UNIT - II

Function of Complex Variable:-

Def:- If x and y are real then a number is of form x+iy is called complex variable and denoted by $z = x+iy$ where $i^2 = -1$

Def:- of function of complex variable:-

For a complex variable $z = x+iy$ their exits another complex variable is of form $w = u+iv$ where u, v are real.

It corresponding to each value of 'z' their exits one or more values in w corresponding to mapping 'f' then $w = f(z)$ is said to be a complex value function.

$$w = f(z) = u+iv$$

$$u = u(x,y) \; ; \; v = v(x,y)$$

Polar form of complex variable:-

A complex variable $z = x+iy$ written is of form $z = r(\cos\theta + i\sin\theta)$ is called Polar form of 'z'. here r is said to be $|z|$ and θ is amplitude of z and is given by $r = |z| = \sqrt{x^2+y^2}$

$$\theta = \tan^{-1}(y/x)$$

Properties:-

1) $z = x+iy$; $\bar{z} = x-iy$
2) $z.\bar{z} = |z|^2$
3) $|z| = |\bar{z}|$

4) $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$

5) $|z_1+z_2| \le |z_1|+|z_2|$

6) $|z_1-z_2| \ge |z_1|+|z_2|$

7) $|z_1+z_2| \ge |z_1|-|z_2|$

* Write the Polare form of following.

a) $z = 1+i$

$\sqrt{2}(\cos \pi/4 + i \sin \pi/4)$

b) $1+\cos\alpha + i\sin\alpha$

$2(\cos^2 \alpha/2 + i\, 2\sin \alpha/2 \cos \alpha/2)$

c) $1+i\cos\alpha + \sin\alpha$

$1 + i\sin\left(\frac{\pi}{2}-\alpha\right) + \cos\left(\frac{\pi}{2}-\alpha\right)$

Limit of a function:-

If a function $\omega = f(z)$ is said to have limit l has $z \to z_0$ if for $\forall x \in (0,\infty)$ there exits a real number $|f(z)-l| < \epsilon \ \forall\ |z-z_0| < \delta$ simillarly we can write $\lim_{z\to z_0} f(z) = l$

Continuity of function:- if function $\omega = f(z)$ is said to be continous $z = z_0$ is $f(z) = f(z_0)$

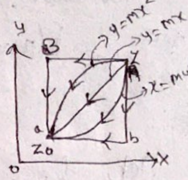
Result:-

1) $f(z)$ is continous in region are then it is continous at every point of region P

2) if $f(z)$ is continous function of z then $|f(z)|$ function is also function of 'z'.

3) $\omega = f(z)$ which is equal to $u+iv$ is continous function of z then u, v are also continous at z

4) the converge is also function.



Note:-

In the above defination $z \to z_0$ exits only in 5 Paths.

1) Along APQ

2) " ABQ

3) " a line $y = mx$

4) " Parabola $y = mx^2$

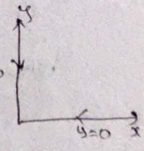
5) " Parabola $x = my^2$

Problems:-

Q:- S.T the limit of the function $f(z) = \begin{cases} \frac{Re(z)}{|z|} & ; z \ne 0 \\ 0 & z = 0 \end{cases}$

By the Def of limit

$\lim_{z\to z_0} f(z) = \lim_{z\to 0} \frac{Re(z)}{|z|} \ (z \ne 0)$



$= \lim_{z\to 0} \frac{x}{\sqrt{x^2+y^2}}$

first assume $z \to 0$ along the direction of x-axis given function is $\lim_{z\to 0} \frac{x}{\sqrt{x^2}} = 1 \to (1)$

Similarly on the direction of y-axis (i.e $x=0$)

$= \lim_{z\to 0} 0 = 0 \to (2)$

$\Rightarrow (1) \neq (2)$

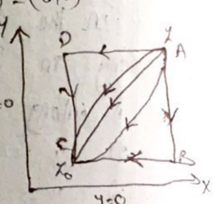
The limit of $f(z)$ is not exists.

P2)* Discuss the continuity of function at origin.

$$f(x,y) = \begin{cases} \frac{2xy(x+y)}{x^2+y^2} & ; (x,y)\neq(0,0) \\ 0 & ; (x,y)=(0,0) \end{cases}$$

Sol:-

let $l = \lim_{(x,y)\to(0,0)} \frac{2xy(x+y)}{x^2+y^2}$



1) along the path ADC

the first $y\to 0$ then $x\to 0$

$$l = \lim_{y\to 0}\left\{\lim_{x\to 0} \frac{2xy(x+y)}{x^2+y^2}\right\} = 0$$

2) along the path ABC

here $x\to 0$ first, then $y\to 0$

$$l = \lim_{y\to 0}\left\{\lim_{x\to 0} \frac{2xy(x+y)}{x^2+y^2}\right\} = 0$$

3) along $y = mx$

$$l = \lim_{x\to 0} \frac{2x(mx)(x+mx)}{x^2+mx^2}$$

$$= \lim_{x\to 0} \frac{2x^3 m(1+m)}{x^2(1+m^2)} = 0$$

4) along $y = mx^2$

$$l = \lim_{x\to 0} \frac{2x(mx^2)(x+mx^2)}{x^2+m^2x^4} = 0$$

5) along $x = my^2$

$$l = \lim_{y\to 0} \frac{2my^2(y)(my^2+y)}{m^2y^4+y^2} = 0$$

6) $\Rightarrow \lim_{z\to 0} f(z)$ is exists

also $f(0,0) = 0$

here $\lim_{z\to 0} f(z) = f(0)$

Sp)
$$f(x,y) = \begin{cases} \frac{x^3-y^3}{x^2+y^2} & ; (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$$

$$f(x,y) = \begin{cases} \frac{xy^3}{x^4+y^4} & ; (x,y)\neq(0,0) \\ 0 & ; (x,y)=(0,0) \end{cases}$$

Def:- Derivative of a function:-

Let $f(z)$ be a complex valued function defined at a point z_0 then the derivative of $f(z)$ at the point $z = z_0$ is denoted by $f'(z_0)$ (or) $f'(z)$ and is given by limit.

$$\lim_{\delta z \to 0} \frac{f(z_0+\delta z)-f(z_0)}{\delta z}$$ is exits.

Result:- Every derivable function is continous ~~and~~ but converse need not be true.

Eg:- Consider $f(z) = \bar{z}$

clearly $\lim_{z\to 0} f(z) = 0 = f(0)$.

checking of derivability :-

By def:- $f'(z_0) = \lim_{\delta z\to 0} \frac{f(z_0+\delta z)-f(z_0)}{\delta z}$.

along x-axis $\therefore y=0$

$$f'(z_0) = \lim_{\delta z\to 0} \frac{x+\Delta x - x}{\delta x} = 1$$

along y-axis $x=0$; $\Delta x = 0$

$$f'(z_0) = \lim_{\delta z = 0} \frac{-iy - i\Delta y + iy}{i\Delta y} = -1$$

here $f(z) = \bar{z}$ is not derivable.

Q) Show that $f(z) = z^n$ where n is positive integer is differentiable $\forall$ values of z.

Sol:- given $f(z) = z^n$; $n>0$

By def of derivability we write

$$f'(z) = \lim_{\delta z\to 0} \frac{(z+\delta z)^n - z^n}{\delta z}$$

$$= \lim_{\delta z\to 0} \frac{\cancel{z^n} + n.z^{n-1}\delta z + \frac{n(n-1)}{2}.z^{n-2}(\delta z)^2 + \dots + (\delta z)^n - \cancel{z^n}}{\delta z}$$

$$= n.z^{n-1} ; \forall z.$$

Anylatic functions:-

Def:- Anylatic at a point:- A single valid function $f(z)$ possess a unique derivative at point z_0 and in some neighbour hood of z_0 then we say $f(z)$ is anylatic at point $z = z_0$

Anylatic function:- A function $f(z)$ is said to be anylatic in a region if it is anylatic at every point in the region.

Singular point:- singular point is a point at which an anylatic function cases to be analytic.

Entire functions:- A function which is analytic every where is called entire function.

Eg. $f(z) = e^z$ is analytic every where in complex function

$\times$ $f(z) = \frac{1}{z}$ $(z \neq 0)$ is " " " " "

The Cauchy – Riemann Eq :-

Statement:- The nessary and suficent condition that function $\omega = F(z) = u(x,y) + iv(x,y)$ is to be anylatic in a region where

(i) $\frac{du}{dx}, \frac{du}{dx}, \frac{dv}{dx}, \frac{dv}{dy}$

(ii) $\frac{du}{dx} = \frac{dv}{dy}$; $\frac{du}{dy} = -\frac{dv}{dx}$.

Proof:- necessary condition:-

Since $F(z)$ is anylatic

$\Rightarrow \omega = F(z) = u(x,y) + iv(x,y)$ is differentiable

$\Rightarrow u_x, u_y, v_x, v_y$ are continous functions

Since the derivibility of $f(z)$, denoted by $F'(z)$ is defined as follows

$$f'(z) = \lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z}$$

$$\Rightarrow f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\{(u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)\} - \{u(x,y) + iv(x,y)\}}{\Delta x + i\Delta y}$$

That means the limits and the same for all the directions along with $\Delta_{x \to 0}$

Case i:- let $\Delta z \to 0$ along x-axis then $\Delta y = 0$ & $\Delta z = \Delta x$

$$f'(z) = \lim_{\Delta \to 0} \frac{u(x+\Delta x, y) + iv(x+\Delta x, y) - u(x,y) - iv(x,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$= \frac{du}{dx} + i\frac{dv}{dx} \to (1)$$

Case (ii):- let $\Delta z \to 0$ along y-axis then $x = 0$; $\Delta x = 0$; $\Delta y = \Delta z = i\Delta y$

$$f'(z) = \lim_{\Delta y \to 0} \left\{ \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i\frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y} \right\}$$

$$= \frac{1}{i}\frac{du}{dx} + \frac{dv}{dy}$$

$$= \frac{dv}{dy} - i\frac{du}{dy} \quad (\because \tfrac{1}{i} = -i) \to (2)$$

$\because f(z)$ is derivable ie (1) = (2).

$$\frac{du}{dx} + i\frac{dv}{dx} = \frac{dv}{dx} + i\frac{du}{dy}$$

$$\frac{du}{dx} = \frac{dv}{dy}, \quad \frac{du}{dy} = -\frac{dv}{dx}$$

Sufficient Condition:-

Conversly suppose $f(z)$ is any function satisfing (1) & (2) conditions in therome.

We have to prove that $f(z)$ is anylatic.

$\Rightarrow f'(z)$ Exits

now $f(z+\Delta z) = u(x+\Delta x, y+\Delta y) + i\,v(x+\Delta x, y+\Delta y)$

Expanding R.H.S. using Taylor's Th to function two variables and neglecting seconds higher order terms in Δx & Δy we get

$$f(z+\Delta z) = u(x,y) + \left(\Delta x \frac{\partial u}{\partial x} + \Delta y \cdot \frac{\partial u}{\partial y}\right) + i\left(v(x,y) + \left(\Delta x \frac{\partial v}{\partial x} + \Delta y \frac{\partial v}{\partial y}\right)\right)$$

$$= u(x,y) + i v(x,y) + \left(\Delta x \frac{\partial u}{\partial x} + \Delta y \frac{\partial u}{\partial y}\right) + i\left(\Delta x \frac{\partial v}{\partial x} + \Delta y \frac{\partial v}{\partial y}\right)$$

$$f(z+\Delta z) = f(z) + \left(\Delta x \frac{\partial u}{\partial x} + \Delta y \frac{\partial u}{\partial y}\right) + i\left(\Delta x \frac{\partial v}{\partial x} + \Delta y \frac{\partial v}{\partial y}\right)$$

using C-R Eq

$$f(z+\Delta z) - f(z) = \Delta x\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)$$

&

$$+ \Delta x\left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)$$

$$= \Delta x\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)$$

$$+ i\Delta y\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)$$

$$= (\Delta x + i\Delta y) - \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)$$

$$\Rightarrow \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

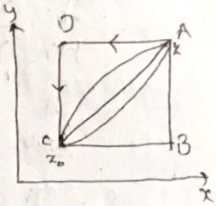
= exits.

H.W:-

3P) $$f(x,y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Sol:-

let $l = \lim_{(x,y) \to (0,0)} \dfrac{x^3 - y^3}{x^2 + y^2}$



(i) Along the Path ADC:-

$$l = \lim_{y \to 0}\left[\lim_{x \to 0} \frac{x^3 - y^3}{x^2 + y^2}\right]$$

$$l = \lim_{y \to 0}\left[\frac{-y^3}{y^2}\right] = \lim_{y \to 0} -y = 0$$

(ii) Along the Path ABC.

here $x \to 0$ first and $y \to 0$ next.

$$l = \lim_{y \to 0}\left[\lim_{x \to 0} \frac{x^3 - y^3}{x^2 + y^2}\right] = 0$$

(iii) Along the line $y = mx$

$$l = \lim_{x \to 0} \frac{x^3 - y^3}{x^2 + y^2}$$

$$= \lim_{x \to 0} \frac{x^3 - m^3x^3}{x^2 + m^2x^2}$$

$$= \lim_{x \to 0} \frac{x^3(1 - m^3)}{x^2(1 + m^2)} = 0$$

(iii) Along the Parabola $y = mx^2$

$$l = \lim_{x \to 0} \frac{x^3 - y^3}{x^2 + y^2}$$

$$= \lim_{x \to 0} \frac{x^3 - (m^3 x^6)}{x^2 + m^2 x^4}$$

$l = \lim_{x\to0} \frac{x^3(1-m^3x^2)}{x^2(1+m^2x^2)} = 0$

(v) Along the Parabola $x = my^2$

$l = \lim_{y\to0}\frac{x^3 - y^3}{x^2+y^2}$

$= \lim_{y\to0}\frac{(m^3y^6)-y^3}{(m^2y^4)+y^2}$

$l = \lim_{y\to0}\frac{y^3(m^3y^2-1)}{y^2(m^2y^2+1)} = 0$

(vi) Along x and x_0 is

$l = \lim_{x\to0} f(x)$ is Exists

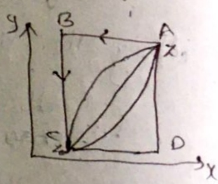
also $f(0,0) = 0$

here $\lim_{x\to0} f(x) = f(0)$

H.W

4P) $f(x,y) = \begin{cases} \frac{xy^3}{x^4+y^4} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$

let. $l = \lim_{(x,y)\to(0,0)} \frac{xy^3}{x^4+y^4}$



i) Along ABC ~~Same~~ Path

$l = \lim_{y\to0}\left[\lim_{x\to0} \frac{xy^3}{x^4+y^4}\right]$

$l = \lim_{y\to0}\left[\frac{0}{y}\right] = 0$

(ii) Along a ~~Same~~ Path ADC:-

$l = \lim_{y\to0}\left[\lim_{x\to0} \frac{xy^3}{x^4+y^4}\right] = 0$

(iii) Along a line $y = mx$

$l = \lim_{x\to0} \frac{xy^3}{x^4+y^4}$

$l = \lim_{x\to0} \frac{mx^2}{x^4+m^4x^4}$

$= \lim_{x\to0} \frac{mx^2}{(1+m^4)x^4} = 0$

(iv) Along the Parabola $y = mx^2$

$l = \lim_{x\to0} \frac{xy^3}{x^4+y^4}$

$l = \lim_{x\to0} \frac{x\,m^3x^6}{x^4+m^4x^8}$

$l = \lim_{x\to0} \frac{x\,m^3x^6}{(1+m^4x^4)x^4} = 0$

(v) Along the Parabola $x = my^2$

$l = \lim_{y\to0} \frac{xy^3}{x^4+y^4}$

$l = \lim_{y\to0} \cancel{\frac{x\,m^3y^6}{y^4+m^4y^8}} \quad \frac{my^5}{m^4y^8+y^4}$

$l = \lim_{y\to0} \frac{my^5}{(y^4m^4+1)y^4} = 0$

(vi) Along x and x_0 is

$l = \lim_{x\to0} f(x)$ is Exists

also $f(0,0) = 0$ here $\lim_{x\to0} f(x) = f(0)$

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Taylor's Theorem:-

Let $f(x,y)$ be a function of two independent variables x and y. If h & k are a small errors in x and y then
$$f(x+h, y+k) = f(x,y) + \frac{1}{1!}\left(h\frac{\partial f}{\partial x} + k\cdot\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)$$

C.R. Equations in Polar form:-

Let (r,θ) be the Polar Co-ordinates

then $x = r\cos\theta$; $y = r\sin\theta$

Since $z = r(\cos\theta + i\sin\theta)$

$z = r.e^{i\theta}$

Now $\omega = f(z)$ becomes

$$u+iv = f(r.e^{i\theta}) \to ①$$

Diff ① Partially w.r.t to r & θ we get.

