Unit - IV

Realization of Filters

Definition of the z-transform :-

The z-transform of a discrete-time signal $x(n)$ is defined as

$\omega = \frac{2\pi k}{N}$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} \longrightarrow (1)$$

Where z is a complex variable.

In polar form z can be expressed as $z = re^{j\omega} \rightarrow (2)$

where r is the radius of the circle. Sub (2) in (1)

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, r^{-n}\, e^{-j\omega n}.$$

For $r=1$,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \longrightarrow \text{Fourier transform equation.}$$

$$X(k) = \sum_{n=-\infty}^{\infty} x(n)\cdot e^{-j\omega n}.$$

If $x(n)$ is a causal sequence i.e $x(n)=0$ for $n<0$

then the z-transform is

$$X_+(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}.$$

The above expression is referred to as one-sided z-transform.

$$X(z) = Z\{x(n)\}$$

Since the z-transform is an infinite power series it exists only for those values of z for which this series converges. The region of convergence (ROC) of X is the set of all values of z for which $X(z)$ attains a finite value.

Z-transform & ROC of finite duration sequences :- (45)

Right hand sequence :-

A right hand sequence for which $x(n) = 0$ for all $n < n_0$ where n_0 is +ve or -ve but finite. If n_0 is greater than or equal to zero, the resulting sequence is causal or a positive time sequence. For such a type of sequence the ROC is entire z-plane except $z=0$.

Example

Find the z-transform and ROC of the causal sequence.

$$x(n) = \{\underset{\uparrow}{\overset{x(0)}{1}}, \overset{x(1)}{0}, \overset{x(2)}{3}, \overset{x(3)}{-1}, \overset{x(4)}{2}\}$$

$$X_+(z) = x(0) + x(1) z^{-1} + x(2) z^{-2} + \dots$$

$$= 1 + 0 z^{-1} + 3 z^{-2} + (-1) z^{-3} + 2 z^{-4} + \dots$$

$$X_+(z) = 1 + 3z^{-2} - z^{-3} + 2z^{-4}$$

The $X_+(z)$ converges for all values of z except at $z=0$.

Left hand sequence :-

A left hand sequence $x(n)$ is one for which $x(n) = 0$ for all $n \geq n_0$ where n_0 is +ve or -ve but finite. If $n_0 \leq 0$ the resulting sequence is anticausal sequence. For such type of sequence the ROC is entire z-plane except at $z = \infty$

Ex

Find the z-transform and ROC of the anticausal sequence. $x(n) = \{-3, -2, -1, 0, \underset{\uparrow}{1}\}$

We know $\qquad x(-4)\ x(-3)\ x(-2)\ x(-1)\ x(0)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$X(z) =$ $x(0) = 1$ $x(-1) = 0$ $x(-2) = -1$ $x(-3) = -2$

$x(-4) = -3$

$$X(z) = 1 - z^2 - 2z^3 - 3z^4$$

The $X(z)$ converges for all values of z except at $z = \infty$.

$X(z)$

Two-Sided sequence :-

A signal that has finite duration on both left and right hand sides is known as two sided sequence. for such type of sequence the ROC is entire z-plane except at $z=0$ and $z=\infty$.

Ex - 2

Ex Find the z-transform of the sequence.

$$x(n) = \{2, -1, 3, 2, \underset{\uparrow}{1}, 0, 2, 3, -1\}$$

$-4\ -3\ -2\ -1\ 0\ 1\ 2\ 3\ 4$

$x(-4) = 2$, $x(-3) = -1$, $x(-2) = 3$, $x(-1) = 2$, $x(0) = 1$ $x(1) = 0$ $x(2) = 2$

$x(3) = 3$, $x(4) = -1$.

$$X(z) = 2z^4 - z^3 + 3z^2 + 2z + 1 + 2z^{-2} + 3z^{-3} - z^{-4}$$

The $X(z)$ converges for all the values of z except at $z=0$ and $z=\infty$.

Z-transform and ROC of infinite duration sequence

Infinite Duration sequence

Ex-1 $x(n) = a^n u(n)$

$u(n) \to u(n) = 1 \Rightarrow n \geq 0$

$u(n) = 0 \Rightarrow$ oth

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}.$$

$$= \sum_{n=-\infty}^{\infty} a^n u(n)\, z^{-n}.$$

$$= \sum_{n=0}^{\infty} a^n z^{-n}.$$

$$= \sum_{n=0}^{\infty} (a z^{-1})^n.$$

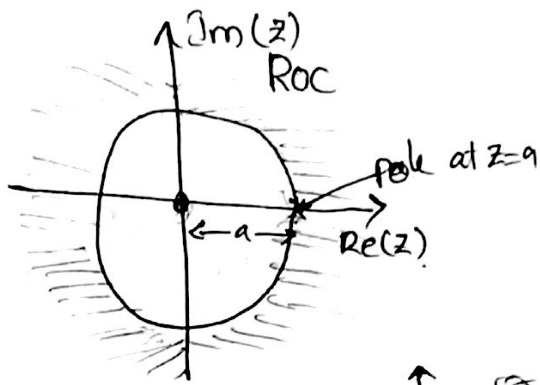
$$= \frac{1}{1-az^{-1}}$$

$$\left[\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}\right]$$ (46)

n_0 = finite

$$= \frac{1}{1-\frac{a}{z}}$$

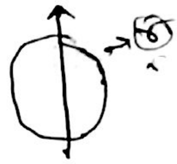
$$X(z) = \frac{z}{z-a}.$$



and for $|az^{-1}| < 1$

for

$\frac{a}{z} < 1 \Rightarrow$ for $z > a$, $X(z)$ converges



$$x(n) = -b^n u(-n-1)$$

$u(-n-1) = 1$

$$X(z) = \sum_{n=-\infty}^{\infty} -b^n u(-n-1) z^{-n}.$$

$$= \sum_{n=-\infty}^{-1} -b^n z^{-n}.$$

$$= -\sum_{n=-\infty}^{-1} (b z^{-1})^n$$

$$= -\left[\sum_{n=0}^{\infty} (b^{-1} z)^{-n} - 1\right]$$

the above series converges for $|b^{-1}z| < 1$

$\frac{z}{b} < 1 \Rightarrow z < b.$

$$X(z) = -\left[\frac{1}{1-b^{-1}z} - 1\right].$$

$$= -1\left[\frac{1}{1-\frac{z}{b}} - 1\right] = -1\left[\frac{1}{\frac{b-z}{b}} - 1\right]$$

$$= -\left[\frac{b}{b-z} - 1\right] = -\left[\frac{b-b+z}{b-z}\right] = -\frac{z}{b-z}$$

$$= \frac{-z}{z-b}$$

Roc of two-sided sequence :-

Ex $x(n) = a^n u(n) + b^n u(-n-1)$

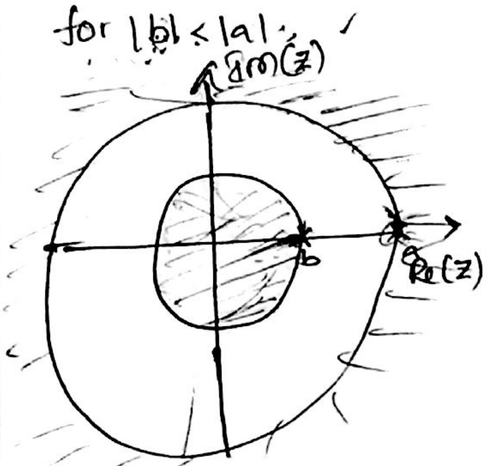
$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} + \sum_{n=-\infty}^{-1} b^n z^{-n}$$

