Fuzzy logic & Set Theory :- ① (7)

Why need :- ? & What ? & How ? (L.A. Zadeh 1965)

fuzzy is a branch of mathematic actually comes under
AI - (Artificial intelligence) (or) Sub computing methods ⟷ Areas
① ANN (Artificial neural N/w)
② GA (Genetic Algorithm)
③ fuzzy
④ SVM (support vector machines)

Why fuzzy :- (Set theory)

Eg :- If Ram is tall (how to define height ?)
if it is a set theory we many know fixed value :

If $X = \{ x \mid x = \text{tall} \}$ → have to define a number exactly

(Hot | Cold)
↓ ↓
1 0

fuzzy means as example :- (less Hot ; more hot ; less cold ; avg more cold) is to be said in term as (fuzzy) (or) uncertainity.

Eg :- Hot range > 35°C (i.e 30° - 45° Hot)

(Logic) :- 0 & 1

How ? :-

① Crispness & Impreciseness

↳ means we know the value of the variable (or) Parameter have exactly then it is crispness.

Impreciseness :- means you have either in variable i.e variables are not exact.

(2) uncertainity & vagueness :-

Eg:- I will come back soon → (vagueness) (regular) (either 1/2 hour (or) ...)

I will come back within in 1 min (it may be 0.1 min, 0.2 min ⇒ fuzzy)

I will come back " " " from 2pm

I " " " " " → (Probability)

→ (fuzzy may be a vagueness but

→ vagueness may not be a fuzzy)

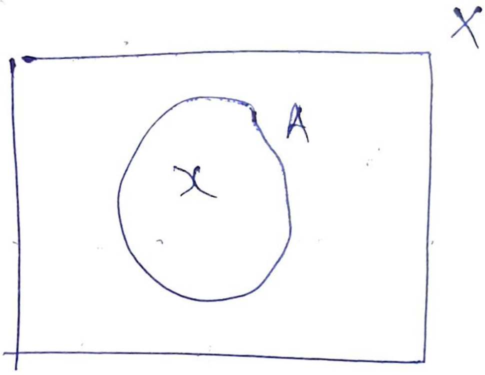
(*) Crisp set theory :-

set means :- if 'A' is set then : Collection of well defined objects

if 'x' is universal set : let

A is set and

x is an element



represents as :-

$x \in A$ (or)

$x \notin A$

characteristic function :- for set theory

ki (symbol)

$$\chi_A(x) = \begin{cases} 1, & \text{if \& only iff } x \in A \\ 0; & \text{if \& only } x \notin A \end{cases}$$

$$\chi_A : X \to \{0, 1\}$$

(*) Relation b/w sets :-

(i) $A \subseteq B$ (subset) ; $B \subseteq A$ (subset) ⇒ $A = B$

(2) $A \subseteq B$ (subset), when we have $A \subset B$ (Proper sub set) ⇒ is $A \neq B$

Operations on set :-

1. $A^c = X - A$ (if X is universal set)
2. $(A^c)^c = A$
3. $\phi^c = X$
   (Empty set) (or) (null set)
4. $X^c = \phi$
5. $A \cup B = \{x / x \in A$ OR $x \in B\}$
   / → such that
6. $A \cap B = \{x / x \in A$ And $x \in B\}$
7. $A \cup X = X$
8. $A \cap X = A$

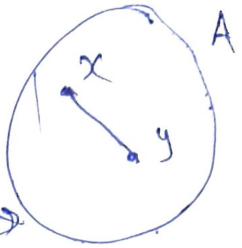
Properties of set theory :-

1. Involution :- $(A^c)^c = A$
2. Commutativity :- $A \cup B = B \cup A$ → union
   $A \cap B = B \cap A$ → intersection.
3. Associativity :- $A \cup (B \cup C) = (A \cup B) \cup C$
   $A \cap (B \cap C) = (A \cap B) \cap C$
4. Distributive :- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   → *not valid in fuzzy (Equality)
5. Idempotency :- $A \cup A = A$; $A \cap A = A$
6. Absorption :- $A \cup (A \cap B) = A$
   $A \cap (A \cup B) = A$
7. Identity :- $A \cup \phi = A$
   (null set).
   $A \cap X = A$
8. Demorgan's law :- $\overline{A \cup B} = \bar{A} \cap \bar{B}$
   $\overline{A \cap B} = \bar{A} \cup \bar{B}$
9. Law of Contradiction :- $\boxed{A \cap \bar{A} = \phi}$
10. Law of Excluded middle :- $\boxed{A \cup \bar{A} = X}$
   → (imp in fuzzy).

* Convex set:-

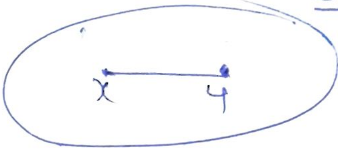
→ set A, is Convex
→ Consider 2-Points and join line (x & y)
this line is set A is Convex

Eg:-



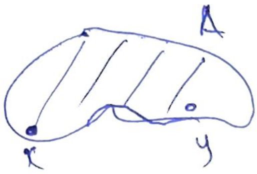
(or)

Eg



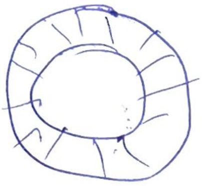
* non Convex set:-

Eg

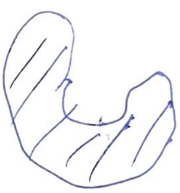


if i join x & y there are many points which are not in a set so it is non Convex set

Eg:-



Eg:-



* Cardinality of set:-

$|A|$ = no of elements in A
if A is finite then $|A|$ is finite
if A is infinite then $|A|$ is infinite

* Power set:-

$P(A)$ = is a Collection of all subsets of A

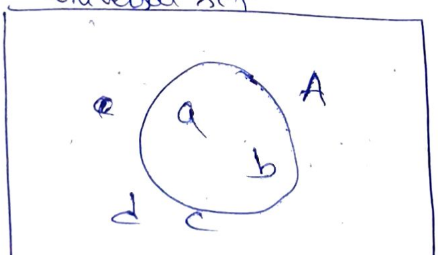
Eg:- $A = \{a, b, c\}$ (3)

$P(A) = \{\phi; \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$

$= 2^{|A|} = 2^3 = 8$

* Crisp set:-

Universal set



(Complete not X) (X Bar)

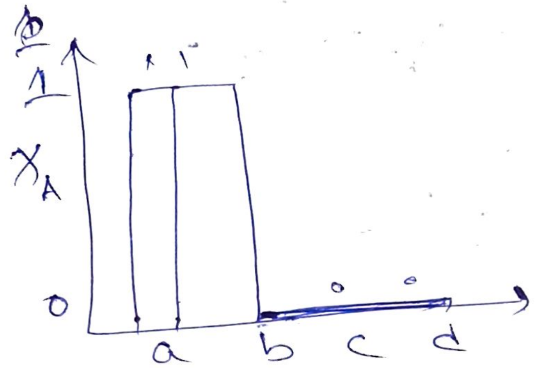
fuzzy set
Universal set → $\bar{X}$

$\chi_A(a) = 1$

$\chi_A(b) = 1$

$\chi_A(c) = 0$

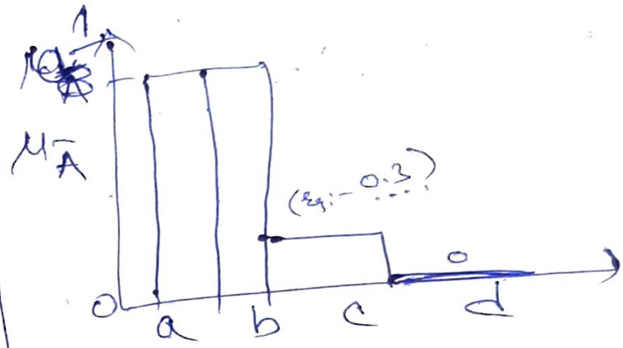
$\chi_A(d) = 0$



Member ship function (what is degree of member lie in that set)

(Coz of uncertainity.)

only (d) = 0 remaining values



Lecturer – 2

Crisp set:- let:- $\mu_A : X \rightarrow \{0, 1\}$

function from (0 → 1)

