24/9/07

# UNIT-6 :-

## The Calculus of Residues:-

### Residue of a function of its pole:-

The Co-efficient of $1/z-a$ be expansion of $F(z)$ about this similar point $z=a$ is called the residue of $F(z)$ at $z=a$ and is denoted by $Res\left[F(z): z=a\right]$ (or) $Res\left[F(z)\right]_{z=a}$

$\therefore$ By larence series we have

$f(z) = \sum a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$ here the Co-efficient of $\frac{1}{z-a}$ is $b_1$ which can be defined as follows

where $b_1 = \frac{1}{2\pi i} \int_c \frac{f(z)}{(z-a)^{-1+1}} \cdot dz = 2\pi i (Res[F(z): z=a])$

where 'c' is closed curve containing a point $z=a$

note:-

i). if $F(z)$ has a simple pole at $z=a$ then resolve

$\left[F(z) \text{ at } z=a\right] = \lim_{z \to a} (z-a) \cdot F(z)$

2) if $F(z)$ has simple pole at $z=a$ and let $f(z) = \frac{\phi(z)}{\psi(z)}$ where $\phi(z)$ ; $\psi(z)$ are analytic in perticular $\phi(z)$ is not equal to zero and ~~~~ so then $\psi(z)$ has simple pole

$Resi\left[F(z) : z=a\right] = \frac{\phi(a)}{\psi'(a)}$

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3) If $f(z)$ has a pole at $z=a$ of order $m (m \geq 1)$ then

$$\operatorname{Res}\left[f(z)\right]_{z=a} = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left[ (z-a)^m \cdot f(z) \right].$$

4) If $z=a$ is essential similarity of $f(z)$ then the above fail to compute residue in such cases we expand the func $f(z)$ about $z=a$ and identify co-efficent of $1/z-a$ which gives the residues of $f(z)$ at $z=a$

Problem:-

Find the Residues of following func:-

1) $f(z) = \dfrac{z^2}{(z-1)^2 (z+2)}$ 2) $f(z) = \dfrac{z \cdot e^z}{(z-1)^3}$

3) $f(z) = \dfrac{z^2 - 2z}{(z+1)^2 (z^2+1)}$ 4) $f(z) = \dfrac{z^2}{z^4 - 1}$

3) $f(z) = \dfrac{z^2 - 2z}{(z+1)^2 (z^2+1)}$

The poles of $f(z)$ are given by

$$(z+1)^2 (z^2+1) = 0$$

$$(z+1)^2 = 0 \; ; \; (z^2+1) = 0$$

$z = -1$ is a pole of order 2

$z = \pm i$ is a simple pole

Residue of $f(z)$ at $z=-1$

$$\operatorname{Res}\left[f(z)\right]_{z=-1} = \frac{1}{(2-1)!} \cdot \lim_{z \to -1} \left\{ \frac{d^{2-1}}{dz^{2-1}} \left( (z+1)^2 \cdot \frac{z^2 - 2z}{(z+1)^2 (z^2+1)} \right) \right\}$$

$$= \lim_{z \to -1} \left\{ \frac{d}{dz} \left( \frac{z^2 - 2z}{z^2+1} \right) \right\}$$

$$= \lim_{z \to -1} \left\{ \frac{(z^2+1)(2z-2) - (z^2-2z)(2z)}{(z^2+1)^2} \right\}$$

$$= \lim_{z \to -1} \left\{ \frac{2z^3 - 2z^2 + 2z - 2 - 2z^3 + 4z^2}{(z^2+1)^2} \right\}$$

$$= \lim_{z \to -1} \left[ \frac{2z^2 + 2z - 2}{(z^2+1)^2} \right] = \frac{\cancel{2} - \cancel{2} - 2}{4} = \frac{-2}{4} = -\frac{1}{2}$$

$$\text{Res}\left[f(z)\right]_{z=+i} = \lim_{z \to i} (z-i) \left[ \frac{z^2 - 2z}{(z+1)^2 (z+i)(z-i)} \right]$$

$$= \lim_{z \to i} \left[ \frac{z^2 - 2z}{(z+1)^2 (z+i)} \right]$$

$$= \left[ \frac{i^2 - 2i}{(i+1)^2 (2i)} \right] = \frac{\cancel{i}(i-2)}{(i+1)^2 (2\cancel{i})} = \frac{i-2}{2(1+i)^2}$$

$$\text{Res}\left[f(z)\right]_{z=-i} = \lim_{z \to -i} (z+i) \left[ \frac{z^2 - 2z}{(z+1)^2 (z-i)(z+i)} \right]$$

$$= \frac{-i^2 - 2(-i)}{(-i+1)^2 (-i-i)} = \frac{1+2i}{(1-i)^2 (-2i)}$$

Q) find Residue of $f(z) = \frac{z^2}{z^4+1}$ at singular points which lies inside circle $|z| = 2$

