DIGITAL SIGNAL PROCESSING

HAND NOTES

BY

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Digital Signal Processing:-

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UNIT - 1

INTRODUCTION TO Digital Signal Processing:-

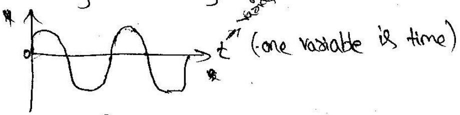
*) **Signal:-** The Physical quality which varies with time Space (or)
The More independent variables $X(t) = F(x_1, x_2, x_3, x_4 \ldots\ldots)$

x_i = time, Space, temperature Etc., ----> (independent variables)

Eg:- Audio, video, ECG (Electro Cardio Graam), AC Power Supply signal.

*) **One - Dimensional Signal:-**
If signal depends on one - variable is called one dimensional signal.

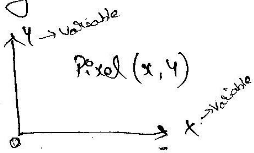
Eg:- Voice signal, AC signal.



*) **Two - Dimensional Signal:-**
If the signal depends on two variable is called two dimensional signal.

Eg:- Pixel has two values which consists of X-dimension & Y-dimension

Eg:- Picture, video signal



*) Multi - Dimensional Signal :-

If the signal depends 1 (or) more than 2-variables is Called Multi dimensional signal.

Eg :- Speed of the winds.

*) Classification of Signals :-

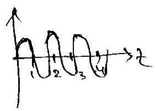
ⓐ Continous time signals
ⓑ Discrete - time signals
ⓒ Digital signals.

ⓐ Continous time signals :- (or) Analog Signal → denoted by $x(t)$

is defined for every instant of time

(or)

Defined over a Continous range of time.

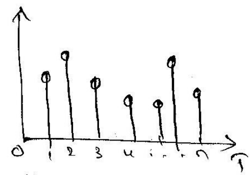


Eg :- A mathematical functions is

$A \sin(\omega t)$ and $a + bt$

ⓑ Discrete - time Signals :- denoted by $x(n)$

The discrete time signals are defined at discrete instants of time.



Eg :- Sampling Period :

$$x(nT) = x(t)\Big|_{t=nT}$$

T = Sampling interval

n = integer ranging from $(-\infty \text{ to } \infty)$. Called time index.

For Convenience we write $x(nT) = x(n) = 0, \pm 1, \pm 2 \ldots$

Thus represented as $x(-2), x(-1), x(0), x(1), x(2) \ldots$

(c) Digital signal:-

Is nothing but the distance time signal which takes the infinite values.

The digital signal is binary signal which takes as whose values Equal to '1' (or) '0'

i.e $x(n) = $ '0' (or) '1' $(-\infty$ to $\infty)$

for $n = -2, -1, 0, 1, 2 \ldots$

Digital signals that are discrete in time and quantized in amplitude are digital signals.

Problem 1:- Sketch the Continous time signal $x(t) = 2e^{-2t}$ for an interval $0 \le t \le 2$. Sample the Continous time signal with a sampling Period $T = 0.2$ sec and sketch the discrete time signal.

Solution:- Given $x(t) = 2e^{-2t}$ at interval $0 \le t \le 2$

at $t = 0$, $x(0) = 2.e^{-2(0)} = 2.e^{0} = 2$

$t = 0.2$, $x(0.2) = 2.e^{-2(0.2)} = 2.e^{-0.4} = 1.34$

"Continous signal"

$t = 0.4$, $x(0.4) = 2.e^{-2(0.4)} = 2.e^{-0.8} = 0.89$

$t = 0.6$, $x(0.6) = 2.e^{-2(0.6)} = 2.e^{-(1.2)} = 0.60$

$t = 0.8$, $x(0.8) = 2.e^{-2(0.8)} = 2.e^{-1.} = 0.40$

$t = 1$, $x(1) = 2.e^{-2(1)} = 0.270$

$t = 1.2$, $x(1.2) = 2.e^{-2(1.2)} = 0.1814$

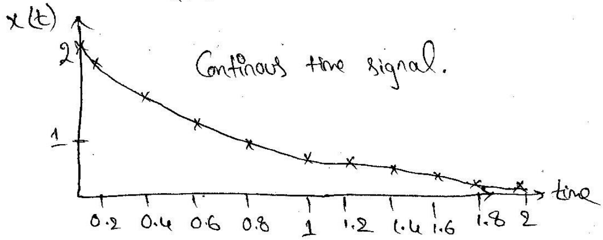
(4)

$t = 1.4, x(1.4) = 2 \cdot e^{-2(1.4)} = 0.1216$

$t = 1.6, x(1.6) = 2 \cdot e^{-2(1.6)} = 0.081$

$t = 1.8, x(1.8) = 2 \cdot e^{-2(1.8)} = 0.054$

$t = 2, x(2) = 2 \cdot e^{-2(2)} = 0.0366$



Given sampling time Period $T = 0.2$

for Discrete time signal. we know.

$$x(nT) = x(t)\Big|_{t=nT} \quad \text{where } n = 0,1,2,3 \ldots \infty$$

$$= x(t)\Big|_{t = 0.2 \times n} \quad (\because T = 0.2)$$

$$= x(0.2 \times n)$$

$$\Rightarrow x(n) = 2 \cdot e^{-2(0.2n)} \quad \text{where "}x(t) = 2 \cdot e^{-2t}\text{ given"}$$

$$x(n) = 2 \cdot e^{-0.4n}$$

$x(0) = 2 \cdot e^{-0.4 \times 0} = 2$

$x(1) = 1.3406$

$x(2) = 0.8987$

$x(3) = 0.6024$

$x(4) = 0.4038$

$x(5) = 0.2707$

$x(6) = 0.1814$

$x(7) = 0.1216$

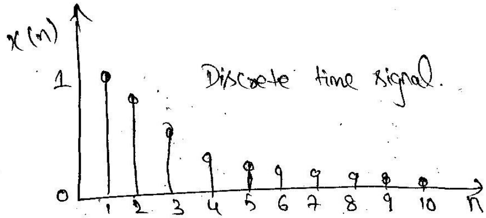
$x(8) = 0.0815$

$x(9) = 0.0546$

$x(10) = 0.0366$

The sequence $x(n)$ can be written as

$x(n) = \{2, 1.34, 0.89, 0.602, 0.403, 0.2707, 0.1814, 0.1216, 0.0815, 0.0546, 0.0366\}$



Problem 2:- Sketch the signal $x(t) = \sin 7t + \sin 10t$ for an interval $0 \le t \le 2$ sample the signal with a sampling Period $T = 0.2$ sec and sketch the discrete time signal.

Given:- $x(t) = \sin 7t + \sin 10t$; $T = 0.2$ sec

for interval of $0 \le t \le 2$

Continuous time signal:- w.k.T $x(t) = \sin 7t + \sin 10t$

$t = 0$; $x(0) = \sin 7(0) + \sin 10(0) = 0$

$t = 0.2$; $x(0.2) = \sin 7(0.2) + \sin 10(0.2) = 1.89$

$t = 0.4$; $x(0.4) = \sin 7(0.4) + \sin 10(0.4) = -0.4218$

$t = 0.6$, $x(0.6) = \sin 7(0.6) + \sin 10(0.6) = -1.1510$

$t = 0.8$; $x(0.8) = \sin 7(0.8) + \sin 10(0.8) = 0.3581$

$t = 1$, $x(1) = \sin 7(1) + \sin 10(1) = 0.1130$

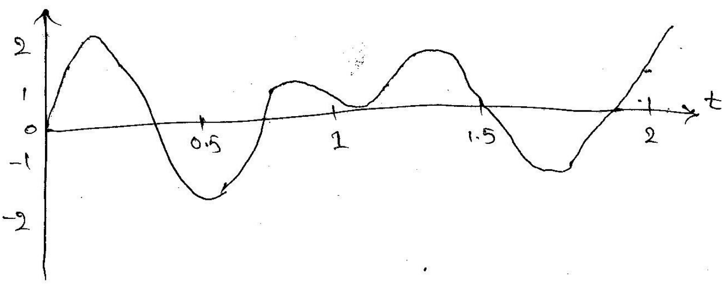
$t = 1.2$; $x(1.2) = \sin 7(1.2) + \sin 10(1.2) = 0.3180$

$t = 1.4$; $x(1.4) = \sin 7(1.4) + \sin 10(1.4) = 0.6241$

$t = 1.6$; $x(1.6) = \sin 7(1.6) + \sin 10(1.6) = -1.2671$

$t = 1.8$; $x(1.8) = \sin 7(1.8) + \sin 10(1.8) = -0.7174$

$t = 2$; $x(2) = \sin 7(2) + \sin 10(2) = 1.9036$



Discrete time signal:-

Given $T = 0.2$ interval

w.k.T $\quad \cancel{x(t)} = x(nT) = x(t)\big|_{t=nT}$

$$\because T = 0.2$$

$$x(nT) = x(t)\big|_{t=n\times 0.2}$$

$$x(nT) = x(0.2n) \quad \text{where } n = 0,1,2,3\ldots$$

at $n=0$; $x(0) = \sin 7(0.2\times 0) + \sin 10(0.2\times 0) = 0$

at $n=1$; $x(1) = \sin 7(0.2\times 1) + \sin 10(0.2\times 1) = 0.059$

at $n=2$; $x(2) = \sin 7(0.2\times 2) + \sin 10(0.2\times 2) = 0.118$

$n=3$; $x(3) = \sin 7(0.2\times 3) + \sin 10(0.2\times 3) = 0.17$

$n=4$; $x(4) = \sin 7(0.2\times 4) + \sin 10(0.2\times 4) = 0.23$

$n=5$; $x(5) = \sin 7(0.2\times 5) + \sin 10(0.2\times 5) = 0.29$

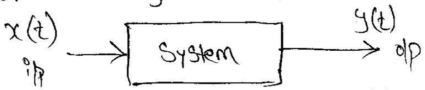
$n=6$; $x(6) = \sin 7(0.2\times 6) + \sin 10(0.2\times 6) =$

$\vdots$

$n=10$; $x(10) = \sin 7(0.2\times 10) + \sin 10(0.2\times 10) =$

3. Problem:- Sketch the signal $x(t) = e^{-t^2/2}$ for $-1 \le t \le 1$. Sample the signal with a sampling period $T = 0.1$ sec and sketch the discrete time signal.