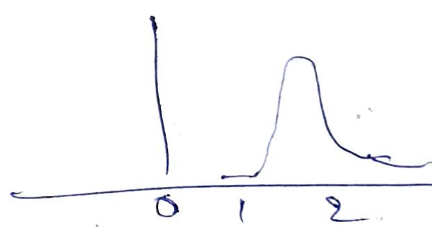
fuzzy:- let $\mu_{\bar{A}} : X \rightarrow [0, 1]$ (has infinite no: of value point include 0 & 1)

member ship function.

fuzzy set:- It is a ordered pair.

$$\{(x, \mu_{\bar{A}}(x)) \mid x \in X\}$$

element, such that, membership func



Eg

if $X = \{1, 2, 3, 4, 5, 6 \ldots\}$

$\bar{A} = \{$'Two or so'$\}$

$\mu_{\bar{A}}(1) = 0.5$, $\mu_{\bar{A}}(2) = 1$; $\mu_{\bar{A}}(3) = 0.5$,

$\mu_{\bar{A}}(4) = 0$

fuzzy set $\bar{A} = \{(1, 0.5), (2, 1)(3, 0.5)(4, 0)(5, 0) \ldots\}$

member ship, element, member ship, element (or) Can be written as

Event → membership

$$\bar{A} = \frac{0.5}{1} + \frac{1}{2} + \frac{0.5}{3} + \frac{0}{4} + \frac{0}{5} \dots$$

membership

note: (not division but it is representation function)
(not Addition but it is Union ")

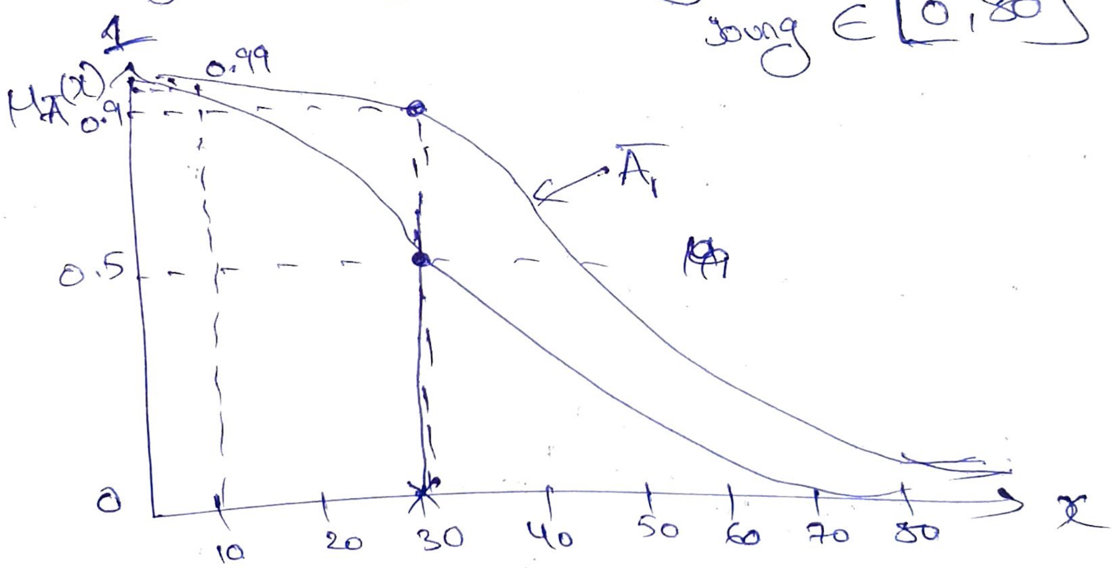
$$\bar{A} = \{x, \mu_{\bar{A}}(x)\} \text{ (or) } \left\{ \sum \mu_{\bar{A}}(x_i)/x_i \right\} \to \text{discrete case}$$

in Continuous

$$\left(\int \mu_{\bar{A}}(x)/x \right) \to \text{representation.}$$

Eg:- if $\bar{A}$ (fuzzy set) = John is young

young $\in [0, 80]$



if 30 is so young means it should be near to 0.9 of range

if 70 is so goes ... 0.1 ...

if i Consider range of young $\in [0, 80]$ Plot $\bar{A}_1$ then

let $\bar{A}_2$ = very young $\in [0, 50]$ another fuzzy set

then plot is a (0.5).

$\therefore \mu_{\bar{A}_1}(30) = 0.9$

$\therefore \mu_{\bar{A}_2}(30) = 0.5$

Eg:- 3

$\bar{A}$ = { Real no:- close to zero }

$$\mu_{\bar{A}}(x) = \frac{1}{1+x^2}$$

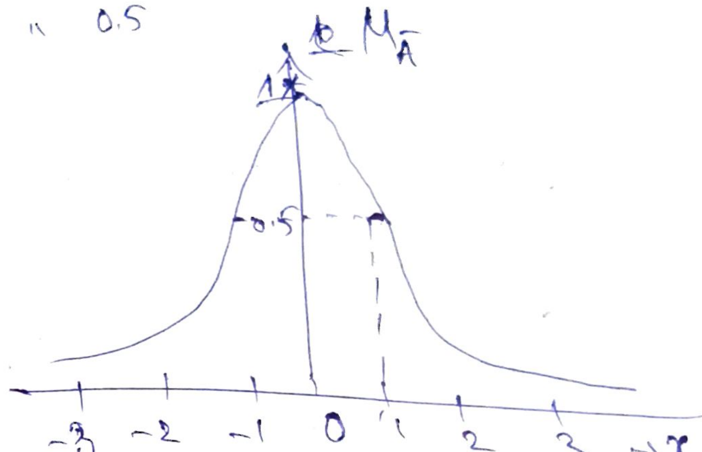
∴ @ $x = 0$ have membership as 1 ✓

$x = 1$ " " " 0.5

$\mu_{\bar{A}}(0) = 1$

for 1 have $\mu_{\bar{A}}(1) = 0.5$

$\mu_{\bar{A}}(2) = 0.2$



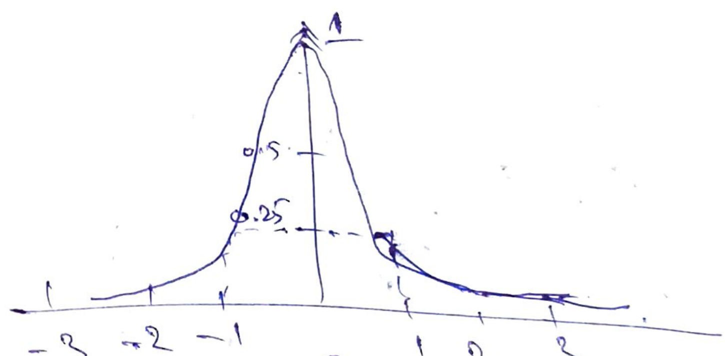
Eg:- (4)

$\bar{A}$ = { Real no very close to zero }

$$\mu_{\bar{A}}(x) = \frac{1}{(1+x^2)^2}$$

$\mu_A(0) = 1$

$\mu_{\bar{A}}(1) = 0.25$



Eg:- 5 Consider a universal set 'X' which define on age domains.

$X = \{5, 15, 25, 35, 45, 55, 65, 75, 85\}$

fuzzy set (1) Infant (2) Young (3) Adult (4) senior

child/baby

Age	Infant	Young	Adult	Senior
5	0	0	0	0
15	0	0.2	0	0
25	0	0.8	0.8	0
35	0	1	0.9	0
45	0	0.6	1	0
55	0	0.5	1	0.3
65	0	0.1	1	0.9
75	0	0	1	1
85	0	0	1	1

Terms

* Suppose of fuzzy set

$\bar{S}(\bar{A}) = \{x \mid \mu_{\bar{A}}(x) > 0\}$ → crisp set

Eg:-

$\bar{S}(\text{young}) = \{15, 25 \ldots 65\}$

$\bar{S}(\text{Adult}) = \{25 \ldots 85\}$

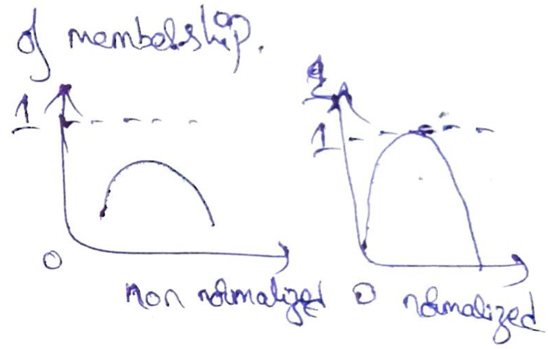
$\bar{S}(\text{senior}) = \{55 \ldots 85\}$

$\bar{S}(\text{infant}) = \phi$.