$f(z) = \frac{z^2}{z^4+1}$

the Pole of $f(z)$ are given by

$z^4+1 = 0$

$z = (-1)^{1/4} = (\cos\pi + i\sin\pi)^{1/4}$

$(\because \cos\theta + i\sin\theta)^n = \cos n\theta + i\sin\theta = \cos n(\theta + 2k\pi) + i\sin n(\theta$

$k = 0, 1, 2 \cdots (n-1)$

$= (\cos \pi/4 + i\sin \pi/4)$

$= \cos\left(\frac{\pi + 2k\pi}{4}\right) + \cos($

for $k=0$ ; $z = \cos \pi/4 + i\sin \pi/4 = e^{i\pi/4}$

$k = 1$ ; $z = e^{3i\pi/4}$

$k=2$ ; $z = e^{5i\pi/4}$ ; $k=3$ ; $z = e^{-i\pi/4}$ all are simple play 2 lies in $|z| = n$

$= \frac{\phi(e^{i\pi/4})}{\psi'(e^{i\pi/4})}$ , where $\phi(z) = z^2$

$\psi(z) = z^4+1$

$= \frac{1}{4 \cdot e^{i\pi/4}} = \frac{1}{4} \cdot e^{-i\pi/4}$

$\text{Res}\{f(z)\}_{z = e^{3i\pi/4}} = \frac{1}{4} \cdot e^{3i\pi/4}$

$\text{Res}\{f(z)\} = \frac{1}{4} \cdot e^{-3i\pi/4}$

$\text{Res}\{f(z)\}_{z = e^{5i\pi/4}} = e^{5i\pi/4}$

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Q) $f(z) = \frac{1 - e^{2z}}{z^4}$

$= \frac{1}{z^4}\left(1 - \left(1 + 2z + \frac{(2z)^2}{2!} + \frac{(2z)^3}{3!} + \cdots\right)\right)$

$= -\frac{1}{z^4}\left[2z + 2z^2 + \frac{4}{3} \cdot z^3 + \cdots\right]$

$= -\left(\frac{2}{z^3} + \frac{2}{z^2} + \frac{4}{3} \cdot \frac{1}{z} + \cdots\right)$

$=$ co-eff of $\frac{1}{z-0}$ is

$\text{Res}\{f(z)\}_{z=0} = -\frac{4}{3}$

Note:-

Residue of $f(z)$ at $z = \infty$ is $\lim_{z\to\infty} -z f(z)$ is exists if $f(z)$ is analytic $z$ = infinite

find the Res of $f(z) = \frac{z}{(z-1)(z-2)}$ at $z = \infty$

$\lim_{z\to\infty} \{-z f(z)\} = \lim_{z\to\infty}\left[-z \cdot \frac{z}{z^2(1-\frac{1}{z})(1-\frac{2}{z})}\right] = -1$

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$$\cancel{\oint_C \frac{x^2-2z}{(z+1)^2(z^2+1)}\,dz \quad c: |z|=2}$$

## Recedue Th:-

if $F(x)$ is anylatic inside on a simple closed curve 'c' except on at a' finite no:- of holes $(x_1, x_2 \ldots \ldots z_n)$ and let $R_1, R_2 \ldots R_n$ be the Co-responding Res of $f(x)$ at these poles then

$$\oint_C F(z)\cdot dz = 2\pi i\,(R_1 + R_2 + \ldots R_n)$$
$$= 2\pi i\,(\text{sum of residues})$$

**Proof**:- draw the small circles $c_1, c_2, c_3$ with Centres at Poles $x_1, z_2, z_3 \ldots z_n$ with very small radius so that they are entirely lies in side 'c' and don't over lap.

