

UNIT - 3

Power Spectrum Estimation:-

- This chapter we concerned estimation of spectral characteristics of signals characterized as random processes.
- Ex:- Meteorological phenomena such as fluctuations in air temperature and pressure are best characterized statistically as random processes.
- ✓ Thermal noise voltages generated in resistors and electronic devices are additional examples of physical signals that are well modeled as random processes.
- Due to random fluctuations in such signals, we must adopt a statistical point of view, which deals with the average characteristics of random signals.
- In particular, the auto correlation function of a random process is the appropriate statistical average that we will use for characterizing random signals in time domain.
- The Fourier transform of auto correlation function which yields the power density spectrum provides transformation from time domain to frequency domain.

Estimation of Spectra from Finite Duration Observations of Signals:-

In this we estimate basic problem of power density spectrum of a signal from observation of signals over a finite time interval.

- When data recorded longer, the better it is to be estimate.
But on other hand we cannot select an arbitrarily long data recorded to estimate the spectrum.

In such cases the length of data recorded depends on time variations in signal statistics.

- The goal is to select the data recorded as short as possible that allows us to resolve the spectral characteristics of different signal components in the data record.

a) Computation of Energy Density Spectrum

Let us consider computation of spectrum of deterministic signal for a finite sequence of data.

The first sequence $x_a(t)$ is usually result of sampling a continuous time signal $x_a(t)$ at some uniform sampling rate T_s .
The objective is to estimate the spectrum from a finite duration sequence $x_a(t)$.
If $x_a(t)$ is a finite energy signal, i.e., finite value of E

$$E = \int_{-\infty}^{\infty} |x_a(t)|^2 dt < \infty \rightarrow 1$$

then Fourier transform exists is

$$X_a(f) = \int_{-\infty}^{\infty} x_a(t) e^{-j2\pi f t} dt$$

From Routh's theorem \rightarrow

$$E = \int_{-\infty}^{\infty} |x_a(t)|^2 dt = \int_{-\infty}^{\infty} |X_a(f)|^2 df \rightarrow 2$$

where the quantity $|X_a(f)|^2$ represents distribution of a signal energy as a function of frequency and hence called as Energy density spectrum.

of signal given as

$$S_{xx}(f) = |X_a(f)|^2 \rightarrow 3$$

The total energy in the signal is integral of $S_{xx}(f)$ over all 'F'

$$\therefore E = \int_{-\infty}^{\infty} S_{xx}(f) df \rightarrow 4$$

Also interesting to note $S_{xx}(f)$ can be viewed as Fourier transform of another function $R_{xx}(r)$.

$R_{xx}(r)$ - called as auto correlation function of finite energy

signal $x_a(t)$.

$$R_{xx}(r) = \int_{-\infty}^{\infty} x_a^*(t) \cdot x_a(t+r) dt \rightarrow 5$$

where

$$\int_{-\infty}^{\infty} R_{xx}(v) \cdot e^{-j2\pi ft} dt = S_{xx}(f) \approx |x_a(f)|^2 \rightarrow 6$$

$\therefore R_{xx}(v)$ and $S_{xx}(F)$ are Fourier Transform Pairs

Suppose the energy density spectrum of signal $x_a(t)$ from its samples taken at rate f_s samples per second.

To ensure no spectral aliasing resulting from sampling process the signal is assumed to be prefiltered. So in practical its bandwidth is limited to 'B' in hertz's.

\Rightarrow The sampled version of $x_a(t)$ is sequence $x(n)$, $-\infty < n < \infty$ which has Fourier Transform

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n} \quad (6)$$

$$X(f) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j2\pi f n} \rightarrow 7$$

$X(f)$ can be expressed in terms of voltage spectrum of analog signal $x_a(t)$ as

$$X\left(\frac{f}{f_s}\right) = f_s \sum_{k=-\infty}^{\infty} x_a(F - k f_s) \rightarrow 8$$

where $f = \frac{F}{f_s}$ is normalized frequency variable.

In absence of aliasing within the fundamental range $|F| \leq \frac{f_s}{2}$ we have

$$X\left(\frac{f}{f_s}\right) = f_s x_a(F), |F| \leq \frac{f_s}{2} \rightarrow 9$$

Hence voltage spectrum of sampled signal is identical to voltage spectrum of analog signal.

\therefore Energy density of sampled signal is

$$S_{xx}\left(\frac{f}{f_s}\right) = \left| X\left(\frac{f}{f_s}\right) \right|^2 = f_s^2 |x_a(F)|^2 \rightarrow 10$$

∴ Auto Correlation of sampled signal is

$$r_{xx}(k) = \sum_{n=-\infty}^{\infty} x^*(n) \cdot x(n+k) \xrightarrow{(1)} \text{Wiener-Kintchine theorem}$$

has Fourier transform (Wiener-Kintchine theorem)

$$S_{xx}(f) = \sum_{k=-\infty}^{\infty} r_{xx}(k) \cdot e^{-j2\pi kf} \xrightarrow{(2)}$$

Hence, the energy density spectrum can be obtained by Fourier transforms of auto correlations of sequence $\{x(n)\}$.

There are 2-methods to conclude the energy density spectrum of signals $X(f)$ from samples of $x(n)$.

Ist method:- Direct method:- which involves computing the Fourier transforms of $\{x(n)\}$ and then

$$S_{xx}(f) = |X(f)|^2 = \left| \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j2\pi fn} \right|^2 \xrightarrow{(3)}$$

IInd method:- Indirect method:- required 2-steps.

Step (i) :- $r_{xx}(k)$ is computed from $x(n)$

Step (ii) :- Fourier transforms of auto correlation function, is computed

as per eq (2) to obtain Energy density spectrum.

5).

Estimation of the Auto Correlation and Power Spectrum of Random Signals :- The Periodogram:-

The finite energy signal considered in the preceding section possess a Fourier transform, and are characterized in the spectral domain by their energy density spectrum.

On the other hand the important class of signals characterized as stationary random processes do not have finite average power and hence are characterized by a power density spectrum.