(2) height terms:- defines as max value of membership.

↳ normalized fuzzy set.

Height is 1.



(3) α - Cut (or) α - level set

$\bar{A}_\alpha = \{x \mid \mu_{\bar{A}}(x) \geq \alpha\}$ α is arbitrary in [0, 1]

$\text{young}_{0.2} = \{15, \ldots 65\}$

$\text{young}_{0.4} = \{25 \ldots 55\}$

(4) strong α - Cut set:- (α')

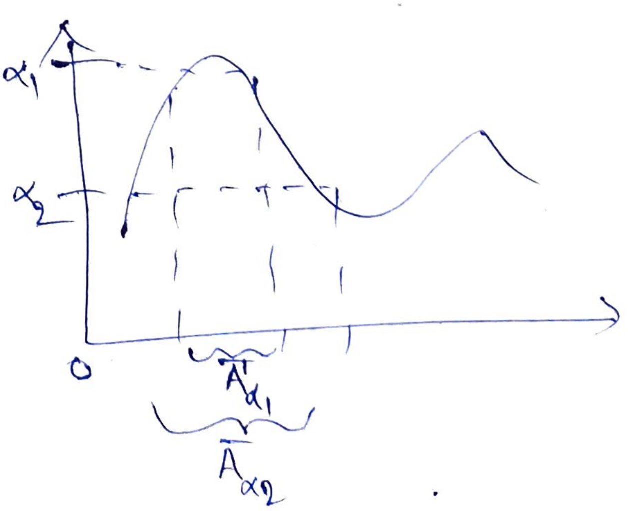
$\bar{A}_{\alpha'} = \{x \mid \mu_{\bar{A}}(x) > \alpha'\}$

$\text{young}_{0.2'} = \{25 \ldots 55\}$

→ if $\alpha_1 > \alpha_2$

then $(\bar{A}_{\alpha_1} \subseteq \bar{A}_{\alpha_2})$

↓ subset



Fuzzy Sets & Systems :-

Concepts in unit :-

→ Fuzziness is an alternative to randomness for describing uncertainty.

→ We develop the new sets as points geometric view of fuzzy sets.

Concepts → This view identifies a fuzzy set with a point in a unit hyper cube (1), a non-fuzzy set with vertex of the cube (2), and fuzzy system as mapping (3) b/w hyper cubes.

Concept → Paradoxes of 2-valued logic and set theory, such as Russell's paradox, corresponds to the midpoint of the fuzzy cube.

Concept → Geometrically answer of fundamental questions of fuzzy theory as -

- How fuzzy is a fuzzy set ?
- How much is one fuzzy set a subset of another ?

→ Answered by $\boxed{\text{fuzzy Entropy theorem}}$ & $\boxed{\text{fuzzy sub sethood theorem}}$.

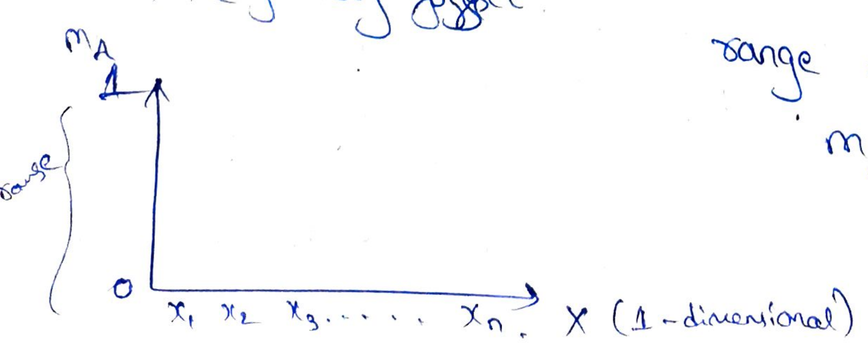
(1) The Geometry of fuzzy sets : Sets as Points :-

→ Fuzzy sets as generalized indicator (or) membership functions, mapping 'm_A' from domain 'X' to range [0, 1] → (a)

→ fuzzy theorist often says membership functions as 2-dimensional graphs, with domain 'X' - as 1-dimensional axis

→ The geometry fuzzy set involves both domain $X = \{x_1, x_2 \dots x_n\}$ and range = [0, 1] of mappings.

$m_A : X \to [0, 1]$. → (b)



→ An odd questions reveals the geometry of fuzzy sets:

(a) what does fuzzy Power set $F(2^X)$, the set of all fuzzy subsets of 'X' look like?

Ans:- it look like a cube. ✓

(b) what does a fuzzy set look like?

Ans:- A Point in a Cube. ✓

→ The sets of all fuzzy sets Equals the unit hypercube $I^n = [0,1]^n$ ↳ (c)

A fuzzy set is any Point in Cube I^n. so (X, I^n) defines the fundamental measurable space of (finite) fuzzy theory.

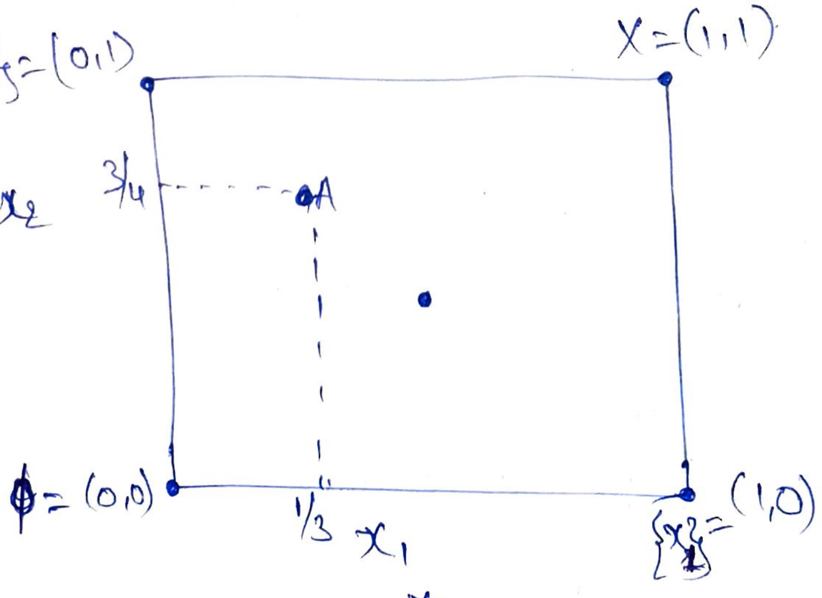
∴ Fuzzy theory Explained on $\boxed{\text{Rubik's Cube}}$

Case-(i):-

Eg:-

↓

$\boxed{\text{Sets as Points}}$



→ the fuzzy subset 'A' is a point in the unit 2-dimensional cube with coordinates $(\frac{1}{3}, \frac{3}{4})$ (or) fit value.

~~1st Co-ordinate value $\frac{1}{3}$ belongs~~ to

1st element x_1 fits to A to ~~belon~~ degree $\frac{1}{3}$ → (d)

" x_2 fits to A to degree $\frac{3}{4}$. → (e)

The Cube Consists of Possible fuzzy Subsets of two Elements $\{x_1, x_2\}$.

→ The four corner represents the Power set 2^X of $\{x_1, x_2\}$ → (f) (11)

→ Vertices of the cube I^n define non fuzzy sets. So ordinary Power set 2^X, the set of all 2^n non fuzzy subsets of 'X', equals the boolean n-cube B^n: $2^X = B^n$. → (g)

fuzzy sets filled in lattice B^n to produce solide cube I^n: $F(2^X) = I^n$. → (h)

Consider the set of 2-elements $X = \{x_1, x_2\}$ → (i)

The non-fuzzy Power set 2^X contain 4-sets: $2^X = \{\phi, X, \{x_1\}, \{x_2\}\}$ → (j)

these four composed respectively to four bit vector

$(0,0)\ (1,1)\ (1,0)\ \&\ (0,1)$ → (k)

→ The 1's and 0's indicate presence (or) absence of i^{th} element x_i in subset.

as uniquely define each subset A as one of 2-valued membership function

$m_A : X \to \{0,1\}$. → (l)

Case (ii):-

In this example

x_1 belongs to subset 'A' a little bit to degree $\frac{1}{3}$ → (m)

x_2 have more membership than not at $\frac{3}{4}$ → (n)

Analogous to bit vector representation of finite (countable) sets, we say the fit vector $\left(\frac{1}{3}, \frac{3}{4}\right)$ represents 'A'. → (o)

The element $m_A(x_i) = i^{th}$ fit (or) fuzzy unit value

→ The sets as points geometrically represents as fuzzy subset 'A' as point in I^2, the unit square → (p)

→ The midpoint of cube I^n is maximally fuzzy. All its membership value equal $\frac{1}{2}$.