*) **System :-** A system is an inter connection of components. It is a physical device that performs an operation on an i/p signal and produces another signal as o/p.



$y(t) =$ operation on $x(t)$

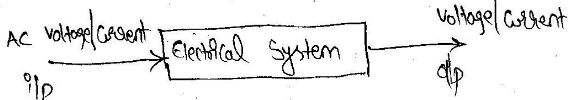
mathematically $\quad y(t) = T[x(t)]$

Represents $x(t)$ is transformed to $y(t)$

(or)

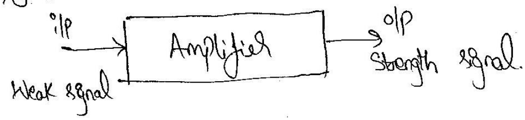
$y(t)$ is the transformed form of $x(t)$

Eg:- ① **An Electrical System :-**
An electrical ckt is a system with i/p's equal to driving voltage/currents and with o/p equal to voltage/currents at various points in circuit



② **Mechanical System :-**

③ **Amplifier System :-**



*) Classification of System :-

(i) Continuous - time System
(ii) Discrete - time System.

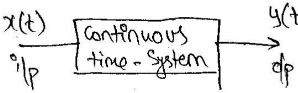
(i) Continuous - time system :-

It is one which operates on continuous time signal and Produces a Continuous - time o/p signal.

If i/p & o/p $x(t)$ & $y(t)$ are Continous then

$x(t)$ is transformed to $y(t)$

$\therefore \; y(t) = T[x(t)]$

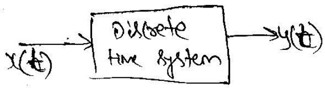


(ii) Discrete - time System :-

Is one which operates on discrete time signal and Produces a discrete - time o/p signal.

If i/p $x(n)$ and o/p $y(n)$ then

$y(n) = T[x(n)]$.



*) Behaviour of a System :-

This is from finding mathematical model of system.

Generally mathematical model Consists of Collection of Equations describing the relationship b/w i/p & o/p signals of system.

There are 2-types of basic's of mathematical models

(a) i/p / o/p Representation describing the relationship b/w the i/p and o/p signals of a system.

(b) The state or internal model describing the relationship among the i/p, state and o/p signals of a system.

The i/p / o/p representation of system can be divided into 4-types:

1. The i/p / o/p differential equation (for continuous-time system) (or) difference equation (for discrete-time systems).
2. The convolution model.
3. The Fourier transform representation
4. The system function representation.

*) Signal Processing:-

* A system is defined as a physical device that performs an operation on a signal.

* Signal processing is any operation that changes the characteristics of a signal.

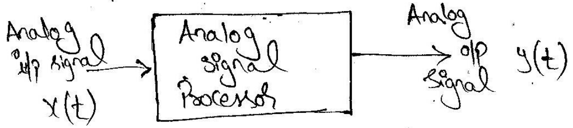
*) These characteristics include the amplitude, shape, phase, and frequency content of the signal.

→ 2 types of signal processings:-

(a) Analog signal processing
(b) Digital signal processing.

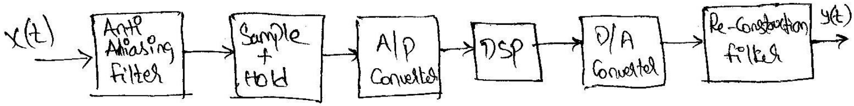
(a) Analog signal processing:-

The system that process the analog signal is known as analog signal processing system.



(b) Digital Signal Processing System :-

The system that Process the digital signal is known as digital signal Processing system.



*) The i/p signal is applied to an antialiasing filter, which is a low Pass filter used to removes high frequencys noise and to band limit the signal.

*) The sample and hold device Provides the i/p to the ADC and will be required. if the i/p signal must remain relatively Constant during the Conversion of Analog signal to digital format.

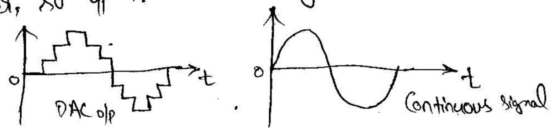
*) The o/p of sample and hold Serves i/p to A/D Converter the o/p of ADC is an N-bit binary no:- depending on the value of analog signal at its i/p. The ADC i/p signal is limited range 0 to +10V if unipolar, and bipolar means -5V to 5V.

*) After Converted digital form, the signal Can be Processed using digital techniques.

*) The DSP Processor may be large Programmable digital Computer/uP's. to Perform desired operation.

*) The Digital signal is applied to DAC. The o/p of DAC is Continous but not smooth, which Contains unwanted high freq Components.

* To Eliminate high freq Components, the o/p of DAC is applied to a reconstruction filter, so o/p is Continous signal.



$x(t)$ → Anti Aliasing filter → Sample + Hold → A/D converter → DSP → D/A converter → Reconstruction Filter →

Advantages & Limitations of Digital signal processing

Advantages:-

Digital signal processing posesses several advantages over analog signal processing.

1. Greater accuracy :- The tolerance of the circuit compo used to design the analog filters affects the accuracy, whereas DSP provides superior control of accuracy.
2. Cheaper :- In many applications, digital realization is comparatively cheaper than its analog counter part.
3. Ease of Data storage :- Digital signals can be easily stored on magnetic media without loss of fidelity and can be processed off-line in a remote laborato
4. Implementations of sophisticated algorithms :- The DSP allows to implement sophisticated algorithms when compared to its analog counterpart.
5. Flexibility in configuration :- A DSP system can be easily reconfigured by changing the program. Reconfi of analog system involves redesign of system hardwa
6. ~~Application~~
6. Applicability of VLF signals :- The very low frequenc signals such as those occuring in seismic applic can be easily processed using a digital signal proces when compared to an analog processing system.
7. Time sharing :- DSP allows the sharing of a given processor among a number of signals by time sharing thus reducing the cost of processing a signa

Limitations :-

1. System comp
digital proces
Such as A/p
2. Bandwidth li
Bo
information los
the bandwidth
bandwidths re
and fast dig
limitation in
and digital
3. power consu
algorithms ca
elements li
do not need
containing
more power.

Applications of

1. Tele commu
Telephone diall
multiplexing,
phone, FAX.
2. consumer e
synthesizer,
sound record
3. Instrumenta
filter, PLL,
control, pr

Limitations :-

System complexity :- System complexity increases in the digital processing of an analog signal because of devices such as A/D & D/A converters and their associated filters.

Bandwidth limited by sampling rate :-

Band limited signals can be sampled without information loss if the sampling rate is more than twice the bandwidth. Therefore, signals having extremely wide bandwidths require fast sampling rate A/D converters and fast digital signal processors. But there is a practical limitation in the speed of operations of A/D converters and digital signal processors.

Power consumption :- A variety of analog processing algorithms can be implemented using passive circuit elements like inductors, capacitors and resistors that do not need much power. Whereas a DSP chip containing over 4 lakh transistors ~~disappears~~ dissipates more power.

Applications of DSP :-

Tele communication :- Echo cancellation in telephone n/w, Telephone dialling application, modems, Line repeaters, channel multiplexing, Data encryption, Video conferencing, cellular phone, FAX.

Consumer electronics :- Digital Audio/TV, electronic music synthesizer, educational toys, FM stereo applications, sound recording applications.

Instrumentation and control : Spectrum analysis, Digital filter, PLL, function generator, servo control, Robot control, process control.

4. Image processing :- Image compression, image enhancem[ent,] image analysis and recognition.

5. Medicine : Medical diagnostic instrumentation such as computerized Tomography (CT), X-ray scanning, Magne[tic] resonance imaging, spectrum analysis of ECG, EEG signals to detect the various disorders in heart and brain, patient monitoring.

6. Speech processing :- speech analysis methods are used in automatic speech recognition, speaker verification and speaker identification. speech synthesis techniques include conversion of written text into speech.

7. Seismology :- DSP techniques are employed in the geoph[ysical] exploration for oil & gas, detection of underground nucle[ar] explosion & earthquake monitoring.