If $x(t)$ is stationary random process its auto correlation function is

$$R_{xx}(\tau) = E[x^*(t) \cdot x(t+\tau)] \rightarrow (1)$$

where $E[\cdot]$ denotes statistical average.
By Khintchine theorem the power density spectrum of stationary random process is the Fourier transform of Auto correlation function

$$P_{xx}(F) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j2\pi F\tau} d\tau \rightarrow (2)$$

On the other hand from a single realization of random process we can compute the time average auto correlation function.

$$\bar{R}_{xx}(\tau) = \frac{1}{T_0} \int_{-T_0}^{T_0} x^*(t) \cdot x(t+\tau) dt \rightarrow (3)$$

where T_0 is the observation interval.
If stationary random process is ergodic in first and second moments (mean and Auto correlation)

$$\begin{aligned} \bar{R}_{xx}(\tau) &= \lim_{T_0 \rightarrow \infty} R_{xx}(\tau) \\ &= \lim_{T_0 \rightarrow \infty} \frac{1}{T_0} \int_{-T_0}^{T_0} x^*(t) \cdot x(t+\tau) dt \rightarrow (4) \end{aligned}$$

The Fourier transform of $\bar{R}_{xx}(\tau)$ provides an estimate $P_{xx}(F)$ of Power density spectrum is

$$P_{xx}(F) = \int_{-T_0}^{T_0} \bar{R}_{xx}(\tau) e^{-j2\pi F\tau} d\tau \rightarrow (5) \quad (\text{indirect method})$$

$$\bar{R}_{xx}(\tau) = \frac{1}{2T_0} \int_{-T_0}^{T_0} \left[\int_{-T_0}^{T_0} x^*(t) \cdot x(t+\tau) dt \right] e^{-j2\pi F\tau} d\tau \rightarrow (6)$$

$$P_{xx}(F) = \frac{1}{2\pi} \left| \int_{-B}^B x(t) e^{-j2\pi f t} dt \right|^2 \rightarrow (7) \quad (\text{direct method})$$

The actual power density spectrum is expected value of

$P_{xx}(F)$ in limit as $T_0 \rightarrow \infty$

$$\Gamma_{xx}^1(F) = \lim_{T_0 \rightarrow \infty} E[P_{xx}(F)]$$

$$= \lim_{T_0 \rightarrow \infty} E \left[\frac{1}{2\pi} \left| \int_{-B}^B x(t) e^{-j2\pi f t} dt \right|^2 \right] \rightarrow (8)$$

∴ from Eq (3) & (4) the 2-Possible approaches to Computing $P_{xx}(F)$ the direct method is as Eq (7) and indirect method as we obtain $P_{xx}(F)$ first and Compute Fourier Transform.

Let us Consider the estimation of Power density spectrum from samples of single realization of random process $x(t)$ is sampled rate $F_a > 2B$ where 'B' is highest frequency contained in Power density spectrum of random process.

The averaged auto correlation sequence is

$$\gamma_{xx}^1(m) = \frac{1}{N-m} \sum_{n=0}^{N-m-1} x^*(n) \cdot x(n+m) ; m = 0, 1, \dots, N-1 \rightarrow (9)$$

for negative value of 'm' we have

$$\gamma_{xx}^1(m) = [\gamma_{xx}^1(-m)]$$

$$\gamma_{xx}^1(m) = \frac{1}{N-|m|} \sum_{n=|m|}^{N-1} x^*(n) \cdot x(n+m) ; m = -1, -2, \dots, 1-N \rightarrow (10)$$

on Compute of Fourier transform.

$$P_{xx}^1(F) = \sum_{m=-N+1}^{N-1} \gamma_{xx}^1(m) e^{-j2\pi fm} \rightarrow (11)$$

∴ The normalization factor $N-|m|$ in Eq (10) eliminated with mean value

$$E[\gamma_{xx}^1(m)] = \frac{1}{N-|m|} \sum_{n=0}^{N-m-1} E[x^*(n) \cdot x(n+m)] \rightarrow (12)$$

The power density spectrum is given by

$$P_{xx}(f) = \sum_{m=-N+1}^{N-1} \gamma_{xx}^1(m) e^{-j2\pi fm}$$

$$= \frac{1}{N} \left| \sum_{m=0}^{N-1} \gamma_{xx}^1(m) e^{-j2\pi fm} \right|^2 = \frac{1}{N} |X(f)|^2 \rightarrow (13)$$

where $X(f)$ is the Fourier transform of $x(n)$ sequence.

This form is Power density spectrum estimate is called Periodogram.

The average value of Periodogram estimate $P_{xx}(f)$ is given by

$$E[P_{xx}(f)] = E \left[\sum_{m=-(N-1)}^{N-1} x_{xx}(m) e^{-j2\pi fm} \right] \rightarrow (14)$$

$$\Rightarrow \sum_{m=-(N-1)}^{N-1} \left(1 - \frac{|m|}{N} \right) \cdot x_{xx}(m) e^{-j2\pi fm} \rightarrow (15)$$

$$E[P_{xx}(f)] = \int_{-1/2}^{1/2} P_{xx}(\alpha) N_B(f-\alpha) d\alpha \rightarrow (16)$$

The variance of estimate $P_{xx}(f)$ is

$$\text{var}[P_{xx}(f)] = P_{xx}^2(f) \left[1 + \left(\frac{\sin 2\pi f N}{N \sin \pi f} \right)^2 \right] \rightarrow (17)$$

which is limit as $N \rightarrow \infty$ becomes

$$\lim_{N \rightarrow \infty} \text{var}[P_{xx}(f)] = P_{xx}^2(f) \rightarrow (18)$$

use of DFT in power spectrum estimation:-

The estimated energy density spectrum $S_{xx}(f)$ and periodogram $P_{xx}(f)$ respectively can be computed by use of DFT which in turn is efficiently computed by an FFT algorithm. in case of N -data points

for eg: the computation yields samples of a Periodogram

$$P_{xx}\left(\frac{k}{N}\right) = \frac{1}{N} \left| \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \right|^2 ; k=0, 1, \dots, N-1 \rightarrow (1)$$

at the frequencies $f_k = \frac{k}{N}$

zero padding & evaluating the DFT at $L > N$ points does not improve the frequency resolution in the spectral estimation.

2) Non-Parametric Methods for Power spectrum Estimation :-

The Power estimation methods that make no assumption about how data were generated and hence are called non parametric.

