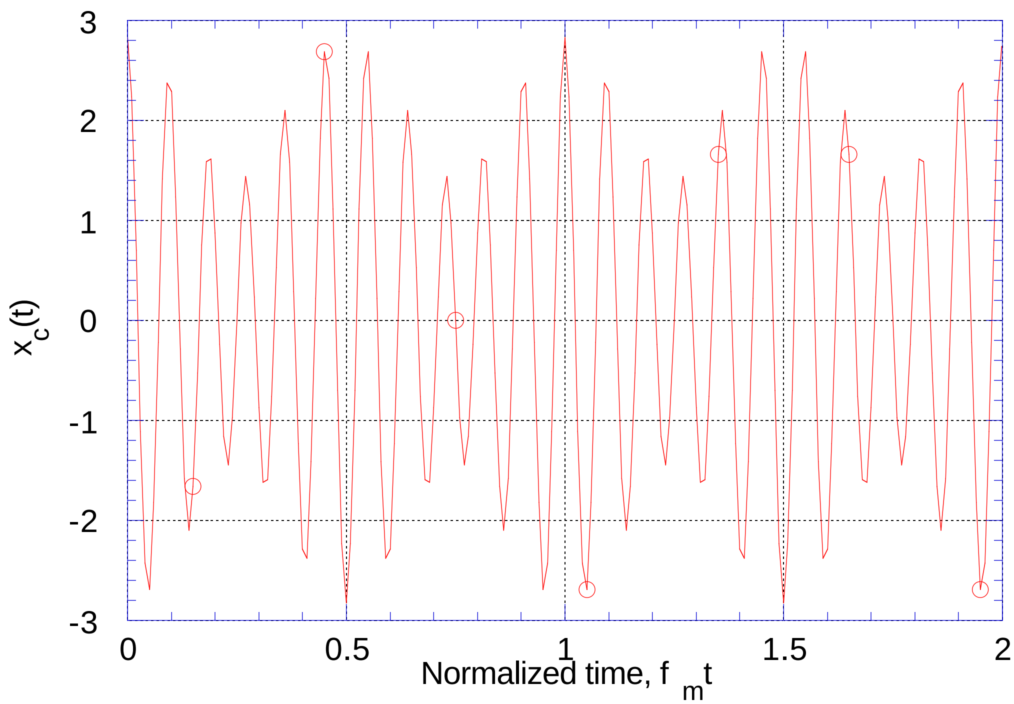


Figure 3.2: Plot of the bandpass signal for Example 3.1.

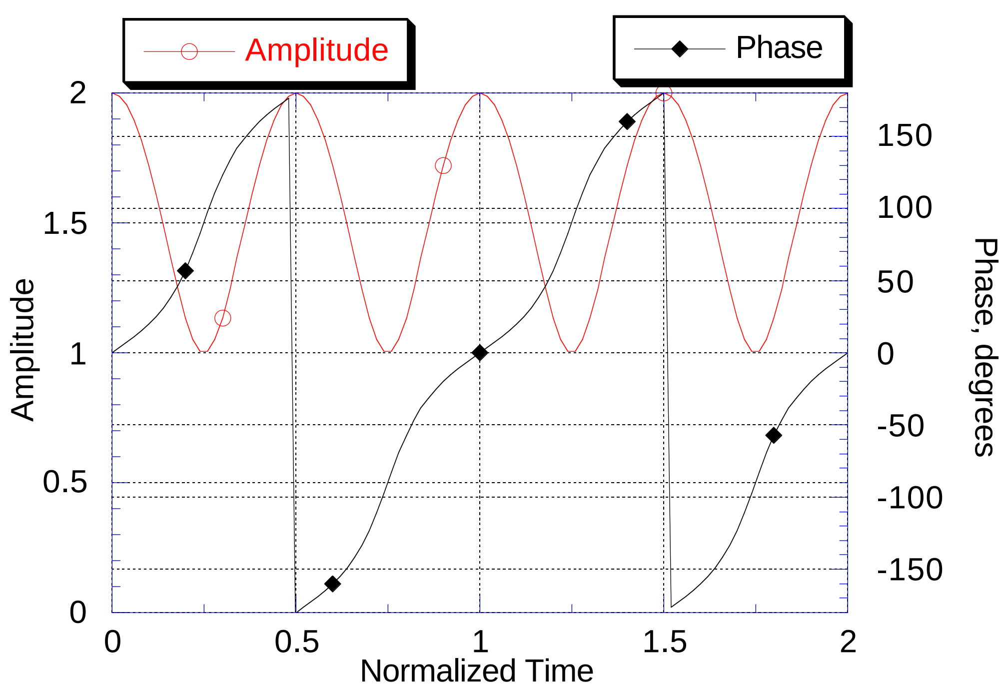


Figure 3.3: Plot of the amplitude and phase for Example 3.1.

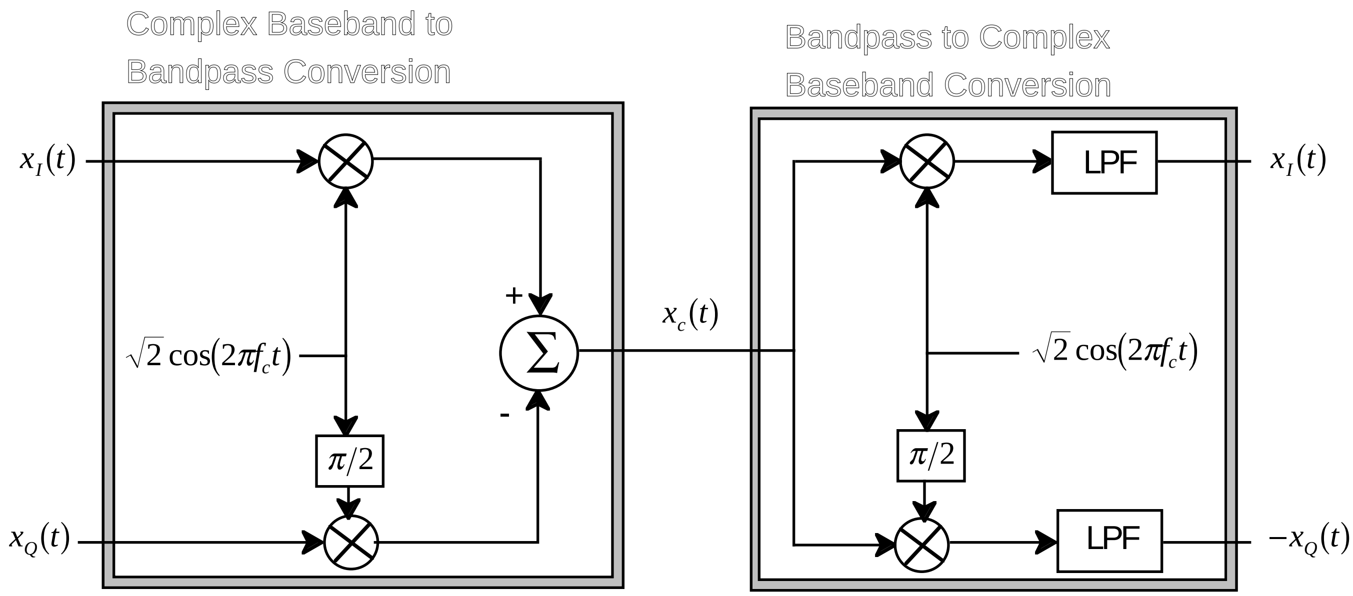


Figure 3.4: Schemes for converting between complex baseband and bandpass representations. Note that the LPF simply removes the double frequency term associated with the down conversion.

3.3 Spectral Characteristics of the Complex Envelope

3.3.1 Basics

It is of interest to derive the spectral representation of the complex baseband signal, $x_z(t)$, and compare it to the spectral representation of the bandpass signal, $x_c(t)$. Assuming $x_z(t)$ is an energy signal, the Fourier transform of $x_z(t)$ is given by

$$X_z(f) = X_I(f) + jX_Q(f) \tag{3.6}$$

where $X_I(f)$ and $X_Q(f)$ are the Fourier transform of $x_I(t)$ and $x_Q(t)$, respectively, and the energy spectrum is given by

$$G_{x_z}(f) = G_{x_I}(f) + G_{x_Q}(f) + 2\Im\left[X_I(f)X_Q^*(f)\right] \tag{3.7}$$

where $G_{x_I}(f)$ and $G_{x_Q}(f)$ are the energy spectrum of $x_I(t)$ and $x_Q(t)$, respectively. The signals $x_I(t)$ and $x_Q(t)$ are lowpass signals with a one-sided bandwidth of less than B_T so consequently $X_z(f)$ and $G_{x_z}(f)$ can only take nonzero values for $|f| < B_T$.

Example 3.2: Consider the case when $x_I(t)$ is set to be the message signal from Example 1.15(computer voice saying "bingo") and $x_Q(t) = \cos(2000\pi t)$. $X_I(f)$ will be a lowpass spectrum with a bandwidth of 2500Hz while $X_Q(f)$ will have two impulses located at ±1000Hz. Fig. 3.5 show the measured complex envelope energy spectrum for these lowpass signals. The complex envelope energy spectrum has a relation to the voice spectrum and the sinusoidal spectrum exactly as predicted in (3.6).

Eq. (3.6) gives a simple way to transform between the lowpass signal spectrums to the complex envelope spectrum. A similar simple formula exists for the opposite transformation. Note that $x_I(t)$ and $x_Q(t)$ are both real signals so that $X_I(f)$ and $X_Q(f)$ are Hermitian symmetric functions of frequency and it is straightforward to show

$$X_z(-f) \quad = \quad X_I^*(f) + jX_Q^*(f)$$

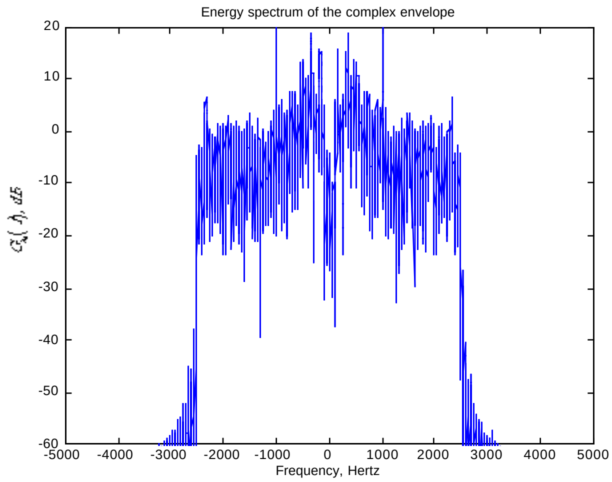


Figure 3.5: The complex envelope resulting from $x_I(t)$ being a computer generated voice signal and $x_Q(t)$ being a sinusoid.

$$X_z^*(-f) \quad = \quad X_I(f) - jX_Q(f). \tag{3.8}$$

This leads directly to

$$\begin{aligned} X_I(f) &= \frac{X_z(f) + X_z^*(-f)}{2} \\ X_Q(f) &= \frac{X_z(f) - X_z^*(-f)}{j2}. \end{aligned} \tag{3.9}$$

Since $x_z(t)$ is a complex signal, in general, the energy spectrum, $G_{x_z}(f)$, has none of the usual properties of real signal spectra (i.e., spectral magnitude is even and the spectral phase is odd).

An analogous derivation produces the spectral characteristics of the bandpass signal. Examining (3.1) and using the Modulation Theorem of the Fourier transform, the Fourier transform of the bandpass signal, $x_c(t)$, is expressed as

$$X_c(f) = \left[\frac{1}{\sqrt{2}}X_I\left(f - f_c\right) + \frac{1}{\sqrt{2}}X_I\left(f + f_c\right)\right] - \left[\frac{1}{\sqrt{2}j}X_Q\left(f - f_c\right) - \frac{1}{\sqrt{2}j}X_Q\left(f + f_c\right)\right].$$

This can be rearranged to give

$$X_c(f) = \left[\frac{X_I(f - f_c) + jX_Q(f - f_c)}{\sqrt{2}}\right] + \left[\frac{X_I(f + f_c) - jX_Q(f + f_c)}{\sqrt{2}}\right] \tag{3.10}$$

Using (3.8) in (3.10) gives

$$X_c(f) = \frac{1}{\sqrt{2}}X_z(f - f_c) + \frac{1}{\sqrt{2}}X_z^*(-f - f_c). \tag{3.11}$$

This is a very fundamental result. Equation (3.11) states that the Fourier transform of a bandpass signal is simply derived from the spectrum of the complex envelope. For positive values of f, $X_c(f)$ is obtained by translating $X_z(f)$ to f_c and scaling the amplitude by $1/\sqrt{2}$. For negative values of f, $X_c(f)$ is obtained by flipping $X_z(f)$ around the origin, taking the complex conjugate, translating the result to

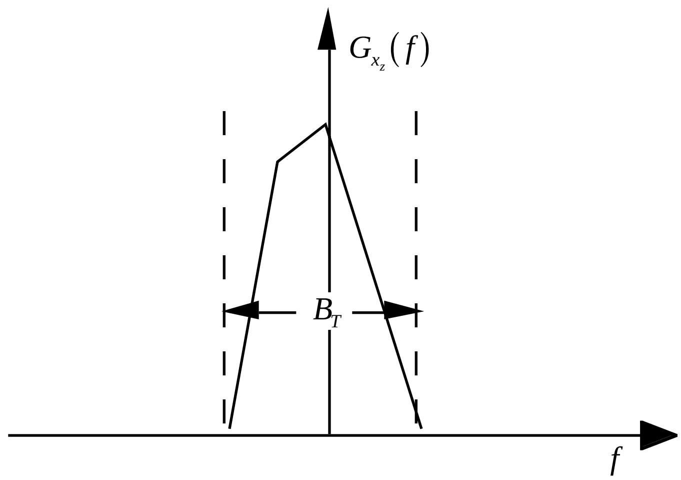


Figure 3.6: The complex envelope energy spectrum of the bandpass signal in Fig. 3.1 with $f_c = f_C$.

$-f_c$, and scaling the amplitude by $1/\sqrt{2}$. This also demonstrates that if $X_c(f)$ only takes values when the absolute value of f is in $[f_c - B_T, f_c + B_T]$, then $X_z(f)$ only takes values in $[-B_T, B_T]$. The energy spectrum of $x_c(t)$ can also be expressed in terms of the energy spectrum of $x_z(t)$ as

$$G_{x_c}(f) = \frac{1}{2}G_{x_z}(f - f_c) + \frac{1}{2}G_{x_z}(-f - f_c). \tag{3.12}$$

Equation (3.12) guarantees that the energy spectrum of the bandpass signal is an even function and that the energy of the complex envelope is identical to the energy of the bandpass signal. Considering these results, the spectrum of the complex envelope of the signal shown in Fig. 3.1 will have a form shown in Fig. 3.6 when $f_c = f_C$. Other values of f_c would produce a different but equivalent complex envelope representation. This discussion of the spectral characteristics of $x_c(t)$ and $x_z(t)$ should reinforce the idea that the complex envelope contains all the information in a bandpass waveform.

Example 3.3: (Example 3.1 continued)

$$x_I(t) = 2\cos(2\pi f_m t) \qquad\qquad x_Q(t) = \sin(2\pi f_m t)$$

$$X_I(f) = \delta(f - f_m) + \delta(f + f_m) \qquad\qquad X_Q(f) = \frac{1}{2j}\delta(f - f_m) - \frac{1}{2j}\delta(f + f_m)$$

$$X_z(f) = 1.5\delta(f - f_m) + 0.5\delta(f + f_m)$$

Note in this example $B_T = 2f_m$.

$$X_c(f) = \frac{1.5}{\sqrt{2}}\delta(f - f_c - f_m) + \frac{1}{2\sqrt{2}}\delta(f - f_c + f_m) + \frac{1.5}{\sqrt{2}}\delta(f + f_c + f_m) + \frac{1}{2\sqrt{2}}\delta(f + f_c - f_m)$$

Example 3.4: For the complex envelope derived in Example 3.2 the measured bandpass energy spectrum for f_c=7000Hz is shown in Fig. 3.7. Again the measured output is exactly predicted by (3.12).

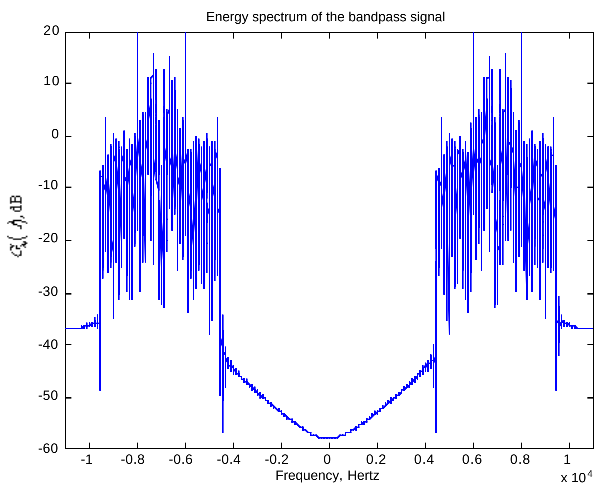


Figure 3.7: The bandpass spectrum corresponding to Fig. 3.5.

3.3.2 Bandwidth of Bandpass Signals

The ideas of bandwidth of a signal extend in an obvious way to bandpass signals. For bandpass energy signals we have the following two definitions

Definition 3.2 *If a signal $x_c(t)$ has an energy spectrum $G_{x_c}(f)$ then B_X is determined as*

$$10\log\left(\max_f G_{x_c}(f)\right) = X + 10\log\left(G_{x_c}(f_1)\right) \tag{3.13}$$

where $G_{x_c}(f_1) > G_{x_c}(f)$ for $0 < f < f_1$ and

$$10\log\left(\max_f G_{x_c}(f)\right) = X + 10\log\left(G_{x_c}(f_2)\right) \tag{3.14}$$

where $G_{x_c}(f_2) > G_{x_c}(f)$ for $f > f_2$ where $f_2 - f_1 = B_x$.

Definition 3.3 *If a signal $x_c(t)$ has an energy spectrum $G_{x_c}(f)$ then B_P is determined as*

$$P = \frac{2\int_{f_1}^{f_2} G_{x_c}(f)df}{E_{x_c}} \tag{3.15}$$

where $B_P = f_2 - f_1$.

Note the reason for the factor of 2 in (3.15) is that half of the energy of the bandpass signal is associated with positive frequencies and half of the energy is associated with negative signals

Again for bandpass power signals similar ideas hold with $G_{x_c}(f)$ being replaced with $S_{x_c}(f,T)$., i.e.,

Definition 3.4 *If a signal $x_c(t)$ has a sampled power spectral density $S_{x_c}(f,T)$ then B_X is determined as*

$$10\log\left(\max_f S_{x_c}(f,T)\right) = X + 10\log\left(S_{x_c}(f_1,T)\right) \tag{3.16}$$

where $S_{x_c}(f_1,T) > S_{x_c}(f,T)$ for $0 < f < f_1$ and

$$10\log\left(\max_f S_{x_c}(f,T)\right) = X + 10\log\left(S_{x_c}(f_2,T)\right) \tag{3.17}$$

where $S_{x_c}(f_2,T) > S_{x_c}(f,T)$ for $f > f_2$ where $f_2 - f_1 = B_x$.

Definition 3.5 *If a signal $x_c(t)$ has an power spectrum $S_{x_c}(f,T)$ then B_P is determined as*

$$P = \frac{2\int_{f_1}^{f_2} S_{x_c}(f)df}{P_{x_c}(T)} \tag{3.18}$$

where $B_P = f_2 - f_1$.

3.4 Linear Systems and Bandpass Signals

This section discusses methods for calculating the output of a linear, time-invariant (LTI) filter with a bandpass input signal using complex envelopes. Linear system outputs are characterized by the convolution integral given as

$$y_c(t) = \int_{-\infty}^{\infty} x_c(\tau)h(t-\tau)d\tau \tag{3.19}$$

where $h(t)$ is the impulse response of the filter. Since the input signal is bandpass, the effects of the filter in (3.19) can be modeled with an equivalent bandpass filter. This bandpass LTI system also has a canonical representation given as

$$h_c(t) = 2h_I(t)\cos(2\pi f_c t) - 2h_Q(t)\sin(2\pi f_c t). \tag{3.20}$$

The complex envelope for this bandpass impulse response is given by

$$h_z(t) = h_I(t) + jh_Q(t)$$

where the bandpass system impulse response is

$$h_c(t) = 2\Re\left[h_z(t)\exp\left[j2\pi f_c t\right]\right].$$

The representation of the bandpass system in (3.20) has a constant factor difference from the bandpass signal representation of (3.1). This factor is a notational convenience that permits a simpler expression for the system output (as is shown shortly). Using similar techniques as in Section 3.3, the transfer function is expressed as

$$H_c(f) = H_z(f-f_c) + H_z^*(-f-f_c).$$

This result and (3.11) combined with the convolution theorem of the Fourier transform produces an expression for the Fourier transform of $y(t)$ given as

$$Y_c(f) = X_c(f)H_c(f) = \frac{1}{\sqrt{2}}\left[X_z(f-f_c) + X_z^*(-f-f_c)\right]\left[H_z(f-f_c) + H_z^*(-f-f_c)\right].$$

Since both $X_z(f)$ and $H_z(f)$ only take values in $[-B_T, B_T]$, the cross terms in this expression will be zero and $Y_c(f)$ is given by

$$Y_c(f) = \frac{1}{\sqrt{2}}\left[X_z(f-f_c)H_z(f-f_c) + X_z^*(-f-f_c)H_z^*(-f-f_c)\right]. \tag{3.21}$$

Since $y_c(t)$ will also be a bandpass signal, it will also have a complex baseband representation. A comparison of (3.21) with (3.11) demonstrates the Fourier transform of the complex envelope of $y(t)$, $y_z(t)$, is given as

$$Y_z(f) = X_z(f)H_z(f).$$

Linear system theory produces the desired form

$$y_z(t) = \int_{-\infty}^{\infty} x_z(\tau)h_z(t-\tau)d\tau = x_z(t)\otimes h_z(t). \tag{3.22}$$

In other words, convolving the complex envelope of the input signal with the complex envelope of the filter response produces the complex envelope of the output signal. The different scale factor was introduced in (3.20) so that (3.22) would have a familiar form. This result is significant since $y_c(t)$ can be derived by computing a convolution of baseband (complex) signals which is generally much simpler than computing the bandpass convolution. Since $x_z(t)$ and $h_z(t)$ are complex, $y_z(t)$ is given in terms of the I/Q components as

$$y_z(t) = y_I(t) + jy_Q(t) = \left[x_I(t) \otimes h_I(t) - x_Q(t) \otimes h_Q(t)\right] + j\left[x_I(t) \otimes h_Q(t) + x_Q(t) \otimes h_I(t)\right].$$

Fig 3.8 shows the lowpass equivalent model of a bandpass system. The two biggest advantages of using the complex baseband representation are that it simplifies the analysis of communication systems and permits accurate digital computer simulation of filters and the effects on communication systems performance.

Example 3.5: (Example 3.1 continued) The input signal and Fourier transform are

$$x_z(t) = 2\cos(2\pi f_m t) + j\sin(2\pi f_m t) \qquad X_z(f) = 1.5\delta(f - f_m) + 0.5\delta(f + f_m).$$

Assume a bandpass filter with

$$\begin{aligned} H_I(f) &= 2 && -2f_m \le f \le 2f_m \\ &= 0 && \text{elsewhere} \end{aligned} \qquad \begin{aligned} H_Q(f) &= j\tfrac{f}{f_m} && -f_m \le f \le f_m \\ &= j && f_m \le f \le 2f_m \\ &= -j && -2f_m \le f \le -f_m \\ &= 0 && \text{elsewhere.} \end{aligned}$$

This produces

$$\begin{aligned} H_z(f) &= 2 - \tfrac{f}{f_m} && -f_m \le f \le f_m \\ &= 1 && f_m \le f \le 2f_m \\ &= 3 && -2f_m \le f \le -f_m \\ &= 0 && \text{elsewhere.} \end{aligned}$$

and now the Fourier transform of the complex envelope of the filter output is

$$Y_z(f) = H_z(f)X_z(f) = 1.5\delta(f - f_m) + 1.5\delta(f + f_m).$$

The complex envelope and bandpass signal are given as

$$y_z(t) = 3\cos(2\pi f_m t) \qquad y_c(t) = 3\cos(2\pi f_m t)\sqrt{2}\cos(2\pi f_c t)$$

3.5 Conclusions

The complex baseband representation of bandpass signals permits accurate characterization and analysis of communication signals independent of the carrier frequency. This greatly simplifies the job of the communication system engineer. A linear system is often an accurate model for a communication system, even with the associated transmitter filtering, channel distortion, and receiver filtering. As demonstrated in Fig 3.9, the complex baseband methodology truly simplifies the models for a communication system performance analysis.

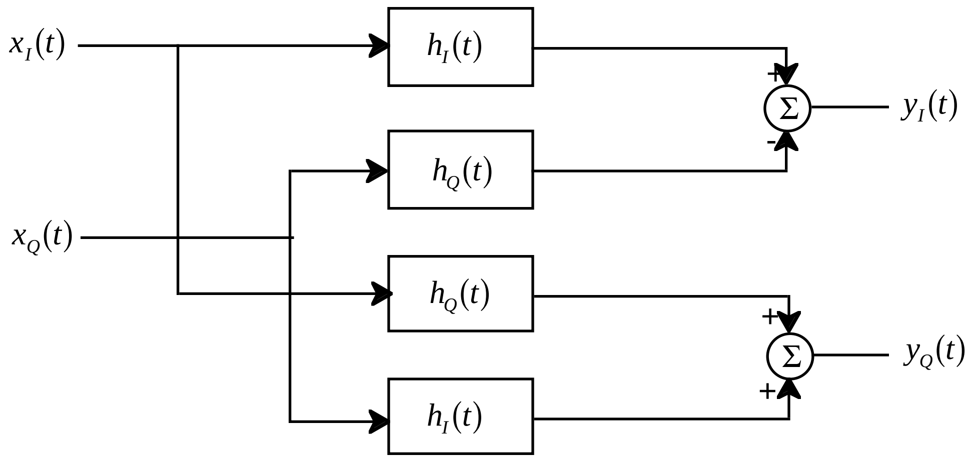


Figure 3.8: Block diagram illustrating the relation between the input and output complex envelope of bandpass signals for a linear time invariant system.

3.6 Homework Problems

Problem 3.1. Many integrated circuit implementations of the quadrature upconverters produce a bandpass signal having a form

$$x_c(t) = x_I(t)\cos(2\pi f_c t) + x_Q(t)\sin(2\pi f_c t) \tag{3.23}$$

from the lowpass signals $x_I(t)$ and $x_Q(t)$ as opposed to (3.1). How does this sign difference affect the transmitted spectrum? Specifically for the complex envelope energy spectrum given in Fig. 3.6 plot the transmitted bandpass energy spectrum.

Problem 3.2. Find the form of $x_I(t)$ and $x_Q(t)$ for the following $x_c(t)$

a) $x_c(t) = \sin(2\pi(f_c - f_m)t)$.

b) $x_c(t) = \cos(2\pi(f_c + f_m)t)$.

c) $x_c(t) = \cos(2\pi f_c t + \phi_p)$

Problem 3.3. If the lowpass components for a bandpass signal are of the form

$$x_I(t) = 12\cos(6\pi t) + 3\cos(10\pi t)$$

and

$$x_Q(t) = 2\sin(6\pi t) + 3\sin(10\pi t)$$

a) Calculate the Fourier series of $x_I(t)$ and $x_Q(t)$.

b) Calculate the Fourier series of $x_z(t)$.

c) Assuming f_c=20Hz calculate the Fourier series of $x_c(t)$

d) Calculate and plot $x_A(t)$. Computer might be useful.

e) Calculate and plot $x_P(t)$. Computer might be useful.

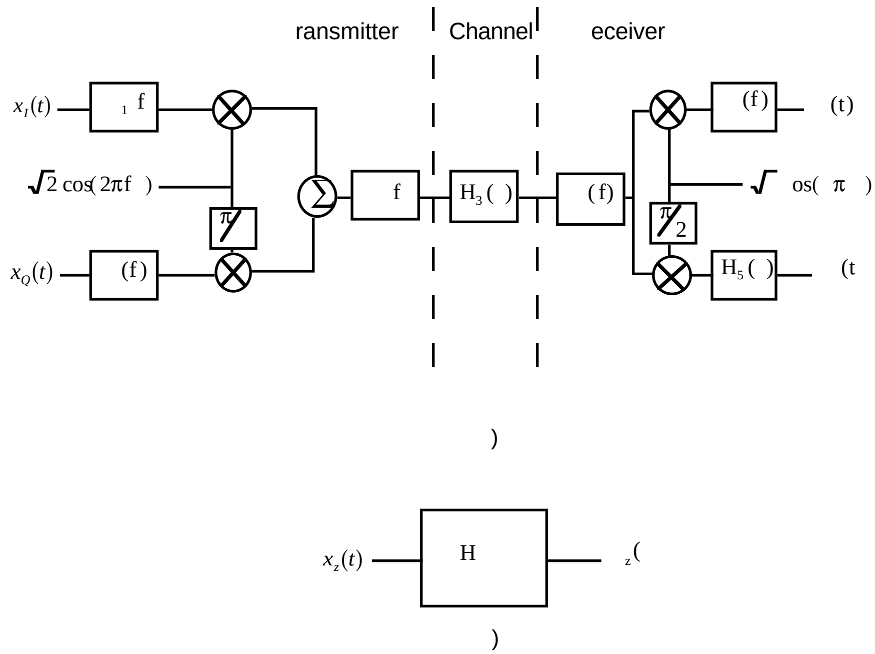


Figure 3.9: A comparison between **a**) the actual communication system model and **b**) the complex baseband equivalent model.

Problem 3.4. A bandpass filter has the following complex envelope representation for the impulse response

$$
\begin{aligned}
h_z(t) &= 2\left(\frac{1}{2}\exp\left[-\frac{t}{2}\right]\right) + j2\left(\frac{1}{4}\exp\left[-\frac{t}{4}\right]\right) \qquad t \geq 0 \\
&= 0 \qquad \text{elsewhere}
\end{aligned}
$$

a) Calculate $H_z(f)$. *Hint: The transforms you need are in a table somewhere.*

b) With $x_z(t)$ from Problem 3.3 as the input, calculate the Fourier series for the filter output, $y_z(t)$.

c) Plot the output amplitude, $y_A(t)$, and phase, $y_P(t)$.

d) Plot the resulting bandpass signal, $y_c(t)$ using f_c=20Hz.

Problem 3.5. The picture of a color television set proposed by the National Television System Committee (NTSC) is composed by scanning in a grid pattern across the screen. The scan is made up of three independent beams (red, green, blue). These independent beams can be combined to make any color at a particular position. In order to make the original color transmission compatible with black and white televisions the three color signals ($x_r(t)$, $x_g(t)$, $x_b(t)$) are transformed into a luminance signal (black and white level), $x_L(t)$, and two independent chrominance signals, $x_I(t)$ and $x_Q(t)$. These chrominance signals are modulated onto a carrier of 3.58MHz to produce a bandpass signal for transmission. A commonly used tool for video engineers to understand this coloring patterns is the vectorscope representation shown in Figure 3.10.

a) If the video picture is making a smooth transition from a blue color (at t=0) to green color (at t=1), make a plot of the waveforms $x_I(t)$ and $x_Q(t)$.

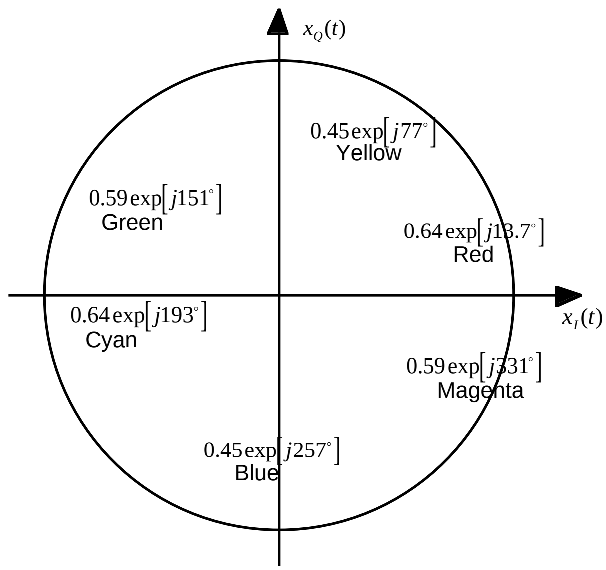


Figure 3.10: Vector scope representation of the complex envelope of the 3.58MHz chrominance carrier.

b) What form would $x_I(t)$ and $x_Q(t)$ have to represents a scan across a red and green striped area. For consistency in the answers assume the red starts and t=0 and extends to $t = 1$, the green starts at $t = 1^+$ and extends to $t = 2, \cdots$.

Problem 3.6. Consider two lowpass spectrum, $X_I(f)$ and $X_Q(f)$ in Figure 3.11 and sketch the energy spectrum of the complex envelope, $G_{x_z}(f)$.

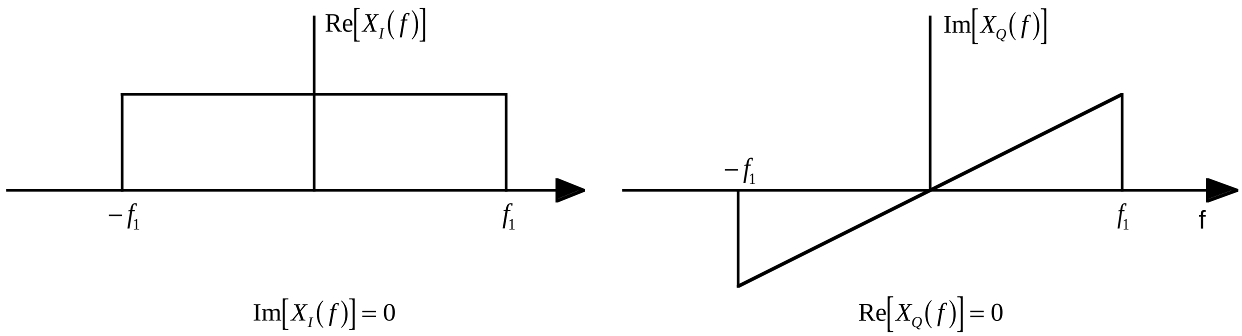


Figure 3.11: Two lowpass Fourier transforms.

Problem 3.7. The lowpass signals, $x_I(t)$ and $x_Q(t)$, which comprise a bandpass signal are given in Figure 3.12.

a) Give the form of $x_c(t)$, the bandpass signal with a carrier frequency f_c, using $x_I(t)$ and $x_Q(t)$.

b) Find the amplitude, $x_A(t)$, and the phase, $x_P(t)$, of the bandpass signal.

c) Give the simplest form for the bandpass signal over $[2T, 3T]$.

Problem 3.8. The amplitude and phase of a bandpass signal is plotted in Figure 3.13. Compute the in-phase and quadrature signals of this baseband representation of a bandpass signal.

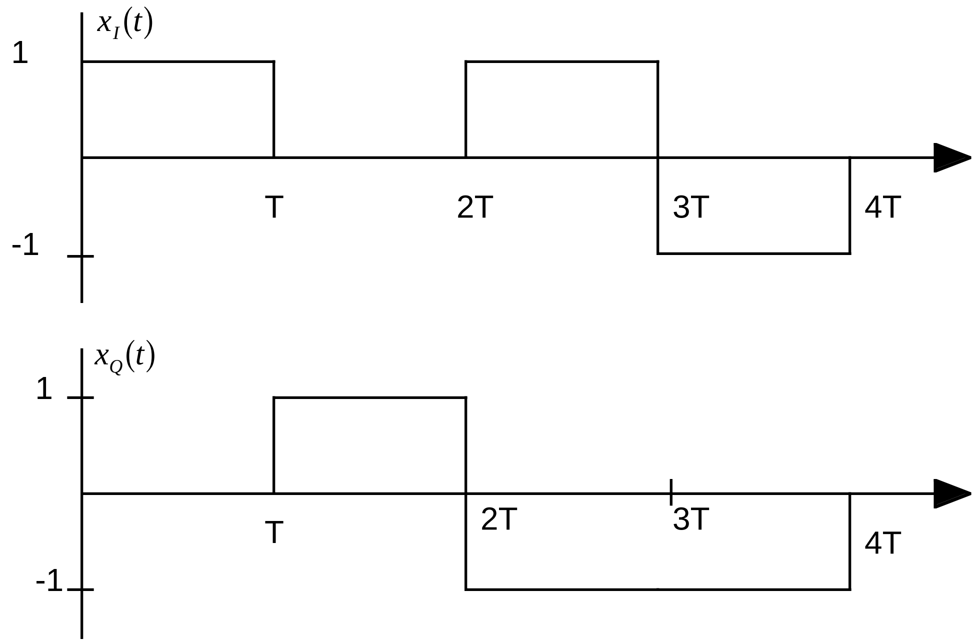


Figure 3.12: $x_I(t)$ and $x_Q(t)$.

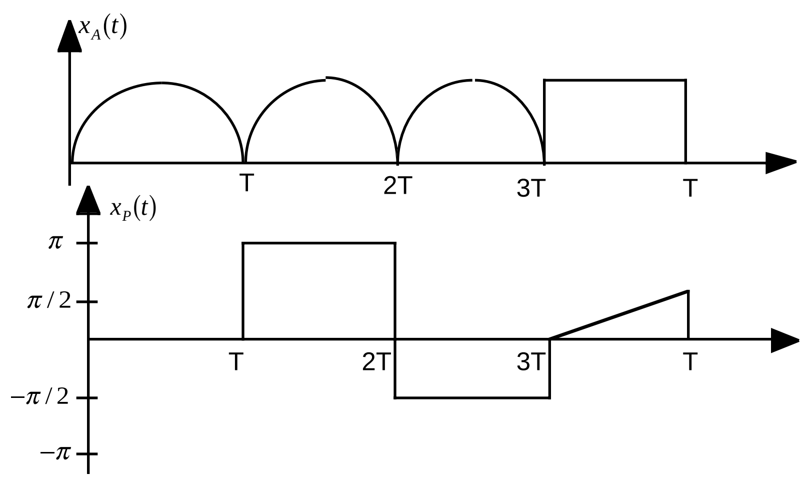


Figure 3.13: The amplitude and phase of a bandpass signal.

Problem 3.9. (Design Problem) A key component in the quadrature up/down converter is the generator of the sine and cosine functions. This processing is represented in Figure 3.14 as a shift in the phase by 90° of a carrier signal. This function is done in digital processing in a trivial way but if the carrier is generated by an analog source the implementation is more tricky. Show that this phase shift can be generated with a time delay as in Figure 3.15. If the carrier frequency is 100MHz find the value of the delay to achieve the 90° shift.

Problem 3.10. The block diagram in Fig. 3.16 shows a cascade of a quadrature upconverter and a quadrature downconverter where the phases of the two (transmit and receive) carriers are not the same.

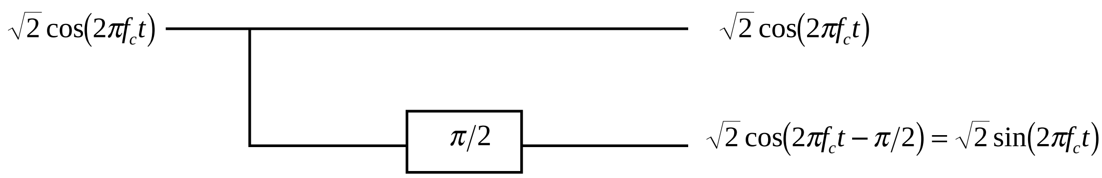


Figure 3.14: Sine and cosine generator.

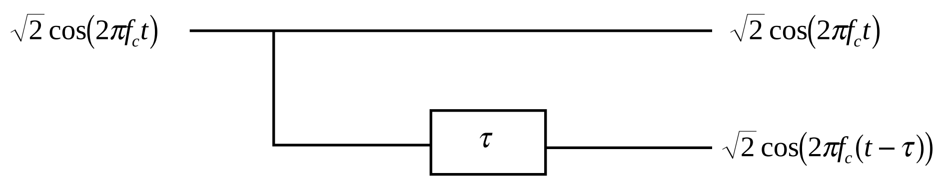


Figure 3.15: Sine and cosine generator implementation for analog signals.

Show that $y_z(t) = x_z(t)\exp[-j\theta(t)]$. Specifically consider the case when the frequencies of the two carriers are not the same and compute the resulting output energy spectrum $G_{Y_z}(f)$.

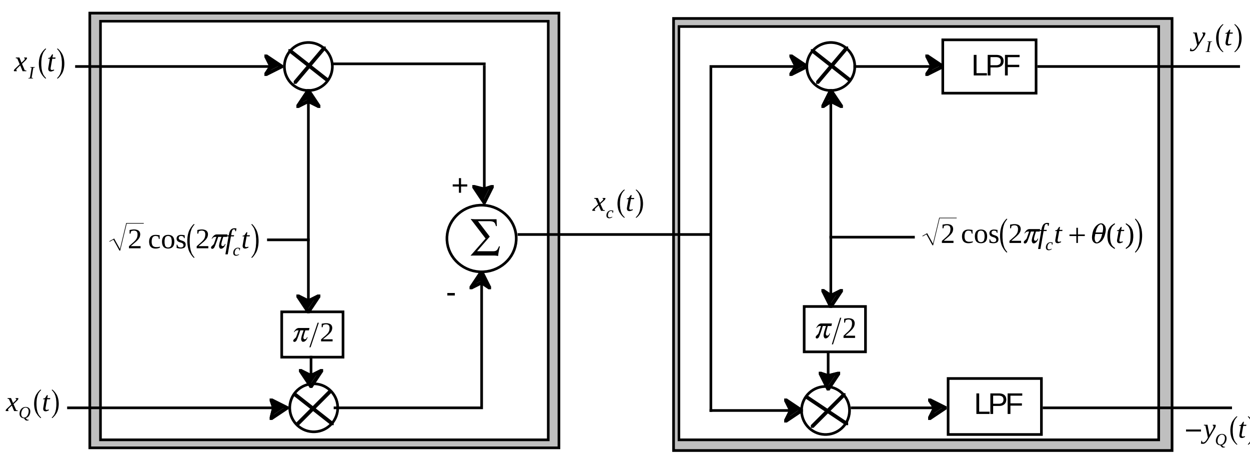


Figure 3.16: A downconverter with a phase offset.

Problem 3.11. A periodic real signal of bandwidth W and period T is $x_I(t)$ and $x_Q(t) = 0$ for a bandpass signal of carrier frequency $f_c > W$.

a) Can the resulting bandpass signal, $x_c(t)$, be periodic with a period of $T_c < T$? If yes give an example.

b) Can the resulting bandpass signal, $x_c(t)$, be periodic with a period of $T_c > T$? If yes give an example.

c) Can the resulting bandpass signal, $x_c(t)$, be periodic with a period of $T_c = T$? If yes give an example.

d) Can the resulting bandpass signal, $x_c(t)$, be aperiodic? If yes give an example.

Problem 3.12. In communication systems bandpass signals are often processed in digital processors. To accomplish the processing the bandpass signal must first be converted from an analog signal to a digital signal. For this problem assume this is done by ideal sampling. Assume the sampling frequency, f_s, is set at four times the carrier frequency.

a) Under what conditions on the complex envelope will this sampling rate be greater than the Nyquist sampling rate (see Section 1.4.1) for the bandpass signal?

b) Give the values for the bandpass signal samples for $x_c(0)$, $x_c\left(\frac{1}{4f_c}\right)$, $x_c\left(\frac{2}{4f_c}\right)$, $x_c\left(\frac{3}{4f_c}\right)$, and $x_c\left(\frac{4}{4f_c}\right)$.

c) By examining the results in b) can you postulate a simple way to downconvert the analog signal when $f_s = 4f_c$ and produce $x_I(t)$ and $x_Q(t)$? This simple idea is frequently used in engineering practice and is known as $f_s/4$ downconversion.

Problem 3.13. A common implementation problem that occurs in an I/Q upconverter is that the sine carrier is not exactly 90° out of phase with the cosine carrier. This situation is depicted in Fig. 3.17.

a) What is the actual complex envelope, $x_z(t)$, produced by this implementation as a function of $\tilde{x}_I(t)$, $\tilde{x}_Q(t)$, and θ?

b) Often in communication systems it is possible to correct this implementation error by preprocessing the baseband signals. If the desired output complex envelope was $x_z(t) = x_I(t) + jx_Q(t)$ what should $\tilde{x}_I(t)$ and $\tilde{x}_Q(t)$ be set to as a function of $x_I(t)$, $x_Q(t)$, and θ to achieve the desired complex envelope with this implementation?

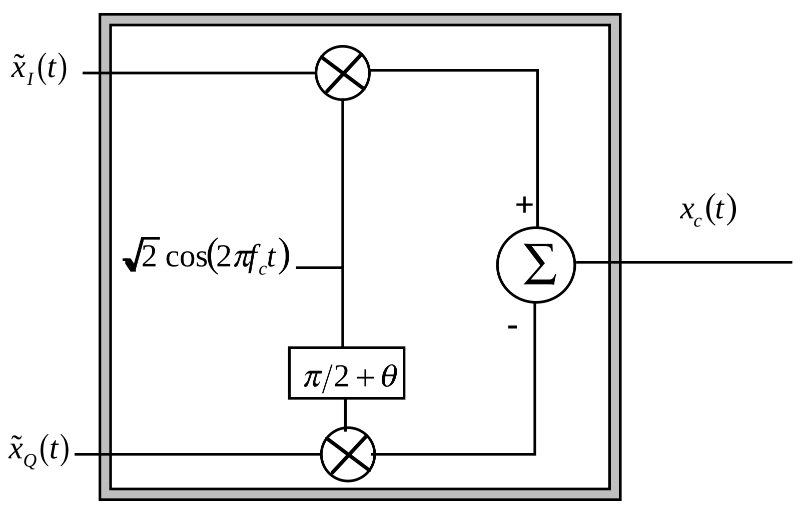


Figure 3.17: The block diagram for Problem 3.13.

Problem 3.14. A commercial airliner is flying 15,000 feet above the ground and pointing its radar down to aid traffic control. A second plane is just leaving the runway as shown in Fig. 3.18. The transmitted waveform is just a carrier tone, $x_z(t) = 1$ or $x_c(t) = \sqrt{2}\cos(2\pi f_c t)$

The received signal return at the radar receiver input has the form

$$y_c(t) = A_P\sqrt{2}\cos(2\pi(f_c + f_P)t + \theta_P) + A_G\sqrt{2}\cos(2\pi(f_c + f_G)t + \theta_G) \tag{3.24}$$

where the P subscript refers to the signal returns from the plane taking off and the G subscript refers to the signal returns from the ground. The frequency shift is due to the Doppler effect you learned about in your physics classes.

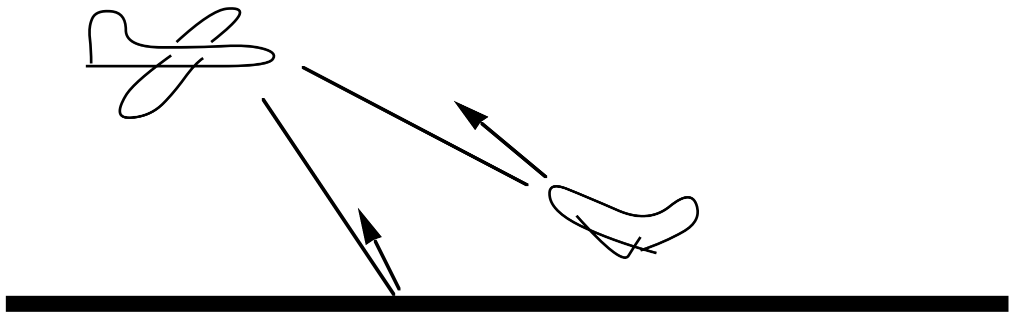


Figure 3.18: An airborne air traffic control radar example.

a) Why does the radar signal bouncing off the ground (obviously stationary) produce a Doppler frequency shift?

b) Give the complex baseband form of this received signal.

c) Assume the radar receiver has a complex baseband impulse response of

$$h(t) = \delta(t) + \beta\delta(t-T) \tag{3.25}$$

where β is a possibly complex constant, find the value of β which eliminates the returns from the ground at the output of the receiver. This system was a common feature in early radar systems and has the common name Moving Target Indicator as stationary target responses will be canceled in the filter given in (3.25).

Modern air traffic control radars are more sophisticated than this problem suggests. An important point of this problem is that radar and communication systems are similar in many ways and use the same analytical techniques for design.

Problem 3.15. A baseband signal (complex exponential) and linear system are shown in Fig. 3.19. The linear system has an impulse response of

$$\begin{aligned} h_z(t) &= \frac{1}{\sqrt{T}} \qquad 0 \le t \le T \\ &= 0 \qquad \text{elsewhere} \end{aligned} \tag{3.26}$$

a) Describe the kind of filter that $h_z(t)$ represents at bandpass.

b) What is the input power? Compute $y_z(t)$.

c) Select a delay, τ_d, such that $\arg\left[y_z(t)\right] = 2\pi f_0(t-\tau_d)$ for all f_0.

d) How large can f_0 be before the output power is reduced by 10dB compared to the input power?

Problem 3.16. The following bandpass filter has been implemented in a communication system that you have been tasked to simulate:

$$H_c(f) = \begin{cases} 1 & f_c + 7500 \le |f| \le f_c + 10000 \\ 2 & f_c + 2500 \le |f| < f_c + 7500 \\ \frac{4}{3} & f_c \le |f| < f_c + 2500 \\ \frac{3}{4} & f_c - 2500 \le |f| < f_c \\ 0 & \text{elsewhere} \end{cases}$$

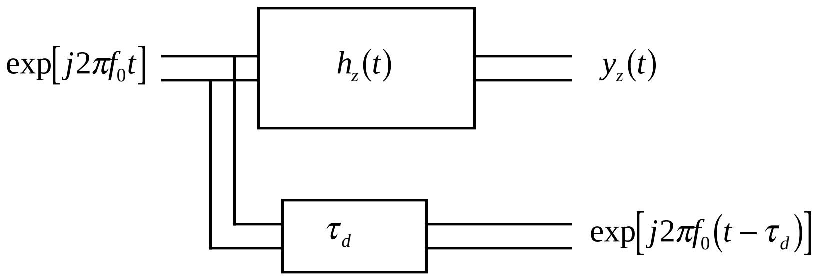


Figure 3.19: The block diagram for Problem 3.15.

You know because of your great engineering education that it will be much easier to simulate the system using complex envelope representation.

a) Find $H_z(f)$.

b) Find $H_I(f)$ and $H_Q(f)$.

c) If $x_z(t) = \exp(j2\pi f_m t)$ compute $y_z(t)$ for $2000 \leq f_m < 9000$.

3.7 Example Solutions

Problem 3.2.

a) Using $\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$ gives

$$x_c(t) = \sin(2\pi f_c t)\cos(2\pi f_m t) - \cos(2\pi f_c t)\sin(2\pi f_m t). \tag{3.27}$$

By inspection we have

$$x_I(t) = \frac{-1}{\sqrt{2}}\sin(2\pi f_m t) \qquad x_Q(t) = \frac{-1}{\sqrt{2}}\cos(2\pi f_m t). \tag{3.28}$$

b) Recall $x_c(t) = x_A(t)\sqrt{2}\cos(2\pi f_c t + x_P(t))$ so by inspection we have

$$x_z(t) = \frac{1}{\sqrt{2}}\exp(j2\pi f_m t) \qquad x_I(t) = \frac{1}{\sqrt{2}}\cos(2\pi f_m t) \qquad x_Q(t) = \frac{1}{\sqrt{2}}\sin(2\pi f_m t). \tag{3.29}$$

b) Recall $x_c(t) = x_A(t)\sqrt{2}\cos(2\pi f_c t + x_P(t))$ so by inspection we have

$$x_z(t) = \frac{1}{\sqrt{2}}\exp(j\phi_p) \qquad x_I(t) = \frac{1}{\sqrt{2}}\cos(\phi_p) \qquad x_Q(t) = \frac{1}{\sqrt{2}}\sin(\phi_p). \tag{3.30}$$

3.8 Mini-Projects

Goal: To give exposure

1. to a small scope engineering design problem in communications
2. to the dynamics of working with a team
3. to the importance of engineering communication skills (in this case oral presentations).