$\because f(z)$ is anylatic in side 'c' hence is also anylatic along $c_1, c_2 \ldots c_n$ hence by Cauchys Th multiple collected regions we have

$$\oint_C F(x)\,dz = \oint_{C_1} F(z)\,dz + \oint_{C_2} F(z)\cdot dz + \ldots \oint_{C_n} F(x)\,dz$$

$\because$ by def of residue we write

$$\oint_{C_1} f(z)\cdot dz = 2\pi i\,\{\text{Res}\,(F(z) : z = z_i)\}$$
$$\vdots$$
$$\oint_{C_2} F(z)\cdot dz = 2\pi i\; R_i$$

$$\oint_{C_3} f(z)\cdot dz = 2\pi i\,R_3$$
$$\vdots$$
$$= 2\pi i\,R_n$$

using (2) in (1)

$$\oint_C F(z)\cdot dz = 2\pi i R_1 + 2\pi i R_2 + \ldots 2\pi i R_n$$
$$\oint_C F(z)\cdot dz = \{2\pi i\,[R_1 + R_2 + R_3 \ldots R_n]\}.$$

## Problems:-

Using Residue Th. Evaluate the following integrals:-

Q) $\oint_C \frac{4-3z}{z(z-1)(z-2)}\cdot dz$ where 'c' is circle $c: |z| = 3/2$

The given fun $F(z) = \frac{4-3z}{z(z-1)(z-2)}$

Poles of $f(x)$ are $z = 0, 1, 2 \ldots$ are sample Poles

here $z = 0, 1$ are the only Pole lies in side $|z| = 3/2$

Residues:-

$$\text{Res}\,[F(z)]_{z=0} = \lim_{z\to 0} (z-0)\,\frac{4-3z}{z(z-1)(z-2)}$$
$$= \frac{4}{(-1)(-2)} = 2 = R_1$$

$$\text{Res}\,[F(z)]_{z=1} = \lim_{z\to 1} (z-1)\cdot\frac{4-3z}{z(z-1)(z-2)}$$
$$= \frac{4-3}{-1} = -1 = R_2$$

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By residues Th we have

$$\oint_C \frac{4-3z}{z(z-1)(z-2)} \cdot dz = 2\pi i(2-1) = 2\pi i$$

2) $\oint_C \frac{(2z+1)^2}{(4z^3+z)} \cdot dz \quad C: |z| = 1$

$$f(z) = \frac{(2z+1)^2}{(4z^3+z)}$$

$(4z^3+z) = z(4z^2+1) = 0$

$= z = 0 \; ; \; z^2 = -1/4$

$\Rightarrow z = \pm i/2$

Poles are $z = 0 \; ; \; z = \pm i/2$

$4z^3 + z \longrightarrow (z + i/2) \cdot (z - i/2)$

$\text{Res}\{f(z)\}_{z=0} = \lim_{z \to 0} (z-0)$

P) $\oint_C \frac{z-3}{z^2+2z+5} \cdot dz$

a) $|z| = 1$   b) $|z+1-i| = 2$   c) $|z+1+i| = 2$

here $f(z) = \frac{z-3}{z^2+2z+5}$

Pole $z^2+2z+5 = 0$

$\Rightarrow z = \frac{-2 \pm \sqrt{4-20}}{2}$

$= -1+2i \; ; \; |-1+2i| = \sqrt{1^2+2^2} = \sqrt{5} > 1$

$z = -1-2i \; ; \; = \sqrt{5} > 1$ (outside)

By Cauchys Th:

$$\oint_C \frac{z-3}{z^2+3z+5} dz = 0$$

b) $|z+1-i| = 2$

for $z = -1+2i \; ; \; |-1+2i+1-i| = |i| < 2$

$z = -1-2i \; ; \; |-1-2i+1-i| = |-3i| = 3 > 2$

$\text{Res}\{f(z)\}_{z=-1+2i} = \lim_{z \to -1+2i} (z+1-2i) \cdot \frac{z-3}{(z+1-2i)(z+1+2i)}$

$= \frac{-1+2i-3}{-1+2i+1+2i} = \frac{2i-4}{4i} = 1/2 + i$ //

$\sin z = 0 \Leftrightarrow z = n\pi$

$n = 0, \pm1, \pm2 \cdots$

$\cos z = 0 \Leftrightarrow z = (2n+1)\pi/2$

$\tan z = 0 \Leftrightarrow z = n\pi + \pi/2$

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4) $\oint_C \frac{\coth z}{(z-i)} \cdot dz$ $|z|=2$

Given integral is

$\oint_C \frac{\cosh z}{(z-i)\sinh z} \cdot dz$ where $C: |z|=2$

here $f(z) = \frac{\cosh z}{(z-i)\sinh z}$ $\sinh(ix) = i(\sin x)$

Poles are $(z-i)\sinh z = 0$

$(z-i) = 0$ ; $\sinh z = 0$

$z = i$ ; $z = n\pi i$

$z = n\pi i$

$n=0 \Rightarrow z=0$

$n=1 \Rightarrow z = \pi i$ ; $>2$

$n=-1 \Rightarrow z = -\pi i$ ; $>2$

here $z=i$ ; $z=0$ are only the poles lies inside

$\text{Res}[f(z)]_{z=i} = \lim_{z \to i} (z-i) \frac{\cosh z}{(z-i) \cdot \sinh z}$