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(r.e^{i\theta}).e^{i\theta} \to ②$$

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(r.e^{i\theta}).r.ie^{i\theta} \to ③$$

from ② & ③

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = ir\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$
$$= ir\left(\frac{\partial u}{\partial r}\right) - r.\frac{\partial v}{\partial r}$$

on comparing

$$\frac{\partial u}{\partial \theta} = -r.\frac{\partial v}{\partial r} \;\&$$
$$\frac{\partial v}{\partial \theta} = r.\frac{\partial u}{\partial r}$$

(or)

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{\partial u}{\partial \theta} = -r.\frac{\partial v}{\partial r}}$$

Note:-

1) C.R. Eq will note give the guaranty of differentiability. Even the function is continous.
2) For a complex value function CR Eq are not satisfied. that means that the function is not differentiable.

Harmonic & conjugate harmonic function:-

Harmonic function:- Any function $\phi(x,y)$ which Possess continous Partical derivatives first & second order and satisfies Laplacian Eq. $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ is called a harmonic function.

Conjugate ~~conjugate~~ harmonic function:- If a function $u\,f(x,y)$ is harmonic and if we can find another harmonic function $f(x,y)$ that they satisfies C-R Eq and Laplace Eq the we say that $v(x,y)$ is said to be conjugate harmonic of function

Note:-

1) for harmonic function can be determined by using CR. function
2) If ~~...~~ v is harmonic of 'u' Than and not be conjugate harmonic of 'v'.

Properties of anylatic function:-

i) an anylatic function with constant Real part is constant

ii) anylatic function with constant modulus is. constant

iii) The anylatic function with a the modus is constant

Both Real and imaginary parts of an Analytic function are orthogonal. $f(z) = \gamma_1 + \gamma_2$

Proof :- let $\omega = f(z) = u + iv$ be anylatic

i.e $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \rightarrow ①$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \rightarrow ②$

Diff. Partically ① w.r.t to x & (2) w.r.t to 'y'

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y^2} \rightarrow ③$

$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x . \partial y} \rightarrow ④$

③ + ④ $\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\Rightarrow u(x,y)$ is harmonic

Differ Partially ① w.r.t. to 'x' we get

$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2} \rightarrow ⑤$

$\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2} \rightarrow ⑥$

$\Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

$\Rightarrow v(x,y)$ is harmonic.

Orthogonal system:- Two family of curves are said to form a orthogonal system if they intersects at right angle at each point of their intersection.

✓ Every anylatic function $\omega = f(z) = u(x,y) + iv(x,y)$ defines two families of curves forming $u(x,y) = c_1$ & $v(x,y) = c_2$ forming an orthogonal systems.

Proof: Since $\omega = f(z) = u(x,y) + iv(x,y)$ is anylatic from C-R-Eq:-

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

(or) $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Given families of curves are $u(x,y) = c_1 \rightarrow ①$

$v(x,y) = c_2 \rightarrow ②$

let m_1 & m_2 are slopes of ① & ②

let $u(x,y) = c_1$

$du = 0$

$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} . dy = 0$

$\Rightarrow \frac{\partial u}{\partial y} . dy = -\frac{\partial u}{\partial x} . dx$

$\frac{dy}{dx} = -\frac{\partial u}{\partial x} / \frac{\partial u}{\partial y} = m_1$

for $v(x,y) = c_2 \Rightarrow dv = 0$

$= \frac{\partial v}{\partial x} . dx + \frac{\partial v}{\partial y} . dy = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{\partial v}{\partial x} / \frac{\partial v}{\partial y} = m_2$

Now:- $m_1 \times m_2 = -\dfrac{\partial u}{\partial x} \Big/ \dfrac{\partial u}{\partial y} \times -\dfrac{\partial v}{\partial x} \Big/ \dfrac{\partial v}{\partial y}$

~~for $v(x,y) = \ldots \Rightarrow \partial v \times 0$~~ $\quad -\dfrac{\partial u}{\partial x} \Big/ \dfrac{\partial u}{\partial y} \times \dfrac{\partial u}{\partial y} \Big/ \dfrac{\partial u}{\partial x}$ (using C-R)

$\Rightarrow 1$

$= -1$

P.T $\omega = f(z) = z^n$ is anylatic and hence find its derivative.

Sol) Given $\omega = f(z) = z^n$

$\because z = r \cdot e^{i\theta}$

hence $\omega = f(z) = r^n \cdot e^{in\theta}$

$= r^n(\cos n\theta + i \sin n\theta)$

here $u = r^n \cos n\theta$; $v = r^n \sin n\theta$

now:- $\dfrac{\partial u}{\partial r} = n \cdot r^{n-1} \cos n\theta$ | $\dfrac{\partial v}{\partial r} = n \cdot r^{n-1} \cdot \sin n\theta$

$\dfrac{\partial u}{\partial \theta} = -r r^n \sin n\theta$ | $\dfrac{\partial v}{\partial \theta} = n \cdot r^n \cos n\theta$

on observing

$\dfrac{\partial u}{\partial r} = \dfrac{1}{r} \cdot \dfrac{\partial v}{\partial \theta}$; $\dfrac{\partial u}{\partial \theta} = -r \cdot \dfrac{\partial v}{\partial r}$

hence $f(z) = z^n$ is an anylatic function.

$\because$ WKT $\boxed{f'(z) = \dfrac{\partial u}{\partial x} + i \dfrac{\partial v}{\partial x}}$

given $\omega = f(z) = z^n$

$\dfrac{\partial f}{\partial r} = f'(z) \cdot \dfrac{\partial z}{\partial r}$

$f'(z) = \dfrac{1}{\partial z/\partial r} \cdot f(r)$

$\Rightarrow f'(z) = \dfrac{1}{e^{i\theta}} \left(\dfrac{\partial u}{\partial r} + i \dfrac{\partial v}{\partial r} \right)$

$f'(z) = \dfrac{1}{e^{i\theta}} \left(n \cdot r^{n-1} \cos n\theta + i n \cdot r^{n-1} \sin n\theta \right)$

$= \dfrac{1}{e^{i\theta}} \cdot n \cdot r^{n-1} \cdot e^{in\theta}$

$= n \cdot r^{n-1} \cdot e^{i(n-1)\theta} = n \cdot (r e^{i\theta})^{(n-1)}$

$= n \cdot z^{n-1}$

find the points where the function $\omega = \dfrac{z+2}{z(z^2+1)}$ fails to be anylatic

$\omega = \dfrac{z+2}{z(z^2+1)}$

here $z(z^2+1) = 0$

$z = 0$; $z^2 + 1 = 0$

$z = 0$; $z = \pm i$

3) H.W P.T $f(z) = \bar{z}$ is not anylatic at any point.

4) Show that the curve $r^n = \alpha \sec n\theta$ & $r^n = \beta \csc n\theta$ cuts orthogonally to each other

$r^n = \alpha \sec n\theta$; $r^n = \beta \csc n\theta$

Given curves are

$r^n \cos n\theta = \alpha$

$r^n \sin n\theta = \beta$

now: $\alpha + i\beta = r^n \cos n\theta + i r^n \sin n\theta$

$= r^n(\cos n\theta + i \sin n\theta)$

$= r^n(e^{in\theta}) = z^n$

$\because f(z) = z^n$ is analytic at any point.

here given curves cuts with each other.

3) if $\omega = f(z) = z^3$ then P.T $u(x,y) = C_1$ & $v(x,y) = C_2$ are orthogonal to each other

$\omega = f(z) = z^3$

$u(x,y) = C_1$

$v(x,y) = C_2$

G.T $\omega = f(z) = z^3 = (x+iy)^3$

$= x^3 - iy^3 + 3x^2 iy - 3xy^2$

$= (x^3 - 3xy^2) + i(3x^2y - y^3)$

here $u(x,y) = x^3 - 3xy^2 = C_1 \rightarrow (1)$

$v(x,y) = 3x^2y - y^3 = C_2 \rightarrow (2)$

let m_1, m_2 are the slopes (1), (2) then

differenciating (1) w.r.t 'x'

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$ | $\frac{\partial v}{\partial x} = 6xy$

$\frac{\partial u}{\partial y} = -6xy$ | $\frac{\partial v}{\partial y} = 3x^2$

$3x^2 - 3\left(x \cdot 2y \frac{dy}{dx} + y^2\right) = 0$

$3x^2 - 6xy \cdot \frac{dy}{dx} - 3y^2 = 0$

$3x^2 - 3y^2 = 6xy \cdot \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{x^2 - y^2}{2xy} = m_1$

differianting (2) w.r.t x

$6xy_1 - 3y^2 y_1 = 0$

$y_1(6x - 3y^2) = 0$

6) if $u = x^2 - y^2$; $v = -\frac{y}{x^2+y^2}$ then S.T both u, v are harmonic but $u+iv$ is not allowed.

8) S.T the function $f(z) = \frac{xy^2(x+iy)}{x^2+y^2}$; $z \neq 0$

0 ; $z = 0$

is not derivable at $z=0$ even CR conditions are satisfies at origin.

Derivability at $z=0$

By the def $f'(z) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$

$\lim_{z \to 0} \frac{\frac{xy^2(x+iy)}{x^2+y^4}}{x+iy} = 0$

$\lim_{z \to 0} \frac{xy^2}{x^2+y^4}$ (or) $(x,y) \to (0,0)$

let $l = \lim_{z \to 0} \frac{xy^2}{x^2+y^4}$

(i) along the direction of co-ordinates axis

$l = \lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2+y^2} = \lim_{\substack{y \to 0 \\ x \to 0}} \frac{xy^2}{x^2+y^4} = 0 = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{xy^2}{x^2+y^2}$

(ii) Along $x = my^2$

$$l = \lim_{y\to 0} \frac{my^2}{m^2 y^4} = \frac{m}{m^2+1} \neq 0$$

$f'(0)$ is not exist (or) $f'(z)\big|_{z=0}$

C-R Conditions:-

Since $f(z) = \frac{x^2 y^2}{x^2+y^4} + i\,\frac{xy^3}{x^2+y^4}$

here $u(x,y) = \frac{x^2+y^2}{x^2+y^4}$; $v(x,y) = \frac{xy^3}{x^2+y^4}$

$$\therefore u(x) = \frac{\partial u}{\partial y}\Big|_{z=0} = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x-0}$$

$$= \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$u(y) = \frac{\partial u}{\partial y}\Big|_{z=0} = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y-0} = 0$$

$$v_x = \frac{\partial v}{\partial x}\Big|_{z=0} = \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x-0} = 0$$

$$v_y = \frac{\partial v}{\partial y}\Big|_{z=0} = \lim_{y\to 0} \frac{v(0,y) - v(0,0)}{y-0} = 0$$

It is clear that

$$u_x = v_y \; ; \; u_y = -v_x$$

here $F(z)$ satisfies CR - conditions at $z=0$

Construction of an analytical function:-

Suppose $u(x,y)$ given

since $F(z)$ is analytic

i.e $u_x = v_y$ (or) $v_y = -v_x$

$$\therefore dv = \frac{\partial v}{\partial x}\, dx + \left(\frac{\partial u}{\partial x}\right) dy$$

Let $M = \frac{\partial u}{\partial y}$; $N = \frac{\partial u}{\partial x}$

$$dv = M\cdot dx + N dy \rightarrow (1)$$

now:- $\frac{\partial u}{\partial y} - \frac{\partial N}{\partial x} = -\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2}\right) = 0$

(1) is a Exact. d.eq

Sol:- on integration (1)

$$v = \int M.dx + \int (\text{terms of } N \text{ free form } x)\, dy + C$$

Milne - Thomson's method:-

Suppose $u(x,y)$ is given

$\therefore$ $F(z)$ is analytic

i.e $u_x = v_y$; $u_y = -v_x$

now:- $F'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}$

$$= \frac{\partial u}{\partial x} + i\left(-\frac{\partial u}{\partial y}\right)$$

Let $\frac{\partial u}{\partial x} = \phi(x,y)$ & $\frac{\partial u}{\partial y} = \phi_2(x,y)$

$$F'(z) = \phi(x,y) - i\,\phi_2(x,y)$$

we expresses $F'(z)$ in terms of z by replacing 'x' by 'z' and 'y' by '0'

we get $F'(z) = \phi(z,0) - i\,\phi_2(z,0)$

on integrating

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$$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\, dz + C$$

similarly:-

Let $\frac{\partial v}{\partial y} = \psi_1(x,y)$ & $\frac{\partial u}{\partial x} = \psi_2(x,y)$

$$f'(z) = \psi_1(x,y) + i\psi_2(x,y)$$

we express $f'(z)$

we get $f'(z) = \psi_1(z,0) + i\psi_2(z,0)$

on integrating.

$$f(z) = \int [\psi_1(z,0) - i\phi_2(z,0)] \cdot dz + C$$

Problems:-

1) find an anylatical fun whose real part is $\frac{\sin 2x}{\cosh 2y - \cos 2x}$ by using miline Thomsons method

Sol:- now. $\frac{\partial u}{\partial x} = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - (0 + 2\sin 2x)(\sin 2x)}{(\cosh 2y - \cos 2x)^2}$

$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$

$$\Rightarrow \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2} = \phi_1(x,y)$$

Now: $\frac{\partial u}{\partial y} = \frac{(\cosh 2y - \cos 2x)(0) - (\sin 2x)(2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$

$$= \frac{-2\sin 2x \cdot \sinh 2y}{(\cosh 2y - \cos 2x)^2} = \phi_2(x,y)$$

By miline thomson method we have $f'(z)$

$$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)\, dz + C]$$

$$= \int \left(\frac{2\cos 2z - 2}{1 - \cos 2z} - i(0) \right) \cdot dz + C$$

$$= -2 \int \frac{1 - \cos 2z}{(1 - \cos 2z)^2} \cdot dz + C$$

$$= -2 \int \frac{1}{2\sin^2 z} \cdot dz + C$$

$$= -\int \operatorname{cosec}^2 z \cdot dz + C$$

$$f(z) = \cot^2 z + C.$$

imp

2) S.T $u(x,y) = e^{2x}(x\cos 2y - y\sin 2y)$ is harmonic and find its conjugate harmonic function.

Sol:-

3) find an anylatic fun imaginary part is $e^{-x}(x\cos y + y\sin y)$

4) if $f(z) = u + iv$ is an anylatic fun then P.T

a) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$

b) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u^p = p(p-1)u^{p-2}|f'(z)|^2$

c) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|\operatorname{Re} f(z)|^2 = 2|f'(z)|^2$

Sol:- since $f(z) = u + iv$ is anylatic

$u_x = v_y$ & $u_y = -v_x$

also $\nabla^2 u = 0$, & $\nabla^2 v = 0$

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(a) since $|f(z)|^2 = u^2+v^2$

now $\frac{\partial}{\partial x}(u^2+v^2) = 2u.\frac{\partial u}{\partial x} + 2v\frac{\partial u}{\partial x}$

$\frac{\partial^2}{\partial x^2}(u^2+v^2) = 2\left\{u.\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + v\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2\right\}$

$\text{Illy } \frac{\partial}{\partial y^2}(u^2+y^2) = 2\left\{u.\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2 + v\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2\right\}$

$① + ② \Rightarrow \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u^2+v^2) = 2\left\{u.\nabla^2 u + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + u.\nabla^2 v + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right\}$

$\Rightarrow 2\left\{u.0 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + v.0 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right\}$

$= 4\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right) \to ③$

since $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$= \frac{\partial u}{\partial x} + \left(-\frac{\partial u}{\partial y}\right)$

$= |f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \to ④$

from ③ & ④

b) $\frac{\partial}{\partial x}(u^p) = p.u^{p-1}\frac{\partial u}{\partial x}$

$\frac{\partial^2}{\partial x^2}(u^p) = p\left\{u^{p-1}.\frac{\partial^2 u}{\partial x^2} + (p-1).u^{p-2}\left(\frac{\partial u}{\partial x}\right)^2\right\}$

$= p(p-1).u^{p-2}|f'(z)|^2.$

4) find the value of P. If $F(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{Px}{y}\right)$ is an anylatic fonction

5) find k such that $u(x,y) = x^3 + 3kxy^2$ be a harmonic and find its conjugate

Sol: for 4th Poblem,

$\frac{\partial u}{\partial x} = \frac{1}{2}\,\frac{1}{x^2+y^2}\times 2x = \frac{x}{x^2+y^2}$

$\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$

$\frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{Px}{y}\right)^2}(P/y) = \frac{Py}{y^2+P^2x^2}$

$\frac{\partial u}{\partial y} = \frac{1}{1+(Px/y)^2}\left(\frac{-Px}{y^2}\right) = \frac{-Px}{y^2+p^2x^2}$

$\because f(z)$ is an anylatic

$u(x) = v(y)$ and $u_y = -v_x$

$\frac{x}{x^2+y^2} = \frac{-Px}{y^2+p^2x^2}$ ① and $\frac{y}{x^2+y^2} = \frac{Py}{y^2+p^2x^2}$ ②

from ① and ②

$(1)+(2) = \frac{x+y}{x^2+y^2} = \frac{-P(x+y)}{y^2+p^2x^2}$

$y^2+p^2x^2 = -P(x^2+y^2)$

$= P = -1$

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if $F(z)$ is an anylatic fun and if
$u-v = e^x(\cos y - \sin y)$ then find $f(z)$

Sol:- $\because F(z) = u+iv$ is anylatic

$\therefore i f(z) = iu - v$

on adding $\Rightarrow (1+i)f(z) = (u-v) + i(u+v)$

let $u-v = U$ & $u+v = V$.

also $(1+i)f(z) = F(z)$

here $F(z) = U+iV$ is also anylatic

given $U = e^x(\cos y - \sin y)$

$$\frac{\partial u}{\partial x} = e^x(\cos y - \sin y) = \phi_1(x,y)$$

$$\frac{\partial u}{\partial y} = e^x(-\sin y - \cos y) = \phi_2(x,y)$$

By milne thom method:-

$$F'(z) = e^x(\cos y - \sin y) + i \cdot e^x(\sin y + \cos y)$$

$$\Rightarrow F(z) = \int [e^z(1-0) + i e^z(0+1)]\,dz + C$$

$$= (1+i)\int e^z \, dz + C$$

$$F(z) = (1+i)e^z + C$$

$\because (1+i)f(z) = F(z)$

$$(1+i)f(z) = (1+i)e^z + c$$

$$f(z) = e^z + c$$

1) if $u+v = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$ then find $f(z)$.