→ →

→ The midpoint is unique in 2-Reflects

(i) 1st midpoint is only set A that not only equals to its own opposite A^c but equals overlap and underlap.

$$A = A \cap A^c = A \cup A^c = A \rightarrow (q)$$

(ii) 2nd midpoint is only point in cube I^n equidistant to each of the 2^n vertices of cube.

→ Set of Elements with Operators :-

Minimum ; Maximum & order reversal

$$m_{A \cap B} = \min(m_A, m_B) \rightarrow (r)$$

$$m_{A \cup B} = \max(m_A, m_B) \rightarrow (s)$$

$$m_{A^c} = 1 - m_A. \rightarrow (t)$$

Eg:-

$$A = (1, 0.8, 0.4, 0.5)$$

$$B = (0.9, 0.4, 0, 0.7)$$

$$A \cap B = \min(0.9, 0.4, 0, 0.5)$$

$$A \cup B = \max(1, 0.8, 0.4, 0.7).$$

$$A^c = (0, 0.2, 0.6, 0.5) \quad (\text{i.e } A^c = 1 - A)$$

$$A \cap A^c = \min(0, 0.2, 0.4, 0.5)$$

$$A \cup A^c = \max(1, 0.8, 0.6, 0.5)$$

→ Overlap set vector $(A \cap A^c) \rightarrow (u)$

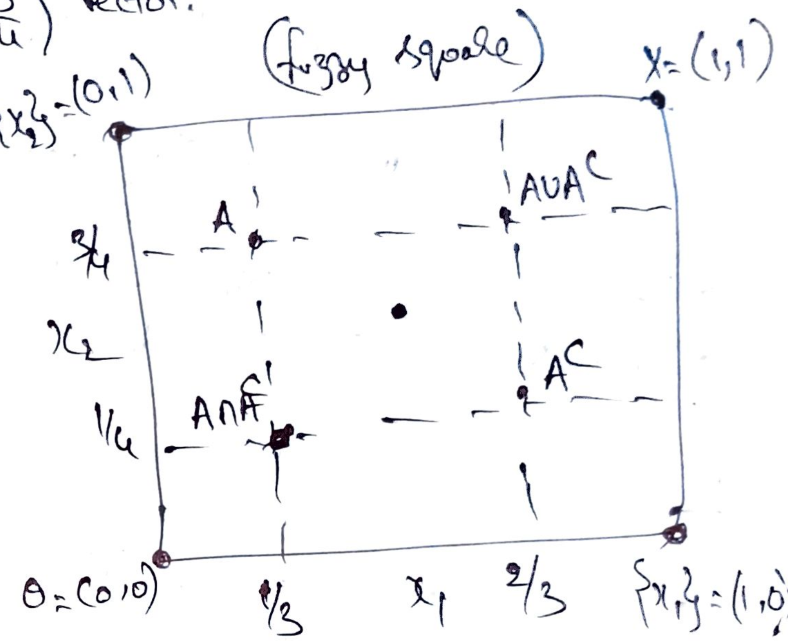
underlap fit vector $(A \cup A^c). \rightarrow (v)$

Proposition:- A is properly fuzzy iff $A \cap A^c \neq \phi$ (null set) → (w)

iff $A \cup A^c \neq X$ (universal set) → (x)

Eg:-2 Consider again 2-dimensional fuzzy set

Case (iii) A defined as $\left(\frac{1}{3}, \frac{3}{4}\right)$ vector.



→ 'A' closer to mid point of fuzzy cube

→ 'A' Approaches the midpoint of all 4-Point $(A, A^c, A \cap A^c, A \cup A^c)$ Contact to midpoint.

→ The less fuzzy 'A' is closer to 'A' is to ~~vertices~~ vertex.

→ As 'A' approaches the vertex 4-Points spread out to the 4-vertices and we cover bivalent power set 2^X.

→ In an n-dimensional fuzzy cube, the 2^n fuzzy set with element a_i (or) $1-a_i$ → (4)

→ ~~Eg~~ Similarly Contact midpoints & expand to 2^n vertices as 'A' approaches total fuzziness (or) bivalent total.

$$A = \left(\frac{1}{3} \; \frac{3}{4}\right)$$

$$A^c = \left(\frac{2}{3} \; \frac{1}{4}\right)$$

$$A \cup A^c = \left(\frac{2}{3} \; \frac{3}{4}\right)$$

$$A \cap A^c = \left(\frac{1}{3} \; \frac{1}{4}\right)$$

The 4-Point in unit square hang together and move together in a very natural way.

→ The 4-fuzzy sets involved in fuzziness of set 'A' — contact to midpoint as A becomes maximally fuzzy and expand out to boolean corner of cube as 'A' becomes minimally fuzzy.

→ The Contraction & Expansion occurs in n-dimensions for 2^n fuzzy sets define all combination of

$m_A(x_1)$ and $m_{A^c}(x_1)$, ... $m_A(x_n)$ and $m_{A^c}(x_n)$ → (3)

→ At mid point nothing is distinguishable
→ At vertices Everything is "

Paradox at the midpoint:-

Eg:- let 'S' be Proposition that statement Ram does work himself and not-S as that he does not.

∴ S implies not-S & not-S implies S

$S \Rightarrow (\text{not-}S)$ & $(\text{not-}S) \Rightarrow S$

$\therefore S = \text{not-}S$

$t(S)$ → as truth interval

solving for $t(S) = t(\text{not-}S)$ → (i)

$= 1 - t(S)$ → (ii)

solving for $t(S)$ gives mid point of the truth interval (1-dimensional) cube $[0,1]$ is $t(S) = \frac{1}{2}$. The midpoint is equidistant to vertices 0 & 1.

Counting with fuzzy sets:-

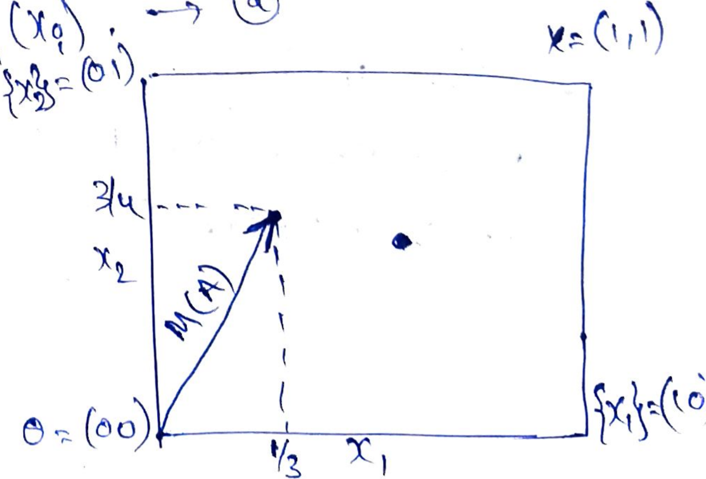
How big is a fuzzy set ?

→ the size (or) Cardinality of A is $M(A)$, equals the sum of fit values of A :

$$M(A) = \sum_{i=1}^{n} m_A(x_i) \rightarrow (a)$$

The Count of $A = \left(\frac{1}{3}, \frac{3}{4}\right)$ equal

$$M(A) = \frac{1}{3} + \frac{3}{4} = \frac{13}{12}$$



The measure M generalizes the classical Counting measure of Combinatorics and measure theory.

so (X, I^n, M) defines fundamental measure space of fuzzy theory. → (b)

→ The measure M has natural geometric interpretation in sets as points frame walks.

M(A) Equals magnitude of vector drawn from origin to the fuzzy set 'A'.

Consider l^p distance b/w fuzzy set A & B in l^n: → (c)

$$l^p(A,B) = \sqrt[p]{\sum_{i=1}^{n} |m_A(x_i) - m_B(x_i)|^p} \rightarrow (d)$$

where $1 \leq P \leq \infty$

→ The l^2 distance is physical Euclidean distance actually illustrated in fig

→ The simplest distance is l^1 (or) fuzzy haming distance, the sum of absolute fit differences.