Non-Parametric methods are:-

- ① Bartlett
- ② Blackman and Tukey
- ③ Welch.

we describe to
Estimate mean and variance

① Bartlett method: Averaging Periodograms:-

" " for reducing the variance in Periodogram involve 3-steps

② N-point sequence is subdivided into K - non overlapping segments where each segment has length M . This results in K -data segments

$$x_i(n) = x(n+iM) \quad i=0, 1, \dots, K-1 \quad n=0, 1, \dots, M-1 \rightarrow ①$$

③ for each segment we compute Periodogram

$$P_{xx}^{(i)}(f) = \frac{1}{M} \left| \sum_{n=0}^{M-1} x_i(n) e^{-j2\pi fn} \right|^2; i=0, 1, \dots, K-1 \rightarrow ②$$

④ finally we average the Periodogram for K - segments to obtain bartlett power spectrum estimate

$$P_{xx}^{(B)}(f) = \frac{1}{K} \sum_{i=0}^{K-1} P_{xx}^{(i)}(f) \rightarrow ③$$

The statistical properties of this estimate are easily obtained. First the mean value is

$$\mathbb{E}\left[P_{xx}^{(B)}(f)\right] = \frac{1}{K} \sum_{i=0}^{K-1} \mathbb{E}\left[P_{xx}^{(i)}(f)\right] \rightarrow ④$$

$$= \mathbb{E}\left[P_{xx}^{(0)}(f)\right]$$

$$\therefore \mathbb{E}\left[P_{xx}^{(0)}(f)\right] = \sum_{m=-M+1}^{M-1} \left(1 - \frac{|m|}{M}\right) \gamma_{xx}(m) e^{-j2\pi fm} \rightarrow ⑤$$

$$E\left[P_{xx}^{(i)}(f)\right] = \frac{1}{M} \int_{-\frac{1}{2}}^{\frac{1}{2}} P_{xx}(a) \left(\frac{\sin \pi(f-a)M}{\sin \pi(f-a)} \right)^2 da \rightarrow ⑥$$

where

$$W_B(f) = \frac{1}{M} \left(\frac{\sin \pi f M}{\sin \pi f} \right)^2 \rightarrow ⑦$$

is the frequency characteristics of Bartlett window

$$w_B(n) = \begin{cases} 1 - \frac{|n|}{M} & |n| \leq M-1 \\ 0 & \text{otherwise} \end{cases} \rightarrow ⑧$$

From eq ⑦ it is observed that true spectrum is convolved with frequency characteristics $w_B(f)$ of Bartlett window.

The effect of reducing the length of data from N -points to $M \ll N$ results in window whose spectral width has been increased by a factor 'K'.

Consequently the frequency resolution has been reduced by a factor 'K'.

Further the reduction in resolution has reduced the variance

\therefore Variance is given as

$$\text{Var}\left\{P_{xx}^{(B)}(f)\right\} = \frac{1}{K} \text{Var}\left\{P_{xx}^{(i)}(f)\right\} \rightarrow ⑨$$

② The Welch Method :- Average Modified Periodogram:-

Welch made 2 modifications to bartlett method.

First he allowed the data segments to overlap thus the data segments can be represented as.

$$x_i(n) = x(n+iD) \quad ; n=0, 1, \dots, M-1 \quad i=0, 1, \dots, L-1 \rightarrow ⑩$$

where iD is starting point for the i^{th} segment.

If $D=M$, the segments do not overlap and number 'L' of data segments is identical to number 'K' in bartlett method.

If $D=\frac{M}{2}$ there is 50% overlap \Rightarrow ~~more~~ fewer successive data segments and $L=2K$ segments obtained.

Alternatively we can form 'K' data segments each of length $2M$.

The second modification is to window the data segments prior to computing the periodogram.

$$\tilde{P}_{xx}^{(i)}(f) = \frac{1}{M_U} \left| \sum_{n=0}^{M-1} x_i(n) \cdot w(n) \cdot e^{-j2\pi fn} \right|^2 ; i=0, 1, \dots, L-1 \rightarrow (2)$$

where U - is normalization factor for power in window function.

$$U = \frac{1}{M} \sum_{n=0}^{M-1} w(n) \rightarrow (3)$$

The Welch Power spectrum estimates is average of modified periodograms

$$P_{xx}^{(W)}(f) = \frac{1}{L} \sum_{i=0}^{L-1} \tilde{P}_{xx}^{(i)}(f) \rightarrow (4)$$

The mean value of modified periodogram is

$$E\left[\tilde{P}_{xx}^{(i)}(f)\right] = \frac{1}{M_U} \sum_{n=0}^{M-1} \sum_{m=0}^{M-1} w(n) \cdot w(m) E\left[x_i(n) \cdot x_i^*(m)\right] e^{-j2\pi f(n-m)} \rightarrow (5)$$

$$= \frac{1}{M_U} \sum_{n=0}^{M-1} \sum_{m=0}^{M-1} w(n) \cdot w(m) \cdot S_{xx}(n-m) e^{-j2\pi f(n-m)} \rightarrow (6)$$

since

$$S_{xx}(f) = \int_{-f/2}^{f/2} P_{xx}(\alpha) \cdot e^{j2\pi f\alpha} d\alpha \rightarrow (7)$$

substituting eq (7) on eq (6) gives,

$$\begin{aligned} E\left[\tilde{P}_{xx}^{(i)}(f)\right] &= \frac{1}{M_U} \int_{-f/2}^{f/2} P_{xx}(\alpha) \left[\sum_{n=0}^{M-1} \sum_{m=0}^{M-1} w(n) \cdot w(m) e^{-j2\pi f(n-m)} e^{j2\pi f(\alpha-\alpha)} \right] d\alpha \\ &= \int_{-f/2}^{f/2} P_{xx}(\alpha) W(f-\alpha) d\alpha \rightarrow (8) \end{aligned}$$

From definition

$$\text{where } W(f-\alpha) = \frac{1}{M_U} \left| \sum_{n=0}^{M-1} w(n) \cdot e^{-j2\pi fn} \right|^2 \rightarrow (9)$$

The normalization factor 'U' ensures that

$$\int_{-f/2}^{f/2} W(f) df = 1 \rightarrow (10)$$

There is no overlap b/w successive data segments i.e. $L=K$ "
variance of

$$\text{Var} \left[\hat{P}_{xx}^{(w)}(f) \right] = \frac{1}{L} \sum_{l=1}^L \text{Var} \left[\hat{P}_{xx}^{(l)}(f) \right] \rightarrow ①$$

$$= \frac{1}{L} \text{Var} \left[\hat{P}_{xx}(f) \right] \rightarrow ②$$

If there is 50% overlap b/w successive data segments, $L=2K$ then

Variance of Welch Power Spectrum Estimate

$$\text{Var} \left[\hat{P}_{xx}^{(w)}(f) \right] = \frac{9}{8L} \sum_{l=1}^L \text{Var} \left[\hat{P}_{xx}^{(l)}(f) \right] \rightarrow ③$$

- ③ The Blackman & Tukey Method Smoothing the Periodogram:-
 " " Proposed and analysed the method in which the sample auto correlation sequence is windowed first and then Fourier transformed to yield the estimate of power spectrum.