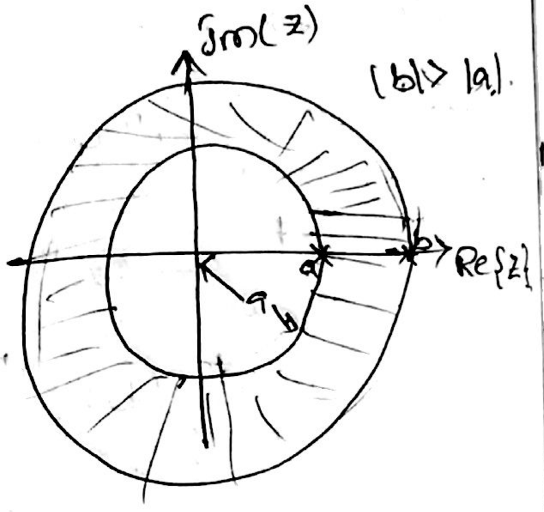
$$X(z) = \sum_{n=0}^{\infty} (a z^{-1})^n + \sum_{n=1}^{\infty} (b^{-1} z)^n$$

$$X(z) = \frac{z}{z-a} - \frac{z}{z-b}$$

Roc $|a| < |z| < |b|$

for $|b| < |a|$



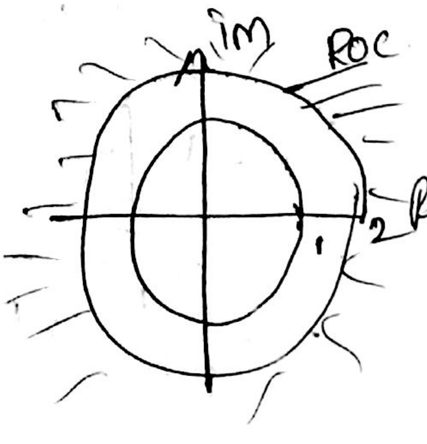


Stability & Roc :

Let h(n) be the impulse Response of a causal of a causal or noncausal linear time-invariant system and H(z) be the z-transform of h(n).

Ex $h(n) = 2^n u(n)$

$$H(z) = \frac{z}{z-2}, \quad |z| > 2$$



Roc is $|z| > 2$ it does not contain unit circle. the system is unstable.

Properties of the z-transform :-

Linearity

if $X_1(z) = Z\{x_1(n)\}$ $X_2(z) = Z\{x_2(n)\}$

$$Z\{a x_1(n) + b x_2(n)\} = a X_1(z) + b X_2(z)$$ ✓

Time shifting

if $X(z) = Z\{x(n)\}$ then

$$Z\{x(n-m)\} = z^{-m} X(z)$$

$k \Rightarrow$ large

$n-m$

$n-k$

if $X_+(z) = Z\{x(n)\}$

(i) $Z\{x(n-m)\} = z^{-m}\left\{X_+(z) + \sum_{k=1}^{m} x(-k) z^{k}\right\}$

(ii) $Z\{x(n+m)\} = z^{m}\left\{X_+(z) - \sum_{k=0}^{m-1} x(k) z^{-k}\right\}$

Multiplication by an exponential sequence

$$Z\{a^n u(n)\} = X(a^{-1} z)$$

Time reversal

$$Z\{x(-n)\} = X(z^{-1})$$

Convolution theorem

$$Z\{x(n) * h(n)\} = X(z) H(z)$$

Parseval's relation

$$\sum_{n=-\infty}^{\infty} x_1(n) x_2^*(n) = \frac{1}{2\pi j} \oint_C X_1(v) X_2^*\left(\frac{1}{v^*}\right) v^{-1} dv$$

Initial value theorem:

$$\lim_{z\to\infty} X_+(z) = \lim_{z\to\infty} \sum_{n=0}^{\infty} x(n) z^{-n} = x(0)$$

Final value theorem

$$x(\infty) = \lim_{z\to 1} (1 - z^{-1}) X_+(z)$$

or

$$x(\infty) = \lim_{z\to 1} (z-1) X_+(z)$$

Inverse z-transform:

There are four methods.

1. Long division method
2. partial fraction expansion method
3. Residue method
4. convolution method.

1. Long division method :-

$$X_+(z) = x(0) + x(1)z^{-1} + \cdots + x(k)z^{-k} + \cdots + x(n)z^{-n} + \cdots$$

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

$$= x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

Ex

$$X(z) = \frac{z+0.2}{(z+0.5)(z-1)} \qquad |z|>1$$

$$\begin{array}{r|l} & z^{-1} + 0.7z^{-2} + 0.85z^{-3} + 0.775z^{-4} \\ \hline z^2 - 0.5z - 0.5 & z + 0.2 \\ & z - 0.5 - 0.5z^{-1} \\ & \quad - \quad + \quad + \\ \hline & 0.7 + 0.5z^{-1} \\ & 0.7 - 0.35z^{-1} - 0.35z^{-2} \\ & - \quad + \quad + \\ \hline & 0.85z^{-1} + 0.35z^{-2} \\ & 0.85z^{-1} - 0.425z^{-2} - 0.425z^{-3} \\ & - \quad + \quad + \\ \hline & 0.775z^{-2} + 0.425z^{-3} \\ & 0.775z^{-2} - 0.387z^{-3} - 0.3875z^{-4} \end{array}$$

$$X(z) = z^{-1} + 0.7z^{-2} + 0.85z^{-3} + 0.775z^{-4} + \cdots$$

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}.$$

Partial fraction Method :-

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

Multiply both numerator & denominator by z^N

$$X(z) = \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_M z^{N-M}}{z^N + a_1 z^{N-1} + \cdots + a_N}$$

$$X(z) = \frac{z\left[b_0 z^{N-1} + b_1 z^{N-2} + \cdots + b_M z^{N-M-1}\right]}{z^N + a_1 z^{N-1} + \cdots + a_N}$$

$$\frac{X(z)}{z} = \frac{\left[b_0 z^{N-1} + b_1 z^{N-2} + \cdots + b_M z^{N-M-1}\right]}{z^N + a_1 z^{N-1} + \cdots + a_N}$$

$$= \frac{b_0 z^{N-1} + b_1 z^{N-2} + \cdots + b_M z^{N-M-1}}{(z-P_1)(z-P_2)\cdots\cdots(z-P_N)}$$

$$\frac{X(z)}{z} = \frac{C_1}{z-P_1} + \frac{C_2}{z-P_2} + \cdots + \frac{C_N}{z-P_N}$$

Where $C_K = \left.\frac{(z-P_K)(X(z))}{z}\right|_{z=P_K}$ $\quad K = 1, 2, \ldots N$

If $X(z)$ has a pole of multiplicity l, that is, it contains in denominator the factor $(z-P_K)^l$

$$\frac{X(z)}{z} = \frac{1}{(z-P_2)^2(z-P_1)}$$

$$\frac{X(z)}{z} = \frac{C_1}{z-P_1} + \frac{C_2}{z-P_2} + \frac{C_3}{(z-P_3)^2}$$

$$C_1 = (z-P_1)\frac{X(z)}{z}\Big|_{z=P_1}$$

$$C_2 = \frac{d}{dz}(z-P_2)^2\frac{X(z)}{z}\Big|_{z=P_2}$$

$$C_3 = (z-P_2)^2\frac{X(z)}{z}\Big|_{z=P_2}$$

Ex

$$X(z) = \frac{1+8z^{-1}}{1+3z^{-1}+2z^{-2}}$$

$$X(z) = \frac{1+\frac{3}{z}}{1+\frac{3}{z}+\frac{2}{z^2}} = \frac{z(z+3)}{(z+1)(z+2)}$$

$$\frac{X(z)}{z} = \frac{z+3}{(z+1)(z+2)} = \frac{C_1}{z+1}+\frac{C_2}{z+2}$$

$$C_1 = (z+1)\frac{X(z)}{z}\Big|_{z=-1}$$

$$= \frac{\cancel{(z+1)}(z+3)}{\cancel{(z+1)}(z+2)}\Big|_{z=-1} = \frac{2}{1} = 2$$

$$\boxed{C_1 = 2}$$

$$C_2 = (z+2)\frac{X(z)}{z}\Big|_{z=-2}$$

$$= \frac{\cancel{(z+2)}(z+3)}{(z+1)\cancel{(z+2)}}\Big|_{z=-2} = \frac{3-2}{-2+1} = \frac{1}{-1}$$

$$C_2 = -1$$

$$\frac{X(z)}{z} = \frac{2}{z+1} - \frac{1}{z+2}$$

$$X(z) = 2\cdot\frac{z}{z+1} - \frac{z}{z+2}$$

$$x(n) = 2(-1)^n u(n) - (-2)^n u(n).$$

Residue Method :

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

multiply z^{K-1} & integral on both sides.

