DIGITAL SIGNAL PROCESSING

HAND NOTES

BY

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Discrete Fourier Series :-

1) Introduction :-

*) The Discrete Fourier Transforms (DFT) is a Powerful Computation tool which allows us to Evaluate the Fourier transform $X(e^{j\omega})$ on digital Computer (or) Specially designed hardware.

*) DTFT (Discrete Time Fourier Transform) is defined for finite (or) infinite Sequences.

*) DFT is defined only for sequences of finite length. Since $X(e^{j\omega})$ is Continuous and Periodic.

*) DFT is obtained by sampling one Period of fourier transform at a finite no:- of frequency Points, where DFT Plays important role in signal Processing algorithms.

2) Discrete Fourier Series :-

Consider a sequence $x_p(n)$ with Period of N - samples so that $x_p(n) = x_p(n + lN)$

where $x_p(n)$ is Periodic, represented by weighted sum of Complex Exponentials whose frequencies are integer multiples of fundamental frequency $\frac{2\pi}{N}$

∴ The Periodic Complex Exponential are

$$e^{j2\pi kn/N} = e^{j2\pi k(n+lN)/N} \longrightarrow (1)$$

Periodic sequence $x_p(n)$ is

$$x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_p(k) \cdot e^{j2\pi kn/N} \longrightarrow (2) \; ; n = 0, 1, \ldots N-1.$$

where $X_p(k)$, $k = 0, 1, \ldots N-1$ are called discrete fourier series Co-efficients.

To obtain fourier Co-efficients multiply B.S. of Eq 2 by $e^{-j(2\pi/N)mn}$ and sum the Products from $n=0$ to $n=N-1$

$$\Rightarrow \sum_{n=0}^{N-1} x_p(n) \cdot e^{-j(2\pi/N)mn} = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} X_p(k) \cdot e^{j(2\pi/N)(k-m)n} \longrightarrow (3)$$

interchanging the order of summation on Right side Eq 3

$$\sum_{n=0}^{N-1} x_p(n) \cdot e^{-j(2\pi/N) \cdot mn} = \frac{1}{N} \cdot \sum_{k=0}^{N-1} X_p(k) \cdot \sum_{n=0}^{N-1} e^{j(2\pi/N)(k-m)n} \longrightarrow (4)$$

using the relation.

$$\sum_{n=0}^{N-1} e^{j(2\pi/N)(k-m)n} = N \quad \text{if } k-m = 0, \pm N, \pm 2N \ldots$$
$$= 0 \quad \text{otherwise.} \longrightarrow (5)$$

we obtain

$$\sum_{n=0}^{N-1} x_p(n)\, e^{-j(2\pi/N)mn} = X_p(m) \rightarrow (6)$$

changing the index from m to k the fourier series Co-efficients $X_p(k)$ in Eq 5 are obtained from $x_p(n)$ by the

$$\boxed{X_p(k) = \sum_{n=0}^{N-1} x_p(n) \cdot e^{-j2\pi kn/N}} \rightarrow (7) \rightarrow \text{DFS}$$

Eq 2 is IDFS $\rightarrow$ inverse discrete fourier series.

$$\boxed{\therefore \; \text{DFS}\left[x_p(n)\right] = X_p(k)} \rightarrow (8)$$

//.

3) Properties of Discrete Fourier Series:-

(i) Linearity:- Consider two periodic sequences $x_{1p}(n)$, $x_{2p}(n)$ with period N, such that

$$DFS\left[x_{1p}(n)\right] = X_{1p}(k) \text{ and}$$
$$DFS\left[x_{2p}(n)\right] = X_{2p}(k) \longrightarrow (1)$$

then.

$$DFS\left[a_1 x_{1p}(n) + a_2 x_{2p}(n)\right] = a_1 X_{1p}(k) + a_2 X_{2p}(k) \longrightarrow (2)$$ //.

(ii) Time shifting:-

If $x_p(n)$ is a periodic sequences with period 'N' samples.

$$DFS\left[x_p(n)\right] = X_p(k) \text{ then.}$$
$$DFS\left[x_p(n-m)\right] = e^{-j(2\pi/N)mk} \cdot X_p(k).$$

where $(n-m)$ is a shifted version of $x_p(n)$.

$$DFS\left[x_p(n-m)\right] = X_p(n-m) = \sum_{n=0}^{N-1} x_p(n-m) \cdot e^{-j\frac{2\pi}{N}k(n-m)}$$

$$= \sum_{n=0}^{N-1} x_p(n-m) \cdot e^{-j\frac{2\pi}{N}kn} \cdot e^{-j\frac{2\pi}{N}km}$$

$$\boxed{DFS\left[x_p(n-m)\right] = X_p(k) \cdot e^{-j\frac{2\pi}{N}\cdot k\cdot m}}$$ //

(iii) Symmetry Property :-

$$x(n) \xrightarrow{DFS} X(k)$$

$$x^*(n) \xrightarrow{DFS} X(-k)$$

$$x^*(-n) \xrightarrow{DFS} X(k)$$

$$X(k) = DFS[x(n)] = \sum_{n=0}^{N-1} x(n) \cdot e^{-j\frac{2\pi kn}{N}}$$

$$= DFS[x^*(n)] = \sum_{n=0}^{N-1} x^*(n) \cdot e^{-j\frac{2\pi kn}{N}}$$

$$= \left[\sum_{n=0}^{N-1} x(n) \cdot e^{\frac{j2\pi kn}{N}}\right]^*$$

$$= \left[\sum_{n=0}^{N-1} x(n) \cdot e^{-\frac{j2\pi(-k)\cdot n}{N}}\right]$$

$$DFS[x^*(n)] = X^*(-k)$$

$$X(k) = DFS(x^*(-n)) = \sum_{n=0}^{N-1} x^*(-n) \cdot e^{-\frac{j2\pi kn}{N}}$$

$$= \left[\sum_{n=0}^{N-1} x(-n) \cdot e^{\frac{j2\pi kn}{N}}\right]^*$$

~~k = 0, 1, 2 ... N-1.~~

∴ ~~Sampling of $X(e^{j\omega})$ is mathematically expressed~~ as

$$\cancel{X(k) = X(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}} \text{ for } k = 0, 1, 2 \ldots N-1}$$

$$X(k) = \left[\sum_{n=0}^{N-1} x(-n) \cdot e^{\frac{j2\pi kn}{N}}\right]^* \quad //.$$

(iv) Periodic Convolution :-

let $x_{1p}(n)$ & $x_{2p}(n)$ be two Periodic Sequence with Period 'N'

$$DFS\left[x_{1p}(n)\right] = X_{1p}(k)$$

$$DFS\left[x_{2p}(n)\right] = X_{2p}(k)$$

if $X_{3p}(k) = X_{1p}(k) \cdot X_{2p}(k)$ then.

Periodic sequence $x_{3p}(n)$ with fourier series is $X_{3p}(k)$

$$x_{3p}(n) = \sum_{m=0}^{N-1} x_{1p}(m) \cdot x_{2p}(n-m)$$

in summary :-

$$DFS\left[\sum_{m=0}^{N-1} x_{1p}(m) \cdot x_{2p}(n-m)\right] = X_{1p}(k) \cdot X_{2p}(k)$$

(v) Multiplication :-

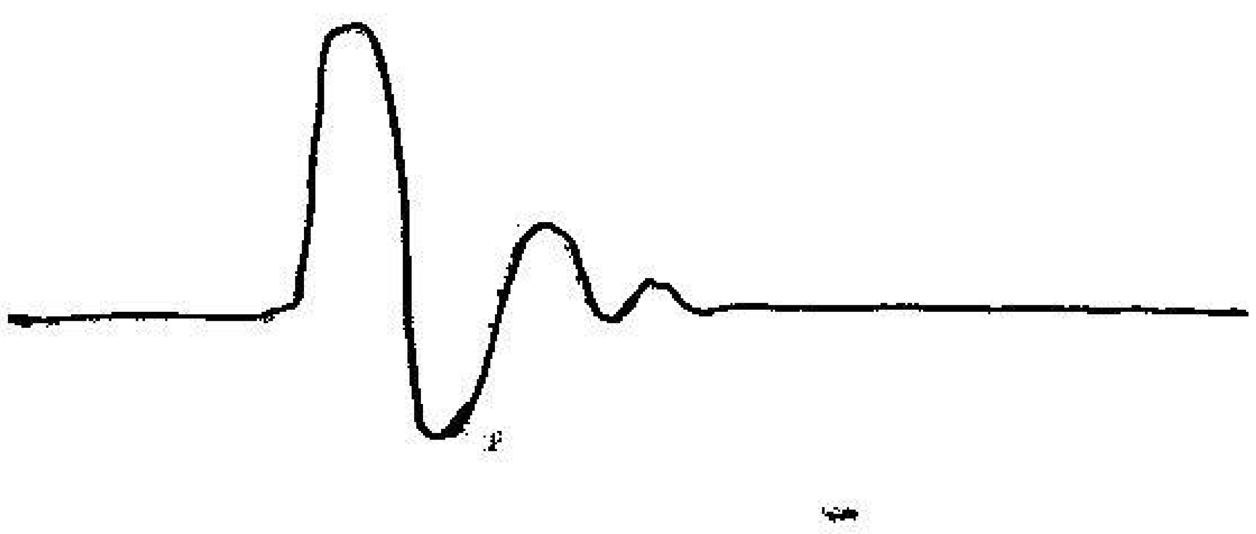
$$x_{1p}(n) \cdot x_{2p}(n) = \frac{1}{N}\sum_{l=0}^{N-1} X_{1p}(l) \cdot X_{2p}(k-l)$$