$= \coth i$

$\text{Res}[f(z)] = \frac{\phi(0)}{\psi'(0)}$. where $\phi(z) = \cosh z$

$\psi(z) = (z-i)\sinh z$

$= \frac{1}{-i} = -1/i$ $\psi'(z) = (z-i)\cosh z + 1 \cdot \sinh z$

$\psi'(0) = (-i) \cdot 1$

$\oint_C \frac{\coth z}{(z-i)} \cdot dz = 2\pi i (\coth i + i)$

Evaluate 1) $\oint_C \frac{dz}{\sinh z}$ $c: |z| = 4$

2) $\oint_C \frac{z^3}{(z-2)(z-1-i)} \cdot dz$ $c: |z| = 3$

3) $\oint_C e^z \sec \pi z$ $c: |z| = 1$

Evalution of different integrals

~~type 1: integral~~

Evaluation along the integral type:-

type 1:- Evaluation of the unit hear Evaluate the integral type

$\oint_0^{2\pi} F(\cos\theta, \sin\theta) \cdot d\theta \to (1)$

Put $z = e^{i\theta}$

$dz = e^{i\theta} \cdot i \cdot d\theta$

$d\theta = \frac{dz}{iz}$

$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = 1/2 (z + 1/z)$

$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2} = 1/2i (z - 1/z)$

on substituting in (1)

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$$= \oint_C F\left(\frac{1}{2}\left(z + \frac{1}{z}\right) ; \frac{1}{2i}\left(z - \frac{1}{z}\right)\right).dz$$

$$\oint_C F(z).dz \to ②$$

which can be Evaluated during residue Th:

S/1:- $\displaystyle\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}} \quad (a>b>0)$

Sol:- Consider $\displaystyle\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$

$z = e^{i\theta}$

$d\theta = \frac{dz}{iz}$

$\cos\theta = \frac{1}{2i}\left(z+\frac{1}{z}\right)$

on sub ① $\Rightarrow \displaystyle\oint_C \frac{1}{a+\frac{b}{2}(z+1/z)} \cdot \frac{dz}{iz}$

$\Rightarrow \displaystyle\oint_C \frac{2z}{bz^2+2az+b} \cdot \frac{dz}{iz}$

$\Rightarrow \displaystyle\oint_C F(z)\,dz$ where $f(z) = \frac{1}{bz^2+2az+b}$

Pole of $F(z)$ are $bz^2+2az+b=0$

$$z = \frac{-2a \pm \sqrt{4a^2-4b^2}}{2b}$$

(Pole of eq $r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$)

$$= \frac{-a \pm \sqrt{a^2+b^2}}{b} \text{ are simply Pole}$$

Let $\alpha = \frac{-a+\sqrt{a^2+b^2}}{b}$ ; $\beta = \frac{-a-\sqrt{a^2-b^2}}{b}$

here for $a>b>0$ ; $|\beta|>1$

WKT $|\alpha\beta| = 1$ & $|\beta|>1$

$\Rightarrow |\alpha|<1$

$\therefore$ here $d\alpha = \frac{-a+\sqrt{a^2-b^2}}{b}$ is lies inside

$$\text{Res}\left[f(z)\right]_{z=\alpha} = \lim_{z\to\alpha} (z-\alpha) \cdot \frac{1}{b(z-\alpha)(z-\beta)}$$

$$= \frac{1}{b(\alpha+\beta)}$$

$$r = \frac{1}{b\left(\frac{2\sqrt{a^2-b^2}}{b}\right)} = \frac{1}{2\sqrt{a^2-b^2}}$$

using Res Th.

$$\oint_C F(z).dz = 2\pi i \left(\frac{1}{2\sqrt{a^2-b^2}}\right) = \frac{\pi i}{\sqrt{a^2-b^2}} \to ③$$

$$\Rightarrow \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} \cdot d\theta = \frac{2}{i}\left(\frac{\pi i}{\sqrt{a^2-b^2}}\right) = \frac{2\pi}{\sqrt{a^2-b^2}}$$