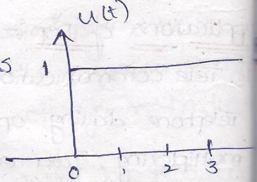
8. Military :- Radar signal processing, sonar signal proces[sing,] Navigation, secure communications.

continuous time signals :-

1. unit step function.

unit step function is defined as

$$u(t) = 1 \quad \text{for } t \geq 0$$
$$= 0 \quad \text{for } t < 0$$



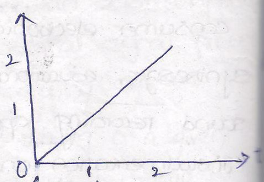
2. unit Ramp Function :-

$$r(t) = t \quad \text{for } t \geq 0$$
$$= 0 \quad \text{for } t < 0$$

(or)

$$r(t) = t\, u(t)$$

$$r(t) = \int_{-\infty}^{t} u(\tau)\, d\tau = \int_{0}^{t} d\tau = t \quad (t>0)$$



Impulse function

$$\int_{-\infty}^{\infty} \delta(t)$$

and $\delta(t)$ =

The 1st conditi[on] the impulse is ... states that $\delta(t$ ... values of t. amplitude ever...

Sinusoidal signal

$x(t) =$

where A = a[mplitude]

Ω = f[requency]

θ = ph[ase]

Real exponential

$x(t) =$

Representation of

1. Graphical r[epresentation]
2. Functional
3. Tabular r[epresentation]
4. Sequence

Graphical repres[entation]

Let us cons[ider]

$x(0) = 2$, $x(1) = 2$

This can

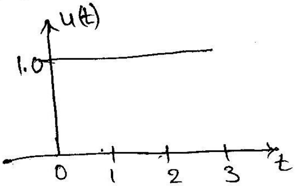
Elementary Continuous time signals:-

1. **unit step function:-** Amplitude of $u(t)$ is Equal to one.

$$u(t) = 1 \text{ for } t \geq 0$$
$$= 0 \text{ for } t < 0$$

*) If the argument 't' in brackets is less than zero (0), the unit step function is zero

*) If the argument 't' inside " " greater " " (or) Equal to zero the unit step function is unity.



2. **unit Ramp function:-**

is defined as

$$r(t) = t \text{ for } t \geq 0$$
$$= 0 \text{ for } t < 0$$

(or)

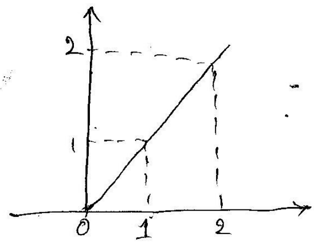
$$\therefore r(t) = t\, u(t)$$

The ramp function can be obtained by applying unit step function to an integrator.

$$r(t) = \int_{-\infty}^{t} u(\tau)\, d\tau = \int_{0}^{t} d\tau = t \text{ (in the interval } t > 0)$$

The unit step function can be obtained by differentiating unit ramp

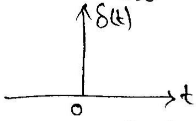
Thus $u(t) = \dfrac{d\, r(t)}{dt}$



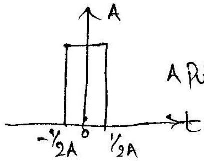
3. Impulse function:- $\delta(t)$

The unit impulse function $\delta(t)$ is defined as

$$\int_{-\infty}^{\infty} \delta(t).dt = 1 \quad \& \quad \delta(t) = 0 \text{ for } t \neq 0$$



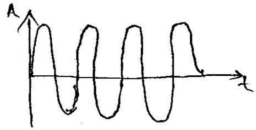
An impulse



A Pulse

*) First Condition states that the area under the impulse is '1'.
*) Second " " " $\delta(t)$ is zero for all non zero value of 't'.
*) An impulse function has zero amplitude every where except at $t=0$.

4. Sinusoidal signal:-



A continuous time sinusoidal signal

$$x(t) = A \sin(\Omega t + \theta)$$

A = Amplitude ; Ω = frequency in radius per second
θ = Phase angle in radians.

5. Real exponential signal:-

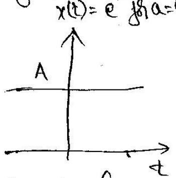
A real exponential signal is defined as

$$x(t) = A . e^{at}$$

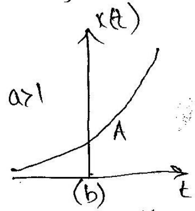
'A' and 'a' both are real. Depending on value of 'a' we get different signals.

If 'a' is positive the signal $x(t)$ is growing exponentially.

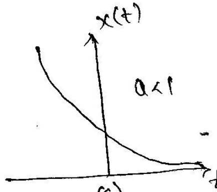
$x(t) = e^{at}$ for $a=0$



dc - signal



(b)
Exponentially growing signal



(c)
exponentially decaying signal.

*) Representation of Discrete-time Signals:-

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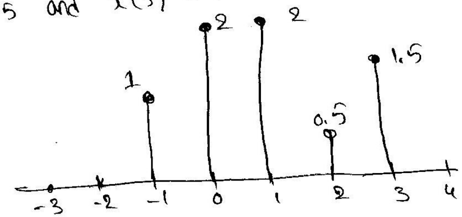
There are 4-types

1) Graphical representation
2) functional "
3) Tabular "
4) Sequence "

1. Graphical Representation:-

Eg:- let us Consider $x(n)$ with $x(-1) = 1$; $x(0) = 2$; $x(1) = 2$; $x(2) = 0.5$ and $x(3) = 1.5$

Sol :-



2. Functional Representation:-

The discrete time signal is represented functionally as

$$x(n) = \begin{cases} 1 & \text{for } n = -1 \\ 2 & \text{for } n = 0, 1 \\ 0.5 & \text{for } n = 2 \\ 1.5 & \text{for } n = 3 \\ 0 & \text{other wise} \end{cases}$$

3. Tabular representation:-

n	-1	0	1	2	3
$x(n)$	1	2	2	0.5	1.5

4. Sequence Representation:-

A finite duration sequence with time origin ($n=0$) indicated by symbol '↑' is

$$x(n) = \{1, \underset{\uparrow}{2}, 2, 0.5, 1.5\}$$

A finite duration sequence can be represented as

$$x(n) = \{\ldots 0, 2, \underset{\uparrow}{1}, -1, 3, 2 \ldots\}$$

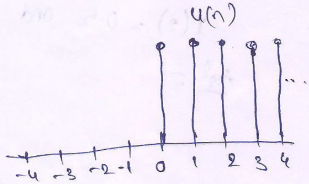
A finite duration sequence that satisfies the condition $x(n) = 0$ for $n < 0$ can be represented as

$$x(n) = \{2, 4, 6, 8, -3\}.$$

12) * Elementary Discrete-time Signals:-

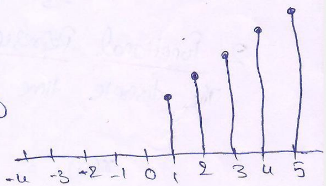
1. Unit step sequence:-

$$u(n) = 1 \text{ for } n \geq 0$$
$$= 0 \text{ for } n < 0$$



2. Unit ramp sequence:-

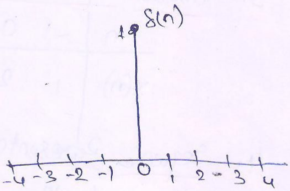
defined $\quad r(n) = n$ for $n \geq 0$
$\quad = 0$ for $n < 0$



unit ramp sequence

3. Unit sample sequence (unit impulse sequence):-

$$\delta(n) = 1 \text{ for } n = 0$$
$$= 0 \text{ for } n \neq 0$$



unit impulse function has following conditions

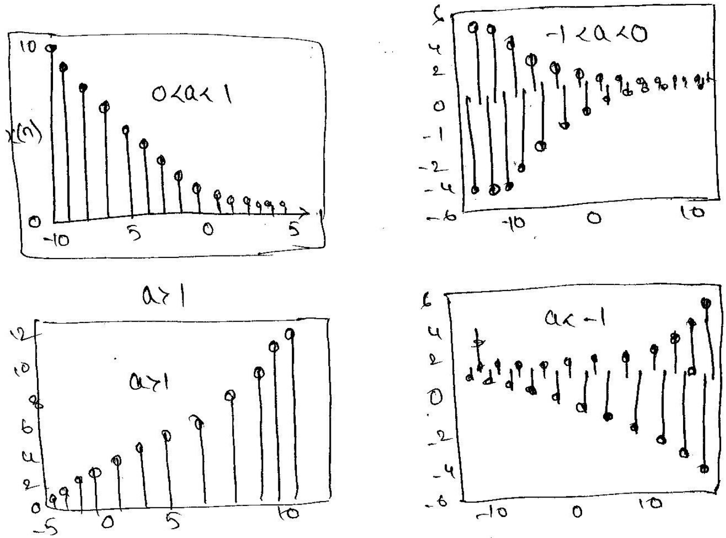
$$\delta(n) = u(n) - u(n-1)$$

$$u(n) = \sum_{k=-\infty}^{n} \delta(k)$$

$$\sum_{n=-\infty}^{\infty} x(n) \cdot \delta(n - n_0) = x(n_0)$$

4. **Exponential sequence:-**

is a sequence of form $x(n) = a^n$ for all 'n'.