4)

*) **Signal form representation** :- Example Signal

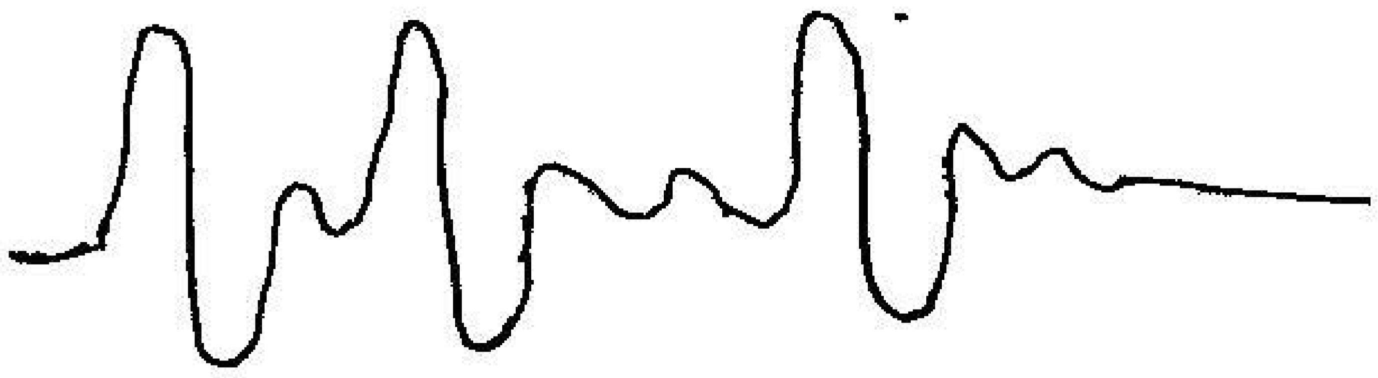
(i) Fourier Transform

(signal that are Continous & aperiodic)



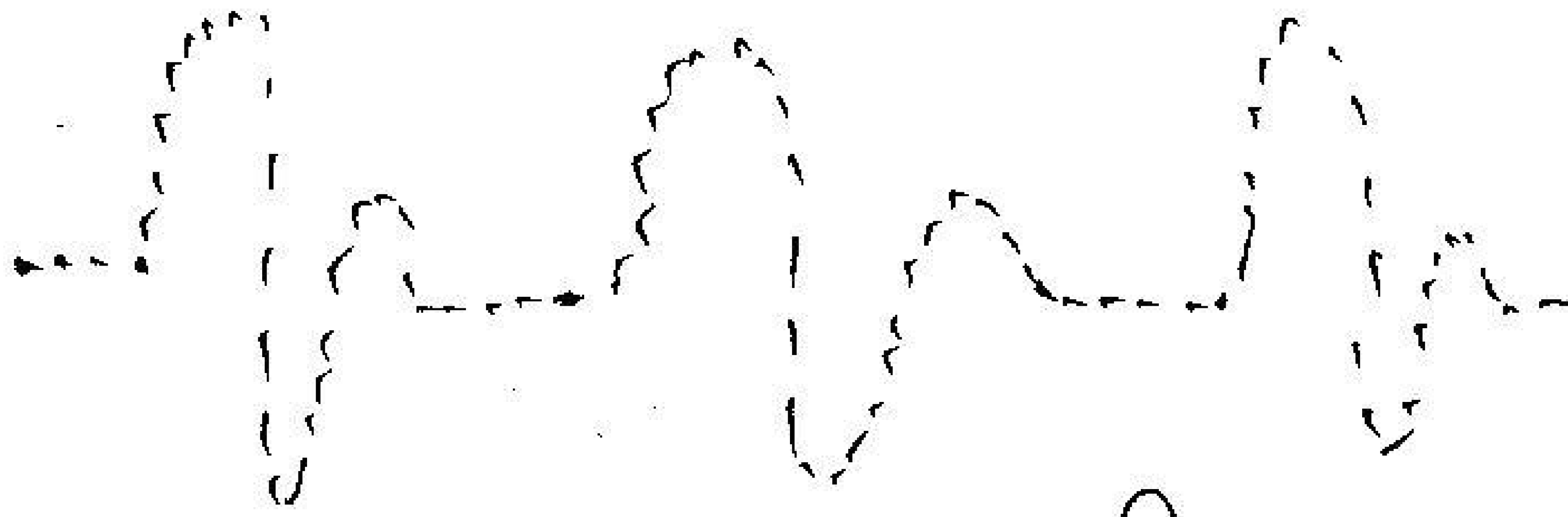
(ii) Fourier Series :-

(signal that are Continous & Periodic)



(iii) Discrete Time fourier Transform
(signal that are discrete and aperiodic)

(iv) Discrete fourier Transforms
(signal that are discrete and Periodic)



5)
*) **Discrete fourier Transforms (DFT) of Discrete Time signal :-**

The discrete time fourier Transforms (DTFT) of a sequence is Periodic, and we are interested in frequency range 0 to 2π. There are infinitely ω in this range.

if we use a digital Computer to Compute N equally spaced Points over the interval $0 \leq \omega < 2\pi$ then the N-points should be located at

$$\omega_k = \frac{2\pi}{N} k \quad ; \quad k = 0, 1, 2 \ldots N-1$$

Where N-Equally spaced frequency samples of DTFT are known as DFT denoted by $X(K)$. is

$$X(K) = X(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N} K} \quad \text{for } K = 0, 1, 2, \ldots N-1$$

The DFT sequences starts at $K=0$, corresponding to $\omega = 0$ but doesnot include $K=N$, corresponding to $\omega = 2\pi$ ($\because$ samples at $\omega = 0$ is same as sample at $\omega = 2\pi$).

Generally, the DFT is defined along with no. of samples and is Called N-Point DFT.

The no:- of samples N for a finite duration sequence $x(n)$ of length L should be such that $N \geq L$, in order to avoid aliasing of frequency spectrum.

Let $x(n)$ = Discrete time signal of length 'L'.

$$X[k] = DFT\{x(n)\}$$

now the N-point DFT of $x(n)$, where $N \geq L$ is defined as

~~now the N-point DFT of x(n), where N≥L~~

$$X[k] = \sum_{n=0}^{N-1} x(n) \cdot e^{-j\frac{2\pi}{N}kn} \quad \text{for } k = 0,1,2 \ldots N-1$$

inverse DFT is

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot e^{j\frac{2\pi}{N}kn} \quad ; n = 0,1,2 \ldots N-1$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot W_N^{-kn} \quad \text{where } W_N = e^{-j\frac{2\pi}{N}}$$

$$X(k] = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn} \quad //.$$

P) Find the DFT of a sequence $x(n) = \{1,1,0,0\}$ & IDFT of $Y(k) = \{1,0,1,0\}$

$$DFT\{x(n)\} = X(k)$$

$$= \sum_{n=0}^{N-1} x(n) \cdot e^{-j\frac{2\pi}{N} \cdot nk}$$

where $k = 0,1,2 \ldots N-1$

now:- for $X(0) = \sum_{n=0}^{4-1} x(n) \cdot e^{-j\frac{2\pi}{4}(0) \cdot n}$ ($\because N \to$ is no of Points are '4') $\forall k = 0$

here $k = 0$ $\quad = \sum_{n=0}^{3} x(n) \cdot e^{0}$ (where given $x(n) = \{1,1,0,0\}$ substitu in eq)

