Here $y(n,k) = y(n-k)$

$$y(n-k) = x(n-k) + x(n-k-1)$$

(ii) $y(n) = T[x(n)] = x(-n)$

$$y(n,k) = T[x(n-k)] = x(-n-k)$$

$$y(n-k) = x(n-k) = x(-n+k)$$

16

Causality :-

A Causal system is one whose o/p depends on Past / Present values of the i/p.

$$y(n) = \sum_{k=-\infty}^{\infty} h(k) \cdot x(n-k)$$

Eg $k=-1$ Advance (-ve)
$k=1$ delay (+ve)

$$= \sum_{k=-\infty}^{-1} h(k) \cdot x(n-k) + \sum_{k=0}^{\infty} h(k) \cdot x(n-k)$$

$$= \underbrace{\cdots h(-2)x(n+2) + h(-1)x(n+1)}_{\text{Depends on future i/p's}} + \underbrace{h(0) \cdot x(n)}_{\text{Present i/p's}} + \underbrace{h(1) \cdot x(n-1)}_{\text{Past i/p.}} + \cdots$$

here we find o/p depends on the Past & Present values of the i/p. if the index $k \geq 0$ if $k < 0$ then o/p depends on future values of i/p.

$\therefore$ For a Causal system whose o/p does not depend on future values of the i/p

$$y(n) = \sum_{k=0}^{\infty} h(k) \cdot x(n-k)$$

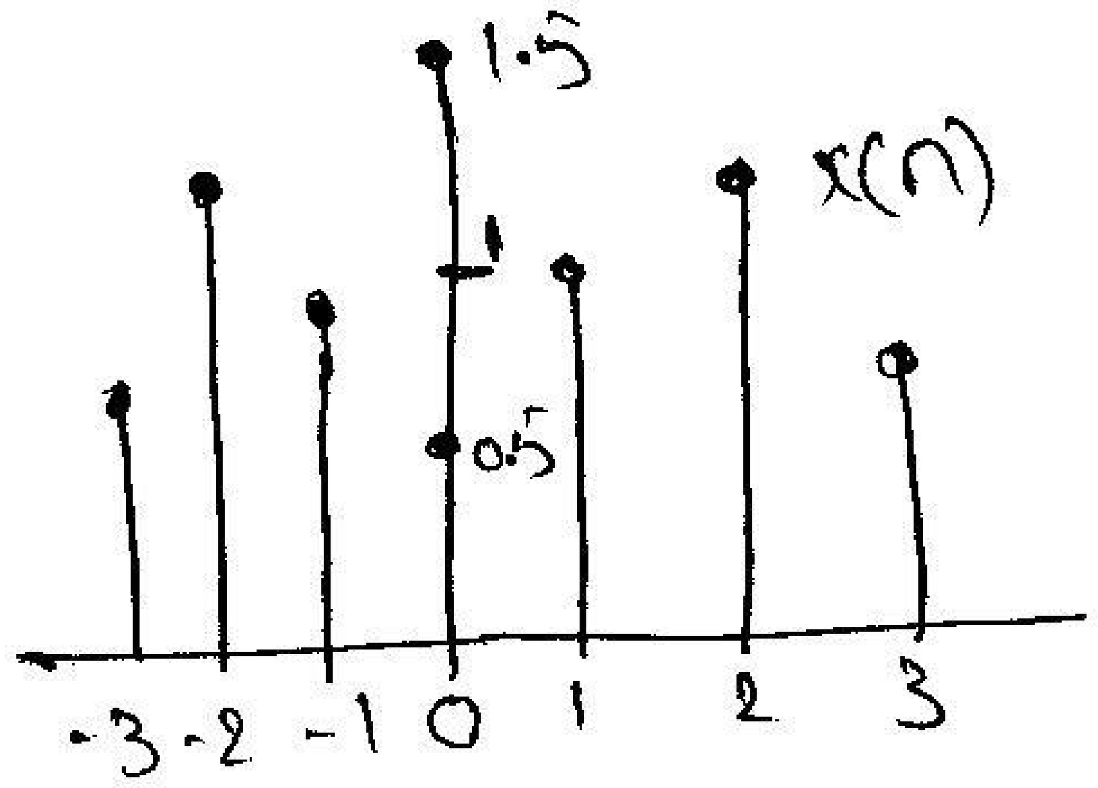
for Causal system $h(k)$ should be zero $k<0$.

*) Representation of an Arbitrary sequence :-

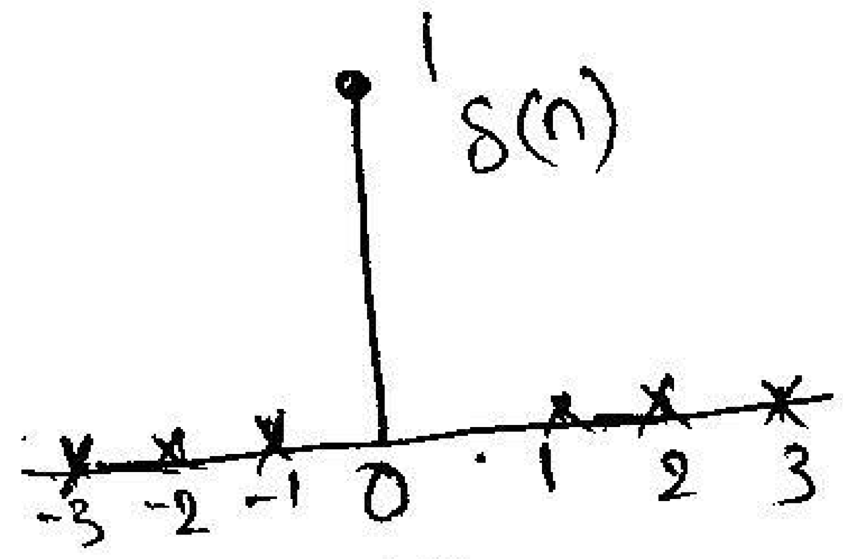
17) Any arbitrary sequence $x(n)$ can be represented in terms of delayed and scaled impulse sequence $\delta(n)$. let $x(n)$ be an infinite sequence.