Presentation: The forum will be similar to a design review at a company (only much shorter) The presentation will be of 5 minutes in length with an overview of the given problem and solution. The presentation will be followed by questions from the audience (your classmates and the professor). All team members should be prepared to give the presentation.

Project 3.1. In engineering often in the course of system design or test anomalous performance characteristics often arise. Your job as an engineer is to identify the causes of these characteristics and correct them. Here is an example of such a case.

Get the Matlab file `chap3ex1.m` from the class web page. In this file the carrier frequency was chosen as 7KHz. If the carrier frequency is chosen as 8KHz an anomalous output is evident from the quadrature downconverter. This is most easily seen in the output energy spectrum, $G_{y_z}(f)$. Postulate a reason why this behavior occurs. *Hint:* It happens at 8KHz but not at 7KHz and Matlab is a sampled data system. What problems might one have in sampled data system? Assume that this upconverter and downconverter were products you were designing how would you specify the performance characteristics such that a customer would never see this anomalous behavior?

Chapter 4

Analog Communications Basics

Analog communication involves transferring an analog waveform containing information (no digitization at any point) between two users. Typical examples where analog information is transmitted in this fashion are:

1. Music - Broadcast radio,
2. Voice - Citizen band radio, Amateur radio, Walkie-Talkies, Cellular radio,
3. Video - Broadcast television.

4.1 Message Signal Characterization

The information bearing analog waveform is denoted $m(t)$ and it is assumed to be an energy signal with a Fourier transform and energy spectral density of $M(f)$ and $G_m(f)$ respectively. Fig. 4.1 and Fig. 4.2 show a short time record and an energy spectral density of an analog message signal that this text will use to demonstrate modulation concepts. Rarely is a DC value an important component in an analog message signal so we will assume that the energy spectrum of the message signal goes to zero for small frequencies and that the message signal will pass through a DC block unchanged. In

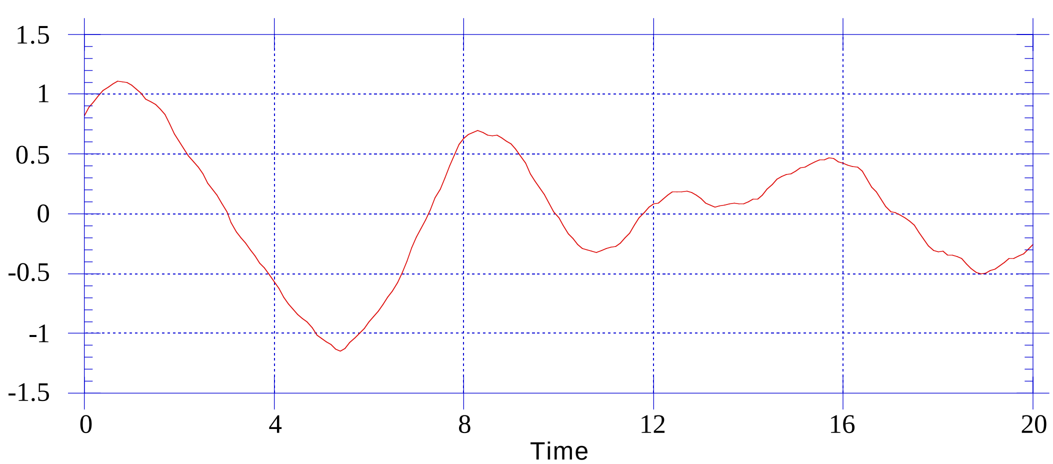


Figure 4.1: An example message signal, $m(t)$.

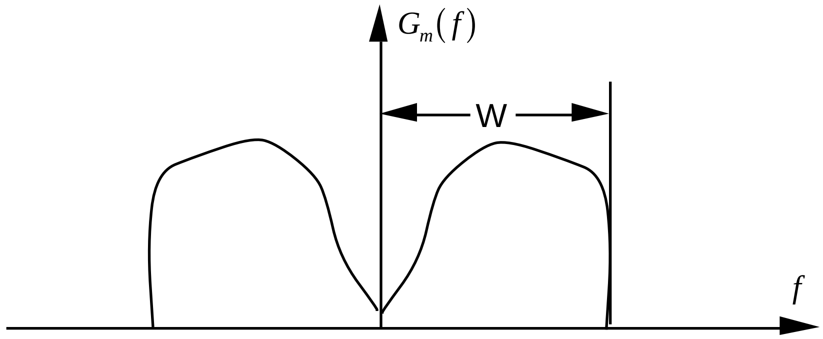


Figure 4.2: An example of a message energy spectral density.

analog modulations it is often important to discuss the signal bandwidth and this will be denoted W. Note that this W could correspond to either relative or integral bandwidth and will not be specified unless necessary for clarity. Also average SNR is important in analog communication and an important quantity to establish this SNR is the time average message power, P_m, defined as

$$P_m = \lim_{T \to \infty} P_m(T) \tag{4.1}$$

where P_m is defined in (1.4) and $P_m(T)$ is defined in (1.37). The message signal is assumed to be a real valued signal.

Example 4.1: The computer generated voice signal whose spectrum was shown in Fig. 1.6. Note this signal has a notch in the energy spectrum at DC so that it will pass through a DC block with little distortion. The bandwidth is given as W=2.5KHz.

4.2 Analog Transmission

The conventional communication system has a modulator producing a signal that is transmitted over a channel (a cable or radio propagation) and demodulator which takes this signal and constructs an estimate of the transmitted message signal. Fig. 4.3 is a block diagram of this system where $r_c(t)$ is the output of the channel and $\hat{m}(t)$ is the estimate of the transmitted message signal. Noise in communication systems will be characterized later (see Chapter 10). Communication systems engineers design and optimize the modulators and demodulators in such systems.

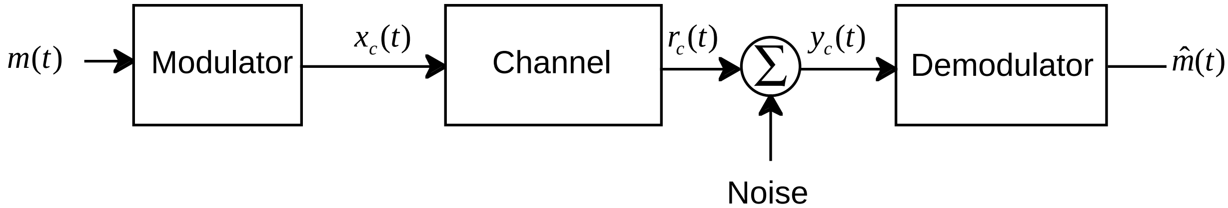


Figure 4.3: An analog communication system block diagram.

4.2.1 Analog Modulation

Definition 4.1 *Analog modulation is a transformation of $m(t)$ into a complex envelope, $x_z(t)$.*

This transformation is equivalent to transforming $m(t)$ into a bandpass signal, $x_c(t)$. The analog modulation process, $x_z(t) = \Gamma_m(m(t))$ is represented in Fig. 4.4[1]. Historically analog modulations were invented long before the invention of the transistor (hence large scale integration) so many of the commonly used analog modulations evolved because of the simplicity of the implementation. While the importance of analog modulation is decreasing in an ever increasingly digital world, analog modulation is still used in many important applications and serves as a good introduction to the idea of the modulation process. Hopefully these comments will become more clear in the remainder of this chapter.

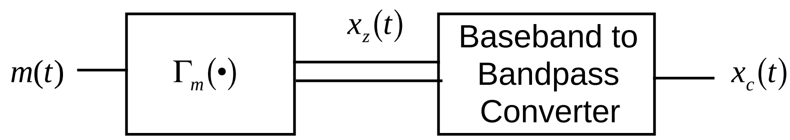


Figure 4.4: The analog modulation process. Note the baseband to bandpass converter is given in Fig. 3.4

4.2.2 Analog Demodulation

Demodulation is essentially taking the received signal, $y_c(t)$, and producing an estimate of the transmitted message signal, $m(t)$. This estimate will be denoted $\hat{m}(t)$. Since this is an introductory course, the channel output is always assumed to be

$$r_c(t) = L_p x_c(t - \tau_p)$$

where L_p is the propagation loss and τ_p is the propagation time delay. Define $\phi_p = -2\pi f_c \tau_p$ so that the channel output is given as

$$\begin{aligned} r_c(t) &= \sqrt{2} L_p x_A(t-\tau_p) \cos\left(2\pi f_c(t-\tau_p) + x_P(t-\tau_p)\right) \\ &= \Re\left[\sqrt{2} L_p x_z(t-\tau_p) \exp[j\phi_p] \exp[j2\pi f_c t]\right]. \end{aligned} \tag{4.2}$$

It is obvious from (4.2) that the received complex envelope is $r_z(t) = L_p x_z(t-\tau_p)\exp[j\phi_p]$. It is important to note that a time delay in a carrier modulated signal will produce a phase offset. Consequently the demodulation process conceptually is a down conversion to baseband and a reconstruction of the transmitted signal from $y_z(t)$. The block diagram for the demodulation process is seen in Fig. 4.5.

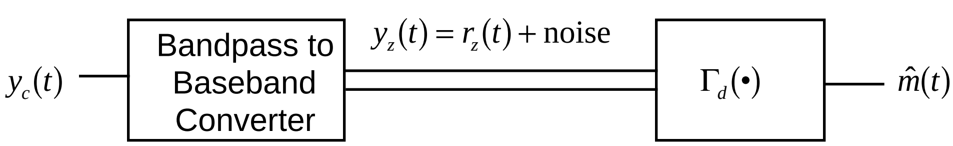


Figure 4.5: The analog demodulation process. Note the bandpass to baseband converter is given in Fig. 3.4.

[1]In figures single lines will denote real signals and double lines will denote complex analytical signals.

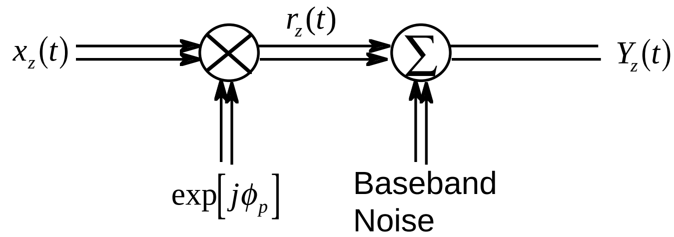


Figure 4.6: The equivalent complex baseband channel model.

Example 4.2: Radio broadcast. For a carrier frequency of 100MHz and a receiver 30 kilometers from the transmitter we have

$$\tau_p = \frac{\text{distance}}{c} = \frac{3 \times 10^4}{3 \times 10^8} = 100\mu\text{s} \qquad \phi_p = -2\pi(10^8)(10^{-4}) = -2\pi \times 10^4\text{radians.} \tag{4.3}$$

For the example of radio broadcast a typical channel produces a relatively short (perhaps imperceivable) delay but a very large phase shift.

Example 4.3: An example that will be used in the sequel has a carrier frequency of 7KHz and a propagation delay of $\tau_p = 45.3\mu$s gives

$$\phi_p = -2\pi(7000)(0.0000453) = -1.995\text{radians} = -114^\circ. \tag{4.4}$$

The following further simplifications will be made when discussing analog demodulation

1. A time delay in the output is unimportant. This is typically true in analog modulation especially since $\tau_p << 1$ second.

2. An amplitude gain in the output is unimportant. Output amplitude control is a function of the receiver circuitry and will not be discussed in detail.

Consequently $r_z(t) = x_z(t)\exp[j\phi_p]$, where ϕ_p is an unknown constant, will be used for the channel output for the remainder of the discussion of analog modulation and demodulation. Fig. 4.6 is a diagram of the equivalent complex baseband channel model. Consequently demodulation can be thought of as the process of producing an $\hat{m}(t)$ from $y_z(t)$ via a function $\Gamma_d(y_z(t))$. The remainder of the discussion on analog modulations in this text will focus on identifying $\Gamma_m(m(t))$ (modulators) and $\Gamma_d(y_z(t))$ (demodulators) and assessing the performance of these modulators and demodulators in the presence of noise.

4.3 Performance Metrics for Analog Communication

In evaluating the efficacy of various designs in this course the performance metrics commonly applied in engineering design must be examined. The most commonly used metrics for analog communications are

- Complexity – This metric almost always translates directly into cost.

- Performance – This metric typically measures how accurate the received message estimate is given the amount of transmitted power.

- Spectral Efficiency – This metric measures how much bandwidth a modulation uses to implement the communication.

Complexity is a quantity that requires engineering judgment to estimate. Often the cost of a certain level of complexity changes over time. What were seen as good design choices in the 1930's when vacuum tubes were prevalent in terms of complexity seem somewhat silly now in the days of integrated circuits.

The performance of a communication system is typically a measure of how well the message is reconstructed at the receiver. The message estimate has the form

$$\hat{m}(t) = Am(t) + N_L(t)$$

where A is an amplitude gain on the signal and $N_L(t)$ is a combination of signal distortion and random noise produced in the demodulation. Engineers have many ways to characterize the fidelity of the estimate, $\hat{m}(t)$. In this introductory course we will concentrate on the idea of signal-to-noise power ratio (SNR). The SNR is defined to be

$$\mathrm{SNR} = \frac{A^2 P_m}{P_N}$$

where P_m is defined in (4.1) and the noise or distortion power, P_N, will require the tools in Chapter 8 to define. Note that SNR is a function of both the received message power and the noise or distortion power. The message power at the output of the demodulator is a function of the processing that happens in the communication system. As a measure of the effectiveness of that processing, the transmission efficiency of an analog communication scheme is defined as

$$E_T = \frac{A^2 P_m}{P_{x_z}}$$

where the numerator is the signal power that arrives at the output of the demodulator and the denominator is the transmitted power. This efficiency measures how much of the transmitted signal power is translated into demodulated signal power.

The spectral efficiency of a communication system is typically a measure of how well the system is using the bandwidth resource. Bandwidth costs money to acquire. Examples are licenses to broadcast radio signals or the installation of copper wires to connect two points. Hence spectral efficiency is very important for people who try to make money selling communication resources. For instance if one communication system has a spectral efficiency that is twice the spectral efficiency of a second system then the first system can support twice the users on the same bandwidth. Twice the users implies twice the revenue. The measure of spectral efficiency that we will use in this class is called bandwidth efficiency and is defined as

$$E_B = \frac{W}{B_T}$$

where W is the message bandwidth and B_T is the transmission bandwidth.

As the the different analog modulation techniques are discussed in this class, constant comparisons will be made to the complexity, the transmission efficiency, the performance, and the spectral efficiency. This will help make the tradeoffs available in the communication system readily apparent.

4.4 Power of Carrier Modulated Signals

The output power of a carrier modulated signal is often important in evaluating the tradeoffs between analog communication options. In this text when power is considered this will refer to the time average power. The time average power is defined as

$$P_{x_c} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x_c^2(t)dt. \tag{4.5}$$

Using the Rayleigh energy theorem and (1.39) the time average power of a bandpass modulated signal is

$$P_{x_c} = \lim_{T\to\infty} \int_{-T}^{T} S_{X_c}(f,T)dt. \tag{4.6}$$

Using the power spectrum analog to (3.12) gives

$$P_{x_c} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x_z^2(t)dt = P_{x_z}. \tag{4.7}$$

Hence we have shown that the power contained in a carrier modulated signal is exactly the power in the complex envelope. This result is the main reason why the notation used in this text uses the $\sqrt{2}$ term in the carrier terms.

4.5 Homework Problems

Problem 4.1. Many characteristics of the message signal appear prominently in the ways we characterize analog modulations. A message signal given as

$$m(t) = -\cos(200\pi t) + \sin(50\pi t)$$

is to be transmitted in an analog modulation. A plot of the signal is given in Figure 4.7.

a) This signal is periodic. What is the period?

b) Give the Fourier series coefficients for $m(t)$.

c) Compute the power of $m(t)$, P_m.

d) Compute the $\min m(t)$.

e) Compute the $\max |m(t)|$.

f) Compute the $\max \left|\frac{d}{dt}m(t)\right|$.

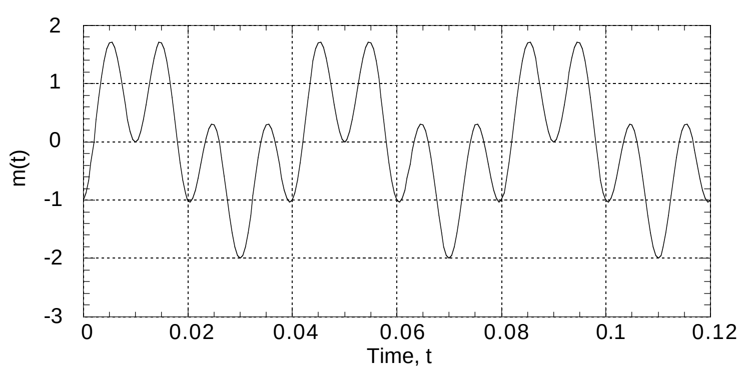


Figure 4.7: An example message signal.

Problem 4.2. A message signal of the form shown in Fig. 4.8 has a series expansion given as

$$m(t) = \frac{8}{\pi^2} \sum_{i=1}^{\infty} \frac{1}{(2n-1)^2} \cos\left(\frac{2\pi(2n-1)t}{T}\right) \tag{4.8}$$

a) Select T such that the fundamental frequency of the Fourier series expansion is 2Hz.

b) If m_k represents the Fourier series coefficients, what is m_0? What is m_1?.

c) Compute the power of $m(t)$, P_m.

d) For transmission in an analog communication system it is desired to limit the bandwidth to 9Hz. Using the T from part a) compute how many terms in the Fourier series will remain after this bandlimiting operation. Give their frequencies and Fourier coefficients.

e) Compute the resulting message power after bandlimiting.

f) Plot the message signal after bandlimiting.

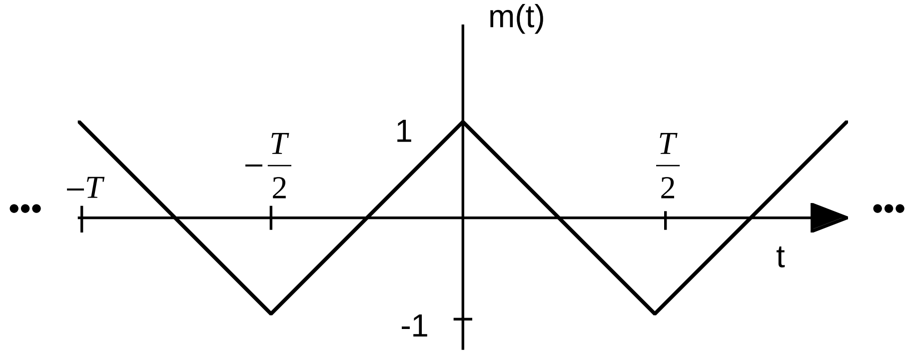


Figure 4.8: A message signal.

Problem 4.3. A message signal is to be transmitted using analog modulation. The message signal Fourier transform has the form

$$\begin{aligned} M(f) &= A\left|\sin\left(\frac{\pi f}{W}\right)\right| \qquad |f| \leq W \\ &= 0 \qquad \text{elsewhere} \end{aligned} \tag{4.9}$$

a) Compute the value of A such that E_m is equal to 1 Joule in a 1Ω system.

b) Compute the $\min m(t)$.

c) Compute the $\max |m(t)|$.

d) Compute the $\max \left|\frac{d}{dt}m(t)\right|$.

Problem 4.4. A message signal given as

$$m(t) = -\cos(200\pi t) + \sin(50\pi t)$$

is input into the system shown in Fig. 4.9 where

$$H_R(f) = \frac{2\pi 20}{j2\pi f + 2\pi 20} \tag{4.10}$$

a) Find the output signal.

b) If $\hat{m}(t) = Am(t) + n_L(t)$ for a chosen value of A find the power of the distortion $n_L(t)$.

c) Find the value of A that minimizes the distortion power.

d) Redesign $H_R(f)$ to reduce the effective distortion power.

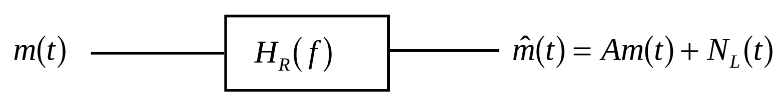


Figure 4.9: An example of filtering a message signal.

Chapter 5

Amplitude Modulation

Amplitude modulation was historically the first modulation developed and conceptually the simplest to understand. Consequently this modulation is developed first.

5.1 Linear Modulation

The simplest analog modulation is to make $\Gamma_m(m(t)) = A_c m(t)$, i.e., a linear function of the message signal. The complex envelope and the spectrum of this modulated signal are given as

$$x_z(t) = A_c m(t) \qquad\qquad G_{x_z}(f) = A_c^2 G_m(f).$$

This modulation has $x_I(t) = A_c m(t)$ and $x_Q(t) = 0$, so the imaginary portion of the complex envelope is not used in a linear analog modulation. The resulting bandpass signal and spectrum are given as

$$x_c(t) = \Re\left[\sqrt{2}x_z(t)\exp[j2\pi f_c t]\right] = A_c m(t)\sqrt{2}\cos(2\pi f_c t) \tag{5.1}$$

$$G_{x_c}(f) = \frac{A_c^2}{2}G_m(f-f_c) + \frac{A_c^2}{2}G_m(-f-f_c) = \frac{A_c^2}{2}G_m(f-f_c) + \frac{A_c^2}{2}G_m(f+f_c) \tag{5.2}$$

where the fact that $m(t)$ was real was used to simplify (5.2). Fig. 5.1 shows the complex envelope and an example bandpass signal for the message signal shown in Fig. 4.1 with $A_c = 1/\sqrt{2}$. It is quite obvious from Fig. 5.1 that the amplitude of the carrier signal is modulated directly proportional to the absolute value of the message signal. Fig. 5.2 shows the resulting energy spectrum of the linearly modulated signal for the message signal shown in Fig. 4.2. A very important characteristic of this modulation is that if the message signal has a bandwidth of W Hz then the bandpass signal will have a transmission bandwidth of $B_T = 2W$. This implies E_B=50%. Because of this characteristic the modulation is often known as double sideband-amplitude modulation (DSB-AM). An efficiency of 50% is wasteful of the precious spectral resources but obviously the simplicity of the modulator is a positive attribute.

The output power of a DSB-AM modulator is often of interest (e.g., to specify the characteristics of amplifiers or to calculate the received signal-to-noise ratio). To this end the power is given as

$$P_{x_c} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x_c^2(t)dt = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} A_c^2 m^2(t) 2\cos^2(2\pi f_c t)dt = P_m A_c^2 \tag{5.3}$$

For a DSB-AM modulated waveform the output power is usually given as the product of the power associated with the carrier amplitude (A_c^2) and the power in the message signal (P_m).

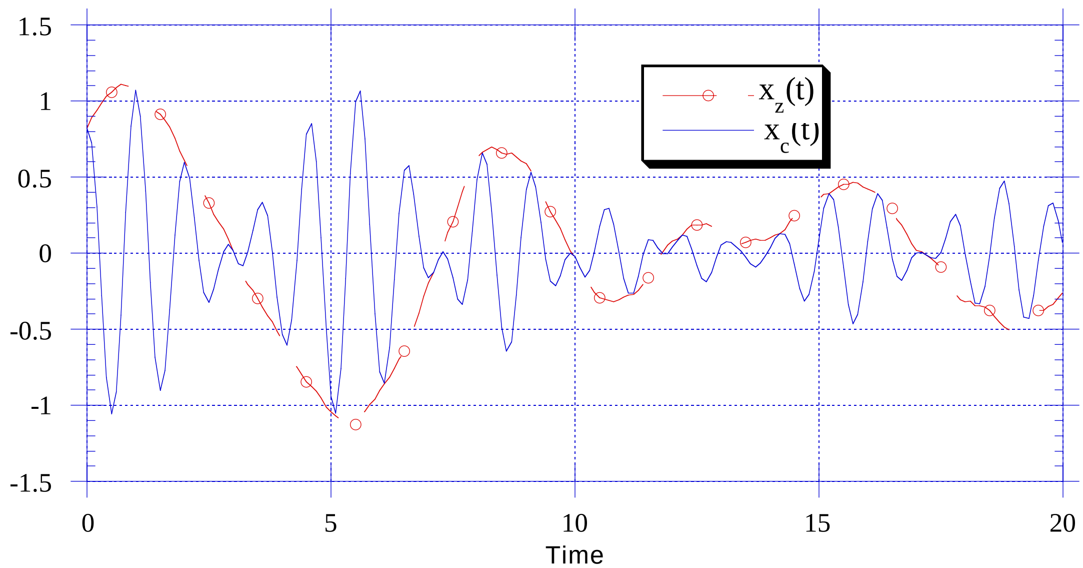


Figure 5.1: Example waveforms for linear analog modulation for the message signal in Fig. 4.1.

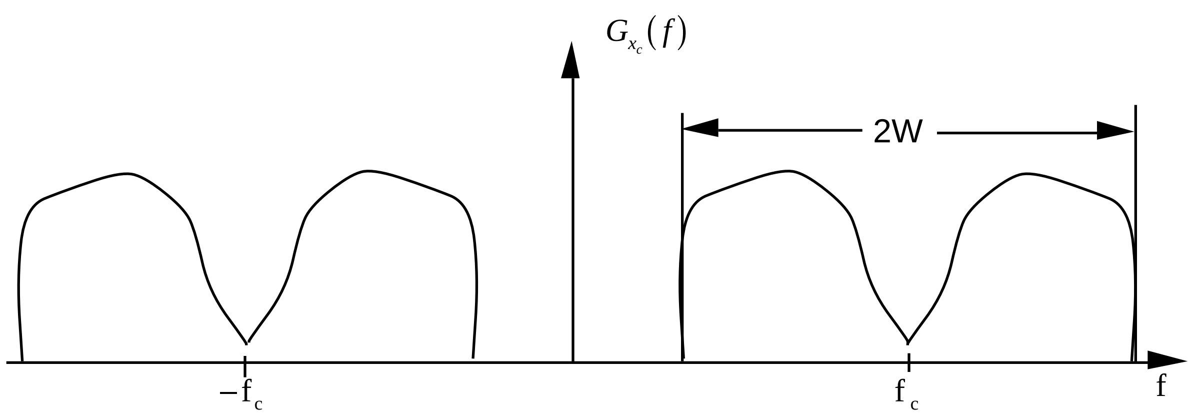


Figure 5.2: The energy spectrum of the linearly modulated signal for the message signal spectrum in Fig. 4.2.

Example 5.1: Linear modulation with

$$m(t) = \beta \sin(2\pi f_m t) \qquad G_m(f) = \frac{\beta^2}{4}\delta(f - f_m) + \frac{\beta^2}{4}\delta(f + f_m)$$

produces

$$x_c(t) = A_c \beta \sin(2\pi f_m t)\sqrt{2}\cos(2\pi f_c t)$$

and

$$G_{x_c}(f) = \frac{A_c^2\beta^2}{8}\left(\delta(f - f_m - f_c) + \delta(f + f_m - f_c) + \delta(f - f_m + f_c) + \delta(f + f_m + f_c)\right).$$

The output power is

$$P_{x_c} = \frac{A_c^2\beta^2}{2}$$

5.1.1 Modulator and Demodulator

The modulator for a DSB-AM signal is simply the structure in Fig. 3.4 except with DSB-AM there is no imaginary part to the complex envelope. Fig. 5.3 shows the simplicity of this modulator.

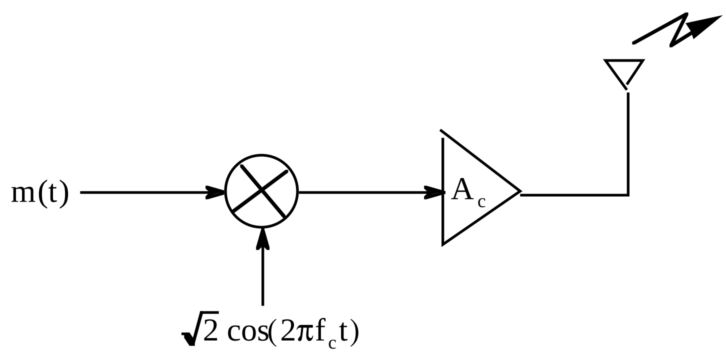


Figure 5.3: A DSB-AM modulator.

Example 5.2: The computer generated voice signal given in Example 1.15(W=2.5KHz) is used to DSB-AM modulate a 7KHz carrier. A short time record of the message signal and the resulting modulated output signal is shown in Fig. 5.5-a). The energy spectrum of the signal is shown in Fig. 5.5-b). Note the bandwidth of the carrier modulated signal is 5KHz.

Demodulation can be accomplished in a very simple configuration for DSB-AM. Given the channel model in Fig. 4.6 a straightforward demodulator is seen in Fig. 5.4. This demodulator simply derotates the received complex envelope by the phase induced by the propagation delay and uses the real part of this derotated signal as the estimate of the message signal. A lowpass filter is added to give noise immunity and the effects of this filter will be discussed later. Note output of the demodulator is given as

$$\hat{m}(t) = A_c m(t) + N_I(t)$$

so that

$$E_T = 100\%.$$

An advantage of DSB-AM is that all transmitted power is directly utilized at the output of the demodulator.

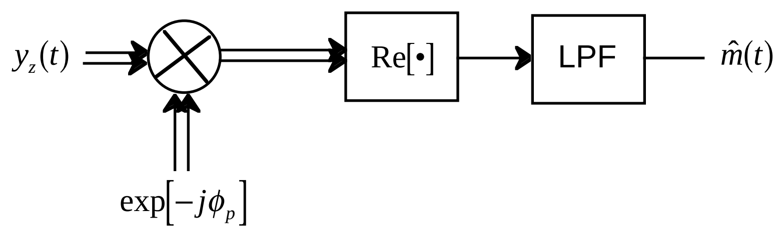
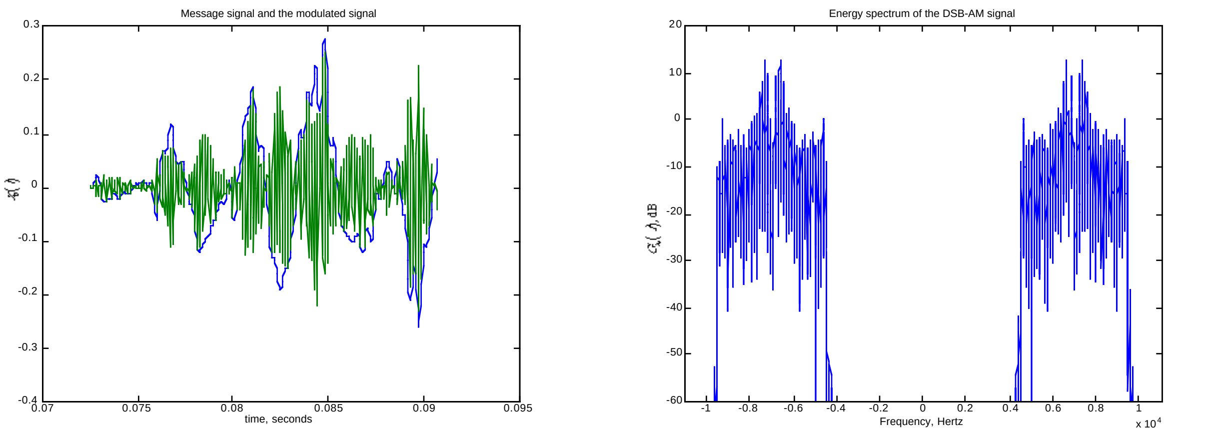


Figure 5.4: The block diagram of an DSB-AM demodulator.



a) A short time record of the message signal, $m(t)$, and the corresponding modulated signal.

b) An energy spectrum of the DSB-AM signal.

Figure 5.5: Example of DSB-AM with the message signal given in Example 1.15. f_c=7KHz.

5.1.2 Coherent Demodulation

An important function of a DSB-AM demodulator is producing the appropriate value of ϕ_p for good message reconstruction. Demodulators that require an accurate phase reference like DSB-AM requires are often call phase **coherent** demodulators. Often in practice this phase reference is obtained manually with a tunable phase shifter. This is unsatisfactory if one or both ends of the link are moving (hence a drifting phase) or if automatic operation is desired.

Automatic phase tracking can be accomplished in a variety of ways. The techniques available for automatic phase tracking are easily divided into two sets of techniques: a phase reference derived from a transmitted reference and a phase reference derived from the received modulated signal. Note a transmitted reference technique will reduce E_T since the transmitted power used in the reference signal is not available at the output of the demodulator. Though a transmitted reference signal is wasteful of transmitted power it is often necessary for more complex modulation schemes (e.g., see Section 5.3.3). For each of these above mentioned techniques two methodologies are typically followed in deriving a coherent phase reference; open loop estimation and closed loop or phase-locked estimation. Consequently four possible architectures are available for coherent demodulation in analog communications.

An additional advantage of DSB-AM is that the coherent reference can easily be derived from the received modulated signal. Consequently in the remainder of this section the focus of the discussion will be on architectures that enable automatic phase tracking from the received modulated signal for DSB-AM. The block diagram of a typical open loop phase estimator for DSB-AM is shown in Fig. 5.6. The essential idea in open loop phase estimation for DSB-AM is that any channel induced phase rotation

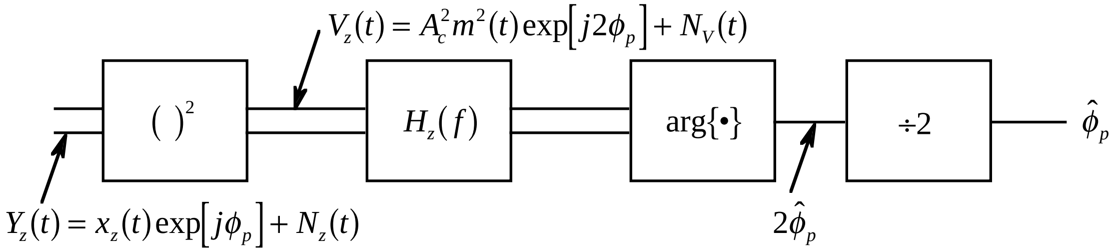


Figure 5.6: An open loop phase estimator for DSB-AM.

can easily be detected since DSB-AM only uses the real part of the complex envelope. Note that the received DSB-AM signal has the form

$$y_z(t) = x_z(t)\exp\left[j\phi_p\right] = A_c m(t)\exp\left[j\phi_p\right]. \tag{5.4}$$

The phase of $y_z(t)$ (in the absence of noise) will either take value of ϕ_p (when $m(t) > 0$) or $\phi_p + \pi$ (when $m(t) < 0$). Squaring the signal gets rid of this bi-modal phase characteristic as can be seen by examining the signal

$$v_z(t) = \left(x_z(t)\exp\left[j\phi_p\right]\right)^2 = A_c^2 m^2(t)\exp\left[j2\phi_p\right] \tag{5.5}$$

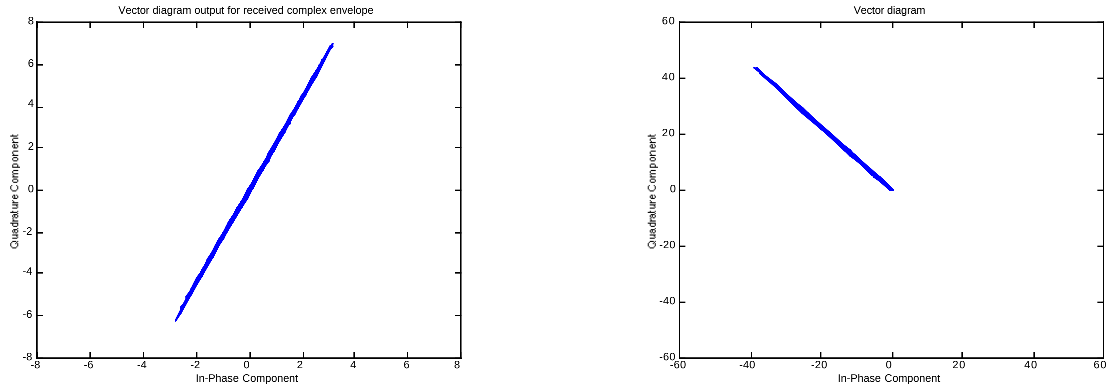
because $A_c^2 m^2(t) > 0$ so the $\arg\left(v_z(t)\right) = 2\phi_p$. In Fig. 5.6 the filtering, $H_z(f)$, is used to smooth the phase estimate in the presence of noise.

Example 5.3: Consider the DSB-AM with the computer generated voice signal given in Example 5.2 with a carrier frequency of 7KHz and a propagation delay in the channel of 45.6μs. This results in a $\phi_p = -114°$ (see Example 5.0.) The vector diagram which is a plot of $x_I(t)$ versus $x_Q(t)$ is a useful tool for understanding the operation of the carrier phase recovery system. The vector diagram was first introduced in Problem 3.6. The vector diagram of the transmitted signal will be entirely on the x-axis since $x_Q(t)$=0. The plot of the vector diagram for the channel output (in the absence of noise) is shown in Fig. 5.7-a). The $-114°$ phase shift is evident from this vector diagram. The output vector diagram from the squaring device is shown in Fig. 5.7-b). This signal now has only one phase angle ($2 \times -114°$) and the coherent phase reference for demodulation can now be easily obtained.

A phase-locked type of phase tracking can be implemented in a DSB-AM demodulator with a Costas loop which has a block diagram given in Fig. 5.8. The Costas loop is a form of a phase-locked loop (PLL). The PLL is a feedback system that consists of three components: a voltage controlled oscillator, a phase detector, and a loop filter. The basic idea in a Costas loop demodulator is the phase detector measures the phase difference between a locally the generated phase reference and the incoming received signal. The feedback system attempts to drive this phase difference to zero and hence implement carrier phase tracking. A simple theory of operation for the PLL will be explored in Chapter 7. The Costas loop has all the components of a PLL and ideas behind a Costas loop synchronous AM demodulator is explored in the homework (see Problem 5.3).

5.1.3 DSB-AM Conclusions

The advantages of DSB-AM are that DSB-AM is very simple to generate, see Fig. 5.3, and it has E_T =100%. The disadvantages of DSB-AM are that phase coherent demodulation is required (relatively complex demodulator) and E_B =50% (wasteful of bandwidth).



a) The vector diagram for the received signal

b) The vector diagram for the output of the squaring device

Figure 5.7: The vector diagram for coherent DSB-AM demodulation. $\phi_p = -114°$.

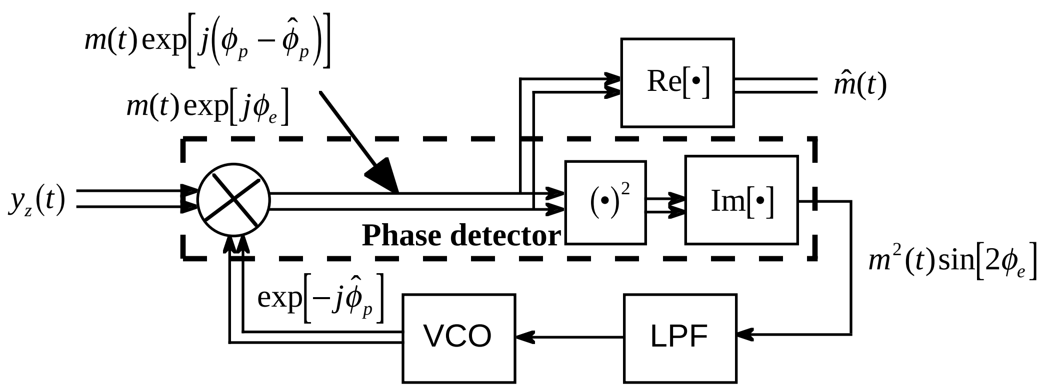


Figure 5.8: The block diagram of a Costas loop for synchronous DSB-AM demodulation.

5.2 Affine Modulation

While now in the age of large scale integrated circuits it may be hard to fathom, when broadcast radio was being developed DSB-AM was determined to have too complex a receiver to be commercially feasible. The coherent demodulator discussed in Section 5.1.2 was too complex in the days of the vacuum tubes. Early designers of radio broadcast systems noted that the message signal modulates the envelope of the bandpass signal, $x_A(t)$. In fact if the message signal never goes negative the envelope of the bandpass signal and the message are identical up to a multiplicative constant. Since an envelope detector is a simple device to build, these early designers formulated a modulation scheme that could use envelope detectors to reconstruct the message signal at the receiver.

This desired characteristic is obtained if a DC signal is added to the message signal to guarantee that the resulting signal always is positive. This implies the complex envelope is an affine[1] function of the message signal, i.e.,

$$x_z(t) = A_c(1 + am(t)) \qquad G_{x_z}(f) = A_c^2[\delta(f) + a^2 G_m(f)]$$

[1]Affine is a linear term plus a constant term.

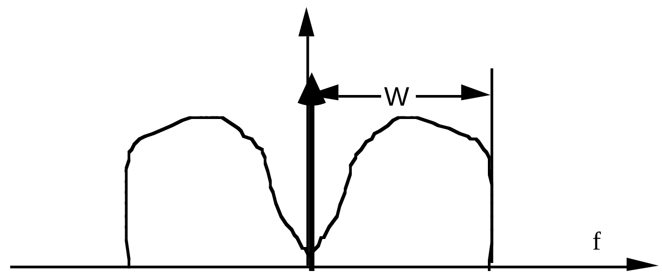


Figure 5.9: An example baseband message spectrum with the DC component added.

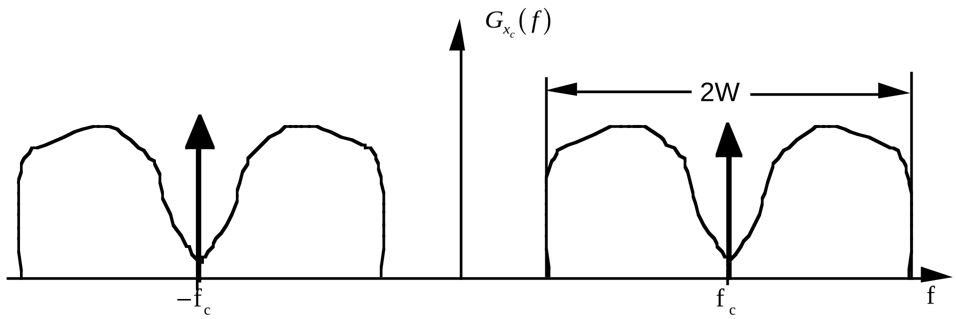


Figure 5.10: The resulting bandpass modulated spectrum with an affine modulation.

where a is a positive number. This modulation has $x_I(t) = A_c + A_c am(t)$ and $x_Q(t) = 0$, so the imaginary portion of the complex envelope is not used again in an affine analog modulation. The resulting bandpass signal and spectrum are given as

$$x_c(t) = \Re\left[\sqrt{2}x_z(t)\exp[j2\pi f_c t]\right] = (A_c + A_c am(t))\sqrt{2}\cos(2\pi f_c t) \tag{5.6}$$

$$G_{x_c}(f) = \frac{A_c^2}{2}\left(\delta(f-f_c) + a^2 G_m(f-f_c)\right) + \frac{A_c^2}{2}\left(\delta(f+f_c) + a^2 G_m(f+f_c)\right) \tag{5.7}$$

Because of the discrete carrier term in the bandpass signal (see (5.7)) this modulation is often referred to as large carrier AM (LC-AM). Fig. 5.9 shows an example message energy spectrum with the DC term added and Fig. 5.10 shows the resulting bandpass energy spectrum.

LC-AM has many of the same characteristics as DSB-AM. This modulation still modulates the amplitude of the carrier and the bandwidth of bandpass signal is still $B_T = 2W$ (E_B =50%). The imaginary part of the complex envelope is also not used in LC-AM. LC-AM differs from DSB-AM in that a DC term is added to the complex envelope. This DC term is chosen such that $x_I(t) > 0$ or equivalently the envelope of the bandpass signal never passes through zero. This implies that $am(t) > -1$. This constant a, here denoted the modulation coefficient, is important in obtaining good performance in a LC-AM system. An example of a LC-AM waveform is shown in Fig. 5.11 where the message waveform is given in Fig. 4.1 ($A_c = 3/\sqrt{8}$ and $a = 2/3$).

Average power is given by

$$P_{x_c} = A_c^2\left(\lim_{T\to\infty}\frac{1}{2T}\int_{-\infty}^{\infty}(1+am(t))^2\,dt\right). \tag{5.8}$$

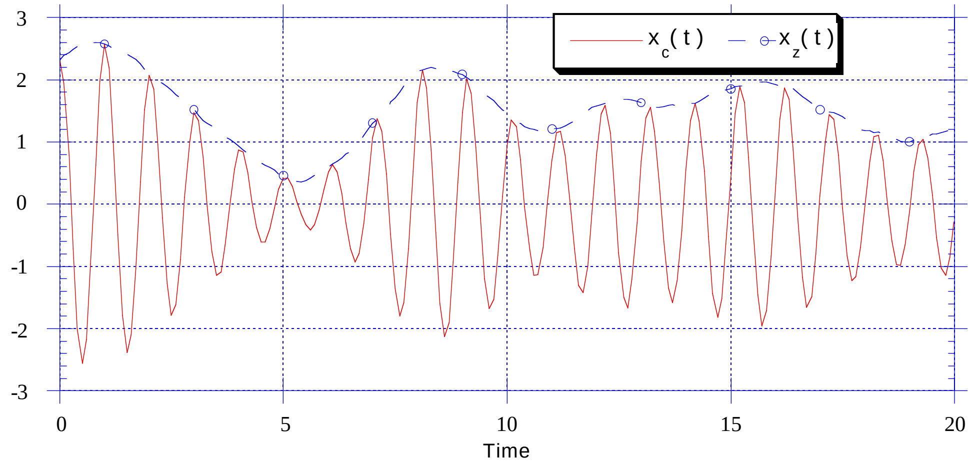


Figure 5.11: A LC-AM time waveform for the example given in Fig. 4.1

Since typically the time average of $m(t)$ is zero the average power simplifies to

$$P_{x_c} = A_c^2 \left(1 + a^2 P_m\right).$$

Example 5.4: Affine modulation with

$$m(t) = \beta \sin(2\pi f_m t) \qquad G_m(f) = \frac{\beta^2}{4}\delta(f - f_m) + \frac{\beta^2}{4}\delta(f + f_m)$$

produces

$$x_c(t) = A_c\left(1 + a\beta \sin(2\pi f_m t)\right)\sqrt{2}\cos(2\pi f_c t)$$

and

$$\begin{aligned} G_{x_c}(f) &= \frac{A_c^2 a^2 \beta^2}{8}\left[\delta(f - f_m - f_c) + \delta(f + f_m - f_c) + \delta(f - f_m + f_c) + \delta(f + f_m + f_c)\right] + \\ &\quad \frac{A_c^2}{2}\left[\delta(f - f_c) + \delta(f + f_c)\right]. \end{aligned}$$

The transmitted power is

$$P_{x_c} = A_c^2\left(1 + \frac{a^2\beta^2}{2}\right)$$

To maintain $x_I(t) > 0$ implies that $a < 1/\beta$ and consequently the maximum efficiency that can be achieved in this example is $E_T = 33\%$.

5.2.1 Modulator and Demodulator

The modulator for LC-AM is very simple. It is just one arm (the in-phase one) of a quadrature modulator where a DC term has been added to the message signal. Fig. 5.12 show a common implementation of this modulator. Again the simplicity of the modulator is an obvious advantage for LC-AM.

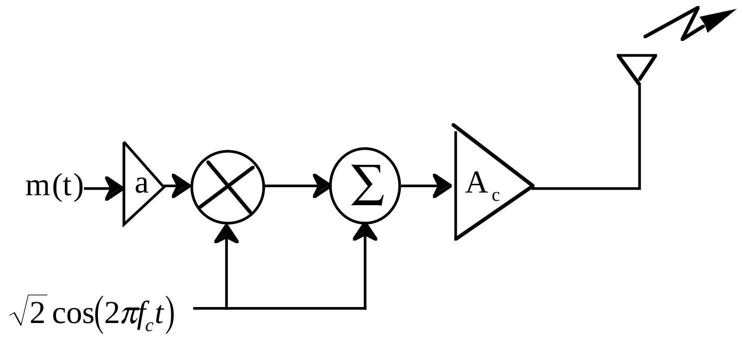
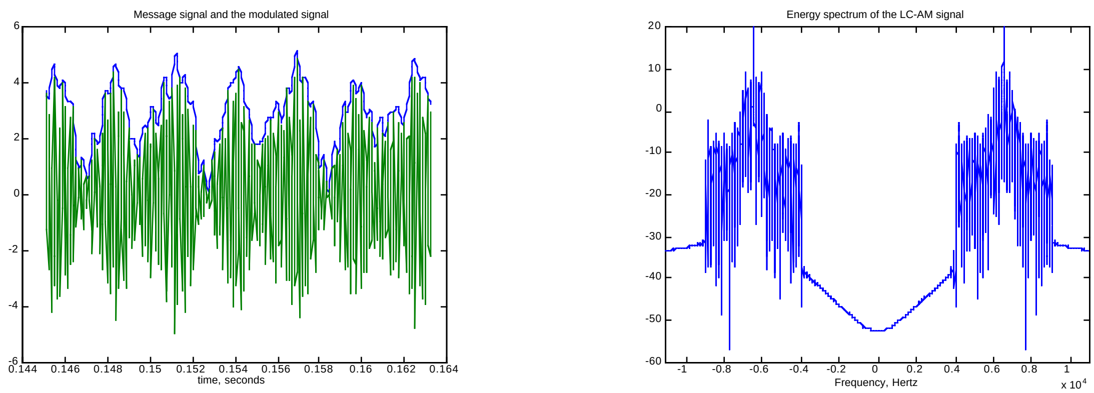


Figure 5.12: The block diagram of a modulator for LC-AM.



a) The time domain signals

b) The bandpass energy spectrum

Figure 5.13: The resulting LC-AM signal for a message signal given in Example 1.15. f_c=7KHz.

Example 5.5: The computer generated voice signal given in Example 1.15 (W=2.5KHz) is used to LC-AM modulate a 7KHz carrier. A short time record of the scaled complex envelope and the resulting output modulated signal is shown in Fig. 5.13-a). Note the minimum value of the voice signal is -3.93 so the modulation coefficient was set to $a = 0.25$. Fig. 5.13-a) shows the modulated signal for large values $m(t)$ and envelope comes close to zero. The energy spectrum of the signal is shown in Fig. 5.13-b). Note the bandwidth of the carrier modulated signal is 5KHz and the large carrier is evident in the plot.

The demodulator for LC-AM is simply an envelope detector followed by a DC block. Following the developed notation, the output of the envelope detector for $1 + am(t) > 0$ in the absence of noise is

$$|y_z(t)| = \left| A_c (1 + am(t)) e^{j\phi_p} \right| = A_c (1 + am(t)).$$

The DC block will remove the DC term to give

$$\hat{m}(t) = A_c am(t).$$

Fig. 5.14 shows the block diagram of a LC-AM demodulator. It is important to note that the demodulation performance is unaffected by the random phase induced by the propagation delay in transmission, ϕ_p. A demodulation structure that does not require an explicit phase reference like that of LC-AM is often denoted a **noncoherent** demodulator.

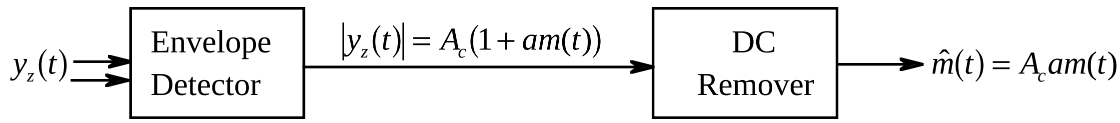


Figure 5.14: The block diagram of a baseband LC-AM demodulator.