∴ we shall use fuzzy hamingin distance thorughout, though all results admit a general l^p formulation.

∴ using fuzzy hamining distance we write Count M as desired l^1 form:

$$M(A) = \sum_{i}^{n} m_A(x_i) \rightarrow (e)$$

$$= \sum_{i} |m_A(x_i) - 0| \rightarrow (f)$$

$$= \sum_{i} |m_A(x_i) - m_\phi(x_i)| \rightarrow (g)$$

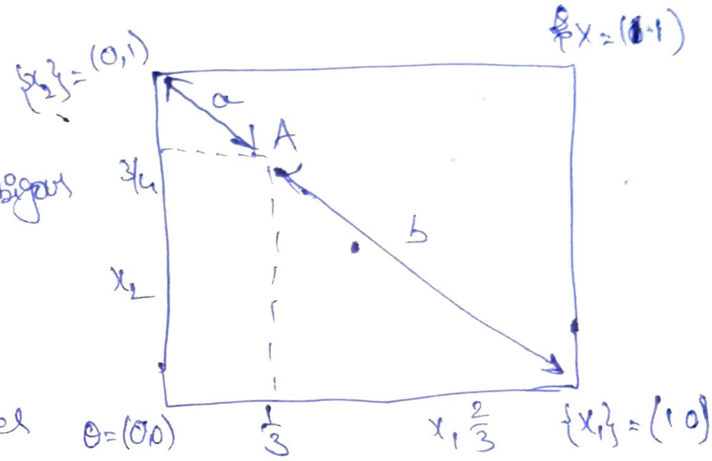
$$M(A) = l^1(A, \phi). \rightarrow (h)$$

Q) The fuzzy Entrop Theorem :- "E(A)"

we to define how fuzzy is a fuzzy set.

Entropy measures uncertainity of system (or) message.

The fuzzy Entrop of A, E(A) varies from 0 to 1 on unit hyper cube I^n.



→ Cube vertices have zero Entropy since non fuzzy sets are unambigous

→ The cube midpoint uniquely has unity (or) max Entropy.

→ fuzzy Entropy smoothly increases as a set point moves from any vertex to the midpoint

→ Geometric Consideration lead to ratio form. for fuzzy Entropy.

The closer the fuzzy set 'A' is nearest vertex A_{near} → (a)

The farther the fuzzy set 'A' is farthest vertex A_{far}. → (b)

let a denote the distance $l^1(A, A_{near})$ to nearest vertex

let b denote the " $l^1(A, A_{far})$ to farthest vertex.

$$\therefore \text{fuzzy Entropy} \quad E(A) = \frac{a}{b} = \frac{l^1(A, A_{near})}{l^1(A, A_{far})} \rightarrow (c)$$

$\therefore$ from fig:- $A = \left(\frac{1}{3}, \frac{3}{4}\right)$ and $A_{far}(1, 0)$ so

$A_{near} = (0, 1)$

$a = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$ and $b = \frac{2}{3} + \frac{3}{4} = \frac{17}{12}$

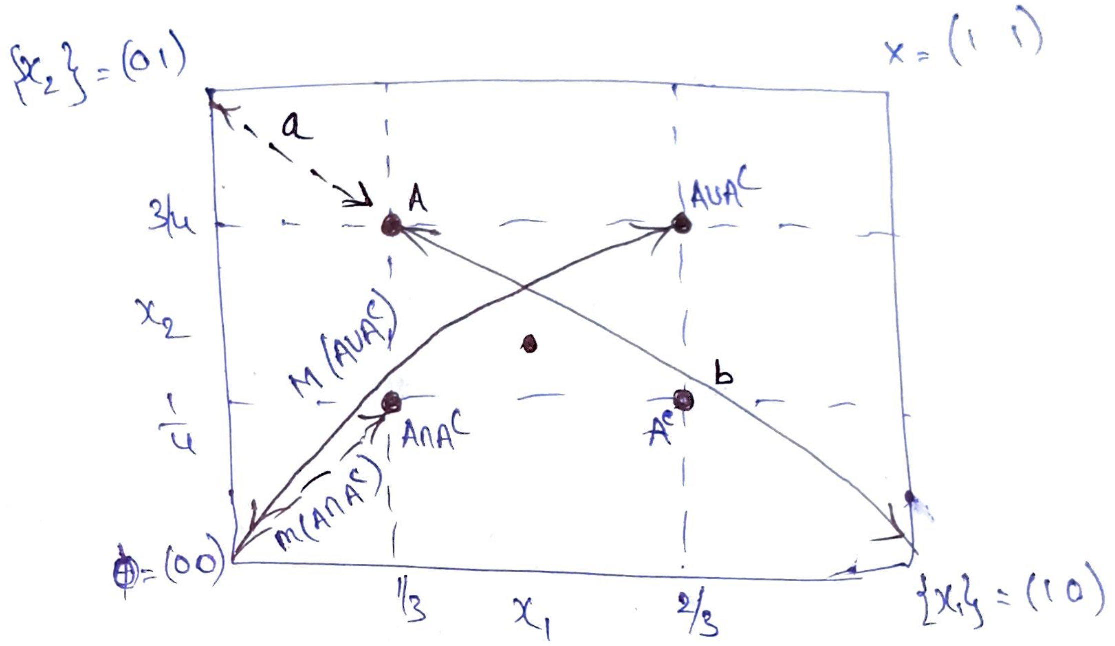
$$\text{so } E(A) = \frac{a}{b} = \frac{\frac{7}{12}}{\frac{17}{12}} = \frac{7}{17}.$$

· overlap & underlap characterize with set fuzziness:-

· ∃ 4-Points A, A^c, $\underset{\text{min}}{A \cup A^c}$, $\underset{\text{Max}}{A \cap A^c}$.

if corner ~~point~~ is near to point A then it is $E(A) = 0$ → (d)

if A is in middle and near/far to corners then $E(A) = 1$ → (e)



$$\therefore \text{Entropy } E(A) = \frac{M(A \cap A^c)}{M(A \cup A^c)} \rightarrow (f)$$

<u>Proof</u>:- The fuzzy Entropy theorem explains when sets (or Propositions) obey the law of non-Contradiction and Excluded ~~so~~ middle, overlap is Empty and underlap is Exhaustive.

so $M(A \cap A^c) = 0$ & $M(A \cup A^c) = n$ and thus $E(A) = 0$ → (g)

→ This result also shows triangular norms (or) T-norms which generalize Conjuction (or) intersection and dual triangular Co-norms (c) which generalize disjunction (or) union, do not have 1st Principle status of min & max.

→ for Triangular-norm inequalities

$$T(x,y) \leq \min(x,y) \leq \max(x,y) \leq C(x,y) \rightarrow (h)$$

∴ replacing min with an 'T' in numerator term of $M(A \cap A^c)$ can only make numerator smaller

→ replacing max with 'C' in term $M(A \cup A^c)$ can only make denominator smaller.

So T & C are not identically min (or) max makes ratio smaller.

→ The Entropy Theorem does not hold and resulting pseudo entropy measure does not equal unity at midpoint though it continues to be maximized there.

∴ we see this with product T-norm

$T(x,y) = xy$ and its Demorgan dual C-norm

$C(x,y) = 1-T(1-x, 1-y) = x+y-xy$ (or) with the bounded sum T-norm

$T(x,y) = \max(0, x+y-1)$ and Demorgan dual $C(x,y) = \min(1, x+y)$

→ The Entropy Theorem similarly fails in general if we replace the negation (or) complement

$N(x) = 1-x$ with parameterized vector

$N_a(x) = \frac{(1-x)}{(1+ax)}$ for non zero $a > -1$

→ For all probability distributions all sets A in I^n with $M(A) = 1$ form an $n-1$ dimensional simplex S^n. In unit square the probability simplex equals the negatively sloped diagonal line.

In the unit 3-cube it equals a solid triangle.

In the unit 4-cube it 〃 a tetrahedron and so...

→ If no probabilistic fit value P_i satisfies $P_i > \frac{1}{2}$ then fuzzy entropy theorem implies that the distribution P has fuzzy entropy

$E(P) = \frac{1}{n-1}$ else $E(P) < \frac{1}{(n-1)}$ so the probability simplex S^n is entropically degenerate for larger dimensions n.