5. **Sinusoidal signal:-**

discrete sinusoidal given as

$$x(n) = A\cos(\omega_0 n + \phi)$$

ω_0 = frequency ; ϕ = Phase ;

By using Euler's identity

$$A\cos(\omega_0 n + \phi) = \frac{A}{2} e^{j\phi} \cdot e^{j\omega_0 n} + \frac{A}{2} \cdot e^{-j\phi} \cdot e^{-j\omega_0 n}$$

since $\left| e^{j\omega_0 n} \right|^2 = 1$.

(6) Complex Exponential Signal:-

given by:-

$$x(n) = a^n \cdot e^{j(\omega_0 n + \phi)}$$

$$= a^n \cdot \cos(\omega_0 n + \phi) + j \cdot a^n \sin(\omega_0 n + \phi).$$

for $|a| = 1$, the real and imaginary parts of complex exponential sequence are of sinusoidal.

$|a| < 1$ the amplitude of sinusoidal decays exponentially

$|a| > 1$ " " " " increases "

13) *) Classification of Discrete Time Signals:-

1. Energy signals and Power signals:-

For discrete-time signal $x(n)$ the energy 'E' is defined as

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

The average Power of discrete signal $x(n)$ is defined as

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

*) A signal is energy signal if and only if total energy of the signal is finite.

*) The signal is said to be Power signal if average Power of signal if finite. $(E = \infty)$

*) A signal which do not satisfy above Properties are neither energy nor Power signals.

Eg:- Determine the values of Power and Energy of the following signals.

(i) $x(n) = \left(\frac{1}{3}\right)^n u(n)$; (ii) $x(n) = e^{j\left(\frac{\pi}{2}n + \frac{\pi}{4}\right)}$

(iii) $x(n) = \sin\left(\frac{\pi}{4}n\right)$ (iv) $x(n) = e^{2n} u(n)$.

(i) $x(n) = \left(\frac{1}{3}\right)^n u(n)$

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \text{ for } a<1$$

The Energy signal

unit step sequence in discrete

$$\left(\because \begin{array}{l} u(n) = 1 \text{ for } n \geq 0 \\ u(n) = 0 \text{ for } n < 0 \end{array} \right)$$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=0}^{\infty} \left[\left(\frac{1}{3}\right)^n\right]^2$$

$$\left[\because 1 + a + a^2 + a^3 + \dots \infty = \frac{1}{1-a}\right]$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{9}\right)^n$$

$$= \frac{1}{1-\frac{1}{9}} = \frac{9}{8}$$

$$\sum_{n=0}^{N} a^n = \frac{1-a^{N+1}}{1-a} \quad \forall a \neq 1$$

The Power signal $\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$

$$\text{no:-} \quad \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} \left(\frac{1}{9}\right)^n$$

$$\left(\frac{1-a^{N+1}}{1-a}\right)$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \left[\frac{1-\left(\frac{1}{9}\right)^{N+1}}{1-\left(\frac{1}{9}\right)}\right]$$

$$= 0$$

The Energy is finite and Power is zero

$\therefore$ signal is Energy signal.

(ii) $x(n) = e^{j\left(\frac{\pi}{2}n + \frac{\pi}{4}\right)}$ (Q1)

for Energy signal

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=-\infty}^{\infty} \left| e^{j\left(\frac{\pi}{2}n + \frac{\pi}{4}\right)} \right|^2 \qquad \boxed{\because \left|e^{j(\omega+\theta)}\right| = 1}$$

$$= \sum_{n=-\infty}^{\infty} 1 = \infty$$

Power signal:- $\boxed{\because \sum_{n=-N}^{N} 1 = 2N+1}$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left| e^{j\left(\frac{\pi}{2}n + \frac{\pi}{4}\right)} \right|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 1$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} (2N+1) = 1$$

$\therefore$ The signal is power signal.

(iii) $x(n) = \sin\left(\frac{\pi}{4}n\right)$

$$E = \sum_{n=-\infty}^{\infty} \left|\sin^2\left(\frac{\pi}{4}n\right)\right| = \sum_{n=-\infty}^{\infty} \left[\frac{1-\cos\left(\frac{\pi}{2}n\right)}{2}\right] = \infty \text{ infinite}$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left|\sin^2\left(\frac{\pi}{4}n\right)\right|$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left[\frac{1-\cos\left(\frac{\pi}{2}n\right)}{2}\right]$$

$$= \frac{1}{2} \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 1$$

$P = \frac{1}{2}$ $\therefore$ finite.

(iv) $x(n) = e^{2n} u(n)$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} e^{4n} \quad \left(\because \begin{array}{l} u(n) = 1 \text{ for } n \geq 0 \\ u(n) = 0 \text{ for } n < 0 \end{array}\right)$$

$$= 1 + e^4 + e^8 + \cdots \infty \quad \left(\because 1 + e^4 + e^8 + e^{16} + \cdots = \infty\right)$$

$$= \infty$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} e^{4n}$$

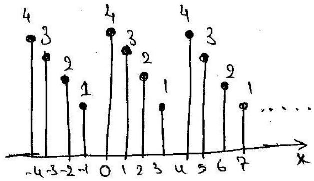
$$= \lim_{N\to\infty} \frac{1}{2N+1} \left[\frac{e^{4(N+1)} - 1}{e^4 - 1}\right] = \infty$$

$\therefore$ The signal Power & Energy are infinite.

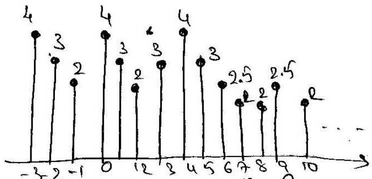
2. Periodic & aperiodic signals:-

A discrete signal $x(n)$ is said to be periodic with period N if

$$x(N+n) = x(n) \text{ for all } n. \rightarrow (a)$$



Periodic Signal.



aperiodic Signal

All continuous time sinusoidal signal are periodic but all the discrete time sinusoidal sequences are not periodic

$$x(n) = A \sin(\omega_0 n + \theta). \rightarrow (b)$$

A discrete time sequence is periodic if it satisfies the condition

$$x(N+n) = x(n) \rightarrow (c)$$

$$x(n+N) = A\sin(\omega_0(n+N)+\theta)$$

$$= A\sin(\omega_0 n + \omega_0 N + \theta) \rightarrow (d)$$

where A = Amplitude ; ω_0 and θ are frequency & phase shift

above Eq (d) satisfies if and only if $\omega_0 N$ is the integer multiple of 2π.

$$\omega_0 N = 2\pi m$$

$$\boxed{\frac{\omega_0}{2\pi} = \frac{m}{N}}$$ rational no:-

(or

$$\boxed{N = 2\pi\left(\frac{m}{\omega_0}\right)}$$

Eg:- Determine whether (or) not each of following signals is periodic. If a signal is periodic. specify fundamental.

(i) $x(n) = \cos\left(\frac{2\pi}{3}\right).n$

$$x(n) = \cos\left(\frac{2\pi}{3}\right).n$$

$$\omega_0 = \frac{2\pi}{3}$$

Compare $\omega_0 = \frac{2\pi m}{N}$

$$\frac{\cancel{2\pi}}{3} = \frac{\cancel{2\pi}m}{N} \Rightarrow \frac{m}{N} = \frac{1}{3}$$ rational no:-

$$N = \frac{3m}{1} \Rightarrow 3m$$

for smallest value of 'm', so that N becomes integer 'm=1'

$$\therefore N = 3$$

3. Causal & Non-Causal signal :-

A signal. $x(n)$ is said to be Causal if its value is zero for $n<0$. other wise signal is non-Causal..

Eg:- $x_1(n) = a^n u(n)$

$x_2(n) = \{\underset{\uparrow}{1}, 2, -3, -1, 2\}$

Eg:- non Causal :- $x_1(n) = a^n u(-n+1)$

$x_2(n) = \{1, -2, \underset{\uparrow}{1}, 4, 3\}$ //.