$x(0) = 1+1+0+0 = 2$

now for $k=1$

$$X(1) = \sum_{n=0}^{3} x(n) \cdot e^{-j\frac{2\pi}{4}n \cdot 1} \quad (\text{where } k=1)$$

$$= x(0) + x(1) \cdot e^{-j\frac{\pi}{2}} + x(2) \cdot e^{-j\frac{\pi}{2}\cdot 2} + x(3) \cdot e^{-j\frac{3\pi}{2}}$$

$$= 1 + 1 \cdot e^{-j\frac{\pi}{2}} + 0 \cdot e^{-j\pi} + 0 \cdot e^{-j\frac{3\pi}{2}}$$

$$= 1 + 1\left[\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right] + 0 + 0$$

$$X(1) = 1 + \left[0 - j\right] + 0 + 0 = 1 + [-j] = 1 - j$$

now for $X(2) = \sum_{n=0}^{3} x(n) \cdot e^{-j\frac{2\pi}{4}\cdot n \cdot 2}$ (where $k=2$)

$$= x(0) \cdot e^{-j\pi 0} + x(1) \cdot e^{-j\pi} + x(2) \cdot e^{-j2} + x(3) \cdot e^{-3j}$$

$$= 1 + \{\cos\pi - j\sin\pi\} + 0 + 0 = 1 - 1 = 0$$

now

$$X(3) = \sum_{n=0}^{3} x(n) \cdot e^{-j\pi/2 \cdot n \cdot 3}$$

$$= x(0) + x(1) \cdot e^{-3j\frac{\pi}{2}} + x(2) \cdot e^{-6j\pi} + x(3) \cdot e^{-9j\pi}$$

$$= 1 + 1\left[\cos\left(\frac{3\pi}{2}\right) - j\sin\left(\frac{3\pi}{2}\right)\right] = 1 + 1\left[0 - j(-1)\right]$$

$$= 1 + j$$

$$\therefore X[k] = \{2, 1-j, 0, 1+j\}.$$

IDFT

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) \cdot e^{j\frac{2\pi}{N} \cdot nk} \quad \forall n = 0 \to N-1$$

$$y(0) = \frac{1}{4} \sum_{k=0}^{3} Y(k) \qquad n = 0,1,2,3$$

$$= \frac{1}{4}\left\{Y(0) + Y(1) + Y(2) + Y(3)\right\}$$

$$= \frac{1}{4}\left\{1 + 0 + 1 + 0\right\} = 0.5$$

$$y(1) = \frac{1}{4} \sum_{k=0}^{3} Y(k) \cdot e^{j\pi k/2}$$

$$= \frac{1}{4}\left\{Y(0) + Y(1) \cdot e^{j\pi/2} + Y(2) \cdot e^{j\pi} + Y(3) \cdot e^{j3\pi/2}\right\}$$

$$= \frac{1}{4}\left[1 + 0 + (\cos\pi + j\sin\pi) + 0\right]$$

$$= \frac{1}{4}\left[1 + 0 - 1 + 0\right] = 0$$

$$y(2) = \frac{1}{4}\left\{Y(0) + Y(1) \cdot e^{j\pi} + Y(2) \cdot e^{j2\pi} + Y(3) \cdot e^{j3\pi}\right\}$$

$$= \frac{1}{4}\left[1 + 0 + \cos 2\pi + j\sin 2\pi + 0\right]$$

$$= \frac{1}{4}\left\{1 + 0 + 1 + 0\right\} = 0.5$$

$$y(3) = \frac{1}{4}\left[Y(0) + Y(1) \cdot e^{j3\pi/2} + Y(2) \cdot e^{j3\pi} + Y(3) \cdot e^{j\frac{9\pi}{2}}\right]$$

$$= \frac{1}{4}\left[1 + 0 + \cos 3\pi + j\sin 3\pi + 0\right]$$

$$= \frac{1}{4}\left[1 + 0 + (-1) + 0\right]$$

$$y(n) = \{0.5, 0, 0.5, 0\}.$$

*) Properties of the Discrete Fourier Transforms:-

The DFT Properties are used to process the finite duration sequences.

(i) Periodicity:- If $X(k)$ is N-point DFT of a finite duration sequence $x(n)$

then $x(n+N) = x(n) \quad \forall n$

$X(K+N) = X(K) \quad \forall K$

(ii) Linearity:- If two finite duration sequences $x_1(n)$ and $x_2(n)$ are linearly combined as.

$$x_3(n) = a\,x_1(n) + b\,x_2(n)$$

the DFT of $x_3(n)$ is

$$X_3(K) = a\,X_1(K) + b\,X_2(K)$$

$$DFT\left[a\,x_1(n) + b\,x_2(n)\right] = a\,DFT\left[x_1(n)\right] + b\,DFT\left[x_2(n)\right]$$

(iii) circular shift of a sequences

The N-Point DFT of a finite duration sequence $x(n)$, of length $L \le N$ is Equivalent to the N-Point DFT of a Periodic sequence $x_p(n)$, of Period N, which is obtained by Periodically Extending $x(n)$

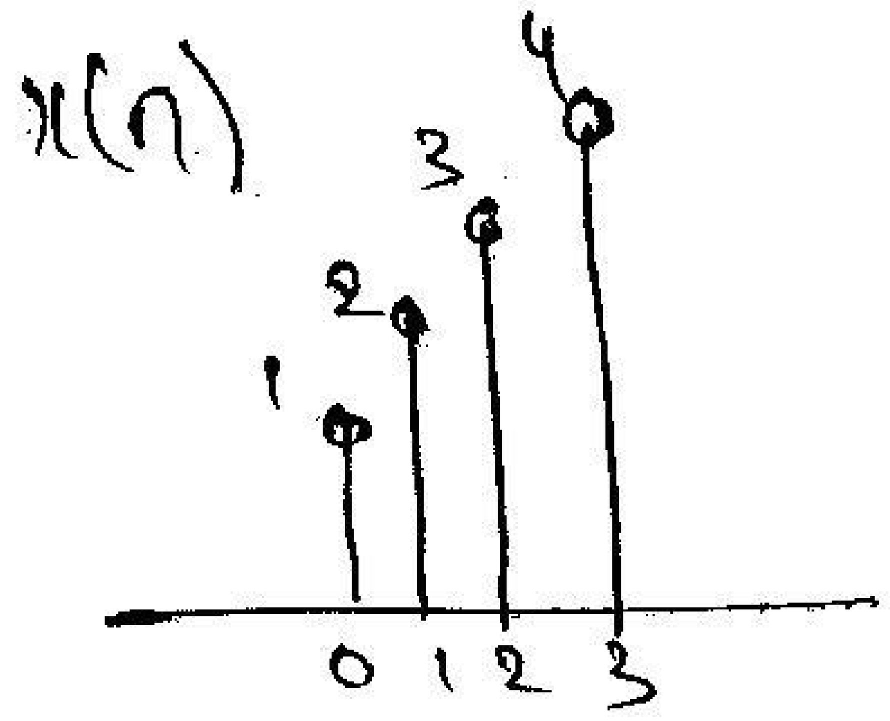
$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN) \rightarrow (a)$$

