UNIT - 3

The Fast Fourier Transform :-

1*) Introduction :-

The ~~DFT~~ Fast fourier Transform (FFT) is a method (or) Algorithm for Computing the discrete fourier transforms of finite sequence and requires less no:- of Computations than that of direct Evaluation of DFT.

It reduces the Computations by taking advantages of the fact that the Calculation of the Co-efficients of DFT Can be carried out iteratively.

By FFT Computation technique is used in Digital spectral Analysis, filter simulation, auto correlation and Pattern recognition.

The FFT is based on the decomputation and breaking the transform in to small transforms and Combining them to get the total transform.

This was Proposed in 1965 by Cooley & Tukey.

2*) Radix - r FFT :- In an N-Point sequence, if N can be ~~decimated~~ expressed expressed as $\boxed{N = r^m}$, then the sequence can be decimated in to r-point sequence.

$N = r^m$. Then sequence can be decimated in to r-point sequence. r-point DFT can be Computed.

*) For Each r-Point sequence, r-point DFT can be Computed.

*) From result of r-point DFT r^2-Point DFT's, the r^3-Point DFT's are Computed so on, untill we get r^m-Point.

*) This FFT algorithm is Called "radix - r" FFT.

3*) **Radix - 2 FFT :-**

for Radix-2 FFT the value of N should be such that $\boxed{N = 2^m}$, so that the N-point sequence is decimated in to 2-points sequences and 2-point DFT for each decimated sequence is computed.

From the results of 2-point DFT's, the 4-point DFT's can be computed. from the results of 4-points DFT's the 8-point DFT's can be computed. and so on untill N-point DFT.

No:- of Calculations in N-point DFT :-

the DFT of an N-length sequence $x(n)$ is given by

$$X[K] = \sum_{n=0}^{N-1} x(n) \cdot e^{-j\frac{2\pi}{N} \cdot kn} \quad ; K = 0, 1, 2 \ldots (N-1)$$

$$= x(0) \cdot e^{0} + x(1) \cdot e^{-j\frac{2\pi}{N} \cdot k} + x(2) \cdot e^{-j\frac{2\pi}{N} 2k} + \ldots x(N-1) \cdot e^{-j\frac{2\pi}{N}(N-1) \cdot k}$$

where $x(n)$ is complex valued N-length sequence.

4*) **Twiddle factor (or) Phase factor :-**

$$X(K) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j\frac{2\pi}{N} \cdot k \cdot n} \quad ; \forall K = 0 \rightarrow N-1$$

To simplify the notation it is desiarable to define the complex valued phase factor W_N (also called twiddle factor) as

$$\boxed{W_N = e^{-j\frac{2\pi}{N}}}$$

re writting the DFT eq

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn}} \quad k = 0, 1, 2 \ldots N-1$$

this eq is popullarly used in FFT.

*) Few important Points of twiddle factor:-

1. $W_N^k = W_N^{k+N}$

$$= \left[e^{-j\frac{2\pi}{N}}\right]^{k+N} = e^{-j\frac{2\pi}{N}\cdot k} \cdot e^{-j\frac{2\pi}{\cancel{N}}\cdot\cancel{N}}$$

$$= W_N^k.$$

2. $W_N^{k+\frac{N}{2}} = -W_N^k$

$$W_N^{k+\frac{N}{2}} = \left(e^{-j\frac{2\pi}{N}}\right)^{k+\frac{N}{2}} = e^{-j\frac{2\pi}{N}\cdot k} \cdot e^{-j\frac{2\pi}{\cancel{N}}\cdot\frac{\cancel{N}}{\cancel{2}}}$$

$$= e^{-j\frac{2\pi}{N}\cdot k}(-1)$$

$$= -e^{-j\frac{2\pi}{N}\cdot k}.$$

$$\boxed{W_N^{k+\frac{N}{2}} = -W_N^k}$$

3. $W_N^2 = W_{N/2}$

$$W_{\frac{N}{2}} = e^{-j\frac{2\pi}{N/2}} = e^{-j\frac{2\pi}{N}\cdot 2} = \left[e^{-j\frac{2\pi}{N}}\right]^2$$

$$\boxed{W_{N/2} = W_N^2} \;//.$$

5) **Decimation in time algorithm:-** (DIT Radix-2 FFT)

In this algorithm, the time domain sequence $x(n)$ is decimated and smaller point DFT's are performed. The result of smaller point DFT's are combined to get the result of N-point DFT.

In this algorithm the no: of o/p points 'N' can be expressed as a power of 2 that is $N = 2^m$ where m is integer.

Let $x(n)$ is an N-point sequence, where N is assumed to be a power of 2. Decimate (or) break this sequence in to 2-sequences of length of $N/2$ where one sequence consisting of the even indexed values of $x(n)$ and other of odd indexed values of $x(n)$.

i.e
$$\boxed{\begin{array}{ll} x_e(n) = x(2n) & n = 0, 1, \ldots \frac{N}{2}-1 \\ x_o(n) = x(2n+1) & n = 0, 1, \ldots N/2 - 1 \end{array}}$$

By Defination of DFT, the N-point DFT of $x(n)$ is given by

$$X[K] = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn} \quad \forall\ k = 0 \to N-1$$

Seperating $x(n)$ in to Even & odd index values of $x(n)$

$$\boxed{X(k) = \sum_{\substack{n=0 \\ (\text{Even})}}^{N-1} x(n) \cdot W_N^{nk} + \sum_{\substack{n=0 \\ (\text{odd})}}^{N-1} x(n) \cdot W_N^{nk}}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(2n) \cdot W_N^{k(2n)} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) \cdot W_N^{k(2n+1)}$$

$$= \sum_{n=0}^{N/2-1} x(2n) \cdot W_N^{k2n} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) \cdot W_N^{k2n} \cdot W_N^{K}$$

The phase factors can be modified as

$$W_N^{k(2n)} = e^{-j\frac{2\pi}{N} \cdot 2kn} = e^{-j\frac{2\pi}{N/2} kn} = W_{N/2}^{kn}$$

$$W_N^{k(2n+1)} = e^{-j\frac{2\pi}{N}k(2n+1)} = e^{-j\frac{2\pi}{N}\cdot k} \cdot e^{-j\frac{2\pi}{N}\cdot k2n}$$

$$= e^{-j\frac{2\pi}{N}\cdot k} \cdot e^{-j\frac{2\pi}{N/2}\cdot kn} \Rightarrow W_N^k \cdot W_{N/2}^{kn}$$

substituting the above relation in the DFT.