4. Symmetric (Even) and Antisymmetric (odd signals) :-

(*) A discrete time signal $x(n)$ is said to be a Symmetric if it satisfies the Condition

$$x(-n) = x(n) \quad \forall n$$

Eg:- $x(n) = \cos \omega n$

(*) The signal is said to be an odd signal if satisfies the Condition

$$x(-n) = -x(n) \quad \forall n$$

Eg:- $x(-n) = A \sin \omega n$

(*) A signal $x(n)$ Can be Expressed as the Sum of Even and odd Components. i.e

$$x(n) = x_e(n) + x_o(n) \rightarrow (1)$$

$x_e(n)$ = Even Component ; $x_o(n)$ = odd Component

Replace n by '$-n$' in Eq (1)

$$\therefore x(-n) = x_e(-n) + x_o(-n)$$

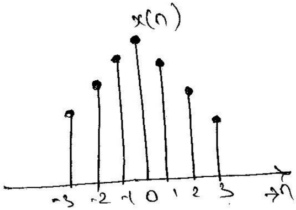
$$= x_e(n) - x_o(n) \rightarrow (2)$$

Adding eq (1) & (2)

$$2x_e(n) = x(n) + x(-n)$$

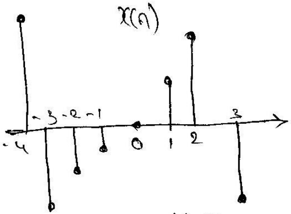
$$\Rightarrow x_e(n) = \frac{1}{2}\left[x(n) + x(-n)\right]$$

Similly:- $$x_o(n) = \frac{1}{2}\left[x(n) - x(-n)\right].$$



Even

Symmetric



odd signals

Antisymetric

14)

*) Operation on Signals:-

The mathematical transform from one signal to another is given

$$y(n) = T\left[x(n)\right].$$

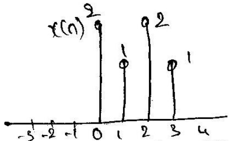
(1) **Shifting**:- The shift operation takes the i/p sequence and shifts the values by an integer increment of the independent variable. The shifting may Delay (or) Advance the sequence in time.

$$y(n) = x(n-k)$$

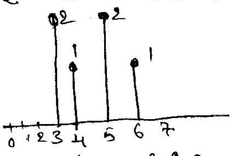
$y(n)$ → o/p, $x(n)$ → i/p

If k is +ve the shifting delays the sequence

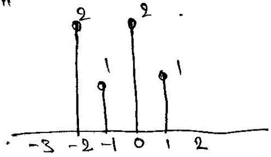
k is -ve the " advances " "



Discrete time signal

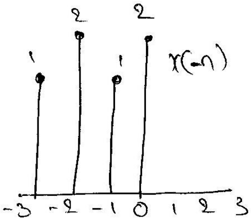


delay version.

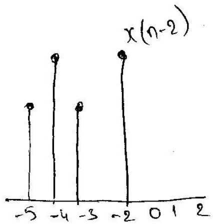


Advance version.

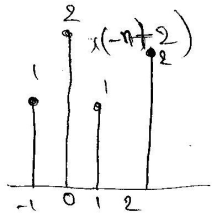
(2) Time reversal:- denoted as $x(-n)$



Shifted version



Advance



delay.

(3) Time Scaling:-

This is obtained by replacing n by λn in sequence $x(n)$

$$\therefore\ y(n) = x(\lambda n)$$

Eg:-
let $x(n)$ is a sequence. If $\lambda = 2$ then

$$y(n) = x(2n)$$

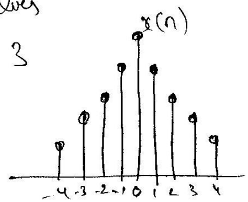
the graph is ploted by using different values

for:- $n = -1$; $y(-1) = x(-2) = 3$

similarly $y(0) = x(0) = 5$

$y(1) = x(2) = 3$

$y(2) = x(4) = 1$



(4) Scalar Multiplication:-

The signal $x(n)$ is multiplied by a scale factor a

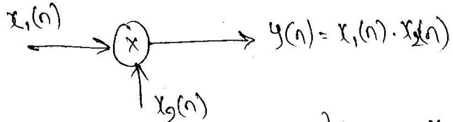
$$x(n) \xrightarrow{a} y(n) = a \cdot x(n)$$

Eg:- $x(n) = \{1, 2, 1, -1\}$; and $a = 2$;

$y(n) = a\,x(n) = \{2, 4, 2, -2\}$.

5) Signal Multiplier:-

Multiplication of 2-signal sequences to form another seq.

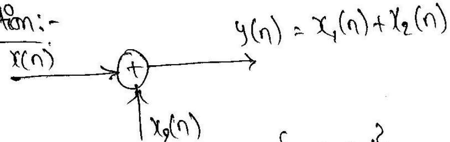


Eg:- $x_1(n) = \{-1, 2, -3, 2\}$ and $x_2 = \{1, -1, -2, 1\}$

then $x_1(n) \cdot x_2(n) = \{(-1 \times 1), (2 \times -1), (-3 \times -2), (-2 \times 1)\}$

$= \{-1, -2, 6, -2\}.$

6) Addition operation:-



$x_1(n) = \{1, 2, 3, 4\}$; $x_2(n) = \{4, 3, 2, 1\}$

$x_1(n) + x_2(n) = \{5, 5, 5, 5\}.$

15) * Classifications of Discrete time Systems:-

1. Static & dynamic systems
2. Causal and Non-Causal systems
3. Linear and Non-linear systems
4. Time variant and Time invariant systems
5. FIR & IIR system
6. Stable and unstable systems.

1. Static & Dynamic Systems :-

A discrete time System is Called Static (or) memoryless if its o/p at any instant depends on the i/p samples at the same time, but not on past (or) future sample of i/p. otherwise the system is dynamic.

$$\left.\begin{aligned} y(n) &= a\,x(n) \\ y(n) &= a\,x^2(n) \end{aligned}\right\} \text{static}$$

$$\left.\begin{aligned} y(n) &= x(n-1) + x(n-2) \\ y(n) &= x(n+1) + x(n) \end{aligned}\right\} \text{dynamic..}$$

2. Causal & Non-Causal Systems :-

A System is said to be Causal if the o/p of system at any time n depends only at Present & Past i/p's but doesnot depends on future i/p's.

$$y(n) = F\left[x(n), x(n-1), x(n-2) \ldots\right].$$

if the o/p of a system depends on future i/p, the system is said to be non-Causal (or) anticipatory

$y(n) = x(n) + x(n-1)$ Causal System

$y(n) = x(2n)$ non-Causal System..

Eg:- $y(n) = x(n) + \dfrac{1}{x(n-1)}$

for $n = -1$ $\quad y(-1) = x(-1) + \dfrac{1}{x(-2)}$

$n = 0$ $\quad y(0) = x(0) + \dfrac{1}{x(-1)}$

$n = 1$ $\quad y(1) = x(1) + \dfrac{1}{x(0)}$

for all values of 'n' the o/p depends on present & past i/p.

$\therefore$ System is Causal.

Eg:- (iii) $y(n) = x(n^2)$

$n = -1$; $y(-1) = x(1)$

$n = 0$; $y(0) = x(0)$

$n = 1$; $y(1) = x(1)$

$\forall$ value of n, (except for $n=0$ & $n=1$), the system depends on future i/p. so the system is non causal.

3. Linear & Non-linear Systems:-

A system satisfies the superposition principle is said to be linear system.

Super position ~~theorem~~ states that the response of the system to a weighted sum of signals should be equal to the corresponding weighted sum of the o/p's of sm to each of i/p signals.

A system is linear if and only if

$$T[a_1 x_1(n) + a_2 x_2(n)] = a_1 T[x_1(n)] + a_2 T[x_2(n)]$$

~~y(n) = 2x(n)~~

Eg:- $y(n) = n\,x(n)$

$$y_1(n) = T[x_1(n)] = n.x_1(n)$$

$$y_2(n) = T[x_2(n)] = n.x_2(n)$$

$$a_1 T[x_1(n)] + a_2 T[x_2(n)] = a_1 n.x_1(n) + a_2 n\,x_2(n)$$

$\therefore$ o/p due to weighted sum of i/p is

$$y_3(n) = T[a_1 x(n) + a_2 x(n)] = a_1 n\,x_1(n) + a_2 n\,x_2(n)$$

(Equal) .//

$\therefore$ super position principle is satisfied.

4. Time Variant & Time invariant system :-

A system is said to be time variant if the characteristics of the system do not change with time

$$y(n,k) = T[x(n-k)].$$

Delay the o/p sequence by k-samples, denote it as $y(n-k)$ if

$$y(n,k) = y(n-k)$$

$\forall$ possible value of k, the system is time invariant on the other hand.

$$y(n,k) \neq y(n-k)$$

Eg:- $y(n) = x(n) + x(n-1)$

Given $y(n) = T[x(n)] = x(n) + x(n-1)$

if the i/p is delayed by k-units in time, we have

$$y(n,k) = T[x(n-k)] = x(n-k) + x(n-k-1)$$

Here $y(n,k) = y(n-k)$

$y(n-k) = x(n-k) + x(n-k-1)$

(ii) $y(n) = T[x(n)] = x(-n)$

$y(n,k) = T[x(n-k)] = x(-n-k)$

$y(n-k) = x(n-k) = x(-n+k)$

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Causality :-

A Causal system is one whose o/p depends on Past / Present values of the i/p.

Eg: $k=-1$ Advance (-ve)
$k=1$ delay (+ve)

$$y(n) = \sum_{k=-\infty}^{\infty} h(k) \cdot x(n-k)$$

$$= \sum_{k=-\infty}^{-1} h(k) \cdot x(n-k) + \sum_{k=0}^{\infty} h(k) \cdot x(n-k)$$

$$= \underbrace{\cdots h(-2)\,x(n+2) + h(-1)\,x(n+1)}_{\text{Depends on Future i/p's}} + \underbrace{h(0) \cdot x(n)}_{\text{Present i/p's}} + \underbrace{h(1) \cdot x(n-1)}_{\text{Past i/p}} + \cdots$$

here we find o/p depends on the Past & Present values of the i/p.
if the index $k \geq 0$ if $k < 0$ then o/p depends on future values of i/p.

$\therefore$ For a Causal system whose o/p does not depend on future values of the i/p

$$y(n) = \sum_{k=0}^{\infty} h(k) \cdot x(n-k)$$

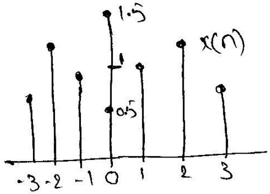
for Causal system $h(k)$ should be zero $k < 0$.