∴ Thus Blackman & Tukey Estimate is given by

$$\hat{P}_{xx}^{BT}(f) = \sum_{m=-(M-1)}^{M-1} x_{xx}(m) \cdot w(m) \cdot e^{-j2\pi f m} \rightarrow ①$$

where window function $w(m)$ has length $2M-1$ and frequency domain

$$\text{Estimation in } \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} P_{xx}(\alpha) \cdot W(f-\alpha) \cdot d\alpha \rightarrow ②$$

where $P_{xx}(f)$ is the Periodogram

$$E \left[\hat{P}_{xx}^{BT}(f) \right] = \int_{-\frac{1}{2}}^{\frac{1}{2}} E \left[P_{xx}(\alpha) \right] W(f-\alpha) \cdot d\alpha \rightarrow ③$$

$$\therefore \text{WKT} \quad E \left[P_{xx}(\alpha) \right] = \int_{-\frac{1}{2}}^{\frac{1}{2}} P_{xx}(\theta) \cdot W_B(\alpha-\theta) \cdot d\theta \rightarrow ④$$

$$E \left[\hat{P}_{xx}^{BT}(f) \right] = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} P_{xx}(\theta) \cdot W_B(\alpha-\theta) \cdot W(f-\alpha) \cdot d\alpha \cdot d\theta \rightarrow ⑤$$

In time domain

$$E\left[P_{xx}^{BF}(f)\right] = \sum_{m=-(M-1)}^{M-1} x_{xx}(m) \cdot W_B(m) \cdot W(m) \cdot e^{-j2\pi fm} \rightarrow ⑥$$

$$\text{Var}\left[P_{xx}^{BF}(f)\right] \approx P_{xx}^2(f) \left\{ \frac{1}{N} \int_{-\frac{1}{2}}^{\frac{1}{2}} w^2(\omega) \cdot d\omega \right\}$$

$$\approx P_{xx}^2(f) \left[\frac{1}{N} \sum_{m=-(M-1)}^{M-1} w^2(m) \right] \rightarrow ⑦$$

3) Performance characteristics of non-parametric Power Spectrum Estimators :-

The Quality of Power Spectrum Estimate is given by

$$Q_A = \frac{\{E\left[P_{xx}^A(f)\right]\}^2}{\text{Var}\left[P_{xx}^A(f)\right]} \rightarrow ①$$

for Bartlett Power spectrum estimate

$$Q_B = K = \frac{N}{M} \rightarrow ②.$$

The frequency resolution of bartlett estimate is measured by taking the 3DB width of the main lobe of the rectangular window.

$$\Delta f = \frac{0.9}{M} \rightarrow ③$$

$$Q_B = 1.1 N \Delta f \rightarrow ④$$

for Welch Power Spectrum Estimate

$$Q_N = \begin{cases} L = \frac{N}{M} & \text{for no overlap} \\ \frac{8L}{9} = \frac{16N}{9M} & \text{for 50% overlap} \end{cases} \rightarrow ⑤$$

for BF Power Spectrum Estimate

$$Q_{BF} = \frac{1.6 \frac{N}{M}}{\Delta f} \quad \Delta f = \frac{1.98}{2M}$$

(13)

The frequency resolution of Welch estimate is measured by taking the SDR width of the main lobe of the window

$$Q_W = \left\{ \begin{array}{l} L = N \cdot \Delta f \left(\frac{1}{1.28} \right) = 0.78 \text{ NAF} \\ \frac{8L}{9} = N \cdot \Delta f \left(\frac{16}{1.28} \right) = 1.39 \text{ NAF} \end{array} \right. \begin{array}{l} \text{non-overlap} \\ \text{overlap} \end{array}$$

for ~~Bartlett~~ Power spectrum estimate
Blackman & Tukey

$$Q_{BT} = 1.5 \frac{N}{M}$$

$$\Delta f = \frac{1.28}{2M} \Rightarrow M = \frac{1.28}{2\Delta f}$$

$$\therefore Q_{BT} = 1.5 \times \left(\frac{N}{\frac{1.28}{2\Delta f}} \right) \times \Delta f = 2.34 \text{ NAF} /$$

Comparing these 3-Bartlett method is better of three computational requirements of non-parametric Power spectrum estimates

Parameter	Bartlett	welch	B.T
FFT length	$M = 0.91 \Delta f$	$M = \frac{1.28}{\Delta f}$	$M = \frac{1.28}{\Delta f}$
No. of FFT's	$\frac{N}{M}$	$\frac{N}{M}$	$\frac{N}{M}$
No. of Computations	$\frac{N}{2} \log_2 \left(\frac{0.9}{\Delta f} \right)$	$N \log_2 \left(\frac{1.12}{\Delta f} \right)$	$N \log_2 \left(\frac{1.28}{\Delta f} \right)$

→ Computational estimates are based on fixed amount of data 'N' and a specified resolution.

(i) Bartlett Power spectrum Estimate :-

$$\text{FFT length} = M = \frac{0.9}{\Delta f}$$

$$\text{No. of FFT's} = \frac{N}{M} = 1.11 \text{ NAF}$$

$$\text{No. of Computations} = \frac{N}{M} \left(\frac{M}{2} \log_2 M \right) = \frac{N}{2} \log_2 \left(\frac{0.9}{\Delta f} \right).$$

(14)

(ii) Welch Power Spectrum Estimate (50% overlap)

$$\text{FFT length } M = \frac{1.28}{\Delta f}$$

$$\therefore \text{# of FFTs} = \frac{2N}{M} = 1.56 N \Delta f$$

$$\begin{aligned} \therefore \text{# of Computations} &= \frac{2N}{M} \left(\frac{M}{2} \log_2 M \right) = N \log_2 \left(\frac{1.28}{\Delta f} \right) \\ \text{Total} \quad " \quad &= 2N + N \log_2 \left(\frac{1.28}{\Delta f} \right) = N \log_2 \left(\frac{5.12}{\Delta f} \right) \end{aligned}$$

(iii) Blackman-Tukey Power Spectrum Estimate:-

$$\text{FFT length } M = \frac{1.28}{\Delta f}$$

$$\therefore \text{# of FFTs} = 2K+1 = 2 \left(\frac{N}{2M} \right) * 1 \approx \frac{N}{M}$$

$$\begin{aligned} \text{Total Computations} &= \frac{N}{M} (M \log_2 M) = N \log_2 \left(\frac{1.28}{\Delta f} \right) \end{aligned}$$

Quality Power Spectrum Estimates:-

Estimated

Bartlett

Welch (50% overlap)

Blackman-Tukey

Quality factor

1.11 N Δf

1.39 N Δf

2.34 N Δf

Q:- Obtain the frequency resolution of Bartlett, Welch, & Blackman-Tukey techniques of Power Spectrum Estimates for a quality factor $Q = 15$. Assume that the overlap in Welch method is 50% and length of sample sequence is 1050.