The sample $x(0)$ can be obtained by multiplying $x(0)$, the magnitude with unit impulse $\delta(n)$

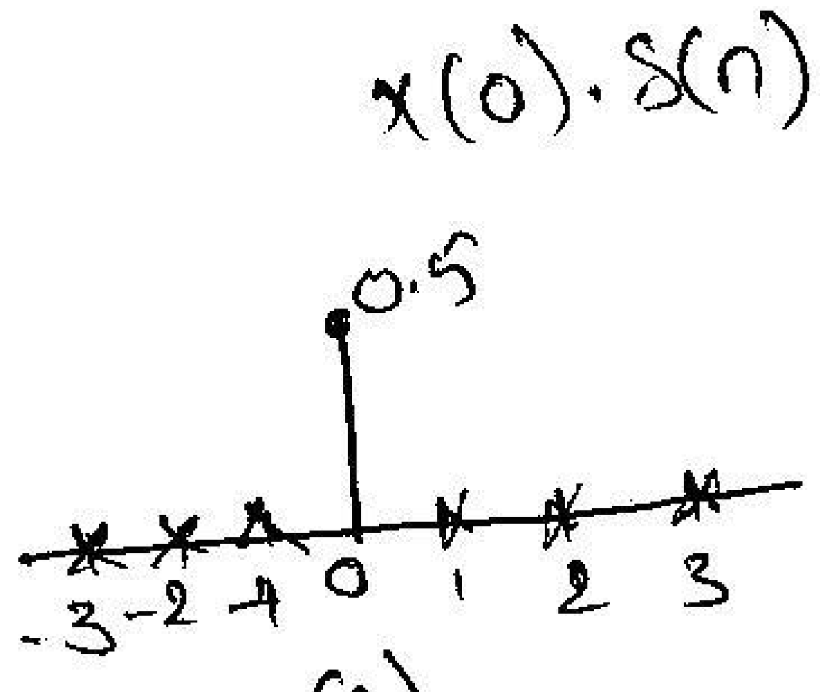
ie $$x(0).\delta(n) = \begin{cases} x(0) & \text{for } n=0 \\ 0 & \text{for } n \neq 0 \end{cases}$$



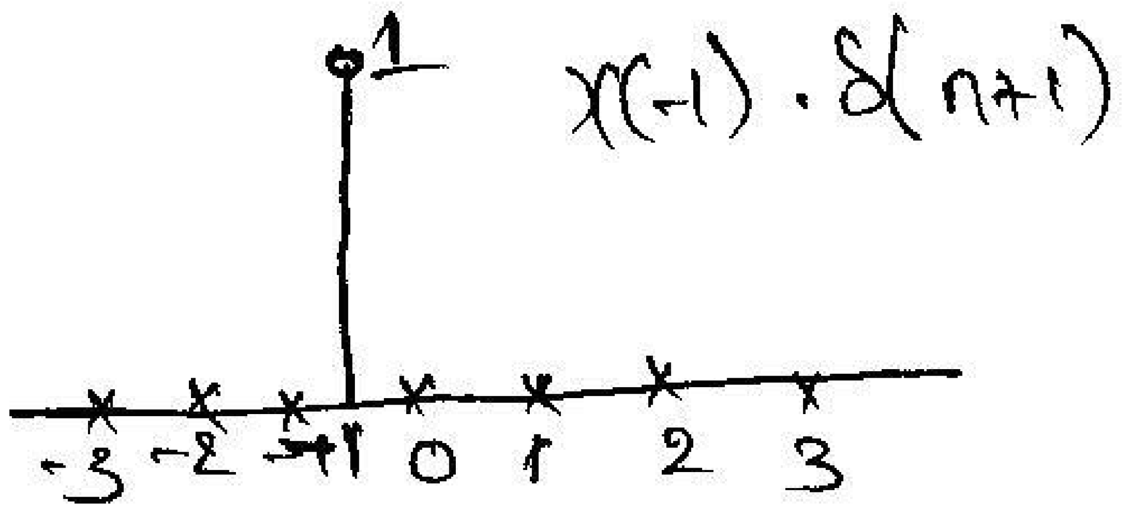
(a)