2) find an anylatic function $f(z) = u(r,\theta) + iv(r,\theta)$.

(a) when $u(r,\theta) = a(1+\cos\theta)$

(b) when $v(r,\theta) = r^2\cos 2\theta - r\cos\theta + 2$

Sol given $f(z)$ is anylatic

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

given $u(r,\theta) = a(1+\cos\theta)$

$$= \frac{\partial u}{\partial r} = 0$$

$$\frac{\partial u}{\partial \theta} = -a\sin\theta$$

$\therefore\ 0 = \frac{1}{r}\frac{\partial v}{\partial \theta} \Rightarrow \frac{\partial v}{\partial \theta} = 0 \rightarrow (1)$

$-a\sin\theta = -r \cdot \frac{\partial v}{\partial r} \Rightarrow \frac{\partial v}{\partial r} = 0 \rightarrow (2)$

on integrating (2) w.r.t to 'r' $\Rightarrow V = a\sin\theta \log r + c(\theta)$

$\Rightarrow V = a\sin\theta \log r + c(\theta) \rightarrow (3)$

diff Partically $\Rightarrow$ w.r.t 'θ'. $\frac{\partial v}{\partial \theta} = a\cos\theta \log r + \frac{\partial c}{\partial \theta}$

$$0 = a\cos\theta \log r + \frac{\partial c}{\partial \theta}$$

$$\Rightarrow \frac{\partial c}{\partial \theta} = a\cos\theta \log r$$

on integrating $\Rightarrow C = a\sin\theta \log r$

on sub in (3) $\Rightarrow V = 2a\sin\theta \log r$

Imp Q:-

1) $v = x^2 - y^2 + \frac{x}{x^2+y^2}$ then find anylatic fon $f(z)$

(2) P.T $f(z) = \begin{cases} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} & ; z \neq 0 \\ 0 & ; z = 0 \end{cases}$ so that fon is not derivable

at $f(z)=0$ ~~so that~~ fun even though CR Condintion not satisfied

3) u, and v are harmonic functions then P.T function

$\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$ then is an anylatic ".

4) Imp Problems:-

1) Problems on anylatic

2) Construction of " "

Q) $U(x,y) = e^{-x}(x\cos y + y\sin y)$

$f(z) = i \cdot z e^{-z} = u + iv$

since $z = x+iy$

$f(z) = i(x+iy) \cdot e^{-(x+iy)}$

$= i(x+iy) \cdot e^{-x} \cdot e^{-iy}$

$ix \cdot e^{-x} \cdot e^{-y} - y \cdot e^{-x} \cdot e^{-iy}$

$= ix \cdot e^{-x}(\cos y - i\sin y) - y e^{-y}(\cos y - i\sin y)$

$= ix \cdot e^{-x}\cos y + e^{-x}\sin y - y \cdot e^{-x}\cos y + iy e^{-x}\sin y$

$= (x \cdot e^{-x}\sin y - y \cdot e^{-x}\cos y) + ie^{-x}(x\cos y + y\sin y)$.

find an anylatic function $f(z)$ such that real part of

$f(z) = 3x^2 - 4y - 3y^2$

—— x ——

UNIT - III

Elementary functions:-

1) Complex Exponential :-
The " " func of a complex variable $z = x+iy$ is given by e^z (or) $exp(z)$ and is defined as $e^z = e^{x+iy}$

Properties:-
* If $x=0$; $e^z = e^{iy} = e^x(\cos y + i\sin y)$ which is called Euler's formulae
also $|e^{iy}| = 1$
* If $y=0$ then $e^z = e^x$ (real)
* e^z is an entire function

$u = e^x\cos y$, $v = e^x\sin y$

$\frac{\partial u}{\partial x} = e^x\cos y$ | $\frac{\partial v}{\partial x} = e^x\sin y$

$\frac{\partial u}{\partial y} = -e^x\sin y$ | $\frac{\partial v}{\partial y} = e^x\cos y$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

* If e^z is a periodic fun e^z. $2\pi i$ if $e^z = e^{z \pm 2n\pi i}$
* $\frac{\partial}{\partial z}(e^z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^z$
* $e^z = 0$ ~~for any~~ is impossible for any z

$\frac{\partial}{\partial z} e^z =$

Trignometric functions:-

1) $\cos z = \frac{e^{iz}+e^{-iz}}{2}$

2) $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$

3) $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$

4) $\cos(z_1 \pm z_2) = \cos z_1 \cdot \cos z_2 \mp \sin z_1 \sin z_2$

5) $\cos^2 z - \sin^2 z = \cos 2z = 1-2\sin^2 z$
$= 2\cos^2 z - 1$

6) $\sin 2z = 2\sin z\cos z$

7) $\tan 3z = \frac{3\tan z - 4\tan^3 z}{1+3\tan^2 z}$

8) $\sin z, \cos z$ are periodic with period 2π
$\tan z, \cot z$ " " " " π

since e^z is analytic
(a) $\sin z$, $\cos z$ are also analytic
(b) $\tan z$, $\sec z$ are analytic except at $\cos z = 0$
(c) $\cot z$, $\csc z$ " " " " $\sin z = 0$

Hyperbolic fun:-

Since $\sinh z = \frac{e^z - e^{-z}}{2}$; $\cosh z = \frac{e^z + e^{-z}}{2}$

$\tanh z = \frac{\sinh z}{\cosh z}$; $\cosh z \neq 0$

$\cosh^2 z - \sinh^2 z = 1$

$\sinh(z_1+z_2) = \sinh z_1 \cdot \cosh z_2 + \cosh z_1 \cdot \sinh z_2$

$\cosh(z_1+z_2) = \cosh z_1 \cdot \cosh z_2 + \sinh z_1 \cdot \sinh z_2$

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$\sinh(-z) = -\sinh z$

$\cosh(-z) = \cosh z$

$\cosh(iz) = \cos hz$

$\cos iz = \cosh z$

$\sinh(iz) = i\sin z$

$\sin(iz) = i\sinh z$

$\because$ e^z is Every where anylatic using this we can say that

$\sinh z$, $\cosh z$ are "

$\tanh z$, $\operatorname{sech} z$ " " Except at $\cosh z = 0$

$\coth z$, $\operatorname{cosech} z$ " " " " $\sinh z = 0$

$\sinh z$, $\cosh z$ are Periodic function with Period $2\pi i$

$\tanh z$, $\coth z$ " " " " πi

$\sinh z = \sinh(x+iy)$

$= \sinh x \cdot \cosh iy + \cosh x \cdot \sinh iy$

$= \sinh x \cdot \cos y + i\cosh x \cdot \sin y$.

$\operatorname{Re}(\sinh z) = \sinh x \cdot \cos y$

$\operatorname{Im}(\sinh z) = \cosh x \cdot \sin y$

absolute value of $\sinh z$:-

$|\sinh z|^2 = \sinh^2 x \cdot \cos^2 y + \cosh^2 x \cdot \sin^2 y$

$= \sinh^2 x(1-\sin^2 y) + (1+\sinh^2 x)\cdot \sin^2 y$

$= \sinh^2 x - \cancel{\sinh^2 x \cdot \sin^2 y} + \sin^2 y + \cancel{\sinh^2 x \cdot \sin^2 y}$.

$|\sinh z|^2 = \sinh^2 x + \sin^2 y$

Simly:- $|\cosh z|^2 = \sinh^2 x + \cos^2 y$.

Logarithemic functions:-

If $z = x+iy$; $\omega = u+iv$ be any two complex valued function such that $z = e^{\omega}$ then ω is called the logarithmic of z and is written as $\omega = \log z$.

Properties:-

1) $\because$ $z = r \cdot e^{i\theta}$ then $\omega = \log z = \log(r \cdot e^{i\theta})$

$= \log r + i\theta$; $r>0$

2) when $z = r \cdot e^{i\theta}$

$\omega = \log z = \log r + i\theta$

$\log z = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$.

3) when $-\pi < \theta \le \pi$

$\omega = \log z = \log r + i(\theta \pm 2n\pi)$; $n = 0,1,2\ldots$

It is clear that the logarithmic func is a multi valued func for $n=0$ the above eq gives the principle value of $\log z$.

$\omega = \log z = \log r + i\theta$; $r>0$; $-\pi < \theta \le \pi$

4) The logarithmic fun is every where anylatic Expect at $z=0$

5) $\frac{d}{dz}\log z = \frac{1}{z}$; $z \neq 0$

6) If z_1, z_2 are any two complex number then

$\log(z_1 \cdot z_2) = \log z_1 + \log z_2$

7) $\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2$

Inverse trignometric and inverse hyperbolic func:-

1) Since w.k.t the general inverse trignometric sin function is $\omega = \sin^{-1} z$ (i.e $z = \sin\omega$)

2) All the inverse trignometric fun are multi valued fun for $\omega = \cos^{-1} z$.

$$z = \cos\omega = \frac{e^{i\omega} + e^{-i\omega}}{2}$$

$$z = \frac{e^{i\omega} + e^{1/i\omega}}{2} \Rightarrow e^{2i\omega} - 2 \cdot z \cdot e^{i\omega} + 1$$

$$e^{i\omega} = \frac{2z \pm \sqrt{4z^2 - 1}}{2} = z \pm \sqrt{z^2 - 1}$$

$$\omega = -i \log (z \pm \sqrt{z^2 - 1}) \quad (\because 1/i = -i)$$

$\because$ wkt logarithemic fun is a multi valued fun then it is clear that $\omega = \cos^{-1} z$ is also a multi valued function.

3) $\frac{d}{dx}(\sin^{-1} z) = \frac{1}{\sqrt{1-z^2}}$; $\frac{d}{dz}(\cos^{-1} z) = \frac{-1}{\sqrt{1-z^2}}$ ($\tan^{-1} z$) $= \frac{1}{1+z^2}$

4) Complex exponense:-

Let α be any complex valued fun then the complex exponent of α is denoted by z^{α} and is defined as follows

$$z^{\alpha} = e^{\log z^{\alpha}} = e^{\alpha \log z}.$$

1) Seperate the real and imaginary parts for the following function

a) $\sin z$, $\cos z$, $\tan z$, $\operatorname{cosec} z$, $\sec z$

b). $\sinh z$, $\cosh z$, $\tanh z$, $\coth z$, $\operatorname{sech} z$.

1) $$\tan z = \frac{\sin(x+iy)}{\cos(x+iy)}$$

$$= \frac{2\sin(x+iy) \cdot \cos(x-iy)}{2\cos(x+iy) \cdot \cos(x-iy)}$$

$$= \frac{\sin 2x + \sin 2iy}{\cos 2x + \cos 2iy} = \frac{\sin 2x + i\sinh 2y}{\cos 2x + i\cosh 2y}$$

$$\tan z = \left(\frac{\sin 2x}{\cos 2x + \cosh 2y}\right) + i\left(\frac{\sinh 2y}{\cos 2x + \cosh 2y}\right)$$

2) $$\operatorname{cosec} z = \frac{1}{\sin(x+iy)} = \frac{2\sin(x-iy)}{2\sin(x+iy) \cdot \sin(x-iy)}$$

$$= \frac{2\{\sin x \cdot \cos iy - \cos x \cdot \sin iy\}}{\cos(2iy) - \cos(2x)}$$

$$= \frac{2\{\sin x \cdot \cosh y - \cos x \cdot i\sinh y\}}{\cosh 2y - \cos 2x}$$

$$= \left(\frac{2\sin x \cdot \cosh y}{\cosh 2y - \cos 2x}\right) - 2i\left(\frac{\cos x \cdot \sinh y}{\cosh 2y - \cos 2x}\right).$$

***

3) $$\tanh z = \frac{1}{i}\tan iz$$

$$= -i \frac{\sin i(x+iy)}{\cos i(x+iy)} = -i \frac{\sin(ix-y)}{\cos(ix-y)}$$

$$= -i\left\{\frac{2\sin(ix-y) \cdot \cos(ix+y)}{2\cos(ix-y) \cdot \cos(ix+y)}\right\}$$

$$= -i\left\{\frac{i\sinh 2x - \sin 2y}{\cosh 2x + \cos 2y}\right\}$$

$$= \left(\frac{\sinh 2x}{\cosh 2x + \cos 2y}\right) + i\left(\frac{\sin 2y}{\cosh 2x + \cos 2y}\right)$$

4) $\operatorname{sech} z = \frac{1}{\cosh z} = \frac{1}{\cosh(x+iy)}$

$\operatorname{sech} z = \frac{1}{\cos i(x+iy)} = \frac{1}{\cos(ix-y)}$

$= \frac{2\cos(ix+y)}{2\cos(ix-y)\cdot\cos(ix+y)}$

$= \frac{2\{\cos ix\cdot\cos y - \sin ix\cdot\sin y\}}{\cos 2ix + \cos(-2y)}$

$= \frac{2\{\cosh x\cdot\cos y - i\sinh x\cdot\sin y\}}{\cosh 2x + \cos 2y}$

$= \frac{2\cosh x\cdot\cos y}{\cosh 2x+\cos 2y} - 2i\frac{\sinh x\cdot\sin y}{\cosh 2x+\cos 2y}$

5) Separate the real & imaginary Parts of $\tan^{-1}z$, $\cot^{-1}z$

let $\tan^{-1}(x+iy) = A+iB$.

$\Rightarrow x+iy = \tan(A+iB)$

$x-iy = \tan(A-iB)$

Consider:- $2A = (A+iB)+(A-iB) \to ①$

$2iB = (A+iB)-(A-iB) \to ②$

Apply tan on B.S of ①

$\tan 2A = \tan\{(A+iB)+(A-iB)\}$

$= \frac{\tan(A+iB)+\tan(A-iB)}{1-\tan(A+iB)\tan(A-iB)}$

$\tan 2A = \frac{x+iy+x-iy}{1-\{(x+iy)(x-iy)\}}$

$= \frac{2x}{1-x^2-y^2}$

$A = \frac{1}{2}\tan^{-1}\left(\frac{2x}{1-x^2-y^2}\right)$

$\tan 2iB = \frac{\tan(A+iB)-\tan(A-iB)}{1+\tan(A+iB)\cdot\tan(A-iB)}$

$i\tanh 2B = \frac{x+iy-(x-iy)}{1+(x+iy)(x-iy)}$

$= \frac{x+iy-x+iy}{1+x^2+y^2} = \frac{2iy}{1+x^2+y^2}$

$B = \frac{1}{2}\tanh^{-1}\left(\frac{2iy}{1+x^2+y^2}\right)$

6) find the general & Principle of value of following func

a) $\log(1+i\sqrt{3})$; b) $\log(-i)$; c) $\log(-1)$

a) $\log(1+i\sqrt{3})$

$x=1$; $y=\sqrt{3}$; $r = \sqrt{3+1} = 2$

since $x = r\cos\theta = 1$ | $y = r\sin\theta = \sqrt{3}$

$\cos\theta = 1/2$ | $\sin\theta = \sqrt{3}/2$

$\therefore$ wkt $\log z = \log r + i(\theta \pm 2n\pi)$; $r>0$; $-\pi<\theta\leq\pi$

$\log(1+i\sqrt{3}) = \log 2 + i(\pi/3 \pm 2n\pi)$; $n=0,1,2\ldots$ is the general value.