let the shift in Period sequence $x_p(n)$ by K-units to right

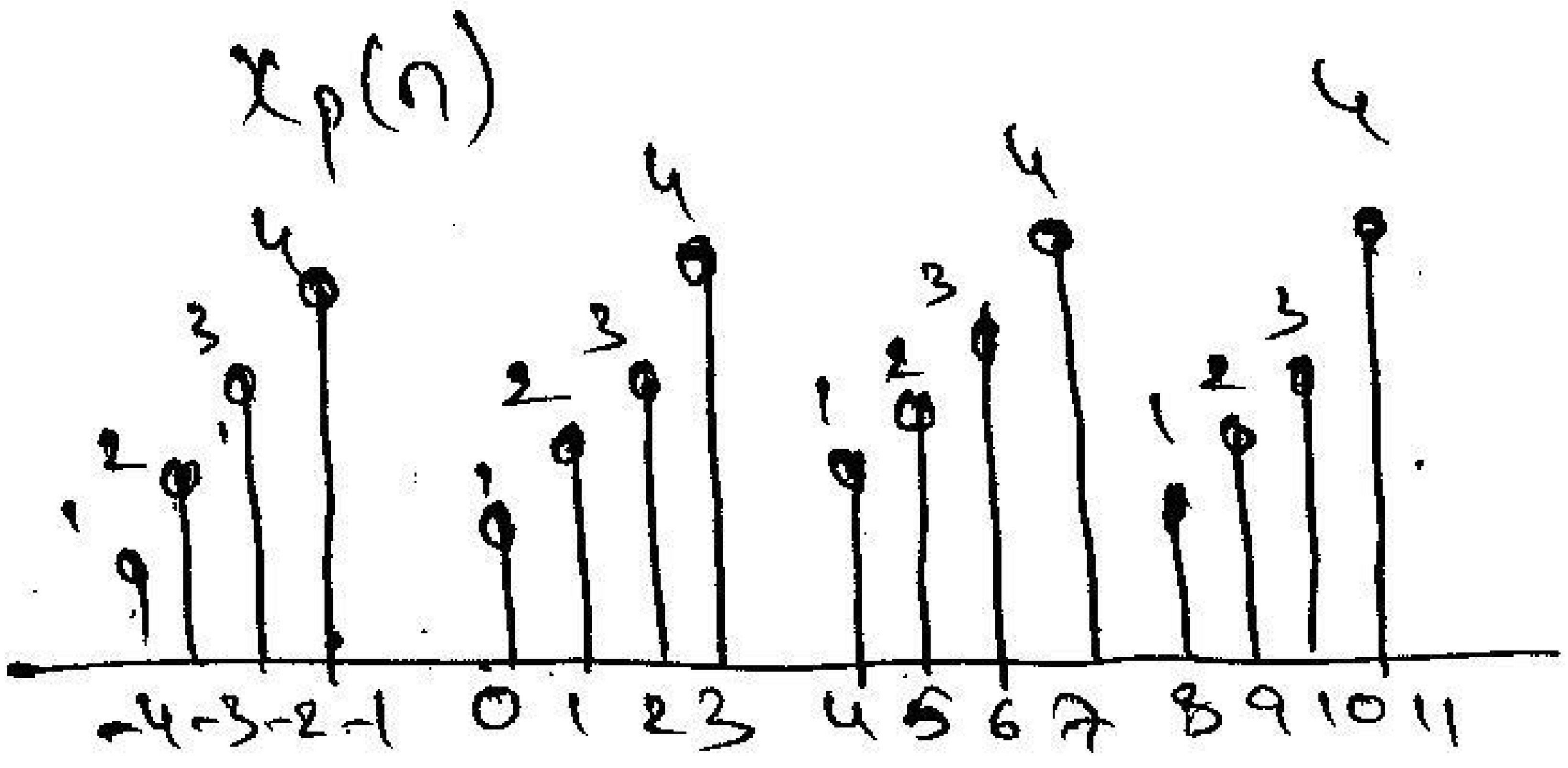
$$x_p'(n) = x_p(n-k) = \sum_{l=-\infty}^{\infty} x(n-k-lN) \rightarrow (b)$$

The finite duration sequence

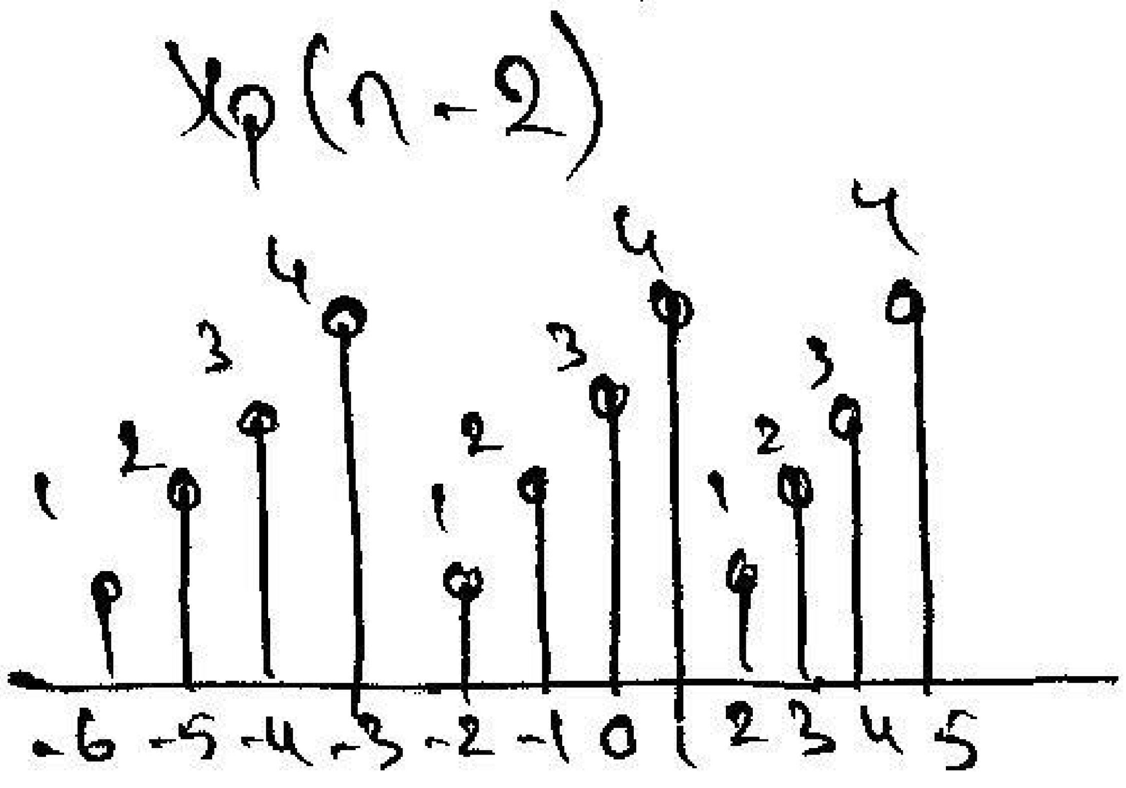
$$x'(n) = \begin{cases} x_p'(n) & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$



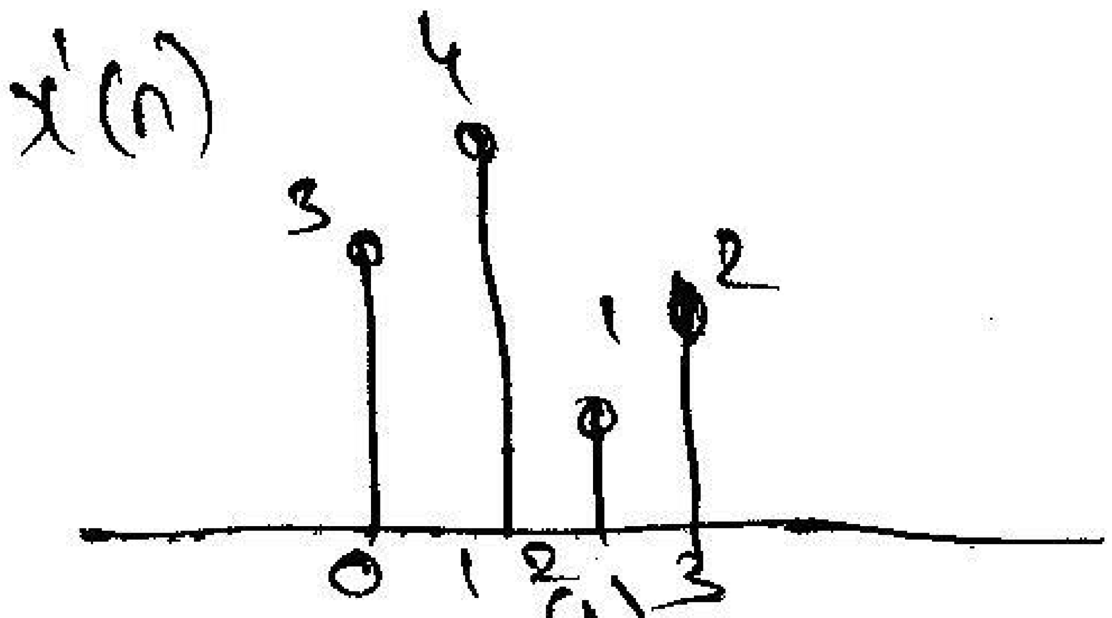
(a)



(b)