Evaluate $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}.d\theta$

Given $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}.d\theta \to ①$

Put $z = e^{i\theta}$

$d\theta = \frac{dz}{iz}$

$\sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right)$

$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$

$$\Rightarrow \oint_C \frac{\left(\frac{1}{2i}\left(z-\frac{1}{z}\right)\right)^2}{a+\frac{b}{2}\left(z+\frac{1}{z}\right)} \cdot \frac{dz}{iz}$$

$$\Rightarrow -\frac{1}{4}\oint_C \frac{(z^2-1)^2}{z^2} \times \frac{2z}{bz^2+2az+b} \times \frac{dz}{iz}$$

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$$\Rightarrow -\frac{1}{2i}\oint \frac{(z^2-1)^2}{z^2(bz^2+2az+b)}\,dz$$

where $f(z) = \dfrac{(z^2-1)^2}{z^2(bz^2+2az+b)}$

Poles of $f(z)$ also $z=0$ of order 2

Let $\alpha = \dfrac{-a+\sqrt{a^2-b^2}}{b}\ (<1)$ ; $\beta = \dfrac{-a-\sqrt{a^2-b^2}}{b}$

Now $[f(z)]_{z=0} = \lim_{z\to 0}\left\{\dfrac{d}{dz}(z-0)^2 \times \dfrac{(z^2-1)^2}{z^2(bz^2+2az+b)}\right\}$

$$= \lim_{z\to 0}\left\{\frac{(bz^2+2az+b)\,2(z^2-1)\cdot 2z-(z^2-1)^2(2bz+2a)}{(bz^2+2az+b)^2}\right\}$$

$$= 0-\frac{1\times 2a}{b^2} = \frac{-2a}{b^2}$$

$$\text{Res}[f(z)]_{z=\alpha} = \lim_{z\to\alpha}\left((z-\alpha)\cdot\frac{(z^2-1)^2}{z^2 b(z-\alpha)(z-\beta)}\right)$$

$$= \frac{(\alpha^2-1)^2}{\alpha^2\times b\times(\alpha-\beta)}$$

$$= \frac{\{\alpha(\alpha-\frac{1}{\alpha})\}^2}{\alpha^2\times b\times(\alpha-\beta)}$$

$$\frac{\alpha^2(\alpha-\beta)^2}{\alpha^2\times b\times(\alpha-\beta)} = \frac{\alpha-\beta}{b}$$

$$= \frac{2\sqrt{a^2-b^2}}{b^2}$$

By def of Res Th

$$\oint_C f(z)\cdot dz = 2\pi i\left[\frac{-2a}{b}+\frac{2\sqrt{a^2-b^2}}{b^2}\right]$$

$$= 2\pi i\times\frac{-2}{b^2}\left(a-\sqrt{a^2-b^2}\right)$$

on sub (2)

$$\int_0^{2\pi} = -\frac{1}{2\pi i}\left\{\frac{-4\pi i}{b^2}\left(a-\sqrt{a^2-b^2}\right)\right\}$$

$$= \frac{2\pi}{b^2}\left(a-\sqrt{a^2-b^2}\right)\quad(a>b>0)$$

P) $\displaystyle\int_0^{2\pi}\frac{d\theta}{\pi+\cos\theta}$

P) Evaluate $\displaystyle\int_0^{2\pi}\frac{\cos 2\theta}{1-a^2-2a\cos\theta}\,d\theta$ when $(a<$ [illegible]$)$

Given $\displaystyle\int_0^{\pi}\frac{\cos\theta}{1-a^2-2a\cos\theta}\,d\theta \to (1)$

on integrating we get

$$\int_0^{2\pi}\frac{\cos 2\theta(1+a^2)+(2a\cos\theta)}{(1+a^2)^2-(2a\cos\theta)^2}\,d\theta$$

$$\Rightarrow (1+a^2)\int_0^{\pi}\frac{\cos 2\theta}{(1+a^2)^2-(2a\cos\theta)^2}\,d\theta+2a\int_0^{\pi}\frac{\cos 2\theta\cdot\cos\theta\,d\theta}{(1+a^2)^2-(2a\cos\theta)^2}$$