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(2n)\cdot W_{N/2}^{kn} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1)\cdot W_{N/2}^{kn}\cdot W_N^k$$

$$X[k] = X_e(k) + W_N^k X_o(k) \rightarrow (1) \quad \forall\ k = 0 \to \frac{N}{2}-1$$

for $k \geq N/2$

W_N^k can be written as

$$W_N^{k+N/2} = W_N^{k-N/2} = -W_N^k \quad \text{(from twiddle points)}$$

$$\Rightarrow e^{-j\frac{2\pi(k+N/2)}{N}} = e^{-j\frac{2\pi(k-N/2)}{N}}$$

$$\Rightarrow e^{-j\frac{2\pi k}{N}} \cdot e^{-j\frac{2\pi N}{2N}} = e^{-j\frac{2\pi k}{N}} \cdot e^{-j2\pi N/2\cdot N}$$

$$\Rightarrow W_N^k \cdot e^{-j\pi} = W_N^k \cdot e^{j\pi}$$

$$W_N^k\{\cos\pi - j\sin\pi\} = W_N^k\cdot\{\cos\pi - j\sin\pi\}$$

$$W_N^k(-1) = W_N^k(-1) \Rightarrow -W_N^k$$

$\therefore$ $X(k)$ can be written as

$$X(k) = X_e\left(k-\frac{N}{2}\right) + W_N^k X_o\left(k-\frac{N}{2}\right)$$

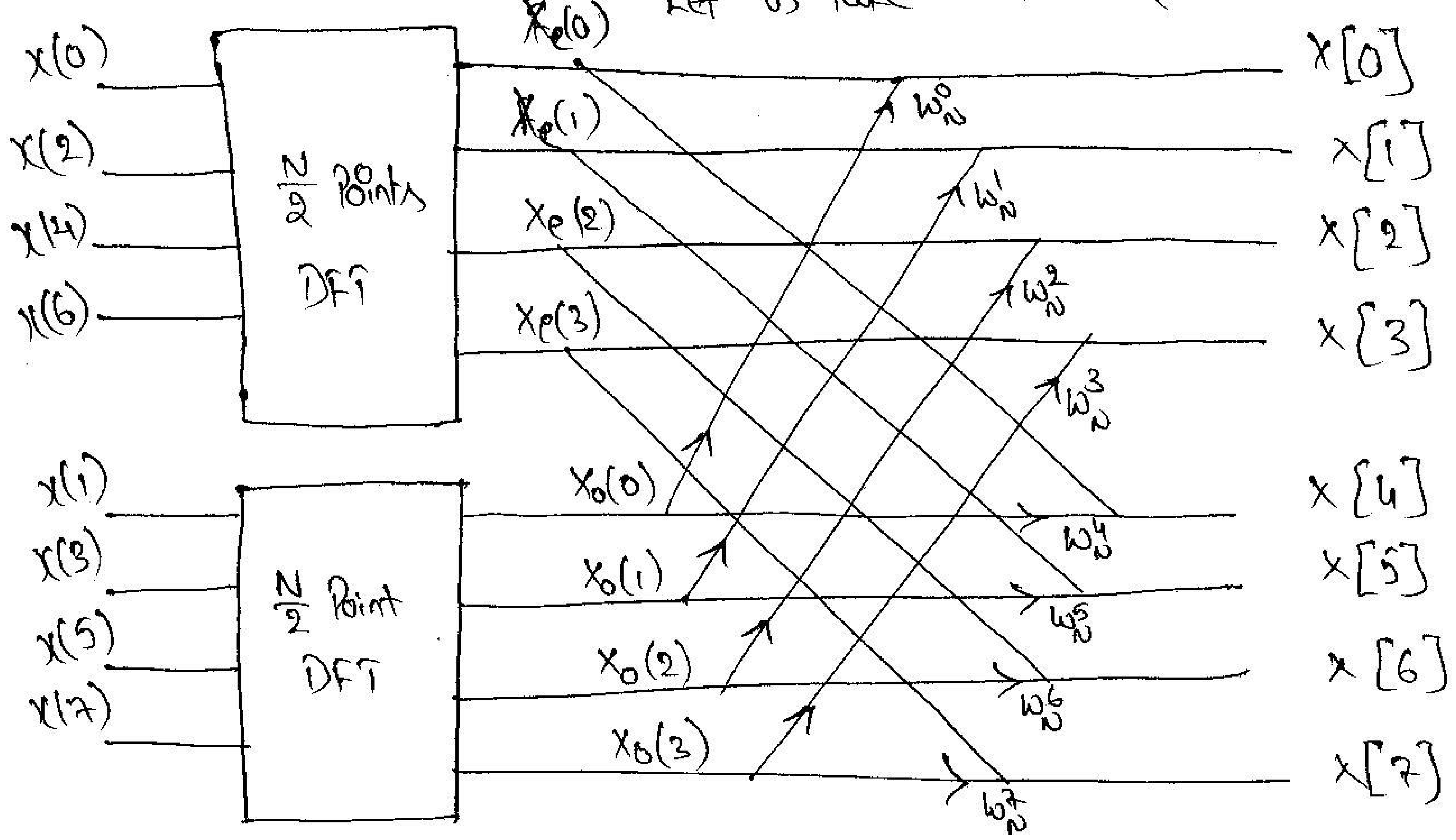
$$= X_e\left(k-\frac{N}{2}\right) - W_N^{k-\frac{N}{2}} X_o\left(k-\frac{N}{2}\right) \rightarrow (2)$$

for $k = \frac{N}{2}, \frac{N}{2}+1 \ldots N-1$

from Eq (1) :-

$$\boxed{X[K] = X_e(K) + W_N^K X_o(K) \quad \forall \; K = 0 \to N-1}$$

the above Eq can be written as
Let us take N = 8 Point $\left(\therefore \frac{N}{2} \text{ is} \Rightarrow \frac{8}{2} = 4 \text{ Point}\right)$



When $N = 8$;

for $K = 0 \quad X[0] = X_e(0) + W_N^0 X_o(0)$

$K = 1 \quad X[1] = X_e(1) + W_N^1 X_o(1)$

$K = 2 \quad X[2] = X_e(2) + W_N^2 X_o(2)$

$K = 3 \quad X[3] = X_e(3) + W_N^3 X_o(3)$

$\vdots \qquad \vdots \qquad \vdots$

$K = 7 \quad X[7] = X_e(7) + W_N^3 X_o(7)$

To find no:- of Complex multiplication & Complex addition required to Compute Eq (1).

for direct Evaluation of DFT we know that no:- of Complex multiplication required is Equal to N^2. Same way to Calculate $\left(\frac{N}{2}\right)$ Points we require $\left(\frac{N}{2}\right)^2$ Complex multiplication & to Compute

$X_o(k)$ we need another $\left(\frac{N}{2}\right)^2$ Complex multiplications.

ie we require $2\left(\frac{N}{2}\right)^2$ Complex multiplications.

$\therefore$ Total no:- of Complex multiplication required is Computing $X(k)$ is

$$\boxed{\left(\frac{N}{2}\right)^2 + \left(\frac{N}{2}\right)^2 + N = N + \frac{N^2}{2}}$$

Simillarly total no:- of complex addition required is

$$\left(\frac{N}{2}\right)\left(\frac{N}{2}-1\right) + \frac{N}{2}\left(\frac{N}{2}-1\right) + N = \frac{N^2}{2} \text{ ''}$$

Eg:- Now let us take $N=8$ then $X_e(k)$ & $X_o(k)$ are 4-Points $\left(\frac{N}{2}\right)$ DFTs of Even index sequence $x_e(n)$ and odd index sequence $x_o(n)$.

where $x_e(0) = x(0)$; $x_o(0) = x(1)$

$x_e(1) = x(2)$; $x_o(1) = x(3)$

$x_e(2) = x(4)$; $x_o(2) = x(5)$

$x_e(3) = x(6)$; $x_o(3) = x(7)$

Eq (1) & (2) Can be written as. $(N=8)$

from eq (1) $X[k] = X_e(k) + W_8^k X_o(k)$ for $k = 0 \to \frac{N}{2}-1$
$= 0 \to 3$

from eq (2)