(15)

Sol: Given data

$$\text{Quality factor } Q = 15$$

$$\text{Sample sequence length } L = N = 1050$$

overlapping in Welch method = 50%.

$$\text{Bartlett method } Q_{\text{Bart}} = + \cancel{1.11} \text{ N AF}$$

$$\Rightarrow \Delta f = \frac{Q_{\text{Bart}}}{1.11 \times N} = \frac{15}{1.11 \times 1050} = 0.0128.$$

$$\text{Welch method: } Q_W = 1.39 \text{ N AF}$$

$$\Delta f = \frac{Q_W}{1.39 \times N} = \frac{15}{1.39 \times 1050} = 0.01027$$

$$\text{Blackman-Tukey method: } Q_{\text{BT}} = 2.34 \cdot \text{N AF}$$

$$\Delta f = \frac{Q_{\text{BT}}}{2.34 \times N} = \frac{15}{2.34 \times 1050} = 0.0061$$

Eg:- Determine the frequency resolution of Bartlett, Welch and Blackman Tukey method of Power Spectrum Estimates for a quality factor $Q = 15$ assume that the length of sample sequence is 1500.

Sol: Given data $Q = 15 ; L = N = 1500$

$$\text{Bartlett method } Q_{\text{Bart}} = 1.11 \text{ N AF}$$

$$\Rightarrow \text{frequency resolution } \Delta f = \frac{Q_{\text{Bart}}}{1.11 \times N} = \frac{15}{1.11 \times 1500} = \cancel{9.009 \times 10^{-3}}$$

$$\text{Welch method } Q_W = 1.39 \text{ N AF}$$

$$\Rightarrow \text{frequency resolution } \Delta f = \frac{Q_W}{1.39 \times N} = \frac{15}{1.39 \times 1500} = \cancel{7.19 \times 10^{-3}}$$

signed x
Date _____

Blackman-Turkey method $Q_{BT} = 2.34 N \Delta f$

freq resolution $\Delta f = \frac{Q_{BT}}{2.34 N} = 0.00462 \text{ (6)} \approx 4.27 \times 10^{-3}$

Q: Obtain the frequency resolution of the Bartlett, Welch and Blackman-Turkey techniques of Power Spectrum Estimates for a quality factor $Q=10$. Assume that the overlap in Welch method is 50% and the length of sample sequence is 1000.

Sol Given data $Q = 10$; $L = N = 1000$

overlap in Welch method = 50%

Bartlett method $Q_{Bart} = 1.11 N \Delta f$

$$\Rightarrow \text{freq resolution } \Delta f = \frac{Q_{Bart}}{1.11 N} = \frac{10}{1.11 \times 1000} = 9.009 \times 10^{-3}$$

Welch method $Q_w = 1.39 N \Delta f$

$$\text{frequency resolution } \Delta f = \frac{Q_w}{1.39 \times N} = \frac{10}{1.39 \times 1000} = 7.19 \times 10^{-3}$$

Blackmann-Turkey method $Q_{BT} = 2.34 N \Delta f$

$$\text{frequency resolution } \Delta f = \frac{Q_{BT}}{2.34 \times N} = \frac{10}{2.34 \times 1000} = 4.27 \times 10^{-3}$$

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A signal with (voltage) spectrum.

$$X(f) = \begin{cases} 1 & |f| \leq 0.1 \\ 0 & \text{otherwise} \end{cases}$$

is convolved with rectangular window of length, $N=61$. Determine the spectrum of $\tilde{X}(f)$ given by

$$\tilde{X}(f) = X(f) * W(f)$$

$$= \int_{-1/2}^{1/2} X(\alpha) \cdot W(f-\alpha) \cdot d\alpha$$

Solution

spectral characteristics $W(f)$ for length of $N=61$ rectangular window

→ The width of main lobe of the window function is $\Delta w = \frac{4f_1}{61}$ (cos)

$$\Delta f = \frac{2}{61} \text{ which is narrow compared to } X(f).$$

∴ The convolution of $X(f)$ with $W(f)$ is given in figure

∴ The energy has leaked in to frequency band $0.1 < |f| \leq 0.5$ where

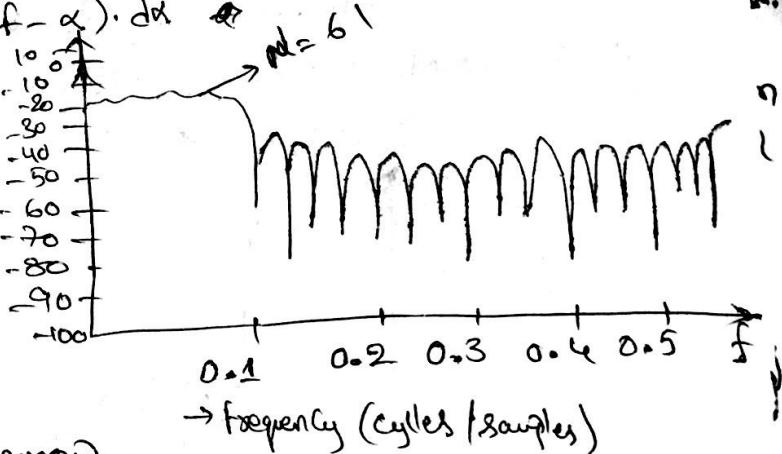
$X(f) = 0$ The part of this is due to width of main lobe in $W(f)$, which causes

broadening (or) smearing of $X(f)$ outside the range $|f| \leq 0.1$ while side lobe energy in $\tilde{X}(f)$ is due to presence of side lobes of $W(f)$, which convolved with $X(f)$.

→ The smearing of $X(f)$ for $|f| > 0.1$ and the sidelobes in range $0.1 \leq |f| \leq 0.5$ constitute the leakage.

→ The side lobe leakage has certainly been reduced but spectral width

has been increased by about 50% ∵ on broadening of spectrum being estimated due to windowing to resolve signals with closely spaced frequency components



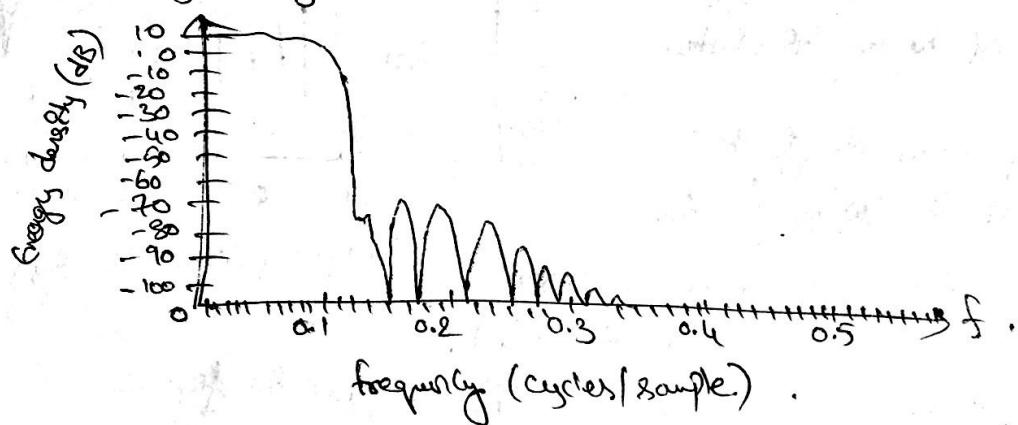
The signal with spectral characteristics given as

$$X(f) = X_1(f) + X_2(f)$$

can not be resolved as two signals unless width of window function is significantly narrower than frequency separation.