*) Representation of an Arbitrary sequence :-

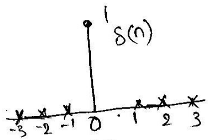
17) Any arbitrary sequence $x(n)$ can be represented in terms of delayed and scaled impulse sequence $\delta(n)$. let $x(n)$ be an infinite sequence.

The sample $x(0)$ can be obtained by multiplying $x(0)$, the magnitude with unit impulse $\delta(n)$

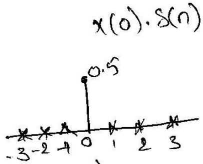
ie $x(0).\delta(n) = \begin{cases} x(0) & \text{for } n=0 \\ 0 & \text{for } n \neq 0 \end{cases}$



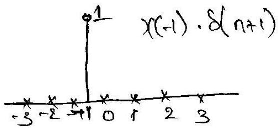
(a)

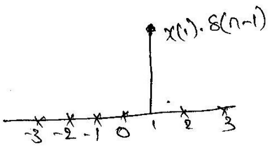


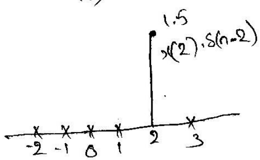
(b)

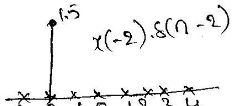


(c)









Simillarly the sample $x(-1)$ can be obtained by multiplying $x(-1)$, the magnitude with one sample advanced unit impulse $\delta(n+1)$.

i.e. $x(-1)\,\delta(n+1) = \begin{cases} x(-1) & \text{for } n = -\phi \\ 0 & \text{for } n \neq -\phi \end{cases}$

In same way $x(-2).\delta(n+2) = \begin{cases} x(-2) & \text{for } n = -2 \\ 0 & \text{for } n \neq -2 \end{cases}$

$x(1).\delta(n-1) = \begin{cases} x(1) & \text{for } n = 1 \\ 0 & \text{for } n \neq 1 \end{cases}$

$x(2).\delta(n-2) = \begin{cases} x(2) & \text{for } n = 2 \\ 0 & \text{for } n \neq 2. \end{cases}$

$\therefore$ The sum of the sequences

$$x(-2)\cdot\delta(n+2) + x(-1)\cdot\delta(n+1) + x(0)\cdot\delta(n) + x(1)\cdot\delta(n-1) + x(2)\cdot\delta(n-2)$$

Equals $x(n)$ for $-2 \le n \le 2$.

In general form $x(n)$ for $-\infty \le n \le \infty$

$$x(n) = +\dots x(-3)\cdot\delta(n+3) + x(-2)\cdot\delta(n+2) + x(-1)\cdot\delta(n+1) + x(0)\cdot\delta(n) + x(1)\cdot\delta(n-1) + x(2)\cdot\delta(n-2) + x(3)\cdot\delta(n-3) + \dots$$

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\cdot\delta(n-k).$$

where $\delta(n-k)$ is unity for $n=k$ and zero $\forall$ others

Eg:- Represent the sequence $x(n) = \{4, 2, -1, \underset{\uparrow}{1}, 3, 2, 1, 5\}$ as sum of shifted unit impulses.

$$x(n) = \{4, 2, -1, \underset{\uparrow}{1}, 3, 2, 1, 5\}$$

$$n = -3, -2, -1, 0, 1, 2, 3, 4$$

$$x(n) = x(-3)\cdot\delta(n+3) + x(-2)\cdot\delta(n+2) + x(-1)\cdot\delta(n+1) + x(0)\cdot\delta(n) + x(1)\cdot\delta(n-1) + x(2)\cdot\delta(n-2) + x(3)\cdot\delta(n-3) + x(4)\cdot\delta(n-4)$$

$$x(n) = +4\cdot\delta(n+3) + 2\cdot\delta(n+2) - 1\cdot\delta(n+1) + 1\cdot\delta(n) + 3\cdot\delta(n-1) + 2\cdot\delta(n-2) + 1\cdot\delta(n-3) + 5\cdot\delta(n-4)$$ //

*) Impulse response & Convolution Sum:-

A discrete time system perform an operation on an i/p signal based on a predefied criteria to produce a modified o/p signal. The i/p signal $x(n)$ is system excitation and $y(n)$ is the system response.

$$x(n) \longrightarrow \boxed{T} \longrightarrow y(n) = T[x(n)]$$

If the i/p to the system is unit impulse i.e $x(n) = \delta(n)$ then the o/p of system is known as impulse response denoted by $h(n)$

$$h(n) = T[\delta(n)]$$

wkt any arbitary sequence $x(n)$ can be represented as a weighted sum of discrete impulses

$$x(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot \delta(n-k) \rightarrow (1)$$

$$y(n) = T[x(n)] = T\left[\sum_{k=-\infty}^{\infty} x(k) \cdot \delta(n-k)\right] \rightarrow (2)$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot T[\delta(n-k)] \rightarrow (3)$$

The response to shifted impulse sequence can denoted by $h(n,k)$

$$h(n,k) = T[\delta(n-k)] \rightarrow (4)$$

for a time invariant system

$$h(n,k) = h(n-k) \rightarrow (5)$$

sub (5) in Eq (4)

$$T[\delta(n-k)] = h(n-k)$$

from Eq (3) we have

$$\boxed{y(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k)}$$

∴ Above Eq is known as Convolution Sum Eq given as

$$y(n) = x(n) \cdot h(n) \quad [* \rightarrow \text{denotes Convolution}].$$

Properties of Convolution :-

(i) Commulative law :- $x(n) * h(n) = h(n) * x(n)$

(ii) Associative law :- $[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$.

(iii) Distributive law :- $x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$

*) FIR and IIR Systems :-

Linear time invariant systems can be classified according to the type of impulse response

1. FIR Systems (FINITE impulse Response)
2. IIR Systems. (INFINITE " ")

FIR System :-

If the impulse response of system is of finite duration, then the system is called a finite impulse Response (FIR system).

Eg :-

$$h(n) = \begin{cases} 1 & \text{for } n = -1, 2 \\ 2 & \text{for } n = 1 \\ 3 & \text{for } n = 0, 3 \\ 0 & \text{Otherwise} \end{cases}$$

IIR System :-

Def :- An infinite impulse Response (IIR) System has an impulse response for infinite duration

$$h(n) = a^n \cdot u(n)$$

*) Stable and unstable Systems:-

An LTI System is stable if it produces a bounded o/p sequence for every bounded i/p sequence.

If for some bounded i/p sequence $x(n)$, the o/p is unbounded (infinite), the system is classified as unstable.

Let $x(n)$ be bounded i/p sequence $h(n)$ be the impulse response of system and $y(n)$ be o/p sequence.

Taking magnitude of the o/p.

we have $$|y(n)| = \left| \sum_{k=-\infty}^{\infty} |h(k)| \cdot |x(n-k)| \right| \rightarrow$$

w.k.t the magnitude of the sum of terms is less than or equal to sum of magnitudes

$$|y(n)| \leq \sum_{k=-\infty}^{\infty} |h(k)| \, |x(n-k)|$$

Let the bounded value of i/p is equal to M

$$y(n) \leq M \sum_{k=-\infty}^{\infty} h(k)$$

The above condition is satisfied when

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

$\therefore$ necessary condition for stability is

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

Eg:- $h(n) = \left(\frac{1}{2}\right)^n \cdot u(n)$

Stability $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n \cdot u(n)$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \qquad \because \left[1+a+a^2+\cdots = \frac{1}{1-a}\right]$$

$$= 1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\cdots\cdots+$$

$$= \frac{1}{1-\frac{1}{2}} = \frac{1}{1/2} = 2 < \infty$$

hence system is stable.

18)

*) **Linear Constant Co-efficient difference Equations :-**

(or)

Time Response Analysis of Discrete time systems :-

The general form of difference equation of an N^{th} order linear time invariant discrete time (LTI - DT) system is

$$y(n) = -\sum_{k=1}^{N} a_k \cdot y(n-k) + \sum_{k=0}^{M} b_k\, x(n-k) \rightarrow (1)$$

where $\{a_k\}$ & $\{b_k\}$ are constants. The response of any discrete time system can be decomposed as

Total response = Zero state response + Zero i/p response

*) The Zero state response of system is response of system due to i/p alone when the intial state of system is zero i.e the system intially relaxed at time $n=0$. (Ideal)

On other hand, the zero i/p response depends only on the intial state of the system. that is the i/p is zero. (when i/p zero)

Eg:- let us consider a first order discrete time system with difference equation.

$$y(n) = a\, y(n-1) + x(n)$$

where $x(n)$ & $y(n)$ are i/p & o/p.

let $x(n)$ i/p sequence is zero for $n<0$ and intially $y(n)$ for $n=-1$ exits. i.e $y(-1) \neq 0$

$\therefore$ The successive value of $y(n)$ for $n \geq 0$ are as follows

for $n=0$

$$y(0) = a\, y(-1) + x(0) \rightarrow (a)$$

$$y(1) = a \cdot y(0) + x(1) \rightarrow (b)$$

$$= a[a\, y(-1) + x(0)] + x(1) \quad \left[\begin{array}{l}\text{substitute } y(0) \text{ eq (a)} \\ \text{in eq (b).}\end{array}\right]$$

$$= a^2 y(-1) + a x(0) + x(1)$$

$$y(2) = a\, y(1) + x(2)$$

$$= a[a^2 y(-1) + a x(0) + x(1)] + x(2)$$

$$= a^3 y(-1) + a^2 x(0) + a.x(1) + x(2)$$

for any n.

$$y(n) = a^{n+1} y(-1) + a^n x(0) + a^{n-1} x(1) + \ldots + x(n)$$

$$= a^{n+1} y(-1) + \sum_{k=0}^{n} a^k . x(n-k) \quad n \geq 0.$$

The response $y(n)$ includes two parts. The first part depends on initial condition of the system and second term on i/p.