$\therefore X(k) = X_e(k-4) - W_N^{k-4} X_o(k-4)$ for $4 \le k \le 7$

$\therefore X(0) = X_e(0) + W_8^0 X_o(0)$; $X(4) = X_e(0) - W_8^0 X_o(0)$

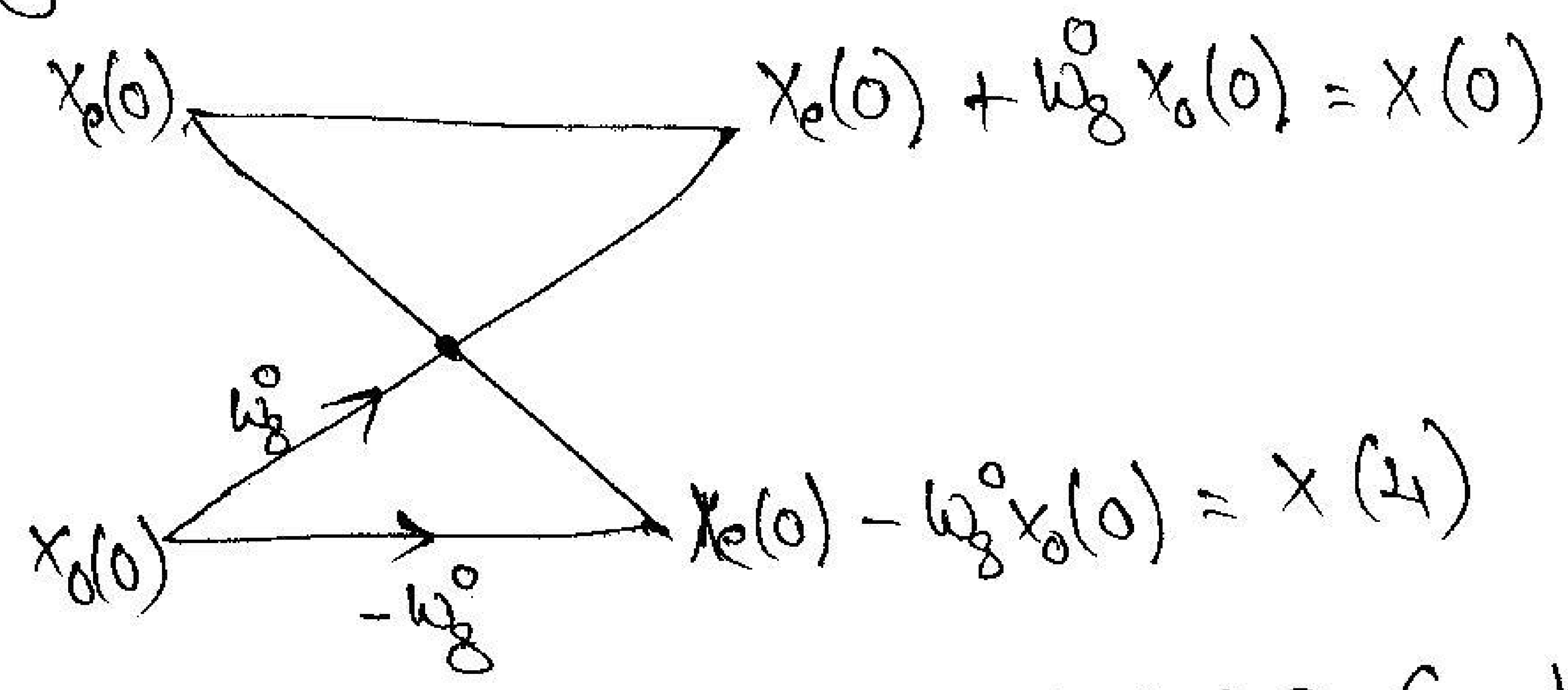
$X(1) = X_e(1) + W_8^1 X_o(1)$; $X(5) = X_e(1) - W_8^1 X_o(1)$

$X(2) = X_e(2) + W_8^2 X_o(2)$; $X(6) = X_e(2) - W_8^2 X_o(2)$

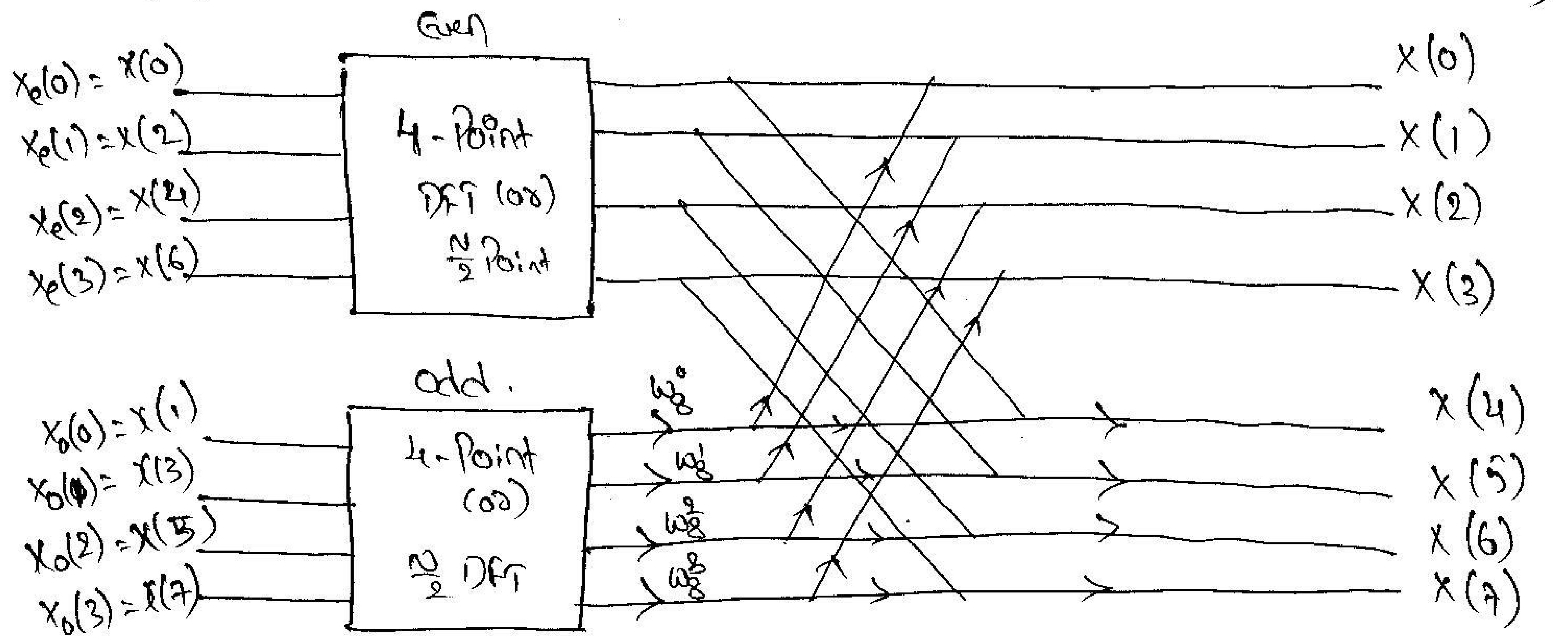
$X(3) = X_e(3) + W_8^2 X_o(3)$; $X(7) = X_e(3) - W_8^3 X_o(3)$

from Eq (1)   from Eq (2)