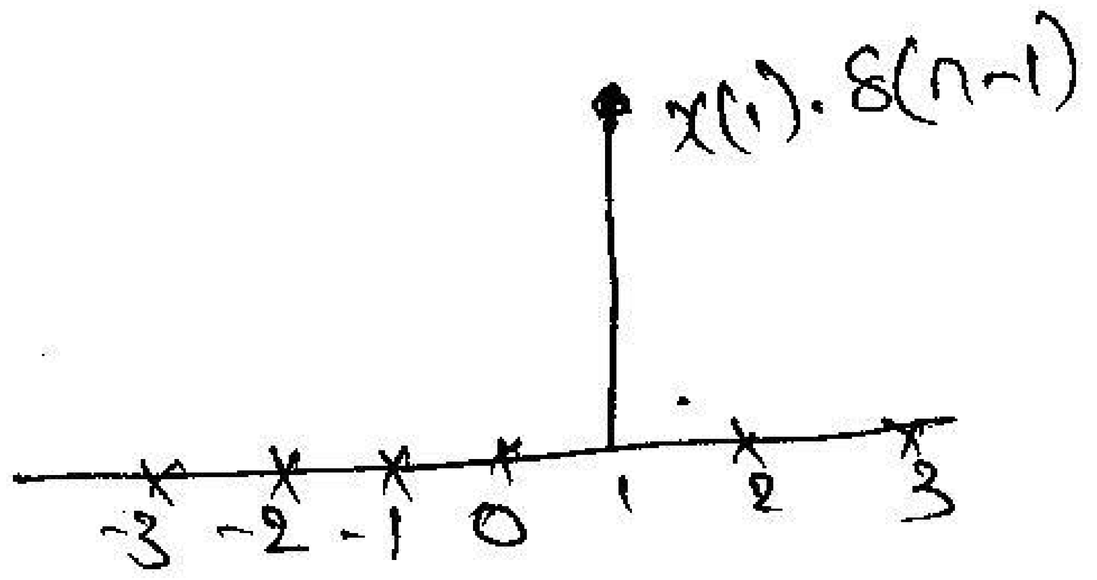


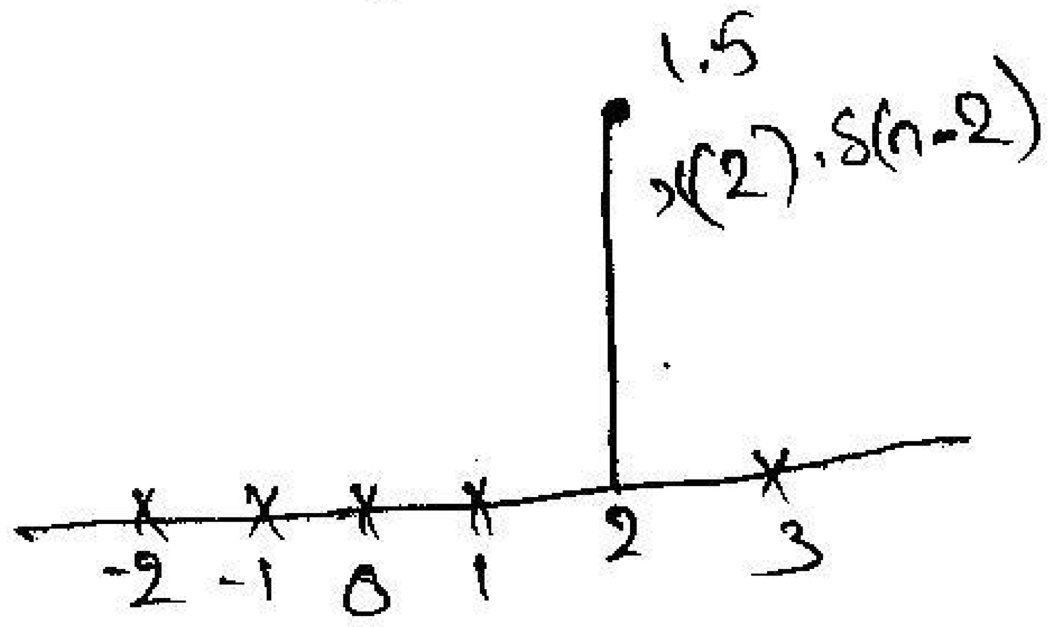
(b)

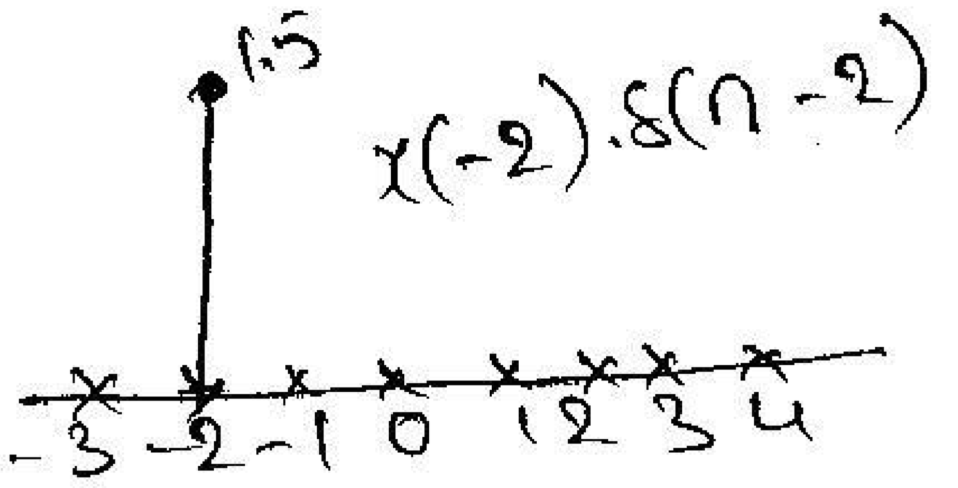


(c)