Fig. 5.15 shows a circuit implementation of the bandpass version of the demodulator. Note that no active devices are contained in this implementation which was a big advantage in the vacuum tube days. Currently with large scale integrated circuits so prevalent, the advantage is not so large.

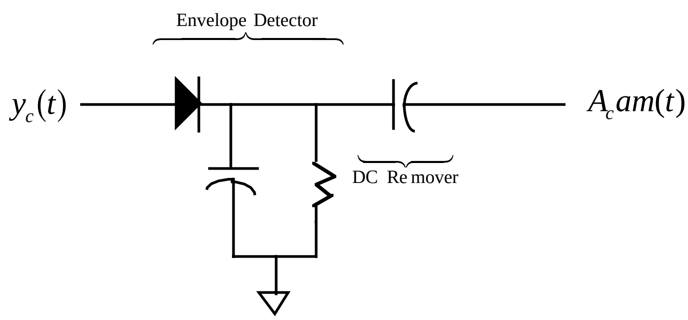


Figure 5.15: Circuit implementation of an LC-AM demodulator.

Since all the transmitted power P_{x_z} does not arrive at the demodulator output, LC-AM is less efficient than DSB-AM. The transmission efficiency is important in LC-AM modulation and is given as

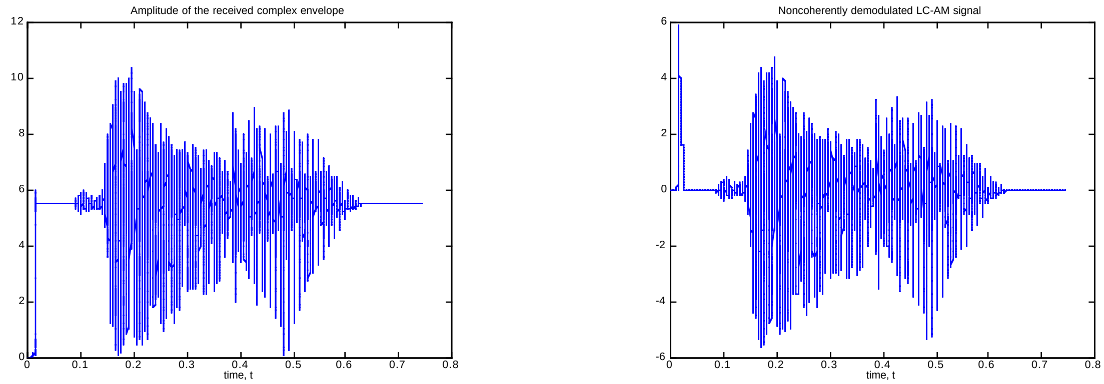
$$E_T = \frac{a^2 P_m}{1 + a^2 P_m}.$$

Note that the choice of a in a LC-AM systems is a compromise between ensuring that the carrier envelope, $x_A(t)$, is proportional to the message signal (small a) and that transmitted power is used for information transmission and not just wasted on carrier power (large a). Thus the selection of a in engineering practice is very important. The best transmission efficiency that can be achieved with any message signal in LC-AM is E_T =50% (what is that waveform?), the best efficiency that can be achieved with a sinusoidal message signal is E_T =33%, and the efficiency achieved with voice broadcasting is around E_T =10%. Hence a great price is paid in performance to achieve a simple demodulator.

Example 5.6: Consider the LC-AM computer generated voice signal given in Example 5.5 with a carrier frequency of 7KHz and a propagation delay in the channel of 45.6μs. This results in a $\phi_p = -114°$ (see Example 5.3.) The plot of the envelope of the received signal,

$$y_A(t) = \sqrt{y_I^2(t) + y_Q^2(t)},$$

is shown in Fig. 5.16-a) (Note the LPFs in the quadrature downconverter did not have unity gain). $y_A(t)$ has a DC offset due to the large carrier component but the message signal is clearly seen riding on top of this DC offset. The demodulated output after the DC block is shown in Fig. 5.16-b). Note the impulse at the start of this plot is due to the transient response of the filters and the DC block.



a) Output of the envelope detector b) Output of the DC block

Figure 5.16: Demodulation example for LC-AM.

5.2.2 LC-AM Conclusions

The advantage of LC-AM is again it is easy to generate and it has a simple cheap modulator and demodulator. The disadvantage is that E_B =50% and that E_T <50%. Later we will learn that noncoherent detection suffers a greater degradation in the presence of noise than coherent detection.

5.3 Quadrature Modulations

Both DSB-AM and LC-AM are very simple modulations to generate but they have E_B =50%. The spectral efficiency of analog modulations can be improved. Note that both DSB-AM and LC-AM only use the real part of the complex envelope. The imaginary component of the complex envelope can be used to shape the spectral characteristics of the analog transmissions. As an example, the bandwidth of analog video signals is approximately 4.5MHz. DSB-AM and LC-AM modulation would produce a bandpass bandwidth for video signals of 9MHz. Broadcast analog television (TV) signals have a bandwidth of approximately 6MHz and this is achieved with a quadrature modulation.

Vestigial sideband amplitude modulation (VSB-AM) is a modulation that improves the spectral efficiency of analog transmission by specially designed linear filters at the modulator or transmitter. The goal with VSB-AM is to achieve $E_B > 50\%$ using the imaginary component of the modulation complex envelope. A VSB-AM signal has a complex envelope of

$$x_z(t) = A_c \left[m(t) + j\left(m(t) \otimes h_Q(t)\right)\right]$$

where $h_Q(t)$ is the impulse response of a real LTI system. Two interpretations of VSB-AM are useful. The first interpretation is that $x_I(t)$ is generated exactly the same as DSB-AM (linear function of the message) and $x_Q(t)$ is generated as a filtered version of the message signal. A block diagram of a baseband VSB-AM modulator using this interpretation is seen in Fig. 5.17. Recalling the results in Section 3.4, the second interpretation is that a bandpass VSB-AM signal is generated by putting a bandpass DSB-AM signal through a bandpass filter (asymmetric around f_c) whose complex envelope impulse response is

$$h_z(t) = \delta(\tau) + jh_Q(t). \tag{5.9}$$

Note the transmitted power of a VSB-AM signal is going to be higher than a similarly modulated DSB-AM signal since the imaginary portion of the complex envelope is not zero. The actual resulting

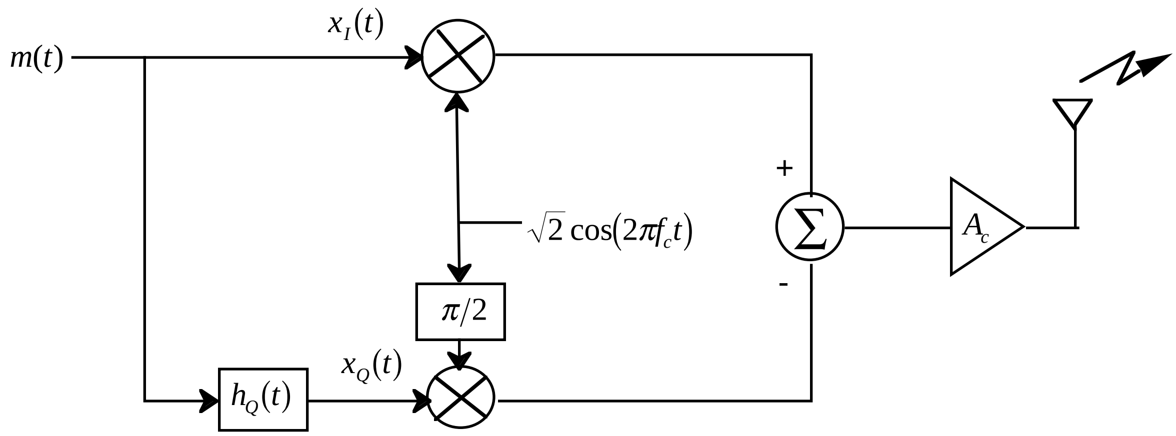


Figure 5.17: A block diagram of a VSB-AM modulator.

output power is a function of the filter response, $h_z(t)$, and an example calculation will be pursued in the homework.

5.3.1 VSB Filter Design

The design of the filter, $h_Q(t)$, is critical to achieving improved spectral efficiency. The Fourier transform of a VSB-AM signal (using (3.6)) is given as

$$\begin{aligned} X_{x_z}(f) &= X_I(f) + jX_Q(f) \\ &= A_c M(f)\left[1 + jH_Q(f)\right]. \end{aligned} \tag{5.10}$$

Additionally note $X_z(f) = A_c H_z(f) M(f)$ with $H_z(f)$ as defined in (5.9). Since the message signal spectrum is non-zero over $-W \leq f \leq W$, reduction of the bandwidth of the bandpass signal requires that $[1 + jH_Q(f)]$ be zero over some part of $f \in [-W, W]$. The remainder of the discussion will examine the conditions which must hold to produce

$$X_z(f) = 0 \qquad -W \leq f \leq -f_v \leq 0$$

where f_v is commonly called the vestigial frequency. In other words what conditions on $H_z(f)$ must hold so that a portion of the lower sideband of the bandpass signal is eliminated to achieve

$$E_B = \frac{W}{W + f_v} > 50\%.$$

Note that VSB-AM is always more bandwidth efficient than either DSB-AM or LC-AM and the improvement in bandwidth efficiency is a function of f_v. Since the lower portion of the signal spectrum is eliminated, this type of transmission is often termed upper side band VSB-AM (the upper sideband is transmitted). Similar results as the sequel can be obtained to produce lower side band VSB-AM transmissions. A set of example spectra for upper sideband VSB-AM is shown in Fig. 5.18 to illustrate the idea of quadrature modulation. The figure uses the energy spectrum of the baseband transmitted signal since it is simple to represent the energy spectrum with a graph two dimensional graph. Note that the remains of the lower sideband in this example illustrates the reason for name VSB-AM[2].

[2]A dictionary definition of vestigial is *pertaining to a mark, trace, or visible evidence of something that is no longer present or in existence.*

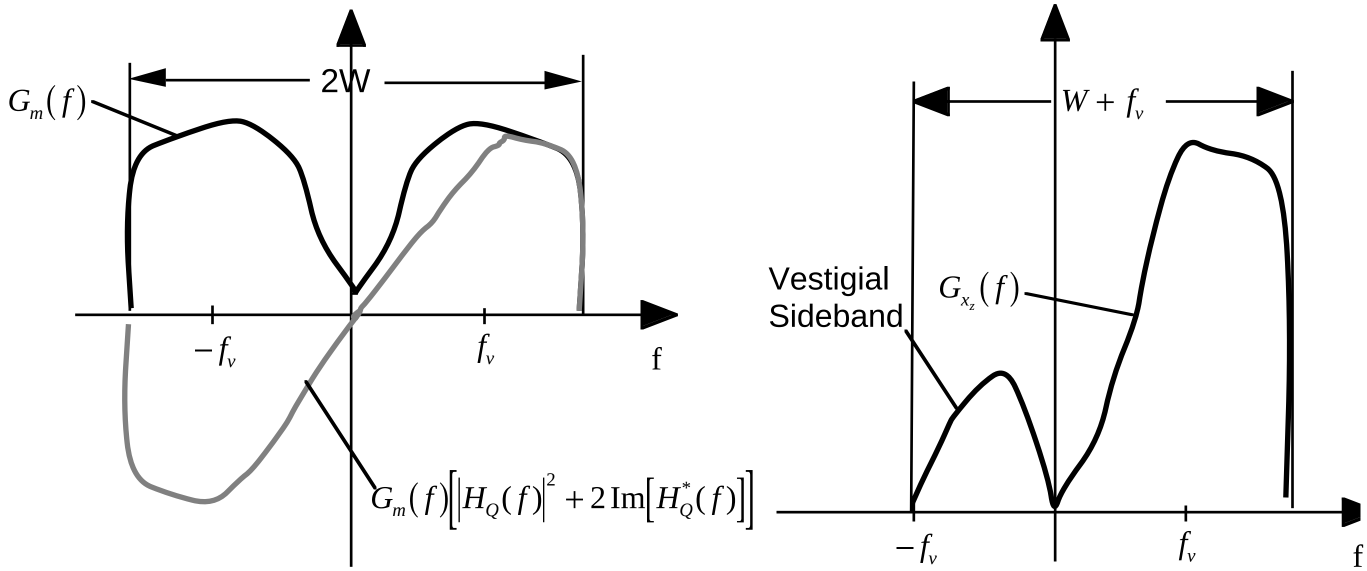


Figure 5.18: An example to show how quadrature modulation can be used to reduce the bandwidth of the transmitted signal. Upper sideband modulation.

If we want

$$\begin{aligned} 0 &= H_z(f) \qquad -W \leq f \leq -f_v \\ &= [1 + jH_Q(f)] \qquad -W \leq f \leq -f_v. \end{aligned} \tag{5.11}$$

Since $H_Q(f) = \Re\left[H_Q(f)\right] + j\Im\left[H_Q(f)\right]$ it is simple to see that (5.11) implies that

$$\Im\left[H_Q(f)\right] = 1 \qquad \Re\left[H_Q(f)\right] = 0 \qquad -W \leq f \leq -f_v. \tag{5.12}$$

Note since $h_Q(t)$ is real then this implies

$$\Im\left[H_Q(f)\right] = -1 \qquad \Re\left[H_Q(f)\right] = 0 \qquad f_v \leq f \leq W \tag{5.13}$$

but the values for $H_Q(f), -f_v \leq f \leq f_v$ are unconstrained. A plot of two possible realizations of the filter $H_Q(f)$ is shown in Fig. 5.19. It should be noted that our discussion focused on eliminating a portion of the lower sideband but similar results hold for eliminating the upper sideband.

Example 5.7: Analog television broadcast in the United State uses a video signal with a bandwidth of approximately 4.5MHz. The transmitted signals are a form of quadrature modulation where f_v=1.25MHz for a total channel bandwidth of <6MHz. Television stations in the United States have 6MHz spacings. Consequently

$$E_B = \frac{4.5\text{MHz}}{6\text{MHz}} = 75\%$$

5.3.2 Single Sideband AM

An interesting special case of VSB-AM occurs when $f_v \to 0$. In this case the $E_B \to 100\%$. This is accomplished by eliminating one of the sidebands and hence this modulation is often termed single

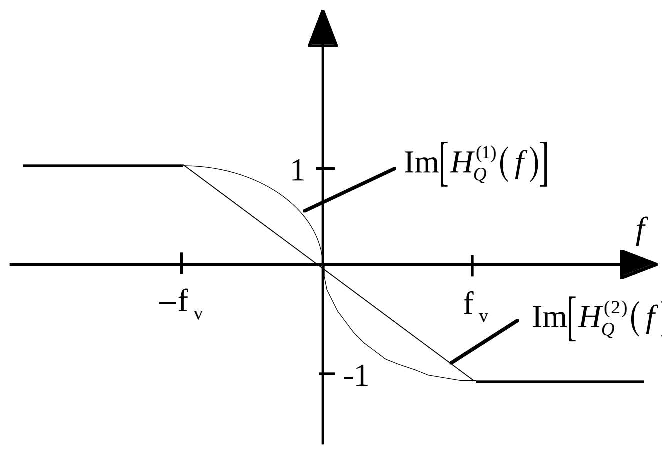


Figure 5.19: Examples of the imaginary part of the quadrature filter for VSB-AM.

sideband amplitude modulation (SSB-AM). The $h_Q(t)$ that results in this case is important enough to get a name, the Hilbert Transform [ZTF89]. The transfer function of the Hilbert transform is

$$H_Q(f) = -j\mathrm{sgn}\,(f) \tag{5.14}$$

where

$$\begin{aligned} \mathrm{sgn}\,(f) &= 1 \qquad f > 0 \\ &= -1 \qquad f < 0. \end{aligned}$$

Note because of the sharp transition in the transfer function at DC it is only possible to use SSB-AM with message signals that do not have significant spectral content near DC.

5.3.3 Modulator and Demodulator

The modulator for VSB-AM is given in Fig. 5.17. Note also an implementation can be achieved with a bandpass filter which has an asymmetric frequency response around f_c. In practice both the baseband and bandpass implementations are used. The high performance digital processing that has emerged over the recent past has pushed the implementations toward the baseband realizations due to the precise nature with which filter responses can be controlled in digital circuits.

Example 5.8: The computer generated voice signal given in Example 1.15(W=2.5KHz) is used to SSB-AM modulate a 7KHz carrier. The quadrature filter needs to be a close approximation to a Hilbert transformer given in (5.14). The magnitude and phase response of a filter designed for a voice signal is shown in Fig. 5.20. The resulting complex envelope energy spectrum is given in Fig. 5.21-a). This implementation provides greater than 80dB of sideband rejection over most of the band. The bandpass energy spectrum is shown in Fig. 5.21-b). This plot clearly shows that SSB-AM uses half the bandwidth of DSB-AM or LC-AM.

The coherent demodulator for VSB-AM is shown in Fig. 5.22. It is obvious from examining Fig. 5.22 that the VSB-AM demodulator is identical to the coherent demodulator for DSB-AM and the output is

$$\hat{m}(t) = A_c m(t) + N_I(t).$$

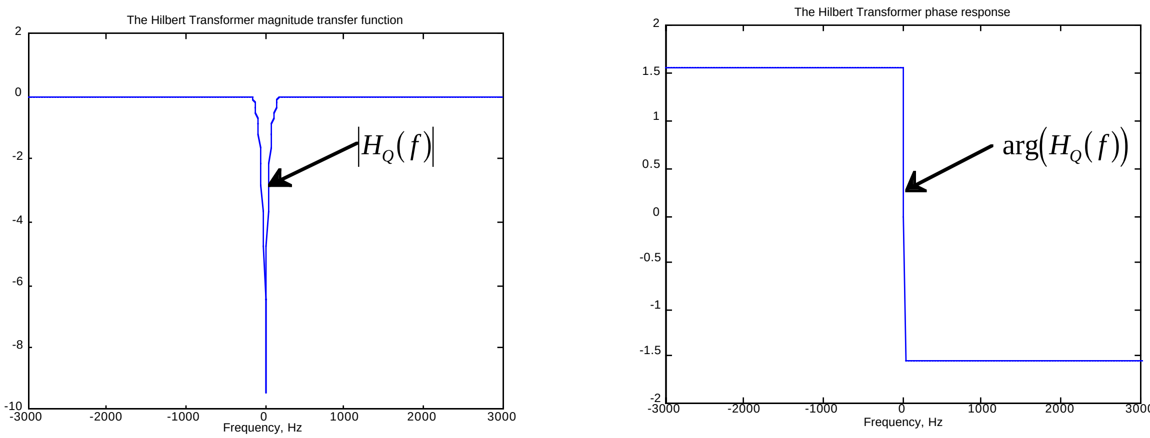
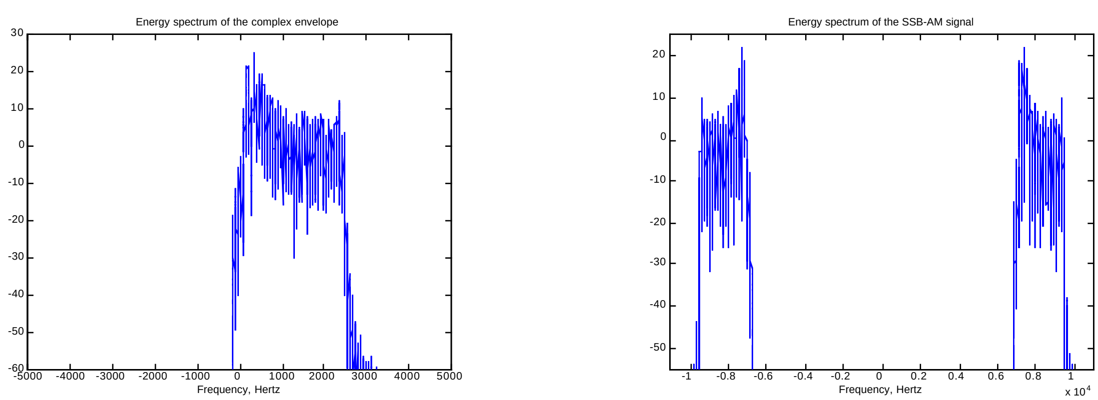


Figure 5.20: The magnitude and phase response of the Hilbert transform implementation used in Example 5.8.



a) Complex envelope b) Bandpass

Figure 5.21: The spectrum for SSB-AM transmission using the quadrature filter given in Fig. 5.20.

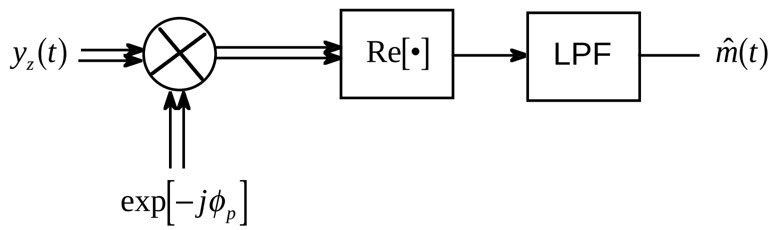


Figure 5.22: A coherent demodulator for VSB-AM. Note it is exactly the same as DSB-AM.

It is obvious that since the imaginary component of the complex envelope is used in the modulation that loss in the transmission efficiency will occur, i.e.,

$$E_T = \frac{A_c^2 P_m}{P_{x_z}} < 100\%.$$

For the specific case of SSB-AM it is clear that $|H_Q(f)|^2 = 1$ and that $E_T = 50\%$. This loss in efficiency can be recovered if the demodulator processing is appropriately designed and this will be examined in Chap. 10.

Example 5.9: Single-sideband modulation with a message signal

$$m(t) = \beta \sin(2\pi f_m t)$$

has an in-phase signal of

$$x_I(t) = A_c\beta \sin(2\pi f_m t) \qquad X_I(f) = \frac{A_c\beta}{2j}\delta(f - f_m) - \frac{A_c\beta}{2j}\delta(f + f_m)$$

Applying the Hilbert transform to $x_I(t)$ produces a quadrature signal with

$$X_Q(f) = \frac{-A_c\beta}{2}\delta(f - f_m) - \frac{A_c\beta}{2}\delta(f + f_m)$$

or

$$x_Q(t) = -A_c\beta \cos(2\pi f_m t).$$

This results in

$$x_z(t) = A_c\beta\left(\sin(2\pi f_m t) - j\cos(2\pi f_m t)\right) = -jA_c\beta \exp(j2\pi f_m t)$$

and

$$G_{x_z}(f) = A_c^2\beta^2\delta(f - f_m).$$

The bandpass spectrum is

$$G_{x_c}(f) = \frac{A_c^2\beta^2}{2}\delta(f - f_m - f_c) + \frac{A_c^2\beta^2}{2}\delta(f + f_m + f_c).$$

Note that the lower sideband of the message signal has been completely eliminated with SSB-AM.

5.3.4 Transmitted Reference Based Demodulation

A coherent demodulator is also necessary for VSB-AM transmission. Unfortunately the phase reference cannot be derived from the received signal like in DSB-AM because VSB-AM uses both the real and

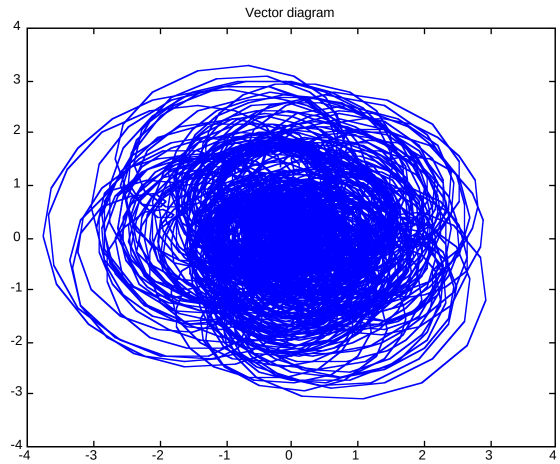


Figure 5.23: Vector diagram for the SSB-AM transmission in Example 5.9.

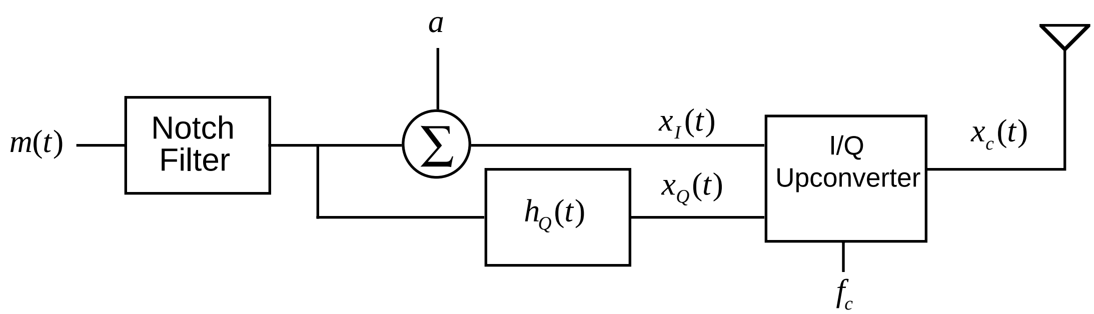


Figure 5.24: A modulator implementation for SSB-AM using a transmitted reference signal.

imaginary components of the complex envelope. This characteristic is best demonstrated by the vector diagram of the complex envelope. Fig. 5.23 shows the vector diagram of the SSB-AM transmission of Example 5.9. Clearly if one compares the vector diagram shown in Fig. 5.7-a) to the vector diagram in Fig. 5.23 it is easy to see why automatic phase recovery from the received signal is tough. So while VSB-AM provides improved spectral efficiency it does it at the cost of increased demodulator complexity.

If automatic phase tracking is desired with VSB-AM, a transmitted reference signal is often used. We will explore this idea in a SSB-AM application. The block diagram of an example SSB-AM transmitter system that uses a transmitted reference is given in Fig. 5.24. The notch filter removes the message signal components from around DC. The in-phase signal consists of this notched out message and a DC term. This DC term when up-converted will result in a carrier signal much like LC-AM. The quadrature signal is a Hilbert transform of the notched out message signal.

The demodulator uses the transmitted reference as a carrier phase reference to recover the message signal. The demodulator block diagram is shown in Fig. 5.25. In essence at the receiver two filters are used to separate the transmitted modulated signal and the transmitted reference signal into two separate path, i.e.,

$$y_H(t) = A_c\left[m(t) + j(m(t)\otimes h_Q(t))\right]\exp\left[j\phi_p\right] \quad (5.15)$$

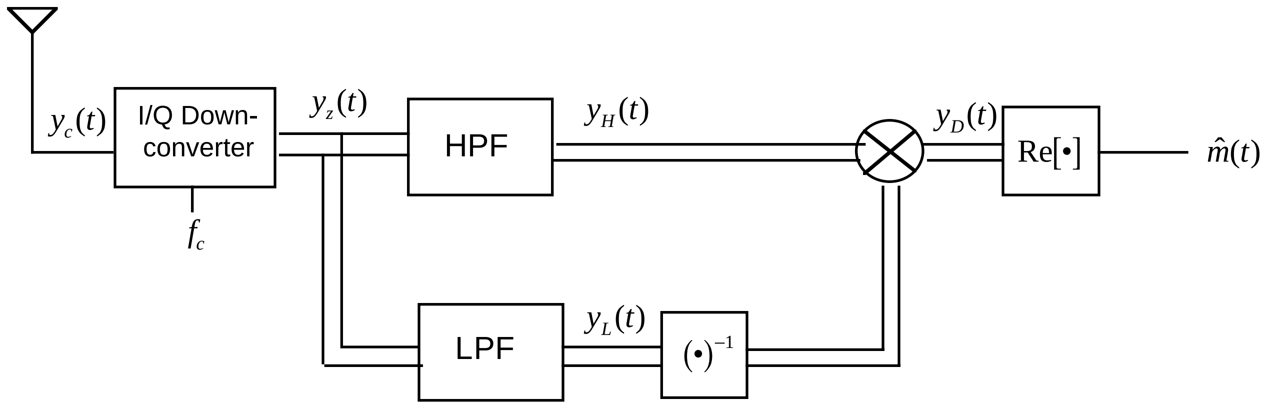


Figure 5.25: A transmitted reference based demodulator implementation for SSB-AM.

$$y_L(t) = A_c a \exp\left[j\phi_p\right]. \tag{5.16}$$

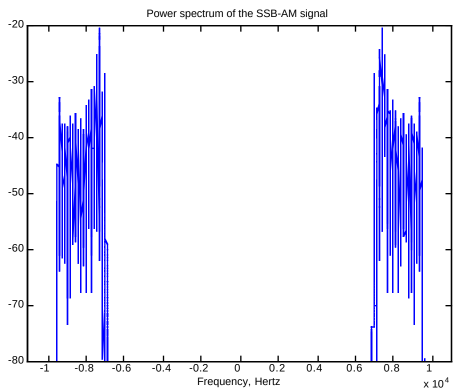
Since each of these paths experience the same channel distortion and phase shift the reference can be used to derotate the demodulated signal and recover the message signal. This is easily seen with

$$y_D(t) = \frac{y_H(t)}{y_L(t)} = \frac{1}{a}\left[m(t) + j(m(t) \otimes h_Q(t))\right]. \tag{5.17}$$

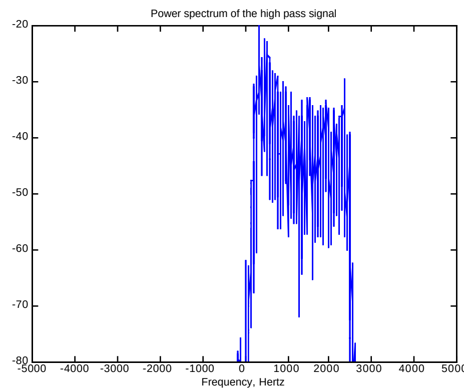
This transmitted reference demodulation scheme is very useful especially in systems where automatic operation is desired or if the channel is varying rapidly. Consequently transmitted reference systems are often used in land mobile radio where multipath and mobility can often cause significant channel variations [Jak74, Lee82]. An important practical design consideration is how large the reference signal power should be in relation to the modulated signal power. This consideration is not important unless noise is considered in the demodulation so this discussion will also be left until Chap. 10.

Example 5.10: In this example we consider again the computer generated voice signal given in Example 1.15(W=2.5KHz). This signal is SSB-AM modulated with a transmitted reference at a 7KHz carrier in a fashion as shown in Fig. 5.24. The notch bandwidth is chosen to be about 200Hz as this does not effect the audio quality of the signal. The measured transmitted signal spectrum is shown in Fig. 5.26-a). The transmitted reference signal is the tone at the carrier frequency. The signal out of the notch filter in the demodulator of Fig. 5.25 is approximately a standard baseband SSB-AM modulation. The measured power spectrum of $y_H(t)$ is shown in Fig. 5.26-b) for a notch bandwidth of 100Hz. The transmitted reference tone has been greatly attenuated but the modulated signal is roughly unaltered. The signal out of the lowpass filter in the demodulator of Fig. 5.25 is approximately the phase shifted reference tone. The measured power spectrum of $y_L(t)$ is shown in Fig. 5.27-a) for a notch bandwidth of 100Hz. The modulated signal has been greatly attenuated. For example in a channel with $\tau_p = 45.3\mu$s a phase shift of $-114°$ occurs. The measured phase of the output tone is shown in Fig. 5.27-b) and it is close to $-114°$ once the filter transients have passed. The imperfections in the filter can be seen to allow the message signal to cause some phase jitter to be induced on the demodulator reference but the overall audio performance is gives high quality audio output.

Noncoherent demodulation (envelope detection) is also used in conjunction with a large carrier version of VSB-AM in analog television reception in the United States. The details of how this is possible with a quadrature modulation will be addressed in the homework.

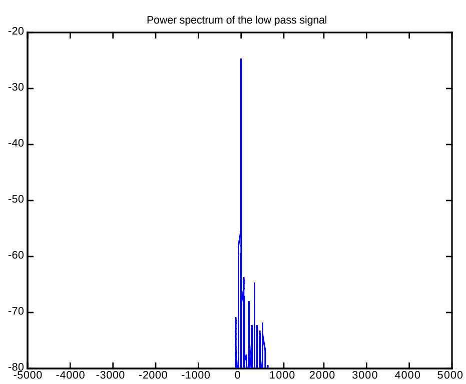


a) The transmitted spectrum

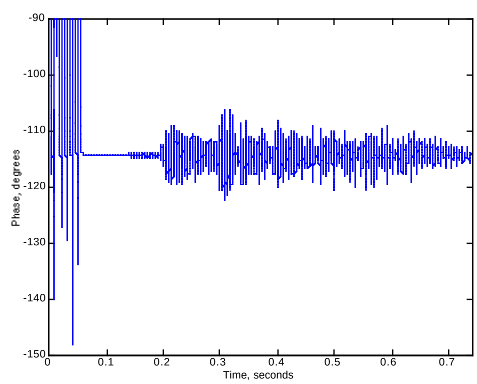


b) The spectrum of the highpass filter output

Figure 5.26: Signals in a SSB-AM transmitted reference system. f_c=7kHz



a) The spectrum of the lowpass filter output



b) Measured phase of the demodulation reference signal

Figure 5.27: Signals in a SSB-AM transmitted reference system.

5.3.5 Quadrature Modulation Conclusions

Quadrature modulation provides better spectral efficiency than either DSB-AM or LC-AM but with a more complex modulator. Quadrature modulation accomplishes this improved spectral efficiency by using the imaginary portion of the complex envelope. Like LC-AM, VSB-AM wastes part of the transmitted power (the imaginary component is not used in demodulation) but for a useful purpose. LC-AM contains a carrier signal that transmits no information while VSB-AM transmits a filtered signal in the quadrature component of the modulation but never uses it in demodulation. VSB-AM is also suitable for use with a large carrier and envelope detection (this is what is used in broadcast TV) and this idea will be explored in the homework.

5.4 Homework Problems

Problem 5.1. The message signal in a DSB-AM system is of the form

$$m(t) = 12\cos(6\pi t) + 3\cos(10\pi t)$$

a) Calculate the message power, P_m.

b) If this message is DSB-AM modulated on a carrier with amplitude A_c, calculate the Fourier series of $x_z(t)$.

c) Assuming $f_c = 20$Hz calculate the Fourier series of $x_c(t)$ and plot the resulting time waveform when $A_c = 1$.

d) Compute the output power of the modulated signal, P_{x_c}.

e) Calculate and plot $x_P(t)$. Computer might be useful.

Problem 5.2. A message signal of the form

$$m(t) = \cos(2\pi f_m t)$$

is to be transmitted by DSB-AM with a carrier frequency f_c.

a) Give the baseband and bandpass forms of the modulated signal for this message signal that has a power of 8 Watts (in a 1 Ω system).

b) Assume $f_c = 10 f_m$ give the baseband and bandpass spectral representation of this modulation. What is the transmission bandwidth, B_T of such a system?

c) Plot the resulting bandpass signal.

d) Give the simplest form of the modulator and demodulator (assume you know ϕ_p).

Problem 5.3. The received bandpass DSB-AM signal has the form

$$y_c(t) = A_c m(t)\sqrt{2}\cos(2\pi f_c t + \phi_p)$$

and the Costas loop used in phase synchronous demodulation of DSB-AM is shown in Figure 5.28

a) What is $y_z(t)$?

b) Find expressions for the signals at points A and B. Assume the LPF are only to remove the double frequency terms.

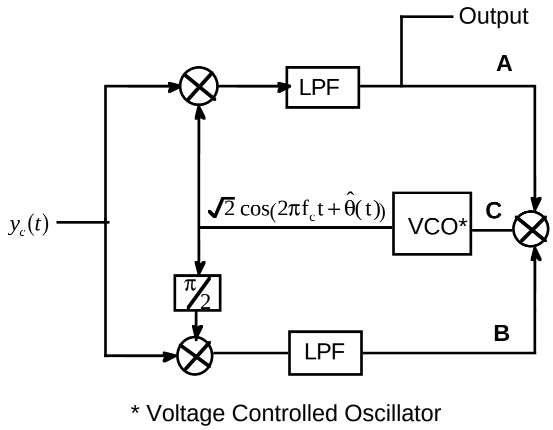


Figure 5.28: A bandpass version of the Costas loop.

c) Define a complex signal $z_1(t) = A - jB$. Show that $z_1(t) = y_z(t)\exp[-j\hat{\theta}(t)]$.

d) Note that $C = 0.5 * Im[z_1(t)^2]$. Find C in terms of the phase error, $\phi_p - \hat{\theta}(t)$.

Problem 5.4. As your first task, your new boss at Fony asks you to design a LC-AM modulator (obviously busy work until he can find something real for you to do since AM is not used much in practice) for a 1KHz, 3 V peak amplitude test tone.

a) He wants a 5 Watt output power across a 1Ω resistor with E_T=7% transmission efficiency. Provide the values of A_c and a to achieve this specification.

b) What is the maximum transmission efficiency that can be achieved in this problem and have envelope detection still possible? Give the values of A_c and a to achieve this efficiency while maintaining a 5 Watt output power.

Problem 5.5. A message signal of the form

$$m(t) = \cos(2\pi f_m t)$$

is to be transmitted by LC-AM with a carrier frequency f_c.

a) Give the baseband and bandpass forms of the modulated signal for this message signal that has a power of $P_{x_z} = 8$ Watts (in a 1 Ω system) and E_T=1/9.

b) Assume $f_c = 10f_m$ give the baseband and bandpass spectral representation of this modulation. What is the transmission bandwidth, B_T of such a system?

c) Give the simplest form of the modulator and a noncoherent demodulator

Problem 5.6. A term often used to characterize LC-AM waveforms is the percent modulation defined as

$$P_{LC} = \frac{\text{Maximum value of the envelope-Minimum value of the envelope}}{\text{Maximum value of the envelope}} \tag{5.18}$$

For this problem consider a sinusoidal message signal of unit amplitude and LC-AM as shown for $P_{LC} = 0.5$ in Fig. 5.29.

a) What is the range for P_{LC}?

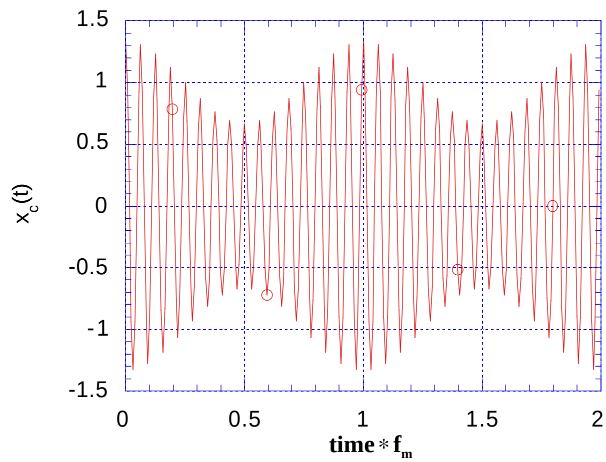


Figure 5.29: A LC-AM signal with $P_{LC} = 0.5$. Sinusoidal signal.

b) Solve for the value of a to give a particular P_{LC}.

c) The bandpass frequency domain representation (Fourier series or Fourier transform) of this signal will have three frequencies (the carrier and two modulation sidebands). Plot the relative power ratio (in dB) between the carrier and one of the modulation sidebands as a function of P_{LC}.

d) Plot the efficiency, E_T , as a function of P_{LC}.

Problem 5.7. The Federal Communications Commission (FCC) makes each station strictly limit their output frequency content. Assume AM stations produce a usable audio bandwidth of 10kHz. AM stations in the same geographical area are normally spaced at least 30kHz apart. As an example of why limits on the frequency content are necessary the following problem is posed.

Consider two stations broadcasting in the same geographic area with a receiver tuned to one of them as shown in Fig. 5.30. Station A is broadcasting a 1V peak sinewave of 2kHz, $m_A(t)$ (a test of the emergency broadcast system), at a center frequency of f_c Hz. Station B is broadcasting a 1V peak 3kHz square wave, $m_B(t)$, at f_c+30kHz with no filtering at the transmitter. Since each transmitted waveform will have a different propagation loss, the received waveform is given by the form

$$r_c(t) = A_A\left[(1.5 + m_A(t))\sqrt{2}\cos(2\pi f_c t + \phi_A)\right] + A_B\left[(1.5 + m_B(t))\sqrt{2}\cos(2\pi(f_c + 30000)t + \phi_B)\right]. \tag{5.19}$$

Assume without loss of generality that the phase shift for the station A is zero, i.e., $\phi_A = 0$.

a) Give the complex envelope of the received signal, $r_z(t)$.

b) Find the spectrum (Fourier series coefficients) of the complex envelope, $r_z(t)$.

c) The demodulator has an ideal bandpass filter with a center frequency of f_c and a two sided bandwidth of $2W$=15kHz followed by an envelope detector as shown in Fig. 5.30. Plot the output demodulated waveform, $\hat{m}_A(t)$, over 5ms of time for several values of ϕ_b in the range $[0, 2\pi]$ and $A_B = 0.1$, 1, 10.

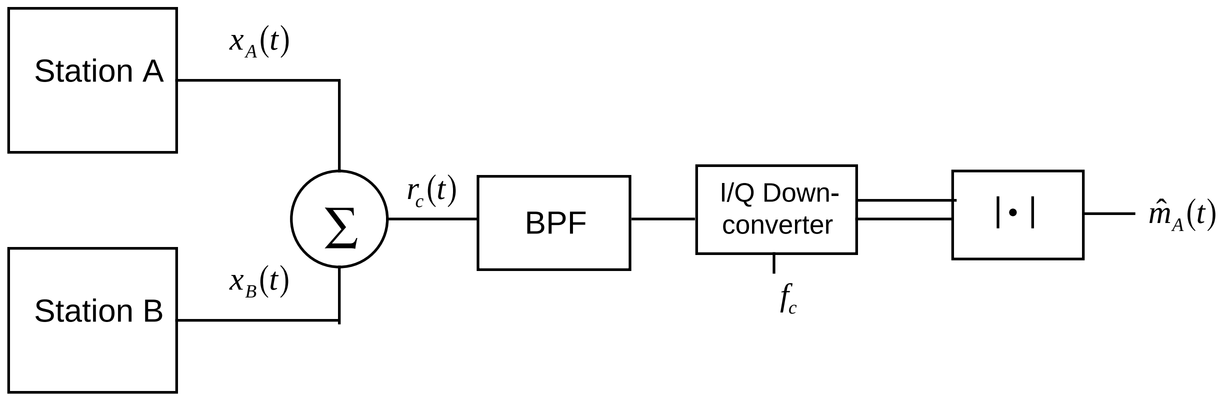


Figure 5.30: A model for two transmitting AM radio stations for Problem 5.7.

The distortion you see in this example is called adjacent channel interference and one of the FCC's functions is to regulate the amount of interference each station produces for people trying to receive another station.

Problem 5.8. Consider the message signal in Fig. 5.31 and a bandpass signal of the form

$$x_c(t) = (5 + am(t))\sqrt{2}\cos(2\pi f_c t).$$

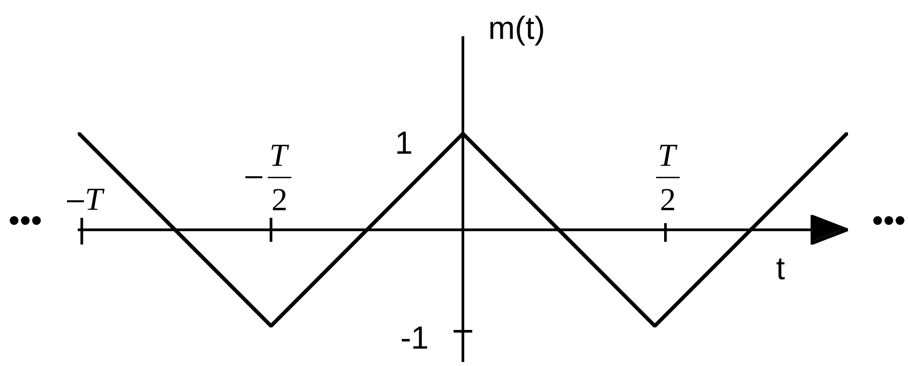


Figure 5.31: A message signal for Problem 5.8.

a) What sort of modulation is this?

b) Plot the bandpass signal with a=2 when $f_c \gg \frac{1}{T}$.

c) How big can a be and still permit envelope detection for message signal recovery.

d) Compute the transmitted power for a=3.

e) Compute E_T for a=3.

Problem 5.9. The complex baseband model for analog communications is given in Figure 4.6. Ignore noise and assume $m(t) = \sin(2\pi f_m t)$. If $y_z(t) = (5 + 3\sin(2\pi f_m t))\exp[j\pi/3]$,

a) What kind of modulation does this represent? Identify all the important parameters.

b) Plot the envelope of the output, $y_A(t)$.

c) Plot the phase of the output, $y_P(t)$

d) Show how to process $y_z(t)$ to recover $m(t)$.

Problem 5.10. A message signal with an energy spectrum given in Fig. 5.32 is to be transmitted with large carrier amplitude modulation. Additionally $\min(m(t)) = -2$ and $P_m = 1$.

a) If the transmission efficiency is set at E_T=20%, what is the modulation index, a.

b) Will the modulation index obtained in part a) be sufficiently small to allow demodulation by envelope detection. *Note if you cannot solve for part a) just give the necessary conditions such that envelope detection is possible.*

c) Plot the modulator output spectrum (either bandpass or baseband is fine) and compute the E_B.

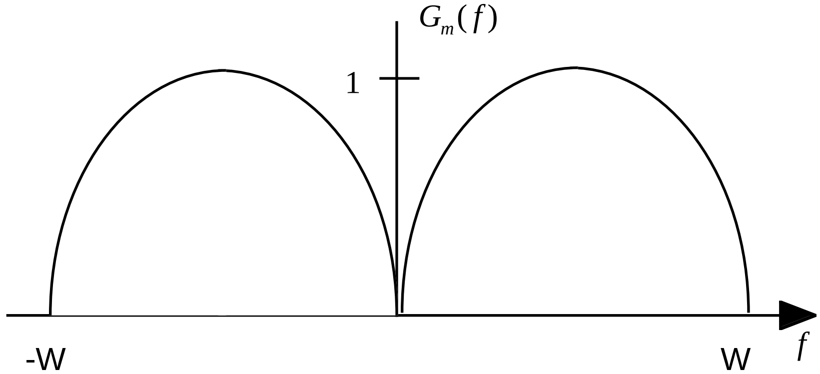


Figure 5.32: A message signal energy spectrum.

Problem 5.11. Commercial TV uses large carrier AM transmission in conjunction with VSB-AM for the intensity signal (black and white levels). A typical TV receiver block diagram for the intensity signal demodulation is shown in Figure 5.33.

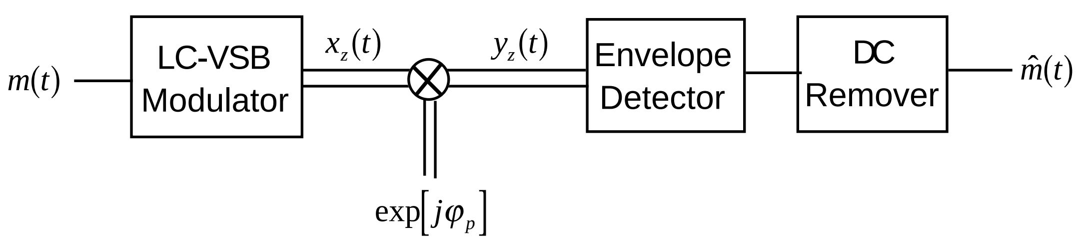


Figure 5.33: Typical TV demodulator.

This problem tries to lead you to an explanation of why the filtering does not significantly effect the envelope detector's performance. Consider the simple case of a sinusoidal message signal

$$m(t) = \cos(2\pi f_m t)$$

with $f_m > 0$ and a VSB modulator that produces a complex envelope signal of the form

$$\begin{aligned} x_z(t) &= A_c\left(1 + a\cos(2\pi f_m t) + ja\sin(2\pi f_m t)\right) \qquad f_m > f_v \\ &= A_c\left(1 + a\cos(2\pi f_m t)\right) \qquad f_m \leq f_v. \end{aligned} \tag{5.20}$$

a) Sketch the modulator that would produce (5.20) and derive the transfer function of the quadrature filter, $H_Q(f)$, necessary.

b) Calculate the output envelope, $|y_z(t)|$.

c) Show if a is chosen such that a^2 is very small compared to a, then the envelope detector output is approximately the desired signal (the message signal plus a DC offset).

d) Compute the efficiency of the transmission, E_T, for $f_m < f_v$ and $f_m > f_v$ assuming a^2 is small.

e) Choose a value of a for which you think this will hold. What does this say about the efficiency of typical TV broadcast?

Problem 5.12. RF engineers that design and build quadrature upconverters (see Fig. 5.34) need tests to estimate how close to ideal their circuits are performing. The standard test used is known as a single sideband rejection test. This test uses an input of $x_I(t) = \cos(2\pi f_m t)$ and $x_Q(t) = \sin(2\pi f_m t)$ and measures the resulting bandpass power spectrum, $|X_c(f)|^2$ on a spectrum analyzer.

a) Compute what the output bandpass spectrum, $|X_c(f)|^2$, should be for an ideal quadrature upconverter.

b) A common design issue in quadrature modulators is that the quadrature carrier has a phase offset compared to the in-phase carrier, i.e.,

$$x_c(t) = x_I(t)\sqrt{2}\cos(2\pi f_c t) - x_Q(t)\sqrt{2}\sin(2\pi f_c t + \theta).$$

For the test signal in a single sideband rejection test what will be the output bandpass spectrum as a function of θ.

c) If $|X_c(f_c + f_m)|^2 = 100\,|X_c(f_c - f_m)|^2$ what is the value of θ that would produce this spectrum.

d) Postulate why this test is known as a single side band rejection test.

Problem 5.13. A message with a Fourier transform of

$$\begin{aligned} M(f) &= \frac{1}{\sqrt{W}}\left|\sin\left(\frac{\pi f}{W}\right)\right| \qquad -W \leq f \leq W \\ &= 0 \qquad \text{elsewhere} \end{aligned} \tag{5.21}$$

is to be transmitted with a quadrature modulation.

a) Calculate the message energy, E_m.

b) For SSB-AM (upper sideband) compute the output transmitted energy, E_{x_z}.

c) Design a simple filter quadrature filter for VSB-AM where $f_v = W/4$. Give either $H_Q(f)$ or $H_z(f)$.

d) Compute the resulting output energy for your design, E_{x_z}, in part c). *Note depending on your design the solution might be easiest done with the aid of a computer*

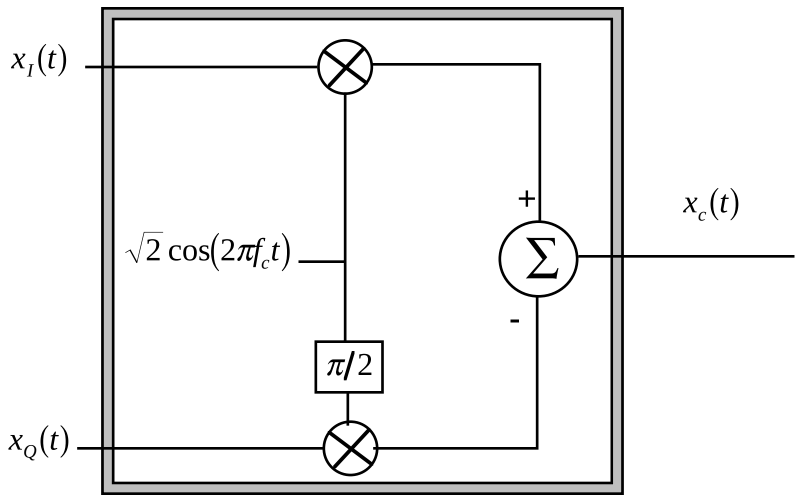


Figure 5.34: A quadrature upconverter.