Butterfly diagram:-



now the values $X(k)$ for $k = 0, 1, 2, 3, 4, 5, 6, 7$ can be explained. and an 8-point DFT flow graph can be constructed from two 4-Point DFT



Now we can apply same approach to decompose Each of $\frac{N}{2}$ sample DFT

*) this can be done by dividing $x_e(n)$ & $x_o(n)$ into two sequence

$$x_e(n) = x(2n) \quad ; \quad x_o(n) = x(2n+1)$$

$$\therefore x_{ee}(n) = x(4n) \quad ; \quad x_{eo}(n) = x(4n+2)$$

now let $N = 8$

we have

Even	Odd
0, 2, 4, 6	1, 3, 5, 7

divide in to two ← 0, 2, 4, 6

ee   eo   oe   oo

$$\therefore X_e(k) = \sum_{n=0}^{N/4-1} x(4n) \cdot W_N^{k4n} + \sum_{n=0}^{\frac{N}{4}-1} x(4n+2) \cdot W_N^{k(4n+2)}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} x(4n) \cdot W_{N/4}^{kn} + \sum_{n=0}^{\frac{N}{4}-1} x_{eo}(4n+2) \cdot W_{\frac{N}{4}}^{kn} \cdot W_N^{2k}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} x_{ee}(n) \cdot W_{\frac{N}{4}}^{kn} + \sum_{n=0}^{\frac{N}{4}-1} x_{eo}(n) \cdot W_{\frac{N}{4}}^{kn} \cdot W_N^{2K}$$

$$X_e(K) = X_{ee}(K) + X_{eo}(K) \cdot W_N^{2k}$$

$$\Rightarrow \boxed{X_e(K) = X_{ee}(K) + W_N^{2K} \cdot X_{eo}(K)} \longrightarrow (3) \quad \text{for } 0 \le K \le \frac{N}{4}-1$$

for $K \ge \frac{N}{4}$

$$X_{ee}\left(K - \frac{N}{4}\right) - W_N^{2\left(K-\frac{N}{4}\right)} \cdot X_{oe}\left(K - \frac{N}{4}\right) \quad \text{for } \frac{N}{4} \le K \le \frac{N}{2}-1$$

$X_{ee}(K)$ is the $\frac{N}{4}$ Point DFT of even members of $x_e(n)$ & $X_{eo}(K)$ is the $\frac{N}{2}$ Point DFT of odd members of $x_e(n)$.

in same way $x_{oe}(n) = x(4n+1)$; $x_{oo}(n) = x(4n+3)$

$$X_o(K) = X_{oe}(K) + W_N^{2k} X_{oo}(K) \quad \text{for } 0 \le K \le \frac{N}{4}-1$$

$$= X_{oe}\left(K - \frac{N}{4}\right) - W_N^{2\left(K-\frac{N}{4}\right)} X_{oo}\left(K - \frac{N}{4}\right) \quad \text{for } \frac{N}{4} \le K \le \frac{N}{2}-1$$

where $X_{oe}(k)$ is $\frac{N}{4}$ Point DFT of even members of $x_o(n)$ & $X_{oo}(k)$ is $\frac{N}{4}$ Point of DFT of the odd member of $x_o(n)$

for $N = 8$

$x_{ee}(0) = x(0) = x_e(0)$; $x_{eo}(0) = x(2) = x_e(1)$

$x_{ee}(1) = x(4) = x_e(2)$; $x_{eo}(1) = x(6) = x_e(3)$

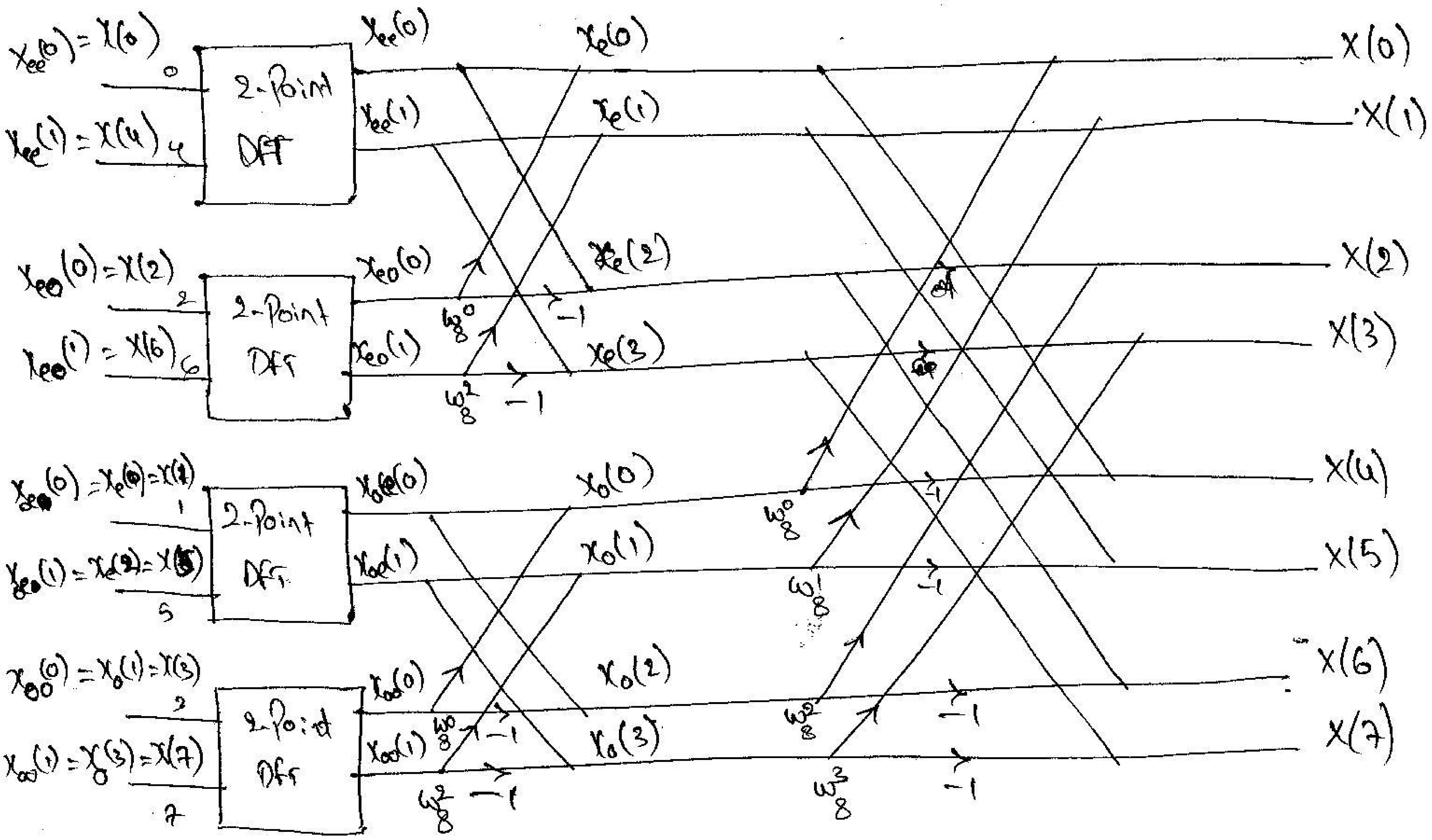
from Eq 3

$$X_e(0) = X_{ee}(0) + W_8^0 X_{eo}(0)$$
$$X_e(1) = X_{ee}(1) + W_8^2 X_{eo}(1)$$
$$X_e(2) = X_{ee}(0) - W_8^0 X_{eo}(0)$$
$$X_e(3) = X_{ee}(1) - W_8^2 X_{eo}(1)$$