→ The Fuzzy Entropy Theorem implies that analogous to $\log\left(\frac{1}{p}\right)$ a unit fuzzy information equals $\frac{f}{(1-f)}$ (or) $\frac{(1-f)}{f}$ depending on whether the fit value 'f' obeys

$$f \leq \frac{1}{2} \quad (\text{or}) \quad f \geq \frac{1}{2}.$$

→ The event 'x' can be ambiguous (or) clear. It is ambiguous if f is approximatly equal to $\frac{1}{2}$ and clear if f equals approximatly 0 (or) 1

→ If an ambiguous event occurs is observed is disambiguated etc., then it is maximally informative

$$E(F) = E\left(\frac{1}{2}\right) = 1$$

If an clear event occurs is observed etc., it is minimally informately

$$E(F) = E(0) = E(1) = 0$$

→ This agrees with information interpretation of Probablistic entropy measure $\log\frac{1}{p}$ where occurence of a sure event $(P=1)$ is minimal informative (zero entropy) and the occurence of a impossible event $(P=0)$ is maximally informative (infinite entropy)

ie: $P=1$ (zero entropy) (minimal)
$P=0$ (infinite ") (maximally).

(3) The subset hood theorem :-

Sets contain subsets.

→ 'A' is subset of B $(A \subset B)$ iff every element in A is a element of 'B'.

ie Power set 2^B contains all of B's subsets.

∴ similly A is subset of B $(A \subset B)$ iff $(A \in B)$ power set

$A \subset B$ if and only if $A \in 2^B$. → (a)

→ The subset relation corresponds to implication relation in logic.

i.e in classical logic truth maps the set of statements $\{s\}$ to two truth values :

$$t : \{s\} \to \{0,1\} . \to (b)$$

Case (i) :-

Let consider truth table defination of implication for bivalent propositions P and Q.

P	Q	P → Q
0	0	1
0	1	1
1	0	0
1	1	1

ie truth implis false hood (as P- is true & Q is false).

while implication is false if and only if the statement - when antecedent P is true and the consequent Q is false. "truth implies falsehood".

The same holds for subsets. Representing sets as bivalent functions (or) indicator functions

$$m_A : X \to \{0,1\} \to (c)$$

'A' is a subset of 'B' iff there is no element 'x' that ∈ 'A' but not to 'B' (or)

$$m_A(x) = 1 \text{ but } m_B(x) = 0 \to (d)$$

by rewriting membership function defination.

$A \subset B$ iff $m_A(x) \leq m_B(x) \ \forall x$.

Ex:- if $A = (0.3 \quad 0 \quad 0.7)$ and $B = (0.4 \quad 0.7 \quad 0.9)$ then

A is fuzzy subset of B. but B is not a fuzzy subset of A

i.e $0.3 < 0.4$, $0 < 0.7$, $0.7 < 0.9$ so $m_A(x) < m_B(x)$

i.e Either fuzzy set A is, & is not a fuzzy subset of B.

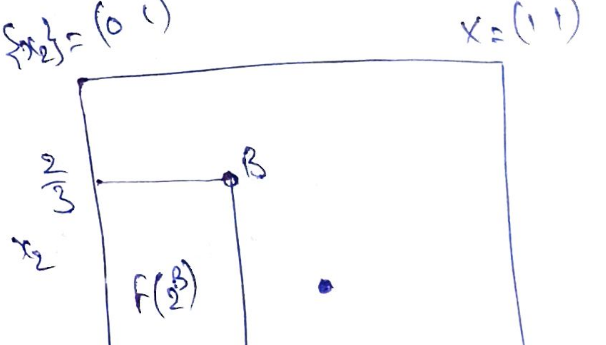
so Zadeh's relation of fuzzy subsethood is not fuzzy.

Case-(ii) :-

from Concept of "sets as points" the geometry questions comes as?

(i) what do all fuzzy sets of 'B' look like?

(ii) what does the fuzzy power set $B - F(2^B)$, the set of all fuzzy subset of 'B' look like?



"Fuzzy Power set $F(2^B)$ as hyperrectangle in fuzzy Cube."

∴ The dominant membership function relationship implies that $F(2^B)$ defines hyper rectangle in fuzzy Cube. with side lengths Equal to fit values $m_A(x_i)$.

when $B = \left(\frac{1}{3}, \frac{2}{3}\right)$, i.e $F(2^B)$ has finite Cardinality if B is not Empty.

→ for finite dimension set we measure size of $F(2^B)$ as the Lebesgue measure (or) volume $V(B)$, the product of fit values.

$$V(B) = \prod_{i=1}^{n} m_B(x_i).$$

Product

from fig:- $F(2^B)$ is not a fuzzy set.

Either a cube point A is (&) is not a hyper rectangle $F(2^B)$.

→ Different point outside the hyper rectangle $f(2^B)$ resemble of B to different degree..

(Thus bivalent defination of subsethood ignores)

in natural generalization defines fuzzy ~~sets~~ subsets on $F(2^B)$.

i.e $F(2^B)$: some sets A belong to $F(2^B)$ to different degree.

then membership function $m_{F(2^B)}(A)$ can equal any nor in $[0,1]$.

defines degree of subsethood.

Let $S(A,B)$ denote degree of A is subset B:

$$S(A,B) = \text{Degree } (A \subset B)$$
$$= m_{F(2^B)}(A).$$

To measure $S(A,B)$

we 1st present with algebraic derivation of subsethood measure ~~(B)~~, $S(A,B)$

and then present new more fundamental geometric derivation.

→ Algebraic derivation is known as fit "violation strategy".

Consider again dominated membership func relationship:

$A \subset B$ iff $m_A(x)$ ∀ ~~x~~ in X.

Let suppose element x_v violates the dominated membership function relationship:

$m_A(x_v) > m_B(x_v)$ then $A \not\subset B$, at least not totally. (not subset).

Suppose

Suppose dominated membership inequality hold for all other element 'x'.

only 'x_v' violates the relationship.

Eg:- let 'X' may consist of one hundred values : $X = \{x_1, \ldots, x_{100}\}$

→ The violation may occur say with first element $x_1 = x_v$. → (a)

Then intuitively 'A' is largely a subset of 'B'. suppose that X contains a thousand elements (or) a trillion elements, the only first element violates.

i.e it seems A is overwhelmingly a subset of B;

$$S(A, B) = 0.9999999999. \rightarrow (b)$$

i.e Example suggest we should count for violations in magnitude and frequency.

→ The greater the violation in magnitude $m_A(x_v) - m_B(x_v)$ and greater the no:- of violations relative to size $M(A)$ of A, the less A is a subset of B. (or) Equivalently more is superset of B.

$$\text{Supersethood}(A, B) = 1 - S(A, B). \rightarrow (c)$$

→ The count of M(A) provide a simple and approximate normalization.

when M(A) examine boundaries cases in geometric approach supersethood.

Assume $M(A) > 0$; ∵ $M(A) = 0$ iff A is Empty

∴ normalization gives minimal measure of non-subsethood

$$\text{Supersethood}(A, B) = \frac{\sum_x \max(0, m_A(x) - m_B(x))}{M(A)}. \rightarrow (d)$$

Then subsethood is a negation of this ratio.

$$S(A, B) = 1 - \frac{\sum_x \max(0, m_A(x) - m_B(x))}{M(A)} \rightarrow (e)$$

Eg:- let fit vector $A = (0.2, 0, 0.4, 0.5)$
$B = (0.7, 0.6, 0.3, 0.7)$ neither set is a proper subset of other.

A is almost subset of B $(A \subset B)$ but not quite

$\frac{0.12}{1.2}$

$\because \; m_A(x_3) - m_B(x_3) = 0.4 - 0.3$
$= 0.1$

where $0.1 > 0$

hence $S(A,B) = 1 - \frac{0.1}{1.1} = \frac{10}{11}$ (i.e:- $1 - \frac{\sum_x (0, m_A(x) - m_B(x))}{M(A)}$ where $M(A)$ = Total of elements)

Similarly $S(B,A) = 1 - \frac{1.3}{2.3} = \frac{10}{23}$.