Similarly the sample $x(-1)$ can be obtained by multiplying $x(-1)$, the magnitude with one sample advanced unit impulse $\delta(n+1)$.

ie. $$x(-1)\,\delta(n+1) = \begin{cases} x(-1) & \text{for } n=-1 \\ 0 & \text{for } n \neq -1 \end{cases}$$

In same way $$x(-2).\delta(n+2) = \begin{cases} x(-2) & \text{for } n=-2 \\ 0 & \text{for } n \neq -2 \end{cases}$$

$$x(1).\delta(n-1) = \begin{cases} x(1) & \text{for } n=1 \\ 0 & \text{for } n \neq 1 \end{cases}$$

$$x(2).\delta(n-2) = \begin{cases} x(2) & \text{for } n=2 \\ 0 & \text{for } n \neq 2. \end{cases}$$

∴ The sum of the sequences

$$x(-2)\cdot\delta(n+2) + x(-1)\cdot\delta(n+1) + x(0)\cdot\delta(n) + x(1)\cdot\delta(n-1) + x(2)\cdot\delta(n-2)$$

Equals $x(n)$ for $-2 \le n \le 2$.

In general form $x(n)$ for $-\infty \le n \le \infty$

$$x(n) = +\dots x(-3)\cdot\delta(n+3) + x(-2)\cdot\delta(n+2) + x(-1)\cdot\delta(n+1) + x(0)\cdot\delta(n) + x(1)\cdot\delta(n-1) + x(2)\cdot\delta(n-2) + x(3)\cdot\delta(n-3) + \dots$$

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\cdot\delta(n-k).$$

where $\delta(n-k)$ is unity for $n=k$ and zero ∀ others

Eg:- Represent the sequence $x(n) = \{4, 2, -1, \underset{\uparrow}{1}, 3, 2, 1, 5\}$ as Sum of shifted unit impulses.

$$x(n) = \{4, 2, -1, \underset{\uparrow}{1}, 3, 2, 1, 5\}$$

$$n = -3, -2, -1, 0, 1, 2, 3, 4$$

$$x(n) = x(-3)\cdot\delta(n+3) + x(-2)\cdot\delta(n+2) + x(-1)\cdot\delta(n+1) + x(0)\cdot\delta(n) + x(1)\cdot\delta(n-1) + x(2)\cdot\delta(n-2) + x(3)\cdot\delta(n-3) + x(4)\cdot\delta(n-4)$$

$$x(n) = +4\cdot\delta(n+3) + 2\cdot\delta(n+2) - 1\cdot\delta(n+1) + 1\cdot\delta(n) + 3\cdot\delta(n-1) + 2\cdot\delta(n-2) + 1\cdot\delta(n-3) + 5\cdot\delta(n-4)$$ //.

*) Impulse response & Convolution sum:-

A discrete time system perform an operation on an i/p signal based on a predefined criteria to produce a modified o/p signal. The i/p signal $x(n)$ is system excitation and $y(n)$ is the system response.

$$x(n) \longrightarrow \boxed{T} \longrightarrow y(n) = T[x(n)]$$

if the i/p to the system is unit impulse ie $x(n) = \delta(n)$ then the o/p of system is known as impulse response denoted by $h(n)$

$$h(n) = T[\delta(n)]$$

WKT any arbitrary sequence $x(n)$ can be represented as a weighted sum of discrete impulses

$$x(n) = \sum_{k=-\infty}^{\infty} x(k).\delta(n-k) \rightarrow (1)$$

$$y(n) = T[x(n)] = T\left[\sum_{k=-\infty}^{\infty} x(k).\delta(n-k)\right] \rightarrow (2)$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k).T[\delta(n-k)] \rightarrow (3)$$

the response to shifted impulse sequence can denoted by $h(n,k)$

$$h(n,k) = T[\delta(n-k)] \rightarrow (4)$$

for a time invariant system

$$h(n,k) = h(n-k) \rightarrow (5)$$

sub (5) in Eq (4)

$$T[\delta(n-k)] = h(n-k)$$

from Eq ③ we have

$$\boxed{y(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k)}$$

∴ Above Eq is known as Convolution Sum & given as

$$y(n) = x(n) \cdot h(n) \quad [\ast \rightarrow \text{denotes Convolution}].$$

Properties of Convolution :-

(i) Commulative law :- $x(n) \ast h(n) = h(n) \ast x(n)$

(ii) Associative law :- $[x(n) \ast h_1(n)] \ast h_2(n) = x(n) \ast [h_1(n) \ast h_2(n)]$

(iii) Distributive law :- $x(n) \ast [h_1(n) + h_2(n)] = x(n) \ast h_1(n) + x(n) \ast h_2(n)$

*) FIR and IIR Systems :-

Linear time invariant systems can be classified according to the type of impulse response

1. FIR Systems (FINITE impulse Response)
2. IIR Systems. (INFINITE " ")

FIR System :-

If the impulse response of system is of finite duration, then the system is called a finite impulse Response (FIR system).

Eg :-
$$h(n) = \begin{cases} 1 & \text{for } n = -1, 2 \\ 2 & \text{for } n = 1 \\ 3 & \text{for } n = 0, 3 \\ 0 & \text{Otherwise} \end{cases}$$

IIR System :-

Def :- An infinite impulse Response (IIR) System has an impulse response for infinite duration

$$h(n) = a^n \cdot u(n)$$

*) Stable and unstable Systems:-

An LTI System is stable if it produces a bounded o/p sequence for every bounded i/p sequence.