When $y(-1)=0$, the o/p $y(n)$ depends only on i/p applied. Hence $y(n)$ is known as the zero state response (or) forced response

$$y_f(n) = \sum_{k=0}^{n} a^k x(n-k) \quad n \geq 0.$$

If the system is initially non-relaxed that is $y(-1) \neq 0$ and i/p $x(n) = 0 \ \forall n$, the o/p of system $y(n)$ depends only on the initial state of system.

Then the response of system is called the zero i/p response (or) natural response and is denoted as

$$y_n(n) = a^{n+1} y(-1) \quad n \geq 0$$

The zero i/p response (natural response) is obtained by letting the i/p signal to zero. It depends on the nature of the system and initial conditions. On the other hand the zero state response depends on the nature of i/p signal.

$\therefore$ The difference eq of N^{th} order discrete time system of eq (1) is also written as

$$\sum_{k=0}^{N} a_k\, y(n-k) = \sum_{k=0}^{M} b_k \cdot x(n-k) \rightarrow (2)$$

Where 'N' is called order of difference equation.

If the co-efficient a_0 is not equal to one we can divide eq (2) through out a_0 to normalize the equation.

The solution of difference equation can be given as sum of two parts

$$y(n) = y_h(n) + y_p(n)$$

$y_h(n)$ = homogenous (or) complimentary solution

$y_p(n)$ = Particular solution //.

$\boxed{\text{Note :}}$ for an LTI system the response $y(n)$ can be expressed as a weighted summation of dependent terms.

so:- $y(n) = -a_1 y(n-1) - a_2 y(n-2) - a_3 y(n-3) - \dots\dots + b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + b_3 x(n-3) + \dots$ → (C)

*) -ve constants are inserted for o/p signals because o/p signals are feed back from o/p to i/p.

*) +ve constants are inserted for i/p signals because i/p signals are ~~feedbacks~~ forward from i/p to o/p.

∴ Practically the response $y(n)$ at any time instant 'n', may depend on 'N' no:- of past o/p's, present i/p & 'M' no:- of past i/ps.

∴ Eq (C) is written as

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) - a_3 y(n-3) - \dots - a_N y(n-N) + b_0 x(n) + b_1 x(n-1) + \dots + b_M x(n-M)$$

$$\therefore y(n) = -\sum_{M=1}^{N} a_m y(n-M) + \sum_{m=0}^{M} b_m x(n-m) \quad //$$

In above Eq value of "N" defines order of system

If $N = 1$ the D.T System is 1st order

$N = 2$ " " " " 2nd "

$N = 3$ " " " " 3rd "

Note

(a) Natural Response (zero i/p Response)

The natural response $y_n(n)$ is the solution of Eq (2) below with $x(n) = 0$

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k \cdot x(n-k)$$

for a discrete time system the natural response is the solution of homogenous Eq.

$$\sum_{k=0}^{N} a_k \cdot y(n-k) = 0 \rightarrow (3)$$

it is in the form $y_h(n) = \lambda^n$

$$\sum_{k=0}^{N} a_k \cdot \lambda^{n-k} = 0 \quad ; a_0 = 1$$

$$\lambda^n + a_1 \lambda^{n-1} + \dots + a_{N-1} \cdot \lambda^{n-N+1} + a_N \cdot \lambda^{n-N} = 0 \quad (\text{where } k = N-1)$$

$$\lambda^{n-N}\left[\lambda^N + a_1 \lambda^{N-1} + \dots + a_{N-1} \cdot \lambda + a_N\right] = 0$$

$$\lambda^N + a_1 \cdot \lambda^{N-1} + \dots + a_{N-1} \cdot \lambda + a_N = 0 \rightarrow (4)$$

Eq (4) is characteristic Eq of system.

Eq (4) Can be Expressed in factorized form as

$$(\lambda-\lambda_1).(\lambda-\lambda_2)\dots(\lambda-\lambda_N)=0$$

$\lambda_1, \lambda_2, \lambda_3 \dots \lambda_N$ are roots of Eq (4). The roots of Characteristic Equation are Called Characteristics roots (or) Eigen values of System.

The nature of natural response depends on type of roots.

Real roots → real Exponential
imaginary roots → Sinusoidal
Complex roots → Exponentially damped sinusoidal.

(i) Distinct roots :-

If the roots $\lambda_1, \lambda_2 \dots \lambda_N$ of Eq (4) are distinct then it has N-solutions $C_1\lambda_1^n, C_2\lambda_2^n, \dots C_N\lambda_N^n$

the general solution is of the form

$$y_h(n) = C_1\lambda_1^n + C_2\lambda_2^n + \dots + C_N.\lambda_N^n$$

where $C_1, C_2, \dots C_N$ are arbitary Constants

Eg:- if $\lambda_1 = 2$; $\lambda_2 = 3$

$$y_h(n) = C_1(2)^n + C_2(3)^n$$

(ii) Repeated roots :-

If the roots λ_1 is repeated m times and the remaining (N-m) roots are distinct then, the characteristic Equation of System is.

$$(\lambda-\lambda_1)^m (\lambda-\lambda_{m+1})\cdot(\lambda-\lambda_{m+2}) \cdots (\lambda-\lambda_N) = 0$$

and general solution is

$$y_h(n) = (c_1 + c_2 \cdot n + c_3 \cdot n^2 + \cdots + c_m \cdot n^{m-1}) \cdot (\lambda_1^n) + c_{m+1}\cdot(\lambda_{m+1})^n + c_{m+2}\cdot(\lambda_{m+2})^n + \cdots + c_N \cdot \lambda_N^n.$$

Eg:-

The roots of characteristic Eq $\lambda_1 = 1$; $\lambda_2 = 1$ & $\lambda_3 = 2$

$$y_h(n) = \{c_1 + c_2 n\}\cdot(1)^n + c_3(2)^n.$$

(iii) Complex roots:-

if roots are complex $\lambda_1 = \lambda = a + jb$

$\lambda_2 = \lambda^* = a - jb$

$$y_h(n) = r^n\left[A_1 \cos n\theta + A_2 \sin n\theta\right]$$

$r = \sqrt{a^2+b^2}$; $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

A_1 & A_2 are constants.

(b) Forced Response (zero state response):-

The forced response is the solution of diff. Eq (DE) for the given i/p, when the initial conditions are zero.

It consists of two parts, homogeneous solution & Particular Solu.

* The homogeneous solution can be obtained from the roots of characteristics Equation.
* The Particular Solution $y_p(n)$ is to satisfy the d.e. for specific i/p signal $x(n)$. $n \geq 0$

$$1 + \sum_{k=1}^{N} a_k\, y_p(n-k) = \sum_{k=0}^{M} b_k\, x(n-k) ..$$

The general form of the Particular solution for several i/p are given in table. From table we can find $x(n) = A \sin \omega n$ then

$$y_p(n) = C_1 \cos \omega n + C_2 \sin \omega n.$$

where C_1 & C_2 are obtained by substituting $y_p(n)$ and $x(n)$ in d.E.

$x(n)$ i/p Signal	$y_p(n)$ Particular Solution
A (Step input)	K
AM^n	$K.M^n$
An^M	$K_0 n^M + K_1 . n^{M-1} + \ldots + K_M$
$A^n . N^M$	$A^n [K_0 . n^M + K_1 n^{M-1} + \ldots + K_M]$
$A \cos \omega n$, $A \sin \omega n$	$C_1 \cos \omega n + C_2 \sin \omega n.$

If the i/p applied to the system and one of the components of homogenous solution are same; then multiply the particular solution by the lower power of 'n' that will give a response component not included in homogenous solution.

ⓒ **Total response** :- It is obtained by adding natural + forced responses

$$y(n) = y_h(n) + y_p(n)$$

ⓓ **Impulse response** :-

The general form of d.E of N^{th} order system is given

$$1 + \sum_{k=1}^{N} a_k . y(n-k) = \sum_{k=0}^{M} b_k \, x(n-k) \quad N > M$$

for the i/p $x(n) = \delta(n)$ obtained as

o/p of unit step impulse

$$1 + \sum_{k=1}^{N} a_k\, y(n-k) = \sum_{k=0}^{M} b_k \cdot \delta(n-k)$$

for $N > M$

$$1 + \sum_{k=1}^{N} a_k\, y(n-k) = 0$$

$$\sum_{k=0}^{N} a_k \cdot y(n-k) = 0$$

if $N = M$ we have to add an impulse functi to homogeneous solution.