$$\oint_c X(z)\, z^{K-1}\, dz = \oint_c \sum_{n=-\infty}^{\infty} x(n)\cdot z^{-n} z^{K-1} dz$$

$$\oint_c X(z)\, z^{K-1}\, dz = \sum_{n=-\infty}^{\infty} x(n) \oint_c z^{-n+K-1}\cdot dz.$$

by cauchy residue theorem.

$$\oint_c z^{-n+K-1} = 2\pi j\, \delta_{Kn}. \quad \begin{cases} \delta_{Kn} = 1 \text{ for } K = n. \\ \delta_{Kn} = 0 \text{ for } K \neq n. \end{cases}$$

Hence $\displaystyle\oint_c X(z)\, z^{K-1}\, dz = 2\pi j \sum_{n=-\infty}^{\infty} x(n)\, \delta_{Kn}$

$$= 2\pi j\, x(K).$$

$$x(K) = \frac{1}{2\pi j}\oint_c X(z)\, z^{K-1}\, dz$$

replace $K = n$.

$$x(n) = \frac{1}{2\pi j}\oint_c X(z)\, z^{n-1}\, dz.$$

$\searrow$ inverse z-transform

$x(n) = \sum$ residue of $X(z)\,z^{n-1}$ at the poles inside

$$= \sum_i (z-z_i)\,X(z)\,z^{n-1}\Big|_{z=z_i}$$

<u>Ex</u>

$$X(z) = \frac{z+1}{(z+0.2)(z-1)} \qquad |z|>1$$

$$x(n) = \frac{1}{2\pi j}\oint_c X(z)\,z^{n-1}\,dz.$$

$= \sum$ residues of $X(z)z^{n-1}$ at poles of $X(z)z^{n-1}$ co...

$= \sum$ residues of $\dfrac{(z+1)\,z^{n-1}}{(z+0.2)(z-1)}$ at poles of same wi...

$x(0) = \sum$ residues of $\dfrac{z+1}{z(z+0.2)(z+1)}$ at pole $z=0$, $z=1$, $z=$...

$$= \cancel{z}\,\frac{z+1}{\cancel{z}(z+0.2)(z+1)}\Bigg|_{z=0} + \cancel{(z+0.2)}\,\frac{z+1}{z\cancel{(z+0.2)}(z+1)}\Bigg|_{z=-0.2} + \cancel{(z+1)}\,\frac{z+1}{z(z+\ldots}$$

$$= -5 + \frac{10}{3} + \frac{5}{3} = 0$$

$x(0) = 0.$

for $n \geq 1$

$x(n) = \sum$ residues of $\dfrac{(z+1)\,z^{n-1}}{(z+0.2)(z-1)}$ at poles $z=-0.2$, $z=1$.

$$= \cancel{(z+0.2)}\,\frac{(z+1)\,z^{n-1}}{\cancel{(z+0.2)}(z-1)}\Bigg|_{z=-0.2} + \cancel{(z-1)}\,\frac{(z+1)\,z^{n-1}}{(z+0.2)\cancel{(z-1)}}$$

$$= \frac{0.8}{-1.2}(-0.2)^{n-1}\,u(n-1) + \frac{5}{3}\,u(n-1)$$

$x(n) = -\frac{2}{3}(-0.2)^{n-1}u(n-1) + \frac{5}{3}u(n-1)$ (50)

Convolution method :

$$Z\{x_1(n) * x_2(n)\} = X_1(z)X_2(z) = X(z)$$

Ex

$$X(z) = \frac{1}{1-3z^{-1}+2z^{-2}}$$

$$X(z) = \frac{1}{(1-z^{-1})(1-2z^{-1})} = X_1(z)X_2(z)$$

$$X_1(z) = \frac{1}{1-z^{-1}} \qquad X_2(z) = \frac{1}{1-2z^{-1}}$$

$$= \frac{z}{z-1} \qquad = \frac{z}{z-2}$$

$x_1(n) = 1^n u(n)$ $\qquad$ $x_2(n) = 2^n \cdot u(n)$

$$x(n) = x_1(n) * x_2(n)$$

$$= \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

$$= \sum_{k=0}^{\infty} u(k)\, 2^{n-k}\, u(n-k)$$

$$= \sum_{k=0}^{n} 2^{n-k} = 2^n \sum_{k=0}^{n} 2^{-k}$$

$$= 2^n \left[\frac{1-(1/2)^{n+1}}{1-1/2}\right] u(n)$$

$$= 2 \cdot 2^n \left[1-(2)^{-(n+1)}\right] = 2^{n+1}\left[1-2^{-(n+1)}\right]$$

$$= \left[2^{n+1}-1\right]u(n).$$

The system function

In general the system is described by a linear constant coefficient difference equation of the form

$$y(n) = -\sum_{k=1}^{N} a_k\, y(n-k) + \sum_{k=0}^{M} b_k\, x(n-k).$$

apply z - transform on both sides.

$$Y(z) = -\sum_{k=1}^{N} a_k\, Y(z)\, z^{-k} + \sum_{k=0}^{M} b_k\, X(z)\, z^{-k}.$$

$$Y(z) + \sum_{k=1}^{N} a_k\, z^{-k}\, Y(z) = \sum_{k=0}^{M} b_k\, X(z)\, z^{-k}$$

$$Y(z)\left[1 + \sum_{k=1}^{N} a_k z^{-k}\right] = \sum_{k=0}^{M} b_k\, z^{-k}\, X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}.$$

Poles & zeros of a system function

We know the system function.

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}.$$

if $a_k = 0$ for $1 \le k \le N$

then $H(z) = \sum_{k=0}^{M} b_k z^{-k}.$

it is called all zero system.

(or) FIR system. [Finite impulse response] if $b_K = 0$ for $1 \le K \le M$.

inear

orm.

$$H(z) = \frac{b_0}{1 + \sum_{K=1}^{N} a_K z^{-K}}.$$

↳ it is called all pole system.

(or) IIR system [infinite impulse response]

Ex Determine pole zero plot for the s/m.

$$y(n) - \frac{3}{4} y(n-1) + \frac{1}{8} y(n-2) = x(n) - x(n-1)$$

apply z-transform.

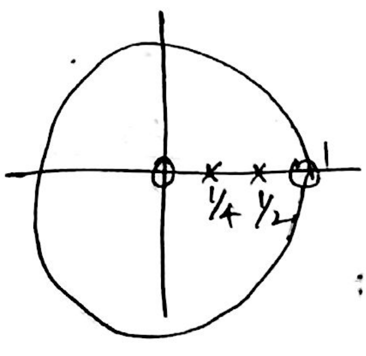
$$Y(z) - \frac{3}{4} z^{-1} Y(z) + \frac{1}{8} Y(z) z^{-2} = X(z) - z^{-1} X(z)$$

$$Y(z)\left[1 - \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2}\right] = X(z)\left[1 - z^{-1}\right]$$

$$\frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 - \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2}}.$$

$$= \frac{1 - z^{-1}}{\left(1 - \frac{1}{4} z^{-1}\right)\left(1 - \frac{1}{2} z^{-1}\right)}$$

$$= \frac{z(z-1) \rightarrow \text{zeros}}{(z - 1/4)(z - 1/2) \rightarrow \text{poles}}.$$



poles $z - 1/4 = 0$ $z - 1/2 = 0$

$z = 1/4$ $z = 1/2$

zeros $z = 0$ $z - 1 = 0$

$z = 1$

Solution of difference equations using one sided z-transform

Same as transfer function. in this method just we will find impulse response also $\text{inv } Z[H(z)] = h(n)$.