Narrow band signal spectra.

\therefore The smooth time domain windows reduced leakage at the expense of a decrease in frequency resolution.



Q:- A sequence of $N=16$ samples is obtained by sampling an analog signal consisting of two frequency components. The resulting discrete time sequence is.

$$x(n) = \sin 2\pi(0.132)n + \cos 2\pi(0.132 + Af)n \quad ; n = 0, 1, \dots, 15$$

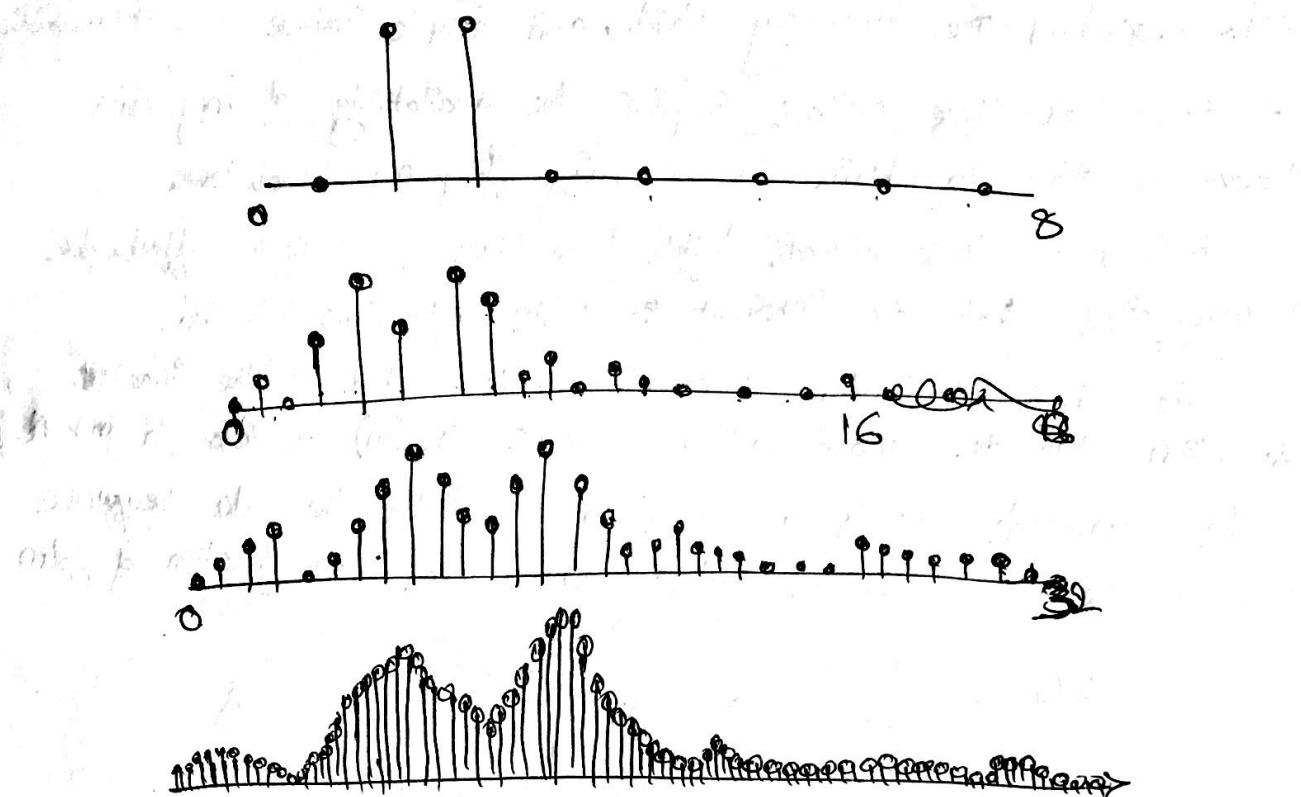
where Af is frequency separation. Evaluate the power spectrum

$$P(f) = \frac{1}{N} |X(f)|^2 \text{ at the frequencies } f_k = \frac{k}{L} ; k = 0, 1, \dots, L-1 \text{ for } L = 8, 16, 32, \text{ and } 128 \text{ for values } Af = 0.06 \text{ and } Af = 0.01.$$

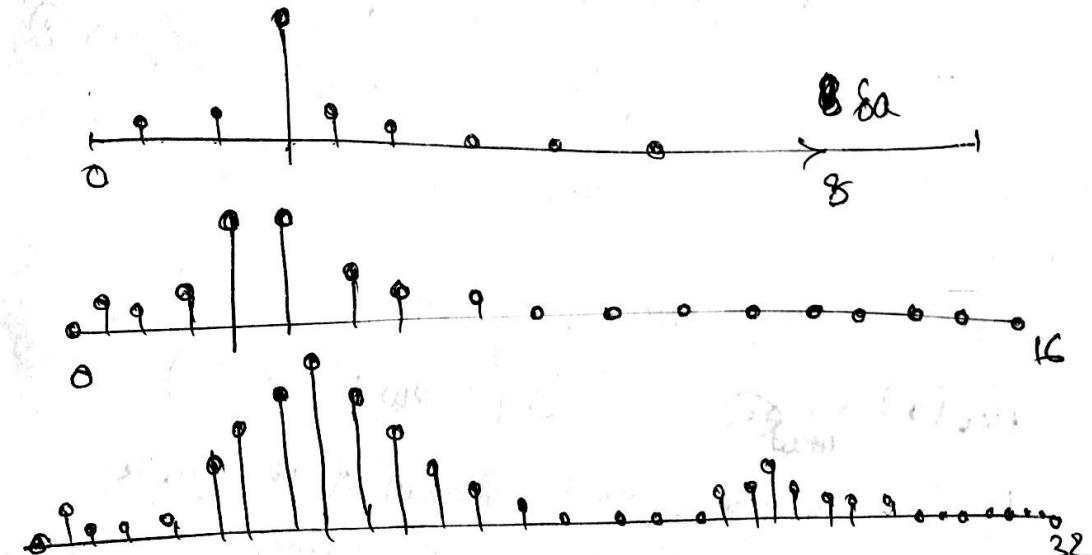
Sol: By zero padding we increase data sequence to obtain Power spectrum estimate $P_{xx}\left(\frac{k}{L}\right)$

where zero padding does not change the resolution, but have effect on interpolating the spectrum $P_{xx}(f)$.