Problem 5.14 (UM). An amplitude modulated (AM) signal has the following form:

$$x_c(t) = [A + 1.5\cos(10\pi t) - 3.0\cos(20\pi t)]\cos(2\pi f_c t),$$

where the message signal is

$$m(t) = \cos(10\pi t) - 2.0\cos(20\pi t)$$

a) What type of amplitude modulation is this?

b) What is the spectrum of $x_c(t)$, $X_c(f)$?

c) Determine the conditions on A such that envelope detection could be used in the demodulation process.

d) Determine the efficiency of this modulation scheme as a function of A.

Problem 5.15. You need to test a VSB-AM system that your company has purchased. The modulation scheme is DSB-AM followed by a bandpass filter, $H_c(f)$. The demodulator is exactly the same as DSB-AM when the phase offset, ϕ_p, is known. The system does not work quite right and testing has led you to suspect the filter $H_c(f)$. After analysis and testing, you determine that the filter has the following bandpass characteristic:

$$H_c(f) = \begin{cases} 1 & f_c + 7500 \le |f| \le f_c + 10000 \\ 2 & f_c + 2500 \le |f| < f_c + 7500 \\ \frac{4}{3} & f_c \le |f| < f_c + 2500 \\ \frac{3}{4} & f_c - 2500 \le |f| < f_c \\ 0 & \text{elsewhere} \end{cases}$$

You will be using the system to transmit voice signals with a bandwidth of 5000Hz.

a) Compute $H_z(f) = H_I(f) + jH_Q(f)$.

b) Does the VSB system produce a distortionless output? That is, does $\hat{m}(t) = Km(t)$, where K is some constant?

c) The filter above consists of five frequency segments with constant gain. Because of cost restrictions, you can change the gain on only **one** of the segments. Change one segment gain such that the system **will** produce a distortionless output.

d) What is the bandwidth efficiency of your resulting VSB system and the resulting savings in transmission bandwidth over DSB-AM?

e) What is the spectrum of the corresponding complex envelope equivalent, $H_z(f)$, of the improved filter?

Problem 5.16. This problem is concerned with double sided band amplitude modulation (DSB-AM) of the message signal given in Problem 4.1. Assume a carrier frequency of f_c=200Hz and a carrier amplitude of A_c.

a) Give the baseband, $x_z(t)$, and bandpass, $x_c(t)$, time waveforms for DSB-AM for this message signal as a function of $m(t)$.

b) The bandpass signal is also periodic. What is the period? Give the Fourier series representation for the bandpass signal

c) Give the value of A_c that will produce a 50 Watt output power in a 1 Ω system.

d) Sketch the demodulation process for a received signal $y_c(t)$.

Problem 5.17. This problem is concerned with large carrier amplitude modulation (LC-AM) by the message signal given in Problem 4.1. Assume a carrier frequency of f_c=200Hz, a carrier amplitude of A_c, and a modulation coefficient of a.

a) Give the baseband, $x_z(t)$, and bandpass, $x_c(t)$, time waveforms for LC-AM for this message signal as a function of $m(t)$.

b) Give the Fourier series representation of the bandpass modulated signal as a function of A_c and a.

c) Give the value of a such that the efficiency is maximized and distortion free envelope detection is still possible.

d) What is the efficiency with the value of a computed in c)?

e) With the value of a computed in c) give the value of A_c that will produce a 50 Watt output power in a 1 Ω system.

f) Sketch the demodulation block diagram for a received signal $y_c(t)$.

Problem 5.18. This problem is concerned with single sideband amplitude modulation (SSB-AM) by the message signal given in Problem 4.1. Assume a carrier frequency of f_c=200Hz, and a carrier amplitude of A_c.

a) Give the baseband, $x_z(t)$, and bandpass, $x_c(t)$, time waveforms for an upper sideband SSB-AM for this message signal.

b) Give the Fourier series representation of the bandpass modulated signal as a function of A_c.

c) Give the value of A_c that will produce a 50 Watt output power in a 1 Ω system.

d) Sketch the demodulation block diagram for a received signal, $y_c(t)$, assuming that you known ϕ_p.

5.5 Example Solutions

Not included this edition

5.6 Mini-Projects

Goal: To give exposure

1. to a small scope engineering design problem in communications
2. to the dynamics of working with a team
3. to the importance of engineering communication skills (in this case oral presentations).

Presentation: The forum will be similar to a design review at a company (only much shorter) The presentation will be of 5 minutes in length with an overview of the given problem and solution. The presentation will be followed by questions from the audience (your classmates and the professor). All team members should be prepared to give the presentation.

Project 5.1. Undergraduate classes often only look at idealized problems. For example one thing that is ignored in a course like this one is that frequency sources at the transmitter and receiver cannot be made to have exactly the same value. Consequently the complex envelope of the received signal will be rotating due to this frequency offset, i.e.,

$$y_z(t) = x_z(t) \exp\left[j(\phi_p + 2\pi f_o t)\right] \tag{5.22}$$

where f_o is the exisiting frequency offset. This mini-project will investigate the effects of this frequency offset on one demodulation algorithm for DSB-AM.

Get the Matlab file `chap4ex1.m` and `chap4ex2.m` from the class web page. In these files a DSB-AM transmitter and receiver is implemented. If the carrier frequency offset is set to 10Hz `deltaf=10` the output of the squaring device will be

$$V_z(t) = A_c^2 m^2(t) \exp\left[j40\pi t + j2\phi_p\right] + N_V(t) \tag{5.23}$$

The demodulator implemented in the m-file still works pretty well with this frequency offset. Explain why. Specifically the filter in the open loop phase estimator, $H_z(f)$, has a specific characteristic that enable the frequency offset to be tracked in such a way as to not cause significant distortion. Note at higher frequency offsets (e.g., 100Hz) the performance suffers noticeable distortion. Extra credit will be given if you can figure out a method to eliminate this distortion.

Project 5.2. LC-AM uses a DC offset in $x_I(t)$ to ensure that envelope detection is possible on the received signal. An envelope detector is a very simple demodulator. This DC offset results in wasted transmitted power and loss in transmission efficiency, E_T. Typically the best E_T that can be achieved with voice signals is around 10% if $x_I(t) > 0$. Better E_T can be achieved if the $x_I(t) < 0$ for a small amount of time. This of course will cause distortion in the demodulated signal.

Get the Matlab file `chap4ex3.m` and `chap4ex4.m` from the class web page. In these files a LC-AM transmitter and receiver is implemented. Find the maximum value of a such that no distortion envelope detection is possible. Increase a beyond this value and see how large it can be made before the distortion becomes significant. Calculate the maximum efficiency for the no distortion case and calculate the efficiency when a small amount of distortion is allowed.

Project 5.3. SSB-AM uses the Hilbert Transform in $x_Q(t)$ to eliminate half of the spectral content of a DSB-AM signal. In class we showed that for demodulation a coherent phase reference signal is needed to recover the $x_I(t)$ after the phase shift in the channel. One might imagine that the an offset in the carrier phase reference signal might result in a distorted demodulated signal. For example if the coherent phase reference is $\phi_p + \varphi_e$ then the demodulated output given in Fig. is

$$\hat{m}(t) = m(t)\cos(\varphi_e) + m(t) \otimes h_Q(t)\sin(\varphi_e). \tag{5.24}$$

Get the Matlab file `chap4ex5.m` and `chap4ex6.m` from the class web page. In these files a SSB-AM transmitter and receiver is implemented (note these systems are much more complicated than you will actually need so prune the code accordingly). Find the maximum value of φ_e such that the amount of perceived distortion in the demodulated audio output is small. Note the answer might surprise you (it did me the first time). *Extra credit will be given if the team can explain the reason for this surprising behavior.*

Chapter 6

Analog Angle Modulation

Analog angle modulation embeds the analog message signal in the phase of the carrier or equivalently in the time varying phase angle of the complex envelope (instead of the amplitude as was the case for AM modulation). The general form of angle modulation is

$$x_z(t) = A_c \exp\left[j\Gamma_a(m(t))\right]$$

It is important to note that analog angle modulated signals have a constant envelope (i.e., $x_A(t) = A_c$). This is a practical advantage since most high power amplifiers are not linear and amplitude variations produce distortion (e.g. AM to PM distortion).

6.1 Angle Modulation

Amplitude modulation was the first modulation type to be considered in analog communication systems. Amplitude modulation has the obvious advantage of being simple and relatively bandwidth efficient. The disadvantages of amplitude modulations are:

1. The message is embedded in the amplitude of the carrier signal. Consequently linear amplifiers are very important to obtaining good performance in AM systems. Linear amplifiers are difficult to achieve in applications when either cost or small size are important.

2. When the message signal goes through a quiet period in DSB-AM or SSB-AM systems very small carrier signals are transmitted. This absence of signal tends to accentuate the noise.

3. The bandpass bandwidth in AM systems is directly dependent on the message signal bandwidth. There is no opportunity to use a wider bandwidth to achieve better performance.

These enumerated shortcomings of AM can be addressed by using angle modulation. Angle modulations modulate the angle of the complex envelope with the message signal.

The first issue to address is what is meant by the phase and frequency of a bandpass signal and how it relates to the complex envelope.

Definition 6.1 *When a signal has the form*

$$x(t) = \sqrt{2}\cos\left(\theta(t)\right)$$

then the instantaneous phase is $\theta(t)$ and the instantaneous frequency is

$$f_i(t) = \frac{1}{2\pi}\frac{d\theta(t)}{dt}. \tag{6.1}$$

Example 6.1: When
$$x(t) = \sqrt{2}\cos\left(2\pi f_m t\right)$$
then $\theta(t) = 2\pi f_m t$ and $f_i(t) = f_m$.

Definition 6.2 *For a bandpass signal having the form*

$$x_c(t) = \sqrt{2} x_A(t)\cos\left(2\pi f_c t + x_P(t)\right)$$

then the instantaneous phase is $2\pi f_c t + x_P(t)$ *and the instantaneous frequency is*

$$f_i(t) = f_c + \frac{1}{2\pi}\frac{dx_P(t)}{dt} = f_c + f_d(t) \tag{6.2}$$

where $f_d(t)$ *is denoted the instantaneous frequency deviation.*

The instantaneous frequency deviation is a measure of how far the instantaneous frequency of the bandpass signal has deviated from the carrier frequency.

There are two prevalent types of analog angle modulated signals; phase modulated (PM) and frequency modulated signals (FM). Analog PM changes the phase angle of the complex envelope in direct proportion to the message signal

$$x_z\left(t\right) = \Gamma_m\left(m(t)\right) = A_c \exp\left[jk_p m\left(t\right)\right]$$

where k_p (units of radians/volt) is the phase deviation constant. Similarly FM changes the instantaneous frequency deviation in direct proportion to the message signal. Note that the instantaneous radian frequency is the derivative of the phase so that the phase of a bandpass signal can be expressed as

$$x_P(t) = \int_{-\infty}^{t} 2\pi f_d(\lambda) d\lambda.$$

Consequently FM signals have the following form

$$x_z(t) = \Gamma_m\left(m(t)\right) = A_c \exp\left[j\int_{-\infty}^{t} k_f m(\lambda) d\lambda\right] = A_c \exp\left[j\int_{-\infty}^{t} 2\pi f_d m(\lambda) d\lambda\right]$$

where k_f is the radian frequency deviation constant (units radians/second/volt) and f_d is the frequency deviation constant (units Hertz/volt).

The bandpass angle modulated signals are constant envelope signals with a phase or a frequency modulation. For PM the baseband signals are

$$x_I(t) = A_c\cos\left(k_p m(t)\right) \qquad x_Q(t) = A_c\sin\left(k_p m(t)\right)$$

and bandpass signal is

$$x_c(t) = A_c\sqrt{2}\cos\left(2\pi f_c t + k_p m(t)\right).$$

For FM the baseband signals are

$$x_I(t) = A_c\cos\left(k_f\int_{-\infty}^{t} m(\lambda) d\lambda\right) \qquad x_Q(t) = A_c\sin\left(k_f\int_{-\infty}^{t} m(\lambda) d\lambda\right)$$

and the bandpass signal is

$$x_c(t) = A_c\sqrt{2}\cos\left(2\pi f_c t + k_f \int_{-\infty}^{t} m(\lambda)d\lambda\right).$$

An example message signal and the integral of this message signal is given in Fig. 6.1. The corresponding bandpass phase modulated signal is shown in Fig. 6.2. The corresponding bandpass FM modulated signal is given in Fig. 6.3. It is obvious from this plot that the instantaneous frequency of the carrier[1] is being modulated in proportion to the message signal.

Example 6.2: Phase modulation with

$$m(t) = A_m \sin(2\pi f_m t)$$

produces a complex envelope of

$$x_z(t) = A_c \exp\left[j\beta_p \sin(2\pi f_m t)\right] \text{ (PM)} \qquad x_z(t) = A_c \exp\left[j\beta_f \cos(2\pi f_m t) + \theta_f\right] \text{ (FM)}$$

where $\beta_p = A_m k_p$, $\beta_f = \frac{-k_f A_m}{2\pi f_m}$ and θ_f is a constant phase angle that depends on the initial conditions of the integration. The in-phase and quadrature signals then have the form

$$x_I(t) = A_c \cos\left[\beta_p \sin(2\pi f_m t)\right] \qquad x_Q(t) = A_c \sin\left[\beta_p \sin(2\pi f_m t)\right]. \qquad \text{(PM)}$$

and

$$x_I(t) = A_c \cos\left[\beta_f \cos(2\pi f_m t) + \theta_f\right] \qquad x_Q(t) = A_c \sin\left[\beta_f \cos(2\pi f_m t) + \theta_f\right]. \qquad \text{(FM)}$$

6.1.1 Angle Modulators

Two methods are typically used to produce analog angle modulation in modern communication systems; the voltage controlled oscillator (VCO) and the the direct digital synthesizer. The VCO is a device which directly generates an FM signal and the block diagram is given in Fig 6.4. The VCO varies the frequency of the oscillator in direct proportion to the input signal and it is very useful in a variety of applications besides FM modulation as well. Note the VCO was first introduced as a component in a Costas Loop in Chapter 5. The direct digital synthesizer is detailed in Fig 6.5. The structure is most often implemented with digital processing.

The output power of an angle modulated signal is of interest. To this end using (4.7), the power in an angle modulated is given as

$$P_{x_c} = A_c^2 \tag{6.3}$$

For an angle modulated waveform the output power is only a function of the power associated with the carrier (A_c^2) and not a function of the message signal at all.

6.2 Spectral Characteristics

Unfortunately a general expression cannot be found for a Fourier transform of the form

$$X_z(f) = \mathcal{F}\left\{\exp\left[ju(t)\right]\right\}$$

so no general results can be given for angle modulated spectrums.

[1]The instantaneous carrier frequency can be estimated by noting the frequency of the peaks of the sinusoid.

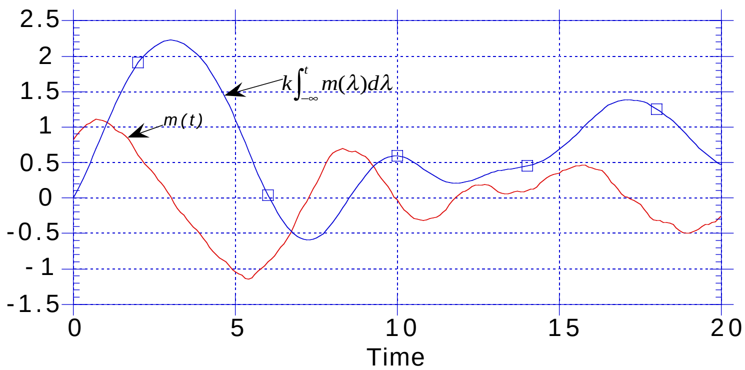


Figure 6.1: An example message signal and message signal integral.

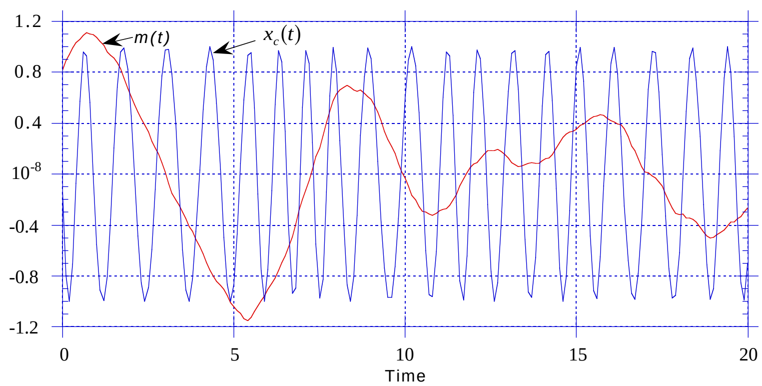


Figure 6.2: The PM modulated bandpass waveform corresponding to the message in Fig. 6.1.

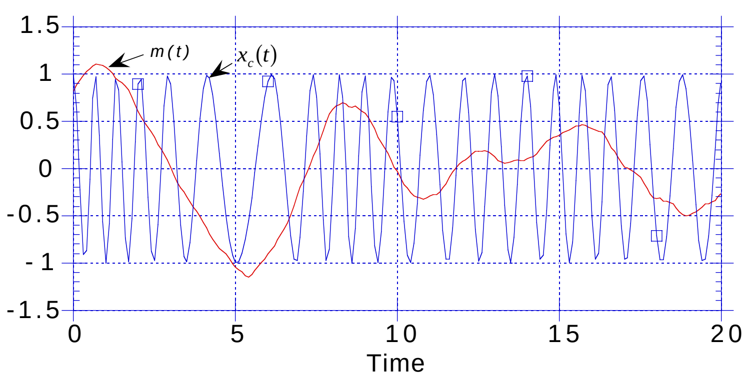


Figure 6.3: The FM modulated bandpass waveform corresponding to the message in Fig. 6.1.

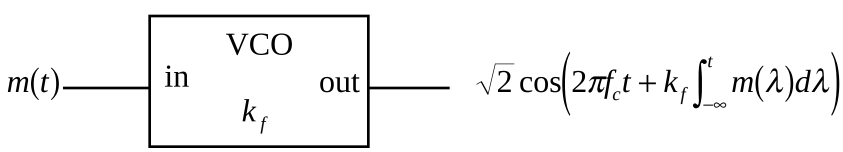


Figure 6.4: A voltage controlled oscillator (VCO).

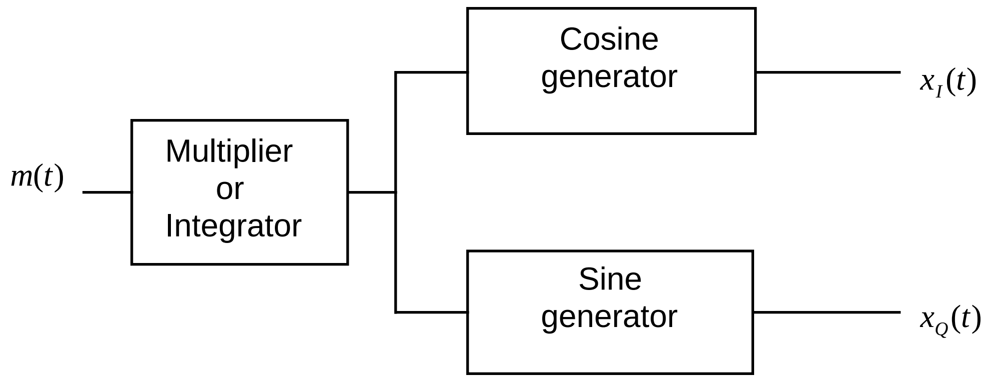


Figure 6.5: A direct digital synthesizer (DDS).

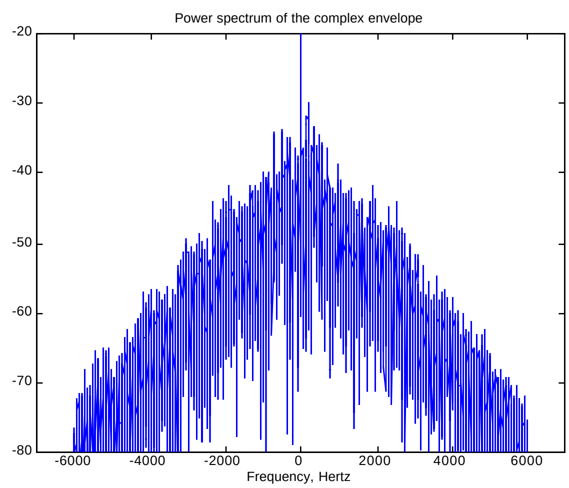


Figure 6.6: The power spectral density of the computer generated voice signal from Example 1.15. k_p=1.

Example 6.3: The computer generated voice signal given in Example 1.15 (W=2.5KHz) is used with phase modulation (k_p=1). The power spectrum of the complex envelope of the PM signal is shown in Fig. 6.6. Note there is no apparent relationship between the baseband spectrum (see Fig. 1.6) and the bandpass modulated spectrum. It is interesting to note that the apparent tone at DC is due to the periods of silence in the message signal ($m(t)$=0).

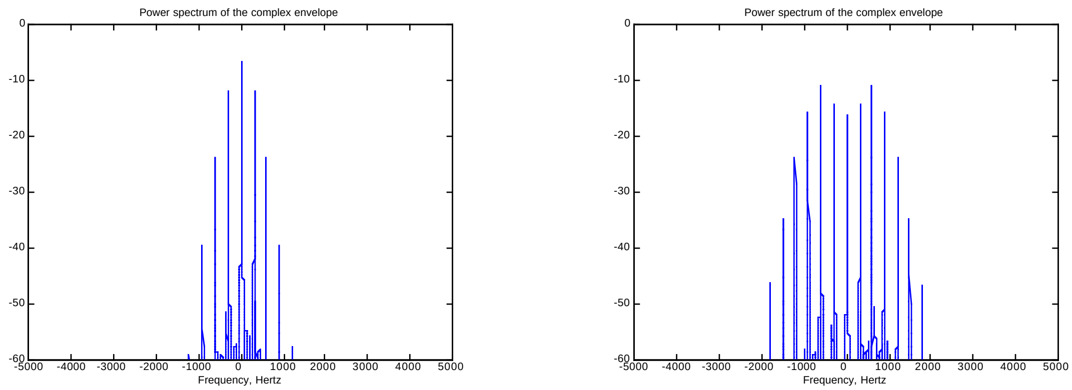
6.2.1 A Sinusoidal Message Signal

As a first step in understanding the spectrum of angle modulated signals let's consider the simple case of

$$x_z(t) = \exp\left[j\beta \sin(2\pi f_m t)\right]. \tag{6.4}$$

This complex envelope corresponds to a phase modulated signal where

$$m(t) = \frac{\beta}{k_p} \sin\left(2\pi f_m t\right)$$



a) $\beta = 1$ b) $\beta = 3$

Figure 6.7: The measured power spectrum of an angle modulation with a sinusoidal message signal.

or a frequency modulated signal where

$$m(t) = \beta \frac{f_m}{f_d} \cos\left(2\pi f_m t\right).$$

It is important in this setup to note that for PM signals β is proportional to the message amplitude and the phase deviation constant while in FM β is directly proportional to the message amplitude and the radian frequency deviation constant while inversely proportional to the message frequency (see Example 6.2). The signal $x_z(t)$ is a periodic signal with period $T = 1/f_m$, i.e., it is obvious that $x_z(t) = x_z\left(t + 1/f_m\right)$. The obvious tool for spectral representation is the Fourier series.

Example 6.4: Consider an angle modulation with a sinusoidal message signal with f_m=300Hz, A_c=1, and β=1,3. The measured power spectrum of the angle modulated complex envelope is shown in Fig. 6.7. The impulsiveness of the spectrum indicates the signal is periodic and the fundamental frequency is f_m=300Hz. It can be deduced from Fig. 6.7 that a larger value of β produces a larger bandwidth.

The Fourier series is given as

$$x_z(t) = \sum_{n=-\infty}^{\infty} z_n \exp\left[j 2\pi f_m n t\right] \tag{6.5}$$

where

$$z_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x_z(t) \exp\left[-j\frac{2\pi n t}{T}\right] dt.$$

A first spectral characteristic obvious on examining (6.5) is that angle modulation can potentially produce a bandwidth expansion compared to AM modulation. Note that a DSB-AM signal with a message signal of

$$m(t) = \sin\left(2\pi f_m t\right)$$

will produce a bandpass bandwidth of $2f_m$. If $z_n \neq 0 \quad \forall \quad |n| \geq 2$ then an angle modulated signal will have a bandwidth greater than that of DSB-AM and $E_B < 50\%$. The actual coefficients of Fourier series can be obtained by noting that

$$z_n = f_m \int_{-\frac{1}{2f_m}}^{\frac{1}{2f_m}} A_c \exp\left[-j\left(2\pi f_m n t - \beta \sin(2\pi f_m t)\right)\right] dt. \tag{6.6}$$

The change of variables

$$\theta = 2\pi f_m t \qquad d\theta = 2\pi f_m dt$$

simplifies (6.6) to give

$$z_n = \frac{A_c}{2\pi} \int_{-\pi}^{\pi} \exp\left[-j(\theta n - \beta \sin\theta)\right] d\theta = A_c J_n(\beta)$$

where $J_n(\bullet)$ is the Bessel function of the first kind order n. A convenient form for the spectrum of the angle modulation for a sinusoidal message signal is now available once we characterize the Bessel function.

The Bessel Function

The Bessel function of the first kind is a transcendental function much like sine or cosine and has the following definition [Ae72]

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left[-j(n\theta - x\sin\theta)\right] d\theta.$$

Plots of the Bessel function of the first kind for various orders are seen in Fig. 6.8. The Bessel function

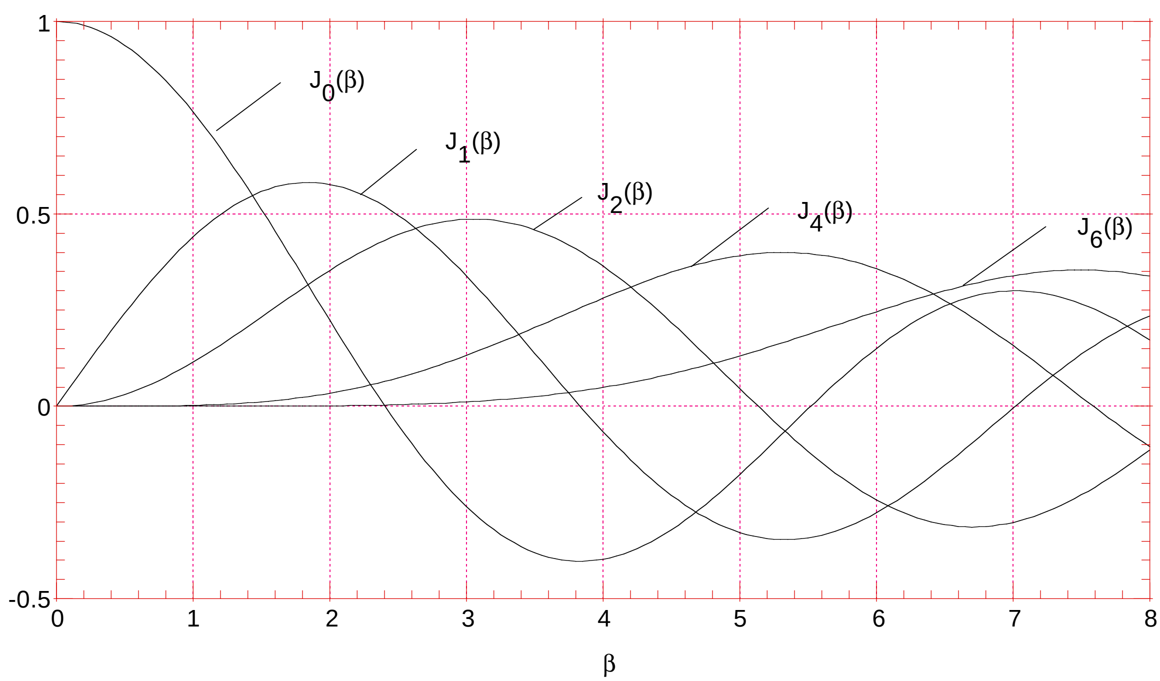


Figure 6.8: The Bessel function of the first kind.

has the following important characteristics

1. $J_n(x)$ is real valued,
2. $J_n(x) = J_{-n}(x) \quad n$ even,
3. $J_n(x) = -J_{-n}(x) \quad n$ odd,
4. $\sum\limits_{n=-\infty}^{\infty} J_n^2(x) = 1$,
5. $\lim\limits_{n\rightarrow\infty} |J_n(x)| = 0$,
6. When $\beta \ll 1$
   (a) $J_0(\beta) \approx 1$,
   (b) $J_1(\beta) \approx \frac{\beta}{2}$,
   (c) $J_n(\beta) \approx 0 \quad \forall \quad |n| > 1$.

The first three characteristics indicate that the power of the angle modulated signal is distributed evenly around zero. The fourth characteristic is simply a consequence of Parseval's theorem, i.e.,

$$P_{x_z} = \frac{1}{T}\int_0^T |x_z(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |z_n|^2 = A_c^2 \sum_{n=-\infty}^{\infty} |J_n(\beta)|^2 = A_c^2$$

The fifth characteristic simply indicates that the angle modulated signal has a finite practical bandwidth (the Fourier coefficients converge to zero). The sixth set of characteristics are important to understanding the case of narrowband angle modulation.

Example 6.5: The previous example considered a sinusoidal message signal. Comparing the measured power spectrum in Fig. 6.7 to the results one could obtain from Fig. 6.8 shows a perfect match between theory and measurement.

Engineering Bandwidth

The bandwidth of angle modulation for a sinusoidal message signal is controlled by β. All terms of the Fourier series expansion of the complex envelope are such that, $z_n = J_n(\beta) \neq 0$, so to transmit this angle modulated waveform without distortion requires an infinite bandwidth. As noted above

$$\lim_{n\rightarrow\infty} J_n(\beta) = 0$$

so a bandwidth can be selected such that the distortion is appropriately small. A traditional design practice sets the transmission bandwidth to be the 98% power bandwidth. This practice selects the bandwidth such that K harmonics of message signal where the value of K satisfies

$$\sum_{n=-K}^{K} J_n^2(\beta) \geq 0.98. \tag{6.7}$$

While this value can be evaluated for each β a good approximation for $\beta > 1$ is given as $K = \beta + 1$ so that the transmission bandwidth accurately approximated as

$$B_T = 2(\beta + 1) f_m. \tag{6.8}$$

and consequently the bandwidth efficiency is given as

$$E_B = \frac{W}{B_T} = \frac{1}{2(\beta + 1)} \tag{6.9}$$

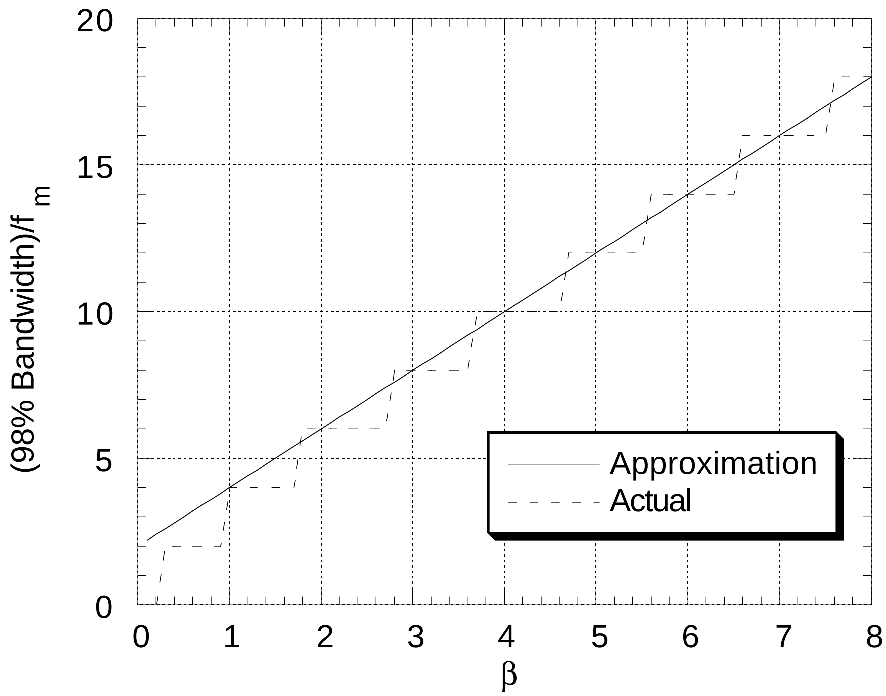


Figure 6.9: A comparison of the actual transmission bandwidth (6.7) and the rule of thumb approximation (6.8) for angle modulation with a sinusoidal message signal.

Fig. 6.9 show a comparison between the actual 98% transmission bandwidth and the simple approximation given in (6.8). Consequently setting β for a particular message frequency, f_m, sets the transmission bandwidth. Commercial FM broadcast can the thought of as having a $0.8 < \beta < 5$, but we will discuss this in more detail later.

Narrowband Angle Modulation

Interesting insights can be obtained by examining $\beta \ll 1$ which is known as narrowband (NB) angle modulation. Using the characteristics of the Bessel function for small arguments only DC and f_m spectral lines are significant, i.e., $z_0 \approx 1$ and $z_{\pm 1} = \pm\beta/2$. The resulting spectrum is given in Fig. 6.10 and we have

$$\begin{aligned} x_z(t) &\approx A_c\left(1 + \beta/2 \exp\left[j2\pi f_m t\right] - \beta/2 \exp\left[-j2\pi f_m t\right]\right) \\ &= A_c\left(1 + j\beta \sin\left[2\pi f_m t\right]\right). \end{aligned} \tag{6.10}$$

Note this implies that

$$x_I(t) = A_c \qquad x_Q(t) = A_c\beta \sin\left[2\pi f_m t\right]. \tag{6.11}$$

It is interesting to note that by examining (6.11) one can see that NB angle modulation is quite similar to LC-AM. Both have a large carrier component but the message signal is modulated on the in-phase component of the complex envelope in LC-AM while the message signal is modulated on the quadrature component in NB angle modulation. Also it should be noted that no bandwidth expansion of an angle modulated signal compared to an AM signal occurs in this case as the bandpass bandwidth in both cases is $2f_m$. This implies that the bandwidth efficiency of narrowband angle modulation is also $E_B = 50\%$.

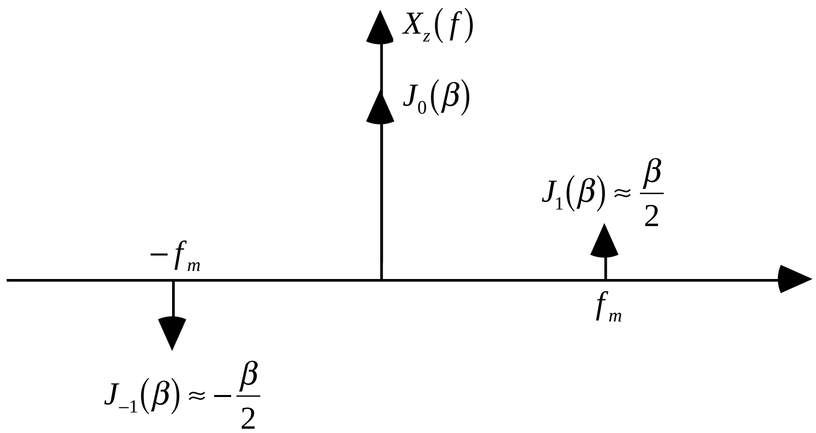


Figure 6.10: The resulting spectrum of a narrowband angle modulation with a sinusoidal message signal.

6.2.2 General Results

We cannot obtain similar analytical results for a general message signal but we can use the results in Section 6.2.1 to provide general guidelines for angle modulation characteristics.

Engineering Bandwidth

The ideas of engineering bandwidth can be extended by defining

$$\begin{aligned} D = \frac{\text{peak frequency deviation}}{W} &= \frac{f_d}{W}\left[\max\ |m(t)|\right] \qquad \text{FM} \\ &= \frac{k_p \max\left|\frac{d}{dt}m(t)\right|}{2\pi W}. \qquad \text{PM} \end{aligned} \tag{6.12}$$

D is roughly the equivalent factor for an arbitrary message signal that β is for a sinusoidal message signal[2]. Consequently an engineering approximation to the 98% power bandwidth of an angle modulated signal is

$$B_T = 2(D+1)W. \tag{6.13}$$

and the bandwidth efficiency is

$$E_B = \frac{1}{2(D+1)}. \tag{6.14}$$

This is known as Carson's rule and is a common rule of thumb used for the engineering bandwidth of angle modulated signals.

[2]In fact if $m(t)$ is a sinusoidal signal it is easy to show $D = \beta$.

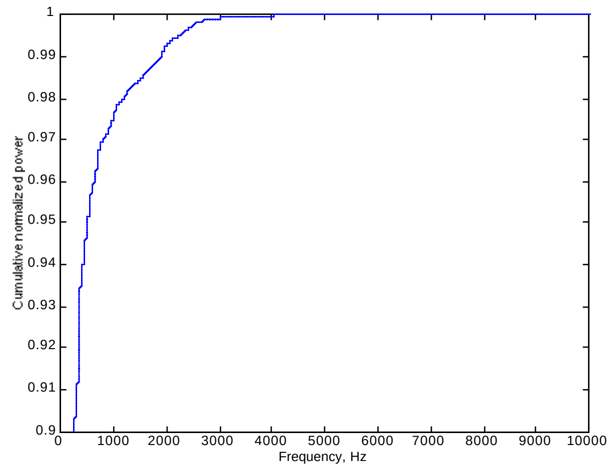


Figure 6.11: The normalized cumulative power of the phase modulated computer generated voice saying "Bingo". $k_p = 0.8$.

Example 6.6: Commercial FM broadcast was originally configured with the following characteristics

1. Channel separation = 200kHz,
2. Peak frequency deviation = $f_d \left[\max |m(t)|\right] \leq$ 75kHz (Set by the FCC),
3. Message bandwidth = W = 15kHz (mono broadcast).

Consequently we have

$$D \leq \frac{75\ KHz}{15\ KHz} = 5$$

and

$$B_T = 2(D+1)\ 15\ KHz = 180\ KHz < 200\ KHz.$$

Modern FM broadcast has two audio channels (stereo) and sometimes even a auxiliary channel so $W \geq$ 15kHz. The later discussion of multiplexing analog signals will provide more details.

Example 6.7: The computer generated voice signal given in Example 1.15 (W=2.5KHz) is used with phase modulation (k_p=0.8). The peak frequency deviation constant is measured to be about 3067Hz in this case. Carson's bandwidth rule of thumb predicts a B_T=11000Hz would be required. The measured cumulative power is plotted in Fig. 6.11. The measured 98% bandwidth (approximately 4KHz) is much less than that predicted by Carson' rule. This characteristic is due to the message signal having fairly long passages of silence ($m(t) \approx 0$). This tends to concentrate more power around DC. Filtering the transmitted signal to a bandwidth less than Carson's bandwidth will distort the bandpass signal when the message signal is large and result in degraded performance. The bandpass PM signal (f_c=5500Hz) filtered to Carson's bandwidth has a measured power spectrum shown in Fig. 6.12.

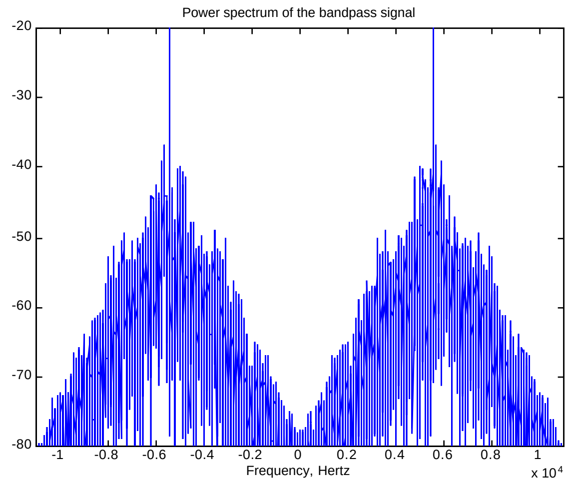


Figure 6.12: The bandpass spectrum for the phase modulated computer generated voice saying "Bingo" filtered at Carson's bandwidth. $k_p = 0.8, f_c = 5500$Hz.

Narrowband Angle Modulation

Recall that PM has the complex envelope

$$x_z(t) = A_c \exp\left[jk_p m(t)\right]$$

and FM has the complex envelope

$$x_z(t) = A_c \exp\left[j \int_{-\infty}^{t} k_f m(\lambda) d\lambda\right].$$

Narrowband angle modulation results when either k_p (PM) or k_f (FM) is small such that the total phase deviation caused by the modulation is small. Consequently we can use the Taylor series expansion and examine the first order approximation, i.e.,

$$x_z(t) \approx A_c\left(1 + jk_p m(t)\right) \quad \text{(PM)} \qquad x_z(t) \approx A_c\left(1 + jk_f \int_{-\infty}^{t} m(\lambda)d\lambda\right) \text{(FM)}.$$

This leads to a bandpass signal of the form

$$\begin{aligned} x_c(t) &\approx A_c\sqrt{2}\cos\left(2\pi f_c t\right) - A_c k_p m(t)\sqrt{2}\sin\left(2\pi f_c t\right) \quad \text{(PM)} \\ x_c(t) &\approx A_c\sqrt{2}\cos\left(2\pi f_c t\right) - A_c k_f \int_{-\infty}^{t} m(\lambda)d\lambda\sqrt{2}\sin\left(2\pi f_c t\right) \quad \text{(FM)}. \end{aligned} \tag{6.15}$$

The bandpass implementation is seen in Fig. 6.13 and is very similar to the LC-AM modulator presented in Fig. 5.12. This modulator was often used in the early days of broadcast as one stage in a wideband angle modulator. This idea will be explored in the homework.

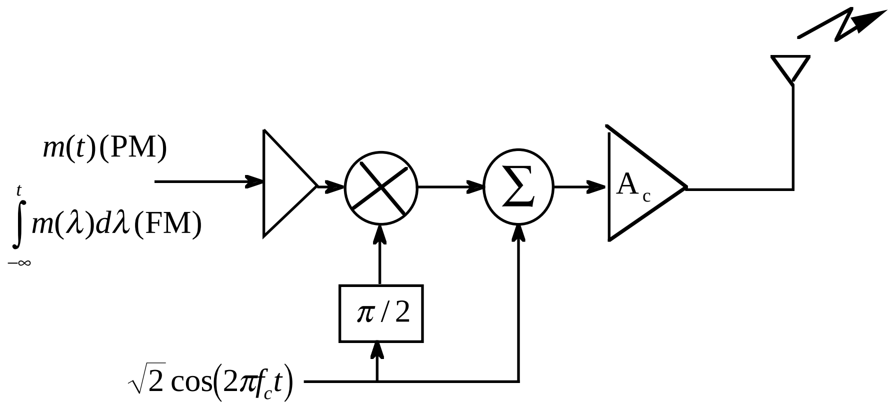


Figure 6.13: A bandpass implementation of a narrowband angle modulator.

6.3 Demodulation of Angle Modulations

Since the modulation only effects the angle of the complex envelope, the demodulation focus on the phase of the received complex envelope. Recall the received complex envelope has the form

$$y_z(t) = x_z(t)\exp\left[j\phi_p\right] + \text{noise}.$$

Consequently, the received complex envelope of the angle modulated signal is given as

$$\begin{aligned} y_z(t) &= A_c \exp\left[j\left(k_p m(t) + \phi_p\right)\right] + \text{Noise} \quad \text{(PM)} \\ y_z(t) &= A_c \exp\left[j\left(k_f \int_{-\infty}^{t} m(\lambda)d\lambda + \phi_p\right)\right] + \text{Noise} \quad \text{(FM)}. \end{aligned} \tag{6.16}$$

The obvious detection approach for PM is to compute the angle of this complex envelope, $y_p(t)$, and implement a DC block to eliminate the phase shift induced by the propagation delay in the channel. The output signal would then be of the form

$$\hat{m}(t) = k_p m(t) + \text{Noise} \tag{6.17}$$

To recover the message signal in the FM case this direct phase detector could be implemented and followed by a differentiator.

$$\hat{m}(t) = \frac{d}{dt}\left(k_f \int_{-\infty}^{t} m(\lambda)d\lambda + \text{Noise}\right) = k_f m(t) + \text{Noise} \tag{6.18}$$

Fig 6.14 shows the block diagram for such a detector. This detector is known as a direct phase detector. Note that the DC block is unnecessary in the case of FM because the constant phase shift will be eliminated by the derivative. This demodulation technique does not have to know or estimate ϕ_p hence it is a **noncoherent** demodulator. Note the demodulated outputs in angle modulations cannot easily be related to the transmitted power. Consequently the notion of a transmission efficiency in angle modulations is not well defined. This notion will be revisited after the effects of noise on angle demodulation is discussed in Chapter 10.

One practical characteristic of a direct phase detector that must be accommodated is that phase is typically measured in the range $[-\pi, \pi]$. A phase shift or a large phase deviation will cause the measured

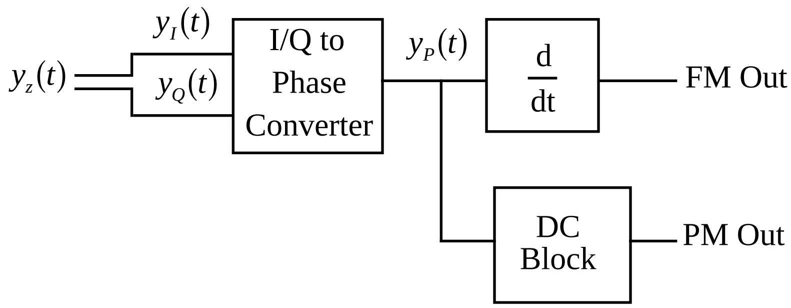


Figure 6.14: The direct phase detector for analog angle modulation.

phase to wrap around the range $[-\pi, \pi]$. Recovery of the message signal requires that measured phase be unwrapped. An example of this phase unwrapping is shown in Fig.6.15. The unwrapping ensures that the output phase from the direct phase detector is a smooth function of time. The unwrapping function is implemented in Matlab in the function `unwrap`.

Example 6.8: Consider the PM signal generated in Example 6.7 with a measured transmitted power spectrum given in Fig. 6.12. In a channel with a propagation delay $\tau_p = 45.3\mu$s a phase shift of $\phi_p = -89.8°$ is produced in transmission. The vector diagram of the received complex envelope is shown in Fig. 6.16. The received signal does not have a constant amplitude due to the filtering to Carson's bandwidth at the transmitted. The phase shift due to the propagation delay in transmission is clearly evident in Fig. 6.16. It is clear that the phase shift incurred in transmission will require phase unwrapping in a direct phase demodulator as the received signal phase crosses the $[-\pi, \pi]$ boundary frequently. The direct detected phase is given in Fig. 6.16 and the unwrapped phase is given in Fig. 6.17. The unwrapped phase has a DC offset due to the propagation delay in the channel. The output of the DC block is also given in Fig. 6.17. The demodulated output signal has little distortion compared to the original computer generated message signal introduced in Chapter 1. Note in this example there is a little glitch at the beginning of the transmission due to the transient response of the DC block.

A second detector called a discriminator has frequently been used in practice because it can be implemented without an active device. The basis for the operation of the discriminator is found in the relation

$$\frac{d}{dt}e^{a(t)} = \frac{d\,a(t)}{dt}e^{a(t)},$$

or equivalently

$$\frac{d}{dt}\cos\left(a(t)\right) = \frac{d\,a(t)}{dt}\cos\left(a(t) + \pi/2\right).$$

For angle modulated signals this implies that the derivative of the complex envelope will have the form

$$\begin{aligned}\frac{d\,y_z(t)}{dt} &= jk_p\frac{d\,m(t)}{dt}A_c\exp\left[j\left(k_p m(t) + \phi_p\right)\right] \quad \text{(PM)}\\ \frac{d\,y_z(t)}{dt} &= jk_f m(t)A_c\exp\left[j\left(k_f\int_{-\infty}^{t} m(\lambda)d\lambda + \phi_p\right)\right] \quad \text{(FM)}.\end{aligned} \tag{6.19}$$

Note that taking the derivative of an angle modulated signal produces an amplitude modulation which is a direct function of the message signal. If we can simply demodulate the AM then we have a simple angle demodulator. Recall the simplest AM demodulator is the envelope detector but this only works for

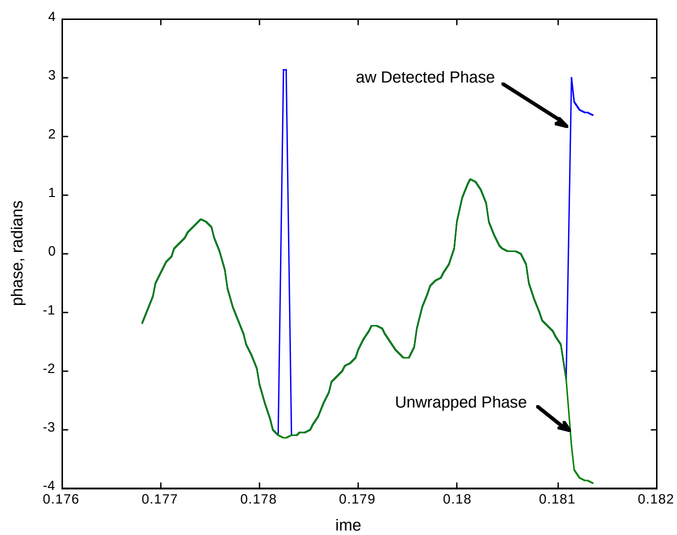
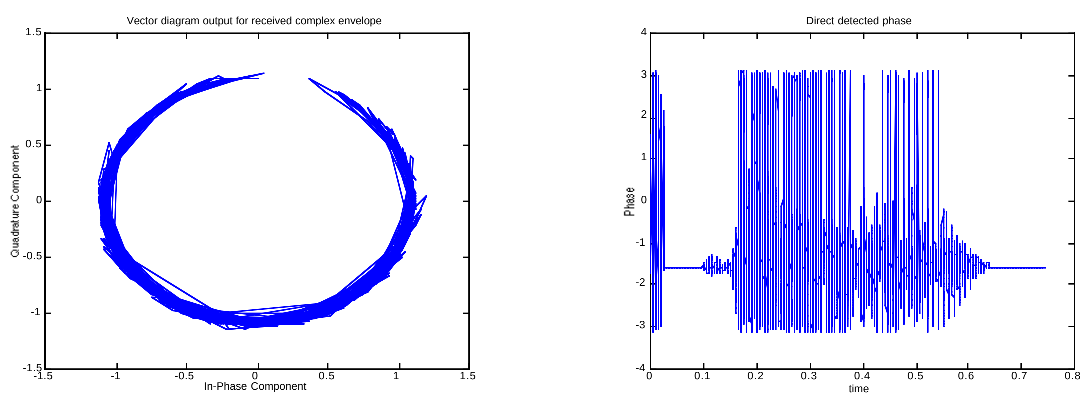


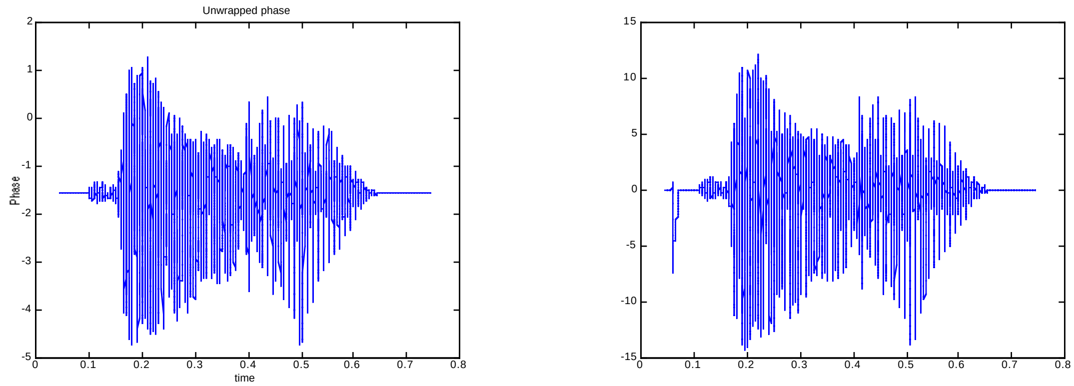
Figure 6.15: An example of phase unwrapping in direct phase detection.



a) Received vector diagram

b) Directly detected phase

Figure 6.16: The signals in a direct phase detector.



a) The unwrapped phase   b) The signal after a DC block

Figure 6.17: The signals in a direct phase detector.