See Previous Example

$$y(n) - \frac{3}{4} y(n-1) + \frac{1}{8} y(n-2) = x(n) - x(n-1)$$

$$\frac{H(z)}{z} = \frac{Y(z)}{X(z)} = \frac{(z-1)}{(z-1/4)(z-1/2)} = \frac{C_1}{z-1/4} + \frac{C_2}{z-1/2}$$

$$C_1 = \cancel{(z-1/4)} \frac{(z-1)}{\cancel{(z-1/4)}(z-1/2)} \Bigg|_{z=1/4} \quad C_2 = \cancel{(z-1/2)} \frac{z-1}{(z-1/4)\cancel{(z-1/2)}}$$

$$= \frac{\left(\frac{1}{4}-1\right)}{\frac{1}{4}-1/2} = \frac{\frac{1-4}{4}}{\frac{1-2}{4}} = \frac{-3}{-1} = 3.$$

$$C_2 = \frac{\frac{1}{2}-1}{\left(\frac{1}{2}-1/4\right)} = \frac{\frac{-1}{2}}{1/4} = \frac{-1}{\cancel{2}} \times \frac{\cancel{4}}{1} = -2.$$

$$\frac{H(z)}{z} = \frac{3}{z-1/4} - \frac{2}{z-1/2}$$

$$H(z) = \frac{3z}{z-1/4} - \frac{2z}{z-1/2}.$$

$$h(n) = 3\,(1/4)^n u(n) - 2\,(1/2)^n u(n)$$

realization of digital filters

1. Recursive
2. Non recursive.

Recursive :- For recursive realization the current o/p $y(n)$ is a function of past outputs, present & past i/p's. This form corresponds to infinite impulse response [IIR]. Digital filter.

Non recursive :- For Nonrecursive realization the current o/p $y(n)$ is a function of only past & present i/p's. This form corresponds to finite impulse response [FIR] Digital filter.

IIR filters can be realized in many forms. They are.

1. Direct form-I realization
2. Direct form-II "
3. Transposed direct form "
4. cascade form "
5. parallel form "
6. Lattice Ladder Structure.

Direct form-I realization

Let us consider an LTI recursive system described by the difference equation.

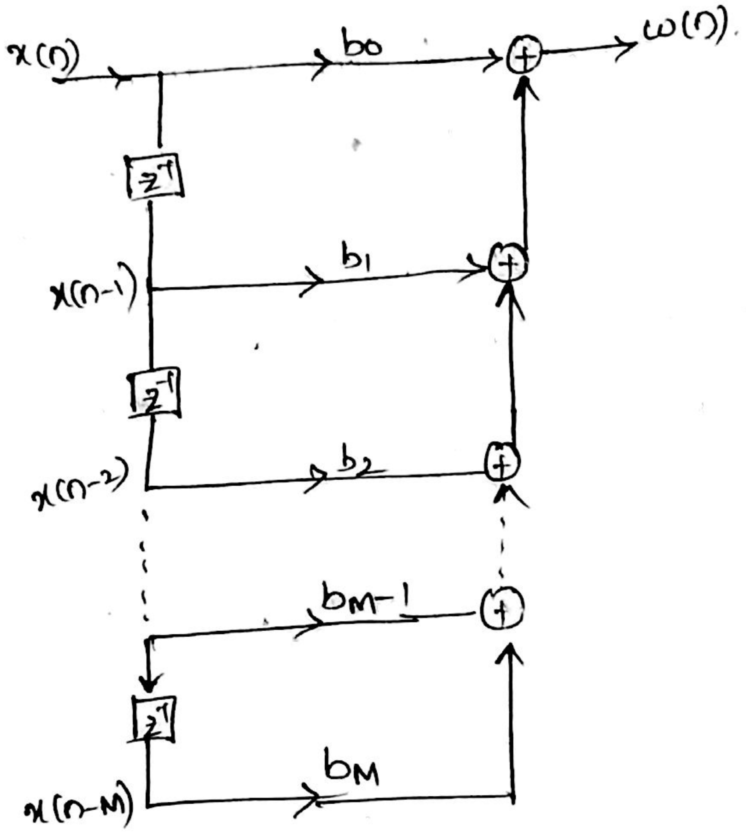
$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

$$= -a_1\, y(n-1) - a_2\, y(n-2) \cdots - a_{N-1}\, y(n-N+1) - a_N\, y(n-N)$$

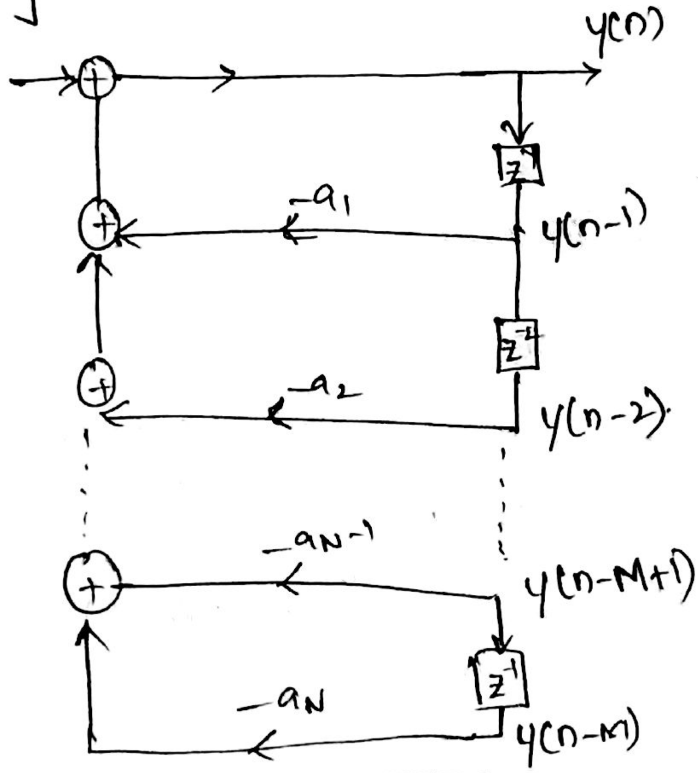
$$+ b_0\, x(n) + b_1 x(n-1) + \cdots + b_M\, x(n-M).$$