(The Second integral vanish $\because f(\pi-\theta)=-f(\theta)$) $\quad 4a^2\left(\frac{1+\cos 2\theta}{2}\right)$

$$= (1+a^2)\int_0^{\pi}\frac{\cos\theta}{1+a^4-2a^2\cos 2\theta}\,d\theta$$

Put $2\theta=\phi$

$d\theta=\phi/2$

$\theta=0\Rightarrow\phi=0$

$\theta=\pi\Rightarrow\phi=\pi$

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$$= (1+a^2)\int_0^{2\pi}$$

$$= (1+a^2)\oint_C \frac{\frac{1}{2}\left(z+\frac{1}{z}\right)}{1+a^4-2a^2\times\frac{1}{2}\left(z+\frac{1}{z}\right)}\cdot\frac{dz}{iz}$$

$$= \frac{(1+a^2)}{2i}\oint_C \frac{\frac{z^2+1}{z}}{\frac{z+a^4\cdot z-a^2z^2-a^2}{z}}\cdot\frac{dz}{z}$$

$$= \frac{(1+a^2)}{2i}\oint_C \frac{z^2+1}{(-a^2\cdot z^2+z(a^4+1))\cdot dz}$$

$$= \frac{(1+a^2)}{2ia^2}\oint \frac{z^2+1}{\left(z^2-\left(\frac{z^4+1}{z}\right)z+1\right)}\cdot dz$$

$$= \frac{(1+a^2)i}{4a^2}\oint_C f(z)\cdot dz \to (2)$$

where $f(z) = \dfrac{z^2+1}{z\left(z^2-\left(a^2+\frac{1}{a^2}\right)z+1\right)}$

$$= \frac{z^2+1}{z(z-a^2)(z-1/a^2)}$$

Poles are $z=0$ ; $a^2$, $1/a^2$.

Evaluate $\int_0^{2\pi}\frac{d\theta}{(a+b\cos\theta)^2}$ ; where $a>b>0$

~~Type :- II~~

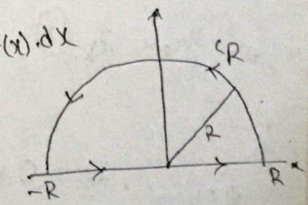
Evaluation of integral is of type $-\infty$ to $\infty$ is of the type $\int_{-\infty}^{\infty} f(x)\cdot dx$

Type II :- integration around semi circles.

To solve these type of integrals $\int_C f(z)dz$ where 'C' is the closed curve consisting the semi circle $C_R : |z| = R$ (or) $z = Re^{i\theta}$ to-gether with real axis $-R$ to $R$.

if $f(z)$ is analytic in the upper half of the plane except at a finite no of poles in it and having no poles on the real axis. then by the residue theorem we have.

$$\therefore \int_C f(z)dz = \int_{C_R} f(z)dz + \int_{-R}^{R} f(x)\cdot dx$$



here $\int_{C_R} f(z)\cdot dz \to 0$ as $R\to\infty$

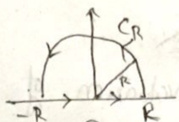
$$\therefore \int_C f(z)\cdot dz = \int_{-\infty}^{\infty} f(x)\cdot dx = \int_{-R}^{R} f(x)\cdot dx.$$

on real axis $z = x$ ; $dz = dx$

$$\therefore \int_{-\infty}^{\infty} f(x)\cdot dx = \int_{-\infty}^{\infty} f(z)\cdot dz$$ which can be evaluated by using residue theorem

Evaluate $\int_{-\infty}^{\infty} \frac{dx}{x^4+1}$

let $f(x) = \frac{1}{x^4+1}$




$$\int_{-\infty}^{\infty} \frac{1}{x^4+1} \cdot dx = \int_{C_R} \frac{1}{x^4+1} \cdot dx + \int_{-R}^{R} \frac{1}{x^4+1} \cdot dx \longrightarrow (1)$$

Consider:- $\int_{C_R} \frac{1}{x^4+1} \cdot dx$ $\quad \because C_R = |z| = R$

$\Rightarrow x = R \cdot e^{i\theta}$

$dx = R \cdot e^{i\theta} \cdot i \cdot d\theta$

$$= \int_0^{\pi} \frac{1}{R^4 e^{4i\theta}+1} \cdot R \cdot e^{i\theta} \cdot i \cdot d\theta$$

$$= \int_0^{\pi} \frac{1}{R^3 \cdot e^{4i\theta} \cdot d\theta} \cdot e^{i\theta} \cdot i d\theta \rightarrow 0 \quad \text{as } R \rightarrow \infty$$

$\Rightarrow$ from Eq (1)