similarly for the sequence $x_o(n)$

$$x_{oe}(0) = x(0) = x_o(0) \qquad x_{oo}(0) = x_o(1)$$
$$x_{oe}(1) = x(5) = x_o(2) \qquad x_{oo}(1) = x_o(3)$$

$$X_o(0) = X_{oe}(0) + W_8^0 X_{oo}(0)$$
$$X_o(1) = X_{oe}(1) + W_8^2 X_{oo}(1)$$
$$X_o(2) = X_{oe}(0) - W_8^0 X_{oo}(0)$$
$$X_o(3) = X_{oe}(1) - W_8^2 X_{oo}(1)$$



where $\Rightarrow X_{ee}(0) = X(0) + X(4)$ $\quad X_{oe}(0) = X(1) + X(5)$

$X_{ee}(1) = X(0) - X(4)$ $\quad X_{oe}(1) = X(1) - X(5)$

$X_{eo}(0) = X(2) + X(6)$ $\quad X_{oo}(0) = X(3) + X(7)$

$X_{eo}(1) = X(2) - X(6)$ $\quad X_{oo}(1) = X(3) - X(7)$ //.

(i) Bit Reversal:-

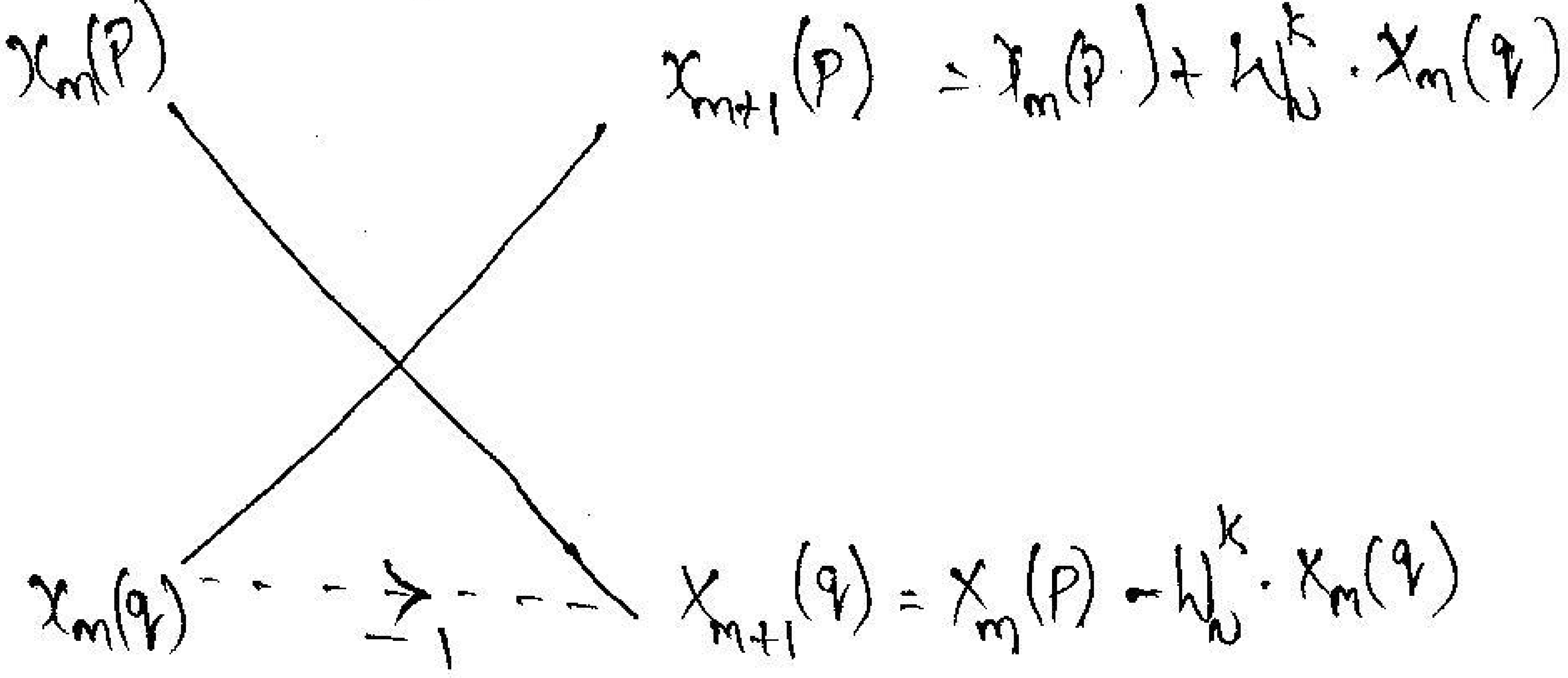
In DFT algorithm we can find that for the o/p sequence to be in a natural order i.e $X(K)$; $K = 0, 1, \dots, N-1$ where as i/p sequence is shuffled.

for $N = 8$ the DIT algorithm i/p sequence is $x(0), x(4), x(2), x(6), x(1), x(5), x(3), x(7)$.

We can see that N is power of '2' the i/p sequence must be stored in bit reversal order for the o/p to be computed natural.

i/p sample index	Bit representation	Bit reversed Binary	Bit Reversed sample index
0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
4	100	001	1
5	101	101	5
6	110	011	3
7	111	111	7. //