7) find the principle value of

a) $(1+i)^{1+i}$ b) $(1+i\sqrt{3})^{(1+i\sqrt{3})}$

b) $(1+i\sqrt{3})^{(1+i\sqrt{3})} = e^{\log_e (1+i\sqrt{3})^{1+i\sqrt{3}}}$

$= \exp\left[(1+i\sqrt{3})\log(1+i\sqrt{3})\right]$

$= \exp\left[(1+i\sqrt{3})(\log 2 + i(\pi/2 \pm 2n\pi))\right]$

$= \exp\left[(1+i\sqrt{3})(\log 2 + i\pi/3)\right] \quad (n=0)$

$= \exp\{\log 2 + i\pi/3 + i\sqrt{3}\log 2 - \pi/\sqrt{3}\}$

$= \exp\left[(\log 2 - \pi/\sqrt{3}) + i(\sqrt{3}\log 2 + \pi/3)\right]$

$= e^{\log 2 - \pi/\sqrt{3}} \cdot e^{i(\sqrt{3}\log 2 + \pi/3)}$

$= e^{\log 2 - \pi/\sqrt{3}}\{\cos(\sqrt{3}\log 2 + \pi/3) + i\sin(\sqrt{3}\log 2 + \pi/3)\}$

$= 2 \cdot e^{-\pi/\sqrt{3}}\{\cos(\sqrt{3}\log 2 + \pi/3) + i\sin(\sqrt{3}\log 2 + \pi/3)\}$

find all sol of (a) $e^{x} = 1+i$ (b) $\tanh z + 2 = 0$ (c) $\sin z = 2$

a) Sol: $e^{x} = 1+i$

$z = \log(1+i) \quad r = \sqrt{2}$

$\theta = \tan^{-1}(1/1) = 0$

$z = \log(\sqrt{2} + i(0)) = \log(\sqrt{2})$

b) $\tanh z + z = 0$

$\frac{\sinh z}{\cosh z} + z = 0$

$\frac{e^{z} - e^{-z}}{e^{z} + e^{-z}} + z = 0$

$3e^{z} + e^{-z} = 0 \Rightarrow 3(e^{z})^2 + 1 = 0$

$(e^{z})^2 = -1/3$

$e^{z} = \pm \frac{i}{\sqrt{3}}$

$z = \log(i/\sqrt{3}) \qquad r = 1/\sqrt{3}$

$z = \log((1/3)^{1/2} + i\pi/2) \qquad \theta = \pi/2$

$z = \frac{1}{2}\log 1/3 + i\pi/2$

$\sin z = 2$

$z = \sin^{-1} 2$

$= \frac{e^{iz} - e^{-iz}}{2} - 2 = 0$

$e^{iz} - e^{iz} - 4 = 0$

$= (e^{iz})^2 - 2e^{iz} + 2i + (2i)^2 - (2i)^2 - 1 = 0$

$(e^{iz} - 2i)^2 + 3 = 0$

$e^{iz} - 2i = i\sqrt{3}$

$e^{iz} = i(\sqrt{3}+2)$

$iz = \log(i(\sqrt{3}+2))$

$z = -i\log[i(\sqrt{3}+2)]$

Q) $\sin(A+iB) = x+iy$ then P.T $\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1$ and also

$\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1$

Sol:- $\sin(A+iB) = x+iy$

$x = \sin A \cdot \cosh B$

$y = \cos A \cdot \sinh B$

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$\frac{x}{\cosh B} = \sin A$; $\frac{y}{\sinh B} = \cos A$

S.O.B.S S.O.B.S

$\frac{x^2}{\cos^2 h B} = \sin^2 A$; $\frac{y^2}{\sin^2 h B} = \cos^2 A$

$\sin^2 A + \cos^2 B = \frac{x^2}{\cos^2 h B} + \frac{y^2}{\sinh B}$

$1 = \frac{x^2}{\cos^2 h B} + \frac{y^2}{\sin^2 h B}$

$\frac{(\sin A \cdot \cosh B)^2}{\sin^2 A} - \frac{(\cos A \cdot \sin h B)^2}{\cos^2 A} = 1$

$\frac{\cancel{\sin^2 A} \cdot \cos^2 h B}{\cancel{\sin^2 A}} - \frac{\cancel{\cos^2 A} \cdot \sinh^2 B}{\cancel{\cos^2 A}} = 1$

$\cosh^2 B - \sin^2 B = 1$

$1 = 1$

L.H.S = R.H.S.

If $\sin(\theta + i\phi) = \cos\alpha + i\sin\alpha$ P.T $\cos^2\theta = \pm \sin\alpha$

Sol: Since $\sin(\theta + i\phi) = \cos\alpha + i\sin\alpha$

$\Rightarrow \sin\theta \cdot \cos i\phi + \cos\theta \cdot \sin i\phi = \cos\alpha + i\sin\alpha$

$\Rightarrow \sin\theta \cosh\phi + i\cos\theta \cdot \sinh\phi = \cos\alpha + i\sin\alpha$

$\Rightarrow \sin\theta \cdot \cosh\phi = \cos\alpha \rightarrow (1)$

$\cos\theta \cdot \sinh\phi = \sin\alpha \rightarrow (2)$

$(1) \Rightarrow \cosh\phi = \frac{\cos\alpha}{\sin\theta}$

$(2) \Rightarrow \sinh\alpha = \frac{\sin\alpha}{\cos\theta}$

$\Rightarrow \cosh^2\phi - \sinh^2\phi = \frac{\cos^2\alpha}{\sin^2\theta} - \frac{\sin^2\alpha}{\cos^2\theta}$

$\Rightarrow 1 = \frac{\cos^2\alpha \cdot \cos^2\theta - \sin^2\alpha \cdot \sin^2\theta}{\sin^2\theta \cdot \cos^2\theta}$

$\Rightarrow \cos^2\alpha \cdot \cos^2\theta - \sin^2\alpha \cdot \sin^2\theta = \sin^2\theta \cdot \cos^2\theta$

$(1 - \sin^2\alpha)\cos^2\theta - \sin^2\alpha(1 - \cos^2\theta) = (1 - \cos^2\theta) \cdot \cos^2\theta$

$\Rightarrow \cos^2\theta - \sin^2\alpha = \cos^2\theta - \cos^4\theta$

$= \sin^2\alpha = \cos^4\theta$. (or) $\cos^2\theta = \pm \sin\phi$.

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Date:- 20/7/07

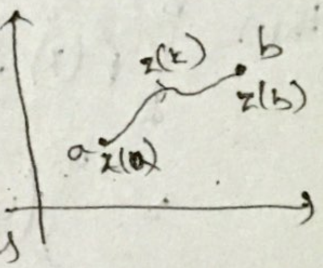
UNIT - IV

Complex integrals:-

Def:- Continous curves:- if $x(t)$ & $y(t)$ are two continous. functions of a variability 't' defined on closed interval a, b then Parametric eq $x = x(t)$; $y = y(t)$ defines the continous curve in the z-plane joining the points a, & b

$z = x + iy$

$= x(t) + iy(t)$ in $[a, b]$



closed curve:- A curve is said to be a " " the intial and final points are coincide

simple closed curve (or) Zordai:- A closed curve which do not intersect its self any were is called a simple closed curve.

Regular (or) smooth curve:- if z(t) have a continous derivatives in closed interval a, b then the curve is said to be a smooth curve.

* A smooth curve has no corners

Contour:- A continous Zodan curve consisting a finite no:- of smooth curves is called a contour.

complex line integral :- Let $\omega = f(z)$ is a continous fun the " variable $z = x + iy$ and let 'c' be any continous curve connecting two points a & b then the integral

$$\int_a^b f(z) \cdot dz \quad (\text{or}) \quad \int_c f(z) \cdot dz$$ is Exits

then it is called the line integral $f(z)$ over the curve c which is also called the path of integration.

If c is a closed curve the line integral is called contour integral and denoted by $\oint_C F(z)\cdot dz$

Since $z = x+iy$; $dz = dx+i\,dy$

$F(z) = u(x,y) + i\,v(x,y)$

then $\int_a^b F(z)\cdot dz = \int_a^b \left(u(x,y)+iv(x,y)\right)(dx+i\,dy)$

$= \int_a^b (u\cdot dx + iu\,dy + iv\,dx - v\,dy)$

$= \int_a^b (u\cdot dx - v\,dy) + i\int_a^b (u\,dy + v\,dx)$

Evaluate $\int_C F(z)\cdot dz$ where $F(z) = x^2 + ixy$ from points A(1,1) to B(2,8) along

(i) The S.L AB

(2) the along curve C; $x = t$ $y = t^3$

Sol: Given $\int_C F(z)\cdot dz = \int_C (x^2+ixy)(dx+i\,dy) \to (1)$

(i) Along the S.L AB

A(1,1) ; B(2,8)

Eq. AB : $y-1 = \frac{8-1}{2-1}(x-1)$

$\Rightarrow y-1 = 7x-7$

$y = 7x-6 \Rightarrow dy = 7dx$

(1) $\Rightarrow \int_C \{(x^2 + ix(7x-6))\}\cdot(dx+i7dx)$

Also $x \to 1$ to 2

$\int_C F(z)\cdot dz = \int_{x=1}^{2} (x^2+7ix^2-6ix)(1+7i)\cdot dx$

$\int_1^2 (x^2+7ix^2-6ix+7ix^2-49x^2+42x)\cdot dx$

$= \left(\frac{x^3}{3}+7i\frac{x^3}{3}-6i\frac{x^2}{2}+7i\frac{x^3}{3}-49\frac{x^3}{3}+42\frac{x^2}{2}\right)_1^2$

(ii) Evaluating C: $x=t$; $y=t^3$

$dx = dt$; $dy = 3t^2\cdot dt$

Also A(1,1) $\Rightarrow 1=t$; $1=t^3 \Rightarrow t=1$

B(2,8) $\Rightarrow 2=t$; $8=t^3 \Rightarrow t=2$

$t \to 1$ to 2

(1) $\Rightarrow \int_C F(z)\cdot dz = \int_{t=1}^{2} (t^2+it^4)(dt+i3t^2\,dt)$

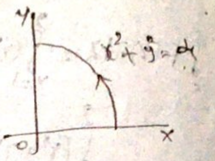
$= \int (t^2+it^4)(1+3it^2)dt$

$= \int_1^2 (t^2+3it^4+it^4-3t^6)\cdot dt$

$= \left(\frac{t^3}{3}+4i\frac{t^5}{5}-3\frac{t^3}{3}\right)_1^2$

$= \left(\frac{8}{3}-\frac{1}{3}+\frac{4i\times 64}{5}-\frac{4i}{5}-\right.$

Evaluate $\int_C (x-2y)\,dx + (y^2-x^2)\,dy$ where 'C' is boundary of circle $x^2+y^2=4$ in first Quadrant.



Sol. $x^2+y^2=4$

$x=2\cos\theta$; $y=2\sin\theta$

$dx=-2\sin\theta\cdot d\theta$; $dy=2\cos\theta\cdot d\theta$

& $\theta \to 0$ to $\pi/2$

$\int_C (x-2y)\,dx + (y^2-x^2)\,dy$

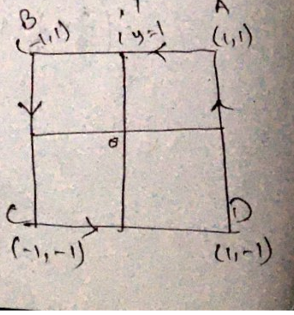
$= \int_{\theta=0}^{\pi/2} (2\cos\theta - 4\sin\theta)(-2\sin\theta\cdot d\theta) + (4\sin^2\theta - 4\cos^2\theta)(2\cos\theta)\,d\theta$

$= \int_0^{\pi/2} (-4\sin\theta\cdot\cos\theta + 8\sin^2\theta + 8\sin^2\theta\cdot\cos\theta - 8\cos^3\theta)\,d\theta.$

$\int_0^{\pi/2} \sin^n\theta\cdot\cos^m\theta\cdot d\theta = \frac{m-1}{n}\cdot\frac{m-3}{n-2}\cdots\frac{\pi}{2}$ (n is even)

$= \frac{m-1}{n}\cdot\frac{m-3}{n-2}\cdots\frac{2}{3}$ (n is odd)

Evaluate $\int_C f(z)\cdot dz$, $f(z) = 3z^2+iz-4$ where c is the unit square with diagonal corners at the points $-1-i$, & $1+i$



Sol.

here $\int_C f(z)\,dz = \left(\int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}\right) f(z)\cdot dz$

Along AB: $y=1$; $dy=0$

$y \to 1$ to -1

$\int_{AB} f(z)\cdot dz = \int_{AB} \{3(x+iy)^2 + i(x+iy) - 4\}(dx+idy)$

$= \int_{x=1}^{-1} (3(x+i)^2 + i(x+i) - 4)\cdot dx$

$= \int_1^{-1} 3(x^2-1+2ix) + ix - 1 - 4)\cdot dx$

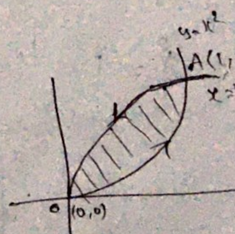
$= \int_1^{-1} (3x^2+7ix-8)\cdot dx$

$= (x^3 + 7ix^2/2 - 8x)_1^{-1}$

$= (-1+7i/2+8) - (1+7i/2-8)$

Now - Adding.

2) Evaluate $\int_C (y^2+2xy)\cdot dx + (x^2-2xy)\cdot dy$ where c is the boundary of region given by $y=x^2$ & $x=y^2$



$\int_C F(z)\cdot dz = \int_{OA} F(z)\cdot dz + \int_{AO} F(z)\cdot dz$

Along OA

∴ OA is $y = \frac{1}{1}(x-0) \Rightarrow y=x$

$\therefore \int_{OA} (y^2+2xy).dx + (x^2-2xy).dy$

$= \int_{x=0}^{1} (x^2+2x^2).dx + (x^2-2x^2).dx$

$= \int_{0}^{1} (x^2+2x^2+x^2-2x^2).dy = 2\int_0^1 x^2 dx = \frac{2}{3}$

Along 'AO':-

$A(1,1)$; $O(0,0)$

Sol:- AO is $x = \frac{1}{1}(y-0) \Rightarrow x = y$

Ans. $\int_C (x+2y).dx + (y-2x).dy$ where C is ellipse

$\frac{x^2}{16} + \frac{y^2}{9} = 1$; $x = 4\cos\theta$, $y = 3\sin\theta$

P) Evaluate:- $\oint_C z^n dz$ where 'n' is an integer and for some closed curve 'C'.

Sol since $z = r.e^{i\theta}$ $\qquad |z| = |r.e^{i\theta}|$

$dz = r.ie^{i\theta}.d\theta$ $\qquad |z| = |r|.1$

$\theta \to 0$ to 2π $\qquad |z-0| = r$

$\int_C z^n.dz = \int_{\theta=0}^{2\pi} (r.e^{i\theta})^n.ir.e^{i\theta}.d\theta$ $\qquad |(x+iy)-0| = r$

$\qquad |(x-0)+i(y-0)| = r$

$= \int_0^{2\pi} r^n.e^{in\theta}.i.r.e^{i\theta}.d\theta$ $\qquad \sqrt{(x-0)^2+(y-0)^2} = r$

since 'r' is an integer

for $n = -1$

$\int_C z^n.dz = i\int_0^{2\pi}.d\theta = 2\pi i$

for $n \neq -1$

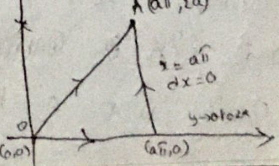
$= ir^{n+1}\left(\frac{e^{i(n+1)\theta}}{i(n+1)}\right)_0^{2\pi}$

$= \frac{r^{n+1}}{n+1}\left(e^{i2\pi(n+1)} - 1\right)$

$= \frac{r^{n+1}}{n+1}\left(\cos 2(n+1)\pi + i\sin 2(n+1)\pi - 1\right)$

$= \frac{r^{n+1}}{n+1}(1+0-1) = 0$

P) Evaluate $\int_C (z^2+3z+2).dz$ where 'c' is cycloide $x = a(\theta + i\sin\theta)$; $y = (1-\cos\theta)$ b/w points $(0,0)$ & $(a\pi, 2a)$



Sol:- Since $f(z) = (z^2+3z+2)$ is an anylatic function hence

$\int f(z)$ b/w any two points is independent of the path

hence $\int_{OA} F(z)\cdot dz = \int_{OB} F(z)\cdot dz + \int_{BA} F(z)\cdot dz$

Along OB : $y=0 \Rightarrow dy=0$

$x \to 0$ to $a\pi$

$\int_{OB} F(z)\cdot dz = \int_{OB} [(x+iy)^2 + 3(x+iy)+2](dx+idy)$

$= \int_{x=0}^{a\pi} (x^2+3x+2)\cdot dx$

$= \left(\frac{x^3}{3} + 3\frac{x^2}{2} + 2x\right)_0^{a\pi}$

$= \frac{a^3\pi^3}{3} + \frac{3}{2}a^2\pi^2 + 2a\pi$

$\int_{BA} F(z)\cdot dz = \int_{y=0}^{2a} (a\pi+iy)^2 + 3(a\pi+iy)+2)\, i\,dy$

$= \int_0^{2a} (a^2\pi^2 - y^2 + i\,2a\pi\, y + 3a\pi + 3iy + 2)\, i\,dy$

$= i a^2\pi^2 \int_0^{2a} dy - i\int_0^{2a} y^2\,dy - 2a\pi\int_0^{2a} y\,dy$

Theorem:- Cauchy's (or) Cauchy's integral Th:-

Statement :- let $f(z) = u+iv$ is an analytic function and $f'(z)$ is continuous with in and on a simple closed curve 'c' then $\oint_c f(z)\cdot dz = 0$.