If for some bounded i/p sequence $x(n)$, the o/p is unbounded (infinite), the system is classified as unstable.

Let $x(n)$ be bounded i/p sequence $h(n)$ be the impulse response of system and $y(n)$ be o/p sequence.

Taking magnitude of the o/p.

we have $$|y(n)| = \left|\sum_{k=-\infty}^{\infty} |h(k)| \cdot |x(n-k)|\right| \rightarrow$$

wkT the magnitude of the sum of terms is less than or equal to sum of magnitudes

$$|y(n)| \leq \sum_{k=-\infty}^{\infty} |h(k)| \, |x(n-k)|$$

let the bounded value of i/p is equal to M

$$y(n) \leq M \sum_{k=-\infty}^{\infty} h(k)$$

The above condition is satisfied when

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

∴ necessary condition for stability is

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

Eg:- $h(n) = \left(\frac{1}{2}\right)^n \cdot u(n)$

Stability $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n \cdot u(n)$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \qquad \because \left[1+a+a^2+\cdots = \frac{1}{1-a}\right]$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots +$$

$$= \frac{1}{1-\frac{1}{2}} = \frac{1}{1/2} = 2 < \infty$$

hence system is stable.

18)
*) **Linear Constant Co-efficient difference Equations :-**

(or)

Time Response Analysis of Discrete time Systems :-

The general form of difference Equation of an N^{th} order linear time invariant discrete time (LTI - DT) system is

$$y(n) = -\sum_{k=1}^{N} a_k \cdot y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \rightarrow (1)$$

where $\{a_k\}$ & $\{b_k\}$ are constants. The response of any discrete time system can be decomposed as

Total response = Zero state response + Zero i/p response

*) The Zero state response of system is response of system due to i/p alone when the intial state of system is zero at time $n=0$. (Ideal)

i.e the system intially relaxed

On other hand, the zero i/p response depends only on the intial state of the system. that is the i/p is zero. (when i/p zero)

Eg:- let us consider a first order discrete time system with difference equation.

$$y(n) = a\,y(n-1) + x(n)$$

where $x(n)$ & $y(n)$ are i/p & o/p.

let $x(n)$ i/p sequence is zero for $n<0$ and intially $y(n)$ for $n=-1$ exits. i.e $y(-1) \neq 0$

$\therefore$ The successive value of $y(n)$ for $n \geq 0$ are as follows

for $n=0$

$$y(0) = a\,y(-1) + x(0) \rightarrow (a)$$

$$y(1) = a.y(0) + x(1) \rightarrow (b)$$

$$= a[a\,y(-1) + x(0)] + x(1) \quad \left[\text{substitute } y(0) \text{ eq } (a) \text{ in eq } (b).\right]$$

$$= a^2 y(-1) + a x(0) + x(1)$$

$$y(2) = a\,y(1) + x(2)$$

$$= a\left[a^2 y(-1) + a x(0) + x(1)\right] + x(2)$$

$$= a^3 y(-1) + a^2 x(0) + a.x(1) + x(2)$$

for any n.

$$y(n) = a^{n+1} y(-1) + a^n x(0) + a^{n-1} x(1) + \dots + x(n)$$

$$= a^{n+1} y(-1) + \sum_{k=0}^{n} a^k . x(n-k) \quad n \geq 0.$$

The response $y(n)$ includes two parts. The first part depends on initial condition of the system and second term on i/p.

When $y(-1)=0$, the o/p $y(n)$ depends only on i/p applied. Hence $y(n)$ is known as the zero state response (or) forced response

$$y_f(n) = \sum_{k=0}^{n} a^k x(n-k) \quad n \geq 0.$$

if the system is initially non-relaxed that is $y(-1) \neq 0$ and i/p $x(n)=0 \ \forall n$, the o/p of system $y(n)$ depends only on the initial state of system.

then the response of system is called the zero i/p response (or) natural response and is denoted as

$$y_n(n) = a^{n+1} y(-1) \quad n \geq 0$$

The zero i/p response (natural response) is obtained by letting the i/p signal to zero. it depends on the nature of the system and initial conditions. on the other hand the zero state response depends on the nature of i/p signal.

∴ The difference eq of N^{th} order discrete time system of eq (1) is also written as

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k \cdot x(n-k) \rightarrow (2)$$

Where 'N' is called order of difference equation.

If the co-efficient a_0 is not equal to one we can divide eq (2) through out a_0 to normalize the equation.

The solution of difference Equation can be given as sum of two Parts

$$y(n) = y_h(n) + y_p(n)$$

$y_h(n)$ = homogenous (or) Complimentary solution

$y_p(n)$ = Particular solution //.

$\boxed{\text{Note :-}}$ for an LTI System the response $y(n)$ can be expressed as a weighted summation of dependent terms.

so:- $y(n) = -a_1 y(n-1) - a_2 y(n-2) - a_3 y(n-3) - \cdots\cdots$
$+ b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + b_3 x(n-3) + \cdots \rightarrow$ (C)

*) -ve Constants are inserted for o/p signals because o/p signals are feedback from o/p to i/p.

*) +ve Constants are inserted for i/p signals because i/p signals are ~~feedback~~ forward from i/p to o/p.

∴ Practically the response $y(n)$ at any time instant 'n', may depend on 'N' no:- of Past o/p's, Present i/p & 'M' no:- of Past i/ps.