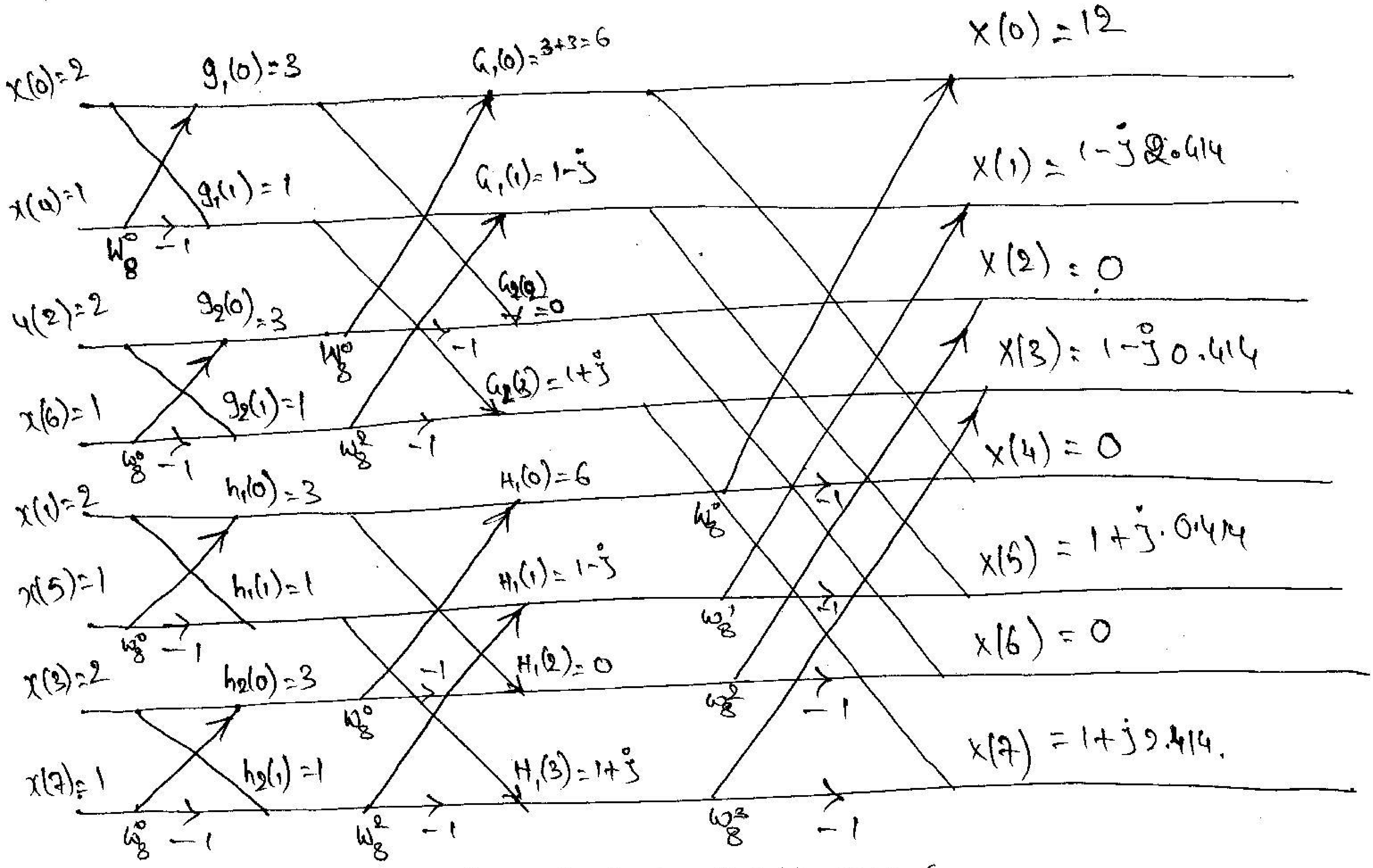
ii) Flow graph of Basic Butterfly diagram for DIT - Algorithm



Eg:-

$$x(n) = \{\underset{0}{2}, \underset{1}{2}, \underset{2}{2}, \underset{3}{2}, \underset{4}{1}, \underset{5}{1}, \underset{6}{1}, \underset{7}{1}\}$$

; $x(0)=2$; $x(2)=2$; $x(4)=1$; $x(6)=1$ | $x(1)=2$; $x(3)=2$; $x(5)=1$; $x(7)=1$.



$g_1(0) = x(0) + x(4) = 2+1 = 3$

$g_1(1) = x(0) - x(4) = 2-1 = 1$

$g_2(0) = x(2) + x(6) = 3$

$g_2(1) = x(2) - x(6) = 1$

$h_1(0) = x(1) + x(5) = 3$

$h_1(1) = x(1) - x(5) = 1$

$h_2(0) = x(3) + x(7) = 3$

$h_2(1) = x(3) - x(7) = 1$

$G_1(0) = g_1(0) + W_8^0 \cdot g_2(0) = 3+1\cdot 3 = 6$

$G_2(2) = g_1(0) - W_8^0\, g_2(0) = 3-1\cdot 3 = 0$

$G_1(1) = g_2(1) + W_8^2 \cdot g_1(1) = 1+(-j)\cdot 1 = 1-j$

$G_2(3) = g_2(1) - W_8^2\, g_1(1) = 1-(-j)\cdot 1 = 1+j$

$H_1(0) = h_2(0) + W_8^0 \cdot h_1(0) = 3+1\cdot 3 = 6$

$H_1(2) = h_2(0) - W_8^0 \cdot h_1(0) = 3-1\cdot 3 = 0$

$H_1(1) = h_2(1) + W_8^2 \cdot h_1(1) = 1+(-j)\cdot 1 = 1-j$

$H_2(3) = h_2(1) - W_8^0 \cdot h_1(1) = 1+j$

6*) Decimation FFT Algorithm:-

In decimation in frequency algorithm the frequency domain sequence $X(K)$ is decimated.

In this algorithm, the N-Point time domain sequence is converted to two no:- of $\frac{N}{2}$ Points sequences.

Then Each $\frac{N}{2}$ Point sequence is converted in two no:- of $\frac{N}{4}$ Point sequences. Thus we get 4 no's of $\frac{N}{4}$ Point sequences. this Process is continued until we get $\frac{N}{2}$ no's of 2-point sequences finally the 2-point DFT of each 2-point sequence is computed here the Equations for forming $\frac{N}{2}$ Point, $\frac{N}{4}$ Points sequence Etc., are obtained by decimation of frequency.

By defination of DFT the N-Point DFT of $x(n)$ is

$$\cancel{X(K) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn} = \sum_{n=N/2}^{N-1} x(n) \cdot W_N^{kn}}$$

$$X(K) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn} = \sum_{n=0}^{\frac{N}{2}-1} x(n) \cdot W_N^{kn} + \sum_{n=\frac{N}{2}}^{N-1} x(n) \cdot W_N^{kn}$$

$$X(k) = \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x(n) \cdot W_N^{kn}}_{\text{I}} + \underbrace{\sum_{n=N/2}^{N-1} x(n) \cdot W_N^{kn}}_{\text{II}}$$

in IInd summation for $n = \frac{N}{2} + n$

for lower limit $n = \frac{N}{2}$

$$\frac{N}{2} = \frac{N}{2} + n \Rightarrow n = 0$$

for $n = N-1$

$$N-1 = \frac{N}{2} + n$$

$$n = N - \frac{N}{2} - 1$$

$$n = \frac{N}{2} - 1$$

$$\therefore \sum_{n=0}^{\frac{N}{2}-1} x(n)\cdot W_N^{kn} + \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right)\cdot W_N^{k\left(n+\frac{N}{2}\right)}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n)\cdot W_N^{kn} + \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right)\cdot W_N^{kn}\cdot W_N^{kN/2}$$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n)\cdot W_N^{kn} + (-1)^k\cdot\sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right)\cdot W_N^{kn}$$

$$\left[W_N^{kN/2} = e^{-j\frac{2\pi}{N}\cdot\frac{N}{2}\cdot k}\right] = e^{-j\pi k} = (-1)^k$$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1}\left[x(n) + (-1)^k\cdot x\left(n+\frac{N}{2}\right)\right]\cdot W_N^{kn}$$

<u>k = Even.</u>

$$X(2k) = X(k) = \sum_{n=0}^{\frac{N}{2}-1}\left[x(n) + x\left(n+\frac{N}{2}\right)\right]\cdot W_N^{2kn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1}\underbrace{\left[x(n) + x\left(n+\frac{N}{2}\right)\right]}_{g_1(n)}\cdot W_{N/2}^{k}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} g_1(n)\cdot W_{N/2}^{k}$$

$$X(2k) = G_1(k)$$

<u>When k = odd</u>

$$X(2k+1) = X(k) = \sum_{n=0}^{\frac{N}{2}-1}\left[x(n) + (-1)^{2k+1}\cdot x\left(n+\frac{N}{2}\right)\right]\cdot W_N^{(2k+1)\cdot n}$$

$$= \sum_{n=0}^{\frac{N}{2}-1}\left[x(n) - x\left(n+\frac{N}{2}\right)\right]\cdot W_N^{2kn}\cdot W_{N/2}^{n}$$

$$= \sum_{n=0}^{\frac{N}{2}-1}\underbrace{\left[x(n) - x\left(n+\frac{N}{2}\right)\right]}_{g_2(n)}\cdot W_N^{n}\cdot W_{N/2}^{kn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} g_2(n)\cdot W_N^{n}\cdot W_{N/2}^{kn}$$

$$\therefore X(2k+1) = G_2(k).$$

$\therefore\ g_1(n) = x(n) + x\left(n+\frac{N}{2}\right)$; $g_2(n) = \left[x(n) - x\left(n+\frac{N}{2}\right)\right] \cdot W_N^n$