let $\quad b_0 x(n) + b_1 x(n-1) + \cdots + b_M\, x(n-M) = w(n)$

then $\quad y(n) = -a_1 y(n-1) - a_2 y(n-2) \cdots - a_N\, y(n-N) + w(n)$



Similarly



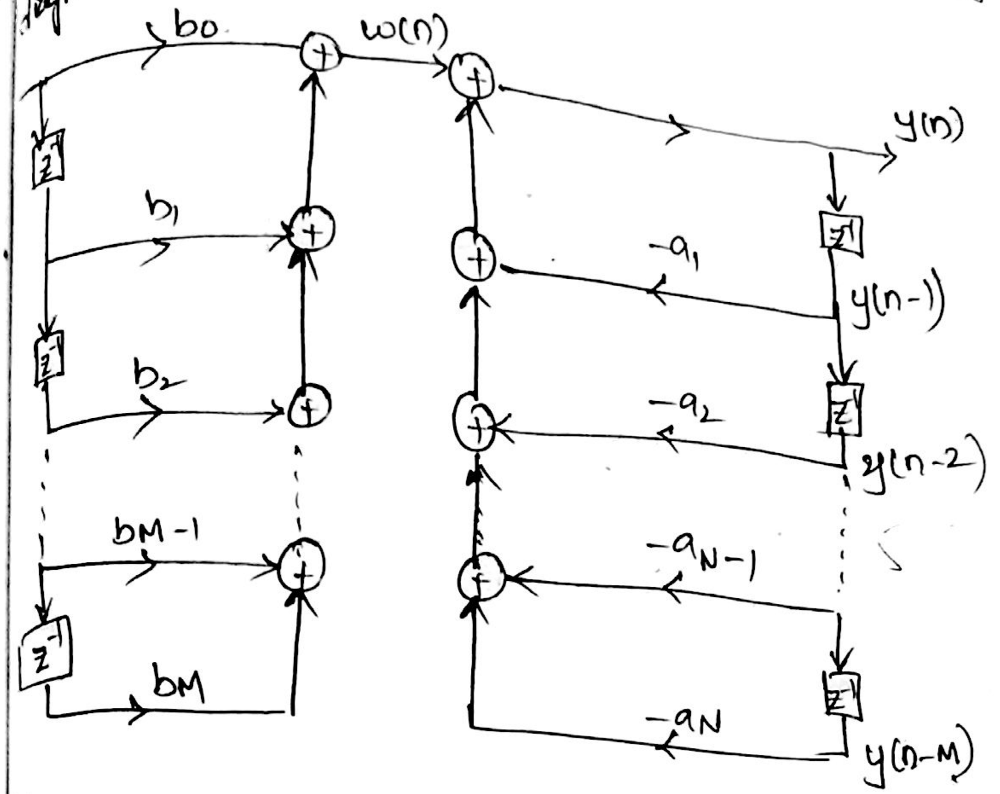
This realization requires

$M+N+1 \rightarrow$ multip

$M+N \rightarrow$ addition

$M+N+1 \rightarrow$ memory location

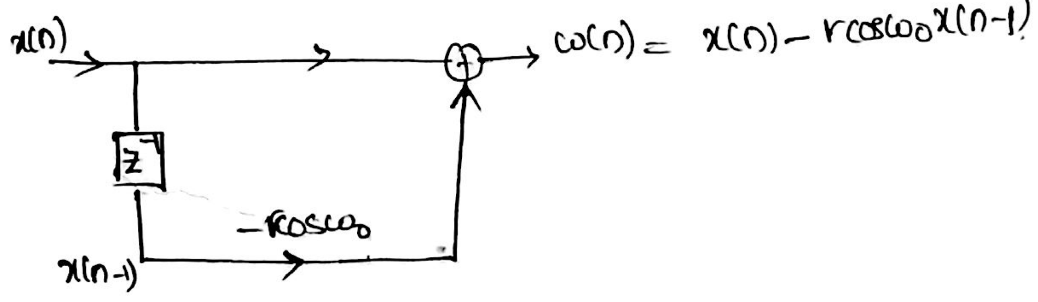
To realize the difference equations combine both the diagrams. (5)

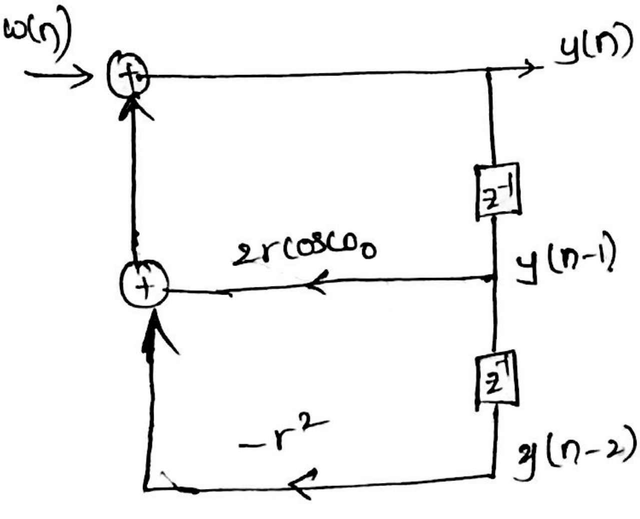


Ex:

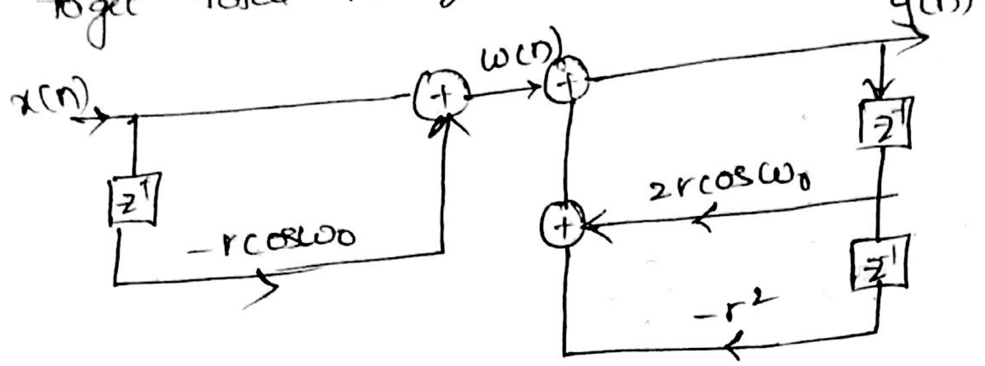
$$y(n) = 2r\cos(\omega_0)\, y(n-1) - r^2 y(n-2) + x(n) - r\cos\omega_0\, x(n-1)$$

$$w(n) = x(n) - r\cos\omega_0\, x(n-1)$$





to get total realization combine both the diagram



Direct form II realization

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k).$$

the system function.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}.$$

Let $\frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)}$.

where $\frac{W(z)}{X(z)} = \frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}}$.

$$W(z) = \frac{X(z)}{1+\sum_{k=1}^{N} a_k z^{-k}}.$$

$$X(z) = W(z)\left[1+\sum_{k=1}^{N} a_k z^{-k}\right]$$

$$= W(z) + W(z)\left[a_1 z^{-1} + a_2 z^{-2} + \dots a_N z^{-N}\right]$$

$$X(z) - a_1 z^{-1} W(z) - a_2 z^{-2} W(z) + \dots a_N z^{-N} W(z) = W(z).$$

↳ inverse z-transform.

$$w(n) = x(n) - a_1 \cancel{w(n)}\, w(n-1) + a_2 w(n-2) \dots - a_N w(n-N) \rightarrow (1)$$

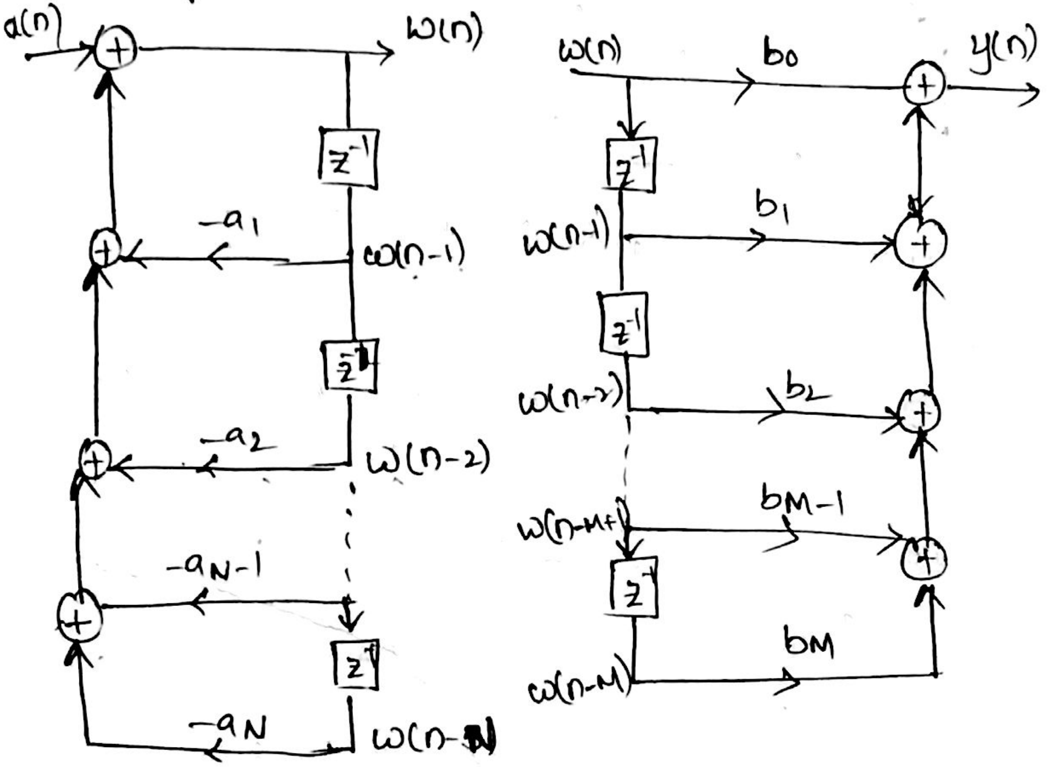
$$\frac{Y(z)}{W(z)} = \sum_{k=0}^{M} b_k z^{-k}$$

$$Y(z) = W(z)\left[b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}\right]$$