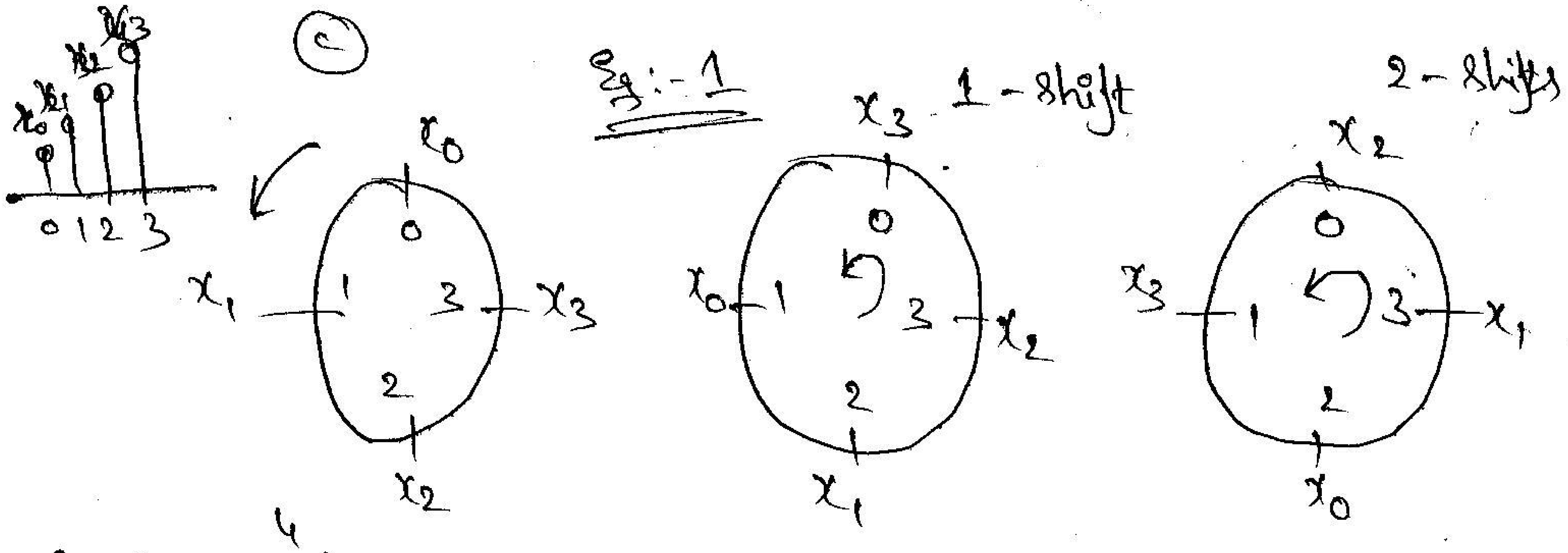


(c)

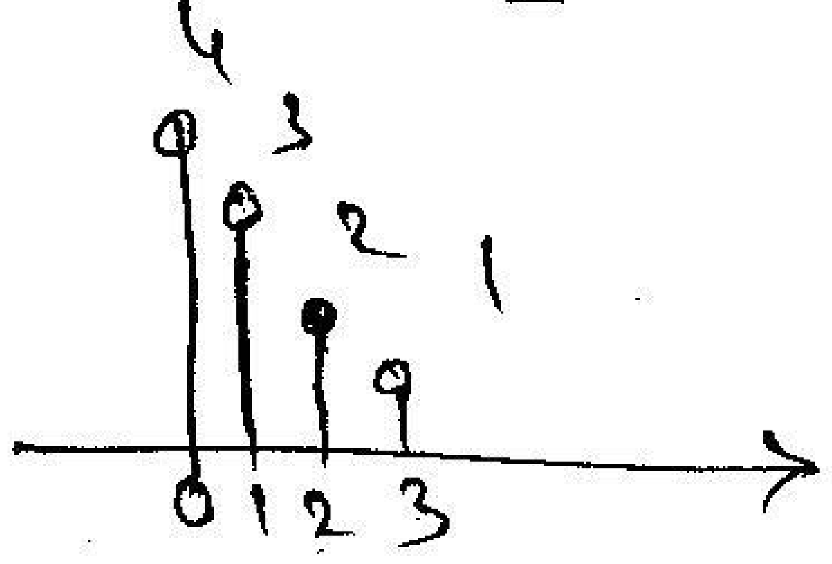


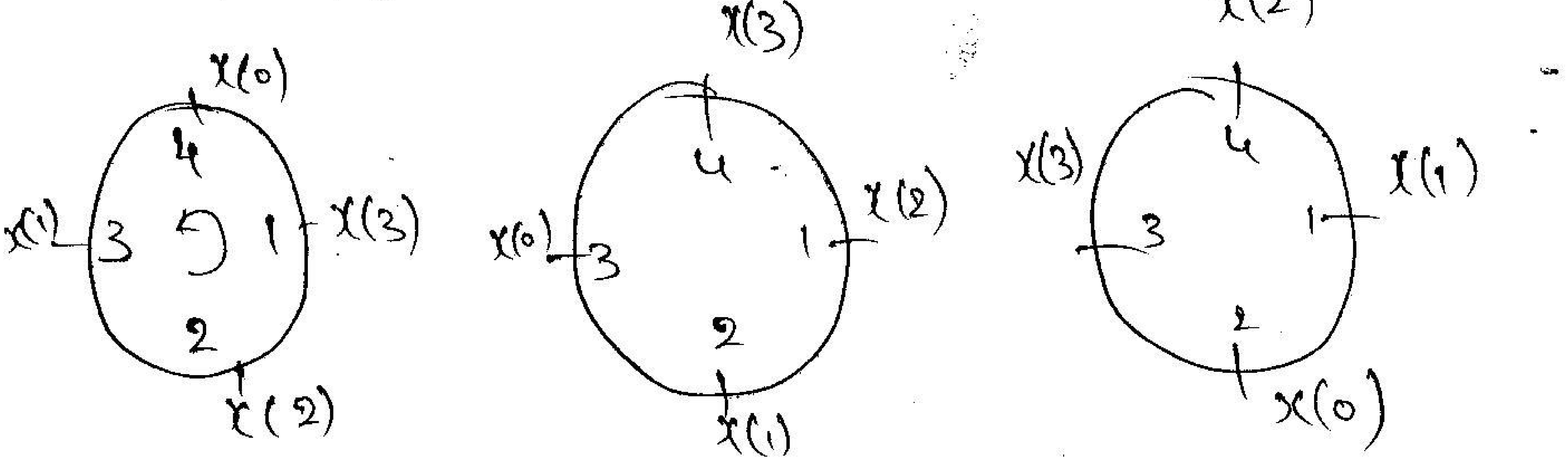
(d)

Eg:- 1



Eg:- 2





As traverse the circumference of circle we see the finite length sequence periodically repeated on a circular (modulo N). time axis.

$$x_p(n) = x\left[(n \text{ modulo } N)\right]$$

$$x_p(n) = x((n))_N \quad \left[\because ((\ )) \text{ indicates circular shift}\right].$$

In general the shifted version $x(n)$ for $n=1,2\ldots N$

$$x(n) = \left[x(0), x(1), x(2), x(3) \ldots\ldots x(N-2), x(N-1)\right]$$

$$x((n-1))_N = \left\{x(N-1), x(0), x(1), \ldots\ldots x(N-3), x(N-2)\right],$$

$$x((x-2))_N = \left\{x(N-2), x(N-1), x(0) \ldots\ldots x(N-4), x(N-3)\right].$$

$$x((n-k))_N = \left[x(N-k), x(N-k+1) \ldots\ldots x(N-k-1)\right]$$

$$x((n-N))_N = \left[x(0), x(1), x(2) \ldots\ldots x(N-2), x(N-1)\right].$$

$$x(x(N+n-N)) = x(n)$$

i.e $x((n-N))_N = x(n) = x(\cancel{N}+n-\cancel{N}) = x(n)$

Simillarly $x((n-m))_N = x(N+n-m)$.

Proof :- $\text{DFT}\left[x(n)\right] = X(K)$

then $\text{DFT}\left[x((n-m))_N\right] = e^{-j2\pi km/N} X(K)$

$$\Rightarrow \text{DFT}\left[x((n-m))_N\right] = \sum_{n=0}^{N-1} x((n-m))_N \cdot e^{-j2\pi kn/N}$$

$$= \underbrace{\sum_{n=0}^{m-1} x((n-m))_N \cdot e^{-j2\pi kn/N}}_{1^{st}} + \underbrace{\sum_{n=m}^{N-1} x((n-m))_N \cdot e^{\frac{-j2\pi kn}{N}}}_{2^{nd}}$$

but $x((n-m))_N = x(N-m+n)$.

Consider 1st Summation

$$\therefore \sum_{n=0}^{m-1} x((n-m))_N \cdot e^{-j2\pi kn/N} = \sum_{n=0}^{m-1} x(N-m+n) \cdot e^{-j2\pi kn/N}$$

let $l = N-m+n$

$n = l+m-N$

$$= \sum_{l=N-m}^{N-1} x(l) \cdot e^{-j2\pi k(-N+m+l)/N}$$

$$= \sum_{l=N-m}^{N-1} x(l) \cdot e^{-j2\pi k(l+m)/N} \longrightarrow (b)$$

when $e^{j2\pi k} = 1$ for $k=0,1,\ldots$

Similarly, second 2nd :- Summation

$$\sum_{n=m}^{N-1} x((n-m))_N \cdot e^{-j2\pi km/N} = \sum_{l=0}^{N-l-m} x(l) \cdot e^{-j2\pi k(m+l)/N} \longrightarrow (c)$$

now from substituting eq (b) in eq (c)

$$= \sum_{l=N-m}^{N-1} x(l) \cdot e^{-j2\pi k(m+l)/N} + \sum_{l=0}^{N-m-1} x(l) \cdot e^{-j2\pi k(m+l)/N}$$

$$= e^{-j2\pi km/N} \sum_{l=0}^{N-1} x(l) \cdot e^{-j2\pi kl/N}$$

$$= e^{-j2\pi km/N} \cdot \sum_{n=0}^{N-1} x(n) \cdot e^{-j2\pi kn/N}$$

$$DFT\left[x((n-m))_N\right] = e^{-j2\pi kn/N} X(K)$$ //.