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1} \cdot dx = 0 + \int_{-\infty}^{\infty} \frac{1}{x^4+1} \cdot dx \quad (\because R \rightarrow -\infty \text{ to } \infty)$$

here $f(x) = \frac{1}{x^4+1}$

Poles of $f(x)$ are $x^4 = -1 \Rightarrow x = (-1)^{1/4}$

Poles are $x = e^{i\pi/4}, e^{3i\pi/4}, e^{5i\pi/4}, e^{7i\pi/4}$

here $x = e^{i\pi/4}, e^{3i\pi/4}$ are only the poles lies in upper half of semi circle

$$\text{Res}\left[f(z)\right]_{z=e^{i\pi/4}} = \frac{\phi(e^{i\pi/4})}{f'(e^{i\pi/4})}$$

$$= \frac{1}{4x^3}\Big|_{x=e^{i\pi/4}}$$

$$= \frac{1}{4} \cdot e^{-3i\pi/4}$$

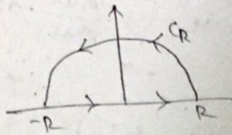
$$\text{Res}\left[f(z)\right]_{z=e^{3i\pi/4}} = \frac{1}{4} \cdot e^{-9i\pi/4}$$

$$\int_{-\infty}^{\infty} f(x) \cdot dx = 2\pi i \left(\frac{1}{4} \cdot e^{-3i\pi/4} + \frac{1}{4} \cdot e^{-9i\pi/4}\right)$$

$$= \pi i/2 \left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4} + \cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$

$$= \frac{\pi i}{2}\left(-\cancel{\frac{1}{\sqrt{2}}} - \frac{i}{\sqrt{2}} + \cancel{\frac{1}{\sqrt{2}}} - \frac{i}{\sqrt{2}}\right) = \frac{\pi i}{\cancel{2}}\left(\frac{-\cancel{2}i}{\sqrt{2}}\right) = \frac{\pi}{\sqrt{2}}$$

P) Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} \cdot dx \; (a>0 \; ; b>0 \; ; a \neq b)$




let $f(x) = \frac{x^2}{(x^2+a^2)(x^2+b^2)}$

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} \cdot dx = \int_{C_R} \frac{x^2}{(x^2+a^2)(x^2+b^2)} \cdot dx + \int_{-R}^{R} \frac{x^2}{(x^2+a^2)(x^2+b^2)} \cdot dx - (1)$$

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Consider :- $\int_{C_R} \frac{z^2}{(z^2+a^2)(z^2+b^2)} dz$ $\left(\because |z| = R \Rightarrow z = R\cdot e^{i\theta},\ dz = R\cdot i e^{i\theta} d\theta\right)$

$$\int_{C_R} \frac{R^2 \cdot e^{i2\theta}}{(R^2 e^{i2\theta}+a^2)(R^2 e^{i2\theta}+b^2)} \cdot R\cdot i e^{i\theta} d\theta$$

$$\int_{C_R} \frac{R^2 \cdot e^{i2\theta}}{R^4 e^{i4\theta}}$$

$z^2 \cdot z^2$ $e^{i2\theta} \cdot e^{i2\theta}$ $e^{-4\theta}$

$(1+a^2)(1+b^2)$

Evaluate :- $\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1} \cdot dx \ ; a>0$

Sol. Since $\cos ax$ is R.P of $e^{iax}$

Consider the func $f(z) = \frac{e^{iaz}}{z^2+1}$

60/8 i.e. $\int_{-\infty}^{\infty} \frac{\cos az}{z^2+1} \cdot dz = R.P\left\{\int_{-\infty}^{\infty} \frac{e^{iaz}}{z^2+1} \cdot dz\right\} \rightarrow (1)$

we evaluate $\int_{-\infty}^{\infty} \frac{e^{iaz}}{z^2+1} \cdot dz$

Pole of $f(z)$ are $z = \pm i$ But only lies on CR

$$Res\{f(z)\}_{z=i} = \lim_{z\to i} (z-i) \frac{e^{iaz}}{(z+i)(z-i)}$$

$$= \frac{e^{-a}}{2i} = -\frac{1}{2} i \cdot e^{-a}$$

on Sub (1)

$$\int_{-\infty}^{\infty} \frac{\cos az}{z^2+1} \cdot dz = R.P\left\{2\pi i\left(-\frac{1}{2} \cdot i \cdot e^{-a}\right)\right\}$$

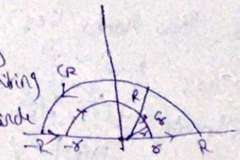
$$= R.P \text{ of } \{\pi \cdot e^{-a}\} = \pi \cdot e^{-a} \ ; a>0$$