LC-AM. Fortunately similar processing can be implemented to produce a large carrier signal amplitude modulated signal. Recall that the form of the bandpass signal is

$$\begin{aligned} y_c(t) &= A_c\sqrt{2}\cos\left(2\pi f_c t + k_p m(t) + \phi_p\right) \quad \text{(PM)} \\ y_c(t) &= A_c\sqrt{2}\cos\left(2\pi f_c t + k_f \int_{-\infty}^{t} m(\lambda)d\lambda + \phi_p\right) \quad \text{(FM)}. \end{aligned} \tag{6.20}$$

so taking a derivative at bandpass actually gives a LC-AM signal, i.e.,

$$\begin{aligned} \frac{d\ y_c(t)}{dt} &= A_c\left(k_p\frac{d\ m(t)}{dt} + 2\pi f_c\right)\sqrt{2}\cos\left(2\pi f_c t + k_p m(t) + \phi_p + \pi/2\right) \quad \text{(PM)} \\ \frac{d\ y_c(t)}{dt} &= A_c\left(k_f m(t) + 2\pi f_c\right)\sqrt{2}\cos\left(2\pi f_c t + k_f \int_{-\infty}^{t} m(\lambda)d\lambda + \phi_p + \pi/2\right) \quad \text{(FM)}. \end{aligned} \tag{6.21}$$

Note that f_c is usually very large in comparison with the frequency deviation of the angle modulation so that the resultant signal is an LC-AM and the modulation can be recovered by envelope detection. Fig. 6.18 shows the block diagram of the discriminator for angle modulations. Implementation of the discriminator can be accomplished without active devices (see Fig. 5.39 in [PS94]) so it was popular in the early days of FM broadcast. The discriminator also does not need to know or estimate ϕ_p so it is also a **noncoherent** demodulation scheme.

6.4 Comparison of Analog Modulation Techniques

A summary of the important characteristics of analog modulations is given in Table 6.1. At the beginning of Chapter 4 the performance metrics for analog communication systems were given as: complexity, performance, and spectral efficiency. At this point in our development we are able to understand the tradeoffs involved in two of the three metrics (spectral efficiency and complexity). The characterization of the performance will have to wait until we develop the tools to characterize noise and interference.

In complexity the angle modulations offer the best characteristics. Transmitters for all of the analog modulations are reasonable simple to implement. The most complicated transmitters are needed for

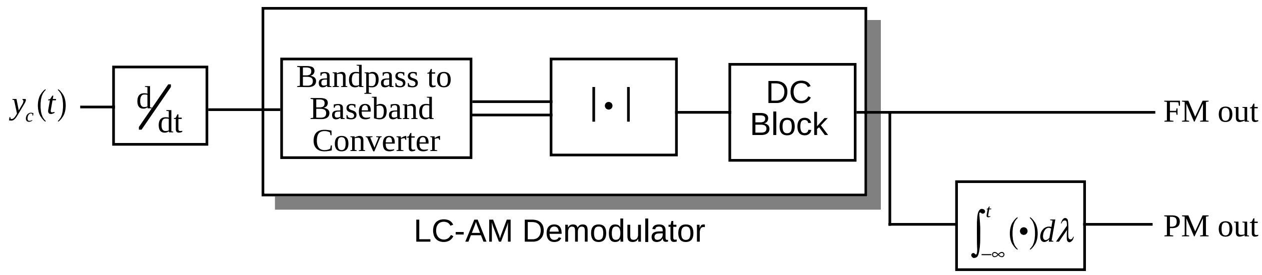


Figure 6.18: Baseband model of a discriminator detector for angle modulation.

Modulation	E_B	E_T	Transmitter Complexity	Receiver Complexity	Performance
DSB-AM	50%	100%	moderate	moderate	?
LC-AM	50%	$<$ 50%	moderate	small	?
VSB-AM	$>$ 50%	$<$ 100%	large	large	?
SSB-AM	100%	50%	large	large	?
PM	$<$ 50%	?	small	moderate	?
FM	$<$ 50%	?	small	moderate	?

Table 6.1: Summary of the important characteristics of analog modulations.

VSB-AM and SSB-AM. The complexity is need to achieve the bandwidth efficiency. The angle modulations have a transmitted signal with a constant envelope. This makes designing the output amplifiers much simpler since the linearity of the amplifier is not a significant issue. Receivers for LC-AM and angle modulation can be implemented in simple noncoherent structures. Receivers for DSB-AM and VSB-AM require the use of a more complex coherent structure. SSB-AM additionally will require a transmitted reference for automatic coherent demodulation and operation.

In spectral efficiency SSB-AM offers the best characteristics. SSB-AM is the only system that achieves 100% efficiency. DSB-AM and LC-AM are 50% efficient. VSB-AM offers performance somewhere between SSB-AM and DSB-AM. Angle modulations are the least bandwidth efficient. At best they achieve $E_B = 50\%$, and often much less.

This discussion demonstrates the tradeoffs typically encountered in engineering practice. Rarely does one option lead in all of the possible performance metric categories. The tradeoffs between analog modulations will become even more complex when the performance is included.

6.5 Homework Problems

Problem 6.1. A message signal is given as

$$\begin{aligned} m(t) &= -1+2t \qquad 0 \le t < 1 \\ &= 2-t \qquad 1 \le t < 2 \\ &= 0 \qquad \text{elsewhere.} \end{aligned} \tag{6.22}$$

Assume A_c=1.

a) If $m(t)$ is phase modulated with $k_p = \pi$ radians/volt, plot $x_P(t)$, $x_I(t)$, and $x_Q(t)$.

b) If $m(t)$ is frequency modulated with $k_f = \pi$ radians/second/volt, plot $x_P(t)$, $x_I(t)$, and $x_Q(t)$.

c) If $m(t)$ is frequency modulated with $k_f = 10\pi$ radians/second/volt, plot $x_P(t)$, $x_I(t)$, and $x_Q(t)$.

Problem 6.2. As a first task at Swensson, your boss asks you to design a PM modulator (obvious busy work) for a 1KHz, 2 V peak amplitude test tone.

a) She wants a 5 Watt output power across a 1Ω resistor with a 98% power bandwidth of 20KHz. Provide the values of A_c and k_p to achieve this specification.

b) If you chose $m(t) = \sin(2\pi f_m t)$, then the Fourier series coefficients are a direct lift from lecture. What if $m(t) = \cos(2\pi f_m t)$?

Problem 6.3. A true angle modulated signal has an infinite bandwidth but the Federal Communications Commission (FCC) makes each station strictly limit their output frequency content as shown in Fig. 6.19. For the case of sinusoidal angle modulation as discussed in class with $\beta = 3$ plot the output waveforms, $\tilde{x}_c(t)$, when the angle modulation is ideally bandlimited before transmission. Consider plots where the bandlimiting is done at 30%, 75%, 100% and 150% of Carson's bandwidth. For uniformity in the answers assume that f_m=1KHz and f_c=20KHz. The plots should have time points no more than 5μs apart.

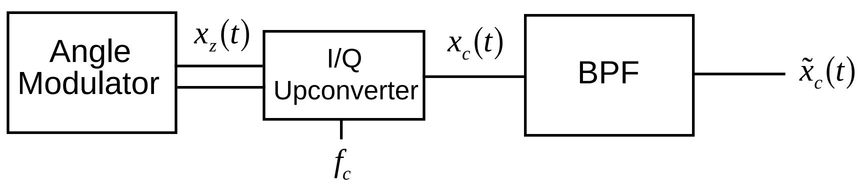


Figure 6.19: Bandlimiting the output of an angle modulator for Problem 6.3.

Problem 6.4. The US company you work for is pathetically behind the foreign competition in building portable PM radio transmitters. Your boss has obtained a transmitter from the competition and has asked you to reverse engineer the product. In the first test you perform, you input a signal into the phase modulator of the form

$$m(t) = \sin(2\pi f_m t)$$

The resulting normalized amplitude spectrum is measured (where $A_c = \sqrt{2}$) and plotted in Fig. 6.20.

a) What is f_m?

b) What is the phase deviation constant, k_p? *Hint: It is a whole number for simplicity.*

c) Draw a block diagram for a demodulator for this signal.

Problem 6.5. In Problem 6.3 you were asked to compute the output waveforms when a sinusoidally modulated FM or PM signal is bandlimited at various fractions of Carson's bandwidth. We return to this problem and look at the distortion in demodulation of this signal for the various bandwidths.

a) Find the functions $y_I(t)$ and $y_Q(t)$ assuming that $\phi_p = \pi/4$.

b) Compute the output from a direct phase detector. Will unwrapping the phase be necessary to reconstruct the signal?

c) Show that the signal can be written in the form $\hat{m}(t) = Am(t) + n(t)$ and identify A and $n(t)$. $n(t)$ here represents the distortion due to nonideal processing at the transmitter and the receiver. This distortion is much different than the noise that will be introduced in Chapters 8 and 9 but produces similar effects.

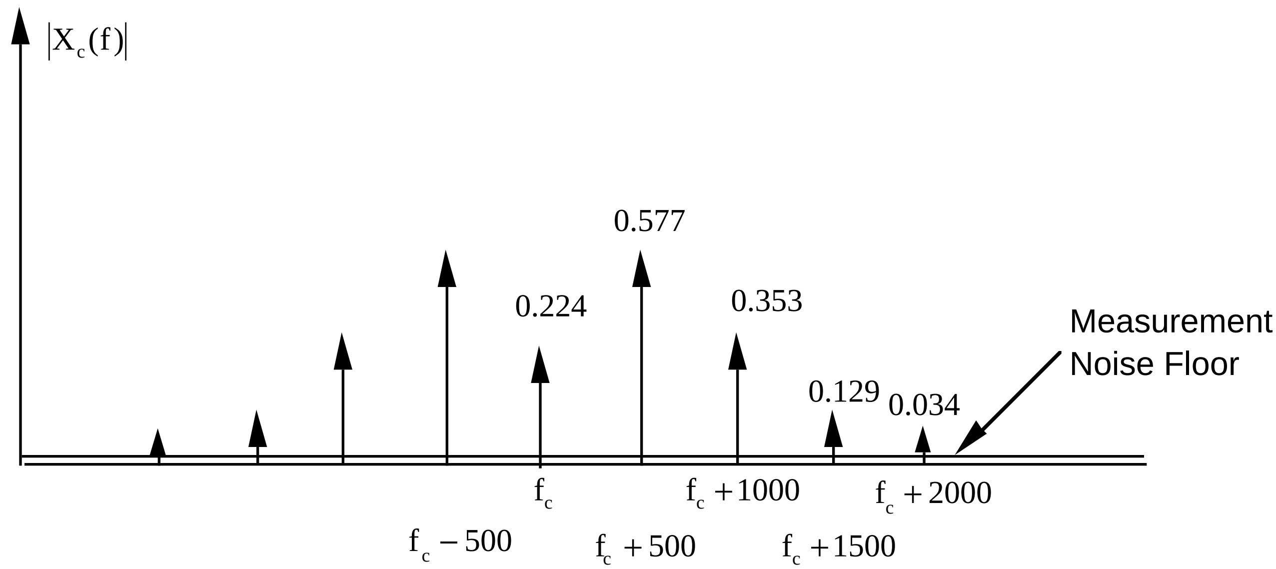


Figure 6.20: The measured bandpass amplitude spectrum for Problem 6.4.

d) With

$$SNR = \frac{A^2 P_m}{P_n}$$

Find B_T such that the SNR>30dB *Hint:* $n(t)$ *is periodic with the same period as the message signal so* P_n *can be computed in a simple way.*

Problem 6.6. A test tone of frequency $f_m = 1000$Hz with an amplitude A_m=2 volts is to be transmitted using both frequency modulation (FM) and phase modulation (PM).

a) Give values of k_p and k_f to achieve a $\beta = 2$. Using Carson's rule what is the resulting transmission bandwidth.

b) Keeping A_m, k_p and k_f the same as in part a) and setting f_m=200Hz compute the resulting transmission bandwidth for both modulations.

c) Keeping A_m, k_p and k_f the same as in part a) and setting f_m=5000Hz compute the resulting transmission bandwidth for both modulations.

d) What characteristic does FM have that would be an advantage in a practical system?

Problem 6.7. A message signal, $m(t)$, with bandwidth $W = 10$KHz and average power P_m is frequency modulated (FM) (the frequency deviation constant is k_f radians/second/volt) onto a carrier (f_c) with amplitude A_c.

a) Give the form of the transmitted signal (either in bandpass form or complex baseband form).

b) Choose the signal parameters such that the transmitted power is $P_{x_c} = 25$W in a 1 Ω system.

c) Define the deviation ratio, D, for an FM signal and choose the signal parameters such that D=4.

d) With $D = 4$ give a transmission bandwidth which will contain approximately 98% of the signal power.

Problem 6.8. As a new engineer at Badcomm, you are given the task to reverse engineer a phase modulator made by your competitor.

a) Given your lab bench contains a signal generator and a spectrum analyzer, describe a test procedure that would completely characterize the phase modulator.

b) Given your lab bench contains a signal generator, a bandpass filter with center frequency f_c and bandwidth 2000Hz, a power meter, and an I/Q down converter describe a test procedure that would completely characterize the phase modulator.

Problem 6.9. This problem is concerned with frequency modulation (FM) by the message signal given in Problem 4.1. Assume a carrier frequency of f_c=1000Hz, a radian frequency deviation constant of k_f and a carrier amplitude of A_c.

a) Give the baseband, $x_z(t)$, and bandpass, $x_c(t)$, bandpass time waveforms.

b) Using the results of Problem 4.1 what is the peak frequency deviation?

c) State Carson's rule for an arbitrarily message signal that is angle modulated. Select a value of k_f that will achieve a Carson's rule bandwidth of 300Hz for FM with a message signal given in Problem 4.1.

d) For the value of k_f selected in c) plot the bandpass time waveforms.

e) Using Matlab to compute a measured power spectrum. How do the results compare to that predicted by the Carson's rule approximation?

f) Give the value of A_c that will produce a 50 Watt output power in a 1 Ω system.

Problem 6.10. This problem is concerned with phase modulation (PM) by the message signal given in Problem 4.1. Assume a carrier frequency of f_c=1000Hz, a phase deviation constant of k_p and a carrier amplitude of A_c.

a) Give the baseband, $x_z(t)$, and bandpass, $x_c(t)$, bandpass time waveforms.

b) Using the results of Problem 4.1 what is the peak frequency deviation?

c) State Carson's rule for an arbitrarily message signal that is angle modulated. Select a value of k_p that will achieve a Carson's rule bandwidth of 300Hz for PM with a message signal given in Problem 4.1.

d) For the value of k_p selected in c) plot the bandpass time waveforms.

e) Using Matlab to compute a measured power spectrum. How do the results compare to that predicted by the Carson's rule approximation?

f) Give the value of A_c that will produce a 50 Watt output power in a 1 Ω system.

Problem 6.11. Angle modulation with a sinusoidal message signal has the form

$$x_z(t) = \exp\left[j\beta \sin\left(2\pi f_m t\right)\right].$$

The vector diagram of the received signal $y_z(t)$ is plotted in Figure 6.21 for $0 \leq t \leq (f_m)^{-1}$. *Each point on the vector diagram represents an instant in time and the X-coordinate is the value of $y_I(t)$ at this time and the Y-coordinate is the value of $y_Q(t)$ at this time.*

a) What is the approximate values of A_c, β and ϕ_p?

b) Give the approximate values of time (or times) associated with each labeled point (A, B, & C).

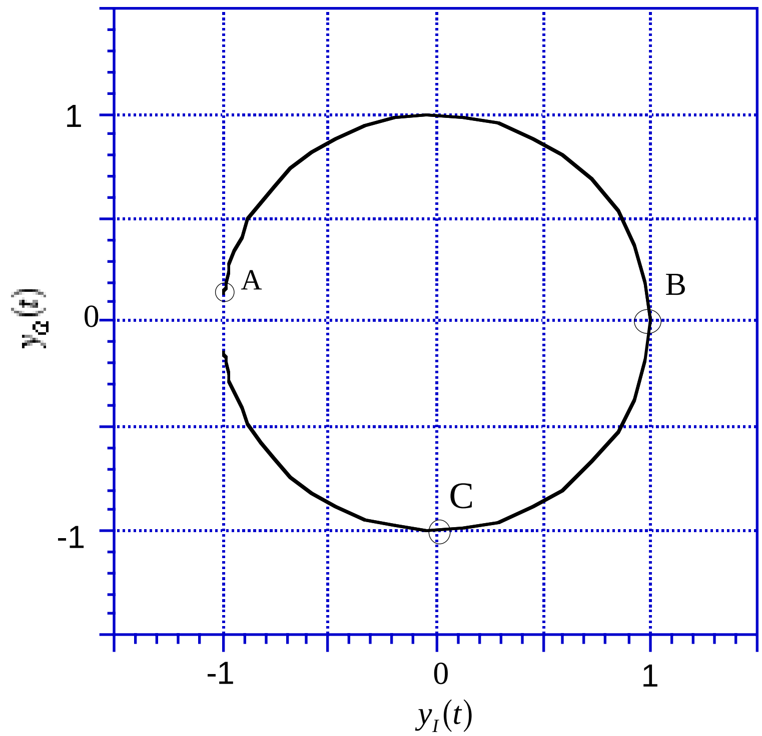


Figure 6.21: A vector diagram for a sinusoidal angle modulation for Problem 6.11.

Problem 6.12. For $m(t) = \sin(2\pi f_m t)$ the received complex envelope is

$$y_z(t) = 5 \exp\left[j\left(0.1\cos(2\pi f_m t - \pi) + \pi/5\right)\right].$$

Identify as many parameters of the modulation and channel as possible.

Problem 6.13. The waveform shown in Figure 6.22 is phase modulated on a carrier with $k_p = \pi/8$.

a) Plot the output bandpass waveform for $f_c = 10 f_m$. For uniformity in the answers assume that f_m=1KHz and f_c=10KHz. The plots should have time points no more than 5μs apart.

b) This same bandpass waveform can be generated with frequency modulation (FM). Specify the message message signal and the f_d that achieves the same waveform as plotted in part a).

Problem 6.14. **(UM)** In this problem, you will consider the effects message amplitude on angle modulation. Consider a message signal, $m(t) = 2\cos(2000\pi t)$. Let the deviation constants be $k_p = 1.5$ radians/volt and $f_d = 3000$ Hz/volt.

a) Determine the modulation indices: β_p for PM and β_f for FM.

b) Determine the engineering transmission bandwidths.

c) Plot the spectrum of the complex envelopes, considering only those components that will lie within the transmission bandwidth determined above. Hint: You'll need to think about what the modulated signal will look like.

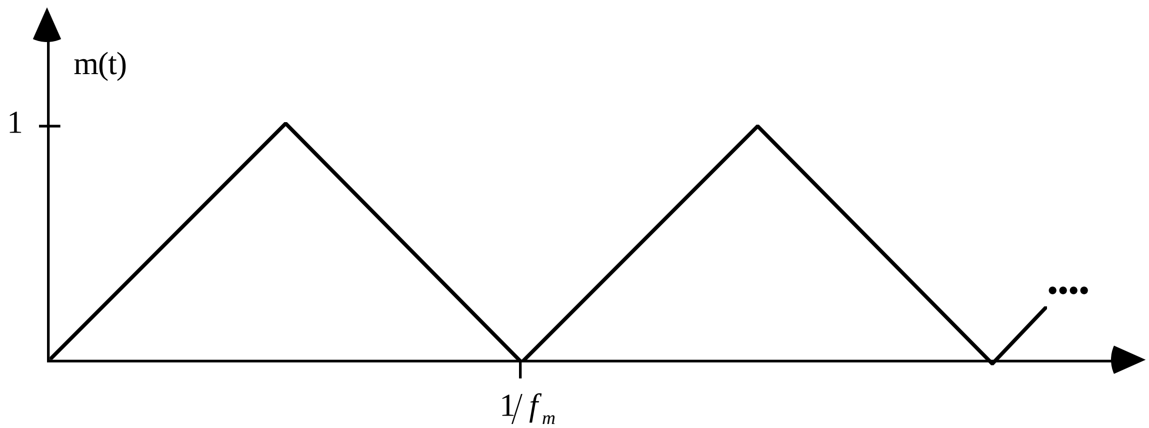


Figure 6.22: Message waveform for Problem 6.13.

d) If the amplitude of $m(t)$ is decreased by a factor of two, how will your previous answers change?

You can use Matlab to compute the Bessel function evaluated for relevant orders and relevant arguments (`besselj(n,x)`).

Problem 6.15. Consider the following complex envelope signal for angle modulation with a sinusoidal message $x_z(t) = \exp\left[j\beta \cos\left(2\pi f_m t\right)\right]$ and compute the Fourier series. Recall that $\sin\left(\theta + \frac{\pi}{2}\right) = \cos(\theta)$. How does the power spectrum of this complex envelope compare to the signal considered in (6.4)?

Problem 6.16. You are making a radio transmitter with a carrier frequency f_c that uses an angle modulation. The message signal is a sinusoid with $A_m = 1$ and the complex envelope of the signal output from your voltage controlled oscillator is

$$x_z(t) = 2\exp\left[j\beta \sin\left(2\pi 500 t\right)\right] \tag{6.23}$$

where β is a constant.

a) Choose β such that B_T=6000Hz (use Carson's rule).

b) The output spectrum for this transmitted signal will have a line (or impulsive) spectrum since the baseband signal is periodic. What is the value of the output bandpass spectrum impulse at $f = f_c$ for the value of β given in part a).

c) What is the message signal and the value of k_p if the signal is phase modulated (PM)?

d) What is the message signal and the value of k_f if the signal is frequency modulated (FM)?

e) For PM if f_m is reduced from 500Hz to 100Hz choose a value of k_p such that the output bandwidth is still B_T=6000Hz according to Carson's rule.

f) For FM if f_m is reduced from 500Hz to 100Hz choose a value of k_f such that the output bandwidth is still B_T=6000Hz according to Carson's rule.

Problem 6.17. Recall that narrowband phase modulation has a very similar form to LC-AM, i.e.,

$$\begin{aligned} x_c(t) &\approx A_c\sqrt{2}\cos\left(2\pi f_c t\right) - A_c k_p m(t)\sqrt{2}\sin\left(2\pi f_c t\right) \quad \text{(PM)} \\ x_c(t) &= A_c\sqrt{2}\cos\left(2\pi f_c t\right) + A_c a m(t)\sqrt{2}\cos\left(2\pi f_c t\right) \quad \text{(AM)} \end{aligned}$$

Because of this similarity one might be tempted to use an envelope detector as a simple demodulator.

a) Compute the envelope, $y_A(t)$, of the received signal.

b) Consider a simple example with $m(t) = \cos(2\pi(100)t)$ and $k_p = 0.1$. Plot the vector diagram of the true PM signal and the approximate form in (6.10).

c) Plot the envelope for the signal in (6.10).

d) Could the envelope detector be used for demodulating narrowband PM?.

Problem 6.18. For a sinusoidal angle modulated signal

$$x_z(t) = A_c \exp\left[j\beta \sin(2\pi 500 t)\right] \tag{6.24}$$

and a propagation phase rotation of ϕ_p.

a) For $\phi_p = 0$, how large could β be and not require phase unwrapping?

b) For $\phi_p = \frac{\pi}{2}$, how large could β be and not require phase unwrapping?

c) For $\phi_p = \pi$, how large could β be and not require phase unwrapping?

Problem 6.19. What condition must hold for the message signal, $m(t)$, and the modulation constants, k_p and k_f, such that the signals in (6.21) are guaranteed to be of the form of a LC-AM signal.

6.6 Example Solutions

Not implemented this edition

6.7 Mini-Projects

Goal: To give exposure

1. to a small scope engineering design problem in communications
2. to the dynamics of working with a team
3. to the importance of engineering communication skills (in this case oral presentations).

Presentation: The forum will be similar to a design review at a company (only much shorter) The presentation will be of 5 minutes in length with an overview of the given problem and solution. The presentation will be followed by questions from the audience (your classmates and the professor). Each team member should be prepared to make the presentation on the due date.

Project 6.1. Undergraduate communication courses often only look at idealized problems. For example one thing that is ignored in a course is that frequency sources at the transmitter and receiver cannot be made to have exactly the same value. Consequently the complex envelope of the received signal will be rotating due to this frequency offset, i.e.,

$$y_z(t) = x_z(t) \exp\left[j(\phi_p + 2\pi f_o t)\right] \tag{6.25}$$

where f_o is the existing frequency offset. This mini-project will investigate the effects of this frequency offset on one phase demodulation algorithm.

Get the Matlab file `chap5ex1.m` and `chap5ex3.m` from the class web page. In these files a PM transmitter (for a sinusoid message signal) and receiver is implemented. If the carrier frequency offset is set to 10Hz `deltaf=10` the demodulator implemented in the m-file still works pretty well with this

frequency offset. Explain why. Note at higher frequency offsets (e.g., 100Hz) the performance suffers noticeable distortion. *Extra credit will be given if you can figure out a method to eliminate this distortion.*
Project 6.2. You have been hired by the National Security Agency of the US Federal government as an engineer. The first job you have been given by your new boss is to decode a radio message signal between two members of a South Albuman drug cartel. This signal was intercepted by a spy satellite and the data has been processed and forwarded on to you for decoding.

The technical details are

1. The sampled data file contains about 1 second worth of data at a 44100Hz sampling rate. (Precisely 44864 samples). This data is available at the class web site.

2. The radio signal is analog modulated. The type of analog modulation is unclear. Human intelligence sources have indicated that the modulation used is either DSB-AM, LC-AM, SSB-AM, or PM.

3. The carrier frequency is unknown. Obviously since the data is provided to you with a sample rate of 44100Hz the carrier frequency must be between 0-22050Hz.

4. The channel includes an unknown delay which produces an unknown phase shift, φ_p.

Your job is to decode the signal. It should be obvious when you are correct as you can play the demodulated output as a sound. The solution should include

- A discussion of the methodology that led you to identifying the modulation,

- A discussion of the methodology that led you to identifying the carrier frequency,

- A discussion of the demodulator architecture and the method used to estimate the phase shift or to operate in the presence of an unknown phase shift,

- Submission of Matlab code for the demodulator.

Chapter 7

More Topics in Analog Communications

This chapter cover two topics which are important for both amplitude and angle types of analog communications: the phase-locked loop and the multiplexing of message signals.

7.1 Phase-Locked Loops

7.1.1 General Concepts

The phase-locked loop (PLL) is a commonly used component in communication systems. It can be used for tracking the phase and frequency of signals, demodulating angle modulated signals, and frequency source synthesis. The PLL is a tracking system which uses feedback. We have seen one example so far in the Costas loop for synchronous demodulation of DSB-AM. This section will investigate the use of the PLL for FM and PM demodulation (feedback demodulation) but the general characteristics are valid for any application of the PLL. The input signal into a PLL has amplitude variation $A(t)$, and a phase variation to be tracked, $\theta(t)$, and a nuisance phase variation, $d(t)$. The typical PLL, seen in Fig. 7.1, has three components: the phase detector, the loop filter and the voltage controlled oscillator (VCO). Recall the VCO has the characteristics shown in Fig. 7.2. The VCO has a quiescent frequency of f_c and a gain of $k_f = 2\pi f_d$ radians/s/volt.

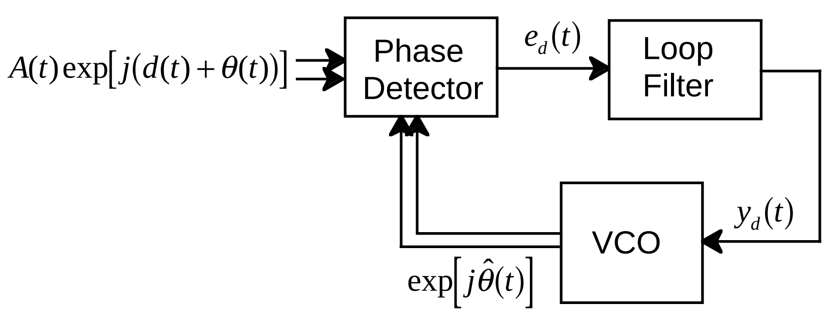


Figure 7.1: Phase-locked loop block diagram.

Example 7.1: For demodulation of PM we have $A(t) = A_c$, $\theta(t) = k_p m(t) + \phi_p$, and $d(t) = 0$.

Example 7.2: For the Costas loop used as a synchronous demodulator for DSB-AM we have $A(t) = A_c|m(t)|$, $\theta(t) = \phi_p$, and $d(t) = (1 - \mathrm{sgn}(m(t)))\frac{\pi}{2}$.

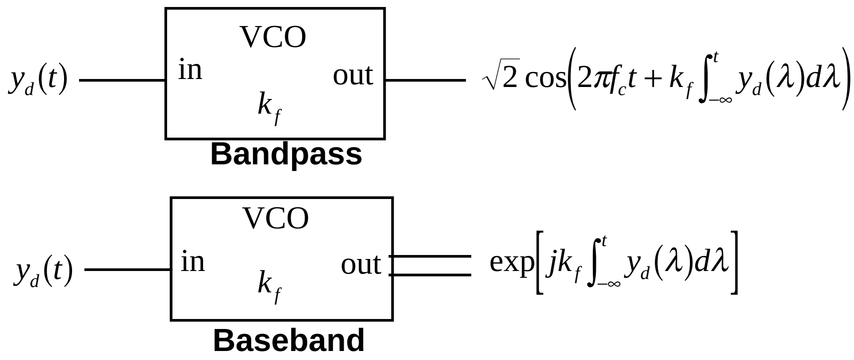


Figure 7.2: The bandpass and baseband block diagrams for a VCO.

The phase detector simply measures the phase difference between the input signal and the locally generated reference. Denoting the input signal to the PLL as $y_z(t)$ the phase detector is a nonlinear function denoted $e_d(t) = g\left(y_z(t), \hat{\theta}(t)\right)$ which is usually designed to be proportional to the phase error $\varphi(t) = \theta(t) - \hat{\theta}(t)$.

Example 7.3: The most common phase detector for an unmodulated sinusoidal input signal ($A(t) = A_c$ and $d(t) = 0$) is the quadrature multiplier and the analytical model is seen in Fig. 7.3. This phase detector is given as

$$g\left(y_z(t), \hat{\theta}(t)\right) = A_c k_m \Im\left[y_z(t)\exp\left[-j\hat{\theta}(t)\right]\right]$$

where k_m is the multiplier gain. The phase detector output can be expressed in terms of the phase error as

$$e_d(t) = A_c k_m \sin\left(\theta(t) - \hat{\theta}(t)\right) = A_c k_m \sin\left(\varphi(t)\right).$$

The signal $e_d(t)$ is a function of the phase error and it can be used as a feedback signal in a closed loop tracking system.

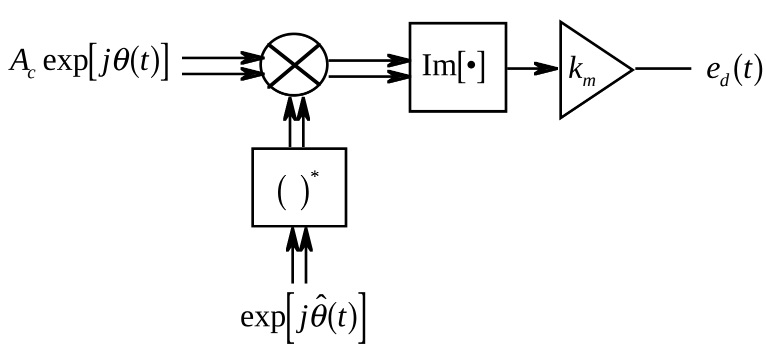


Figure 7.3: The quadrature multiplier phase detector block diagram.

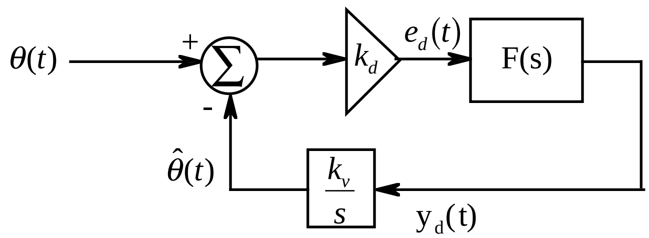


Figure 7.4: The linear model of the phase locked loop.

Example 7.4: Note Fig. 5.8 shows another type of a phase detector which is useful for phase tracking with DSB-AM signals ($A(t) \neq A_c$ and $d(t) \neq 0$). The phase detector is given as

$$\begin{aligned} g\left(y_z(t), \hat{\theta}(t)\right) &= k_m \Im\left[\left(y_z(t)\exp[-j\hat{\theta}(t)]\right)^2\right] \\ &= 2k_m \Im\left[y_z(t)\exp[-j\hat{\theta}(t)]\right]\Re\left[y_z(t)\exp[-j\hat{\theta}(t)]\right]. \end{aligned} \tag{7.1}$$

Expressing the phase detector output in terms of the phase error gives

$$e_d(t) = k_m A_c^2 m^2(t)\sin(2\varphi(t)).$$

The loop filter is a linear filter and it's design is often critical in getting the desired performance out of a PLL. For this development the loop filter is represented with $F(s)$. Two common types of loops are the first and second order PLL. The first-order PLL loop filter has a transfer function given as

$$F(s) = 1$$

and the second-order PLL loop filter is given as

$$F(s) = \frac{s+a}{s+b}.$$

It should be noted that the most common type of second order loop is one with a perfect integrator ($b = 0$).

7.1.2 PLL Linear Model

The linear analysis of the PLL is valid when the loop is closely tracking the input signal ($\varphi(t) \approx 0$). Close tracking implies that $\varphi(t)$ is small and that the phase detector can be linearized around $\varphi(t) = 0$ so that

$$g\left(y_z(t), \hat{\theta}(t)\right) \approx k_d \varphi(t)$$

where k_d is denoted the phase detector gain. This results in a linear model for the PLL that is shown in Fig 7.4. Note since the phase of the VCO output is proportional to the integral of the input signal the VCO can be modeled by a transfer function with a Laplace transform of k_f/s where k_f is the gain of the VCO.

Example 7.5: The quadrature multiplier has $e_d(t) = A_c k_m \sin(\varphi(t))$ so that the linear approximation has $k_d = A_c k_m$ and $\sin(\varphi(t)) \approx \varphi(t)$.

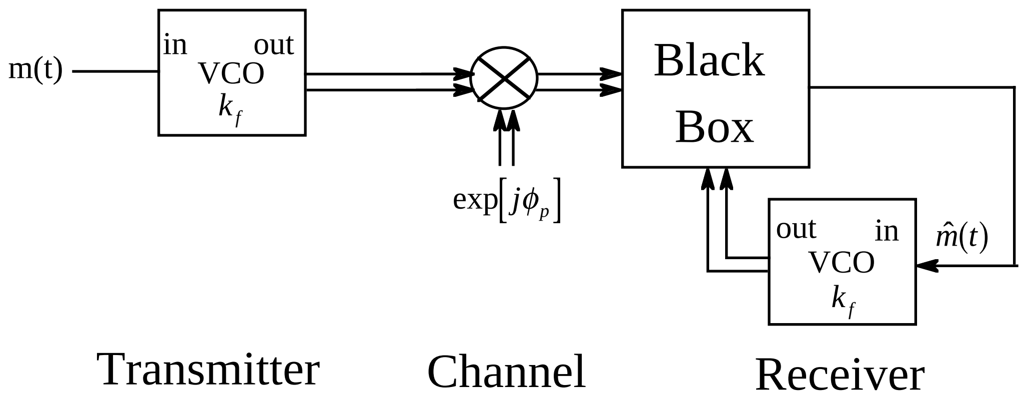


Figure 7.5: The conceptual diagram for FM demodulation.

7.2 PLL Based Angle Demodulation

7.2.1 General Concepts

Demodulation of angle modulations with a PLL is actually quite simple conceptually. This section will concentrate on demodulation of FM signals and the differences for PM will be pointed out when appropriate. Recall a received FM signal has the form

$$y_z(t) = A_c \exp\left[jk_f \int_{-\infty}^{t} m(\lambda)d\lambda + j\phi_p\right] \tag{7.2}$$

Fig. 7.5 shows a block diagram of an FM system with a feedback demodulator. A VCO is used to produce the FM signal and the proposed receiver consists of a two input, one output black box and a VCO identical to the one used for modulation. After a transient period if $\theta(t) = \hat{\theta}(t)$ then we know $\hat{m}(t) = m(t)$. The job of the black box in the diagram is to measure the phase difference $\varphi(t) = \theta(t) - \hat{\theta}(t)$ and update $\hat{m}(t)$ in such a way as to drive $\varphi(t)$ to zero. Note that if

$$\begin{aligned}
\varphi(t) &= \theta(t) - \hat{\theta}(t) < 0 \qquad \Rightarrow \qquad \int_{-\infty}^{t} m(\lambda)d\lambda < \int_{-\infty}^{t} \hat{m}(\lambda)d\lambda, \\
\varphi(t) &= \theta(t) - \hat{\theta}(t) > 0 \qquad \Rightarrow \qquad \int_{-\infty}^{t} m(\lambda)d\lambda > \int_{-\infty}^{t} \hat{m}(\lambda)d\lambda,
\end{aligned} \tag{7.3}$$

Equation (7.3) indicates that $\varphi(t)$ can be driven to zero if $\hat{m}(t)$ is updated in proportion to $\varphi(t)$. Thus the black box in Fig. 7.5 is composed of the other two elements of the PLL; the phase detector and the loop filter.

Since the FM signal has a constant amplitude the quadrature multiplier (see Figure 7.3) can be used as the phase detector. Consequently the overall block diagram is seen in Figure 7.6 and the equation defining the operation is

$$\hat{m}(t) = f(t) \otimes k_d \sin\left(k_f \int_{-\infty}^{t} (m(\lambda) - \hat{m}(\lambda))\, d\lambda + \phi_p\right) \tag{7.4}$$

where $k_d = A_c k_m$ is the quadrature multiplier phase detector gain. It is important to note that the phase detector gain is a function of the received signal amplitude. While this amplitude is treated as a known constant in these notes in practical systems accounting for this amplitude is an important function in a FM demodulator based on a PLL.

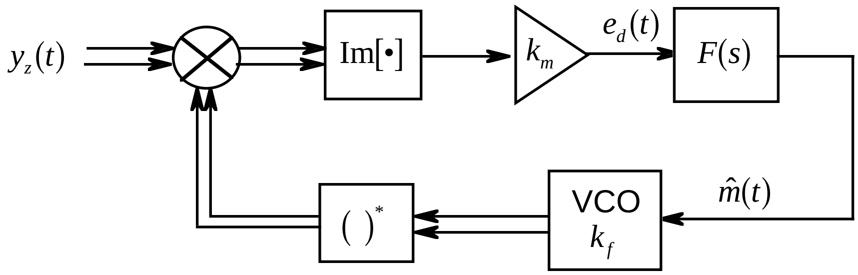


Figure 7.6: The block diagram of a PLL for FM demodulation.

Equation (7.4) is a nonlinear integral equation that can be solved but only with mathematical tools that undergraduate engineering students typically have not been exposed to. Consequently approximations will be made so tools undergraduate engineering students should have can be brought to bear to obtain some design insights and performance characterization.

7.2.2 PLL Linear Model

The linear analysis of the PLL is valid when the loop is closely tracking the input signal. Close tracking implies that $\varphi(t)$ is small and for the quadrature multiplier phase detector

$$e_d(t) = A_c k_m \sin\left(\varphi(t)\right) \approx A_c k_m \varphi(t) = k_d \varphi(t)$$

This results in a linear model for a PLL used in FM demodulation that is shown in Fig. 7.7. Rearranging and defining $k = k_f k_d$ gives the block diagram shown in Fig. 7.8. It is quite obvious from Fig. 7.8 that the linear model for the PLL reduces to a simple linear feedback control system commonly studied in an undergraduate program. Taking Laplace transforms yield

$$\left(M(s) - \hat{M}(s)\right)\frac{kF(s)}{s} = \hat{M}(s).$$

Solving for the transfer function gives

$$H_L(s) = \frac{\hat{M}(s)}{M(s)} = \frac{kF(s)}{s + kF(s)} \tag{7.5}$$

Often the error between the demodulated output and the message signal is of interest. Defining $e(t) = m(t) - y_d(t)$ and using identical techniques a transfer function for the error is given as

$$H_E(s) = \frac{E(s)}{M(s)} = \frac{s}{s + kF(s)}$$

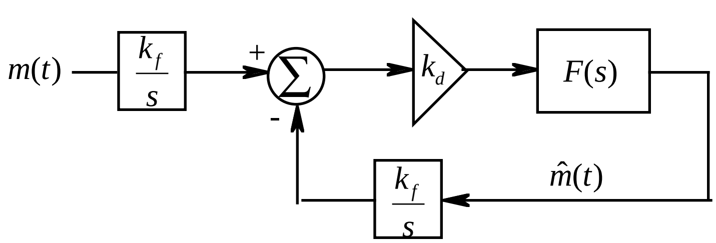


Figure 7.7: The PLL linear model for FM demodulation.

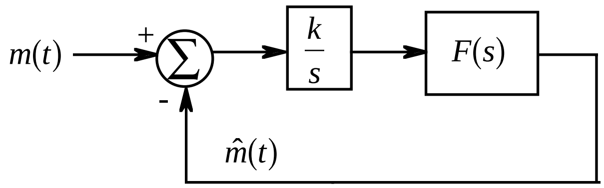


Figure 7.8: The PLL linear model rearranged to have a classic feedback control systems structure.

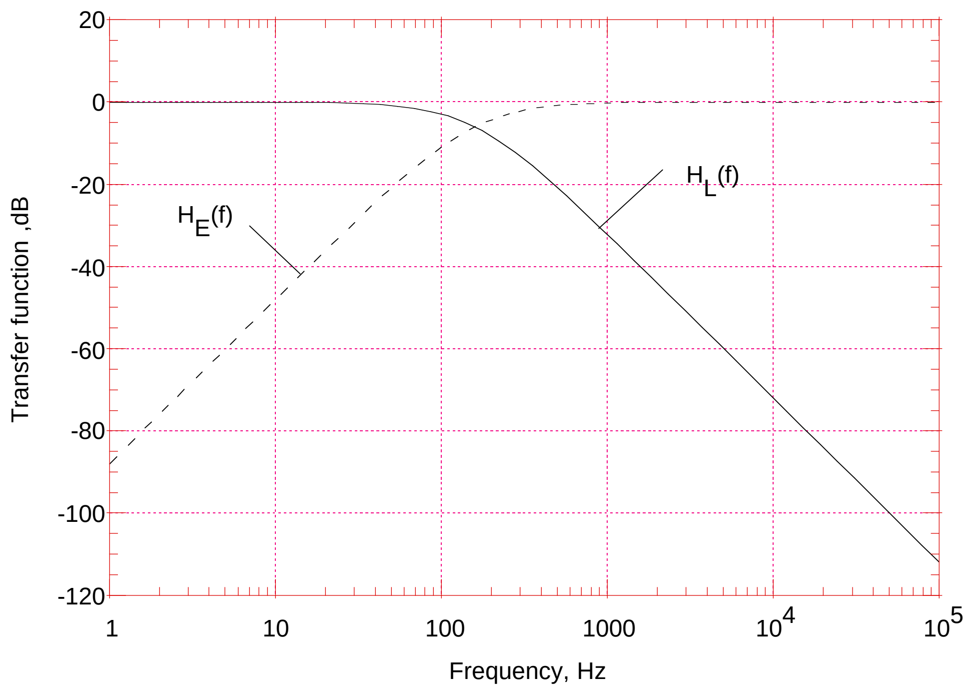


Figure 7.9: The frequency response of a first-order PLL used for FM demodulation. k=1000

Frequency Response

The frequency response of the loop is of interest. The frequency response is given as

$$H(f) = H(s)|_{s=j2\pi f}\ .$$

The loop and error transfer function are plotted Fig. 7.9 for a first-order loop with k=1000. Consequently when the linear model is valid the PLL appears much like a lowpass filter to the message signal. For example for the case considered in Fig. 7.9 (k=1000) the 3dB loop bandwidth is about 100Hz. Consequently design of PLLs for FM demodulation must carefully consider the bandwidth of the message signal in choosing the bandwidth of the loop.

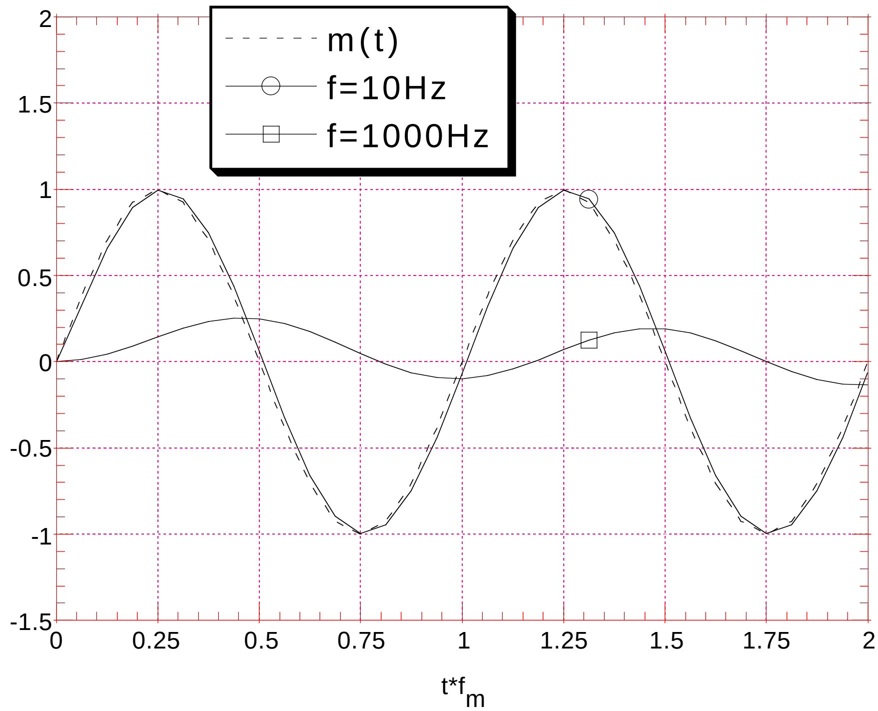


Figure 7.10: The time response of the linear model of a PLL for FM demodulation.

Acquisition

Let's consider the transient response produced by a first-order loop ($F(s) = 1$) to a sinusoidal input signal, $m(t) = \sin(2\pi f_m t)$. The Laplace transform of this signal is

$$M(s) = \frac{2\pi f_m}{s^2 + (2\pi f_m)^2} \tag{7.6}$$

Using (7.6) in (7.5) produces a loop output Laplace transform of

$$\hat{M}(s) = \frac{2\pi f_m}{s^2 + (2\pi f_m)^2} \frac{k}{s+k}.$$

Inverse transforming this signal gives

$$\hat{m}(t) = \frac{k(2\pi f_m)}{k^2 + (2\pi f_m)^2} e^{-kt} + \frac{k}{\sqrt{k^2 + (2\pi f_m)^2}} \sin\left(2\pi f_m t + \tan^{-1}\left(\frac{2\pi f_m}{k}\right)\right).$$

Fig. 7.10 shows a plot of the time response for k=1000 and f_m=10 and f_m=100. Note that if the signal is inside the passband of the loop, the loop quickly locks onto the signal and tracks it very closely. If the signal is outside the loop bandwidth the signal is not tracked and the FM demodulation performance is poor.

7.3 Multiplexing Analog Signals

Definition 7.1 *Multiplexing for analog message signals is the common transmission of multiple message signals using a single communications resource.*

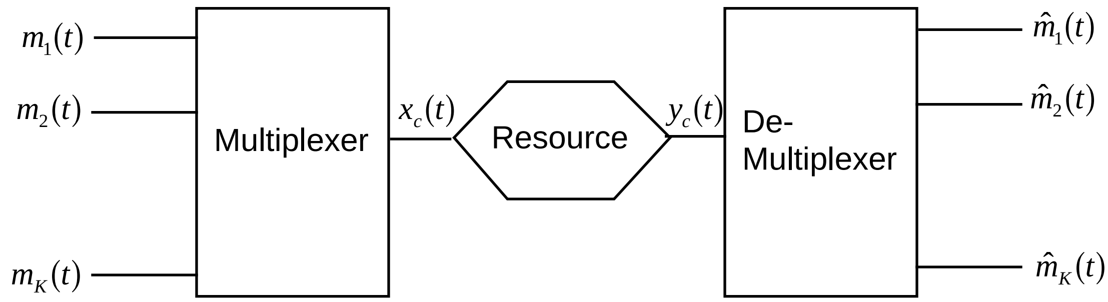


Figure 7.11: The multiplexing concept.

The common resource is typically spectrum (e.g., broadcast radio), expensive electronic equipment (e.g., satellites), or both (e.g., analog cellular telephony). For this discussion we will denote the i^{th} message signal as $m_i(t)$ and K as the number of users of the resource. Consequently the idea of multiplexing is shown in Figure 7.11. As a measure of how efficiently we are using the resource we define the multiplexing efficiency as

$$E_M = \frac{\sum_{i=1}^{K} W_i}{B_T} \tag{7.7}$$

where W_i is the i^{th} message bandwidth and B_T is the transmission bandwidth of the composite signal. Note the best multiplexing efficiency without significant distortion for analog communications is unity. For analog signals there are two basic types of multiplexing techniques: 1) quadrature carrier multiplexing and 2) frequency division multiplexing.

Example 7.6: For those homes that subscribe to cable television services a single cable enters the home with an electronic signal. Typically greater than 100 channels are supported with that one cable.

7.3.1 Quadrature Carrier Multiplexing

Quadrature carrier multiplexing (QCM) uses the two degrees of freedom in a bandpass signal to transmit two message signals on the same carrier. This type of multiplexing is achieved by setting $x_I(t) = A_c m_1(t)$ and $x_Q(t) = A_c m_2(t)$. The block diagram is shown in Figure 7.12. The efficiency of QCM is

$$E_M = \frac{W_1 + W_2}{2 \max W_1, W_2}. \tag{7.8}$$

Note if $W_1 = W_2$ then $E_M = 1$ which is the best that can be accomplished in analog communications. The advantage of QCM is that it is a very efficient method of multiplexing message signals. The disadvantage is that the applicability is limited to only two message signals and that coherent demodulation must be used (typically with a transmitted reference carrier which will reduce the bandwidth efficiency somewhat).

7.3.2 Frequency Division Multiplexing

Frequency division multiplexing (FDM) assigns each user a different carrier frequency for transmission. Typically the carrier frequencies for each user are chosen so that the transmissions do not overlap in frequency. Figure 7.13 shows a typical energy spectrum for a FDM system. Typically each user in the system is separated from his spectral neighbor by a guard band. The size of this guard band limits

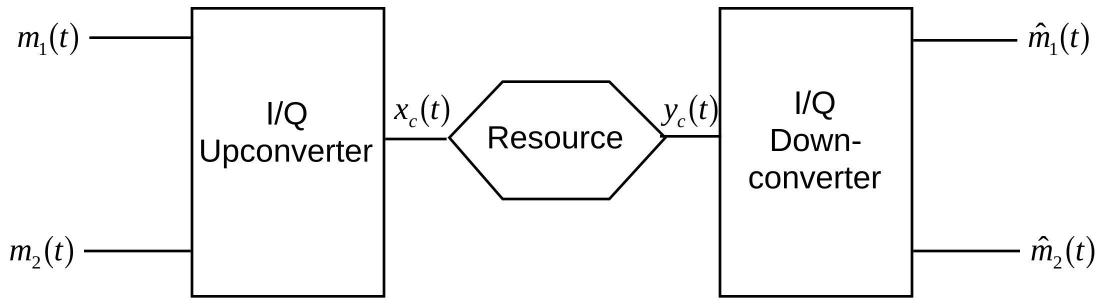


Figure 7.12: Quadrature carrier multiplexing.