(19) \therefore In case of frequency separation Af is sufficiently large so that the two frequency components are resolvable.



Spectra of two sinusoids with frequency separation $Af = 0.06 \dots$



Spectra of two sinusoidal with frequency separation of $Af = 0.01$.

Parametric Methods for Power Spectrum Estimation:-

From previous - non-parametric power spectrum estimation methods described are relatively simple, and easy to compute using FFT algorithm. However these methods require the availability of long data records in order to obtain the necessary frequency resolution. Furthermore, these methods suffer from spectral leakage effects due to windowing that are inherent in finite length data recorded.

The basic limitation of non-parametric method is the inherent assumption that the auto-correlation estimate $r_{xx}(m)$ is zero for $m \neq 0$. → The parametric methods are based on modelling the data sequence $x(n)$ as the O/P of linear system characterized by function of form

$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^q b_k z^{-k}}{1 + \sum_{k=1}^p a_k z^{-k}} \rightarrow \textcircled{1}$$

The corresponding difference equation is

$$x(n) = -\sum_{k=1}^p a_k x(n-k) + \sum_{k=0}^q b_k w(n-k) \rightarrow \textcircled{2}$$

$w(n)$ is O/P sequence to system and

$x(n)$ is O/P sequence.

The power density spectrum of data is

$$\Gamma_{xx}(f) = |H(f)|^2 \cdot \Gamma_{ww}(f) \rightarrow \textcircled{3}$$

$\Gamma_{ww}(f)$ is power density spectrum of O/P sequence

$H(f)$ is frequency response of model.

∴ The objective to estimate the power density spectrum $\Gamma_{xx}(f)$ is convenient to assume that O/P sequence $w(n)$ is zero mean white noise sequence with auto-correlation.

$$Y_{ww}(m) = \sigma_w^2 \delta(m) \rightarrow \textcircled{4}$$

where σ_w^2 is variance (i.e. $\sigma_w^2 = E[(\omega(n))^2]$) $\rightarrow \textcircled{5}$ \textcircled{21}

then Power density spectrum observed data is given by

$$P_{xx}(f) = \sigma_w^2 |H(f)|^2 = \sigma_w^2 \frac{|B(f)|^2}{|A(f)|^2} \rightarrow \textcircled{6}.$$

\therefore The eq \textcircled{6} is represented as stationary random process.

\rightarrow In the model based approach the spectrum estimation procedure consists 2-steps.

① For data sequence $x(n)$, $0 \leq n \leq N-1$, we estimate the

parameters $\{a_k\}$ and $\{b_k\}$ are estimated.

From these estimate the power spectrum is calculated acc to eq \textcircled{6}

② From these estimate the power spectrum is calculated acc to eq \textcircled{6}

~~2nd model~~ from recall of random process $x(n)$ generated by pole zero model in eq ① & ② is called Auto regressive - moving average (ARMA)

Process of order (P, q)

ARMA (P, q) .

~~3rd model~~ denoted as if $q=0$ and $b_0 = 1$ the resulting system model has system function $H(z) = \frac{1}{A(z)}$ and op $x(n)$ is called Auto regressive (AR) process.

of order P .

denoted as AR (P) .

~~3rd model~~ denoted as the third possible model is obtained by setting $A(z) = 1$ so that $H(z) = B(z)$ its op $x(n)$ is called moving average (MA) Process

of order q .

denoted as MA (q) .

→ Of these 3-models AR-model is by far the mostly widely used.

~~1st~~ AR model used to representing \rightarrow spectra with narrow peaks (resonances).

~~2nd~~ " is very simple AR-parametric linear Equations.

~~3rd~~ MA model is general does requires more Co-efficients to represent a narrow spectrum.

Relationships b/w the AutoCorrelation and Model Parameters:-

The basic relationship b/w auto correlation $\{\gamma_{xx}(m)\}$ and model parameters $\{a_k\}$ and $\{b_k\}$ is

$$\gamma_{xx}(m) = \begin{cases} - \sum_{k=1}^P a_k \gamma_{xx}(m-k) & ; m > q \\ - \sum_{k=1}^P a_k \gamma_{xx}(m-k) + \sigma_w^2 \sum_{k=0}^{q-m} h(k) \cdot b_{k+m} & ; 0 \leq m \leq q \\ \gamma_{xx}^*(-m) & ; m < 0 \end{cases} \rightarrow ①$$

The relationship of Eq ① provide a formula for determining the model parameters $\{a_k\}$ by considering case $m > q$.

∴ The set of linear equations

$$\begin{bmatrix} \gamma_{xx}(q_r) & \gamma_{xx}(q_r-1) & \dots & \gamma_{xx}(q_r-p+1) \\ \gamma_{xx}(q_r+1) & \gamma_{xx}(q_r) & \dots & \gamma_{xx}(q_r+p+2) \\ \vdots & \vdots & & \vdots \\ \gamma_{xx}(q_r+p-1) & \gamma_{xx}(q_r+p-2) & \dots & \gamma_{xx}(q_r) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix}$$

$$= - \begin{bmatrix} \gamma_{xx}(q_r+1) \\ \gamma_{xx}(q_r+2) \\ \vdots \\ \gamma_{xx}(q_r+p) \end{bmatrix} \rightarrow ②$$

To solve model parameters $\{a_k\}$ by using estimates of autocorrelation sequence in place of $\gamma_{xx}(m)$ for $m \geq q$.