P) Evaluate $\int_0^{\infty} \frac{dx}{(x^2+a^2)^2}$

P) $\int_0^{\infty} \frac{\cos x}{(1+x^2)^2} \cdot dx$

Type - 3:-

In this case the poles of $f(z)$ lies on real axis then in evaluation omiting that pole by drawing another semic circle whose centre is at that pole




S.T all the Eq ~~...~~ are lies inside the circle $|z| = 2$

$z^4+6z+3=0$

Sol Let choose $f(z) = z^4$

$g(z) = 6z+3$

also $\frac{|g(z)|}{|f(z)|} = \frac{|6z+3|}{|z^4|}$

$\frac{|6(2)+3|}{|2^4|}$ on $|z|=2$

$= \frac{15}{16} < 1$ on $|z| = 2$.

By using Rouche's Th:-

Both $f(z)$ & $f(z)+g(z)$ have the same no. of zero in side $|z|=2$

but $f(z)=z^4$ has four roots in $|z|=2$ &

where $f(z)+g(z) = z^4+6z+3$ have four roots

in side $|z|=2$

finally $z^4+6z+3=0$ has four zeros insides of $|z|=2$.

Here it is sufficient to prove all 5 roots of given to prove inside $|z|<2$ and next we prove one root is lies inside of $|z|<\frac{3}{2}$ then automatically the remaining four roots are lies in the $\frac{3}{2}<|z|$ is less than

All the roots lies inside of $|z|<2$

Consider $f(z)=z^5$. $g(z)=15z+1$

Clearly $f(z), g(z)$ are analytic

Also $\frac{|g(z)|}{|f(z)|} = \frac{|15(2)+1|}{|2^5|}$ on $|z|=2$ $= \frac{31}{32}<1$

By rouche's Th have same no of zero inside of $|z|<2$ $f(z)+g(z)=z^5+15z+1$

but $f(z)=z^5$ has 5 zeros and hence

in side of $|z|<$

$\therefore z^5+15z+1=0$ has 5 roots inside of $|z|<2$ we prove

one root is lies clearly $f(z); g(z)$ are analytic

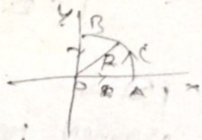
$$\frac{|g(z)|}{|f(z)|} = \frac{|(\frac{3}{2})^5+1|}{|15\times 3/2|} < 1 \text{ on } |z|=3/2$$

By rouche's Th $f(z)=g(z)$ & $f(z)+g(z)=z^5+15z+1$ has same no of 0 in side $|z|<3/2$ but $f(z)=15z$ has 1 root '0' inside of $|z|<0$

2 hence $z^5+15z+1=0$ has 1 root

Given & $f(z)=u+iv$ ; $z^5+z^3+2z+3$

$C: |z|=R$ (or) $z=R.e^{i\theta}$




here fol

Along OA:-

$z=x$. & $x \to 0$ to $\infty$

here $f(z) = x^5+x^3+2x+3$

$$\text{along } f = \tan^{-1}\left(\frac{0}{x^5+x^3+2x+3}\right)\Big]_0^\infty = 0$$

$\therefore \Delta_{OA}$ along $f=0$

Along AS:- $z=Re^{i\theta}$ ; $0<\theta<\frac{\pi}{2}$

here $f(z) = R^5.e^{5i\theta}+R^3.e^{3i\theta}+2Re^{i\theta}+3$

$$= R^5.e^{5i\theta}\left\{1+\frac{1}{R^2.e^{2i\theta}}+\frac{2}{R^4.e^{4i\theta}}+\frac{3}{R^5.e^{5i\theta}}\right\}$$

$f(z) \to R^5.e^{5i\theta}$ as $R\to\infty$

$$\text{along } f = \tan^{-1}\left(\frac{\sin 5\theta}{\cos 5\theta}\right) = 5\theta\Big)_0^{\pi/2} = 5\pi/2$$

$\Delta_{AB}$ arg $f = 5\pi/2$

Along BO:- $z=iy$ ; & $y\to\infty$ to $0$

here $f(z)=iy^5-iy^3+2iy+3$

$$\arg f = \tan^{-1}\left(\frac{y^5-y^3+2y}{3}\right)\Big]_\infty^0$$

$\Delta_{BO}\arg = \tan^{-1}(0)-\tan^{-1}(\infty) = -\pi/2$

~~Along AB = $z=R.e^{i\theta}$ ; $0<\theta<\pi/2$~~

~~here $f(z)=R^5.e^{5i\theta}+R^3e^{3i\theta}+2.R.e^{i\theta}+3$~~

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