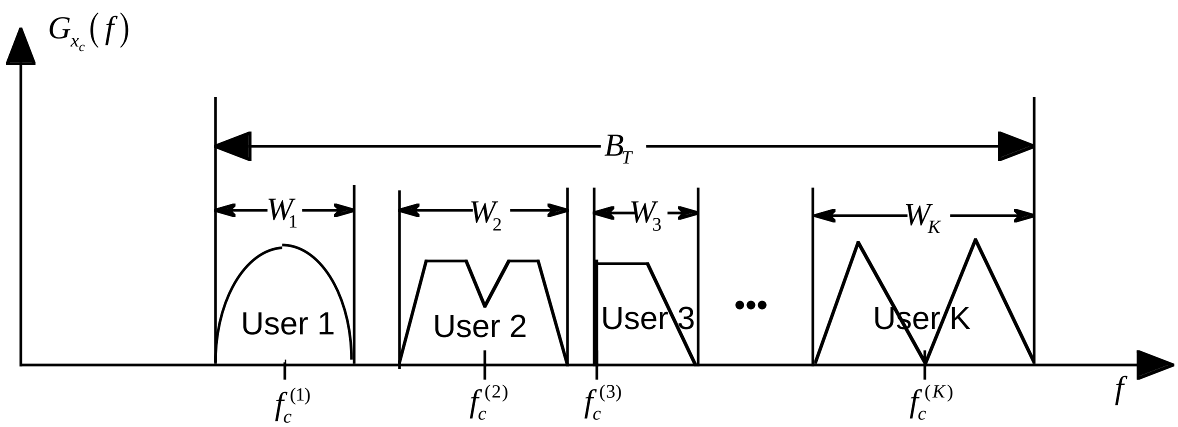


Figure 7.13: A typical energy spectrum for a frequency division multiplexed signal.

the spectral efficiency. The multiplexing for FDM is accomplished by simply summing together appropriately modulated bandpass waveforms as shown in Figure 7.14. Demultiplexing for FDM, seen in Figure 7.15, simply corresponds to bandpass filtering to isolate each user followed by a standard I/Q demodulator. Note the guard bands between users are usually chosen to be compatible with the bandpass filtering operation (i.e., the guard band is made large enough to ensure each user can be adequately separated at the demultiplexer).

Example 7.7: Broadcast communications in the United States uses the resource of free space radio wave propagation. The multiplexing is typically implemented with a combination of frequency multiplexing and location multiplexing (stations far enough away from each other can use the same transmission frequency). AM radio broadcast in the United States has been assigned the spectrum of 535-1605KHz, B_T=1070KHz. There are 107 FDM channels spaced about 10KHz apart ($f_c^{(i)} = 540 + 10i$ KHz). Each AM station provides an audio signal of about 4KHz bandwidth (W_i=4KHz). The efficiency of AM broadcast is then given as

$$E_M = \frac{107 \times 4}{1070} = 40\%.$$

Note 20% of the loss in efficiency is due to guard bands and 40% of the loss is due to using spectrally inefficient LC-AM as the modulation.

There can be several levels of multiplexing that are implemented for a given system. The first level often groups related messages together and the subsequent levels multiplex several groups of messages

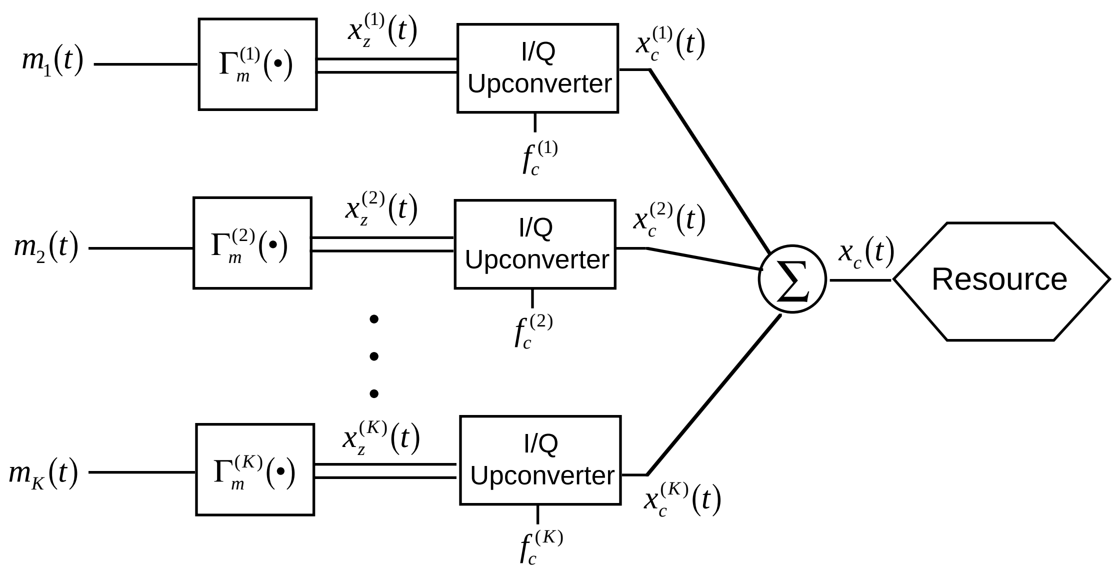


Figure 7.14: A FDM multiplexing system.

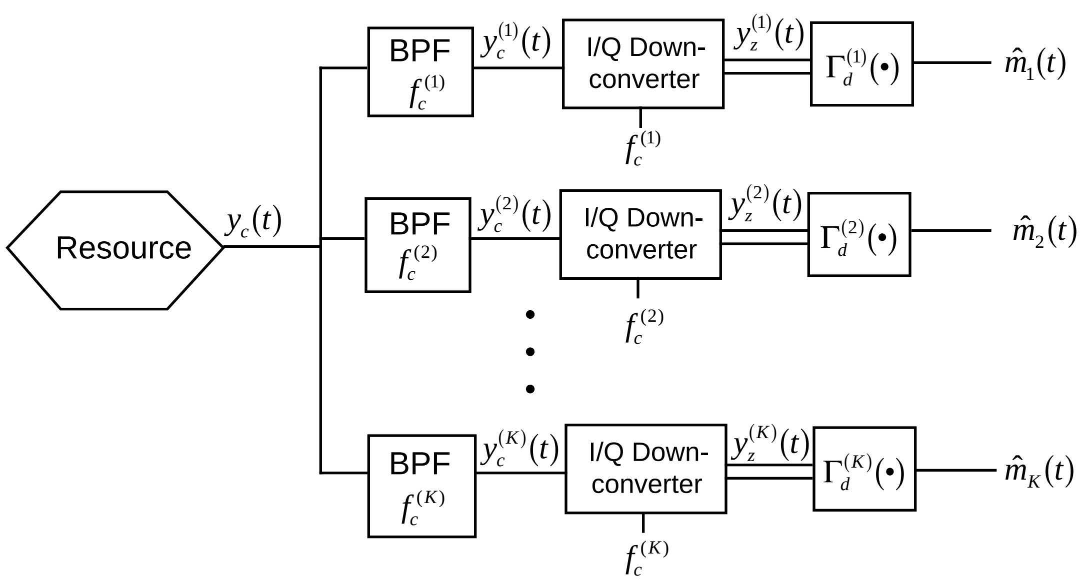


Figure 7.15: A FDM demultiplexing system.

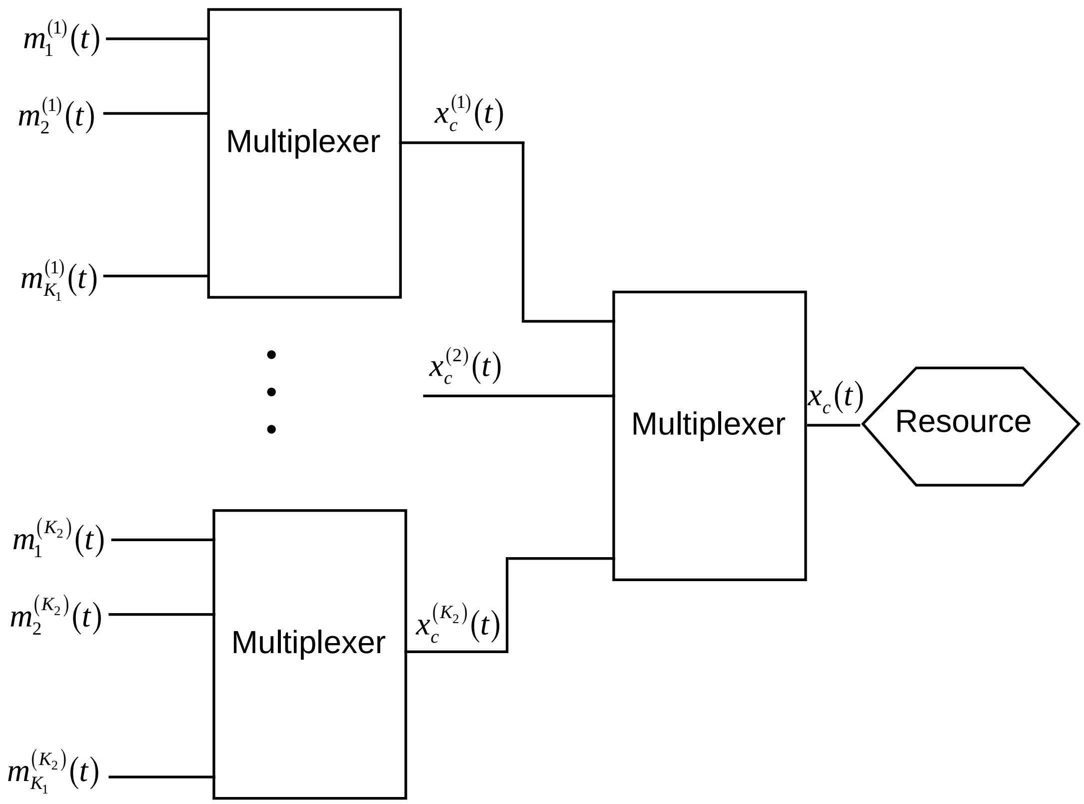


Figure 7.16: A two level multiplexing scheme.

together. The block diagram for a system using two levels of multiplexing is shown in Fig 7.16.

Example 7.8: An example of multi-level multiplexing is seen in broadcast communications in the FM band in the United States. Each FM station in the United States typically broadcasts three message signals. Two are audio channels: the left channel, $m_L(t)$ and the right channel, $m_R(t)$ for stereo broadcast, and one is an auxiliary channel, $m_A(t)$ (e.g., Musak, or Radio Data Services). These three message signals are multiplexing onto one composite message signal which is FM modulated and transmitted. For discussion purposes the bandwidth of these message signals can be assumed to be W_i=15KHz. The first level of multiplexing FDM with $m_1(t) = m_L(t) + m_R(t)$, $m_2(t) = m_L(t) - m_R(t)$, and $m_3(t) = m_A(t)$, $\Gamma_m^{(1)} = \Gamma_m^{(2)} = \Gamma_m^{(3)}$=DSB-AM, and $f_c^{(1)} = 0$, $f_c^{(2)}$=38KHz, and $f_c^{(3)}$=76KHz.

7.4 Homework Problems

Problem 7.1. Your boss at Enginola has assigned you to be the system engineer in charge of an FM demodulator. Unfortunately, this design must be done in one day. The demodulator is required to be a PLL-based system and since you are pressed for time, you will use the linear approximation for the loop. Figure 7.17 is the model for your system

The transfer function, $H_L(f)$, from the input $(m(t))$ to the output $(y\hat{m}(t))$ has the form

$$H_L(s) = \frac{2\zeta\omega_n s + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

a) Select values of k_f, k_1, and a such that ζ=1.00 and $|H_L(12kHz)| > 0.707$ (i.e., the 3dB bandwidth

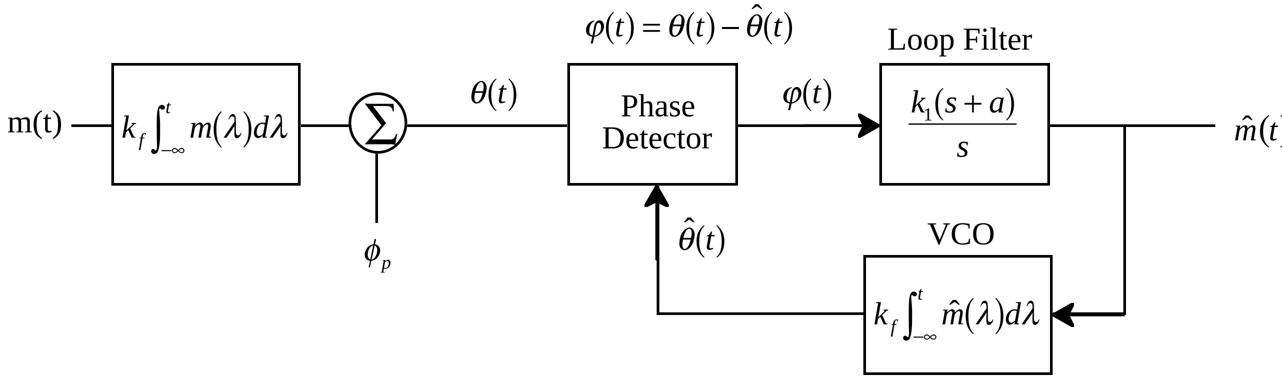


Figure 7.17: Block diagram of a PLL demodulator for FM.

is 12kHz). Note: While this problem can be completed with pencil and paper, a proficiency with a computer would be useful.

b) If the input signal is $m(t) = \sin(2\pi(10\text{kHz})t)U(t)$ where $U(t)$ is the unit step function, calculate the output response to this signal and plot the response.

Problem 7.2. As a communication engineer you need to transmit 8 voice signals (W=4kHz) across a common communication resource at a carrier frequency of 1GHz. For hardware reasons you need to maintain a 1kHz guard band between frequency multiplexed transmissions.

a) If you use DSB-AM and FDM, what is the total used bandwidth, B_T, and the bandwidth efficiency, E_M?

b) If you use SSB-AM and FDM what is the total used bandwidth and the bandwidth efficiency, E_M?

c) If a two stage multiplexing scheme is employed such that two voice signals are grouped together with QCM and the resulting 4 signals are multiplexed using FDM, what is the total bandwidth and the bandwidth efficiency, E_M?

7.5 Example Solutions

Not included this edition

7.6 Mini-Projects

Not included this edition

Chapter 8

Random Processes

An additive noise is characteristic of almost all communication systems. This additive noise typically arises from thermal noise generated by random motion of electrons in the conductors comprising the receiver. In a communication system the thermal noise having the greatest effect on system performance is generated at and before the first stage of amplification. This point in a communication system is where the desired signal takes the lowest power level and consequently the thermal noise has the greatest impact on the performance. This characteristic is discussed in more detail in Chapter 9. This chapter's goal is to introduce the mathematical techniques used by communication system engineers to characterize and predict the performance of communication systems in the presence of this additive noise. The characterization of noise in electrical systems could comprise a course in itself and often does at the graduate level. Textbooks that provide more detailed characterization of noise in electrical systems are [DR87, Hel91, LG89, Pap84]

A canonical problem formulation needed for the analysis of the performance of a communication systems design is given in Fig. 8.1. The thermal noise generated within the receiver is denoted $W(t)$. This noise is then processed in the receiver and will experience some level of filtering, represented with the transfer function $H_R(f)$. The simplest analysis problem is examining a particular point in time, t_s and characterize the resulting noise sample, $N(t_s)$, to extract a parameter of interest (e.g., average signal–to–noise ratio (SNR)). Additionally we might want to characterize two or more samples, e.g., $N(t_1)$ and $N(t_2)$, output from this filter.

To accomplish this analysis task this chapter first characterizes the thermal noise, $W(t)$. It turns out that $W(t)$ is accurately characterized as a stationary, Gaussian, and white random process. Consequently our first task is to define a random process (see Section 8.1). The exposition of the characteristics of a Gaussian random process (Section 8.2) and a stationary random process (Section 8.3) then will follow. A brief discussion of the characteristics of thermal noise is then followed by an analysis of stationary random processes and linear systems. In conclusion, we return and solve the canonical problem posed in Fig. 8.1 in Section 8.6.

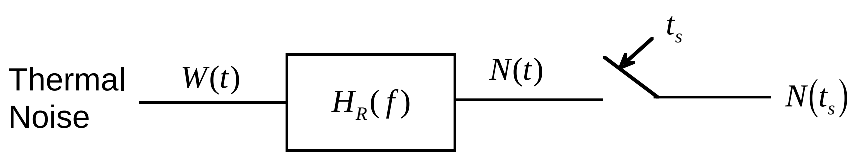


Figure 8.1: The canonical communications noise formulation.

8.1 Basic Definitions

Understanding random processes is fundamental to communications engineering. Random or stochastic processes are indexed families of random variables where the index is normally associated with different time instances.

Definition 8.1 *Let $(\Omega, \mathcal{F}, P)$ be a probability space. A real random process, $X(\omega, t)$, is a single-valued function or mapping from Ω to real valued functions of an index set variable t.*

The index set in this text will always be time but generalizations of the concept of a random process to other index sets is possible (i.e., space in image processing). The idea behind the definition of a random process is shown in Fig. 8.2. Essentially there is an underlying random experiment. The outcome of this random experiment is then mapped to a function of time. The example in Fig. 8.2 shows the mapping of three of the possible experiment outcomes into three different functions of time. The function $X(\omega_1, t)$ for ω_1 fixed is a called a **sample path** of the random process. Communication engineers observe the sample paths of random processes and the goal of this chapter is to develop the tools to characterize these sample paths. From this point forward in the text the experimental outcome index will be dropped and random processes will be represented as $X(t)$.

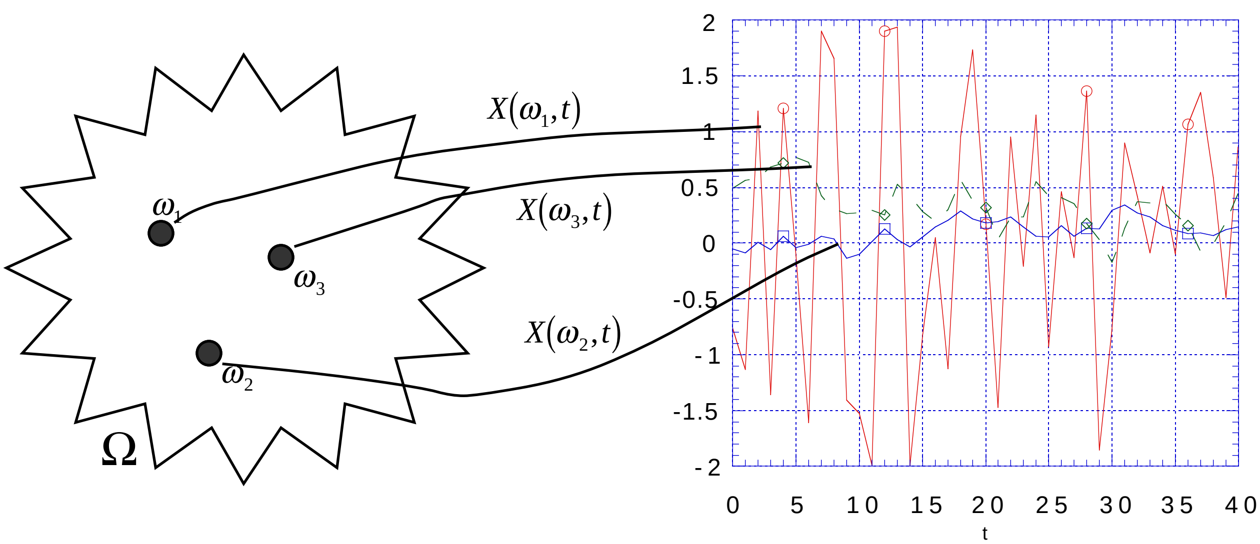


Figure 8.2: A pictorial explanation of the definition of a random process .

Property 8.1 *A sample of a random process is a random variable.*

If we focus on a random process at a particular time instance then we have a mapping from a random experiment to a real number. This is exactly the definition of a random variable (see Section 2.2). Consequently, the first important question in the canonical problem stated at the beginning of this chapter has been answered: *the sample $N(t_s)$ of the random process, $N(t)$, is a random variable.* Consequently to characterize this sample of a random process we use all the tools that characterize a random variable (distribution function, density function, etc.). If random processes are indexed sets of random variables then a complete statistical characterization of the random process would be available if the random variables comprising the random process were completely characterized. In other words for a random process, $N(t)$, if M samples of the random process are taken at various times and if the PDF given by

$$f_{N(t_1),N(t_2),\cdots,N(t_M)}(n_1, n_2, \cdots, n_M)$$

for arbitrary M is known then the random process is completely characterized. This full characterization, in general, is very difficult to do. Fortunately, the case of the Gaussian random process is the exception to the rule since Gaussian random processes are very often accurate models for noise in communication systems.

Point 1: The canonical problem can be solved when the random variable $N(t_s)$ or set of random variables, $N(t_1)$ and $N(t_2)$, can be statistically characterized with a PDF or a CDF

8.2 Gaussian Random Processes

Definition 8.2 *A Gaussian random process is a random process where any set of samples taken from the random process are jointly Gaussian random variables.*

Many important random processes in communication system analysis are well modeled as Gaussian random processes. This is important since Gaussian random variables are simply characterized. A complete characterization of Gaussian random variables (their joint PDF) is obtained by knowing their first and second order moments.

Property 8.2 *If $N(t)$ is a Gaussian random process then one sample of this process, $N(t_s)$, is completely characterized with the PDF*

$$f_{N(t_s)}(n_s) = \frac{1}{\sqrt{2\pi\sigma_s^2}} \exp\left(-\frac{(n_s - m_s)^2}{2\sigma_s^2}\right) \tag{8.1}$$

where

$$m_N(t_s) = E\left[N(t_s)\right] \qquad \sigma_N^2(t_s) = var\left(N(t_s)\right).$$

Consequently the complete characterization of one sample of a random process only requires the mean value and the variance of $N(t_s)$ to be evaluated.

Property 8.3 *Most thermally generated noise corrupting a communication system typically has a zero mean.*

This is motivated by the physics of thermal noise generation. If the average voltage or current in a conductor is not zero that means that electrons are on average leaving the device. Physically it is not possible without an external force to induce an average voltage or current flow. All random processes will be assumed to have a *zero mean* throughout the remainder of this chapter.

Property 8.4 *If $N(t)$ is a Gaussian random process then two samples of this process, $N(t_1)$ and $N(t_2)$, are completely characterized with the PDF*

$$f_{N(t_1)N(t_2)}(n_1, n_2) = \frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2(1-\rho_N(t_1,t_2)^2)}} \exp\left[\frac{-1}{1-\rho_N(t_1,t_2)^2}\left(\frac{n_1^2}{2\sigma_1^2} - \frac{2\rho_N(t_1,t_2)n_1n_2}{2\sigma_1\sigma_2} + \frac{n_2^2}{2\sigma_2^2}\right)\right]$$

where

$$\sigma_i^2 = E\left[N^2(t_i)\right] \qquad \rho_N(t_1,t_2) = \frac{E\left[N(t_1)N(t_2)\right]}{\sigma(t_1)\sigma(t_2)}.$$

The joint PDF of two Gaussian random variables can be characterized by evaluating the variance of each of the samples and the correlation coefficient between the two samples. Recall this correlation coefficient describes how similar two random variables obtained by sampling the random process at different times are. Similarly three or more samples of a Gaussian random process can be characterized by evaluating the variance of each of the samples and the correlation coefficient between each of the samples [LG89, DR87].

Property 8.5 *A function that contains all the information needed to characterize the joint distribution of any set of samples from a zero mean Gaussian random process is the correlation function,*

$$R_N(t_1, t_2) = E\left[N(t_1)N(t_2)\right]. \tag{8.2}$$

Proof: Property 8.4 shows that only $\sigma^2(t)$ and $\rho(t_1, t_2)$ needs to be identified to characterize the joint PDF of Gaussian random variables. To this end

$$\sigma_N^2(t) = R_N(t,t) = E\left[N^2(t)\right] \qquad \rho_N(t_1,t_2) = \frac{R_N(t_1,t_2)}{\sqrt{R_N(t_1,t_1)R_N(t_2,t_2)}}. \qquad \square$$

The correlation function plays a key role in the analysis of Gaussian random processes.

Point 2: When the noise, $N(t)$, is a Gaussian random process the canonical problem can be solved with knowledge of the correlation function, $R_N(t_1, t_2)$. The correlation function completely characterizes any joint PDF of samples of the process $N(t)$.

8.3 Stationary Random Processes

Stationary random processes are random processes that statistically have the same description as a function of time. For example Fig. 8.3-a) shows a sample function of a nonstationary noise and Fig. 8.3-b) shows a sample function of a stationary noise. The random process plotted in Fig. 8.3-a) appears to have a variance, $\sigma_N^2(t)$, that is growing with time, while the random process in Fig. 8.3-b) appears to have a constant variance. Examples of random processes which are not stationary are prevalent

- Temperature in Columbus, OH
- The level of the Dow-Jone's Industrial Average.

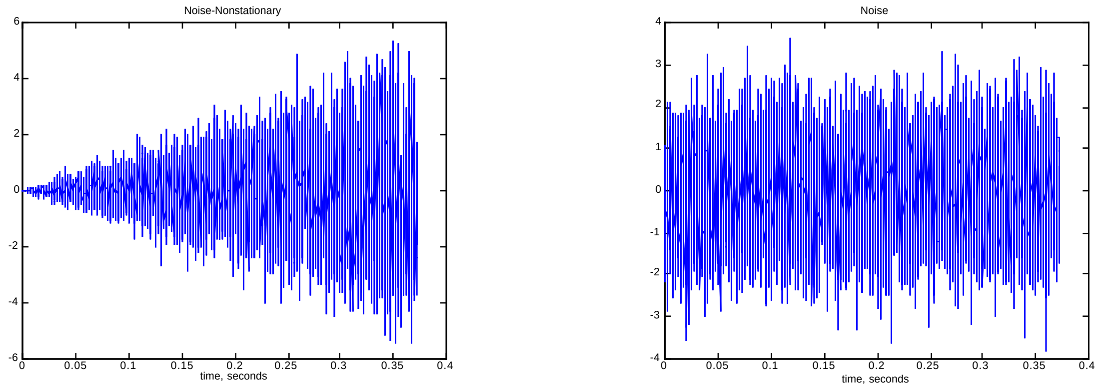
In these nonstationary random processes the statistical description of the resulting random variables will change greatly depending on when the process is sampled. For example the average temperature in Columbus, OH is significantly different in July and in January. Nonstationary random processes are much more difficult to characterize than stationary processes. Fortunately the thermal noise that is often seen in communication systems is well modeled as stationary over the time spans which are of interest in understanding communication system's performance.

8.3.1 Basics

Definition 8.3 *A random process, $N(t)$, is called stationary if the density function describing the samples of the random process has*

$$f_{N(t_1),N(t_2),\cdots,N(t_M)}(n_1, n_2, \cdots, n_M) = f_{N(t_1+t_0),N(t_2+t_0),\cdots,N(t_M+t_0)}(n_1, n_2, \cdots, n_M)$$

for any value of M and t_0.



a) Nonstationary noise b) Stationary noise

Figure 8.3: Sample functions of random processes.

In particular the random variable obtained by sampling stationary random processes is statistically identical regardless of the selected sample time, i.e., $f_{N(t)}(n_1) = f_{N(t+t_0)}(n_1)$. Also two samples taken from a stationary random process will have a statistical description that is a function of the time difference between the two time samples but not the absolute location of the time samples, i.e., $f_{N(0),N(t_1)}(n_1, n_2) = f_{N(t_0),N(t_1+t_0)}(n_1, n_2)$.

8.3.2 Gaussian Processes

Stationarity for Gaussian random processes implies a fairly simple complete description for the random process. Recall a Gaussian random process is completely characterized with $R_N(t_1, t_2) = E[N(t_1)N(t_2)]$. Since $f_{N(t)}(n_1) = f_{N(t+t_0)}(n_1)$ this implies that the mean ($E[N(t)] = 0$) and the variance ($\sigma^2(t_i) = \sigma_n^2$) are constants. Since $f_{N(0),N(t_1)}(n_1, n_2) = f_{N(t_0),N(t_1+t_0)}(n_1, n_2)$ we know from Property 8.4 that $R_N(0, t_1) = R_N(t_0, t_0 + t_1)$ for all t_0. This implies that the correlation function is essentially a function of only one variable, $R_N(t_1, t_2) = R_N(\tau) = E[N(t)N(t-\tau)]$ where $\tau = t_1 - t_2$. For a stationary Gaussian process this correlation function $R_N(\tau)$ completely characterizes any statistical event associated with the noise $N(t)$.

The correlation function gives a description of how rapidly the random process is changing with time. For a stationary random process the correlation function can be expressed as

$$R_N(\tau) = \sigma_N^2 \rho_N(\tau) \tag{8.3}$$

where $\rho(\tau)$ is the correlation coefficient between two samples taken τ seconds apart. Recall that if $\rho(\tau) \approx 0$ then the two samples will behave statistically in an independent fashion. If $\rho(\tau) \approx 1$ then the two samples will be very close in value (statistically dependent).

The idea of a correlation function has been introduced before in the context of deterministic signal analysis. For deterministic signals the correlation function of a signal $x(t)$ was defined as

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)dt \tag{8.4}$$

This correlation function also was a measure of how rapidly a signal varied in time just like the correlation function for random processes measures the time variations of a noise. Table 8.1 summarizes the comparison between the correlation functions for random processes and deterministic energy signals.

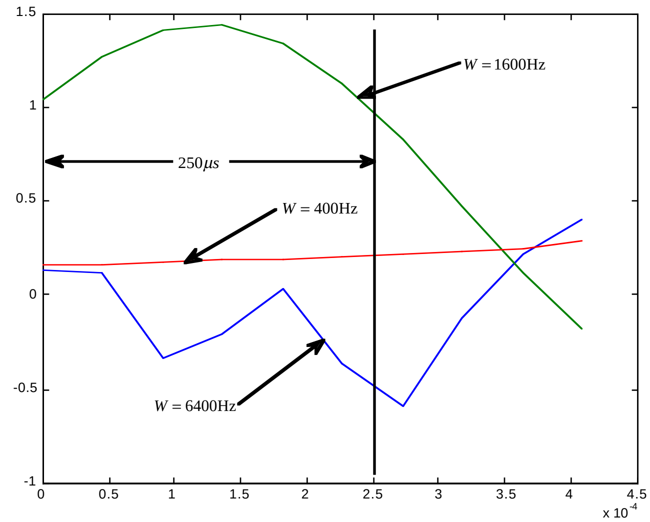


Figure 8.4: Sample functions for several Gaussian random processes parameterized by (8.5). Note this plot show samples of random processes where the sample rate is $f_s = 22050$Hz. The jaggedness and piecewise linear nature of the high bandwidth noise is due to this sampling. A higher sampling rate would result in a smoother appearance.

Signal	Definition	Units	$R_\bullet(0)$
Deterministic	$R_x(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)dt$	Joules	$R_x(0) = E_x$ (Energy of $x(t)$)
Random	$R_N(\tau) = E\left[N(t)N(t-\tau)\right]$	Watts	$R_N(0) = \sigma_N^2$ (Average power of $N(t)$)

Table 8.1: Summary of the important characteristics of the correlation function.

Example 8.1: Consider a correlation coefficient parameterized as

$$\rho_N(\tau) = \mathrm{sinc}\left(2W\tau\right). \tag{8.5}$$

For two samples taken τ=250μs apart, a process characterized with $W = 400$Hz would have these samples behaving very similar ($\rho_N(.00025) = 0.935$) while a process characterized with $W = 6400$Hz would have $\rho_N(.00025) = -0.058$ and essentially the two samples will be independent. Fig. 8.4 shows a plot of several sample functions of Gaussian random processes whose correlation function is given (8.5) with different values of W. The process parameterized with $W = 6400$Hz varies rapidly over 250μs while the process parameterized with $W = 400$Hz is nearly constant.

8.3.3 Frequency Domain Representation

Stationary noise can also be described in the frequency domain. Frequency domain analysis has been very useful for providing an alternate view (compared to time domain) in deterministic signal analysis. This characteristic holds true for stationary random processes. Using the definition of a finite time

Fourier transform given in (1.7), i.e.,

$$N_T(f) = \int_{-T}^{T} N(t) \exp\left[-j2\pi f t\right] dt, \tag{8.6}$$

the average power spectral density (units Watts per Hertz) is given as

$$S_N(f,T) = \frac{1}{2T} E\left[|N_T(f)|^2\right]. \tag{8.7}$$

Definition 8.4 *The power spectral density of a random process $N(t)$ is*

$$S_N(f) = \lim_{T\to\infty} S_N(f,T).$$

Intuitively, random processes that vary rapidly must have power at higher frequencies in a very similar way as deterministic signals behave. This definition of the power spectral density puts that intuition on a solid mathematical basis. The power spectral density is a function which defines how the average power of a random process is distributed as a function of frequency. An additional relation to the correlation function can be made.

Property 8.6 *The power spectrum of a real noise is always a non-negative and even function of frequency, i.e., $S_N(f) \geq 0$ and $S_N(f) = S_N(-f)$.*

Proof: The positivity of $S_N(f)$ is a direct consequence of Definition 8.4 and (8.7). The evenness is due to Property 1.4. □

Property 8.7 (Wiener-Khinchin) *The power spectral density for a stationary random process is given by*

$$S_N(f) = \mathcal{F}\left\{R_N(\tau)\right\}.$$

Proof: The power spectral density (see Definition 8.4) can be rewritten as

$$S_N(f) = \lim_{T\to\infty} \frac{1}{2T} E\left[|N_T(f)|^2\right].$$

Using the definition in (8.6) gives

$$\begin{aligned} E\left[|N_T(f)|^2\right] &= E\left[\int_{-T}^{T} N(t_1)\exp\left[-j2\pi f t_1\right] dt_1 \int_{-T}^{T} N(t_2)\exp\left[j2\pi f t_2\right] dt_2\right] \\ &= \int_{-T}^{T}\int_{-T}^{T} E\left[N(t_1)N(t_2)\right]\exp\left[-j2\pi f(t_1-t_2)\right] dt_1 dt_2. \end{aligned} \tag{8.8}$$

Making a change of variables $\tau = t_1 - t_2$ and using the stationarity of $N(t)$ reduces (8.8) to

$$E\left[|N_T(f)|^2\right] = \int_{-T}^{T}\int_{-T-t_2}^{T-t_2} R_N(\tau)\exp\left[-j2\pi f(\tau)\right] d\tau dt_2. \tag{8.9}$$

Taking the limit gives the desired result. □

Actually the true Wiener-Khinchin theorem is a bit more general but this form is all that is needed for this course.

The power spectral density can be used to find the average power of a random process with a direct analogy to Rayleigh's energy theorem. Recall that

$$R_N(\tau) = \mathcal{F}^{-1}\left\{S_N(f)\right\} = \int_{-\infty}^{\infty} S_N(f)\exp\left(j2\pi f\tau\right) df \tag{8.10}$$

so that the total average power of a random process is given as

$$R_N(0) = \sigma_N^2 = \int_{-\infty}^{\infty} S_N(f) df. \tag{8.11}$$

Equation (8.11) demonstrates that the average power of a random process is the area under the power spectral density curve.

Example 8.2: If

$$R_N(\tau) = \sigma_N^2 \mathrm{sinc}\left(W\tau\right). \tag{8.12}$$

then

$$\begin{aligned} S_N(f) &= \frac{\sigma_N^2}{2W} \qquad |f| \leq W \\ &= 0 \qquad \text{elsewhere.} \end{aligned} \tag{8.13}$$

A random process having this correlation–spectral density pair has power uniformly distributed over the frequencies from $-W$ to W.

The duality of the power spectral density of random signals, $S_N(f)$ and the energy spectral density of deterministic energy signals, $G_x(f)$, is also strong. Table A.1 summarizes the comparison between the correlation functions for random processes and deterministic energy signals.

Point 3: When the noise, $N(t)$, is a stationary Gaussian random process the canonical problem can be solved with knowledge of the power spectral density of the random process, $S_N(f)$. The power spectral density determines the correlation function, $R_N(\tau)$ through an inverse Fourier transform. The correlation function completely characterizes any joint PDF of samples of the process $N(t)$.

Signal	Definition	Units	$R_\bullet(0)$	
Deterministic	$G_x(f) = \mathcal{F}\left\{R_x(\tau)\right\}$	Joules/Hz	$R_x(0) = E_x = \int_{-\infty}^{\infty} G_x(f)df$	$G_x(f) \geq 0$
Random	$S_N(f) = \mathcal{F}\left\{R_N(\tau)\right\}$	Watts/Hz	$R_N(0) = \sigma_N^2 = \int_{-\infty}^{\infty} S_N(f)df$	$S_N(f) \geq 0$

Table 8.2: Summary of the duality of spectral densities.

8.4 Thermal Noise

An additive noise is characteristic of almost all communication systems. This additive noise typically arises from thermal noise generated by random motion of electrons in conductors. From the kinetic theory of particles, given a resistor R at a physical temperature of T degrees Kelvin, there exists a random additive noise voltage, $W(t)$, across the resistor. This noise is zero mean, stationary[1], and Gaussian distributed (due to the large number of randomly moving electrons and the Central Limit

[1] As long as the temperature is constant.

Theorem), with a spectral density given as [Zie86]

$$S_W(f) = \frac{2Rh|f|}{\exp\left[\frac{h|f|}{KT}\right] - 1} \tag{8.14}$$

where $h = 6.62 \times 10^{-34}$ is Planck's constant and $K = 1.3799 \times 10^{-23}$ is Boltzmann's constant. The major characteristic to note is that this spectral density is constant up to about 10^{12} Hz at a value of $2KTR$. Since a majority of communications occurs at frequencies much lower than 10^{12} Hz[2] an accurate model of the thermal noise is to assume a flat spectral density (white noise). This spectral density is a function of the receiver temperature. The lower the temperature of the receiver the lower the noise power.

Additive white Gaussian noise (AWGN) is a basic assumption made about the corrupting noise in most communication systems performance analyses. White noise has a constant spectral density given as

$$S_W(f) = \frac{N_0}{2} \tag{8.15}$$

where N_0 is the one-sided noise power spectral density. Again assuming a resistance of 1Ω and using the maximum power transfer theorem of linear systems and the discussion for (8.14) that $N_0 = KT$. For room temperature ($T = 290°K$) $N_0 = 4 \times 10^{-21}$ Watts/Hz (-174dBm/Hz). This noise spectral density is very small and if this thermal noise is to corrupt the communication signal then the signal must take a small value. Consequently it is apparent from this discussion that thermal noise is only going to be significant when the signal at the receiver has been attenuated to a relatively small power level. Note that since

$$E[W^2(t)] = \int_{-\infty}^{\infty} S_W(f)df$$

this thermal noise model has an infinite average power. This characteristic is only a result of modeling (8.14) with the simpler form in (8.15).

The autocorrelation function of AWGN is given by

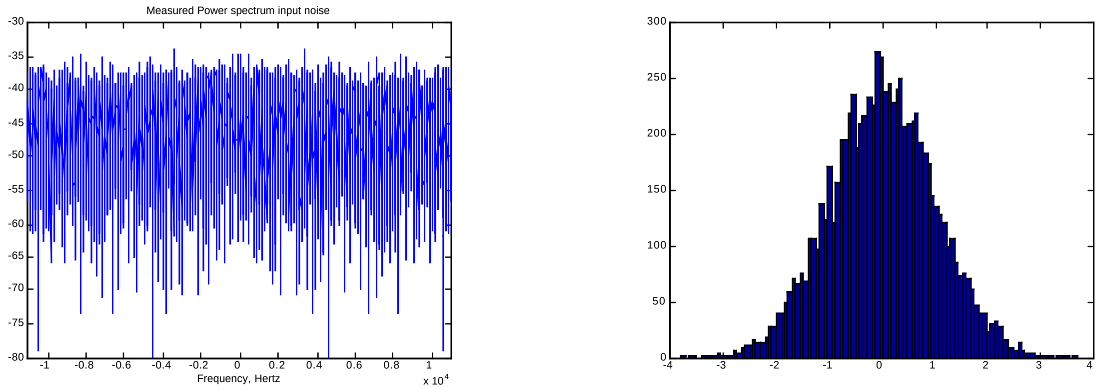
$$R_W(\tau) = \frac{N_0}{2}\delta(\tau).$$

AWGN is a stationary random process so a great majority of the analysis in communication systems design can utilize the extensive theory of stationary random processes. White noise has the interesting characteristic that two samples of the process $W(t)$ will always be independent no matter how closely the samples are taken. Consequently white noise can be thought of as a very rapidly varying noise with no predictability.

Example 8.3: The noise plotted in Fig. 8.3-b) could result from sampling an AWGN at a sample rate of $f_s = 22050$Hz. The rapidly varying characteristic is evident from Fig. 8.3-b). The measured power spectrum (see (8.7)) for this process is shown in Fig. 8.5-a). This PSD, while extremely jagged due to the consideration of a finite time interval, clearly shows the whiteness of the noise. Fig. 8.5-a) show the histogram of the samples of the AWGN. The Gaussian nature of this histogram is also clearly evident.

Point 4: The dominant noise in communication systems is often generated by the random motion of electrons in conductors. This noise process is accurately modeled as stationary, Gaussian, and white.

[2] Fiber optic communication is a notable exception to this statement.



a) The measured PSD. b) The measured histogram.

Figure 8.5: Characteristics of sampled white Gaussian noise.

8.5 Linear Systems and Random Processes

The final step in solving the canonical problem is to characterize the output of a linear time-invariant filter when the input is a white Gaussian stationary random process.

Property 8.8 *Random processes that result from linear filtering of Gaussian random processes are also Gaussian processes.*

Proof: The technical machinery needed to prove this property is beyond the scope of this class but most advanced texts concerning random processes will prove this. For example [DR87]. $\square$

Property 8.9 *A Gaussian random process that results from linear time-invariant filtering of another stationary Gaussian random process is also a stationary process.*

Proof: Assume the filter input is $W(t)$, the filter impulse response is $h_R(t)$, and the filter output is $N(t)$. The proof requires that the correlation function of the process $N(t)$ be a function of τ only. The two argument correlation function of $N(t)$ is given as

$$\begin{aligned} R_N(t_1,t_2) &= E\left[N(t_1)N(t_2)\right] \\ &= E\left[\int_{-\infty}^{\infty} h_R(\lambda_1)W(\lambda_1-t_1)d\lambda_1 \int_{-\infty}^{\infty} h_R(\lambda_2)W(\lambda_2-t_2)d\lambda_2\right]. \end{aligned} \tag{8.16}$$

Rearranging gives

$$\begin{aligned} R_N(t_1,t_2) &= \int_{-\infty}^{\infty} h_R(\lambda_1)\int_{-\infty}^{\infty} h_R(\lambda_2)E\left[W(\lambda_1-t_1)W(\lambda_2-t_2)\right]d\lambda_1 d\lambda_2 \\ &= \int_{-\infty}^{\infty} h_R(\lambda_1)\int_{-\infty}^{\infty} h_R(\lambda_2)R_W(\lambda_1-t_1-\lambda_2+t_2)d\lambda_1 d\lambda_2 \\ &= \int_{-\infty}^{\infty} h_R(\lambda_1)\int_{-\infty}^{\infty} h_R(\lambda_2)R_W(\lambda_1-\lambda_2-\tau)d\lambda_1 d\lambda_2 \\ &= g(\tau). \quad \square \end{aligned} \tag{8.17}$$

The canonical problem given in Fig. 8.1 can now be solved by finding the correlation function or spectral density of the stationary Gaussian random process $N(t)$. For the canonical problem $W(t)$ is a white noise so that (8.17) can be rewritten as

$$R_N(\tau) = \int_{-\infty}^{\infty} h_R(\lambda_1) \int_{-\infty}^{\infty} h_R(\lambda_2) \frac{N_0}{2} \delta(\lambda_1 - \lambda_2 - \tau) d\lambda_1 d\lambda_2. \tag{8.18}$$

Using the sifting property of the delta function gives

$$R_N(\tau) = \frac{N_0}{2} \int_{-\infty}^{\infty} h_R(\lambda_1) h_R(\lambda_1 - \tau) d\lambda_1 = \frac{N_0}{2} R_{h_R}(\tau) \tag{8.19}$$

where $R_{h_R}(\tau)$ is the correlation function of the deterministic impulse response $h_R(t)$ as defined in Chapter 1. This demonstrates that the output correlation function from a linear filter with a white noise input is only a function of the filter impulse response and the white noise spectral density.

The frequency domain description of the random process $N(t)$ is equally simple. The power spectral density is given as

$$S_N(f) = \mathcal{F}\left\{R_N(\tau)\right\} = \frac{N_0}{2} \mathcal{F}\left\{R_{h_R}(\tau)\right\} = \frac{N_0}{2} G_{h_R}(f) = \frac{N_0}{2} \left|H_R(f)\right|^2. \tag{8.20}$$

The average output noise power is given as

$$\sigma_N^2 = R_N(0) = \frac{N_0}{2} \int_{-\infty}^{\infty} \left|H_R(f)\right|^2 df.$$

Example 8.4: Consider an ideal low pass filter of bandwidth W, i.e.,

$$\begin{aligned} H_R(f) &= 1 \qquad |f| \leq W \\ &= 0 \qquad \text{elsewhere.} \end{aligned}$$

The output spectral density is given as

$$\begin{aligned} S_N(f) &= \frac{N_0}{2} \qquad |f| \leq W \\ &= 0 \qquad \text{elsewhere} \end{aligned}$$

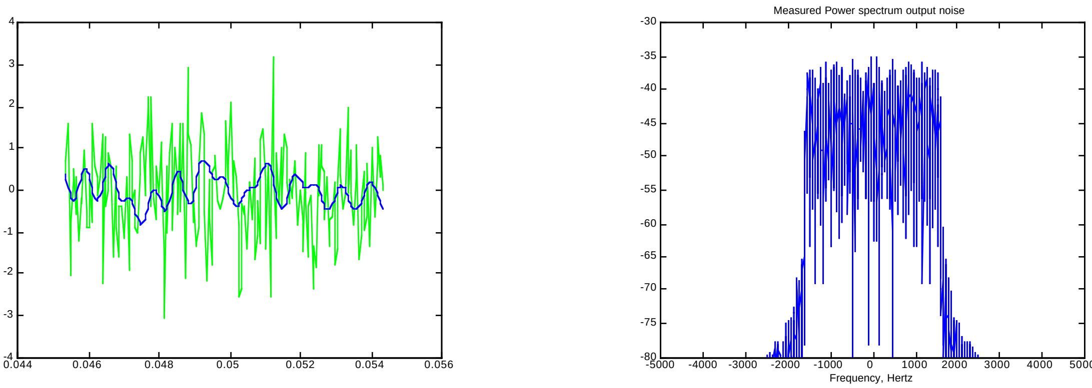
and the average output power is given as $\sigma_N^2 = N_0 W$.

Engineers noted the simple form for the average output noise power expression in Example 8.4 and decided to parameterize the output noise power of all filters by a single number, the noise equivalent bandwidth.

Definition 8.5 *The noise equivalent bandwidth of a filter is*

$$B_N = \frac{1}{2|H_R(0)|^2} \int_{-\infty}^{\infty} \left|H_R(f)\right|^2 df.$$

For an arbitrary filter the average output noise power will be $\sigma_N^2 = N_0 B_N |H_R(0)|^2$. Note for a constant $|H_R(0)|$ the smaller the noise equivalent bandwidth the smaller the noise power. This implies that noise power can be minimized by making B_N as small as possible. In the homework problems we will show that the output signal to noise ratio is not a strong function of $|H_R(0)|$ so that performance in the presence of noise is well parameterized by B_N. In general, $H_R(f)$ is chosen as a compromise between complexity of implementation, signal distortion, and noise mitigation.



a) Sample paths of white and filtered noise b) Measured PSD

Figure 8.6: Measured characteristics of filtered noise.

Example 8.5: The sampled white Gaussian noise shown in Fig. 8.3-b) is put into a lowpass filter of bandwith 1600Hz. This lowpass filter will significantly smooth out the sampled noise and reduce the variance. Two output sample functions; one for the wideband noise and one for the filtered noise, are shown in Fig. 8.6-a). The lowering of the output power of the noise and the resulting smoother time variations are readily apparent in this figure. The output measured PSD of the filtered noise is shown in Fig. 8.6-b). This measured PSD validates (8.20), as this PSD is clearly the result of multiplying a white noise spectrum with a lowpass filter transfer function. The histogram of the output samples, shown in Fig. 8.7, again demonstrates that this noise is well modeled as a zero mean Gaussian random process.

Point 5. A stationary, white, and Gaussian noise of two sided spectral density $N_0/2$ when passed through a filter will produce a stationary Gaussian noise whose power spectral density is given as

$$S_N(f) = \frac{N_0}{2} \left| H_R(f) \right|^2 .$$

8.6 The Solution of the Canonical Problem

Putting together the results of the previous five sections leads to a solution of the canonical problem posed in this chapter.

First, the characterization of one sample, $N(t_s)$, of the random process $N(t)$ is considered. This case requires a four step process summarized as

1. Identify N_0 and $H_R(f)$.

2. Compute $S_N(f) = \frac{N_0}{2} \left| H_R(f) \right|^2$.

3. $\sigma_N^2 = R_N(0) = \frac{N_0}{2} \int_{-\infty}^{\infty} \left| H_R(f) \right|^2 df = N_0 B_N |H_R(0)|^2$

4. $f_{N(t_s)}(n_1) = \frac{1}{\sqrt{2\pi\sigma_N^2}} \exp\left(-\frac{n_1^2}{2\sigma_n^2}\right)$

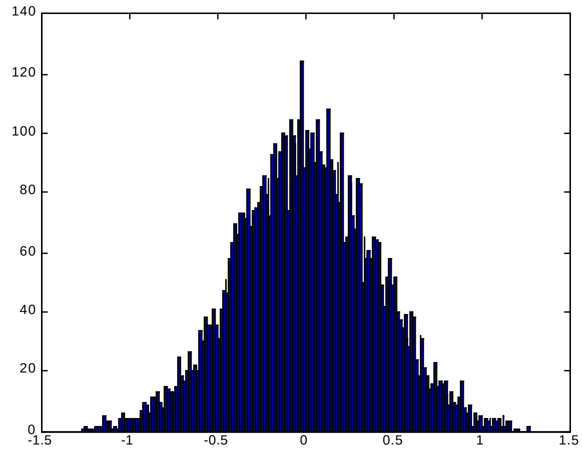


Figure 8.7: A histogram of the samples of the filtered noise.

Example 8.6: Consider two ideal lowpass filters with

$$\begin{aligned} H_{R(i)}(f) &= 1 \qquad |f| \leq W_i \\ &= 0 \qquad \text{elsewhere.} \end{aligned}$$

with W_1=400Hz and W_2=6400Hz and a noise spectral density of

$$N_0 = \frac{1}{1600}.$$

Step 2 gives

$$\begin{aligned} S_{N_i}(f) &= \frac{1}{1600} \qquad |f| \leq W_i \\ &= 0 \qquad \text{elsewhere.} \end{aligned}$$

Step 3 gives $\sigma^2_{N_1} = 0.25$ and $\sigma^2_{N_2} = 4$. The average noise power between the two filter outputs is different by a factor of 16. The PDFs in step 4 are plotted in Fig. 8.8.