∴ Eq (C) is written as

$y(n) = -a_1 y(n-1) - a_2 y(n-2) - a_3 y(n-3) \cdots\cdots - a_N y(n-N) +$
$b_0 x(n) + b_1 x(n-1) + \cdots + b_M x(n-M)$

$$\therefore y(n) = -\sum_{M=1}^{N} a_m y(n-M) + \sum_{m=0}^{M} b_m x(n-m). \quad //$$

in above Eq value of "N" defines order of system

Eg $N = 1$ the D.T system is 1^{st} order

$N = 2$ " " " " 2^{nd} "

$N = 3$ " " " " 3^{rd} "

Note

(a) Natural Response (zero i/p Response)

The natural response $y_n(n)$ is the solution of Eq (2) below with $x(n) = 0$

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k \cdot x(n-k)$$

for a discrete time system the natural response is the solution of homogenous Eq.

$$\sum_{k=0}^{N} a_k \cdot y(n-k) = 0 \rightarrow (3)$$

it is in the form $y_h(n) = \lambda^n$

$$\sum_{k=0}^{N} a_k \cdot \lambda^{n-k} = 0 \quad ; a_0 = 1$$

$$\lambda^n + a_1 \lambda^{n-1} + \dots + a_{N-1} \cdot \lambda^{n-N+1} + a_N \cdot \lambda^{n-N} = 0 \quad (\text{where } k = N-1$$

$$\lambda^{n-N}\left[\lambda^N + a_1 \lambda^{N-1} + \dots + a_{N-1} \cdot \lambda + a_N\right] = 0$$

$$\lambda^N + a_1 \cdot \lambda^{N-1} + \dots + a_{N-1} \cdot \lambda + a_N = 0 \longrightarrow (4)$$

Eq (4) is characteristic Eq of system.

Eq (4) can be expressed in factorized form as

$$(\lambda-\lambda_1).(\lambda-\lambda_2) \ldots (\lambda-\lambda_N) = 0$$

$\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_N$ are roots of Eq (4). The roots of characteristic equation are called characteristics roots (or) Eigen values of system.

The nature of natural response depends on type of roots.

Real roots $\rightarrow$ real exponential
imaginary roots $\rightarrow$ sinusoidal
Complex roots $\rightarrow$ exponentially damped sinusoidal.

(i) Distinct roots :-

If the roots $\lambda_1, \lambda_2 \ldots \lambda_N$ of Eq (4) are distinct then it has N-solutions $C_1\lambda_1^n, C_2\lambda_2^n, \ldots C_N\lambda_N^n$

the general solution is of the form

$$y_h(n) = C_1\lambda_1^n + C_2\lambda_2^n + \ldots + C_N.\lambda_N^n$$

where $C_1, C_2, \ldots C_N$ are arbitary constants

Eg:- If $\lambda_1 = 2$; $\lambda_2 = 3$

$$y_h(n) = C_1(2)^n + C_2(3)^n$$

(ii) Repeated roots :-

If the roots λ_1 is repeated m times and the remaining (N-m) roots are distinct then, the characteristic equation of system is.

$(\lambda - \lambda_1)^m (\lambda - \lambda_{m+1}) \cdot (\lambda - \lambda_{m+2}) \dots (\lambda - \lambda_N) = 0$

and general solution is

$$y_h(n) = \left(c_1 + c_2 \cdot n + c_3 \cdot n^2 + \dots + c_m \cdot n^{m-1}\right) \cdot (\lambda_1^n)$$
$$+ c_{m+1} \cdot (\lambda_{m+1})^n + c_{m+2} \cdot (\lambda_{m+2})^n + \dots + c_N \cdot \lambda^N.$$

Eg:-

The roots of characteristic Eq $\lambda_1 = 1$; $\lambda_2 = 1$ & $\lambda_2 = 2$

$$y_h(n) = \left[c_1 + c_2 n\right] \cdot (1)^n + c_3 (2)^n.$$

(iii) Complex roots:-

if roots are Complex $\lambda_1 = \lambda = a + jb$

$\lambda_2 = \lambda^* = a - jb$

$$y_h(n) = r^n \left[A_1 \cos n\theta + A_2 \sin n\theta\right]$$

$r = \sqrt{a^2 + b^2}$; $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

A_1 & A_2 are constants.

(b) Forced Response (zero state response):-

The forced response is the solution of diff. Eq (DE) for the given i/p, when the initial Conditions are zero.

It Consists of two parts, homogeneous solution & Particular Solu.

* The homogeneous solution Can be obtained from the roots of characteristics Equation.
* The Particular Solution $y_p(n)$ is to satisfy the d.e. for specific i/p signal $x(n)$. $n \geq 0$

$$1 + \sum_{k=1}^{N} a_k \, y_p(n-k) = \sum_{k=0}^{M} b_k \, x(n-k) \, ..$$

The general form of the particular solution for several i/p are given in table. From table we can find $x(n) = A \sin \omega n$ then

$$y_p(n) = C_1 \cos \omega n + C_2 \sin \omega n.$$

where C_1 & C_2 are obtained by substituting $y_p(n)$ and $x(n)$ in d.E.

$x(n)$ i/p signal	$y_p(n)$ Particular solution
A (step input)	K
AM^n	$K.M^n$
An^M	$K_0 n^M + K_1 . n^{M-1} + \dots + K_M$
$A^n . N^M$	$A^n [K_0 . n^M + K_1 n^{M-1} + \dots + K_M]$
$A \cos \omega n$, $A \sin \omega n$	$C_1 \cos \omega n + C_2 \sin \omega n$.