$N = 8$

for $n = 0$

$g_1(0) = x(0) + x(0+4)$
$= x(0) + x(4)$

for $n = 1$

$g_1(1) = x(1) + x(5)$

$g_1(2) = x(2) + x(6)$

$g_1(3) = x(3) + x(7)$

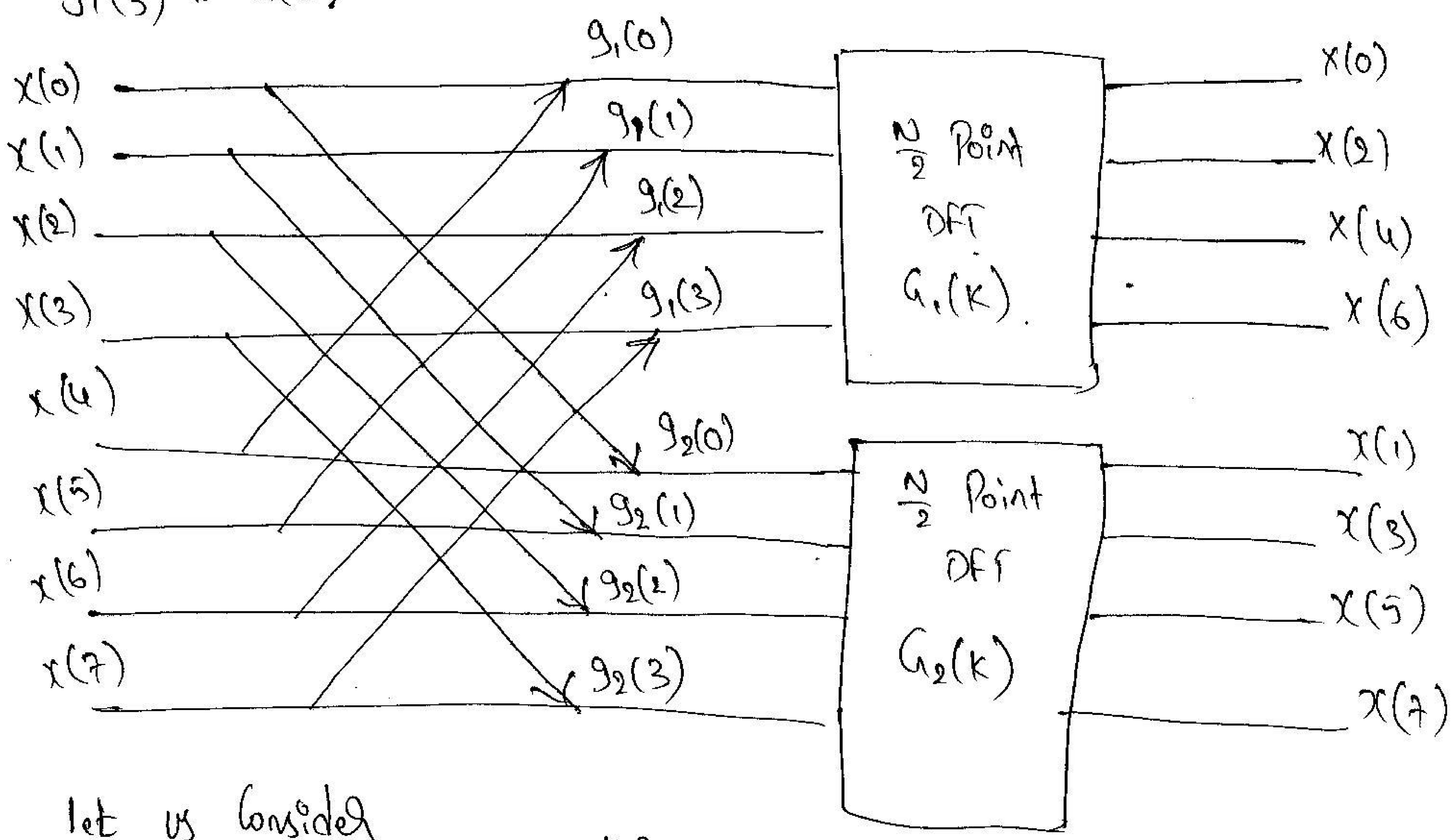
$n = 0$

$g_2(0) = [x(0) - x(4)] \cdot W_8^0$

$g_2(1) = [x(1) - x(5)] \cdot W_8^1$

$g_2(2) = [x(2) - x(6)] \cdot W_8^2$

$g_2(3) = [x(3) - x(7)] \cdot W_8^3$



let us consider

$$G_1(k) = \sum_{n=0}^{\frac{N}{2}-1} g_1(n) \cdot W_{N/2}^{k \cdot n}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} g_1(n) \cdot W_{N/2}^{kn} + \sum_{n=\frac{N}{4}}^{\frac{N}{2}-1} g_1(n) \cdot W_{N/2}^{kn}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} g_1(n) \cdot W_{N/2}^{kn} + \sum_{n=0}^{\frac{N}{4}-1} g_1\left(n+\frac{N}{4}\right) \cdot W_{N/2}^{k\left(n+\frac{N}{4}\right)}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} g_1(n) \cdot W_{N/2}^{kn} + \sum_{n=0}^{\frac{N}{4}-1} g_1\left(n+\frac{N}{4}\right) \cdot W_{N/2}^{kn} \cdot W_{N/2}^{k \cdot \frac{N}{4}}$$

$$\left(\text{where } W_{N/2}^{k \cdot \frac{N}{4}} = e^{\frac{-\cancel{2\pi} k \cancel{N/4} \times 2}{N/2}} = e^{-j\pi k} = (-1)^k\right)$$

$$G_1(k) = \sum_{n=0}^{\frac{N}{4}-1} g_1(n) \cdot W_{N/2}^{kn} + (-1)^k \cdot \sum_{n=0}^{\frac{N}{4}-1} g\left(n+\frac{N}{4}\right) \cdot W_{N/2}^{kn}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} \left[g_1(n) + (-1)^k \cdot g_1(n) + \frac{N}{4} \right] \cdot W_{N/2}^{kn}$$