(iv) Time reversal of sequence:-

" " " an N-Point sequence $x(n)$ is attained by wrapping the sequence $x(n)$ around the circle in clock wise direction.

denoted as $x((-n))_N$ and

$$x((-n))_N = x(N-n) \qquad 0 \le n \le N-1 \rightarrow (a)$$

if DFT

$$DFT[x(n)] = X(k) \text{ then}$$

$$DFT[x((-n))_N] = DFT[x(N-m)]$$

$$= X((-k))_N = X(N-k) \rightarrow (b)$$

changing index from n to $m = N-n$ then

$$DFT[x(N-m)] = \sum_{m=0}^{N-1} x(m) \cdot e^{-j2\pi k(N-m)/N}$$

$$= \sum_{m=0}^{N-1} x(m) \cdot e^{j2\pi km/N}$$

$$= \sum_{m=0}^{N-1} x(m) \cdot e^{-j2\pi m(N-k)/N} \qquad (\because e^{j2\pi k} = 1 \ \forall k = 0,1,2..)$$

$$= X(N-k) \ \text{"}.$$

(v) Circular frequency shift :-

If DFT $[x(n)] = X(k)$ then

$$DFT[x(n) \cdot e^{j2\pi ln/N}] = X((k-l))_N$$

Proof:- $$DFT[x(n) \cdot e^{j2\pi ln/N}] = \sum_{n=0}^{N-1} x(n) \cdot e^{j2\pi ln/N} \cdot e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{N-1} x(n) \cdot e^{-j2\pi n(k-l)/N}$$

$$= \sum_{n=0}^{N-1} x(n) \cdot e^{-j2\pi n(N+k-l)/N}$$

$$= X(N+k-l)$$

$$= X((k-l))_N \ \text{"}.$$

(vi) Complex Conjugate Property :-

If $DFT[x(n)] = X(k)$ then

$$DFT[x^*(n)] = X^*(N-k) = X^*((-k))_N$$

Proof:- $$DFT[x^*(n)] = \sum_{n=0}^{N-1} x^*(n) \cdot e^{-j2\pi kn/N}$$

$$= \left[\sum_{n=0}^{N-1} x(n) \cdot e^{j2\pi kn/N}\right]^*$$

$$= \left[\sum_{n=0}^{N-1} x(n) \cdot e^{-j2\pi n(N-k)/N}\right]^*$$

$$DFT[x^*(n)] = X^*(N-k)$$

Proof:- $$IDFT[X^*(k)] = \frac{1}{N}\sum_{k=0}^{N-1} X^*(k)\, e^{j2\pi kn/N}$$

$$= \frac{1}{N}\left[\sum_{k=0}^{N-1} X(k) \cdot e^{-j2\pi kn/N}\right]^*$$

$$= \frac{1}{N}\left[\sum_{k=0}^{N-1} X(k) \cdot e^{j2\pi k(N-n)/N}\right]^*$$

$$x^*(N-n)$$

$$DFT[X^*(k)] = DFT[x^*(N-n)]$$

(vii) Circular Convolution :-

let $x_1(n)$ & $x_2(n)$ are finite duration sequence both of length 'N' with DFT $X_1(k)$ and $X_2(k)$. Now we find a sequence $x_3(n)$ for which the DFT $X_3(k)$.

where $X_3(k) = X_1(k) \cdot X_2(k) \rightarrow$ (a)

we have $$x_{3p}(n) = \sum_{m=0}^{N-1} x_{1p}(m) \cdot x_{2p}(n-m)$$

(or)

$$x_3((n))_N = \sum_{m=0}^{N-1} x_1((m))_N \cdot x_2((n-m))_N.$$

for $0 \le n \le N-1$; $x_3((n))_N = x_3(n)$

simily $x_1((m))_N = x_1(m)$

$$x_3(n) = \sum_{m=0}^{N-1} x_1(m) \cdot x_2((n-m))_N. \rightarrow (b)$$

The above eq (b) represents circular convolution $x_1(n)$ and $x_2(n)$

$$x_3(n) = x_1(n) \; (N) \; . x_2(n) \rightarrow (c)$$

from (a) and (c) we find

$$DFT\left[x_1(n) \; (N) \cdot x_2(n)\right] = X_1(k) . X_2(k) \rightarrow (d)$$

(viii) Circular Correlation:-

for complex valued sequences $x(n)$ and $y(n)$. if

$$DFT\{x(n)\} = X(K) \text{ and}$$

$$DFT\{y(n)\} = Y(K) \text{ then}$$

$$DFT\left[\tilde{r}_{xy}(l)\right] = DFT\left[\sum_{n=0}^{N-1} x(n) . y^*((n-l))_N\right] = X(K) . Y^*(K)$$

$\tilde{r}_{xy}(l)$ is circular cross correlation sequence.

(ix) Multiplication of two sequences:-

If $DFT\{x_1(n)\} = X_1(K)$ and.

$DFT\left[x_2(n)\right] = X_2(K)$ then

$$DFT\left[x_1(n) . x_2(n)\right] = \frac{1}{N} \cdot \left[X_1(K) \; (N) . X_2(K)\right].$$

(x) Parseval's theorem:-

If $DFT\left[x(n)\right] = X(k)$

$DFT\left[y(n)\right] = Y(K)$ then

$$\sum_{n=0}^{N-1} x(n) \cdot y^*(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) \cdot Y^*(K)$$

*) Methods to Evaluate Circular Convolution of two sequences:-

(a) Concentric Circle method
(b) Matrix Multiplication Method.

(a) Concentric Circle Method:-

Given two sequences $x_1(n)$ and $x_2(n)$ the circular convolution of 2- sequences $x_3(n) = x_1(n) \, (N) \, x_2(n)$ can be found by following steps.

(i) Graph N- samples of $x_1(n)$ as equally spaced points around an outer circle in counter clock wise direction.

(ii) Start at the same point as $x(n)$ graph N samples of $x_2(n)$ as equally spaced points around an inner circle in clockwise direction.

(iii) Multiply corresponding samples on the two circles and sum the products to produce output.

(iv) Rotate the inner circle one sample at a time in counter clock wise direction and go to step 3 to obtain the next value of o/p.

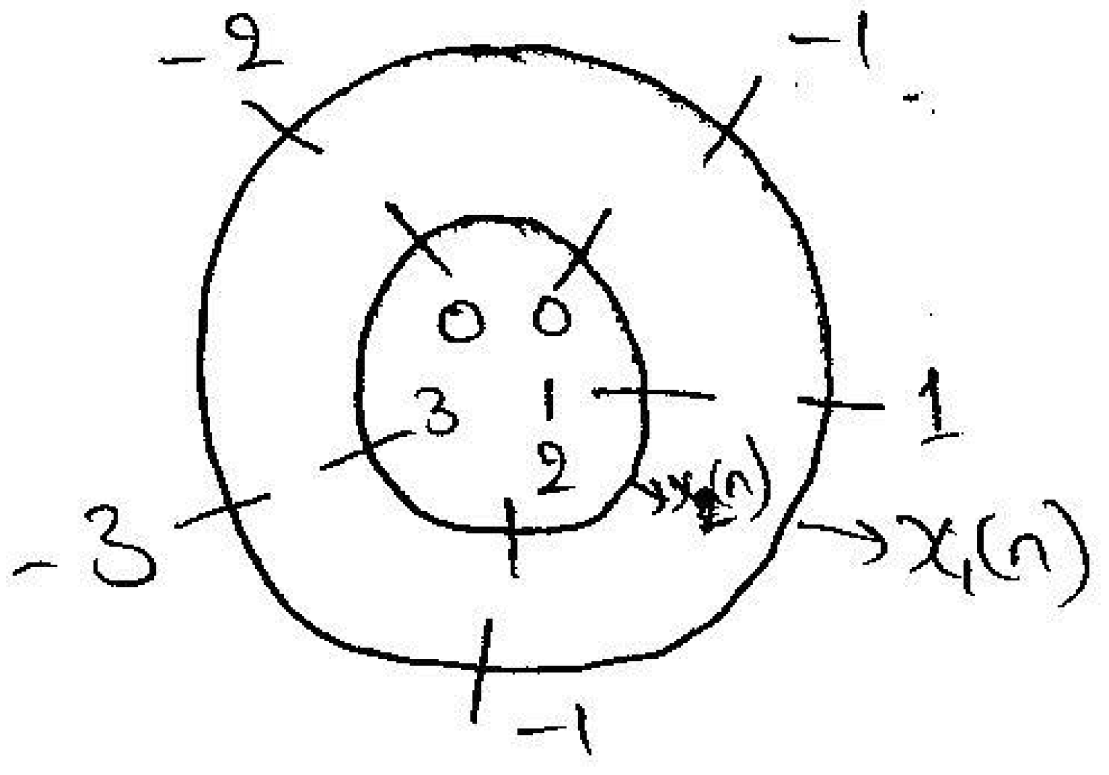
(v) Repeat step No:-4 untill the inner circle first sample lines up the first sample of the exterior circle once again.