Second, the characterization of two samples, $N(t_1)$ and $N(t_2)$, of the random process $N(t)$ is considered. This case requires a five step process summarized as

1. Identify N_0 and $H_R(f)$.
2. Compute $S_N(f) = \frac{N_0}{2} |H_R(f)|^2$.
3. $R_N(\tau) = \mathcal{F}^{-1}\{S_N(f)\}$.
4. $\sigma^2_N = R_N(0)$ and $\rho_N(\tau) = R_N(\tau)/\sigma^2_N$.
5. $f_{N(t_1)N(t_2)}(n_1, n_2) = \frac{1}{2\pi\sigma^2_N\sqrt{(1-\rho_N(\tau)^2)}} \exp\left[\frac{-1}{2\sigma^2_N(1-\rho_N(\tau)^2)}\left(n_1^2 - 2\rho_N(\tau)n_1n_2 + n_2^2\right)\right]$.

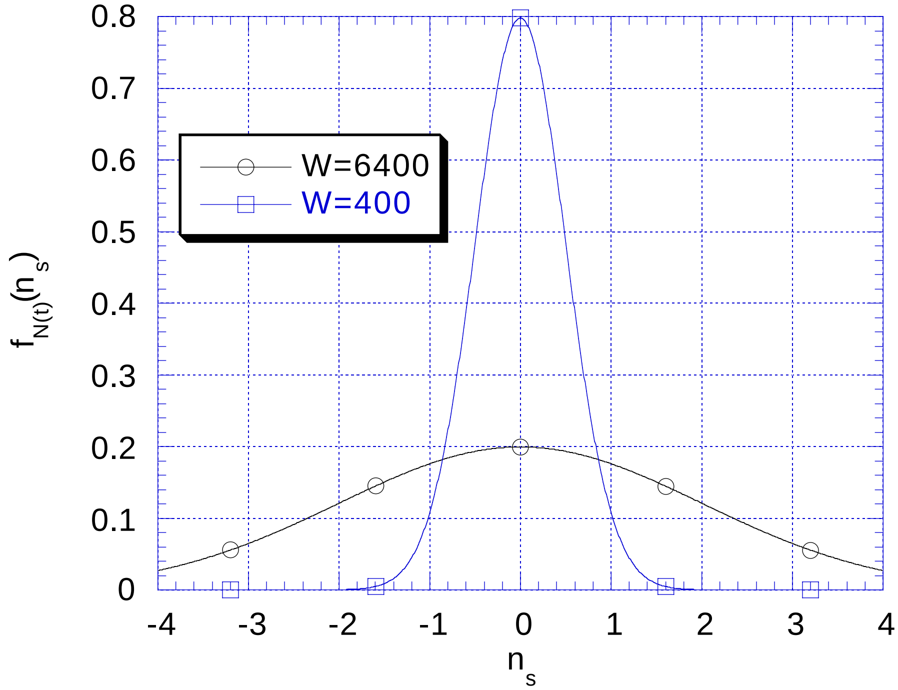


Figure 8.8: The PDF of a filter output sample for W=400 and W=6400Hz.

Example 8.7: Consider two ideal lowpass filters with

$$\begin{aligned} H_{R(i)}(f) &= 1 \qquad |f| \leq W_i \\ &= 0 \qquad \text{elsewhere.} \end{aligned}$$

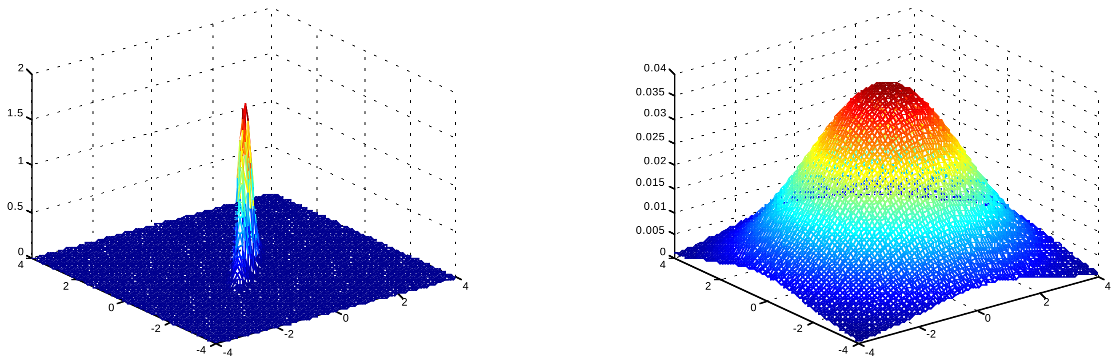
with W_1=400Hz and W_2=6400Hz and a noise spectral density of

$$N_0 = \frac{1}{1600}.$$

Step 2 gives

$$\begin{aligned} S_{N_i}(f) &= \frac{1}{1600} \qquad |f| \leq W_i \\ &= 0 \qquad \text{elsewhere.} \end{aligned}$$

Step 3 gives $R_{N_1}(\tau) = 0.25\text{sinc}\,(800\tau)$ and $R_{N_2}(\tau) = 4\text{sinc}\,(12800\tau)$. Step 4 has the average noise power between the two filter outputs being different by a factor of 16 since $\sigma^2_{N_1} = 0.25$ and $\sigma^2_{N_2} = 4$. For an offset of τ seconds the correlation coefficients are $\rho_{N_1}(\tau) = \text{sinc}\,(800\tau)$ and $\rho_{N_2}(\tau) = \text{sinc}\,(12800\tau)$. The PDFs in step 5 are plotted in Fig. 8.9-a) and Fig. 8.9-b) for $\tau = 250\mu$ s. With the filter having a bandwidth of 400Hz the variance of the output samples is less and the correlation between samples is greater. This higher correlation implies that the probability that the two random variables take significantly different values is very small. This correlation is evident due to the knife edge like joint PDF. These two joint PDFs show the characteristics in an analytical form that were evident in the noise sample paths shown in Fig. 8.4.



a) W=400Hz. b) W=6400Hz.

Figure 8.9: The PDF of two samples separated by $\tau = 250\mu$s.

Similarly three of more samples of the filter output could be characterized in a very similar fashion. The tools developed in this chapter give a student the ability to analyze many of the important properties of noise that are of interest in communication system design.

8.7 Homework Problems

Problem 8.1. For this problem consider the canonical block diagram shown in Fig. 8.10 with $s_i(t) = \cos(2\pi(200)t)$ and with $W(t)$ being an additive white Gaussian noise with two-sided spectral density of $N_0/2$. Assume $H_R(f)$ is an ideal lowpass filter with bandwidth of 400Hz. Two samples are taken from the filter output at time $t = 0$ and $t = \tau$, i.e., $Y_o(0)$ and $Y_o(\tau)$.

a) Choose N_0 such that the output noise power, $E\left[N^2(t)\right] = 1$.

b) $Y_o(0)$ is a random variable. Compute $E\left[Y_o(0)\right]$ and $E\left[Y_o^2(0)\right]$, and $\mathrm{var}\left(Y_o(0)\right)$.

c) $Y_o(\tau)$ is a random variable. Compute $E\left[Y_o(\tau)\right]$ and $E\left[Y_o^2(\tau)\right]$, and $\mathrm{var}\left(Y_o(\tau)\right)$.

d) Find the correlation coefficient between $Y_o(0)$ and $Y_o(\tau)$i.e.,

$$\rho_{Y_o}(\tau) = \frac{E\left[(Y_o(0) - E\left[Y_o(0)\right])(Y_o(\tau) - E\left[Y_o(\tau)\right])\right]}{\sqrt{\mathrm{var}\left(Y_o(0)\right)\mathrm{var}\left(Y_o(\tau)\right)}}. \tag{8.21}$$

e) Plot the joint density function, $f_{Y_o(0)Y_o(\tau)}(y_1, y_2)$, of these two samples for $\tau = 2.5$ms.

f) Plot the joint density function, $f_{Y_o(0)Y_o(\tau)}(y_1, y_2)$, of these two samples for $\tau = 25\mu$s.

g) Compute the time average signal to noise ratio where the instantaneous signal power is defined as $E\left[Y_o(t)\right]^2$.

Problem 8.2. Real valued additive white Gaussian noise (two sided spectral density $N_0/2$) is input into a linear time invariant filter with a real valued impulse response $h_R(t)$ where $h_R(t)$ is constrained to have a unit energy (see Fig. 8.10 where $s_i(t) = 0$).

a) Calculate the output correlation function, $R_N(\tau)$.

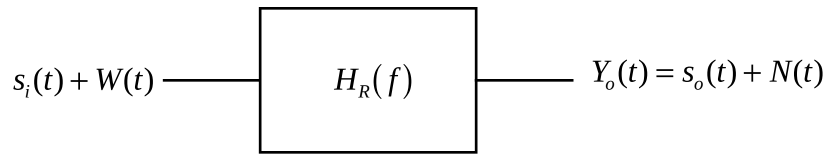


Figure 8.10: Block diagram for signals, noise, and linear systems.

b) The output signal is sampled at a rate of f_s. Give conditions on the impulse response, $h_R(t)$, that will make these samples independent. Give one example of a filter where the output samples will be independent.

c) Give the PDF of one sample, e.g., $f_{N(t_1)}(n_1)$.

d) Give the PDF of any two sample, e.g., $f_{N(t_1)N(t_1+k/f_s)}(n_1, n_2)$ where k is a nonzero integer and the filter satisfies the conditions in b).

e) Give an expression for $P\left(N(t_1) > 2\right)$.

Problem 8.3. Real valued additive white Gaussian noise, $W(t)$ with a two sided spectral density $N_0/2 = 0.1$ is input into a linear time invariant filter with a transfer function of

$$\begin{aligned} H(f) &= 2 \qquad |f| \leq 10 \\ &= 0 \qquad \text{elsewhere.} \end{aligned} \tag{8.22}$$

The output is denoted $N(t)$.

a) What is the correlation function, $R_W(\tau)$, that is used to model $W(t)$?

b) What is $E\left[N(t)\right]$?

c) Calculate the output power spectral density, $S_N(f)$.

d) What is $E\left[N^2(t)\right]$?

e) Give the PDF of one sample, e.g., $f_{N(t_1)}(n_1)$.

f) Given an expression for $P\left(N(t_1) > 3\right)$.

Problem 8.4. For this problem consider the canonical block diagram shown in Fig. 8.10 with $s_i(t) = A$ and with $W(t)$ being an additive white Gaussian noise with two-sided spectral density of $N_0/2$. Assume a 1Ω system and break the filter response into the DC gain term, $H_R(0)$ and the normalized filter response, $H_N(f)$, i.e., $H_R(f) = H_R(0)H_N(f)$.

a) What is the input signal power?

b) What is the output signal power?

c) What is the output noise power?

d) Show that the output SNR is not a function of $H_R(0)$.

e) Give the output SNR as a function of A, N_0, and B_N.

Problem 8.5. For the canonical problem given in Fig. 9.1, the output noise spectral density is given as

$$\begin{aligned} S_N(f) &= N_0 f^2 \qquad |f| \leq W \\ &= 0 \qquad \text{elsewhere.} \end{aligned} \tag{8.23}$$

a) What is the ouput noise power?

b) Give a possible $H_R(f)$ that would result in this output noise?

c) Think of a way to implement this $H_R(f)$ using a lowpass filter cascaded with another linear system.

Problem 8.6. Show that $S_N(f) = S_W(f) |H_R(f)|^2$ by taking the Fourier transform of (8.17).

8.8 Example Solutions

Not included this edition

8.9 Mini-Projects

Goal: To give exposure

1. to a small scope engineering design problem in communications
2. to the dynamics of working with a team
3. to the importance of engineering communication skills (in this case oral presentations).

Presentation: The forum will be similar to a design review at a company (only much shorter) The presentation will be of 5 minutes in length with an overview of the given problem and solution. The presentation will be followed by questions from the audience (your classmates and the professor). Each team member should be prepared to give the presentation.

Project 8.1. You have a voice signal with average power of -10dBm and bandwidth of 4kHz that you want to transmit over a cable from one building on campus to another building on campus. The system block diagram is shown in Fig. 8.11. The noise in the receiver electronics is accurately modeled as an AWGN with a one sided noise spectral density of N_0=-174dBm/Hz. The cable is accurately modeled as a filter with the following impulse response

$$h_c(t) = L_p \delta(t - \tau_p). \tag{8.24}$$

where L_p is the cable loss. You are using cable with a loss of 2dB/1000ft. How long of a cable can be laid and still achieve at least 10dB SNR? If the signal was changed to a video signal with -10dBm average power and bandwidth of 6MHz, how long of a cable can be laid and still achieve at least 10dB SNR?

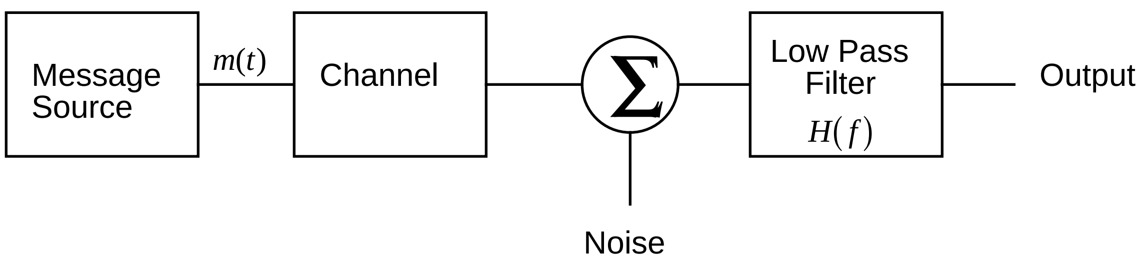


Figure 8.11: Block diagram of a baseband (cable) communication system.

Chapter 9

Noise in Bandpass Communication Systems

Noise in bandpass communication systems is typically produced by filtering wideband noise that appears at the receiver input. Chapter 8 showed that the input noise in a communication system is well modeled as an additive white Gaussian noise. Consequently, the canonical model for noise in a bandpass communication system is given in Fig. 9.1. The white noise, $W(t)$, models a wide band noise which is present at the receiver input. This noise can come from a combination of the receiver electronics, man-made noise sources, or galactic noise sources. This noise is often well modeled as a zero mean, stationary, and Gaussian random process with a power spectral density of $N_0/2$ Watts/Hz. The filter, $H_R(f)$, represents the frequency response of the receiver system. This frequency response is often designed to have a bandpass characteristic to match the transmission band and is bandpass around the carrier frequency, f_c. Chapter 8 showed that $N_c(t)$ is a stationary Gaussian process with a power spectral density (PSD) of

$$S_{N_c}(f) = |H_R(f)|^2 \frac{N_0}{2}. \tag{9.1}$$

The noise, $N_c(t)$, will consequently have all it's power concentrated around the carrier frequency, f_c. Noise with this characteristic is denoted a bandpass noise. This noise in a bandpass communication system will then be passed through a I/Q downconverter to produce an in-phase noise, $N_I(t)$, and a quadrature noise, $N_Q(t)$. This chapter provides the necessary tools to characterize the low pass noise processes that result in this situation.

Example 9.1: Consider a receiver system with an ideal bandpass filter response of bandwidth B_T centered at f_c, i.e.,

$$\begin{aligned} H_R(f) &= A \qquad ||f| - f_c| \leq \frac{B_T}{2} \\ &= 0 \qquad \text{elsewhere} \end{aligned} \tag{9.2}$$

where A is a real positive constant. The bandpass noise will have a spectral density of

$$\begin{aligned} S_{N_c}(f) &= \frac{A^2 N_0}{2} \qquad ||f| - f_c| \leq \frac{B_T}{2} \\ &= 0 \qquad \text{elsewhere.} \end{aligned} \tag{9.3}$$

These tools will enable an analysis of the performance of bandpass communication systems in the presence of noise. This ability to analyze the effects of noise is what separates the competent com-

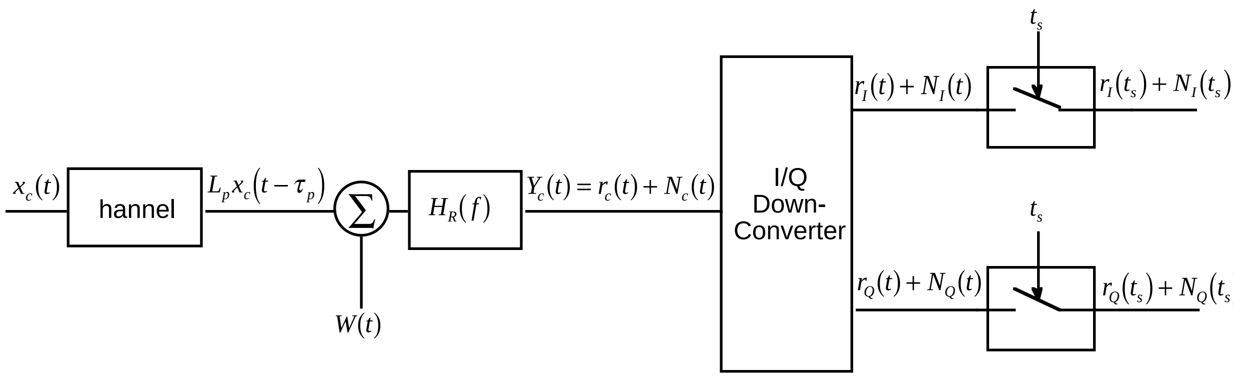


Figure 9.1: The canonical model for communication system analysis.

munication system engineer from the hardware designers and the technicians they work with. The simplest analysis problem is examining a particular point in time, t_s, and characterizing the resulting noise samples, $N_I(t_s)$ and $N_Q(t_s)$, to extract a parameter of interest (e.g., average signal–to–noise ratio (SNR)). Bandpass communication system performance can be characterized completely if probability density functions (PDF) of the noise samples of interest can be characterized, e.g.,

$$f_{N_I}(n_i), \qquad f_{N_I(t)N_I(t+\tau)}(n_{i1}, n_{i2}), \quad \text{and/or} \quad f_{N_Q(t)N_I(t+\tau)}(n_{q1}, n_{i2}). \tag{9.4}$$

To accomplish this analysis task this chapter first introduces notation for discussing the bandpass noise and the lowpass noise. The resulting lowpass noise processes are shown to be zero mean, jointly stationary, and jointly Gaussian. Consequently the PDFs like those detailed in (9.4) will entirely be a function of the second order statistics of the lowpass noise processes. These second order statistics can be produced from the PSD given in (9.1).

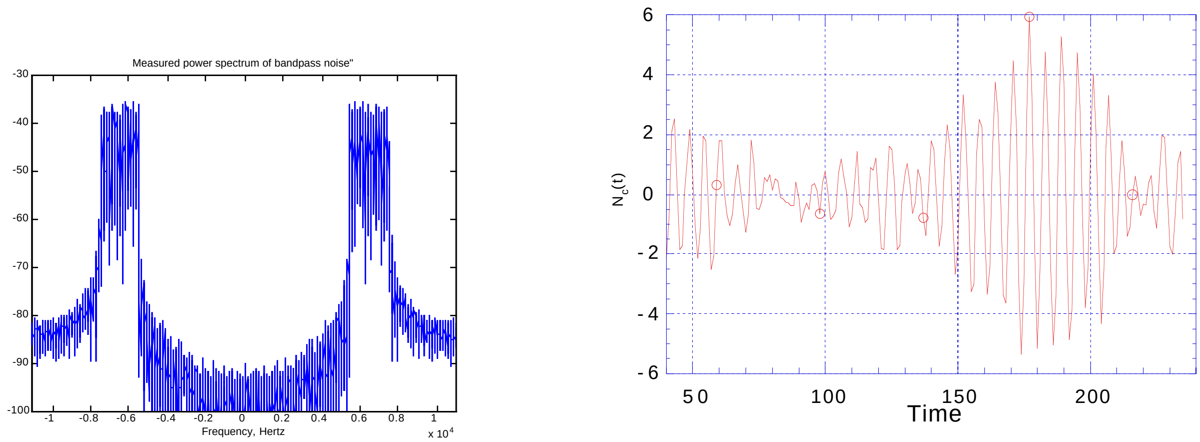
Example 9.2: If a sampled white Gaussian noise like that considered in Fig. 8.3-b) with a measured spectrum like that given in Fig.8.5-a) is put through a bandpass filter a bandpass noise will result. For the case of a bandpass filter with a center frequency of 6500Hz and a bandwidth of 2000Hz a measured output PSD is shown in Fig. 9.2-a). The validity of (9.1) is clearly evident in this output PSD. A resulting sample function of this output bandpass noise is given in Fig.9.2-b). A histogram of output samples of this noise process is shown in Fig.9.3. This histogram again demonstrates that a bandpass noise is well modeled as a zero mean Gaussian random process.

Point 1: In bandpass communications the input noise, $N_c(t)$, is a stationary Gaussian random process with power spectral density given in (9.1). The effect of noise on the performance of a bandpass communication system can be analyzed if the PDFs of the noise samples of interest can be characterized as in (9.4).

9.1 Notation

A bandpass random process will have the same form as a bandpass signal. The forms most often used for this text are

$$\begin{aligned} N_c(t) &= N_I(t)\sqrt{2}\cos(2\pi f_c t) - N_Q(t)\sqrt{2}\sin(2\pi f_c t) & (9.5)\\ &= N_A(t)\sqrt{2}\cos(2\pi f_c t + N_P(t)). & (9.6) \end{aligned}$$



a) The measured PSD

b) The sample function

Figure 9.2: The measured characteristics of bandpass filtered white noise.

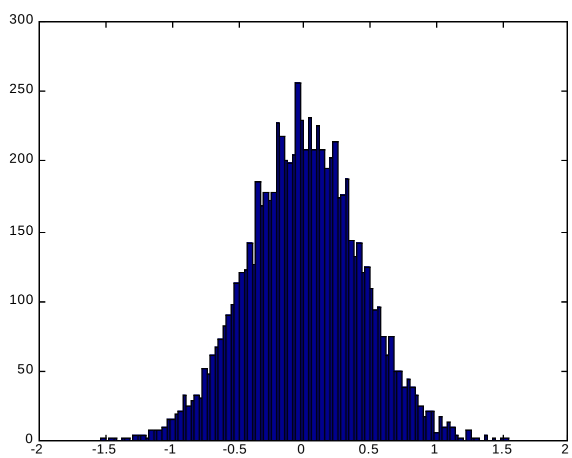


Figure 9.3: A histogram of the samples taken from the bandpass noise resulting from bandpass filtering a white noise.

$N_I(t)$ in (9.5) is normally referred to as the **in-phase (I)** component of the noise and $N_Q(t)$ is normally referred to as the **quadrature (Q)** component of the bandpass noise. The **amplitude** of the bandpass noise is $N_A(t)$ and **phase** of the bandpass noise is $N_P(t)$. As in the deterministic case the transformation between the two representations are given by

$$N_A(t) = \sqrt{N_I(t)^2 + N_Q(t)^2} \qquad N_P(t) = \tan^{-1}\left[\frac{N_Q(t)}{N_I(t)}\right]$$

and

$$N_I(t) = N_A(t)\cos(N_P(t)) \qquad N_Q(t) = N_A(t)\sin(N_P(t)).$$

A bandpass random process has sample functions which appear to be a sinewave of frequency f_c with a slowly varying amplitude and phase. An example of a bandpass random process is shown in Fig. **??**. As in the deterministic signal case, a method of characterizing a bandpass random process which is independent of the carrier frequency is desired. The complex envelope representation provides such a vehicle.

The **complex envelope** of a bandpass random process is defined as

$$N_z(t) = N_I(t) + jN_Q(t) = N_A(t)\exp\left[jN_P(t)\right]. \tag{9.7}$$

The original bandpass random process can be obtained from the complex envelope by

$$N_c(t) = \sqrt{2}\Re\left[N_z(t)\exp\left[j2\pi f_c t\right]\right].$$

Since the complex exponential is a deterministic function, the complex random process $N_z(t)$ contains all the randomness in $N_c(t)$. In a similar fashion as a bandpass signal, a bandpass random process can be generated from its I and Q components and a complex baseband random process can be generated from the bandpass random process. Fig. 9.4 shows these transformations. Fig. 9.5 shows a bandpass noise and the resulting in-phase and quadrature components that are output from a downconverter structure shown in Fig. 9.4.

Correlation functions are important in characterizing random processes.

Definition 9.1 *Given two real random processes, $N_I(t)$ and $N_Q(t)$, the crosscorrelation function between these two random processes is given as*

$$R_{N_Q N_I}(t_1, t_2) = E\left[N_I(t_1)N_Q(t_2)\right]. \tag{9.8}$$

The crosscorrelation function is a measure of how similarly the two random processes behave. In an analogous manner to the discussion in Chapter 8 a crosscorrelation coefficient can be defined.

The correlation function of a bandpass random process, which is derived using (9.5), is given by

$$\begin{aligned} R_{N_c}(t_1, t_2) &= E\left[N_c(t_1)N_c(t_2)\right] \\ &= 2R_{N_I}(t_1,t_2)\cos(2\pi f_c t_1)\cos(2\pi f_c t_2) - 2R_{N_I N_Q}(t_1,t_2)\cos(2\pi f_c t_1)\sin(2\pi f_c t_2) \\ &\quad -2R_{N_Q N_I}(t_1,t_2)\sin(2\pi f_c t_1)\cos(2\pi f_c t_2) \\ &\quad +2R_{N_Q}(t_1,t_2)\sin(2\pi f_c t_1)\sin(2\pi f_c t_2). \end{aligned} \tag{9.9}$$

Consequently the correlation function of the bandpass noise is a function of both the correlation function of the two lowpass noise processes and the crosscorrelation between the two lowpass noise processes.

Definition 9.2 *The correlation function of the complex envelope of a bandpass random process is*

$$R_{N_z}(t_1, t_2) = E\left[N_z(t_1)N_z^*(t_2)\right] \tag{9.10}$$

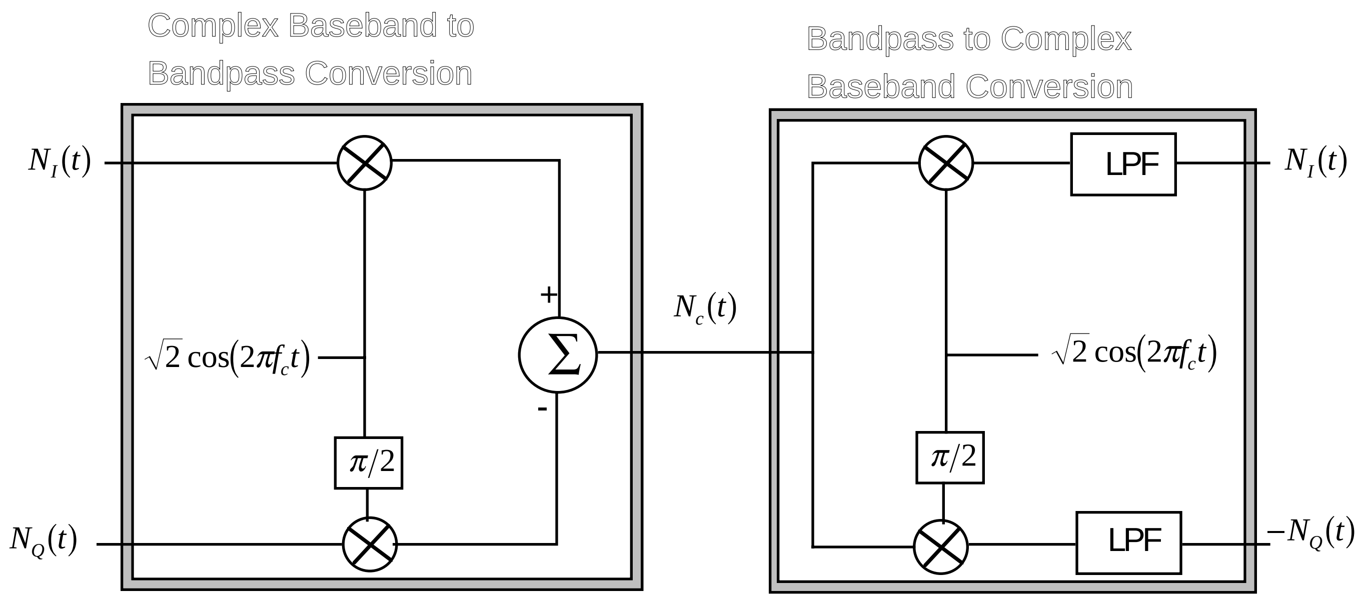


Figure 9.4: The transformations between bandpass noise and the baseband components.

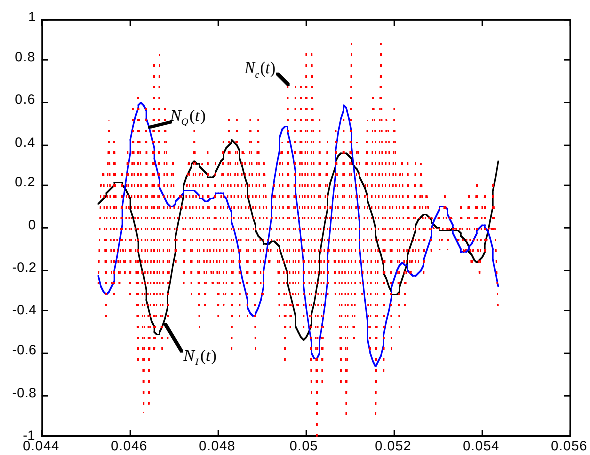


Figure 9.5: A bandpass noise and the resulting in-phase and quadrature noises.

Using the definition of the complex envelope given in (9.7) produces

$$R_{N_z}(t_1,t_2) = R_{N_I}(t_1,t_2) + R_{N_Q}(t_1,t_2) + j\left[-R_{N_I N_Q}(t_1,t_2) + R_{N_Q N_I}(t_1,t_2)\right] \tag{9.11}$$

The correlation function of the bandpass signal, $R_{N_c}(t_1,t_2)$, is derived from the complex envelope, via

$$R_{N_c}(t_1,t_2) = 2E(\Re\left[N_z(t_1)\exp\left[j2\pi f_c t_1\right]\right]\Re\left[N_z^*(t_2)\exp\left[-j2\pi f_c t_2\right]\right]). \tag{9.12}$$

This complicated function can be simplified in some practical cases. The case when the bandpass random process, $N_c(t)$, is a stationary random process is one of them.

9.2 Characteristics of the Complex Envelope

9.2.1 Three Important Results

This section shows that the lowpass noise, $N_I(t)$ and $N_Q(t)$ derived from a bandpass noise, $N_c(t)$, are zero mean, jointly Gaussian, and jointly stationary when $N_c(t)$ is zero mean, Gaussian, and stationary. This will be shown to simplify the description of $N_I(t)$ and $N_Q(t)$ considerably.

Property 9.1 *If the bandpass noise, $N_c(t)$, has a zero mean, then $N_I(t)$ and $N_Q(t)$ both have a zero mean.*

Proof: This property's validity can be proved by considering how $N_I(t)$ (or $N_Q(t)$) is generated from $N_c(t)$ as shown in Fig. 9.4. The I component of the noise is expressed as

$$N_I(t) = N_c(t)\otimes h_L(t) = \sqrt{2}\int_{-\infty}^{\infty} h_L(t-\tau)N_c(\tau)\cos(2\pi f_c\tau)d\tau$$

where $h_L(t)$ is the impulse response of the lowpass filter. Since neither $h_L()$ or the cos() term are random, the linearity property of the expectation operator can be used to get

$$E(N_I(t)) = \sqrt{2}\int_{-\infty}^{\infty} h_L(t-\tau)E(N_c(\tau))\cos(2\pi f_c\tau)d\tau = 0 \qquad \square$$

This property is important since the input thermal noise to a communication system is typically zero mean consequently the I and Q components of the resulting bandpass noise will also be zero mean.

Definition 9.3 *Two random processes $N_I(t)$ and $N_Q(t)$ are jointly Gaussian random processes if any set of samples taken from the two processes are a set of joint Gaussian random variables.*

Property 9.2 *If the bandpass noise, $N_c(t)$, is a Gaussian random process then $N_I(t)$ and $N_Q(t)$ are jointly Gaussian random processes.*

Proof: The detailed proof techniques are beyond the scope of this course but are contained in most advanced texts concerning random processes (e.g., [DR87]). A sketch of the ideas needed in the proof is given here. A random process which is a product of a deterministic waveform and a Gaussian random process is still a Gaussian random process. Hence

$$\begin{aligned} N_1(t) &= N_c(t)\sqrt{2}\cos(2\pi f_c t) \\ N_2(t) &= -N_c(t)\sqrt{2}\sin(2\pi f_c t) \end{aligned} \tag{9.13}$$

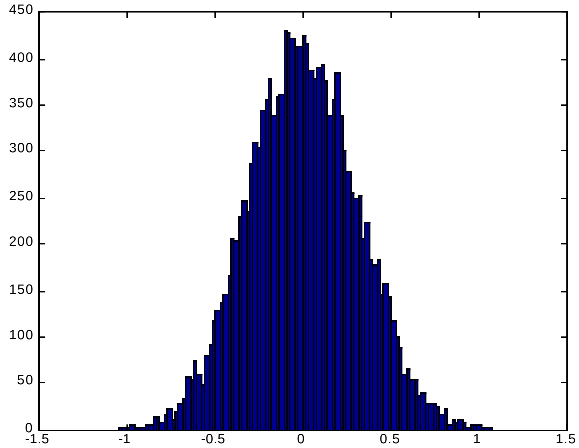


Figure 9.6: A histogram of the samples taken from $N_I(t)$ after downconverting a bandpass noise process.

are jointly Gaussian random processes. $N_I(t)$ and $N_Q(t)$ are also jointly Gaussian processes since they are linear functionals of $N_1(t)$ and $N_2(t)$ (i.e., $N_I(t) = N_1(t) \otimes h_L(t)$). $\square$

Again this property implies that the I and Q components of the bandpass noise in most communication systems will be well modeled as Gaussian random processes. Since $N_c(t)$ is a Gaussian random process and $N_c(k/(2f_c)) = (-1)^k N_I(k/(2f_c))$ it is easy to see that many samples of the lowpass process are Gaussian random variables. Property 8.2 simply implies that all jointly considered samples of $N_I(t)$ and $N_Q(t)$ are jointly Gaussian random variables.

Example 9.3: Consider the previous bandpass filtered noise example where f_c=6500Hz and the bandwidth is 2000Hz. Fig. 9.6 shows a histogram of the samples taken from $N_I(t)$ after downconverting a bandpass noise process. Again is it apparent from this figure that the lowpass noise is well modeled as a zero mean Gaussian random process.

Property 9.3 *If a bandpass signal, $N_c(t)$, is a stationary Gaussian random process, then $N_I(t)$ and $N_Q(t)$ are also jointly stationary, jointly Gaussian random processes.*

Proof: Define the random process $N_{z1}(t) = N_1(t) + jN_2(t) = N_c(t)\exp[-j2\pi f_c t]$. Since $N_c(t)$ is a stationary Gaussian random process then $R_{N_{z1}}(t_1, t_2) = R_{N_c}(\tau)\exp[-j2\pi f_c \tau] = R_{N_{z1}}(\tau)$ where $\tau = t_1 - t_2$. Since $N_z(t) = N_{z1}(t) \otimes h_L(t)$, the correlation function of $N_z(t)$ is also only a function of τ, $R_{N_z}(\tau)$. If $N_{z1}(t)$ is stationary then the stationarity of the output complex envelope is due to Property 8.8. Using (9.11) gives

$$R_{N_z}(\tau) = R_{N_I}(t_1, t_2) + R_{N_Q}(t_1, t_2) + j\left[-R_{N_I N_Q}(t_1, t_2) + R_{N_Q N_I}(t_1, t_2)\right] \tag{9.14}$$

which implies that

$$R_{N_I}(t_1, t_2) + R_{N_Q}(t_1, t_2) = g_1(\tau) \qquad -R_{N_I N_Q}(t_1, t_2) + R_{N_Q N_I}(t_1, t_2) = g_2(\tau). \tag{9.15}$$

Since $R_{N_z}(t_1, t_2)$ is a function only of τ the constraints given in (9.15) must hold.

Alternately, since $N_c(t)$ has the form given in (9.9) a rearrangement (by using trigonometric identities) gives

$$R_{N_c}(\tau) = \left[R_{N_I}(t_1, t_2) + R_{N_Q}(t_1, t_2)\right]\cos(2\pi f_c \tau) + \left[R_{N_I}(t_1, t_2) - R_{N_Q}(t_1, t_2)\right]\cos(2\pi f_c(2t_2 + \tau))$$

$$+\left[R_{N_I N_Q}(t_1,t_2) - R_{N_Q N_I}(t_1,t_2)\right]\sin(2\pi f_c\tau)$$
$$+\left[R_{N_I N_Q}(t_1,t_2) + R_{N_Q N_I}(t_1,t_2)\right]\sin(2\pi f_c(2t_2+\tau)) \quad (9.16)$$

Since $N_c(t)$ is Gaussian and stationary this implies that the right hand side of (9.16) is a function of the time difference, τ. Consequently the factors multiplying the sinusoidal terms having arguments containing t_2 must be zero. Consequently a second set of constraints is

$$R_{N_I}(t_1,t_2) = R_{N_Q}(t_1,t_2) \qquad R_{N_I N_Q}(t_1,t_2) = -R_{N_Q N_I}(t_1,t_2). \quad (9.17)$$

The only way for (9.15) and (9.17) to be satisfied is if

$$R_{N_I}(t_1,t_2) = R_{N_Q}(t_1,t_2) = R_{N_I}(\tau) = R_{N_Q}(\tau) \quad (9.18)$$
$$R_{N_I N_Q}(t_1,t_2) = -R_{N_Q N_I}(t_1,t_2) = R_{N_I N_Q}(\tau) = -R_{N_Q N_I}(\tau) \quad (9.19)$$

Since all correlation functions and crosscorrelation functions are functions of τ then $N_I(t)$ and $N_Q(t)$ are jointly stationary. □

Point 2: If the input noise, $N_c(t)$, is a stationary Gaussian random process then $N_I(t)$ and $N_Q(t)$ are zero mean, jointly Gaussian, and jointly stationary.

9.2.2 Important Corollaries

This section discusses several important corollaries to the important results derived in the last section. The important result from the last section is summarized as if $N_c(t)$ is zero mean, Gaussian and stationary then $N_I(t)$ and $N_Q(t)$ are zero mean, jointly Gaussian, and jointly stationary.

Property 9.4

$$R_{N_I}(\tau) = R_{N_Q}(\tau) \quad (9.20)$$

Surprisingly, this property, given in (9.18), implies that both $N_I(t)$ and $N_Q(t)$ behave in a statistically identical manner.

Property 9.5

$$R_{N_I N_Q}(\tau) = -R_{N_I N_Q}(-\tau) \quad (9.21)$$

This property, due to (9.19), implies $R_{N_I N_Q}(\tau)$ is an odd function with respect to τ.

Property 9.6 $R_{N_z}(\tau) = 2R_{N_I}(\tau) - j2R_{N_I N_Q}(\tau)$

Proof: This is shown by using (9.20) and (9.21) in (9.11). □

This implies that the real part of $R_{N_z}(\tau)$ is an even function of τ and the imaginary part of $R_{N_z}(\tau)$ is an odd function of τ.

Property 9.7

$$R_{N_c}(\tau) = 2R_{N_I}(\tau)\cos(2\pi f_c\tau) + 2R_{N_I N_Q}(\tau)\sin(2\pi f_c\tau) = \Re\left[R_{N_z}(\tau)\exp(j2\pi f_c\tau)\right] \quad (9.22)$$

Proof: This is shown by using (9.20) and (9.21) in (9.16). □

This property implies there is a simple relationship between the correlation function of the stationary Gaussian bandpass noise and the correlation function of the complex envelope of the bandpass noise. This relationship has significant parallels to the relationships given in Chapter 3.

Property 9.8 $var(N_c(t)) = var(N_I(t)) + var(N_Q(t)) = 2var(N_I(t)) = var(N_z(t))$

Proof: This is shown by using (9.20) in (9.22) for $\tau = 0$. $\square$

This property stat that the power in the bandpass noise is the same as the power in complex envelope. This power in the bandpass noise is also the sum of the powers in the two lowpass noises which comprise the complex envelope.

Property 9.9 *For the canonical problem considered in this chapter, $N_I(t)$ and $N_Q(t)$ are completely characterized by the functions $R_{N_I}(\tau)$ and $R_{N_I N_Q}(\tau)$.*

Proof: Any joint PDF of samples taken from jointly stationary and jointly Gaussian processes are completely characterized by the first and second order moments. Note first that $N_I(t)$ and $N_Q(t)$ are zero mean processes. Consequently the characterization of the process only requires the identification of the variance, $\sigma_N = R_{N_I}(0)$, and the correlation coefficient between samples. The correlation coefficient between samples from the same process is given as $\rho_{N_I}(\tau) = R_{N_I}(\tau)/R_{N_I}(0)$ while the correlation coefficient between samples taken from $N_I(t)$ and $N_Q(t)$ is $\rho_{N_I N_Q}(\tau) = R_{N_I N_Q}(\tau)/R_{N_I}(0)$. $\square$

Example 9.4: If one sample from the in-phase channel is considered then

$$f_{N_I}(n_i) = \frac{1}{\sqrt{2\pi\sigma_N^2}} \exp\left(-\frac{n_i^2}{2\sigma_N^2}\right) \tag{9.23}$$

where $\sigma_N^2 = R_{N_I}(0)$.

Example 9.5: If two samples from the in-phase channel are considered then

$$f_{N_I(t)N_I(t-\tau)}(n_1, n_2) = \frac{1}{2\pi\sigma_N^2\sqrt{(1-\rho_{N_I}^2(\tau))}} \exp\left[\frac{-\left(n_1^2 - 2\rho_{N_I}(\tau)n_1 n_2 + n_2^2\right)}{2\sigma_N^2(1-\rho_{N_I}^2(\tau))}\right]$$

where $\rho_{N_I}(\tau) = R_{N_I}(\tau)/R_{N_I}(0)$.

Example 9.6: If one sample each from the in-phase noise and the quadrature noise are considered then

$$f_{N_I(t)N_Q(t-\tau)}(n_1, n_2) = \frac{1}{2\pi\sigma_N^2\sqrt{(1-\rho_{N_I N_Q}^2(\tau))}} \exp\left[\frac{-\left(n_1^2 - 2\rho_{N_I N_Q}(\tau)n_1 n_2 + n_2^2\right)}{2\sigma_N^2(1-\rho_{N_I N_Q}^2(\tau))}\right]$$

where $\rho_{N_I}(\tau) = R_{N_I N_Q}(\tau)/R_{N_I}(0)$.

Property 9.10 *The random variables $N_I(t_s)$ and $N_Q(t_s)$ are independent random variables for any value of t_s.*

Proof: From (9.21) we know $R_{N_I N_Q}(\tau)$ is an odd function. Consequently $R_{N_I N_Q}(0) = 0$ $\square$

Any joint PDF of samples of $N_I(t_s)$ and $N_Q(t_s)$ taken at the same time have the simple PDF

$$f_{N_I(t)N_Q(t)}(n_1, n_2) = \frac{1}{2\pi\sigma_N^2} \exp\left[\frac{-\left(n_1^2 + n_2^2\right)}{2\sigma_N^2}\right]. \tag{9.24}$$

This simple PDF will prove useful in our performance analysis of bandpass communication systems.

Point 3: Since $N_I(t)$ and $N_Q(t)$ are zero mean, jointly Gaussian, and jointly stationary then a complete statistical description of $N_I(t)$ and $N_Q(t)$ is available from $R_{N_I}(\tau)$ and $R_{N_I N_Q}(\tau)$.

9.3 Spectral Characteristics

At this point we need a methodology to translate the known PSD of the bandpass noise (9.1) into the required correlation function, $R_{N_I}(\tau)$, and the required crosscorrelation function, $R_{N_I N_Q}(\tau)$. To accomplish this goal we need a definition.

Definition 9.4 *For two random processes $N_I(t)$ and $N_Q(t)$ whose cross correlation function is given as $R_{N_I N_Q}(\tau)$ the cross spectral density is*

$$S_{N_I N_Q}(f) = \mathcal{F}\left\{ R_{N_I N_Q}(\tau) \right\}. \tag{9.25}$$

Property 9.11 *The PSD of $N_z(t)$ is given by*

$$S_{N_z}(f) = \mathcal{F}\left\{ R_{N_z}(\tau) \right\} = 2S_{N_I}(f) - j2S_{N_I N_Q}(f) \tag{9.26}$$

where $S_{N_I}(f)$ and $S_{N_I N_Q}(f)$ are the power spectrum of $N_I(t)$ and the cross power spectrum of $N_I(t)$ and $N_Q(t)$, respectively.

Proof: This is seen taking the Fourier transform of $R_{N_z}(\tau)$ as given in Property 9.6. □

Property 9.12 *$S_{N_I N_Q}(f)$ is purely an imaginary function and an odd function of τ.*

Proof: This is true since $R_{N_I N_Q}(\tau)$ is an odd real valued function (see Property 1.6). Also the imaginary part must be true for $S_{N_z}(f)$ to be a valid PSD. □

Property 9.13 *The even part of $S_{N_z}(f)$ is due to $S_{N_I}(f)$ and the odd part is due to $S_{N_I N_Q}(f)$*

Proof: A spectral density of a real random process is always even. $S_{N_I}(f)$ is a spectral density. By Property 9.12, $S_{N_I N_Q}(f)$ is a purely imaginary and odd function of frequency. □

Property 9.14

$$S_{N_I}(f) = \frac{S_{N_z}(f) + S_{N_z}(-f)}{4} \qquad S_{N_I N_Q}(f) = \frac{S_{N_z}(-f) - S_{N_z}(f)}{j4}.$$

Proof: This is a trivial result of Property 9.13 □

Consequently Property 9.14 provides a simple method to compute $S_{N_I}(f)$ and $S_{N_I N_Q}(f)$ once $S_{N_z}(f)$ is known.

$S_{N_z}(f)$ can be computed from $S_{N_c}(f)$ given in (9.1) in a simple fashion as well.

Property 9.15

$$S_{N_c}(f) = \frac{1}{2} S_{N_z}(f - f_c) + \frac{1}{2} S_{N_z}(-f - f_c).$$

Proof: Examining Property 9.22 and using some of the fundamental properties of the Fourier transform, the spectral density of the bandpass noise, $N_z(t)$, is expressed as

$$S_{N_z}(f) = 2S_{N_I}(f) \otimes \left[\frac{1}{2}\delta(f - f_c) + \frac{1}{2}\delta(f + f_c) \right] + 2S_{N_I N_Q}(f) \otimes \left[\frac{1}{2j}\delta(f - f_c) - \frac{1}{2j}\delta(f + f_c) \right]$$

where $\otimes$ again denotes convolution. This equation can be rearranged to give

$$S_{N_c}(f) = \left[S_{N_I}(f - f_c) - jS_{N_I N_Q}(f - f_c) \right] + \left[S_{N_I}(f + f_c) + jS_{N_I N_Q}(f + f_c) \right]. \tag{9.27}$$

Noting that due to Property 9.12

$$S_{N_z}(-f) = 2S_{N_I}(f) + j2S_{N_I N_Q}(f), \tag{9.28}$$

(9.27) reduces to the result in Property 9.15. □

This is a very fundamental result. Property 9.15 states that the power spectrum of a bandpass random process is simply derived from the power spectrum of the complex envelope and vice versa. For positive values of f, $S_{N_c}(f)$ is obtained by translating $S_{N_z}(f)$ to f_c and scaling the amplitude by 0.5 and for negative values of f, $S_{N_c}(f)$ is obtained by flipping $S_{N_z}(f)$ around the origin, translating the result to $-f_c$, and scaling the amplitude by 0.5. Likewise $S_{N_z}(f)$ is obtained from $S_{N_c}(f)$ by taking the positive frequency PSD which is centered at f_c and translating it to baseband ($f = 0$) and multiplying it by 2. Property 9.15 also demonstrates in another manner that the average power of the bandpass and baseband noises are identical since the area under the PSD is the same (this was previously shown in Property 9.8).

Example 9.7: Example 8.1 showed a receiver system with an ideal bandpass filter had a bandpass noise PSD of

$$\begin{aligned} S_{N_c}(f) &= \frac{A^2 N_0}{2} \qquad ||f| - f_c| \leq \frac{B_T}{2} \\ &= 0 \qquad \text{elsewhere.} \end{aligned} \tag{9.29}$$

For this case

$$\begin{aligned} S_{N_z}(f) &= A^2 N_0 \qquad |f| \leq \frac{B_T}{2} \\ &= 0 \qquad \text{elsewhere.} \end{aligned} \tag{9.30}$$

and

$$\begin{aligned} S_{N_I}(f) &= \frac{A^2 N_0}{2} \qquad |f| \leq \frac{B_T}{2} \qquad\qquad S_{N_I N_Q}(f) = 0 \\ &= 0 \qquad \text{elsewhere.} \end{aligned} \tag{9.31}$$

Again considering the previous example of bandpass filtered white noise with f_c=6500Hz and a bandwidth of 2000Hz a resulting measured power spectral density of the complex envelope is given in Fig. 9.7. This measured PSD demonstrates the validity of the analytical results given in (9.31).

Point 4: $S_{N_I}(f)$ and $S_{N_I N_Q}(f)$ can be computed in a straightforward fashion from $S_{N_c}(f)$ given in (9.1). A Fourier transform gives $R_{N_I}(\tau)$ and $R_{N_I N_Q}(\tau)$ and a complete statistical description of the complex envelope noise process.

9.4 The Solution of the Canonical Bandpass Problem

The tools and results are now in place to completely characterize the complex envelope of the bandpass noise typically encountered in a bandpass communication system. First, the characterization of one sample, $N(t_s)$, of the random process $N(t)$ is considered. This case requires a six step process summarized as

1. Identify N_0 and $H_R(f)$.
2. Compute $S_{N_c}(f) = \frac{N_0}{2} |H_R(f)|^2$.

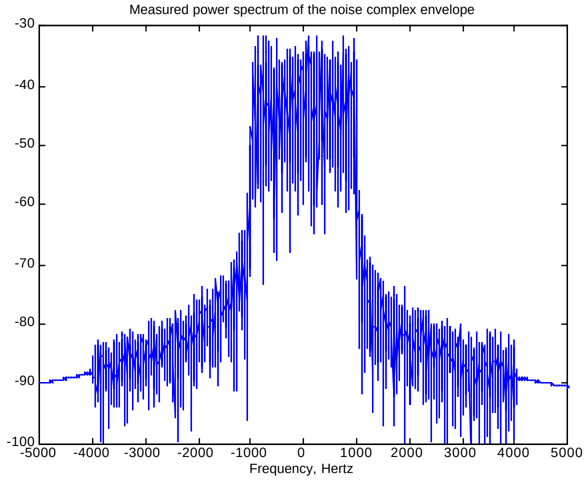


Figure 9.7: The measured PSD of the complex envelope of a bandpass process resulting from bandpass filtering a white noise.

3. Compute $S_{N_z}(f)$.

4. Compute
$$S_{N_I}(f) = \frac{S_{N_z}(f) + S_{N_z}(-f)}{4}.$$

5. $\sigma_N^2 = R_N(0) = \int_{-\infty}^{\infty} S_{N_I}(f) df$.

6. $f_{N(t_s)}(n_1) = \frac{1}{\sqrt{2\pi\sigma_N^2}} \exp\left(-\frac{n_1^2}{2\sigma_N^2}\right)$.

The only difference between this process and the process used for lowpass processes as highlighted in Section 8.6 is step 3–4. These two steps simply are the transformation of the bandpass PSD into the PSD for one channel of the lowpass complex envelope noise.

Example 9.8: The previous examples showed a receiver system with an ideal bandpass filter had after the completion of steps 1–4

$$\begin{aligned} S_{N_I}(f) &= \frac{A^2 N_0}{2} \qquad |f| \leq \frac{B_T}{2} \\ &= 0 \qquad \text{elsewhere.} \end{aligned} \tag{9.32}$$

Consequently $\sigma_N^2 = R_N(0) = \frac{A^2 N_0 B_T}{2}$.