Proof:- let R be region enclosed by curve C

$\because f(z) = u+iv$ is analytic hence satisfies CR conditions

$u_x = v_y$ and $u_y = -v_x$

by def

$\oint_c f(z)\cdot dz = \oint_c (u+iv)(dx+idy) = \oint_c (u\cdot dx - v dy) + i\oint_c (v\,dx + u\,dy)$ → ①

$\because f'(z)$ is continuous then partial derivative

$\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$ are also continuous in the region 'R'.

Applying green's th in a plane for R.H.S of ②

we get

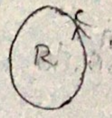
$\oint_c u\,dx + iv\,dy = \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial y}{\partial y}\right)\cdot dx\,dy$

$= \iint_R \left(\frac{-\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\cdot dx\cdot dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\cdot dx\cdot dy$ → ③

using Eq ① in ③ we get

$\iint_R u\cdot dx + iv\cdot dy = 0$.

Proved



Formulaes:-

$$\int_0^{\pi/2}\cos^n\theta\,.d\theta = \int_0^{\pi/2}\sin^n\theta\,.d\theta = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{2}{3}\cdot 1 \ (n\ \text{odd})$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\ (n\ \text{Even})$$

$$\int_0^{\pi/2}\sin^m\theta\,\cos^n\theta\,.d\theta = \frac{m-1}{m+n}\cdot\frac{m-3}{m+n-2}\cdots\frac{1}{n+1}$$

(m odd , n Even (or) odd)

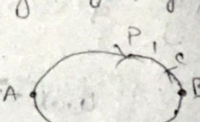
$$= \frac{m-1}{m+n}\cdot\frac{m-3}{m+n-2}\cdots\frac{2}{3}$$

(m Even, 'n' odd)

$$= \frac{m-1}{m+n}\cdots\frac{1}{2}\cdot\frac{\pi}{2}\ (m.\text{odd}, n\ \text{Even}).$$

Q:- Evaluate: $\int_C$ c is circle $|z|$ hear $f(z) = \frac{z}{z-3}$ it is anylatic Every where but $z=3$ is lise out side. A $|z|=2$ that is $f(z)$ is analytic $f(z)=3$. and by using cachys therome $\oint_C \frac{z}{z-3} = 0$.

Q:- So that $\int_A^B f(z).dz$ is independent of the form if $f(z)$ is anylatic.



$$\int_A^B f(z)\,.dz$$

$$\int_{APB} f(z).dz = \int_{AQB} f(z).dz$$

$\because f(z)$ is anylatic

$$\int_C f(z).dz = 0 \Rightarrow \int_{APBQA} f(z).dz = 0$$

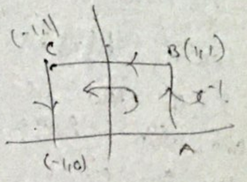
$$\Rightarrow \int_{APB} f(z).dz + \int_{BQA} f(z).dz = 0$$

$$\Rightarrow \int_{APB} f(z).dz = -\int_{BQA} f(z).dz = \int_{AQB} f(z).dz$$

Q) PT $\int_C \frac{dz}{z-a} = 2\pi i$ then $C: |z-a| = r$

1) verify cauchy's theorme for $\int f(z) = z^3$ taken over the boundary of reactangle with vertices $-1, 1, 1+i, -1+i$

$$\oint_C z^3.dz = \left(\int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}\right) z^3.dz =$$



$$\int_{AB} z^3.dz = \int_{AB} (x+iy)^3\,(dx+idy)$$

S:- $\int_C \frac{x.dy - y.dx}{\sqrt{x^2+y^2}}$ is independent of any Path of integration which does not passes trough origin.

Consider $f(z) = \frac{z}{|z|}$; $z \neq 0$.

here $\int f(z).dz = \int_C \frac{x.dx - y.dy}{\sqrt{x^2+y^2}} + i\int_C \frac{x.dy + y.dx}{\sqrt{x^2+y^2}}$.

$\because f(z) = \frac{z}{|z|}$ is anylatic if $z \neq 0$

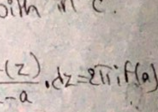
hence it is clear that the above two integrals are independent of any path of integration which do not passes through origin

Theorem:- Cauchy's Integral formula:-

If $f(z)$ is analytic with in and on a simple closed curve 'c' and if a is a point with in 'c' then $f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$ or $\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$

Proof:- Consider a function $\phi(z) = \frac{f(z)}{z-a}$

we, $\phi(z)$ is analytic every where except for $z = a$. now we draw a circle c_1 centre at a and with some radius 'r' which entirely lies in 'c'.



∴ $\phi(z)$ is analytic the region enclosed by curves c_1 & c

then $\oint_C \phi(z)\,dz = \oint_{C_1} \phi(z)\,dz \to (1)$

now we consider $\oint_C \frac{f(z)}{z-a}\,dz$

since $C_1 : |z-a| = r$

$z-a = re^{i\theta}$

$dz = rie^{i\theta}\,d\theta$

$$\Rightarrow \oint_{C_1} \frac{f(z)}{z-a}\,dz = \int_{\theta=0}^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}} \cdot rie^{i\theta}\,d\theta$$

$$= i\int_0^{2\pi} f(a+re^{i\theta})\,d\theta$$

As $r \to 0$ the circle subsets to point $z=a$

$$\therefore \oint_{C_1} \frac{f(z)}{z-a}\,dz = i\int_0^{2\pi} f(a)\,d\theta$$

$$= 2\pi i$$

$$\Rightarrow f(a) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{z-a}\,dz$$

using (1).

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz.$$

Note:- let $f(z)$ be any fun which is analytic in region bounded by 2 simple closed curves C_1 & C_2 the line integration along C_2 & line integration along C_1 =

$$\oint_C f(z)\,dz = \oint f(z)\,dz$$

Generalisation of Cauchy's integral formulae & derivatives of analytic function

we have $f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$

By def $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

$$= \lim_{h\to 0} \frac{\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a-h}\,dz - \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz}{h}$$

$$= \lim_{h\to 0} \frac{1}{2\pi i}\left\{\frac{\oint_C \left(\frac{1}{z-a-h} - \frac{1}{z-a}\right) f(z)\,dz}{h}\right\}$$

$$= \frac{1}{2\pi i}\lim_{h\to 0} \frac{1}{h}\left\{\oint_C \frac{z-a-z+a+h}{(z-a-h)(z-a)}\right\} f(z)\,dz$$

$$= \frac{1}{2\pi i}\lim_{h\to 0}\left\{\oint_C \frac{1}{(z-a-h)(z-a)} \cdot f(z)\,dz\right.$$

$$\therefore f'(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}\,dz$$

(&)

$$F'(a) = \frac{1!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{1+1}} dz$$

$$\text{llr } f''(a) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{2+1}} dz$$

$$f^{n}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz$$

(p) using cauchys integral formulae evalued e

$\oint_C \frac{\log z}{(z-1)^3}$. $c: |z-1| = 1/2$ where 'c' is a circle.

Sol

$$\int_C \frac{\log z}{(z-1)^3} . dz \quad ; \quad c: |z-1| = 1/2$$

here $z=1$ is lies inside of $|z-1| = 1/2$

$$\int_C \frac{\log z}{(z-1)^{2+1}} . dz = \frac{2\pi i}{2!} f''(a)$$

when $f(z) = \log z$ & $a=1$

$F'(z) = \frac{1}{z}$

$F''(z) = \frac{-1}{z^2}$

$F''(z)\big|_{z=1} = -1$

$$= \frac{2\pi i}{2!} (-1)$$

$$= -\pi i.$$

Evaluate $\int_C \frac{z+4}{z^2+2z+5}$ where a) $c: |z|=1$

b) $|z+1-i| = 2$

c $|z+1+i| = 2$

Sol $\int_C \frac{z+4}{z^2+2z+5}$.

$$\therefore z^2+2z+5 = (z+1)^2 + 2^2$$

$$= (z+1)^2 - (2i)^2$$

$$= (z+1+2i)(z+1-2i)$$

$$z = -1-2i \; ; -1+2i$$

a) $|z| = 1$

for $z = -1-2i$; $|z| = |-1-2i| = \sqrt{5} > 1$ (lies out side $|z|=1$)

$z = -1+2i$; $|z| = \sqrt{5} > 1$ (lies out side of $|z|=1$)

By Cauchys Th.

$$\oint_C \frac{z+4}{z^2+2z+5} \cdot dz = 0$$

b) $|z+1-i| = 2$

for $z = -1-2i$; $|-1-2i+1-i| = |-3i| = 3 > 2$ (lies out side of $|z+1-i|=2$)

$z = -1+2i$; $|-1+2i+1-i| = |i| = 1 < 2$ (lies inside $|z+1-i|=2$)

$$\oint_C \frac{z+4}{z^2+2z+5} \cdot dz = \oint_C \frac{z+4}{(z+1+2i)(z+1-2i)} \cdot dz$$

$$= \oint_C \frac{\frac{z+4}{z+1+2i}}{(z-(-1+2i))} \cdot dz$$

$= 2\pi i F(a)$ where $F(z) = \frac{z+4}{z+1+2i}$ $a = -1+2i$

$$2\pi i\left(\frac{2i+3}{4i}\right) \qquad F(a)=\frac{-1+2i+4}{-1+2i+1+2i}$$

$$=\frac{\pi}{2}(2i+3).$$

$$\int_C \frac{1}{(z^2+4)^2}dz \quad ; \quad c: |z-i|=2$$

Evaluate $\oint_C \frac{\cos\pi z^2}{(z-1)(z-2)} \cdot dz$ & $c: |z|=4$.

$$\therefore \frac{\cos\pi z^2}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$-\oint_C \frac{1}{(z-1)} \cdot dz + \oint_C \frac{1}{(z-1)} \cdot dz$$

$$\cos\pi z^2 = A(z-2)+B(z-1)$$

Put $z=1 \Rightarrow \cos$

Imp

1. Evaluate $\oint_C \frac{dz}{(z^2+4)^2}$ where 'c' is the circumference of the Ellipse $x^2+4(y-2)^2=4$

Given Ellipse :- $\frac{x^2}{4}+\frac{4(y-2)^2}{4}=1$

$$\frac{x^2}{4}+\frac{(y-2)^2}{1}=1$$

$x=2i, -2i$
$\downarrow \quad \downarrow$
$(0,2) \quad (0,-2)$

$$\int_C \frac{\frac{1}{(z+2i)^2}}{(z-2i)^2} \cdot dz = \frac{2\pi i}{1!} f'(a)$$

where $f(z)=\frac{1}{(z+2i)^2}$

$a=2i$

Evaluate $\oint_C \left\{\frac{e^z}{z^3}+\frac{z^4}{(z+i)^2}\right\}$ where $c: |z|=2$

Sol:-

$$\int_C \frac{e^z}{z^3} \cdot dz + \int_C \frac{z^4}{(z+i)^2} \cdot dz$$

$z=0 \quad ; \quad z=-i$

$$\Rightarrow \frac{2\pi i}{2!} f''(a) + \frac{2\pi i}{1!} f'(a)$$

here $f(z)=e^z$, $z=a=0$ | here $f(z)=z^4$, $z=a=-i$

$$\Rightarrow \pi i + 2\pi i \; 4(-i)^3$$

$$\Rightarrow \pi i + 8\pi i \times i = \pi i \times 8\pi.$$

Evaluate $\int_C \frac{e^z}{z(1-z)^3} \cdot dz$ when (a) 0 lies inside c' 'i' lies outside c
(b) 1 " " c' 0 " " " c
(c) both lies " ".

a) $\oint_C \frac{\frac{e^z}{(1-z)^3}}{z} \cdot dz = 2\pi i f(a)$

b) $-\int_C \frac{e^z/z}{(z-1)^3} \cdot dz$ (c)

Sequences, Series and Power series:-

Def:- A series is of the form $\sum_{n=0}^{\infty} a_n(z-a)^n$ $= a_0+a_1(z-a)+a_2(z-a)^2+\ldots$ is called Power series where 'a' is any fixed point in complex plane and is known as centre of Power series and also $a_0, a_1, \ldots a_n$ are said to be the Co-efficients of Power series.

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Def:- Region of Convergence (or) radius of convergence of P.S

This is the set of all points for which given Power series is convergent.

(or)

If $\sum a_n z^n$ is converges for $|z|<R$ and diverges for $|z|>R$ then 'R' is called the radius of convergence of power series and in particular $|z|=R$ is called the circle of convergences.

Taylor's series:-

Statement:- If a function $f(z)$ is anylatic inside a circle 'c' whose centre 'a' then $\forall z$ in 'c' we have ~~f(z) = f(a)~~

$$f(z) = f(a) + (z-a)\cdot f'(a) + \frac{(z-a)^2}{2!} f''(a) + \dots + \frac{(z-a)^n}{n!} f^n(a) + \dots$$

is called the taylor series of $f(z)$ about the point $z=a$

(or) Taylor's series of $f(z)$ in power of $(z-a)$.

Result:- 1. Put $z = a+h$ then

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \dots$$

2) In taylor's series of $f(z)$ = ~~at~~ $z=a$, for getting the maclaren series we put $a=0$

$$f(z) = f(0) + z\cdot f'(0) + \frac{z^2}{2!} f''(0) + \dots$$

3) $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots \quad \forall z$

4) $e^{-z} = 1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \dots \quad \forall z$

5) $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots \quad |z|<\infty$

6) $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots \quad |z|<\infty$

$\sinh z = z + \frac{z^3}{3!} + \dots \quad |z|<\infty$

$\cosh z = 1 + \frac{z^2}{2!} + \dots \quad |z|<\infty$

$\tanh z = z + \frac{z^3}{3} + \frac{2}{15} z^5 + \dots \quad |z|<\pi/2$

$(1+z)^{-1} = 1 - z + z^2 - z^3 + \dots \quad |z|<1$

$(1-z)^{-1} = 1 + z + z^2 + \dots \quad |z|<1$

$(1+z)^{-2} = 1 - 2z + 3z^2 - 4z^3 + \dots \quad |z|<1$

$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots \quad |z|<1$

$\log(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \dots \quad |z|<1$

Note:- for finding the taylor's series of $f(z)$ at $z=a$ we put $z-a=\omega$ then $f(z) = f(a+\omega)$

3d-lat:-

1) Expand $f(z) = \sinh z$ and $z = \pi i$.

Sol: $\because$ By Taylor's series we have

$$f(z) = f(a) + (z-a)\cdot f'(a) + \dots$$

here $f(z) = \sinh z$; & $z = \pi i$

Put $z - \pi i = \omega \Rightarrow z = \omega + \pi i$

$$f(z) = \sinh z = \sinh(\omega + \pi i)$$

$$= \sinh\omega \cosh \pi i + \cosh\omega \sinh \pi i$$

Since $\cosh \pi i = \frac{e^{\pi i} + e^{-\pi i}}{2} = \frac{2\cos\pi}{2} = -1$

$\sinh \pi i = 0$

$$\therefore f(z) = -\sinh\omega$$

$$= -\left\{\omega + \frac{\omega^3}{3!} + \frac{\omega^5}{5!} - \dots\right\}$$

$f(z) = -\left\{ (z-\pi i) + \frac{1}{3!}(z-\pi i)^3 + \cdots \right\}$

Q) Expand $f(x) = \sin x$ using Taylor's series about the point $x = \frac{\pi}{4}$

Q) Expand $f(x) = \frac{1}{x}$ about the point $x = 1$.