Eg:- from Probability subsethood applies to nonfuzzy set.

let consider

$C = \{x_1, x_2, x_3, x_5, x_7, x_9, x_{10}, x_{12}, x_{14}\}$
$D = \{x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{12}, x_{13}, x_{14}\}$

with corresponding bit vector

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}
$C =$	1	1	1	0	1	0	1	0	1	1	0	1	0	1
$D =$	0	1	1	1	1	1	1	1	1	1	0	1	1	1

$S(C,D) = 1 - \frac{1}{9} = \frac{8}{9}$ (i.e $\underset{C}{1} \to \underset{D}{0}$ is only 1).

where $m(C)$ has 9 (elements)

Similarly:- $S(D,C) = 1 - \frac{4}{12} = \frac{2}{3}$ (here $\underset{D}{1} \to \underset{C}{0}$ is 4 elements)

where $D \to C$ in this sense $1 \to 0$ terms are 4 (violation)

and total of $M(D) = 12$ (elements).

from Table:-

P	Q	P→Q
0	0	1
0	1	1
1	0	0
1	1	1

(25)

Case (iii) :- Geometric derivation of subsethood operator $S(A,B)$.

(i) Consider sets as points geometry of subsethood

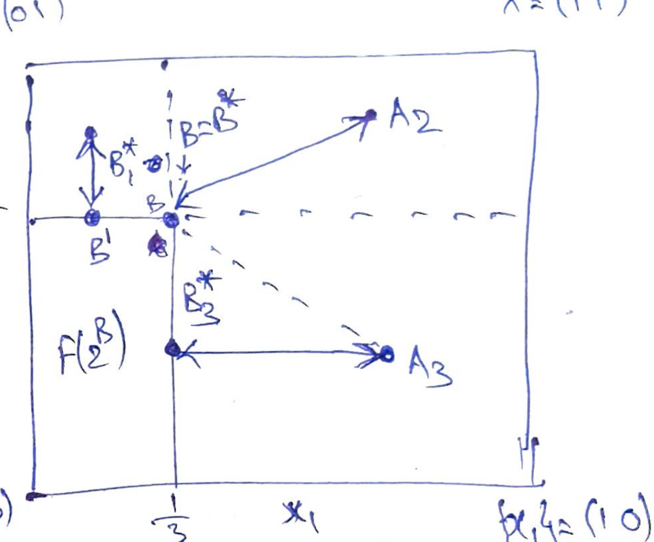
Let 'A' is either lies in hyperrectangle $F(2^B)$ (or) not in it.

i.e it should approach unity as A approaches fuzzier power set $F(2^B)$

$S(A,B)$ should decrease &

$1-S(A,B)$ should increase as 'A' moves away from $F(2^B)$.

Supersethood measure



Explanation :- How close is A to $F(2^B)$?

Let $d(A, F(2^B))$ denote l^p distance

→ $d(A, B')$ denote distance b/w A & point B' in hyper rectangle. and $B' \subset B$.

→ Distance $d(A, F(2^B))$ = Smallest such distance

∴ $d(A,B^*) : d(A, F(2^B)) = \inf\{d(A,B') : B' \in F(2^B)\}$

$= d(A, B^*)$.

inf → infimum distance

i.e Easy to locate closest set B^* in hyper cube geometry.

if $A \subseteq B$ and 'A' is hyperrectangle of $F(2^B)$ - then

$$A = B^*$$

Case (iv) :- Consider to slice the cube I^n into 2^n hyperrectangles by extending the sides of $F(2^B)$ to hyperplanes.

- where interior corresponds 2^n cases where

$m_A(x_i) < m_B(x_i)$ (or) $m_A(x_i) > m_B(x_i)$ for fixed B and arbitrary 'A'.

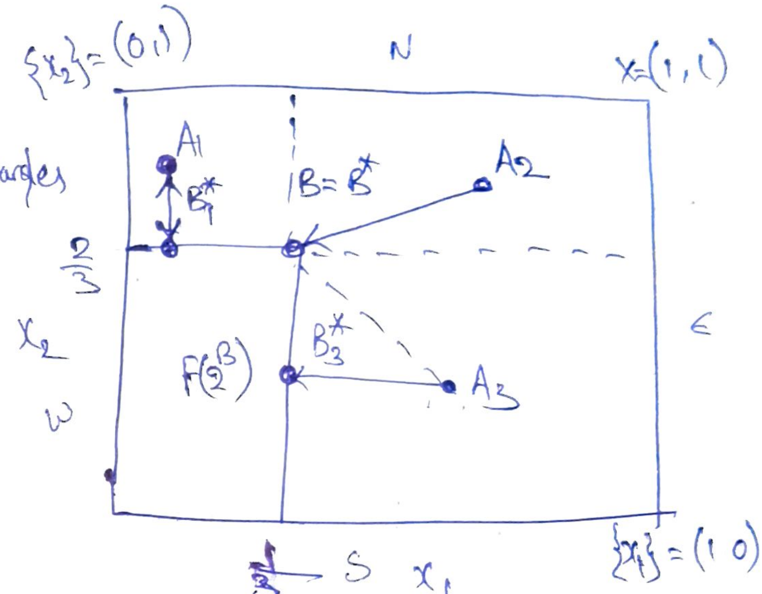
→ The edges corresponds to loci of points where $m_A(x_i) = m_B(x_i)$.

→ The 2^n hyperrectangles classify as mixed (or) pure membership dominations

→ in pure case either $m_A < m_B$ (or) $m_A > m_B$ holds coordinates x_3'

in interior ∀ interior 'A'

fig:- hypercube I^n into 2^n hyper rectangles by linearly extending edges of $F(2^B)$.



we define points B_1^* and B_3^* to point A_1 and A_3

→ The figure also illustrates how fuzzy power set $F(2^B)$ of $B = \left(\frac{1}{3}\ \frac{2}{3}\right)$. linearly extended to partition the unit square into 2^2 rectangles.

→ where the northwest and southeast quadrants define mixed membership dominant rectangles

→ The southwest and north east quadrants define pure rectangles.

The orthogonality condition invokes in l^p version of Pythagorean theorem for our l^1 purpose

$$d(A,B) = d(A,B^*) + d(B,B^*).$$

for l^2 version requires squaring these distances

for general l^p case :-

$$\|A-B\|^p = \|A-B^*\|^p + \|B^*-B\|^p \quad (\text{or})$$

$$\sum_{i=1}^{n} |a_i - b_i|^p = \sum_{i=1}^{n} |a_i - b_i^*|^p + \sum_{i=1}^{n} |b_i^* - b_i|^p.$$

where equality holds $p \geq 1$.

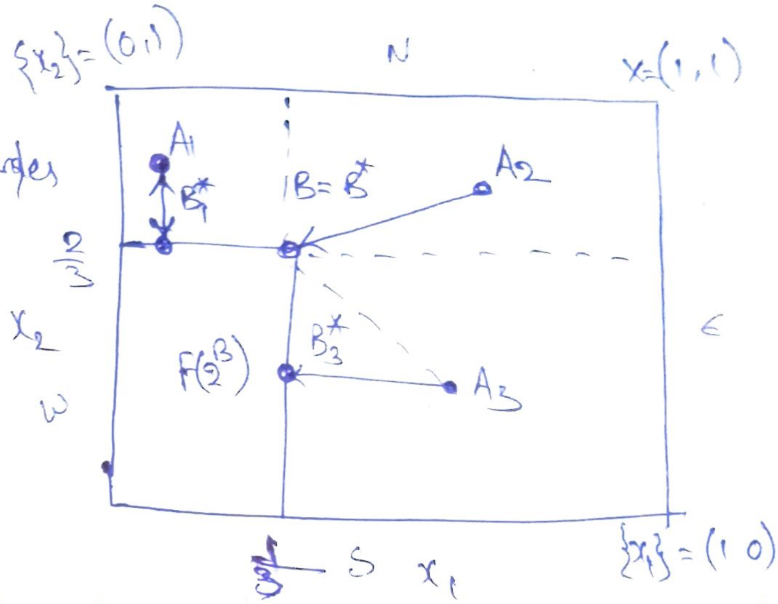
→ The 2^n hyperrectangles classify as mixed (or) pure membership domination

→ In pure case either $m_A < m_B$ (or) $m_A > m_B$ holds coordinates x_3

in exterior ∀ interior 'A'

Fig:- hypercube I^n into 2^n hyperrectangles by linearly extending edges of $F(2^B)$.

we define points B_1^* and B_3^* to point A_1 and A_3



→ The figure also illustrates how fuzzy power set $F(2^B)$ of $B = (\frac{1}{3}\ \frac{2}{3})$. linearly extended to partition the unit square into 2^2 rectangles.