Second, the characterization of two samples from one of the channels of the complex envelope, $N_I(t_1)$ and $N_I(t_2)$, is considered. This case requires a seven step process summarized as

1. Identify N_0 and $H_R(f)$.

2. Compute $S_{N_c}(f) = \frac{N_0}{2} |H_R(f)|^2$.

3. Compute $S_{N_z}(f)$.

4. Compute
$$S_{N_I}(f) = \frac{S_{N_z}(f) + S_{N_z}(-f)}{4}.$$

5. $R_{N_I}(\tau) = \mathcal{F}^{-1}\left\{S_{N_I}(f)\right\}$.
6. $\sigma_N^2 = R_{N_I}(0)$ and $\rho_{N_I}(\tau) = R_{N_I}(\tau)/\sigma_N^2$.
7. $f_{N_{N_I}(t_1)N_{N_I}(t_2)}(n_1,n_2) = \frac{1}{2\pi\sigma_N^2\sqrt{(1-\rho_{N_I}^2(\tau))}} \exp\left[\frac{-1}{2\sigma_N^2(1-\rho_{N_I}^2(\tau))}\left(n_1^2 - 2\rho_{N_I}(\tau)n_1n_2 + n_2^2\right)\right]$.

The only difference between this process and the process used for lowpass processes as highlighted in Section 8.6 is step 3–4. These two steps again are the transformation of the bandpass PSD into the PSD for one channel of the lowpass complex envelope noise.

Example 9.9: The previous examples showed a receiver system with an ideal bandpass filter had after the completion of steps 1–4

$$\begin{aligned} S_{N_I}(f) &= \frac{A^2N_0}{2} \qquad |f| \leq \frac{B_T}{2} \\ &= 0 \qquad \text{elsewhere.} \end{aligned} \tag{9.33}$$

Consequently Step 5 gives $\sigma_N^2 = R_N(0) = \frac{A^2N_0B_T}{2}$ and $R_{N_I}(\tau) = \frac{A^2N_0B_T}{2}\mathrm{sinc}\left(B_T\tau\right)$.

Finally, the characterization of two samples one from each of the channels of the complex envelope, $N_I(t_1)$ and $N_Q(t_2)$, is considered. This case requires a seven step process summarized as

1. Identify N_0 and $H_R(f)$.
2. Compute $S_{N_c}(f) = \frac{N_0}{2}\left|H_R(f)\right|^2$.
3. Compute $S_{N_z}(f)$.
4. Compute
$$S_{N_I}(f) = \frac{S_{N_z}(f) + S_{N_z}(-f)}{4} \qquad S_{N_IN_Q}(f) = \frac{S_{N_z}(-f) - S_{N_z}(f)}{j4}.$$
5. $R_{N_IN_Q}(\tau) = \mathcal{F}^{-1}\left\{S_{N_IN_Q}(f)\right\}$.
6. $\sigma_N^2 = R_{N_I}(0)$ and $\rho_{N_IN_Q}(\tau) = R_{N_IN_Q}(\tau)/\sigma_N^2$.
7. $f_{N_{N_I}(t_1)N_{N_I}(t_2)}(n_1,n_2) = \frac{1}{2\pi\sigma_N^2\sqrt{(1-\rho_{N_IN_Q}^2(\tau))}} \exp\left[\frac{-1}{2\sigma_N^2(1-\rho_{N_IN_Q}^2(\tau))}\left(n_1^2 - 2\rho_{N_IN_Q}(\tau)n_1n_2 + n_2^2\right)\right]$.

The only difference between this process and the process for two samples from the same channel is the computation of the cross correlation function, $R_{N_IN_Q}(\tau)$.

Example 9.10: The previous examples showed a receiver system with an ideal bandpass filter had after the completion of steps 1–4

$$\begin{aligned} S_{N_I}(f) &= \frac{A^2N_0}{2} \qquad |f| \leq \frac{B_T}{2} \qquad\qquad S_{N_IN_Q}(f) = 0 \\ &= 0 \qquad \text{elsewhere.} \end{aligned} \tag{9.34}$$

Consequently Step 5 gives $\sigma_N^2 = R_N(0) = \frac{A^2N_0B_T}{2}$ and $R_{N_IN_Q}(\tau) = 0$. This implies that $N_I(t_1)$ and $N_Q(t_2)$ are independent random variables irregardless of t_1 and t_2.

Similarly, three of more samples from either channel of the complex envelope could be characterized in a very similar fashion. The tools developed in this chapter give a student the ability to analyze many of the important properties of noise that are of interest in a bandpass communication system design.

9.5 Homework Problems

Problem 9.1. What conditions on the bandpass filter characteristic of the receiver, $H_R(f)$, must be satisfied such that $N_I(t_1)$ and $N_Q(t_2)$ are independent random variables for all t_1 and t_2 when the input is AWGN?

Problem 9.2. Consider a bandpass Gaussian noise at the input to a demodulator with a spectrum given as

$$\begin{aligned} S_{N_c}(f) &= 2 \qquad f_c - 1000 \leq |f| \leq f_c + 3000 \\ &= 0 \qquad \text{elsewhere} \end{aligned} \tag{9.35}$$

Assume operation is in a 1Ω system.

a) What bandpass filter, $H_c(f)$, would produce this spectrum from a white noise input with a two sided noise spectral density of $N_0/2$=0.5?

b) What is the spectral density of $N_I(t)$?

c) What is $E\left[N_I^2(t)\right]$?

d) Give the joint PDF of $N_I(t_0)$ and $N_Q(t_0)$ in a 1Ω system for a fixed t_0.

e) Compute $S_{N_I N_Q}(f)$.

Problem 9.3. This problem considers noise which might be seen in a single sideband system. Consider a bandpass Gaussian noise at the input to a demodulator with a spectrum given as

$$\begin{aligned} S_{N_c}(f) &= 3.92 \times 10^{-3} \qquad f_c \leq |f| \leq f_c + 3000 \\ &= 0 \qquad \text{elsewhere} \end{aligned} \tag{9.36}$$

Assume operation is in a 1Ω system.

a) What is the spectral density of $N_I(t)$?

b) What is $E\left[N_I^2(t)\right]$?

c) Give the joint PDF of $N_I(t_0)$ and $N_Q(t_0)$.

d) Give the joint PDF of $N_I(t_1)$ and $N_Q(t_1 - \tau)$.

e) Plot the PDF in d) for $\tau = 0.0001$ and $\tau = 0.001$.

Problem 9.4. Consider a bandpass Gaussian noise at the input to a demodulator with a spectrum given as

$$\begin{aligned} S_{N_c}(f) &= 0.001 \qquad f_c - 2000 \leq |f| \leq f_c + 2000 \\ &= 0 \qquad \text{elsewhere} \end{aligned} \tag{9.37}$$

Assume operation is in a 1Ω system.

a) Find the joint density function of $N_I(t_0)$ and $N_Q(t_0)$, $f_{N_I(t)N_Q(t)}(n_i, n_q)$.

b) Plot $f_{N_I(t)N_Q(t)}(n_i, n_q)$.

c) Consider the complex random variable $\tilde{N}_z(t) = \tilde{N}_I(t) + j\tilde{N}_Q(t) = N_z(t)\exp(-j\phi_p)$ and find an inverse mapping such that

$$\tilde{N}_I(t) = g_1\left(N_I(t_0), N_Q(t_0)\right) \qquad \tilde{N}_Q(t) = g_2\left(N_I(t_0), N_Q(t_0)\right). \tag{9.38}$$

d) Using the results of Section 2.3.4 show that $f_{\tilde{N}_I(t)\tilde{N}_Q(t)}(n_i, n_q) = f_{N_I(t)N_Q(t)}(n_i, n_q)$. In other words the noise distribution is unchanged by a phase rotation. This result is very important for the study of coherent receiver performance in the presence of noise detailed in the sequel.

9.6 Example Solutions

Not included this edition.

9.7 Mini-Projects

Goal: To give exposure

1. to a small scope engineering design problem in communications
2. to the dynamics of working with a team
3. to the importance of engineering communication skills (in this case oral presentations).

Presentation: The forum will be similar to a design review at a company (only much shorter) The presentation will be of 5 minutes in length with an overview of the given problem and solution. The presentation will be followed by questions from the audience (your classmates and the professor). All team members should be prepared to give the presentation.

Project 9.1. A colleague at your company, *Wireless.com*, is working on characterizing the noise in the frontend of an intermediate frequency (IF) receiver that the company is trying to make it's first billion on. The design carrier frequency, f_c, was not documented well by the designer (he left for another startup!) but you do know that the LPF in the downconverters have a cutoff frequency of f_c. The frequency is known to lie somewhere between f_c=2.5kHz and f_c=8kHz. Your colleague is getting some very anomalous results in his testing and has come to you since he knows you had the famous Prof. Fitz (:-)) for analog communications theory. The bandpass noise is received and processed in an I/Q down converter as shown in Fig. 9.8. The anomalous results he sees are

1. if he chooses f_c=3000Hz then the output noise, $N_I(t)$ and $N_Q(t)$, has a bandwidth of 5000Hz,
2. if f_c=4000Hz then the output noise, $N_I(t)$ and $N_Q(t)$, has a bandwidth of 4000Hz.

Explain this anomaly.

Assuming that DSB-AM is the design modulation for the receiver, try and identify the probable design carrier frequency. A sample function of the noise is available for downloading at

`/rcc4/faculty/~fitz/public_html/EE501`

as `noizin.mat`. Examining this file will be necessary to complete this final part of the project.

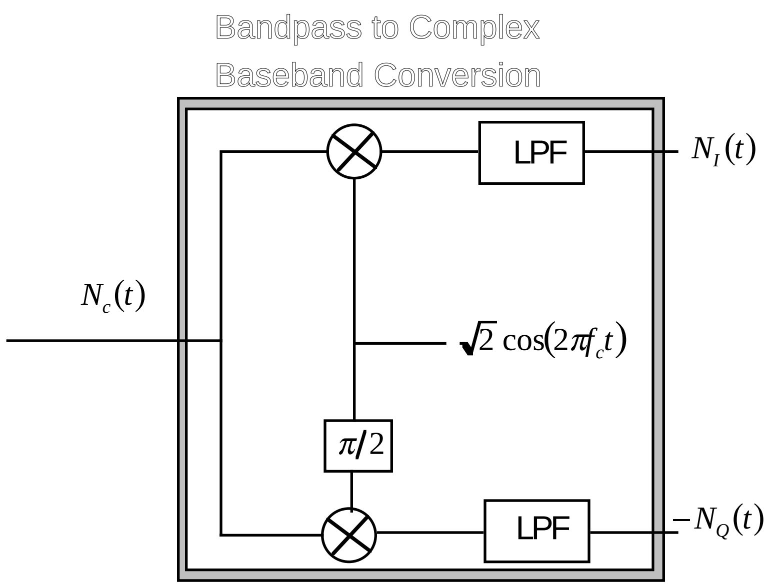


Figure 9.8: Block diagram of a baseband (cable) communication system.

Chapter 10

Performance of Analog Demodulators

This chapter examines all the different analog modulations in the presence of noise that were considered in this text and compares the resulting performance.

10.1 Unmodulated Signals

A first performance evaluation that is useful is to examine the performance of an unmodulated message signal in the presence of an AWGN, $W(t)$. This situation is shown in Fig. 10.1. This situation represents a direct connection of the message signal over a wire/cable from the message source to the message sink. The filter, $H_R(f)$, represents the signal processing at the receiver to limit the noise power. We consider this situation to give a baseline for the performance comparisons in the sequel.

For analog communications the performance metric will be signal to noise ratio (SNR). Recall the output SNR in the system in Fig. 10.1 is defined as

$$\mathrm{SNR} = \frac{P_{x_z}}{P_N} = \frac{A^2 P_m}{P_N}. \tag{10.1}$$

The noise power, P_N, can be calculated with methods described in Chap. 8. Specifically

$$P_N = \sigma_N^2 = N_0 B_N |H_R(0)|^2 = \frac{N_0}{2}\int_{-\infty}^{\infty} |H_R(f)|^2 \tag{10.2}$$

For ease of discussion and comparison this chapter will consider only ideal lowpass filters, e.g.,

$$\begin{aligned} H_R(f) &= 1 \qquad |f| \leq B_f \\ &= 0 \qquad \text{elsewhere} \end{aligned} \tag{10.3}$$

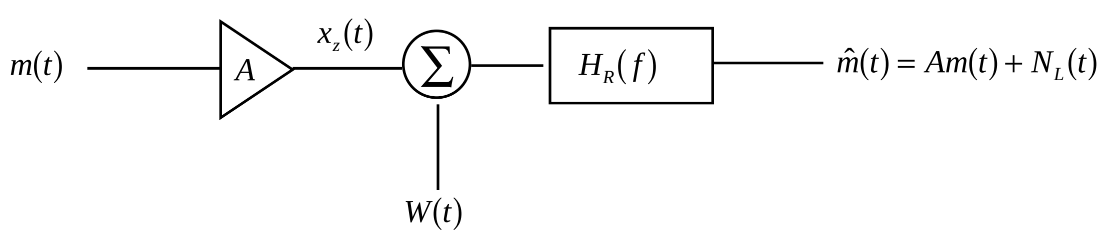


Figure 10.1: A model for baseband transmission.

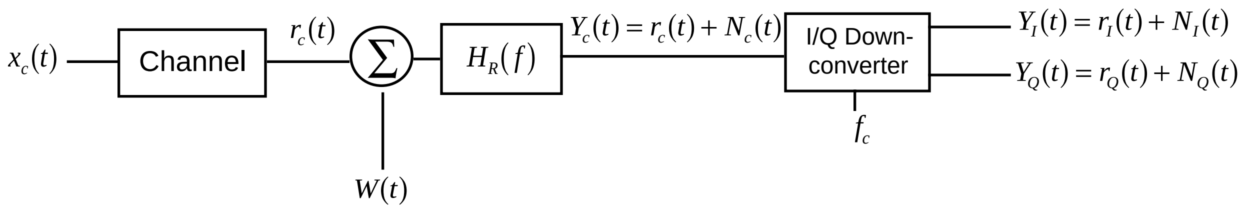


Figure 10.2: A model for bandpass modulated transmission.

Clearly the smaller B_f, the smaller the noise power, but B_f cannot be made too small or else signal distortion will occur and contribute to $N_L(t)$. Again for this chapter to simplify the presentation we will assume $B_f = W$ and consequently signal distortion is not an issue. The output SNR is defined as

$$\mathrm{SNR}_o = \frac{P_{x_z}}{N_0 W} = \frac{A^2 P_m}{N_0 W} = \mathrm{SNR}_b. \tag{10.4}$$

Equation (10.4) is surprisingly simple but it is still instructive to interpret it. In general the noise spectral density is determined by the environment that you are communicating in and is not something the communication system designer has control over. For a fixed noise spectral density the SNR can be made larger by a communication system designer by increasing the message power, P_m, decreasing the loss in the transmission, here A, or decreasing the filter bandwidth, W. Some interesting insights include

1. Transmitting farther distances (smaller A) will produce lower output SNR,
2. Transmitting wider bandwidth signal (video .vs. audio) will produce lower output SNR.

In general we will find the same tradeoffs exist with modulated systems as well. Note this text denotes the output SNR achieved with baseband (unmodulated) transmission as SNR_b. This will be a common reference point to compare analog communication system's performance.

10.2 Bandpass Demodulation

Now let us consider a performance evaluation of bandpass modulated message signal in the presence of an AWGN. This situation is shown in Fig. 10.2. The signal components and the noise components were characterized in the prior chapters so now an integration of this information is needed to provide a characterization of the performance of bandpass communication systems. Again for ease of discussion and comparison this chapter will consider only ideal bandpass filters, e.g.,

$$\begin{aligned} H_R(f) &= 1 \qquad f_c - \frac{B_f}{2} \leq |f| \leq f_c + \frac{B_f}{2} \\ &= 0 \qquad \text{elsewhere.} \end{aligned} \tag{10.5}$$

The output complex envelope has the form

$$Y_z(t) = x_z(t)\exp(j\phi_p) + N_z(t) \tag{10.6}$$

where $N_z(t) = N_I(t) + jN_Q(t)$. The resulting lowpass noise at the output of the downconverter is well characterized via the tools developed in Chapter 9. For example for the filter given in (10.5) it can be shown that

$$\begin{aligned} S_{N_I}(f) &= \frac{N_0}{2} \qquad |f| \leq \frac{B_f}{2} \\ &= 0 \qquad \text{elsewhere} \end{aligned} \tag{10.7}$$

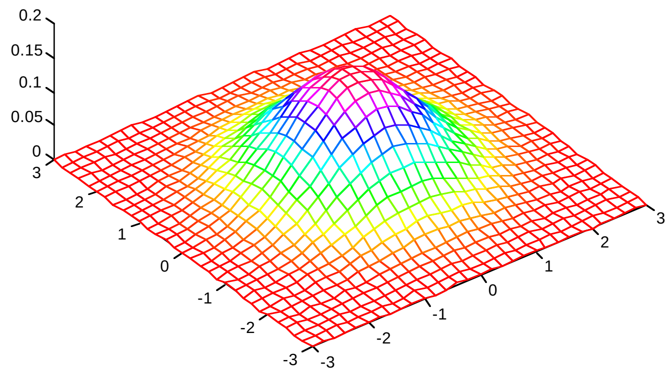


Figure 10.3: Plots of the joint PDF of $N_I(t)$ and $N_Q(t)$ for $N_0 B_f = 2$.

and $S_{N_I N_Q}(f) = 0$. An inverse Fourier transform gives $R_{N_I}(\tau) = \frac{N_0 B_f}{2} \mathrm{sinc}\,(B_f \tau)$. The variance of the lowpass noise is $\mathrm{var}\,(N_I(t)) = \sigma_N^2 = \frac{N_0 B_f}{2}$ and the joint PDF is

$$f_{N_I(t) N_Q(t)}(n_1, n_2) = \frac{1}{2\pi\sigma_N^2} \exp\left[\frac{-\left(n_1^2 + n_2^2\right)}{2\sigma_N^2}\right] = \frac{1}{\pi N_0 B_f} \exp\left[\frac{-\left(n_1^2 + n_2^2\right)}{N_0 B_f}\right]. \tag{10.8}$$

A plot of the density function for $N_0 B_f = 2$ is shown in Fig. 10.3. The input SNR to the demodulator is defined as

$$\mathrm{SNR}_i = \frac{P_{x_z}}{P_{N_z}} = \frac{P_{x_z}}{N_0 B_f}. \tag{10.9}$$

10.2.1 Coherent Demodulation

Many of the analog modulations considered in this text are demodulated with coherent demodulation structures and it is worth examining the noise characteristics of coherent demodulation structures. The coherent demodulator has a block diagram as given in Fig. 10.4. The derotated noise, $\tilde{N}_z(t) = N_z(t) \exp\left(-j\phi_p\right)$ in the coherent demodulator has the same characteristics as $N_z(t)$. This is easy to show mathematically by

$$\begin{aligned} R_{\tilde{N}_z}(\tau) &= E\left[\tilde{N}_z(t)\tilde{N}_z^*(t-\tau)\right] = E\left[N_z(t)\exp\left(-j\phi_p\right) N_z^*(t-\tau)\exp\left(j\phi_p\right)\right] \\ &= E\left[N_z(t) N_z^*(t-\tau)\right] = R_{N_z}(\tau) \end{aligned} \tag{10.10}$$

Equation (10.10) shows that rotating the noise does not change the statistical characteristics of the noise. Intuitively this makes sense because in examining the PDF plotted in Fig. 10.3, the joint PDF of the lowpass noise is clearly circularly symmetric. Any rotation of this PDF would not change the form of the PDF. This is mathematically proven in Problem 9.4. This circularly symmetric noise characteristic and the results of Chapter 9 will allow us to rigorously characterize the performance of coherent demodulation schemes.

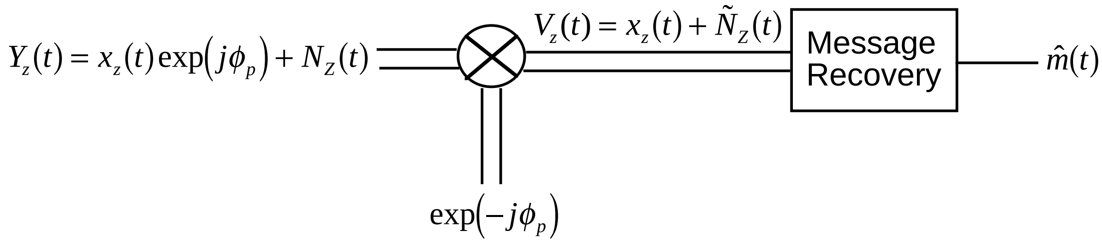


Figure 10.4: The coherent demodulation structure.

10.3 Amplitude Modulation

10.3.1 Coherent Demodulation

Given the tools developed so far the performance analysis of AM modulated systems is straightforward.

Coherent DSB-AM

DSB-AM uses coherent demodulation. Recall the complex envelope form of DSB-AM is $x_z(t) = A_c m(t)$ so consequently the representative block diagram of DSB-AM in noise is given in Fig. 10.5. The output of the *Real* operator has the form $V_I(t) = A_c m(t) + \tilde{N}_I(t)$. This form is exactly the same form as seen in Section 10.1, consequently the choice of the lowpass filter, $H_l(f)$, is subject to the same tradeoffs between minimizing the noise power and minimizing the signal distortion. For simplicity in discussion the lowpass filter will be assume to have the form

$$\begin{aligned} H_L(f) &= 1 \qquad |f| \leq W \\ &= 0 \qquad \text{elsewhere.} \end{aligned} \tag{10.11}$$

The resulting message estimate for DSB-AM has the form $\hat{m}(t) = A_c m(t) + N_L(t)$ where for the bandpass filter given in (10.5)

$$\begin{aligned} S_{N_L}(f) = S_{N_I}(f)\,|H_L(f)|^2 &= \frac{N_0}{2} \qquad |f| \leq W \\ &= 0 \qquad \text{elsewhere.} \end{aligned} \tag{10.12}$$

The output noise power is $\text{var}(N_L(t)) = N_0 W$ and the output SNR is

$$\text{SNR}_o = \frac{A_c^2 P_m}{N_0 W} = \frac{P_{x_z} E_T}{N_0 W} = \frac{P_{x_z}}{N_0 W} = \text{SNR}_b. \tag{10.13}$$

Since for DSB-AM $E_T = 1$ two interesting insights are

1. The output SNR is exactly the same as the case where the signal was transmitted unmodulated. This implies that a coherent modulation and demodulation process can result in no loss of performance compared to baseband transmission while greatly expanding the spectrum that can be used for transmission.

2. Again the two things that affect the output SNR are the transmitted power[1] and the message signal bandwidth. Consequently transmitting video will cost more in terms of resources (transmitted power) than transmitting voice for the same level of output performance.

[1] Also the channel attenuation but we have chosen to normalize this to unity to simplify the discussion

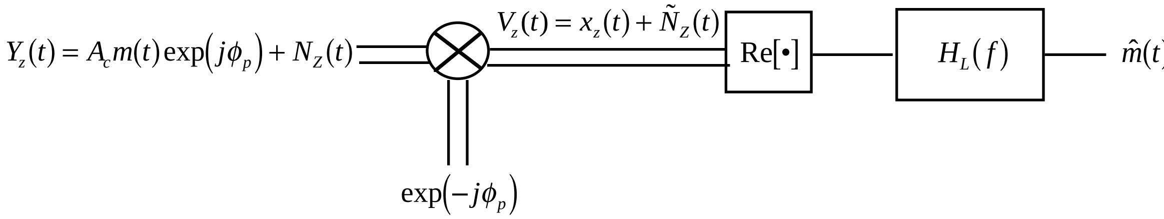


Figure 10.5: The coherent demodulation structure for DSB-AM.

Coherent SSB-AM

SSB-AM also uses coherent demodulation. Recall the complex envelope form of SSB-AM is $x_z(t) = A_c\left(m(t) + jm(t) \otimes h_Q(t)\right)$ so consequently the representative block diagram of SSB-AM in noise is very similar to that in Fig. 10.5. The output of the *Real* operator has the form $V_I(t) = A_c m(t) + \tilde{N}_I(t)$. This form is exactly the same form as DSB-AM. The resulting message estimate for DSB-AM has the form $\hat{m}(t) = A_c m(t) + N_L(t)$ where for the bandpass filter given in (10.5)

$$\begin{aligned} S_{N_L}(f) = S_{N_I}(f)\left|H_L(f)\right|^2 &= \frac{N_0}{2} \qquad |f| \leq W \\ &= 0 \qquad \text{elsewhere.} \end{aligned} \tag{10.14}$$

The output noise power is $\text{var}\left(N_L(t)\right) = N_0 W$ and the output signal to noise ratio is

$$\text{SNR}_o = \frac{A_c^2 P_m}{N_0 W} = \frac{P_{x_z} E_T}{N_0 W} = E_T \text{SNR}_b. \tag{10.15}$$

Since $E_T = 0.5$ for SSB-AM (i.e., half the transmitted power goes into the quadrature channel to shape the spectrum) it appears at first glance that SSB-AM is less efficient in using the transmitted signal power.

This loss in efficiency can be recovered by better signal processing in the receiver. The methodology to improve this efficiency is apparent when examining the signal and noise spectrum of an upper sideband transmission shown in Fig. 10.6. The bandpass filter in (10.5) is passing all signal and noise within B_fHz of the carrier frequency. This bandwidth is required in DSB-AM but clearly the receiver filter can be reduced in bandwidth and have a form

$$\begin{aligned} H_R(f) &= 1 \qquad f_c \leq |f| \leq f_c + \frac{B_f}{2} \\ &= 0 \qquad \text{elsewhere} \end{aligned} \tag{10.16}$$

which results in

$$\begin{aligned} S_{N_c}(f) &= \frac{N_0}{2} \qquad f_c \leq |f| \leq f_c + \frac{B_f}{2} \\ &= 0 \qquad \text{elsewhere.} \end{aligned} \tag{10.17}$$

Using the results of Chapter 9 gives

$$\begin{aligned} S_{N_I}(f) = \frac{S_{N_z}(f) + S_{N_z}(-f)}{4} &= \frac{N_0}{4} \qquad |f| \leq \frac{B_f}{2} \\ &= 0 \qquad \text{elsewhere} \end{aligned} \tag{10.18}$$

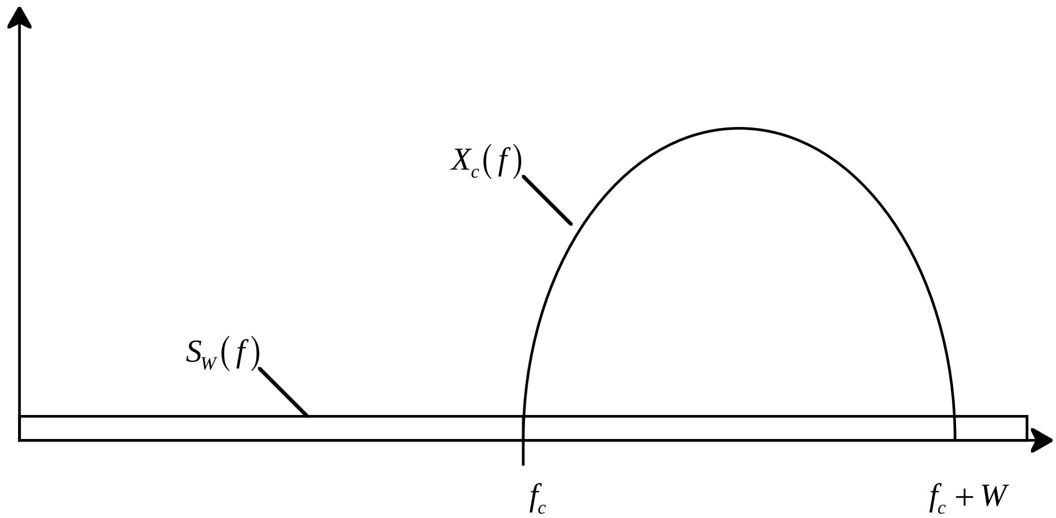


Figure 10.6: The signal and noise spectrum in SSB-AM reception.

The effect of the choosing a filter that only passes the frequencies where the modulated signal is present instead of a filter like that in (10.5) is a reduction by a factor of two in the output noise spectral density. By choosing the LPF to have bandwidth W the output SNR is then given as

$$\mathrm{SNR}_o = \frac{2A_c^2 P_m}{N_0 W} = \frac{2P_{x_z} E_T}{N_0 W} = \frac{P_{x_z}}{N_0 W} = \mathrm{SNR}_b. \tag{10.19}$$

The output SNR is exactly the same as the case where the signal was transmitted unmodulated. This implies SSB-AM can also result in no loss of performance using only the W Hz of bandwidth.

Effect of a Noisy Phase Reference

Not included with this edition.

10.3.2 Noncoherent Demodulation

LC-AM uses noncoherent demodulation. Recall the complex envelope form of DSB-AM is $x_z(t) = A_c(1+am(t))$ with a chosen to allow distortion free envelope detection. Consequently the representative block diagram of LC-AM demodulation in noise is given in Fig. 10.7. Noting

$$Y_z(t) = \exp(j\phi_p)\left(A_c(1+am(t)) + N_z(t)\exp(-j\phi_p)\right) = \exp(j\phi_p)\left(A_c(1+am(t)) + \tilde{N}_z(t)\right), \tag{10.20}$$

the output of the envelope detector has the form

$$Y_A(t) = |Y_z(t)| = \sqrt{\left(A_c(1+am(t)) + \tilde{N}_I(t)\right)^2 + \tilde{N}_Q^2(t)}. \tag{10.21}$$

Using the results of Section 2.3.4 a transformation of random variables could be accomplished and the output could be completely characterized as a Ricean random variable [DR87]. While rigorous the procedure lacks intuition into the effects of noise on noncoherent AM demodulation.

To gain some insight into the operation of a LC-AM system a large SNR approximation is invoked. At high SNR $A_c \gg |\tilde{N}_I(t)|$ and $A_c \gg |\tilde{N}_Q(t)|$ which implies that

$$Y_A(t) \approx \sqrt{\left(A_c(1+am(t)) + \tilde{N}_I(t)\right)^2} = A_c(1+am(t)) + \tilde{N}_I(t) \tag{10.22}$$

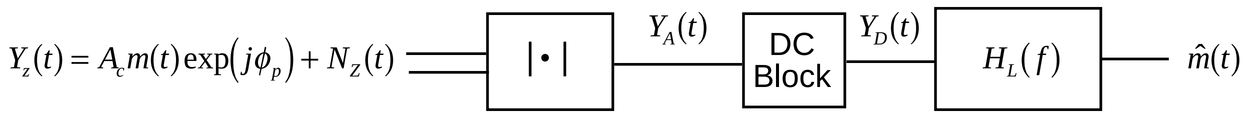


Figure 10.7: The noncoherent demodulation structure for LC-AM.

Consequently at high SNR only the noise in-phase with the carrier will make a difference in the output envelope and the output looks much like that given in for DSB-AM with the exception that the signal format is different. The output of the DC block will be $y_D(t) = A_c a m(t) + \tilde{N}_I(t)$ and with a lowpass filter bandwidth of W and using the same tools as above the output SNR is

$$\mathrm{SNR}_o = \frac{A_c^2 a^2 P_m}{N_0 W} = \frac{P_{x_z} E_T}{N_0 W} = E_T \mathrm{SNR}_b. \tag{10.23}$$

This SNR expression is the same as for coherent AM demodulation. The important thing to remember is that $E_T \ll 1$ for LC-AM to enable envelope detection. Consequently at high SNR the only loss in performance due to using a noncoherent demodulator is the loss in transmission efficiency. The average efficiency of commercial LC-AM is less than 10% so the loss in SNR is greater than 10dB! As SNR_i gets smaller a threshold effect will be experienced in envelope detection and this characteristic is investigated in the sequel.

10.4 Angle Modulations

The analysis of the performance of angle modulation demodulation algorithms is quite complexity and a rich literature exists (e.g., [SBS66, Ric48]). The tack that will be taken in this section is to simplify the analysis to provide the student a good overview of the effects of noise on demodulation.

10.4.1 Phase Modulation

PM uses noncoherent demodulation. Recall the complex envelope of PM is $x_z(t) = A_c \exp(jk_p m(t))$ so consequently the representative block diagram of PM demodulation in noise is given in Fig. 10.8. The output of the phase detector has the form

$$Y_P(t) = \tan^{-1}(Y_Q(t), Y_I(t)) = k_p m(t) + \phi_p + \varphi_e(t) = \varphi_s(t) + \varphi_e(t). \tag{10.24}$$

Defining $\breve{N}_z(t) = N_z(t)\exp(-j\varphi_s(t))$ in a similar way as was done for coherent demodulation it can be shown that

$$\varphi_e(t) = \tan^{-1}\left(\breve{N}_Q(t), A_c + \breve{N}_I(t)\right) = \tan^{-1}\left(\frac{\breve{N}_Q(t)}{A_c + \breve{N}_I(t)}\right). \tag{10.25}$$

Again using the results of Section 2.3.4 a transformation of random variables could be accomplished and the output could be completely characterized [DR87]. While rigorous the procedure lacks intuition into the effects of noise on noncoherent PM demodulation.

Again to gain some insight into the operation of a PM system a large SNR approximation is invoked. At high SNR $A_c \gg |\breve{N}_I(t)|$ and $A_c \gg |\breve{N}_I(t)|$ which implies that

$$\varphi_e(t) \approx \tan^{-1}\left(\frac{\breve{N}_Q(t)}{A_c}\right) \approx \frac{\breve{N}_Q(t)}{A_c}. \tag{10.26}$$

The important insight gained from this high SNR approximation is that the phase noise produced is a function of the noise in quadrature to the modulated signal. At high SNR the output phase noise is

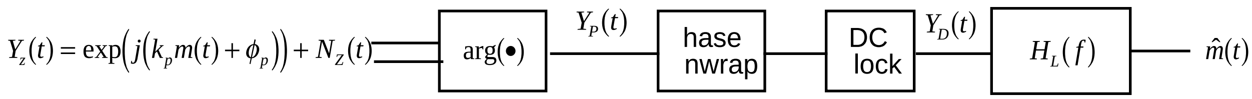


Figure 10.8: The noncoherent demodulation structure for PM.

approximately Gaussian, zero mean (i.e., $E\left[\breve{N}_Q(t)\right] = 0$) and the output noise variance is a function of both the input noise and the input signal (i.e., $E\left[\varphi_e^2(t)\right] = E\left[\breve{N}_Q^2(t)\right]/A_c^2$). The following approximation is also valid at high SNR

$$R_{\breve{N}_Q}(t,\tau) = R_{N_Q}(\tau) = R_{N_I}(\tau). \tag{10.27}$$

Consequently the output phase noise is Gaussian, zero mean and has a PSD of $S_{\varphi_e}(f) = S_{N_I}(f)/A_c^2$. The DC block removes the term due to ϕ_p. Again assuming the lowpass filter has a bandwidth of W then output message estimate has the form $\hat{m}(t) = k_p m(t) + N_L(t)$ where for the bandpass filter given in (10.5)

$$\begin{aligned} S_{N_L}(f) = \frac{S_{N_I}(f)}{A_c^2}\left|H_L(f)\right|^2 &= \frac{N_0}{2A_c^2} \qquad |f| \leq W \\ &= 0 \qquad \text{elsewhere.} \end{aligned} \tag{10.28}$$

The output noise power is $\operatorname{var}\left(N_L(t)\right) = N_0 W/A_c^2$ and the output signal to noise ratio is (recall $P_{x_z} = A_c^2$ for PM)

$$SNR = \frac{A_c^2 k_p^2 P_m}{N_0 W} = k_p^2 P_m \frac{P_{x_z}}{N_0 W} = k_p^2 P_m \mathrm{SNR}_b. \tag{10.29}$$

The resulting SNR performance for PM has two interesting insights

1. At high SNR the effective performance can be improved an arbitrary amount by making increasing k_p. Recall k_p also increases the output bandwidth of a PM signal. Hence PM allows an engineer to trade spectral efficiency for SNR performance. Such a tradeoff is not possible with AM.

2. Again two parameters that affect SNR_o are the transmitted power and the message signal bandwidth much like AM systems. In addition there is a third parameter $k_p^2 P_m$ that can be chosen by the designer.

As SNR_i gets smaller a threshold effect will be experienced in PM demodulation and that is investigated in the sequel.

10.4.2 Frequency Modulation

FM has a noncoherent demodulation structure much like that of PM. This structure is shown in Fig. 10.9. The output of the phase detector has the form

$$Y_P(t) = \tan^{-1}\left(Y_Q(t), Y_I(t)\right) = k_f \int_{-\infty}^{t} m(\lambda)d\lambda + \phi_p + \varphi_e(t) = \varphi_s(t) + \varphi_e(t). \tag{10.30}$$

Again defining $\breve{N}_z(t) = N_z(t)\exp\left(-j\varphi_s(t)\right)$ it can be shown that

$$\varphi_e(t) = \tan^{-1}\left(\breve{N}_Q(t), A_c + \breve{N}_I(t)\right) = \tan^{-1}\left(\frac{\breve{N}_Q(t)}{A_c + \breve{N}_I(t)}\right). \tag{10.31}$$

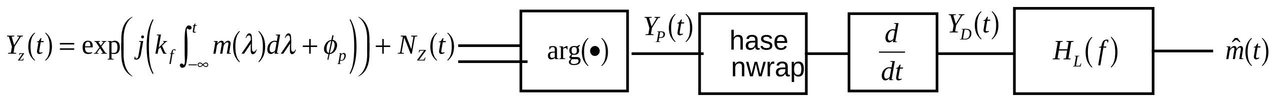


Figure 10.9: The noncoherent demodulation structure for FM.

Again a complete statistical characterization is possible but lacks the intuition that we want to achieve in this course.

We can use the same high SNR approximations as with PM to gain insight into the performance of FM. Recall at high SNR

$$\varphi_e(t) \approx \tan^{-1}\left(\frac{\breve{N}_Q(t)}{A_c}\right) \approx \frac{\breve{N}_Q(t)}{A_c} \tag{10.32}$$

so that the output phase noise is Gaussian, zero mean and has a PSD of $S_{\varphi_e}(f) = S_{N_I}(f)/A_c^2$. The derivative removes the phase offset, ϕ_p so the $Y_D(t) = k_f m(t) + d\varphi_e(t)/dt = k_f m(t) + N_f(t)$. The derivative is a linear system with a transfer function of

$$H_d(f) = j2\pi f \tag{10.33}$$

so at high SNR the output power spectrum of $N_f(t)$ is approximated as

$$\begin{aligned} S_{N_f}(f) = \frac{S_{N_I}(f)}{A_c^2}\left|H_d(f)\right|^2 &= \frac{2N_0\pi^2 f^2}{A_c^2} \qquad |f| \leq \frac{B_f}{2} \\ &= 0 \qquad \text{elsewhere.} \end{aligned} \tag{10.34}$$

Interestingly the derivative operation has colored the noise (PSD is not flat with frequency) as the derivative operation tends to accentuate high frequencies and attenuate low frequencies. The next section will demonstrate how engineers used this colored noise to their advantage in high fidelity FM based communications. The final output after the lowpass filter (assumed ideal with a bandwidth of W) has the form $\hat{m}(t) = k_f m(t) + N_L(t)$ where

$$\begin{aligned} S_{N_L}(f) &= \frac{2N_0\pi^2 f^2}{A_c^2} \qquad |f| \leq W \\ &= 0 \qquad \text{elsewhere.} \end{aligned} \tag{10.35}$$

The resulting signal power is $P_s = k_f^2 P_m$ and the noise power is given as

$$P_N = \int_{-\infty}^{\infty} S_{N_L}(f)df = \frac{N_0\pi^2}{A_c^2}\int_{-W}^{W} f^2 df = \frac{4N_0\pi^2 W^3}{3A_c^2} \tag{10.36}$$

so that the resulting output SNR is

$$\text{SNR}_o = \frac{3A_c^2 k_f^2 P_m}{4N_0\pi^2 W^3} = \frac{3k_f^2}{4\pi^2 W^2}\text{SNR}_b. \tag{10.37}$$

Recalling that $k_f = 2\pi f_d$ so a more succinct form for the output SNR is

$$\text{SNR}_o = \frac{3k_f^2}{4\pi^2 W^2}\text{SNR}_b = \frac{3f_d^2}{W^2}\text{SNR}_b. \tag{10.38}$$

The resulting SNR performance for FM has two interesting insights

Modulation	E_B	TX Complexity	RX Complexity	Performance
DSB-AM	50%	moderate	moderate	Can be same as baseband
LC-AM	50%	moderate	small	Loss compared to baseband due to E_T Experiences a threshold effect
VSB-AM	$> 50\%$	large	large	Can be same as baseband
SSB-AM	100%	large	large	Can be same as baseband
PM	$< 50\%$	small	moderate	Can tradeoff BW for SNR by choice of k_p Experiences a threshold effect
FM	$< 50\%$	small	moderate	Can tradeoff BW for SNR by choice of k_p Experiences a threshold effect Colored noise in demodulation can be exploited

Table 10.1: Summary of the important characteristics of analog modulations.

1. At high SNR the effective performance can be improved an arbitrary amount by making increasing f_d. Recall f_d also increases the output bandwidth of a FM signal. Hence FM also allows an engineer to trade spectral efficiency for SNR performance.

2. The output noise from an FM demodulator is not white (flat with frequency). The sequel will demonstrate how this colored noise characteristic can be exploited.

As SNR_i gets smaller a threshold effect will be experienced in FM demodulation and that is investigated in the sequel.

10.5 Improving Performance with Pre-Emphasis

Not included in this edition

10.6 Threshold Effects in Demodulation

Not included in this addition.

10.7 Final Comparisons

A summary of the important characteristics of analog modulations is given in Table 10.1. At the beginning of Chapter 4 the performance metrics for analog communication systems were given as: complexity, performance, and spectral efficiency.

In complexity the angle modulations offer the best characteristics. Transmitters for all of the analog modulations are reasonable simple to implement. The most complicated transmitters are needed for VSB-AM and SSB-AM. The complexity is need to achieve the bandwidth efficiency. The angle modulations have a transmitted signal with a constant envelope. This makes designing the output power amplifiers much simpler since the linearity of the amplifier is not a significant issue. Receivers for LC-AM and angle modulation can be implemented in simple noncoherent structures. Receivers for DSB-AM and VSB-AM require the use of a more complex coherent structure. SSB-AM additionally will require a transmitted reference for automatic coherent demodulation and operation.

In spectral efficiency SSB-AM offers the best characteristics. SSB-AM is the only system that achieves 100% efficiency. DSB-AM and LC-AM are 50% efficient. VSB-AM offers performance somewhere between SSB-AM and DSB-AM. Angle modulations are the least bandwidth efficient. At best they achieve $E_B = 50\%$, and often much less.

In terms of performance there is a variety of options. Coherent demodulation offers performance at the level that could be achieved with baseband transmission. Noncoherent demodulation schemes can have a threshold effect at low SNR but above threshold provide very good performance. The only difference between coherent demodulation and noncoherent demodulation of LC-AM at high SNR is the transmission inefficiency needed to make envelope detection possible. Phase modulation when operated above threshold offers a tradeoff between output SNR at the demodulator and the transmitted signal bandwidth. Angle modulations offer the ability to trade bandwidth efficiency for output SNR. This ability is not available in AM modulation. Finally the noise spectrum at the output of an FM demodulator is nonwhite. This nonwhite spectral characteristics can be taken advantage of by appropriate signal processing to improve the output SNR even further.

10.8 Homework Problems

Not included this edition.

10.9 Example Solutions

Not included this edition.

10.10 Mini-Projects

Not included this edition.

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Appendix A

Cheat Sheets for Tests

Trigonometric Identities

$$\exp(j\theta) = \cos(\theta) + j\sin(\theta)$$

$$\cos(\theta) = \frac{1}{2}\left[\exp[j\theta] + \exp[-j\theta]\right]$$

$$\sin(\theta) = \frac{1}{2j}\left[\exp[j\theta] - \exp[-j\theta]\right]$$

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$$

$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$$

$$\cos(a)\cos(b) = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]$$

$$\sin(a)\sin(b) = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$$

$$\sin(a)\cos(b) = \frac{1}{2}\left[\sin(a-b) + \sin(a+b)\right]$$

Transcendental Functions

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_{-\infty}^{x} \exp(-t^2)dt$$

$$\mathrm{erf}(x) = -\mathrm{erf}(-x) \qquad \mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$$

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left[-j(n\theta - x\sin(\theta))\right] d\theta$$

Statistics Formulas

Gaussian density function

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}}\exp\left(-\frac{(x-m_x)^2}{2\sigma_x^2}\right) \qquad m_x = E[X] \qquad \sigma_x^2 = E\left[(X-m_x)^2\right]$$

Bivariate Gaussian density function

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{(x_1-m_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-m_1)(x_2-m_2)}{\sigma_1\sigma_2} + \frac{(x_2-m_2)^2}{\sigma_2^2}\right)\right)$$

$$m_i = E\left[X_i\right], \quad \sigma_i^2 = E\left[(X_i-m_i)^2\right], \quad i=1,2 \qquad \rho = E\left[(X_1-m_1)(X_2-m_2)\right]$$

Random Processes Formulas

Stationary Processes

$$R_N(\tau) = E\left[N(t)N(t-\tau)\right] \qquad S_N(f) = \mathcal{F}\left\{R_N(\tau)\right\}$$

Linear Systems and Stationary Processes

$$S_N(f) = \left|H_R(f)\right|^2 S_W(f)$$

Stationary Bandpass Noise

$$S_{N_c}(f) = \frac{1}{2}S_{N_z}(f-f_c) + \frac{1}{2}S_{N_z}(-f-f_c)$$

$$S_{N_I}(f) = \frac{S_{N_z}(f) + S_{N_z}(-f)}{4} \qquad S_{N_I N_Q}(f) = \frac{S_{N_z}(-f) - S_{N_z}(f)}{j4}$$

Fourier Transforms

Time Function	Transform
$x(t) = 1 \quad -T/2 \le t \le T/2$; $\quad = 0 \quad$ elsewhere	$X(f) = T\frac{\sin(\pi f T)}{\pi f T}$
$x(t) = 1 - \frac{\|t\|}{T} \quad -T \le t \le T$; $\quad = 0 \quad$ elsewhere	$X(f) = T\left(\frac{\sin(\pi f T)}{\pi f T}\right)^2$
$x(t) = 2W\frac{\sin(2\pi W t)}{2\pi W t}$	$X(f) = 1 \quad -W \le f \le W$; $\quad = 0 \quad$ elsewhere
$x(t) = \exp(-at) \quad t \ge 0$; $\quad = 0 \quad$ elsewhere	$\frac{1}{j2\pi f + a}$

Table A.1: Fourier Transform Table.

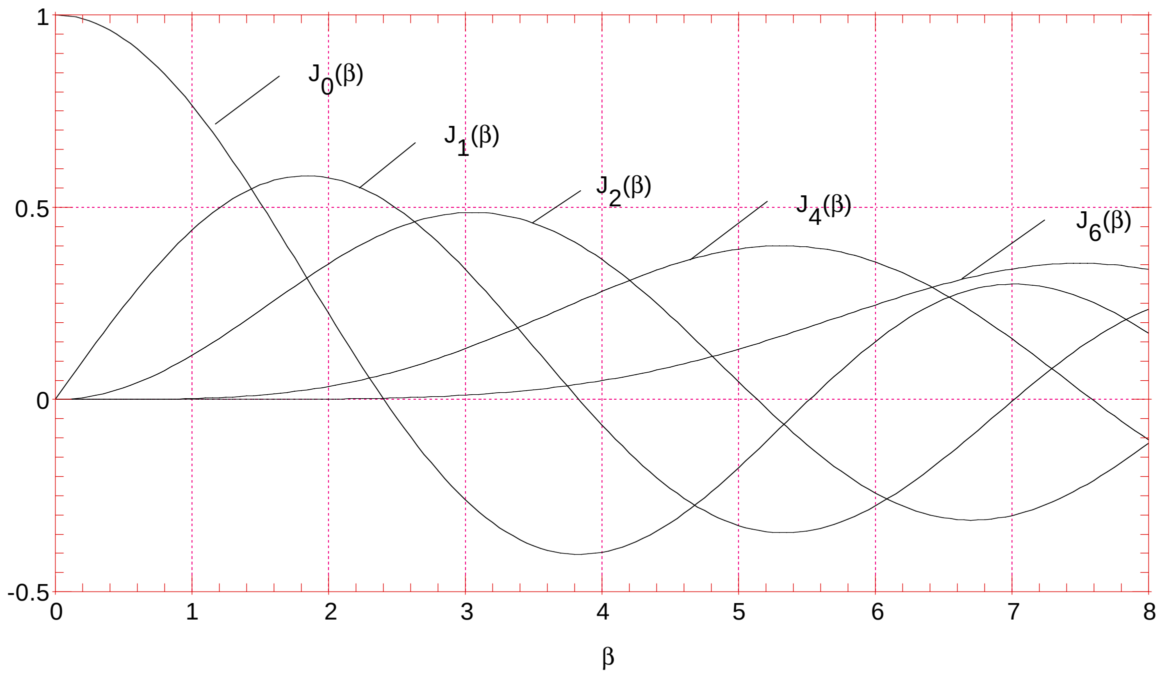

Figure A.1: A plot of the low order Bessel functions of the first kind, $J_n(\beta)$ versus β.

Appendix B

Fourier Transforms: f versus ω

This course and communications engineers in general use the variable f as opposed to $\omega = 2\pi f$ when examining the frequency domain representations of signals and systems. The reason is that units of Hertz have more of an intuitive feel than the units of radians per second. Often undergraduate courses in signals and systems develop the theory with the notation ω. This appendix is provided to show the differences in the results that arise from these two views of frequency domain analysis. First we shall define the Fourier transform for the two variables

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt = \mathcal{F}\{x(t)\} \qquad X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt. \tag{B.1}$$

The inverse Fourier transform is given as

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}dt = \mathcal{F}^{-1}\{X(f)\} \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}dt. \tag{B.2}$$

The other convenient thing about working with the variable f is that the term $(2\pi)^{-1}$ does not need to be included in the inverse transform. A list of Fourier transform properties of different mathematical operations are listed in Table B.1. Additionally Fourier transforms of some commonly used energy signals are listed in Table B.2.

Operation	Time Function, $x(t)$	Transform, $X(f)$	Transform, $X(\omega)$
Reversal	$x(-t)$	$X(-f)$	$X(-\omega)$
Symmetry	$X(t)$	$x(-f)$	$2\pi x(-\omega)$
Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{f}{a}\right)$	$\frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$
Time Delay	$x(t-t_0)$	$X(f)e^{-j2\pi ft_0}$	$X(\omega)e^{-j\omega t_0}$
Time Differentiation	$\frac{d^n}{dt^n}x(t)$	$(j2\pi f)^n X(f)$	$(j\omega)^n X(\omega)$
Energy	$E_x = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt$	$E_x = \int_{-\infty}^{\infty} \lvert X(f)\rvert^2 df$	$E_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 dt$
Frequency Translation	$x(t)e^{-j2\pi f_c t} = x(t)e^{-j\omega_c t}$	$X(f-f_c)$	$X(\omega-\omega_c)$
Convolution	$x(t)\otimes h(t)$	$X(f)H(f)$	$X(\omega)H(\omega)$
Multiplication	$x(t)y(t)$	$X(f)\otimes Y(f)$	$\frac{1}{2\pi}X(\omega)\otimes Y(\omega)$

Table B.1: Fourier transforms for some common mathematical operations.

Time Function, $x(t)$	Transform, $X(f)$	Transform, $X(\omega)$
$\begin{aligned} x(t) &= 1 \qquad -T/2 \leq t \leq T/2 \\ &= 0 \qquad \text{elsewhere} \end{aligned}$	$T\frac{\sin(\pi f T)}{\pi f T}$	$T\frac{\sin(\omega T/2)}{\omega T/2}$
$\begin{aligned} x(t) &= \exp(-at) \qquad t \geq 0 \\ &= 0 \qquad \text{elsewhere} \end{aligned}$	$\frac{1}{j2\pi f+a}$	$\frac{1}{j\omega+a}$

Table B.2: Fourier transforms for some common energy signals.