If the i/p applied to the system and one of the components of homogenous solution are same; then multiply the particular solution by the lower power of 'n' that will give a response component not included in homogenous solution.

ⓒ **Total response** :- It is obtained by adding natural + forced responses

$$y(n) = y_h(n) + y_p(n)$$

ⓓ **Impulse response** :-

The general form of d.E of N^{th} order system is given

$$1 + \sum_{k=1}^{N} a_k \cdot y(n-k) = \sum_{k=0}^{M} b_k \, x(n-k) \quad N > M$$

for the i/p $x(n) = \delta(n)$ obtained as

$\downarrow$

o/p of unit step impulse

$$1 + \sum_{k=1}^{N} a_k\, y(n-k) = \sum_{k=0}^{M} b_k \cdot \delta(n-k)$$

for $N > M$

$$1 + \sum_{k=1}^{N} a_k\, y(n-k) = 0$$

$$\sum_{k=0}^{N} a_k \cdot y(n-k) = 0$$

if $N = M$ we have to add an impulse functi to homogenous solution.

19). <u>Frequency Response analysis of Discrete time system:-</u>

The o/p of any linear time invariant system to an i/p signal $x(n)$ can be obtained using convolution sum

$$y(n) = \sum_{k=-\infty}^{\infty} h(k) \cdot x(n-k).$$

$h(n)$ is impulse response of the system.

Let us consider a complex exponential signal $x(n)$ as i/p to system.

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, e^{j\omega(n-k)}$$

$$= e^{j\omega n}\left[\sum_{k=-\infty}^{\infty} h(k) \cdot e^{-j\omega k}\right]$$

$$\underbrace{e^{j\omega n}}_{\text{i/p}} \underbrace{\left[H \cdot (e^{j\omega})\right]}_{\text{freq response.}} \longrightarrow (1)$$

where $\left[H \cdot (e^{j\omega})\right] = \sum_{k=-\infty}^{\infty} h(k) \cdot e^{-j\omega k} \longrightarrow (2)$

The function $H(e^{Jw})$ is exits if the impulse response is absolutely summable.

(*) from eq (1) we say that for i/p $x(n) = e^{Jwn}$ the o/p of an LTI DT system is also exponential signal of the same frequency, multiplied by factor $H(e^{Jw})$.

(*) these type of signal that produce a response which differ from the i/p signal by a complex constant are known as Eigen functions.

from eq (2) we can find frequency response $H(e^{Jw})$ of system is discrete time fourier transform of impulse response $h(n)$ of system.

$\because$ The frequency response $H(e^{Jw})$ is a complex valued function 'it' can be expressed in polar form as

$$H(e^{Jw}) = |H(e^{Jw})| \cdot e^{J\theta\omega}$$

$$|H(e^{Jw})| \Rightarrow \text{magnitude} \; ; \; \theta(\omega) = \angle H(e^{Jw}).$$

and phase response is an odd function of ω

$$\text{i.e} \quad \theta(\omega) = -\theta(-\omega).$$

$$\because \; e^{J(\omega + 2k\pi)} = e^{J\omega} \cdot e^{J\omega 2k\pi} = e^{J\omega}$$

for an integer k the frequency response $H(e^{Jw})$ is periodic with period 2π.

(a) Frequency response of first order system :-

The d.e for 1st order system is given by

$$y(n) - a \cdot y(n-1) = x(n).$$

$x(n)$ is i/p ; $y(n) = o/p$

Taking Fourier transform on b.s

$$Y(e^{jw}) - a \cdot e^{-jw} \cdot Y(e^{jw}) = X(e^{jw})$$

$$Y(e^{jw})\left[1 - a \cdot e^{-jw}\right] = X(e^{jw}).$$

The freq response.

$$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \frac{1}{1 - a \cdot e^{-jw}}$$

the impulse response

$$h(n) = F^{-1}\left(H \cdot (e^{jw})\right)$$

$$= F^{-1}\left[\frac{1}{1 - a \cdot e^{-jw}}\right]$$

$$h(n) = a^n \cdot u(n) \qquad H(e^{jw}) = \frac{1}{1 - a \cdot e^{-jw}}$$

now:. $H.(e^{jw})$ is called frequency response of the LTI system whose impulse response is $h(n)$

$$\therefore H.(e^{jw}) = \frac{1}{1 - a[\cos w - j\sin w]}$$

$$= \frac{1}{1 - a\cos\omega + aj\sin\omega}$$

$$|H \cdot (e^{j\omega})| = \frac{1}{\sqrt{(1 - a\cos\omega)^2 + a^2\sin^2\omega}}$$

$$= \frac{1}{\sqrt{1 + a^2\cos^2\omega - 2a\cos\omega + a^2\sin^2\omega}}$$

$$= \frac{1}{\sqrt{1 + a^2 - 2a\cos\omega}} = \frac{1}{\sqrt{1 + a^2 - 2a\cos\omega}}$$

$$\angle H \cdot e^{j\omega} = \tan^{-1}\left(\frac{a\sin\omega}{1 - a\cos\omega}\right) //.$$

if λ_1 and λ_2 are ~~real~~ complex.