Eg:- Find the circular convolution of two finite duration sequences

$x_1(n) = \{1, -1, -2, 3, -1\}$; $x_2(n) = \{1, 2, 3\}$

sol: To find circular convolution both sequences must be of same length. Therefore we append two zero's to the sequence $x_2(n)$ and use concentric circle method to find circular convolution.

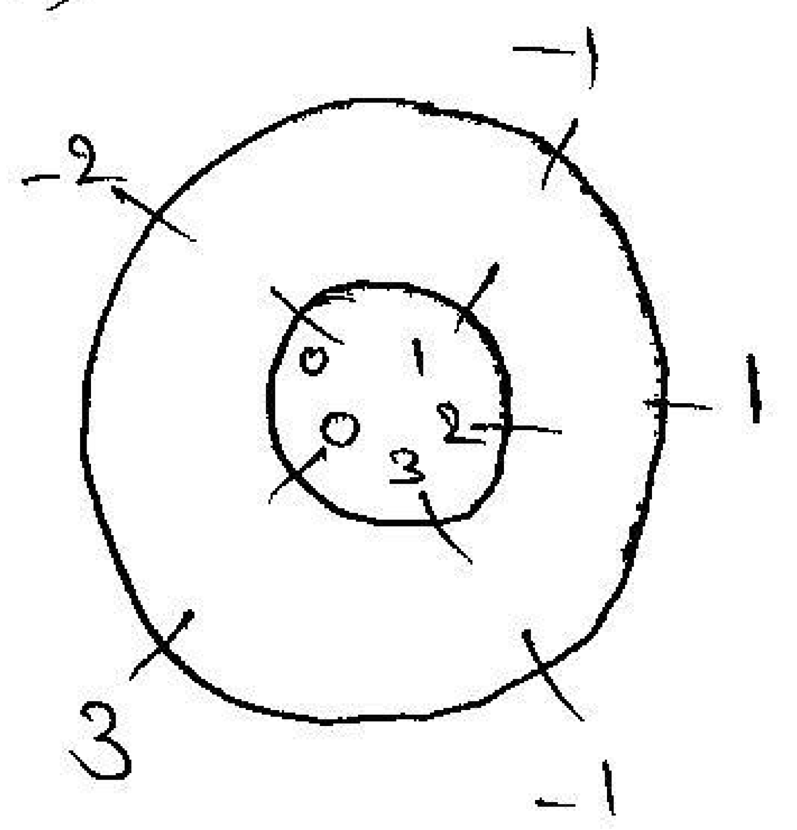
$x_1(n) = \{1, -1, -2, 3, -1\}$; $x_2(n) = \{1, 2, 3, 0, 0\}$

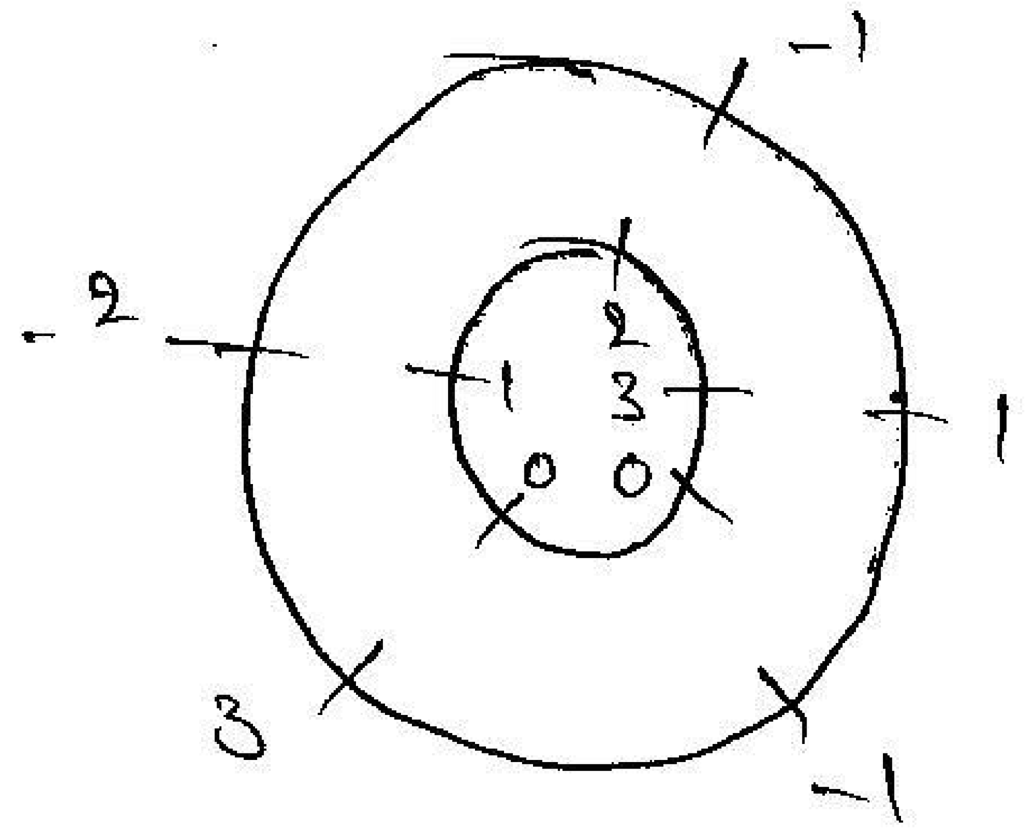


$\Rightarrow y(0) = 1.1 + 0.(-1) + 0(-2) + 3.3 + 2(-1)$

$= 1 + 9 - 2 = 8$

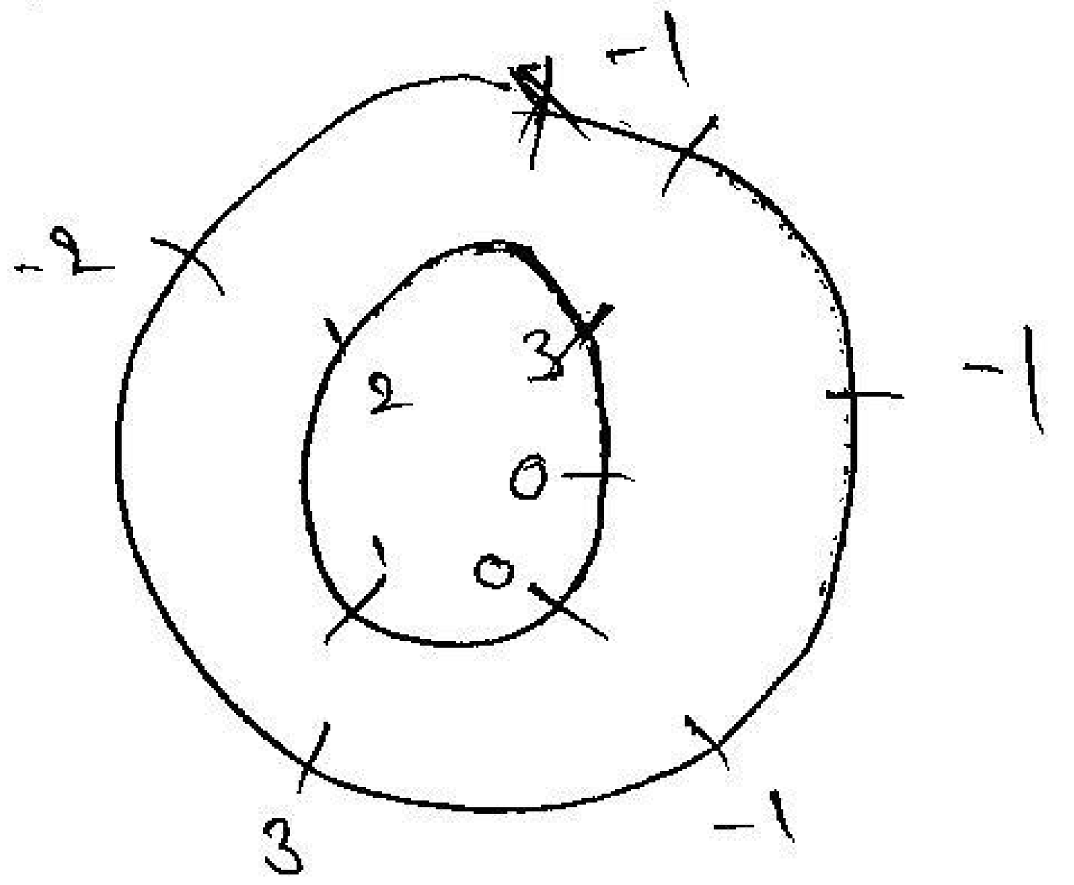
Rotate the inner circle in Counter Clock wise direction by one sample multiply the corresponding samples to obtain $y(1)$.