(The relationship b/w AR Parameters and the autocorrelation sequence is obtained by setting $a_r = 0$ in Eq ①)

If the pole parameters $\{a_k\}$ are obtained from Eq ② the result does not determine MA parameters $\{b_k\}$ bcs.

$$\sigma_w^2 \sum_{k=0}^{q-m} h(k) \cdot b_{k+m} = \gamma_{xx}(m) + \sum_{k=1}^P a_k \gamma_{xx}(m-k) ; 0 \leq m \leq q.$$

Relationship b/w AR Parameters and autocorrelation

(23)

Sequence $\gamma_{xx}(m)$ obtained by setting $q_r = 0$.

$$\gamma_{xx}(m) = \begin{cases} - \sum_{k=1}^P a_k \cdot \gamma_{xx}(m-k) & \text{if } m > 0 \\ - \sum_{k=1}^P a_k \cdot \gamma_{xx}(m-k) + \sigma_w^2 & \text{if } m = 0 \\ \gamma_{xx}^*(-m) & \text{if } m < 0 \end{cases}$$

\therefore The AR Parameters $\{a_k\}$ are obtained from solution of Yule-Walked (or) normal equation.

$$\begin{bmatrix} \gamma_{xx}(0) & \gamma_{xx}(-1) & \dots & \gamma_{xx}(-P+1) \\ \gamma_{xx}(1) & \gamma_{xx}(0) & \dots & \gamma_{xx}(-P+2) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{xx}(P-1) & \gamma_{xx}(P-2) & \dots & \gamma_{xx}(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} = \begin{bmatrix} \gamma_{xx}(1) \\ \gamma_{xx}(2) \\ \vdots \\ \gamma_{xx}(P) \end{bmatrix} \rightarrow (4)$$

and variance σ_w^2 can be obtained from eq

$$\sigma_w^2 = \gamma_{xx}(0) + \sum_{k=1}^P a_k \gamma_{xx}(-k) \rightarrow (5)$$

The eq (4) & (5) combined into single matrix eq as

$$\begin{bmatrix} \gamma_{xx}(0) & \gamma_{xx}(-1) & \dots & \gamma_{xx}(-P) \\ \gamma_{xx}(1) & \gamma_{xx}(0) & \dots & \gamma_{xx}(-P+1) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{xx}(P) & \gamma_{xx}(P-1) & \dots & \gamma_{xx}(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_p \end{bmatrix} = \begin{bmatrix} \sigma_w^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \rightarrow (6)$$

finally we indicate that MA(q) model for observed data the autocorrelation sequence $\gamma_{xx}(m)$ is related to MA Parameters $\{b_k\}$ by eq

$$\gamma_{xx}(m) = \begin{cases} \sigma_w^2 \sum_{k=0}^q b_k b_{k+m} & 0 \leq m \leq q \\ 0 & m > q \\ \gamma_{xx}^*(-m) & m < 0 \end{cases} \rightarrow (7)$$

- * The Yule-Walker Method for the AR Model parameters:-
- We simply estimate the auto correlation from the data and we are estimated to solve for AR model parameters.
- In this method it is desirable to use the biased form of auto correlation estimate

$$\hat{\gamma}_{xx}(m) = \frac{1}{N} \sum_{n=0}^{N-m-1} x^*(n) \cdot x(n+m) ; m \geq 0 \quad \rightarrow ①$$

∴ The corresponding power spectrum estimate is

$$\hat{P}_{xx}^W(f) = \frac{\hat{\sigma}_{wp}^2}{1 + \sum_{k=1}^p \hat{a}_k(k) \cdot e^{-j2\pi f k}} \quad \rightarrow ②$$

$\hat{a}_k(k)$ are estimates of AR Parameters

$$\text{and } \hat{\sigma}_{wp}^2 = \hat{\sigma}_p^2 = \hat{\sigma}_p^2 \hat{\gamma}_{xx}(0) \prod_{k=1}^p [1 - |\hat{a}_k(k)|^2] \quad \rightarrow ③$$

- * The Burg method for the AR model parameters:-
- This method is based on the minimization of forward and backward errors in linear predictors

(Let Given data $x(n) ; n=0, 1, \dots, N-1$ and let us consider forward and backward linear prediction estimate of order 'm' given as

$$\hat{x}(n) = - \sum_{k=1}^m a_m(k) \cdot x(n-k) \quad \rightarrow ④$$

$$\hat{x}(n-m) = - \sum_{k=1}^m a_m^*(k) \cdot x(n+k-m)$$

and corresponding forward and backward errors

$$f.e. f_m(n) = x(n) - \hat{x}(n) \quad \rightarrow ⑤$$

$$b.e. g_m(n) = x(n-m) - \hat{x}(n-m)$$

where $a_m(k), 0 \leq k \leq m-1 ; m=1, 2, \dots, p$ are prediction co-efficients

The least square error is E_m (25)

$$E_m = \sum_{n=m}^{N-1} \left[|f_m(n)|^2 + |g_m(n)|^2 \right] \rightarrow (3)$$

This error is to be minimized by selecting the prediction Co-efficients subject to constraint

$$Q_m(k) = Q_{m-1}(k) + k_m q_{m-1}^*(m-k) \quad 1 \leq k \leq m-1 \quad 1 \leq m \leq P \rightarrow (4)$$

from estimator of AR Parameters the power spectrum estimates

is $P_{xx}^{BU}(f) = \frac{\hat{E}_p}{\left| 1 + \sum_{k=1}^P q_p(k) e^{-j2\pi fk} \right|^2} \rightarrow (5)$

Advantages of Burg:-

- (i) It result in high frequency resolution.
- (ii) It yields a stable AR model
- (iii) It is computationally efficient.

Disadvantages :-

- (i) It exhibit spectral line splitting at high (S/N)
- (ii) It can overcome by the use of "windowing" methods.

* Unconstrained least squares method for the AR model parameters

The burg method for determining the parameters of AR-model is basically a least squares lattice algorithm with the constraints that the Predictor Co-efficients satisfy the recursion relation

In this method we use unconstrained least squares algorithm to determine the AR parameters

The least square error is given by

$$\epsilon_p = \sum_{n=p}^{N-1} \left[|f_p(n)|^2 + |g_p(n)|^2 \right]$$

$$= \sum_{n=p}^{N-1} \left[|x(n) + \sum_{k=1}^p \hat{a}_p(k) \cdot x(n-k)|^2 + |x(n-p) + \sum_{k=1}^{p-1} \hat{a}_p(k) \cdot x(n+k-p)|^2 \right] \quad \rightarrow ①$$

The unconstrained least squares power spectrum estimate is given

$$P_{xx}(f) = \frac{\sum_p}{\left| 1 + \sum_{k=1}^p \hat{a}_p(k) \cdot e^{-j2\pi fk} \right|^2} \quad \rightarrow ②$$

fixed p, minimize

notable property also at slide 2.63

minimize with respect to $\hat{a}_p(k)$

through differentiation of Eq (2)

(2) applied to update and obtained results of (1)

Roots of equation (2) are set to consider no direct

residuals. This is called Levenberg - Marquardt

Algorithm (2) also known as minimum of function part

of the algorithm which was first proposed as related with Projected Gradient part for optimization

multiple except that it is based on modified gradient

relatedly in next section of