if $y(n) = \lambda^n$ from d.e we find from eq (1)

$$\lambda^2 - a_1\lambda - a_2 = 0$$

if roots are complex:-

$$\lambda_1 = \frac{a_1}{2} + j\sqrt{-\frac{a_1^2}{4} - a_2} = \alpha + j\beta = r.e^{j\theta} = r\cos\theta + jr\sin\theta$$

$$\lambda_2 = \frac{a_1}{2} - j\sqrt{-\frac{a_1^2}{4} - a_2} = \alpha - j\beta = r.e^{-j\theta} = r\cos\theta - jr\sin\theta.$$

where $\alpha = \frac{a_1}{2} = r\cos\theta$; $\beta = \sqrt{-\frac{a_1^2}{4} - a_2} = r\sin\theta$

$$\therefore \alpha^2 + \beta^2 = r^2\cos^2\theta + r^2\sin^2\theta$$

$$r^2 = \frac{a_1^2}{4} + \left(-\frac{a_1^2}{4} - a_2\right) = -a_2$$

$$\Rightarrow r = \sqrt{-a_2}.$$

if the intial condition are zero.

$$y(-1) = h(-1) = 0$$

from eq (3)

$$h(-1) = A.r^n \sin(-\theta + \phi) = 0$$

gives $\theta = \phi$

$$\therefore y(n) = h(n) = A.r^n \sin(n\theta + \theta) \rightarrow (4)$$

from eq (1)

$$y(0) = a_1 y(-1) + a_2 y(-2) + x(0) = 1 \rightarrow (5)$$

$$h(0) = y(0) = A\sin\phi = 1$$

$$\Rightarrow A = \frac{1}{\sin\phi} = \frac{1}{\sin\theta} \quad (\because \theta = \phi) \rightarrow (6)$$

now:-

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n) \cdot e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} A \cdot r^n \sin(n\theta + \theta) \cdot e^{-j\omega n}$$

$$= A \cdot \sum_{n=0}^{\infty} r^n \left(\frac{e^{j(n+1)\theta} - e^{-j(n+1)\theta}}{2j} \right) \cdot e^{-j\omega n}$$

$$= \frac{A}{2j} \left\{ \sum_{n=0}^{\infty} r^n \cdot e^{j\theta} \cdot e^{jn\theta} \cdot e^{-j\omega n} - \sum_{n=0}^{\infty} r^n \cdot e^{-j\theta} \cdot e^{-jn\theta} \cdot e^{-j\omega n} \right\}$$

$$= \frac{A}{2j} \left\{ \frac{e^{j\theta}}{1 - r \cdot e^{-j(\omega-\theta)}} - \frac{e^{-j\theta}}{1 - r e^{-j(\omega+\theta)}} \right\}$$

$$= \frac{A}{2j} \left[\frac{e^{j\theta} - r \cdot e^{-j\omega} \cdot e^{-j\theta} \cdot e^{j\theta} - e^{-j\theta} + r \cdot e^{-j\omega} \cdot e^{j\theta} \cdot e^{-j\theta}}{1 - r \cdot e^{-j\omega}\left(e^{j\theta} + e^{-j\theta}\right) + r^2 \cdot e^{-2j\omega}} \right]$$

$$= \frac{A}{2j} \left[\frac{e^{j\theta} - e^{-j\theta}}{1 - 2 \cdot r \cdot e^{-j\omega} \cos\theta + r^2 \cdot e^{-2j\omega}} \right] \rightarrow \cancel{(8)}$$

$$= \frac{A}{2j} \left\{ \frac{\sin\theta}{1 - 2 \cdot r^{-j\omega} \cdot \cos\theta + r^2 \cdot e^{-2j\omega}} \right\} \rightarrow (7)$$

Substitute Eq (6) in Eq (7)

$$= \frac{\cancel{A}}{2j} \left[\frac{\frac{1}{\cancel{A}}}{1 - 2 \cdot r^{-2j\omega} \cdot \cos\theta + r^2 \cdot e^{-2j\omega}} \right]$$

$$\Rightarrow \boxed{H\cdot(e^{j\omega}) = \frac{1}{1-2.r.e^{-j\omega}\cos\theta + r^2.e^{-2j\omega}}} \;//.$$

*) Transfer function:-

If $H(e^{j\omega})$ is fourier Transform of impulse response $h(n)$ an $X(e^{j\omega})$ is F.T of i/p sequence $x(n)$, we can derive the relationship b/w $Y(e^{j\omega})$.

the fourier transform of o/p in terms $X(e^{j\omega})$ & $H(e^{j\omega})$

An arbitary sequence can be represented in the form

i/p $$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}).e^{j\omega n}.d\omega \rightarrow (1)$$

where $$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n).e^{-j\omega n}$$

$$y(n) = e^{j\omega n}.H(e^{j\omega}) \quad ; \; x(n) = e^{j\omega n}$$

o/p $$y(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}).e^{j\omega n}.H(e^{j\omega}).d\omega \rightarrow (a)$$

W.K.T

$$y(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y.(e^{j\omega}).e^{j\omega n}.d\omega \rightarrow (b)$$

now eq (a) = (b) ($\because$ y(n) is common)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} Y.(e^{j\omega}).e^{j\omega n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}).e^{j\omega n}.H(e^{j\omega})$$

$$Y(e^{j\omega}) = X.(e^{j\omega}).H(e^{j\omega})$$

$$\Rightarrow H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} \; // \; \text{Transfer function.}$$