When $\underline{k = \text{even}}$

$$G_1(2k) = G_1(k) = \sum_{n=0}^{\frac{N}{4}-1} \left[g_1(n) + g_1\left(n+\frac{N}{4}\right) \right] \cdot W_{N/2}^{2kn}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} \underbrace{\left\{ g_1(n) + g_1\left(n+\frac{N}{4}\right) \right]}_{P_1(n)} \cdot W_{N/4}^{kn}$$

$$G_1(2k) = \sum_{n=0}^{\frac{N}{4}-1} P_1(n) \cdot W_{N/4}^{kn} = P_1(k)$$

When $\underline{k = \text{odd}}$

$$G_1(2k+1) = G_1(k) = \sum_{n=0}^{\frac{N}{4}-1} \left[g_1(n) + (-1)^{2k+1} g_1\left(N/4+n\right) \right] \cdot W_{N/2}^{(2k+1)\cdot n}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} \left[g_1(n) - g_1\left(N/4+n\right) \right] \cdot W_{N/2}^{2k} \cdot W_{N/2}^{n}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} \underbrace{\left\{ g_1(n) - g_1\left(n+\frac{N}{4}\right) \right]}_{P_2} \cdot W_{N/2}^{n} \cdot W_{N/4}^{k}$$

$$G_1(2k+1) = \sum_{n=0}^{\frac{N}{4}-1} P_2(n) \cdot W_{N/4}^{k.n}$$

$G_1(2k+1) = P_2(k)$ [Do the same for $G_2(k)$]

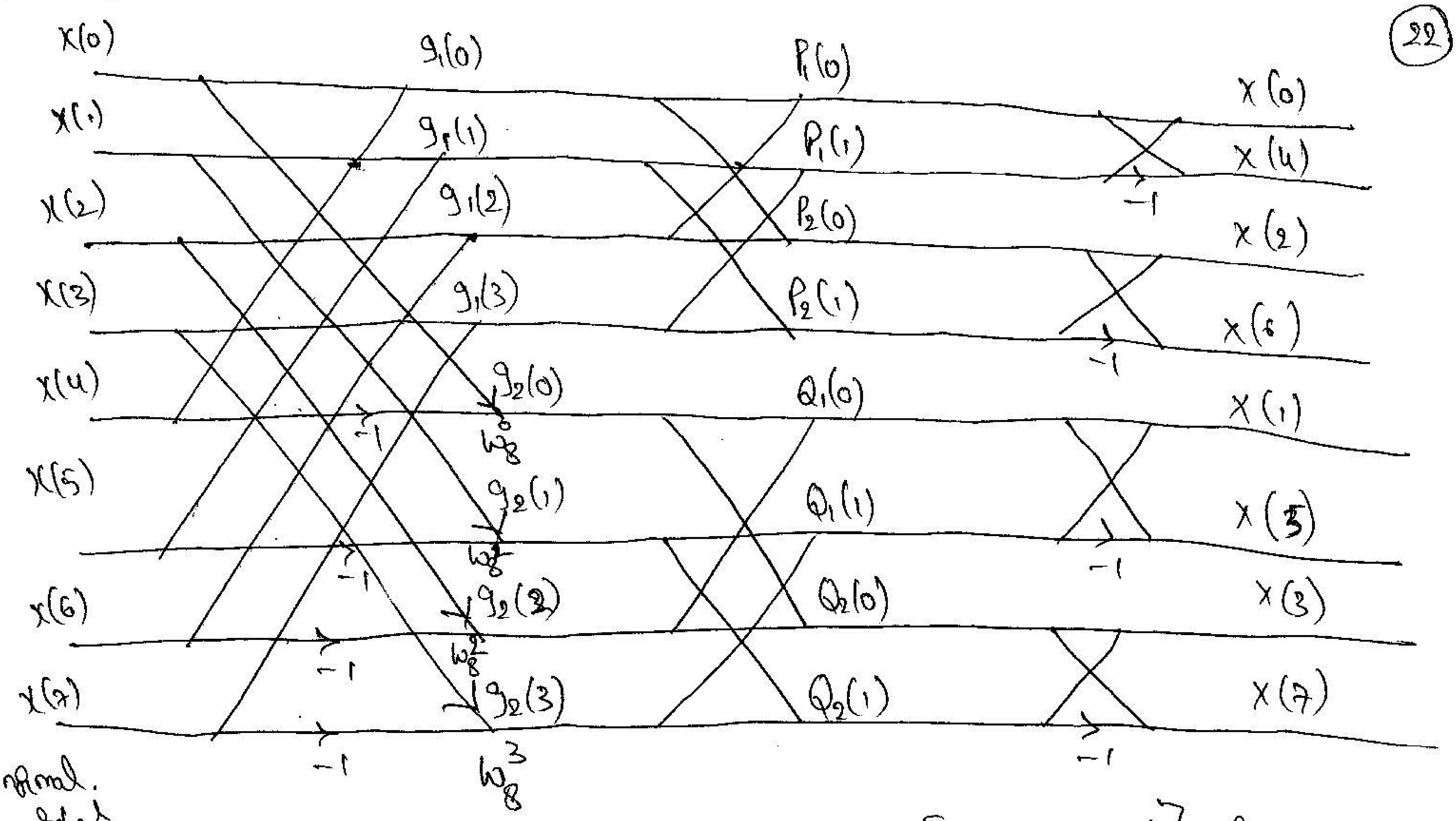
$$P_1(n) = g_1(n) + g_1\left(n+\frac{N}{4}\right)$$

for $n = 0$; $N = 8$

$$\left.\begin{aligned} P_1(0) &= g_1(0) + g_1(2) \\ g_1(1) &= g_1(1) + g_1(3) \end{aligned}\right\}$$

for $n=0$; $N=8$

$$\left.\begin{aligned} P_2(0) &= [g_1(0) - g_1(2)] \cdot W_4^0 \\ P_2(1) &= \{g_1(1) - g_1(3)\ W_4^1 \end{aligned}\right\}$$



$G_2(k)$

$Q_1(0) = g_2(0) + g_2(2)$; $Q_2(0) = [g_2(0) - g_2(2)]\, W_8^0$

$Q_1(1) = g_2(1) + g_2(3)$; $Q_2(1) = [g_2(1) - g_2(3)] \cdot W_8^2$

Difference between DIT & DIF.

DIT	DIF
1. In DIT the time domain Sequence is decimated.	1. In DIF the frequency domain sequence is decimated
2. In DIT the i/p should be in bit reversed & o/p will be in normal order.	2. In DIF i/p is normal order o/p is bit reversed order.
3. In DIT the complex multiplication takes place before the add-substract operation,	3. In DIF the complex multiplication takes place after the add-sub operation.

Similarities in DIT & DIF

1. For both the algorithms the value of N should be $N=2^m$ & there will be m stages of butterfly computations.

2. Both algorithms involve same no. of operations. The total no. of complex additions are $N\log_2^N$ & total no. of complex multiplications are $\frac{N}{2}\log_2 N$.

Computation of Inverse DFT using FFT:-

Let $x(n)$ & $X(K)$ be N-point DFT pair.

by the definition of inverse DFT

$$x(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(K).e^{\frac{j2\pi Kn}{N}} \quad \text{for } n=0,1,2 \cdots N-1$$

$$= \frac{1}{N}\sum_{K=0}^{N-1} X(K).\left[e^{\frac{-j2\pi Kn}{N}}\right]^{*}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot \left[W_N^{kn}\right]^*$$

$$x(n) = \frac{1}{N} \left[\sum_{k=0}^{N-1} X(k) \left[W_N^{kn}\right]^* \right]^* \longrightarrow (1)$$

$$X(k) = \sum_{n=0}^{N-1} x(n) \, e^{-j\frac{2\pi}{N} \cdot k \cdot n}$$

$$= \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn}$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1}$$

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$\frac{N}{2} \log_2 N$:
g FFT :-

DFT pair.

DFT

for $n = 0, 1, 2 \ldots$