Q) Expand $f(x) = \frac{e^x}{x(x+1)}$ about the point $x = 2$

Put $x - 2 = \omega \Rightarrow x = \omega + 2$

$f(x) = \frac{e^x}{x(x+1)} = \frac{e^{\omega+2}}{(\omega+2)(\omega+3)}$

Now: $\frac{e^{\omega}}{(\omega+2)(\omega+3)} = \frac{A}{\omega+2} + \frac{B}{\omega+3}$

$\Rightarrow e^{\omega} = A(\omega+3) + B(\omega+2)$

Put $\omega = -3 \Rightarrow e^{-3} = -B \Rightarrow B = e^{-3}$

$\omega = -2 \Rightarrow e^{-2} = A \Rightarrow A = e^{-2}$

$\therefore f(x) = e^2 \left\{ \frac{e^{-2}}{\omega+2} + \frac{e^{-3}}{\omega+3} \right\}$

$e^2 \left\{ e^{-2}(2+\omega)^{-1} - e^{-3}(\omega+3)^{-1} \right\}$

$= e^2 \left\{ e^{-2} \times \frac{1}{2}\left(1+\frac{\omega}{2}\right)^{-1} - e^{-3}\frac{1}{3}\left(1+\frac{\omega}{3}\right)^{-1} \right\}$ where $\left|\frac{\omega}{2}\right| < 1$; $\left|\frac{\omega}{3}\right| < 1$

$= \frac{1}{2}\left\{ 1 - \frac{\omega}{2} + \left(\frac{\omega}{2}\right)^2 - \cdots \right\} - \frac{e^{-1}}{3}\left\{ 1 - \left(\frac{\omega}{3}\right) + \left(\frac{\omega}{3}\right)^2 \cdots \right\}$

$= \frac{1}{2}\left\{ 1 - (x-2)/2 + \cdots \right\} - \frac{1}{3e}\left\{ 1 - \frac{(x-2)}{3} + \cdots \right\}$

$|x-2| < 2$

$|x-2| < 3$

Taylor's expansion of $f(x) = \frac{2x^3+1}{x^2+x}$ about the point

Observing $f(x)$ we split the func by using general division.

$\therefore$ $(x^2+x)\ 2x^3+1\ (2x+2$

$2x^3 + 2x^2$

$(-)\quad (-)$

$1 - 2x^2$

$2x + 2x^2$

$1 + 2x$

$f(x) = \frac{2x^3+1}{x^2+x} = 2x + 2 + \frac{1+2x}{x^2+x}$

$= 2(x+1) + \frac{(1+2x)}{x(x+1)}$

$f(x) = 2(x+1) + \frac{1}{x+1} + \frac{1}{x}$

By Taylor's series of expansion we have

$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots \rightarrow (1)$

$f(x) = 2(x+1) + \frac{1}{x+1} + \frac{1}{x}$

$f'(x) = 2 + \frac{-1}{(x+1)^2} + \frac{-1}{x^2}$

$f'(x)\Big|_{x=i} = 2 - \frac{1}{(1+i)^2} - \frac{1}{i^2}$

$f''(x) = \frac{2}{(x+1)^3} + \frac{2}{x^3}$

$f''(x)\Big|_{x=i} = \frac{2}{(i+1)^3} + \frac{2}{i^3}$

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Substituting in eq we get:-
$f(x) = 2(i+1) + \frac{1}{i+1} + \frac{1}{i} + (z-i)^2\left(2 - \frac{1}{(i-1)^2} + 1\right)$

1) Expand $f(z) = e^{1+z}$ in $f(z)$ is powers of $z-1$

2) Expand if $f(z) = \frac{1}{(z-a)(z-b)}$; ① $|a| < |z| < |b|$ ② $|z| > |b| (a<b)$

$\boxed{8|8|07}$

Sol:

Put $z = a$

$A = \frac{1}{a-b}$; $B = \frac{1}{b-a}$

$f(z) = \frac{1}{(z-a)(z-b)} = \frac{A}{z-a} + \frac{B}{z-b}$

$A \Rightarrow z = a \Rightarrow A = \frac{1}{a-b}$

$z = b \Rightarrow B = \frac{1}{b-a}$

$\Rightarrow \frac{1}{a-b}\left\{\frac{1}{z-a} - \frac{1}{z-b}\right\}$

(i) $|a| < |z| < |b|$

$\frac{|a|}{|z|} < 1$ $\frac{|z|}{|b|} < 1$.

$\Rightarrow \frac{1}{a-b}\left\{\frac{1}{z(1-a/z)} + \frac{1}{b(1-z/b)}\right\}$

$\Rightarrow \frac{1}{a-b}\left[\frac{1}{z}\left(1-a/z\right)^{-1} + \frac{1}{b}\left(1-\frac{z}{b}\right)^{-1}\right]$

ii) $|z| > |b|$ $(a<b)$

$\Rightarrow |b/z| < 1$

$f(z) = \frac{1}{a-b}\left[\frac{1}{z-a} - \frac{1}{z-b}\right]$

$\left|\frac{z}{a}\right| < 1$; $\left|\frac{z}{b}\right| < 1$

$f(z) = \frac{1}{a-b}\left[\frac{1}{a\left(\frac{z}{a}-1\right)} - \frac{1}{b}\left[\frac{1}{(z/b-1)}\right]\right]$

$\Rightarrow \frac{1}{a-b}\left[\frac{1}{b}\left[\frac{1}{[1-z/b]} - \frac{1}{a(1-z/a)}\right]\right]$

(ii) $f(z) = e^{1+z}$

let $z-1 = \omega$ $z = 1+\omega$

* find Taylor's Expansion at $\frac{1}{z^2-3z+2}$ the region

$0 < |z-1| < 1$

Sol let $z-1 = t \Rightarrow 0 < t < 1$

$f(z) = \frac{1}{(z-1)(z-2)}$

$f(t) = \frac{1}{t(t-1)}$: $0 < t < 1$

$\Rightarrow -\frac{1}{t} + \frac{1}{t-1}$ $(\because t < 1)$

$\Rightarrow -\frac{1}{t} + \frac{1}{-(1-t)} \Rightarrow -\left\{\frac{1}{t} + (1-t)^{-1}\right\}$

$\Rightarrow -\left[\frac{1}{t} + (1+t+t^2+\cdots)\right] t<1$

replace $t = z-1$,

$\Rightarrow -\left[\frac{1}{z-1} + \left(1+(z-1)+(z-1)^2+\cdots\right)\right] |z-1|<1$

Note:-

In taylor's series we find the expansions of $f(z)$ if $f(z)$ must be anylatic inside c. suppose if $f(z)$ has a singularity at some point $z=a$ then taylor's series is not useful to finding the expansion of such functions in such cases we use the Laren's series i.e we now expanding the func even it has a singularity at some $z=a$ by using lariens series.

Laren series:-

if $f(z)$ is anylatic inside and the boundary of the Ring shaped region R bounded by two concentric circles c_1 and c_2 with radius r_1, r_2 $(r_1>r_2)$ then $\forall z$ in R we have $F(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$

$$a_n = \frac{1}{2\pi i} \oint_{c_1} \frac{f(\omega)}{(\omega-a)^{n+1}} \cdot d\omega \quad ; n = 0,1,2\ldots$$

$$b_n = \frac{1}{2\pi i} \oint_{c_2} \frac{f(\omega)}{f(\omega-a)^{n+1}} \, d\omega \quad ; i = n = 1,2,3\ldots$$

Note:- It is rembered, that

$$\oint_C \frac{f(\omega)}{(\omega-a)^{n+1}} \cdot d\omega \neq \frac{2\pi i}{n!} f^n(a)$$

In particular if $f(z)$ is analytic inside c_1 then $b_n = 0$ hence in such cases the larens series reduce to taylor series.

2) find the laren expansion of

$f(z) = \frac{1}{(z+1)(z+1)^2}$ in the region (a) $|z|<2$ (b) $|1+z|>1$

sol $f(z) = \frac{1}{z+2} - \frac{1}{z+1} + \frac{1}{(z+1)^2}$

(a) $|z|<2 \Rightarrow \left|\frac{z}{2}\right|<1$

$f(z) = \frac{1}{2\left(1+\frac{z}{2}\right)} - \frac{1}{z+1} + \frac{1}{(z+1)^2}$

$\Rightarrow \frac{1}{2}\left(1 - \frac{2}{z} + \left(\frac{z}{2}\right)^2 - \cdots\right) - \left(1-z+z^2+\cdots\right) + \left(1-2z+3z^2-\cdots\right)$

(b) $1+z = t \Rightarrow |t|>1 \Rightarrow \frac{1}{|t|}<1$

$f(z) = \frac{1}{(z+2)(z+1)^2} \Rightarrow f(t) = \frac{1}{t^2(t+1)} : |t|>1$

$f(t) = \frac{1}{t^2 \times t\left(1+\frac{1}{t}\right)} \quad ; \left|\frac{1}{t}\right|<1$

$\Rightarrow \frac{1}{t^3}\left(1+\frac{1}{t}\right)^{-1} \Rightarrow \frac{1}{t^3}\left(1 - \frac{1}{t} + \frac{1}{t^2} \cdots\right)$

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$$f(t) = \frac{1}{(1+z)^3}\left(1 - \frac{1}{1+z} + \frac{1}{(1+z)^2} \cdots\cdots\right)$$

P) $f(z) = \frac{7z-2}{z(z+1)(z-2)}$ in $1 < |z+1| < 3$

$\frac{1}{t} < 1$ & $\frac{t}{3} < 1$

$$f(t) = \frac{7t-9}{(t-1)(t)(t-3)}$$

Q) Represent the fun $f(z) = \frac{z+1}{z-1}$ has a

1) maclaren series in the region of validity

(2) lauren series in the region $|z| > 1$

Sol given $f(z) = \frac{z+1}{z-1}$

$$= 1 + \frac{2}{z-1}$$

(∵ $f(z)$ in analytic every where expect $z=1$ & also in exist)

For $|z| > 1$ lauren series

$$f(z) = 1 - \frac{2}{1-z} = 1 - 2(1 + z + z^2 + \cdots) \; ; |z| < 1$$

(b) laurens series in $|z| > 1 \Rightarrow \frac{1}{|z|} < 1$

$$f(z) = 1 + \frac{2}{z\left(1 - \frac{1}{z}\right)} \Rightarrow 1 + \frac{2}{z}\left(1 - \frac{1}{z}\right)^{-1} ; \left|\frac{1}{z}\right| < 1$$

$$\Rightarrow 1 + \frac{2}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right)$$

$$\Rightarrow 1 + \left(\frac{2}{z} + \frac{2}{z^2} + \frac{2}{z^3} + \cdots\right)$$

Q) find the taylor's (or) lauren series expansions of

$f(z) = \frac{1}{(z^2+1)(z+2)}$ when (a) $|z| < 1$; (b) $1 < |z| < 2$.

Given $f(z) = \frac{1}{(z^2+1)(z+2)}$

$$\frac{1}{(z^2+1)(z+2)} = \frac{Ax+B}{(z^2+1)} + \frac{C}{z+2}$$

$$= \frac{-\frac{1}{5}z + \frac{2}{5}}{z^2+1} + \frac{\frac{1}{5}}{z+2}$$

$$\Rightarrow \frac{1}{5}\left[\frac{1}{z+2} - \frac{z-2}{z^2+1}\right]$$

(a) $|z| < 1$

$$f(z) = \frac{1}{5}\left[\frac{1}{2\left(1+\frac{z}{2}\right)} - \frac{z-2}{(1+z^2)}\right]$$

$$\Rightarrow \frac{1}{5}\left[\frac{1}{2}\left(1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 - \cdots\right) - (z-2)(1 + z^2 + (z^2)^2 \cdots)\right]$$

$z/2 < 1$ $\quad |z|^2 < 1$

Here $f(z)$ contains the +ve power of z.

Here $f(z)$ has taylor series $|z| < 1$.

(b) $1 < |z| < 2 \Rightarrow \frac{1}{|z|} < 1 < \frac{2}{|z|} \Rightarrow \frac{|z|}{2} < 1$

$$f(z) = \frac{1}{5}\left(\frac{1}{2\left(1+\frac{z}{2}\right)} - \frac{z-2}{z^2\left(1+\frac{1}{z^2}\right)}\right)$$

$$\Rightarrow \frac{1}{5}\left[\frac{1}{2}\left(1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 \cdots\right) - \frac{z-2}{z^2}\left(1 - \frac{1}{z^2} + \left(\frac{1}{z^2}\right)^2 \cdots\right)\right]$$

Here F(z) has both +ve and -ve Power of z

hence F(z) has larence in $1 < |z| < 2$.

Q) find a larane Series Expansions in Power of 'z' for $f(z) = \frac{1}{z^2(1-z)}$ and specify the region in which Expansions are valid.

$$\frac{1}{z^2(1-z)} = f(z)$$

$$\Rightarrow \frac{1}{z^2}(1-z)^{-1} \Rightarrow \frac{1}{z^2}(1+z+z^2+\dots)$$

$$\Rightarrow \frac{1}{z^2}+\frac{1}{z}+1+z+\dots \quad |z|<1$$

Suppose we take

$$F(z) = \frac{1}{z^2 \cdot z(1-\frac{1}{z})}$$

$$\Rightarrow -\frac{1}{z^3}\left(1+\frac{1}{z}+\frac{1}{z^2}+\dots\right)$$

$$\Rightarrow -\left(-\frac{1}{z^3}+\frac{1}{z^4}+\dots\right) ; \left|\frac{1}{z}\right|<1 \cdot \; |z|>1$$

6/8/20*

Unit - I :-

Special functions:-

def:- β func:- for the +ve values of (n, m) β func is denoted by β(m, n) and is defined by integral $\beta(m,n) = \int_0^1 x^{m-1} \cdot (1-x)^{n-1} \cdot dx$

this is also called the Eulerian integral of first kind.

Properties of β func:-

Symmetric Property:- $\beta(m,n) = \beta(n,m)$

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} \cdot dx$$

$$\int_0^1 (1-x)^{m-1} (1-(1-x))^{n-1} \cdot dx$$

$$\int_0^1 x^{n-1}(1-x)^{m-1}$$

$$\beta(n,m)$$

β func interms of trignometric func:-

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \cdot d\theta$$

By def:- $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} \cdot dx$

Put $x = \sin^2\theta$
$dx = 2\sin\theta \cdot \cos\theta \, d\theta$
$x=0 \Rightarrow \theta = 0$
$x=1 \Rightarrow \theta = \pi/2$

$$= \int_0^{\pi/2} (\sin^2\theta)^{m-1} (1-\sin^2\theta)^{n-1}$$

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$$2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \cdot d\theta.$$

3) $\beta(m+1, n) + \beta(m, n+1)$

4) for the +ve values of (m,n)

$$\beta(m,n) = \frac{(m-1)!\,(n-1)!}{(m+n-1)!}$$

other forms of β: These are

1) S.T $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$

By defn $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

Put $x = \frac{1}{1+y}$; $dx = \frac{-1}{(1+y)^2}\,dy$

when $x=0 \Rightarrow y=\infty$

$x=1 \Rightarrow y=0$

$$\beta(m,n) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1}\left(1-\frac{1}{1+y}\right)^{n-1} \cdot \frac{-dy}{(1+y)^2}$$

$$= \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx \quad \left[\int_a^b f(x)\,dx = \int_a^b f(t)\,dt\right]$$

$\therefore \beta(m,n) = \beta(n,m)$

$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx.$$

2) S.T $\beta(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$

Sol We k that $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

Put $x = \frac{1}{y}$; $dx = \frac{-1}{y^2}\,dy$

$x=1 \Rightarrow y=1$

$x=\infty \Rightarrow y=0$

$$\int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_1^0 \frac{(1/y)^{m-1}}{\left(1+\frac{1}{y}\right)^{m+n}} \cdot \frac{-1}{y^2}\,dy$$

$$+ \int_0^1 \frac{\frac{1}{y^{m-1}}}{\frac{(1+y)^{m+n}}{y^{m+n}}} \cdot \frac{1}{y^2}\,dy + \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dy + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dy$$

$$= \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dy$$

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3) S.T $\beta(m,n) = a^m \cdot b^n \int_0^{a} \frac{x^{m-1}}{(ax+b)^{m+n}} \, dx$

4) $B(m,n) = a^n(1+a)^m \int_0^1 \frac{x^{m-1}(1+x)^{n-1}}{(x+a)^{m+n}} \cdot dy$

(or)

$$\frac{\beta(m,n)}{a^n(1+a)^m} = \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}} \cdot dx.$$

$$\int_b^a (x-1)^{m-1} \cdot (a-x)^{n-1} \cdot dx = (a-b)^{n-1} \cdot dx \, (a-b)^{m+n-1} \, B(m,n).$$

By def:- $B(m,n) \int_0^1 x^{m-1}(1-x)^{n-1} \cdot dx$

$$x = \frac{y-b}{a-b} \Rightarrow dx = \frac{dy}{a-b}$$

where $x = 0 \Rightarrow y = b$

$x = 1 \Rightarrow y-b = a-b \Rightarrow y = a$

$$\beta(m,n) = \int_b^a \left(\frac{y-b}{a-b}\right)^{m-1} \left(1 - \frac{y-b}{a-b}\right)^{n-1} \frac{dy}{a-b}$$

$$= \frac{1}{(a-b)^{m+n-1}} \int_b^a (y-b)^{m-1} \cdot (a-y)^{n-1} \cdot dy$$

$$\Rightarrow \int_b^a (x-b)^{m-1} (a-x)^{n-1} \cdot dx = (a-b)^{m+n-1} \, B(m,n)$$

(a) $\cancel{\int_0^1 \frac{x}{\sqrt{1-x^2}}}$.