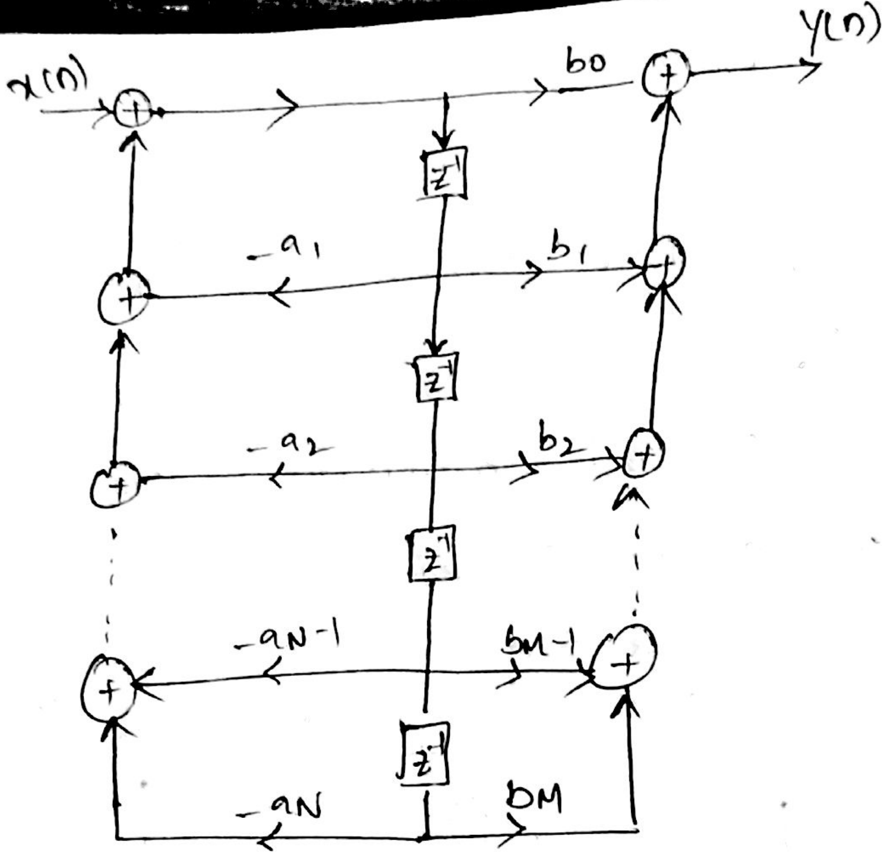
$$Y(z) = W(z)\, b_0 + b_1 z^{-1} W(z) + b_2 z^{-2} W(z) + \cdots + b_M z^{-M} W(z)$$

$$y(n) = b_0 w(n) + b_1 w(n-1) + b_2 w(n-2) + \cdots + b_M w(n-M) \rightarrow (2)$$

we can observe that for equ(1) & equ(2) the delay terms are same.



from fig we observe that the two delay elements contain the same input w(n) & hence the same o/p w(n-1). so we can merge (or) combine these delays into one delay



This structure requires $M+N+1$ multiplication
$M+N$ additions
& $Max\{M,N\}$ memory location.

Ex

$$y(n) = 2r\cos\omega_0\, y(n-1) - r^2 y(n-2) + x(n) - r\cos\omega_0\, x(n-1)$$

$$Y(z) = 2r\cos\omega_0\, z^{-1} Y(z) - r^2 z^{-2} Y(z) + X(z) - r\cos\omega_0\, X(z) z^{-1}$$

$$Y(z) - 2r\cos\omega_0\, z^{-1} Y(z) + r^2 z^{-2} Y(z) = X(z) - r\cos\omega_0\, X(z) z^{-1}$$

$$Y(z)\left[1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2}\right] = X(z)\left[1 - r\cos\omega_0 z^{-1}\right]$$

$$\frac{Y(z)}{X(z)} = \frac{1 - r\cos\omega_0 z^{-1}}{1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2}}.$$

Let $\frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)}$

$$\frac{Y(z)}{W(z)} = 1 - r\cos\omega_0 z^{-1}, \quad \frac{W(z)}{X(z)} = \frac{1}{1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2}}$$

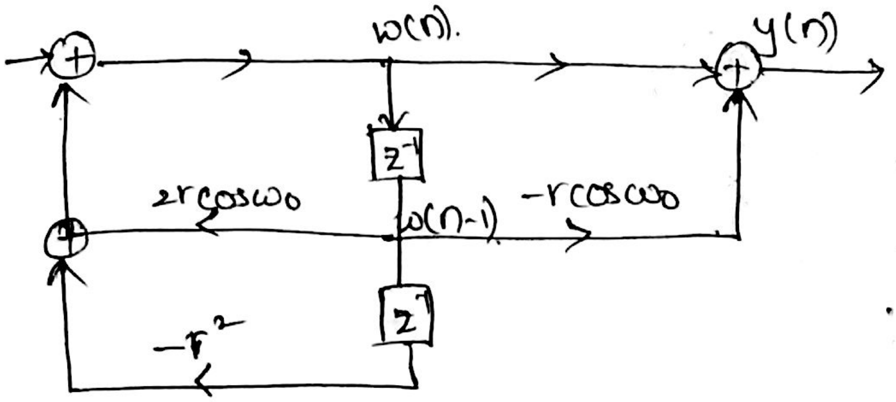
$y(z) = W(z) - r\cos\omega_0 z^{-1} W(z)$

$y(n) = \omega(n) - r\cos\omega_0\, \omega(n-1)$

$W(z) = X(z) + 2r\cos\omega_0 z^{-1} W(z) - r^2 z^{-2} W(z)$

$\omega(n) = x(n) + 2r\cos\omega_0\, \omega(n-1) - r^2 \omega(n-2)$

$\cancel{x(n) = \omega(n) - 2r\cos\omega_0\, \omega(n-1) + r^2 \omega(n-2)}$



Signal flow graph :-

A signal flow graph is a graphical representation of the relationship b/w the variables of a set of linear difference equations.

* The basic elements of a signal flow graph are branches and nodes.

* The signal flow graph is basically a set of directed branches that connect at nodes.

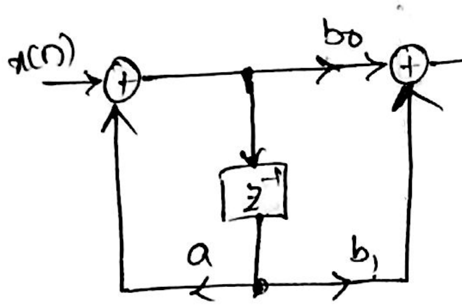
* A node represents a system variable, which is equal to the sum of incoming signals from all branches connecting to the nodes.

* There are two types of nodes 1. Source nodes
 2. Sink nodes

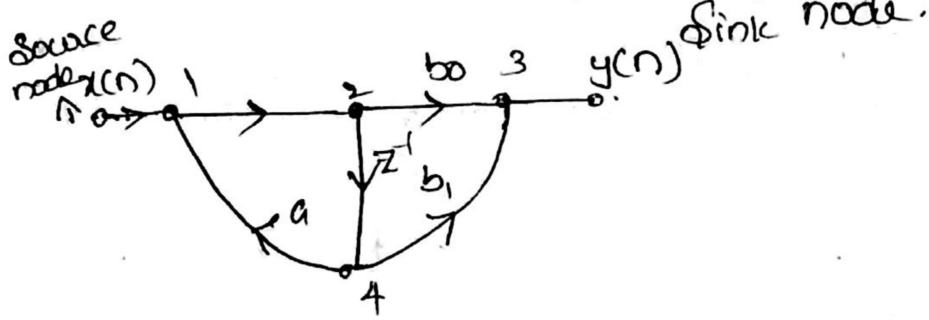
* Source nodes are nodes that have no entering branches

* Sink " " " " only "

* A signal travels along a branch from one node to another node.
* The signal out of a branch is equal to the branch gain times the signal into the branch.
* The delay is indicated by the branch transmittance z^{-1}.
* Let us consider a block diagram.



It can be converted into signal flowgraph.