19) frequency Response analysis of Discrete time System:-

The o/p of any linear time invariant system to an i/p signal $x(n)$ can be obtained using convolution sum

$$y(n) = \sum_{k=-\infty}^{\infty} h(k) \cdot x(n-k).$$

$h(n)$ is impulse response of the system.

Let us consider a complex exponential signal $x(n)$ as i/p to system.

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, e^{j\omega(n-k)}$$

$$= e^{j\omega n}\left[\sum_{k=-\infty}^{\infty} h(k) \cdot e^{-j\omega k}\right]$$

$$e^{j\omega n}\left[H \cdot (e^{j\omega})\right] \rightarrow (1)$$

$e^{j\omega n}$ → i/p ; $H(e^{j\omega})$ → freq response.

where

$$\left[H \cdot (e^{j\omega})\right] = \sum_{k=-\infty}^{\infty} h(k) \cdot e^{-j\omega k} \rightarrow (2)$$

The function $H(e^{jw})$ is exits if the impulse response is absolutely summable.

(*) from eq (1) we say that for i/p $x(n) = e^{jwn}$ the o/p of an LTI DT system is also exponential signal of the same frequency, multiplied by factor $H(e^{jw})$.

(*) these type of signal that produce a response which differ from the i/p signal by a complex constant are known as Eigen functions.

from eq (2) we can find frequency response $H(e^{jw})$ of system is discrete time fourier transform of impulse response $h(n)$ of system.

∵ The frequency response $H(e^{jw})$ is a complex valued function 'it' can be expressed in polar form as

$$H(e^{jw}) = |H(e^{jw})| \cdot e^{j\theta\omega}$$

$|H(e^{jw})| \Rightarrow$ magnitude ; $\theta(\omega) = \angle H(e^{jw})$.

and phase response is an odd function of ω

i.e $\theta(\omega) = -\theta(-\omega)$.

$$\because \quad e^{j(\omega + 2k\pi)} = e^{j\omega} \cdot e^{j\omega 2k\pi} = e^{j\omega}$$

for an integer k the frequency response $H(e^{jw})$ is periodic with period 2π.

(a) Frequency response of first order system :-

The d.E for 1st order system is given by

$$y(n) - a \cdot y(n-1) = x(n).$$

$x(n)$ is i/p ; $y(n) = $ o/p

Taking Fourier transform on b.s

$$Y(e^{j\omega}) - a \cdot e^{-j\omega} \cdot Y(e^{j\omega}) = X(e^{j\omega})$$

$$Y(e^{j\omega})\left[1 - a \cdot e^{-j\omega}\right] = X(e^{j\omega}).$$

The freq response.

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 - a \cdot e^{-j\omega}}$$

the impulse response

$$h(n) = F^{-1}\left(H \cdot (e^{j\omega})\right)$$

$$= F^{-1}\left[\frac{1}{1 - a \cdot e^{-j\omega}}\right]$$

$$h(n) = a^n \cdot u(n) \qquad H(e^{j\omega}) = \frac{1}{1 - a \cdot e^{-j\omega}}$$

now:. $H \cdot (e^{j\omega})$ is called frequency response of the LTI system whose impulse response is $h(n)$

$$\therefore \quad H \cdot (e^{j\omega}) = \frac{1}{1 - a\left[\cos\omega - j\sin\omega\right]}$$

$$= \frac{1}{1 - a\cos\omega + aj\sin\omega}$$

$$|H \cdot (e^{j\omega})| = \frac{1}{\sqrt{(1 - a\cos\omega)^2 + a^2\sin^2\omega}}$$

$$= \frac{1}{\sqrt{1 + a^2\cos^2\omega - 2a\cos\omega + a^2\sin^2\omega}}$$

$$= \frac{1}{\sqrt{1 + a^2 - 2a\cos\omega}} = \frac{1}{\sqrt{1 + a^2 - 2a\cos\omega}}$$

$$\angle H \cdot e^{j\omega} = \tan^{-1}\left(\frac{a\sin\omega}{1 - a\cos\omega}\right) //.$$

if λ_1 and λ_2 are ~~real~~ complex.

if $y(n) = \lambda^n$ from d.e we find, from Eq ①

$$\lambda^2 - a_1\lambda - a_2 = 0$$

If roots are complex :-

$$\lambda_1 = \frac{a_1}{2} + j\sqrt{-\frac{a_1^2}{4} - a_2} = \alpha + j\beta = r.e^{j\theta} = r\cos\theta + jr\sin\theta$$

$$\lambda_2 = \frac{a_1}{2} - j\sqrt{-\frac{a_1^2}{4} - a_2} = \alpha - j\beta = r.e^{-j\theta} = r\cos\theta - jr\sin\theta.$$

where $\alpha = \frac{a_1}{2} = r\cos\theta$; $\beta = \sqrt{-\frac{a_1^2}{4} - a_2} = r\sin\theta$

$$\therefore \quad \alpha^2 + \beta^2 = r^2\cos^2\theta + r^2\sin^2\theta$$

$$r^2 = \frac{a_1^2}{4} + \left(-\frac{a_1^2}{4} - a_2\right) = -a_2$$

$$\Rightarrow \quad r = \sqrt{-a_2}.$$

if the initial condition are zero.

$$y(-1) = h(-1) = 0$$

from Eq ③

$$h(-1) = A.r^n \sin(-\theta + \phi) = 0$$

gives $\theta = \phi$

$$\therefore \quad y(n) = h(n) = A.r^n \sin(n\theta + \theta) \rightarrow ④$$

from Eq ①

$$y(0) = a_1 y(-1) + a_2 y(-2) + x(0) = 1 \rightarrow ⑤$$

$$h(0) = y(0) = A\sin\phi = 1$$

$$\Rightarrow \quad A = \frac{1}{\sin\phi} = \frac{1}{\sin\theta} \quad (\because \theta = \phi) \rightarrow ⑥$$

Now:-

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n) \cdot e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} A.\gamma^n \sin(n\theta + \theta) \cdot e^{-j\omega n}$$

$$= A. \sum_{n=0}^{\infty} \gamma^n \left(\frac{e^{j(n+1)\theta} - e^{-j(n+1)\theta}}{2j} \right) . e^{-j\omega n}$$

$$= \frac{A}{2j} \left\{ \sum_{n=0}^{\infty} \gamma^n . e^{j\theta} . e^{jn\theta} . e^{-j\omega n} - \sum_{n=0}^{\infty} \gamma^n . e^{-j\theta} . e^{-jn\theta} . e^{-j\omega n} \right\}$$

$$= \frac{A}{2j} \left\{ \frac{e^{j\theta}}{1-\gamma . e^{-j(\omega-\theta)}} - \frac{e^{-j\theta}}{1-\gamma e^{-j(\omega+\theta)}} \right\}$$

$$= \frac{A}{2j} \left[\frac{e^{j\theta} - \gamma . e^{-j\omega} . e^{-j\theta} . e^{j\theta} - e^{-j\theta} + \gamma . e^{-j\omega} . e^{j\theta} . e^{-j\theta}}{1 - \gamma . e^{-j\omega}(e^{j\theta} + e^{-j\theta}) + \gamma^2 . e^{-2j\omega}} \right]$$

$$= \frac{A}{2j} \left[\frac{e^{j\theta} - e^{-j\theta}}{1 - 2.\gamma . e^{-j\omega} \cos\theta + \gamma^2 . e^{-2j\omega}} \right] \rightarrow (\cancel{A})$$

$$= \frac{A}{2j} \left[\frac{\sin\theta}{1 - 2.\gamma^{-j\omega} . \cos\theta + \gamma^2 . e^{-2j\omega}} \right] \rightarrow (7)$$

Substitute Eq (6) in Eq (7)

$$= \frac{\cancel{A}}{2j} \left[\frac{\frac{1}{\cancel{A}}}{1 - 2.\gamma^{-2j\omega} . \cos\theta + \gamma^2 . e^{-2j\omega}} \right]$$

$$\Rightarrow \boxed{H\cdot\left(e^{jw}\right) = \frac{1}{1-2.r.e^{-jw}\cos\theta + r^2.e^{-2jw}}} \quad //.$$

*) Transfer function :-

If $H(e^{jw})$ is fourier Transform of impulse response $h(n)$ an $X(e^{jw})$ is F.T of i/p sequence $x(n)$, we can derive the relationship b/w $Y(e^{jw})$.

the fourier transform of o/p in terms $X(e^{jw})$. & $H(e^{jw})$

An arbitary sequence can be represented in the form

i/p $$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{jw}).e^{jwn}.dw \longrightarrow (1)$$

where $$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n).e^{-jwn}$$

$$y(n) = e^{jwn}.H(e^{jw}) \quad ; \; x(n) = e^{jwn}$$

o/p $$y(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{jw}).e^{jwn}.H(e^{jw}).dw \rightarrow (a)$$

W.k.T

$$y(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y.(e^{jw}).e^{jwn}.dw \rightarrow (b)$$

now eq (a) = (b) ($\because$ y(n) is common)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} Y.(e^{jw}).e^{jwn} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{jw}).e^{jwn}.H(e^{jw})$$

$$Y(e^{jw}) = X.(e^{jw}).H(e^{jw})$$

$$\Rightarrow H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} \quad //. \text{ Transfer function.}$$