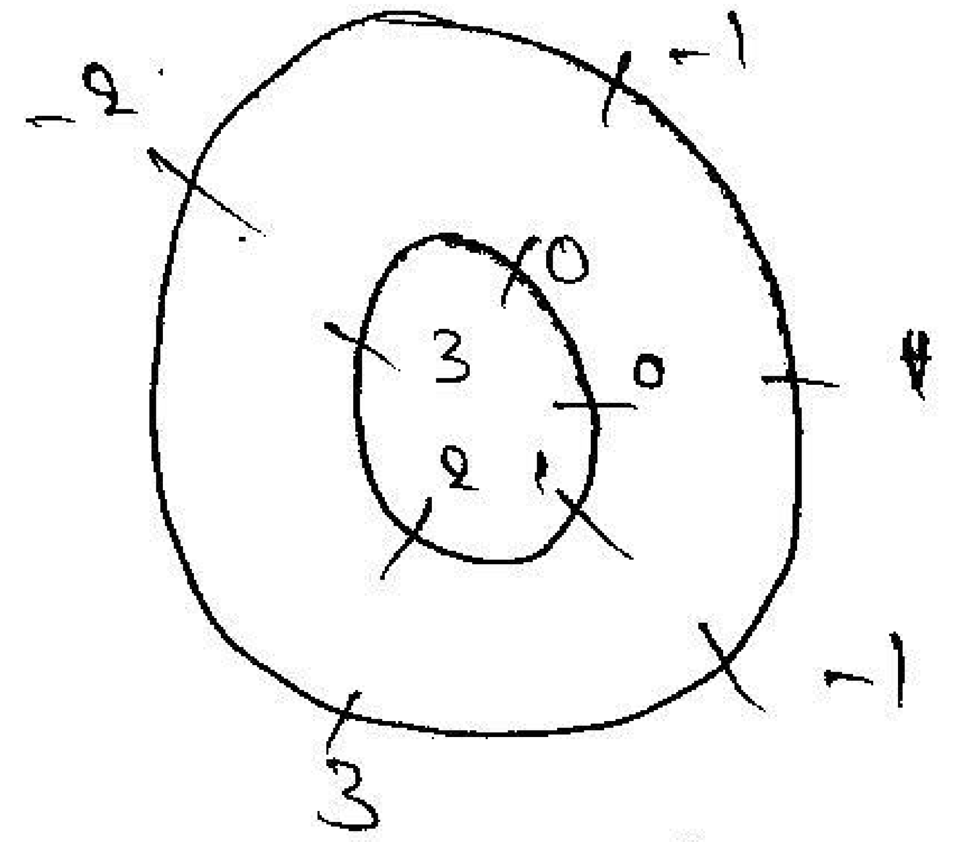




$$y(1) = 1(2) + (-1)\cdot(1) + (-2)\cdot(0) + 3(0) + 3(-1)$$
$$= -2$$

$$y(2) = 3(1) + 2(-1) + (-2) + (0)3 + (-1)0$$
$$= -1$$





$$y(4) = -1$$

$$y(3) = -4$$

$$\therefore y(n) = \{8, -2, -1, -4, -1\}$$

(b) Matrix Multiplication Method:-

Sol:- Given $\quad x_1(n) = \{1, -1, -2, 3, -1\}$

$$x_2(n) = \{1, 2, 3\}$$

By adding two Zero's in $x_2(n)$ we bring length of Sequence $x_2(n)$ to '5'.

Now:- $\quad x_2(n) = \{1, 2, 3, 0, 0\}$

The matrix form Can be written by Substituting $N = 5$.

Represent the sequence $x_2(n)$ in $N \times N$ matrix form and $x_1(n)$ in Column matrix form and multiply to get $y(n)$.

$$x_2(n) \cdot x_1(n) = y(n).$$

$$\Rightarrow \begin{bmatrix} x_2(0) & x_2(4) & x_2(3) & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(4) & x_2(3) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) & x_2(4) & x_2(3) \\ x_2(3) & x_2(2) & x_2(1) & x_2(0) & x_2(4) \\ x_2(4) & x_2(3) & x_2(2) & x_2(1) & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ x_1(3) \\ x_1(4) \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 2 & 1 & 0 & 0 & 3 \\ 3 & 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ -1 \\ -4 \\ -1 \end{bmatrix}.$$

$$\therefore y(n) = \{8, -2, -1, -4, -1\} //.$$

Matrix method Representation:-

The circular convolution of two sequence $x_1(n)$ & $x_2(n)$ can be obtained by representing sequence in matrix as.

$$\begin{bmatrix} x_2(0) & x_2(N-1) & x_2(N-2) & \cdots & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(N-1) & \cdots & x_2(3) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) & \cdots & x_2(4) & x_2(3) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & & x_2(0) & x_2(N-1) \\ x_2(N-2) & x_2(N-3) & x_2(N-4) & \cdots & & \\ x_2(N-1) & x_2(N-2) & x_2(N-3) & \cdots & x_2(1) & x_2(0) \end{bmatrix} \cdot \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_2(2) \\ \vdots \\ x_2(N-2) \\ x_2(N-1) \end{bmatrix} = \begin{bmatrix} x_3(0) \\ x_3(1) \\ \vdots \\ \vdots \\ x_3(N-2) \\ x_3(N-1) \end{bmatrix}$$

$$\Rightarrow \boxed{x_2(n) \, N \, x_1(n) = x_3(n)} //$$

*) Relationship of the DFT to other Transforms

(i) Relationship to the fourier transform:-

The fourier transform $X(e^{j\omega})$ of finite duration sequence $x(n)$ having length 'N' is given by

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j\omega n} \rightarrow (1)$$

where $X(e^{j\omega})$ is a continous function of 'ω'.

The DFT of $x(n)$ is given by

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j\frac{2\pi}{N} \cdot n \cdot k} \; ; \forall \; k = 0, 1, 2 \ldots N-1 \rightarrow (2)$$

Comparing Eq (1) & Eq (2) we find that DFT of $x(n)$ is sampled version of the fourier transform of sequence and is given by

$$X(k) = X(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}} \quad k = 0, 1, 2 \ldots N-1$$

(2) Relationship to Z-transform:-

let us consider a sequence $x(n)$ of finite duration N-width Z-transform,

$$X(z) = \sum_{n=0}^{N-1} x(n) \cdot z^{-n} \rightarrow (1)$$

we have IDFT

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot e^{j\frac{2\pi}{N} k \cdot n} \rightarrow (2)$$

Substituting Eq (2) in Eq (1)

$$X(z) = \frac{1}{N} \cdot \sum_{n=0}^{N-1} \left[\sum_{k=0}^{N-1} X(k) \cdot e^{j\frac{2\pi k}{N} \cdot n} \right] \cdot z^{-n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot \sum_{n=0}^{N-1} \left[e^{j\frac{2\pi k}{N}} \cdot z^{-1} \right]^{n}$$

using $\left(\because \sum_{n=0}^{N-1} a^n = \frac{1 - a^N}{1 - a} \right)$..

$$\therefore X(z) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\cdot\left[\frac{1 - e^{j\frac{2\pi}{N}\cdot k\cdot N}\cdot z^{-N}}{1 - e^{j\frac{2\pi}{N}\cdot k\cdot 1}\cdot z^{-1}}\right]$$

$$\boxed{X(z) = \frac{1 - z^{-N}}{N}\sum_{K=0}^{N-1}\frac{X(K)}{1 - e^{j\frac{2\pi}{N}K}\cdot z^{-1}}}$$

//