5) Evaluate the following integrals in terms β func

a) $\int_0^1 \frac{x}{\sqrt{1-x^2}} \cdot dx$ (b). $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

c) $\int_0^1 \frac{x}{\sqrt{1-x^5}} \cdot dx$

Given. $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$. Put $x^2 = 9y$

$x = 3y^{1/2}$

$dx = \frac{3}{2} y^{1/2 - 1} \cdot dy$

$\int_0^1 \frac{3/2 \, y^{1/2 - 1} \cdot dy}{\sqrt{9-9y}}$

$x = 0 \Rightarrow y = 0$

$x = 3 \Rightarrow y = 1$

$$\frac{3}{2\times 3} \int_0^1 y^{1/2 - 1} \cdot (1-y)^{-1/2} \cdot dy$$

$$\Rightarrow \frac{1}{2} \int_0^1 y^{1/2 - 1} (1-y)^{1/2 - 1} dy = \frac{1}{2} \beta\left(\frac{1}{2}, \frac{1}{2}\right)$$

P) $\int_0^{\pi/2} \sin^m\theta \cdot \cos^n\theta \, d\theta = \frac{1}{2} \beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right); m, n > -1$

WKT. $\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \cdot d\theta = \frac{1}{2}\beta(m,n)$

$m = \frac{m+1}{2}, n = \frac{n+1}{2}$

$$\int_0^{\pi/2} \sin^m\theta \cdot \cos^n\theta \cdot d\theta = \frac{1}{2} \beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

Given $\int_0^{\pi/2} \sin^m\theta \cdot \cos^n\theta \cdot d\theta$

Put $\sin^2\theta = x$

$2\sin\theta \cdot \cos\theta \cdot d\theta = dx$

$\int_0^{\pi/2} \sin^{m-1}\theta \cdot \cos^{n-1}\theta \, (\sin\theta \cdot \cos\theta) \cdot d\theta$

$\theta = 0 \Rightarrow x = 0$

$\theta = \pi/2 \Rightarrow x = 1$

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$$= \int_0^{\pi/2} (\sin^2\theta)^{\frac{m-1}{2}} \cdot (\cos^2\theta)^{\frac{n-1}{2}} \sin\theta \cdot \cos\theta \cdot d\theta$$

$$= \int_0^{\pi/2} (\sin^2\theta)^{\frac{m-1}{2}} (1-\sin^2\theta)^{\frac{n-1}{2}} \sin\theta \cdot \cos\theta \cdot d\theta$$

$$\frac{1}{2}\int_0^1 x^{m-1/2} (1-x)^{n-1/2} \cdot dx$$

$$= \frac{1}{2}\int_0^1 x^{\left(\frac{m+1}{2}\right)-1} (1-x)^{\left(\frac{n+1}{2}\right)-1} \cdot dx = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

P) find $\int_0^a (a-x)^{m-1} \cdot x^{n-1} \cdot dx = a^{m+n} \cdot \beta(m,n)$.

$$\left\{ \begin{array}{l} \beta(m,n) = \int_0^1 x^{m-1} \cdot (1-x)^{n-1} \cdot dx \\ x = \frac{a-y}{a} \end{array} \right\}$$

P) $\int_0^1 x^m (1-x^n)^p \cdot dx = \frac{1}{n}\beta\left(\frac{m+1}{n}, p+1\right)$.

$x^n = y \Rightarrow x = y^{1/n}$

$dx = \frac{1}{n} y^{1/n-1} \cdot dy$

$$\Rightarrow \int_0^1 y^{m/n} (1-y)^p \cdot \frac{1}{n} y^{1/n-1} \cdot dy$$

$$= \frac{1}{n} \cdot \int_0^1 y^{\left(\frac{m+1}{n}\right)-1} (1-y)^{(p+1)-1} \cdot dy$$

$$\frac{1}{n}\beta\left(\frac{m+1}{n}, p+1\right).$$

$$\int_{-1}^{1} (1+x)^{m-1} (1-x)^{n-1} \cdot dx = 2^{m+n-1} \beta(m,n)$$

Gamma func (Γ):

1) $\Gamma(1) = 1$

2) Pf. $\Gamma(n+1) = n\Gamma(n)$

By defn $\Gamma(n+1) = \int_0^\infty e^{-x} \cdot x^n \cdot dx$

$$= \left(x^n\left(\frac{e^{-x}}{-1}\right)\right)_0^\infty - \int_0^\infty n \cdot x^{n-1}(-e)^{-x} \cdot dx$$

$$= 0 + n\int_0^\infty e^{-x} \cdot x^{n-1} \cdot dx$$

$$= n\Gamma(n)$$

$$= n\,\Gamma((n-1)-1)$$

$$= n(n-1)\Gamma(n-1)$$

$$\vdots$$

$$= n(n-1)(n-2)\ldots\Gamma(1)$$

$$= n!$$

3) $\Gamma(n) = (n-1)!$

4) $\Gamma(1/2) = \sqrt{\pi}$

$$\Gamma(3/2) = \Gamma\left(\frac{1}{2}+1\right)$$

$$= \frac{1}{2}\Gamma(1/2) = \frac{1}{2}\sqrt{\pi}$$

$$\Gamma(5/2) = 3/2 \cdot 1/2 \cdot \sqrt{\pi}.$$

for $n \le 0$ the Γ func is not defined.

relation b/w β & Γ :-

i.e $\beta(m,n) = \dfrac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)}$; $m,n>0$

By def:- $\Gamma(m) = \int_0^\infty e^{-x}.x^{m-1}.dx$

Put $x = yt$, $dx = y.dt$

$\Gamma(m) = \int_0^\infty e^{-y t}.(yt)^{m-1}.y.dt$

$\Gamma(m) = y^m \int_0^\infty e^{-yt}.e^{m-1}.dt$

$$\Rightarrow \frac{\Gamma(m)}{y^m} = \int_0^\infty e^{-y}(x)^{m-1}.dx \to (1)$$

multiplying (1) by $e^{-y}.y^{m+n-1}$ & integrating w.r.t 'y' from

$$\Gamma(x)\int_0^\infty e^{-y}.y^{n-1}.dy = \int_0^\infty \left(\int_0^\infty e^{-yx}.x^{m-1}.dx.e^{-y}.y^{m+n-1} \right) dy$$

$\Gamma(m).\Gamma(n)$

$$= \int_0^\infty \int_0^\infty \left(e^{-y(1+x)}.y^{(m+n)-1}.dy \right).x^{m-1}.dx$$

$$= \int_0^\infty \frac{\Gamma(m+n)}{(1+x)^{m+n}}.x^{m-1}.dx \quad (\text{by } (1))$$

$$\Gamma(m+n)\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}.dx = \Gamma(m+n) + \beta(m,n).$$

$$\beta(m,n) = \frac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)} \quad (m,n>0)$$

Put $m = n = 1/2$

$$\beta(1/2, 1/2) = \frac{\Gamma(1/2).\Gamma(1/2)}{\Gamma(1/2+1/2)}$$

$$\therefore \left(\Gamma\left(\tfrac{1}{2}\right)\right)^2 = \beta\left(\tfrac{1}{2},\tfrac{1}{2}\right)$$

$$= \int_0^1 x^{1/2}(1-x)^{-1/2}.dx$$

Put $x = \sin^2\theta$

$dx = 2\sin\theta.\cos\theta.d\theta$

$x = 0 ; \theta = 0$

$x = 1 \Rightarrow \theta = \pi/2$

$$\int_0^{\pi/2} \sin^{-1}\theta.\cos^{-1}\theta(2\sin\theta.\cos\theta).d\theta = 2\int_0^{\pi/2} d\theta = 2\times\frac{\pi}{2} = \pi.$$

Express following integral in the form of Gamma func

a) $\int_0^\infty e^{-x^4}.dx$; b) $\int_0^\infty e^{-x^{1/m}}.dx$

c) $\int_0^\infty e^{-x^3}\sqrt{x}.dx$ d) $\int_0^\infty e^{-x}.x^{7/2}.dx$

e) $\int_0^\infty e^{-ax^n}.x^m.dx$

b) Put $x^{1/m} = y$

$x = y^m$

$dx = m \cdot y^{m-1} \cdot dy$

$$\int_0^\infty e^{-x^{1/m}} \cdot dx = \int_0^\infty e^{-y} \cdot m \cdot y^{m-1} \cdot dy$$

$$= m \int_0^\infty e^{-y} \cdot y^{m-1} \cdot dy$$

$$= m \cdot \Gamma(m)$$

c) Put $ax^n = y \Rightarrow x = (y/a)^{1/n}$

$$dx = 1/n \left(y/a\right)^{1/n - 1} \cdot \frac{1}{a} \cdot dy$$

$$\int_0^\infty e^{-ax^n} \cdot x^m \cdot dx$$

$$= \int_0^\infty e^{-y} \cdot \left(y/a\right)^{m/n} \cdot \frac{1}{n} \left(y/a\right)^{1/n - 1} \cdot \frac{1}{a} \cdot dy$$

$$= \frac{1}{n \cdot a^{\frac{m+1}{n}}} \int_0^\infty e^{-y} \cdot y^{\left(\frac{m+1}{n}\right) - 1} \cdot dy$$

$$= \frac{1}{n \cdot a^{\frac{m+1}{n}}} \cdot \Gamma\left(\frac{m+1}{n}\right)$$

S.T $\int_0^1 y^{q-1} \left(\log \frac{1}{y}\right)^{p-1} \cdot dy = \frac{\Gamma(p)}{q^p}; p > 0; q > 0$

then S.T $\int_0^1 \left(\log \frac{1}{y}\right)^{n-1} \cdot dy = \Gamma(n)$. $(p = n ; q = 1)$

Given $\int_0^1 y^{q-1} \left(\log 1/y\right)^{p-1} \cdot dy$

Put $\log 1/y = t$

$y = e^{-t}, dy = -e^{-t} \cdot dt$

$y = 0 \Rightarrow t = \infty$

$y = 1 \Rightarrow t = 0$

$$\Rightarrow \int_\infty^0 e^{-t(q-1)} \cdot t^{p-1} (-e^{-t}) \cdot dt$$

$$\Rightarrow \int_0^\infty e^{-tq + t - t} \cdot t^{p-1} \cdot dt$$

$$\Rightarrow \int_0^\infty e^{-tq} \cdot t^{p-1} \cdot dt \quad (\text{Or}) \quad tq = x, \quad dt = \frac{dx}{q}$$

$$\int_0^\infty e^{-x} \cdot (x/q)^{p-1} \cdot dx/q$$

$$\Rightarrow \frac{1}{q^p} \int_0^\infty e^{-x} \cdot x^{p-1} dx$$

$$= \frac{1}{q^p} \Gamma(p)$$

a) 1 $\int_0^\infty \sin^m\theta \cdot \cos^n\theta \cdot d\theta = \frac{1}{2} \beta\left(\frac{m+1}{2}; \frac{n+1}{2}\right)$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{m+1}{2}\right) \cdot \Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m+n+2}{2}\right)}$$

2) $\int_0^{\pi/2} \sin^k x \cdot dx = \int_0^{\pi/2} \cdot \cos^k x \cdot dx$

$$\frac{1}{2} \frac{\Gamma\left(\frac{k+1}{2}\right)\sqrt{\pi}}{\Gamma\left(\frac{k+2}{2}\right)} \quad \left(\begin{array}{l} m=k\ ;\ n=0 \\ m=0\ ;\ n=k \end{array}\right)$$

3) $\Gamma(n) \cdot \Gamma(1-n) = \frac{\pi}{\sin n\pi}$

Evaluate the following integrals (a) $\int_0^{\pi/2} \sin^5\theta \cdot \cos^{7/2}\theta \cdot d\theta$

(b) $\int_0^{\pi/2} \sin^7\theta \cdot d\theta$ (c) $\int_0^{\pi/2} \cos^{11}\theta \cdot d\theta$

(d) $\int_0^{\pi/2} \sqrt{\cos^2\theta} \cdot d\theta$ (e) $\int_0^{\pi/2} \sqrt{\cos\theta} \cdot d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos\theta}}$

a) $\int_0^{\pi/2} \sin^5\theta \cdot \cos^{7/2}\theta \cdot d\theta$

here $m = 5$; $n = 7/2$

$$\int_0^{\pi/2} \sin^5\theta \cdot \cos^{7/2}\theta \cdot d\theta = \frac{1}{2} \frac{\Gamma\left(\frac{5+1}{2}\right)\Gamma\left(\frac{7/2+1}{2}\right)}{\Gamma\left(\frac{5+7/2+2}{2}\right)}$$

$$= \frac{1}{2} \frac{\Gamma(3) \cdot \Gamma(9/4)}{\Gamma(21/4)}$$

$$= \frac{1}{2} \frac{2! \cdot 5/4 \cdot 1/4\, \Gamma(1/4)}{17/4 \cdot 13/4 \cdot 9/4 \cdot 5/4 \cdot 1/4 \cdot \Gamma(1/4)}$$

b) $\int_0^{\pi/2} \sqrt{\cos^2\theta} \cdot d\theta = \int_0^{\pi/2} \cos^{1/2}\theta \cdot \sin^{-1/2}\theta \cdot d\theta$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{1/2+1}{2}\right) \cdot \Gamma\left(\frac{-1/2+1}{2}\right)}{\Gamma\left(\frac{1/2 - 1/2 + 2}{2}\right)}$$

$$= \frac{1}{2} \frac{\Gamma(3/4) \cdot \Gamma(1/4)}{\Gamma(1)}$$

$$= \frac{1}{2} \Gamma(1/4) \cdot \Gamma(1 - 1/4)$$

$$= \frac{1}{2} \frac{\pi}{\sin \pi/4} = \frac{1}{2} \cdot \frac{\pi}{1/\sqrt{2}} = \pi/\sqrt{2}$$

P) P.T $\int_0^1 \frac{x^n}{\sqrt{1-x^2}} \cdot dx = \frac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdots n} \cdot \frac{\pi}{2}$ where n is an Even Integer.

Given $\int_0^1 \frac{x^n}{\sqrt{1-x^2}} \cdot dx$ Put $x^2 = y \Rightarrow x = y^{1/2}$

$dx = 1/2 \cdot y^{1/2 - 1} \cdot dy$

$$\Rightarrow \int_0^1 \frac{y^{n/2}}{\sqrt{1-y}} \cdot 1/2 \cdot y^{1/2-1} \cdot dy$$

$$= \frac{1}{2} \int_0^1 y^{\left(\frac{n+1}{2}\right)-1} \cdot (1-y)^{1/2-1} \cdot dy$$

$$= 1/2\, \beta\left(\frac{n+1}{2}, 1/2\right)$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{n+1}{2}\right) \cdot \Gamma(1/2)}{\Gamma\left(\frac{n+1+1}{2}\right)}$$

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$$= \frac{1}{2}\, \frac{\Gamma\left(\frac{n+1}{2}\right) \cdot \sqrt{\pi}}{\Gamma\left(\frac{n+2}{2}\right)}$$

since n is an even integer ($n = 2k$)

$$\frac{1}{2}\, \frac{\Gamma\left(\frac{2k+1}{2}\right)\sqrt{\pi}}{\Gamma\left(\frac{2k+2}{2}\right)}$$

$$= \frac{1}{2}\, \frac{\Gamma\left(k+\frac{1}{2}\right)\cdot\sqrt{\pi}}{k!} \quad (\Gamma(k+1) = k!)$$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\left(k-\frac{1}{2}\right)+1\right)\sqrt{\pi}}{k!}$$

$$= \frac{1}{2} \cdot \frac{\left(k-\frac{1}{2}\right)\cdot\Gamma\left(k-\frac{1}{2}\right)\cdot\sqrt{\pi}}{k!}$$

$$= \frac{1}{2}\, \frac{\left(k-\frac{1}{2}\right)\cdot\left(k-\frac{3}{2}\right)\Gamma\left(k-\frac{3}{2}\right)\cdots\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)}{k!}\cdot\sqrt{\pi}$$

$$= \frac{(2k-1)(2k-3)\cdots 3\cdot 1\cdot\sqrt{\pi}}{2^k k!}\cdot\frac{\sqrt{\pi}}{2}$$

$$= \frac{(2k-1)(2k-3)\cdots 3\cdot 1}{2k(2k-2)(2k-4)\cdots 4\cdot 2}\cdot\frac{\pi}{2}$$

$$= \frac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdots n}\cdot\frac{\pi}{2}$$

P) Evaluate:- a) $\int_0^1 x^5(1-x)^3\,dx$ (b) $\int_0^2 x(8-x^3)^{1/3}\,dx$

(c) $\int_0^1 (1-x^n)^{1/n}\,dx$ (d) $\int_0^1 x^m\left(\log \frac{1}{x}\right)^n dx$

(e) $\int_0^1 \frac{dx}{\sqrt{-\log y}}$

P) If $m, n > 0$ then $\frac{1}{n}\beta(m, n+1) = \frac{1}{m}\beta(m+1, n) = \frac{\beta(m,n)}{m+n}$.