3) <u>Transposition theorem & Transposed Structure</u>:-

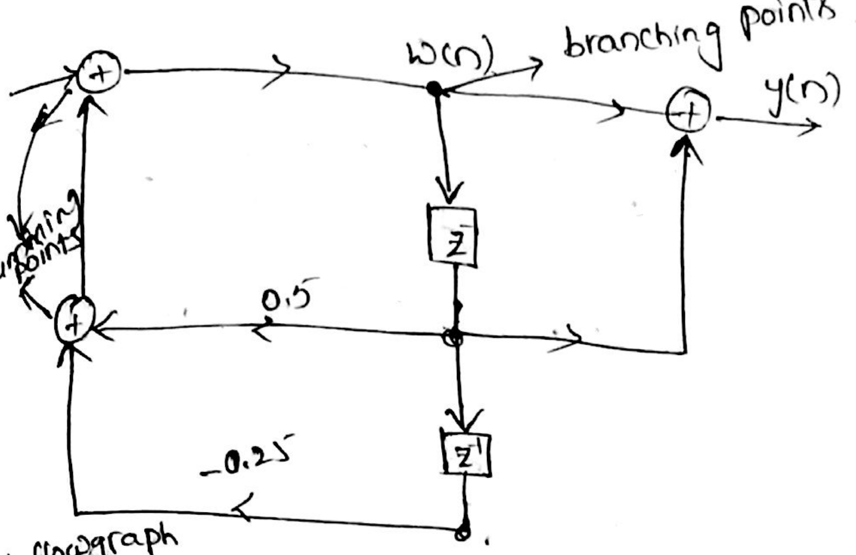
(i) Reverse the direction of all branches in the signal flowgraph
(ii) Interchang the i/p's & o/p's
(iii) Reverse the roles of all nodes in the flowgraph
(iv) Summing points become branching points.
(v) Branching points become Summing points.

Ex

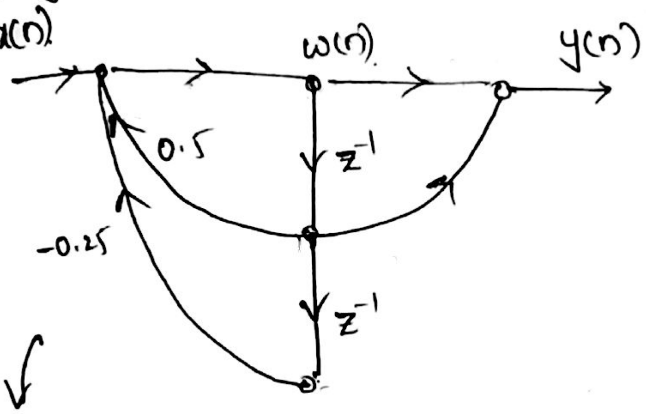
$$y(n) = \frac{1}{2} y(n-1) - \frac{1}{4} y(n-2) + x(n) + x(n-1)$$

$$Y(z) = \frac{1}{2} z^{-1} Y(z) - \frac{1}{4} z^{-2} Y(z) + X(z) + z^{-1} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{1-0.5z^{-1}+0.25z^{-2}}$$



signal flowgraph

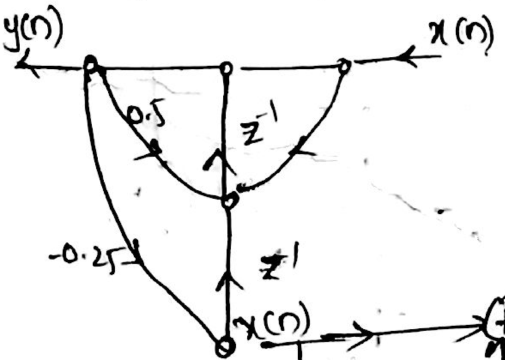


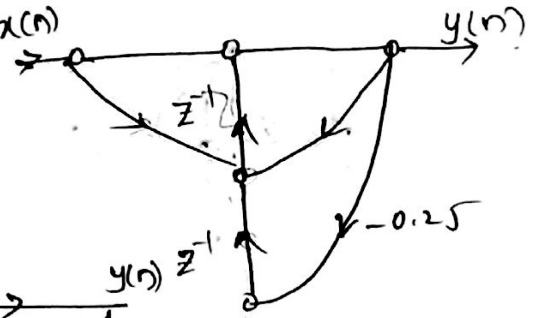
convert to transposed.

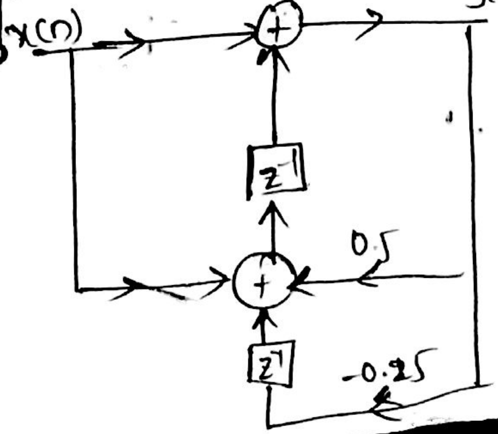
(i) change the direction of all branches

(ii) Interchange the i/p & o/p.

(iii) change summing point to branching point & vice-versa



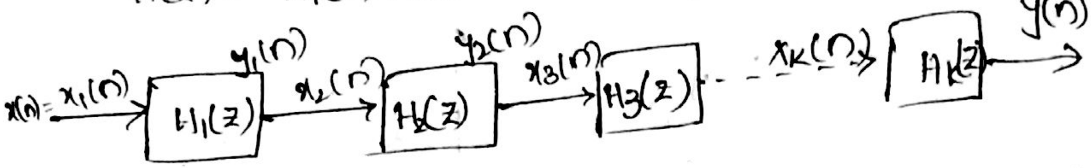




Cascade form :-

Let us consider an IIR system with system func

$$H(z) = H_1(z)\,H_2(z)\cdots H_K(z).$$



Now realize each $H_K(z)$ in direct form II and cascade all structures.

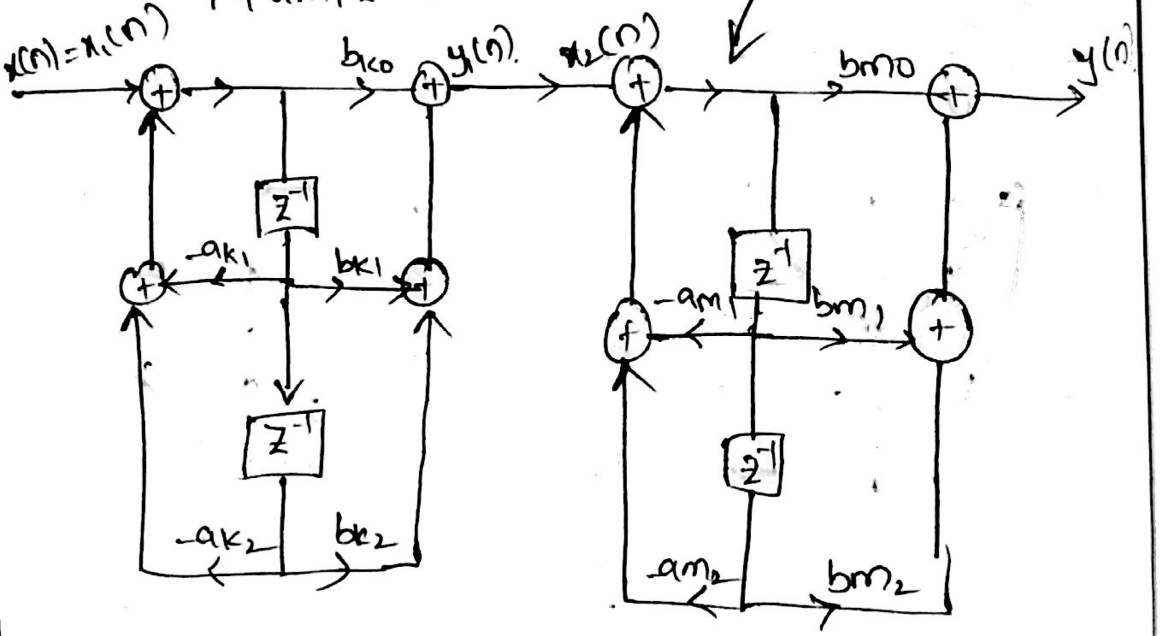
For example let us take a system whose transfer function

$$H(z) = \frac{(b_{K0} + b_{K1}z^{-1} + b_{K2}z^{-2})(b_{m0} + b_{m1}z^{-1} + b_{m2}z^{-2})}{(1 + a_{K1}z^{-1} + a_{K2}z^{-2})(1 + a_{m1}z^{-1} + a_{m2}z^{-2})}$$

$$= H_1(z)\,H_2(z).$$

$$H_1(z) = \frac{b_{K0} + b_{K1}z^{-1} + b_{K2}z^{-2}}{1 + a_K z^{-1} + a_{K_2} z^{-2}}$$

$$H_2(z) = \frac{b_{m0} + b_{m1}z^{-1} + b_{m2}z^{-2}}{1 + a_{m1}z^{-1} + a_{m2}z^{-2}}$$ → from direct form II