→ where the northwest and ~~southwest~~ southeast quadrants define mixed membership dominant rectangles

→ The southwest and northeast quadrants define pure rectangles.

The orthogonality condition invokes in l^p version of Pythagorean theorem for our l^1 purpose

$$d(A,B) = d(A,B^*) + d(B,B^*).$$

for l^2 version requires squaring these distances

for general l^p case:-

$$\|A-B\|^p = \|A-B^*\|^p + \|B^*-B\|^p \quad \text{(or)}$$

$$\sum_{i=1}^{n} |a_i - b_i|^p = \sum_{i=1}^{n} |a_i - b_i^*|^p + \sum_{i=1}^{n} |b_i^* - b_i|^p.$$

where equality holds $p \geq 1$.

The supersethood measure:

$$S(A,B) = 1 - \frac{d(A,B^*)}{n}.$$

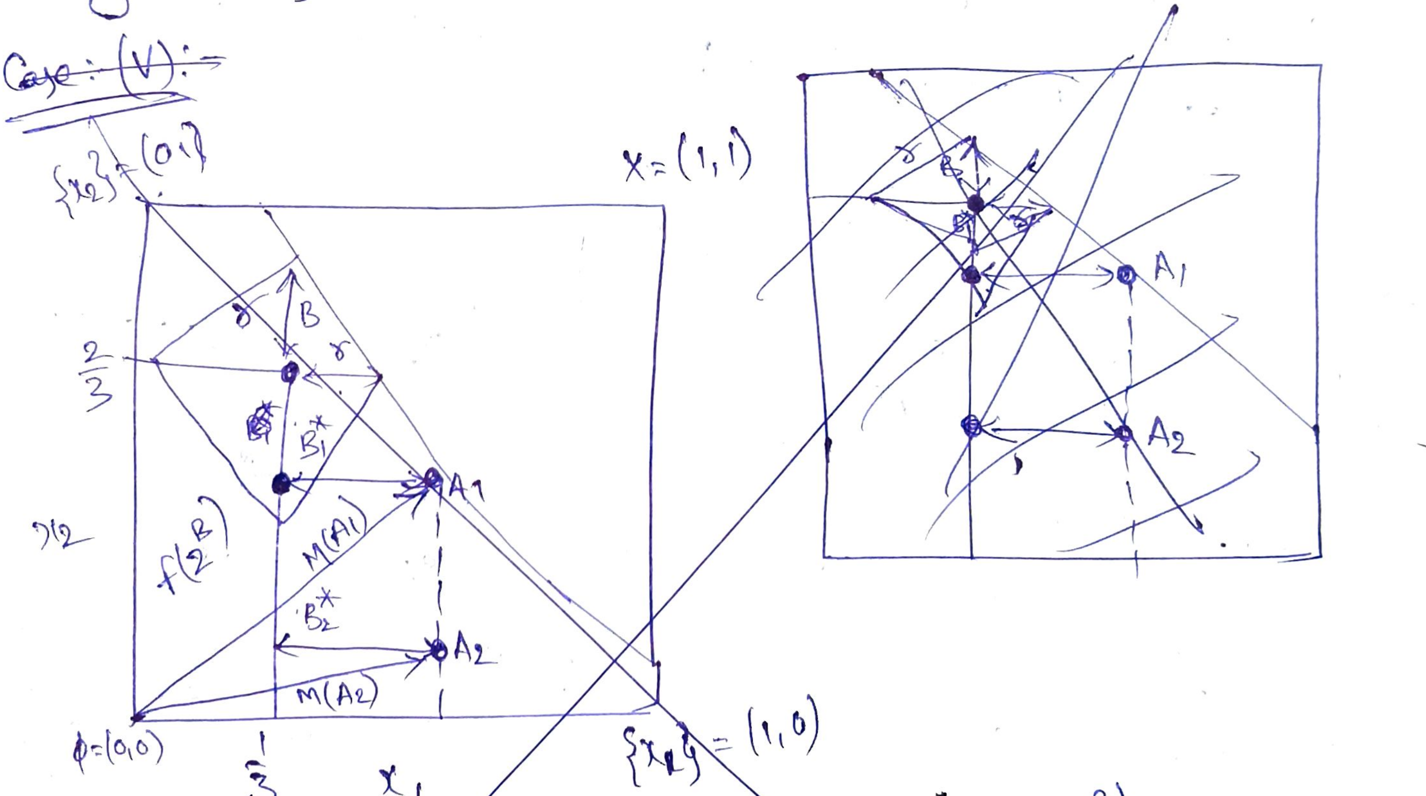
The candidate subsethood fails measure in boundary case when 'B' is the empty set.

then $d(A,B^*) = d(A,B) = M(A)$

$\therefore$ measure $S(A,\phi) = 1 - \frac{M(A)}{n} > 0.$

Equality holds exactly when $A = X$. But empty set has no subsets. Only normalization factor $M(A)$ satisfies this boundary condition.

Case: (V):



A_1 & A_2 are points equidistant to nearest $F(2^B)$ point

$F(2^B)$: $d(A_1, B_1^*) = d(A_2, B_2^*)$

(4) The Entropy-Subsethood Theorem :-

from sets as points unit cube geometry. both theorems involve in cardinalities.

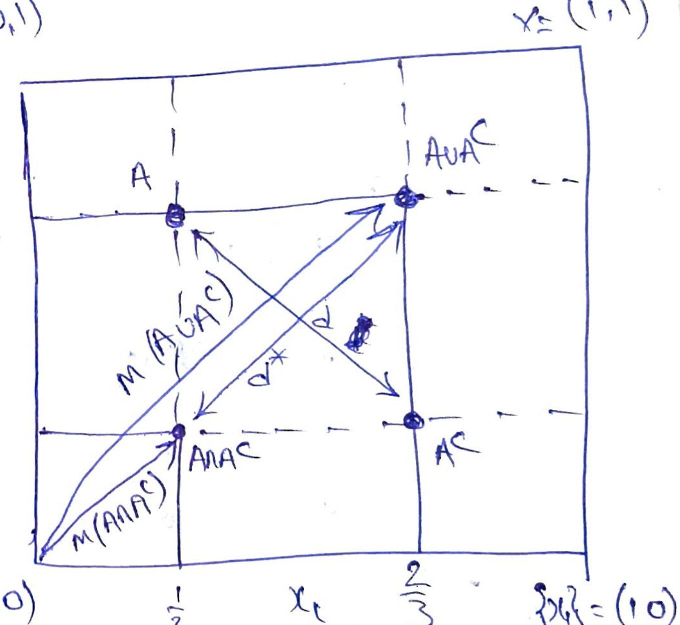
→ This shows connection involves overlap ($A \cap A^c$) and underlap ($A \cup A^c$)

The theorem eliminates fuzzy entropy in favor of subsethood.

$$E(A) = S(A \cup A^c, A \cap A^c)$$

Proof:- If we replace B and A in subsethood theorem respectively overlap ($A \cap A^c$) and underlap ($A \cup A^c$)

$\therefore S(A \cap A^c, A \cup A^c) = 1$ the intersection of two sets equal to overlap.



→ d^* is shortest distance from underlap $A \cup A^c$ to hyperrectangle define fuzzy power set of overlap $A \cap A^c$.

i.e $d(A \cup A^c, A \cap A^c) = d(A, A^c)$ and equals a difference of vector magnitudes :

$$d^* = M(A \cup A^c) - M(A \cap A^c)$$

→ The entropy subsethood theorem defines no probability measures measures fuzziness.

probability measure P such that $P = E$. ($P \neq 0$ every where blog $P(X) = 1$)

For Eg:- if some probability measure $P(A) = E(A) > 0$.

$\therefore$ there is some 'A' in such that $P(A) = E(A) > 0$. overlap & underlap degenerates $A \cap A^c = \phi$ and $A \cup A^c = X$

But in probability space overlap & underlap theorem implies $\boxed{0 < P(A) = E(A) = S(A \cup A^c, A \cap A^c) = S(X, \phi)}$.

$\therefore$ The Entropy subsethood theorem implies X can be subset to non zero degree of empty set only if X itself is empty.

$X = A = \phi$.

Then sole X is impossible $P(X) = P(\phi) = 0$; (or) impossible event is sole $P(\phi) = 1$ //.