Consider $\frac{1}{n}\beta(m, n+1)$

$$= \frac{1}{n}\int_0^1 x^{m-1}\cdot(1-x)^n\,dx$$

$$= \frac{1}{n}\int_0^1 (1-x)^n\cdot x^{m-1}\,dx$$

$$= \frac{1}{n}\left\{\left((1-x)^n\cdot\frac{x^m}{m}\right)_0^1 - \int_0^1 n(1-x)^{n-1}(-1)\frac{x^m}{m}\,dx\right\}$$

$$= \frac{1}{n}\left\{0 + \frac{n}{m}\int_0^1 x^{(m+1)-1}(1-x)^{n-1}\,dx\right\}$$

$$= \frac{1}{m}\beta((m+1),(n))$$

Hy :- P.T $\frac{1}{m}\beta(m+1, n) = \frac{\beta(m,n)}{m+n}$ then P.T

$$\frac{\beta(m,n)}{m+n} = \frac{1}{n}\beta(m, n+1)$$

Evaluate :- $\int_0^\infty y^{-1/2} (1-e^{-y}) \, dy$

Bessel's func :- The d.Eq of form

$$x^2 \cdot \frac{d^2y}{dx^2} + x \cdot \frac{dy}{dx} + (x^2 - n^2)y = 0$$ is called Bessel's func

and its particular sol are called bessels func of another n where n must be +ve constants

Bessel's func of first kind of order :-

It is denoted by $J_n(x)$ and defined as follows

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{r! \, \Gamma(n+r+1)} \; , n=0,1,2$$ for particular

Bessels func of order 0,1. is given by $J_0(x)$:-

$$J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2r}}{r! \, \Gamma(r+1)} = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} \ldots,$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^2 \cdot 4} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6}$$

Recurrence formulae for $J_n(x)$:-

1) $x \cdot J_n'(x) = n \, J_n(x) - x \, J_{n+1}(x)$

(or)

$$J_{n+1}(x) = \frac{n}{x} J_n(x) - J_n'(x)$$

Proof :- we have $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r}}{r! \, \Gamma(n+r+1)}$

Diff w.r.t 'x' ;

L.H.S

$$J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r) \left(\frac{x}{2}\right)^{n+2r-1} \cdot \frac{1}{2}}{r! \, \Gamma(n+r+1)}$$

$$x \, J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \cdot (n+2r) \left(\frac{x}{2}\right)^{n+2r}}{r! \, \Gamma(n+r+1)}$$

R.H.S

$$= n \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{r! \, \Gamma(n+r+1)} + \sum_{r=0}^{\infty} \frac{(-1)^r \, 2r \left(\frac{x}{2}\right)^{n+2r}}{r! \, \Gamma(n+r+1)}$$

$$= n \, J_n(x) + \sum_{r=1}^{\infty} \frac{(-1)^r \cdot r \left(\frac{x}{2}\right)^{n+2r-1} \cdot x}{r(r-1)! \, \Gamma(n+r+1)}$$

Put $r-1 = s$; $r = s+1$

$$= n \, J_n(x) + x \sum_{s=0}^{\infty} \frac{(-1)^{s+1} \left(\frac{x}{2}\right)}{s! \, \Gamma(n+s+2)}$$

$$= n \, J_n(x) - x \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{(n+1)+2s}}{s! \, \Gamma((n+1)+s+1)}$$

$$x \, J_n'(x) = n \, J_n(x) - x \, J_{n+1}(x) \; \rightarrow (1)$$

2) $x \, J_n'(x) = -n \, J_n(x) + J_n(x) + x \, J_{n+1}(x)$ $\rightarrow (2)$

3) $J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$

from Adding ① & ② we get

$\frac{1}{2}\left\{J_{n-1}(x) - J_{n+1}(x)\right]$

4) $J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$

By subtracking ① & ② we get

$\frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$.

5) $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

6) $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$

5) $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

By def $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{r!\,\Gamma(n+r+1)}$

$x^n J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{2n+2r}}{r!\,\Gamma(n+r+1)}$

$\frac{d}{dx}\left[x^n J_n(x)\right] = \sum_{r=0}^{\infty} \frac{(-1)^r (2n+2r)(x/2)^{2n+2r-1} \cdot 1/2 \cdot 2^n}{r!\,\Gamma(n+r+1)}$

$= \sum_{r=0}^{\infty} \frac{(-1)^r \cdot 2(n+r)(x/2)^{2n+2r} \cdot 1/2 \cdot 2^n}{r!\,(n+r)\,\Gamma(n+r)}$

$= \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r-1} \cdot \frac{x^n}{2^n} \cdot 2^n}{r!\,\Gamma(n+r)}$

$= x^n \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{(n-1)+2r}}{r!\,\Gamma((n-1)+r+1)}$

$= x^n J_{n-1}(x)$.

6) Pt ⓐ $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$

ⓑ $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$

Sol:- By def $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{r!\,\Gamma(n+r+1)}$

Put $n = 1/2$ $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{1/2+2r}}{r!\,\Gamma(r+3/2)}$

$\sqrt{x/2} \sum_{r=0}^{\infty} \frac{(x/2)^{2r}}{r+3/2}$

$= \sqrt{\frac{x}{2}}\left\{\frac{1}{\Gamma(3/2)} - \frac{1}{\Gamma(5/2)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma(7/2)}\left(\frac{x}{2}\right)^4 \ldots\right\}$

$= \sqrt{\frac{x}{2}}\left\{\frac{1}{\frac{1}{2}\sqrt{\pi}} - \frac{1}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}\left(\frac{x}{2}\right)^4 \ldots\right\}$

$$\sqrt{\frac{x}{2}} \cdot \frac{2}{\sqrt{\pi}} \left\{1 - \frac{2}{3}\left(\frac{x}{2}\right)^2 + \frac{4}{2!\,15}\left(\frac{x}{2}\right)^4 \cdots\right\}$$

$$= \sqrt{\frac{2x}{\pi}} \times \frac{1}{x}\left\{x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right\}$$

$$= \sqrt{\frac{2}{\pi}}\, \sin x$$

P.T (a) $J_{-n}(x) = (-1)^n J_n(x)$

(b) $J_n(-x) = (-1)^n \cdot J_n(x)$

By def:- $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{-n+2r}}{r!\,\Gamma(-n+r+1)}$

The above summation is exists if $r \geq n$

$$= \sum_{r=n}^{\infty} \frac{(-1)^r \cdot (x/2)^{-n+2r}}{r!\,\Gamma(-n+r+1)}$$

Since $r \geq n \Rightarrow r = n+s$

$$= \sum_{s=0}^{\infty} \frac{(-1)^{n+s} \left(x/2\right)^{-n+2n+2s}}{(n+s)!\,\Gamma(-n+n+s+1)}$$

$$= \frac{(-1)^n \sum_{s=0}^{\infty} (x/2)^{n+2s}}{s!\,\Gamma(n+s+1)} \quad \left(\because s! = \Gamma(s+1),\ (n+s)! = \Gamma(n+s+1)\right)$$

$$= (-1)^n J_n(x)$$

P.T (a) $\frac{d}{dx}\left[x \cdot J_1(x)\right] = x J_0(x)$

(b) $\frac{d}{dx}\left[J_0(x)\right] = -J_1(x)$

P.T (a) $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left\{\frac{1}{x}\sin x - \cos x\right\}$

(b) $J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left\{\frac{3-x^2}{x^2}\sin x - \frac{3}{x}\cos x\right\}$

Sol:- $J_n(x) = \frac{x}{2n}\left\{J_{n-1}(x) + J_{n+1}(x)\right\}$

Put $n = 1/2 \Rightarrow J_{1/2}(x) = x\left\{J_{-1/2}(x) + J_{3/2}(x)\right\}$

$$\Rightarrow J_{3/2}(x) = \frac{1}{x} \cdot J_{1/2}(x) - J_{-1/2}(x)$$

$$= \frac{1}{x}\left\{\sqrt{\frac{2}{\pi x}}\sin x\right\} - \sqrt{\frac{2}{\pi x}}\cos x$$

$$= \sqrt{\frac{2}{\pi x}}\left\{\frac{1}{x}\sin x - \cos x\right\}$$

(b) $J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$

Put $n = \frac{3}{2} \Rightarrow J_{3/2}(x) = x\left\{J_{1/2}(x) + J_{5/2}(x)\right\}$

$$J_{5/2}(x) = \frac{1}{3x} \cdot J_{3/2}(x) - J_{1/2}(x)$$

$$= \frac{1}{3x}\left\{\sqrt{\frac{2}{\pi x}}\sin x\right\} - \left\{\sqrt{\frac{2}{\pi x}}\cos x\right\}$$

✓ P.T $2J_0''(x) = J_2(x) - J_0(x)$

Since wkt $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x) \rightarrow (1)$

Diff w.r.t 'x'

$2J_n''(x) = J_{n-1}'(x) - J_{n+1}'(x)$.

Multiplying by 2 we get

$4J_n''(x) = 2J_{n-1}'(x) - 2J_{n+1}'(x) \rightarrow (2)$

In (1) put $n = n+1$ & $n-1$ we get

$2J_{n+1}'(x) = J_n(x) - J_{n+2}(x) \rightarrow (3)$

$2J_{n-1}'(x) = J_{n-2}(x) - J_n(x) \rightarrow (4)$

Sub. (3), (4) in (2) we get

$4J_n''(x) = J_{n-2}(x) - J_n(x) - J_n(x) + J_{n+2}(x)$

$4J_n''(x) = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$

put $n=0$

$4J_0''(x) = J_{-2}(x) - 2J_0(x) + J_2(x)$

$= (-1)^2 J_2(x) - 2J_0(x) + J_2(x)$

$4J_0''(x) = 2J_2(x) - 2J_0(x)$

$\Rightarrow 2J_0''(x) = J_2(x) - J_0(x)$

P) 1) $J_2(x) = J_0''(x) - \frac{1}{x}J_0'(x)$

P. 2) $\frac{d}{dx}\left[J_n^2 + J_{n+1}^2\right] = \frac{2}{x}\left[nJ_n^2 - (n+1)J_{n+1}^2\right]$

$\frac{d}{dx}\left[J_0^2 + 2\{J_1^2 + J_2^2 + \cdots\}\right] = 1$

$\frac{d}{dx}\left[J_n^2 + J_{n+1}^2\right] = 2J_nJ_n' + 2J_{n+1}J_{n+1}' \rightarrow (1)$

Since $xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$

$\Rightarrow J_n'(x) = \frac{n}{x}J_n(x) - J_{n+1}(x)$

$\Rightarrow xJ_n'(x) = -nJ_n(x) + xJ_{n-1}(x)$

$\Rightarrow J_n'(x) = -\frac{n}{x}J_n(x) + J_{n-1}(x)$

$\Rightarrow J_{n+1}'(x) = \frac{-(n+1)}{x}J_{n+1}(x) + J_n(x) \rightarrow (3)$

sub (2) & 3 in (1)

$\Rightarrow \frac{d}{dx}\left[J_n^2 + J_{n+1}^2\right]$

$= 2J_n\left[\frac{n}{x}J_n - J_{n+1}\right] + 2J_{n+1}\left[\frac{-(n+1)}{x}J_{n+1} + J_n(x)\right]$

$= \frac{2n}{x}J_n^2 - 2J_nJ_{n+1} - \frac{2(n+1)}{x}J_{n+1}^2$

$= \frac{2}{x}\left\{nJ_n^2 - (n+1)J_{n+1}^2\right\}$.

~~Evaluate (a) $\int x\, J_0(x)$~~

$$\int J_3(x)\,dx = \int \left(x^2 J_3(x)\right) dx$$

$$= x^2 \int x^2 J_0(x)\,dx$$

$$= x^2 \left\{ x J_1(x) \right\} - 2\left(\int x J_1(x)\,dx\right)$$

$$= x^2 \left\{ J_1(x) \right\} - 2\left\{ x^2 J_1(x)\,dx \right\}$$

$$= x^2 \left\{ x J_1(x) \right\} - 2\left\{ x^2 \right.$$

P.T. $\int J_0(x) J_1(x)\,dx = -\frac{1}{2}\left[J_0(x)\right]^2$

$\because J_0'(x) = -J_1(x) = -J_1(x)$

(or)

$J_1(x) = -J_0'(x)$

Q)

a) $\int x^3 \cdot J_0(x)\,dx$

b) $\int J_3(x)\,dx$

Sol.

(a) Since $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

(or)

$$\int x^n J_{n-1}(x)\,dx = x^n J_n(x).$$

given $\int x^3 \cdot J_0(x)\,dx = \left(\int x^2 \left(J_0(x)\right) \cdot x\,dx\right)$

$\Rightarrow \int x^2 \left(x \cdot J_0(x)\right) dx$

$\Rightarrow x^2 \int x \cdot J_0(x)\,dx - \int 2x \left[\left[\int x J_0(x)\,dx\right]\right] dx$

$\Rightarrow x^2 \left(x \cdot J_1(x)\right) - 2\int x \left(x \cdot J_1(x)\right) dx$

$\Rightarrow x^3 J_1(x) - 2\int x^2 J_1(x)\,dx.$

$\Rightarrow x^3 J_1(x) - 2\left(x^2 J_2(x)\right)$

$\Rightarrow x^3 J_1(x) - 2x^2 J_2(x).$

$$\Rightarrow \boxed{\begin{array}{c} \frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x) \\ \int x^{-n} J_{n+1}(x) = -x^{-n} J_n(x) \end{array}}$$

b) $\int J_3(x)\,dx \Rightarrow \int x^2 \left(x^{-2} J_3(x)\right) dx$

$x^2 \int x^{-2} J_3(x)\,dx$

$\Rightarrow x^2 \left[-x^{-2} J_2(x)\right]$

$\Rightarrow \int x^2 \left[-x^{-2} J_2(x)\right] + \int 2x \left[\int x^{-2} J_2(x)\right]$

$\Rightarrow \int x^2 \left[-x^{-2} J_2(x)\right] + \int 2x \left[-x^{-2} J_1(x)\right]$

Q) $\int J_0(x)\, J_1(x)\, dx = -\frac{1}{2}\left[J_0(x)\right]^2$.

Sol Since $J_0'(x) = -J_1(x)$

(or)

$J_1(x) = -J_0'(x)$

$\Rightarrow$ Given $\int J_0(x) \cdot J_1(x) \cdot dx = -\int J_0(x) \cdot J_0'(x) \cdot dx$

Consider $\int J_0(x)\, J_0'(x) \cdot dx$.

$\Rightarrow J_0(x) \int J_0'(x)\, dx - \int J_0'(x) \left(\int J_0'(x) \cdot dx\right) \cdot dx$

$\Rightarrow J_0(x)\, J_0(x) - \int J_0'(x)\, J_0(x) \cdot dx$

$\Rightarrow 2\int J_0(x)\, J_0'(x) \cdot dx = \left[J_0(x)\right]^2 \Rightarrow \int J_0(x) J_1(x) \cdot dx$

$= -\frac{1}{2}\left(J_0(x)\right)^2$.

Q) Since $\frac{d}{dx}\left[J_n^2 + J_{n+1}^2\right] = \frac{2}{x}\left[n J_n^2 - (n+1) J_{n+1}^2\right]$

Put $n = 0, 1, 2$

$\frac{d}{dx}\left[J_0^2 + J_1^2\right] = \frac{2}{x}\left[-J_1^2\right]$

$\frac{d}{dx}\left[J_1^2 + J_2^2\right] = \frac{2}{x}\left[J_1^2 - 2J_2^2\right]$

$\frac{d}{dx}\left[J_2^2 + J_3^2\right] = \frac{2}{x}\left[2J_2^2 - 3J_3^2\right]$

$\vdots \qquad \vdots$

on adding $\frac{d}{dx}\left[J_0^2 + 2(J_1^2 + \cdots)\right] = \frac{2}{x}[0] = 0$

$J_0^2 + 2(J_1^2 + J_2^2 + \cdots) = C \quad (\int 0 = C)$

$\because J_0(0) = 1 \;;\; J_n(0) = 0 \;;\; n \geq 1$

$\therefore$ on substituting $(1)^2 + 2(0+0+0 \ldots) = C \Rightarrow C = 1$

$\therefore J_0^2 + 2(J_1^2 + J_2^2 + \cdots) = 1$.

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