Ex

$$y(n) = \frac{3}{4}y(n-1) - \frac{1}{8}y(n-2) + x(n) + \frac{1}{3}x(n-1)$$

Parallel form Structure :-

$$\cancel{H(z) = C + \sum_{k=1}^{N} \frac{C_k}{1 - p_k z^{-1}}}$$

$$H(z) = \frac{1 + 3z^{-1} + 2z^{-2}}{1 + \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3}}$$

$$H(z) = \frac{(1+z^{-1})(1+2z^{-1})}{\left(1+\frac{1}{8}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$$

$$H(z) = \frac{A}{1+\frac{1}{8}z^{-1}} + \frac{B}{1+\frac{1}{2}z^{-1}} + \frac{C}{1-\frac{1}{4}z^{-1}}$$

$$A = \left(1+\frac{1}{8}z^{-1}\right) H(z) \Big|_{z^{-1} = -8}$$

$$= \cancel{\left(1+\frac{1}{8}z^{-1}\right)} \frac{(1+z^{-1})(1+2z^{-1})}{\cancel{\left(1+\frac{1}{8}z^{-1}\right)}\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)} \Bigg|_{z^{-1} = -8}$$

$$= \frac{(1-8)(1-16)}{\left(1+\frac{1}{2}(-8)\right)\left(1+\frac{1}{4}(8)\right)}$$

$$= \frac{(-7)(-15)}{(-3)(3)} = \frac{-35}{3}.$$

$$B = \left(1+\frac{1}{2}\right) z^{-1} (H(z)) \Big|_{z^{-1} = -2.}$$

$$B = 8/3.$$

$$C = 0.$$

$$H(z) = \frac{-\frac{35}{3}}{1+\frac{1}{8}z^{-1}} + \frac{8/3}{1+\frac{1}{2}z^{-1}} + \frac{10}{1-\frac{1}{4}z^{-1}}.$$

$$H(z) = H_1(z) + H_2(z) + H_3(z)$$

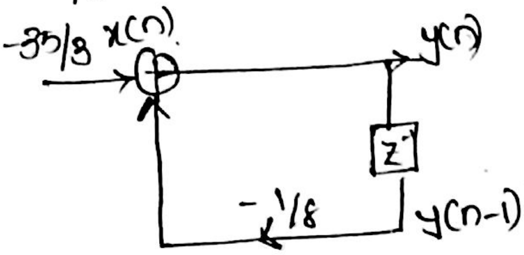
$$H_1(z) = \frac{-\frac{35}{3}}{1+\frac{1}{8}z^{-1}} = \frac{Y(z)}{X(z)}$$

$$-35/3\, X_1(z) = Y_1(z) + \frac{1}{8} z^{-1} Y_1(z)$$

applying inverse z-transform.

$$-35/3\, x_1(n) = y_1(n) + \frac{1}{8}\, y(n-1)$$

$$y_1(n) = -\frac{35}{3}x(n) - \frac{1}{8}y(n$$



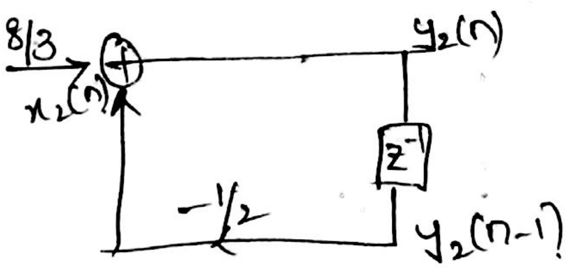
$$H_2(z) = \frac{8/3}{1+\frac{1}{2}z^{-1}} = \frac{Y_2(z)}{X_2(z)}$$

$$Y_2(z) + Y_2(z)\,\frac{1}{2}z^{-1} = X_2(z)\ 8/3.$$

applying inverse z-transform.

$$y_2(n) + y_2(n-1)\,\frac{1}{2} = x_2(n)\ 8/3.$$

$$y_2(n) = \frac{8}{3}x_2(n) - \frac{1}{2}\,y_2(n-1)$$



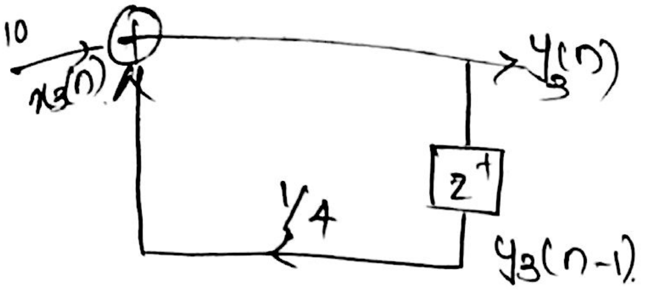
$$H_3(z) = \frac{Y_3(z)}{X_3(z)} = \frac{10}{1-\frac{1}{4}z^{-1}}.$$

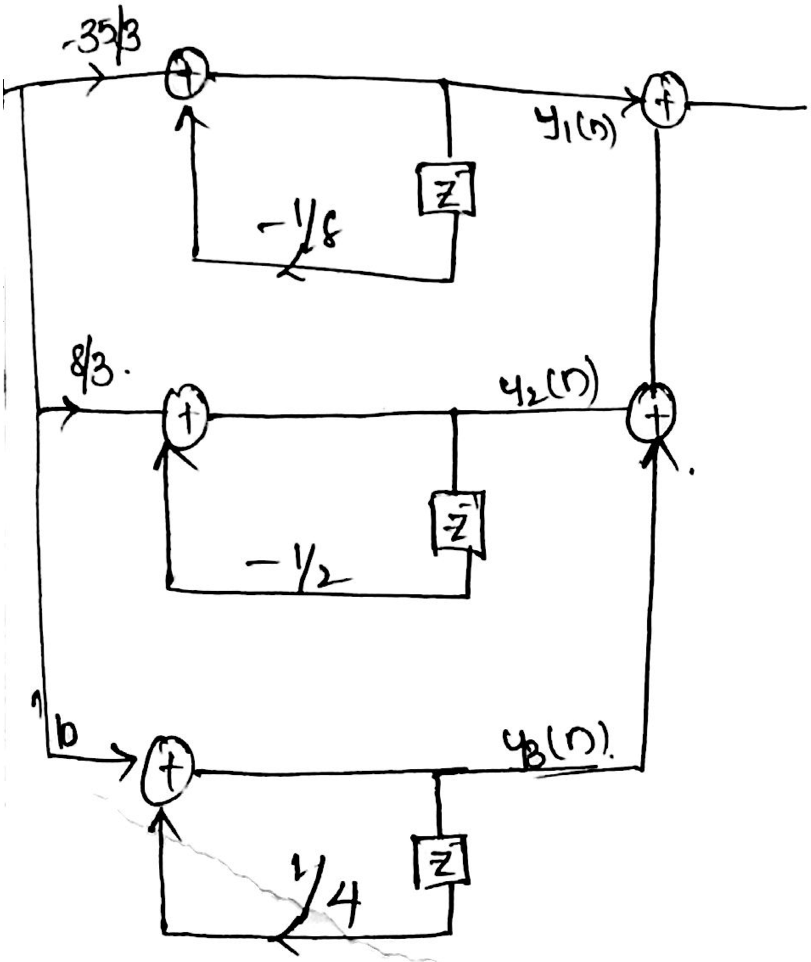
$$Y_3(z) - \frac{1}{4} z^{-1} Y_3(z) = 10\, X_3(z)$$

appling inverse z-transform.

$$y_3(n) - \frac{1}{4}\, y_3(n-1) = 10\, x_3(n).$$

$$\therefore y_3(n) = 10\, x_3(n) + y_3(n-1)\, \frac{1}{4}.$$





Basic FIR Structures :-