

find the weather the system is stable or unstable?

$$y(n) = 2^n u(-n)$$

$$y(n) = 2^n u(-n)$$

$$\sum_{n=-\infty}^{\infty} 2^n u(-n)$$

$$\sum_{n=-\infty}^{\infty} h(n) < \infty$$

$$= \sum_{n=-\infty}^0 2^n$$

$$= 1 + 2 + 4 + \dots = 2^0 + 2^1 + 2^2 + \dots \rightarrow$$

$$= 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots \rightarrow$$

$$= \frac{1}{1-2} = \frac{1}{1-\frac{1}{2}} = \frac{2}{2-1} = 2, < \infty$$

∴ This system is stable

$$h(n) = 5 \sin \frac{n\pi}{2}$$

$$= \sum_{n=-\infty}^{\infty} 5 \sin \frac{n\pi}{2} \quad \text{or} \quad \sum_{n=-\infty}^{\infty} h(n) < \infty$$

∴ This system is unstable

$$h(n) = e^{2n} u(n-1)$$

$$= \sum_{n=-\infty}^{\infty} h(n) < \infty$$

$$= \sum_{n=-\infty}^{\infty} e^{2n} u(n-1) < \infty$$

$$= e^2 + e^4 + e^6 + \dots < \infty$$

∴ This system is unstable

formula for energy of a signal, power of a signal
energy of a signal is defined as

$$E_B = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T [x(t)]^2 dt$$

power of a signal is defined as

$$P = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T [x(t)]^2 dt$$

for discrete channels energy and power of a signal is defined as.

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$P = \lim_{N \rightarrow \infty} \frac{1}{2NH} \sum_{n=-N}^N |x(n)|^2$$

find which of the following signals are energy signals

power signal.

$$1) x(t) = e^{-at} u(t)$$

$$E = \lim_{T \rightarrow \infty} \int_{-T}^T |x(t)|^2 dt$$

$$= \lim_{T \rightarrow \infty} \int_0^T (e^{-3t})^2 dt$$

$$= \lim_{T \rightarrow \infty} \int_0^T e^{-6t} dt$$

$$\left[\frac{e^{-6t}}{-6} \right]_0^T$$

$$= \frac{1}{6} \left[e^{-6T} - e^0 \right]$$

$$= \frac{1}{6} \left[\frac{e^{-6\infty}}{-6} + \frac{1}{6} \right] = 0 + \frac{1}{6} = \frac{1}{6}$$

Power of a signal =

$$P = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T [x(t)]^2 dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_0^T e^{-6t} dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \left[\frac{e^{-6t}}{-6} \right]_0^T$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \left[\frac{e^{-6T}}{-6} + \frac{e^0}{-6} \right]$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \left[\frac{e^{-6\infty}}{-6} + \frac{e^0}{-6} \right]$$

$$= \frac{1}{2T} \left[\frac{e^{-\infty}}{-6} + \frac{e^0}{-6} \right]$$

$$= \frac{1}{2T} \left[0 + \frac{1}{6} \right]$$

$$= \frac{1}{2 \times 6 \times 2T} = \frac{1}{12T} = 0$$

$$x_2(t) = e^{j(2t + \frac{\pi}{4})}$$

$$E = \lim_{T \rightarrow \infty} \int_{-T}^T |x(t)|^2 dt = \int_{-T}^T |e^{j(2t + \frac{\pi}{4})}|^2 dt = 1$$

$$= \lim_{T \rightarrow \infty} \int_{-T}^T [e^{j(2t + \frac{\pi}{4})}]^2 dt = \frac{1}{2}$$

$$= \lim_{T \rightarrow \infty} \int_{-T}^T (1)^2 dt$$

$$= \lim_{T \rightarrow \infty} [T + T]$$

$$E = \lim_{T \rightarrow \infty} \frac{1}{2}T = \infty \quad E = \text{Not Finite}$$

$$P = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T e^{j(2t + \frac{\pi}{4})} dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T (1) dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} [T + T] dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} [2T] dt$$

$$P = \frac{1}{2}$$

$$x(t) = \cos t$$

$$\cos t = e^{jt}$$

$$E = \lim_{T \rightarrow \infty} \int_{-T}^T |x(t)|^2 dt$$

$$= \lim_{T \rightarrow \infty} \int_{-T}^T |\cos t|^2 dt$$

$$= \lim_{T \rightarrow \infty} \int_{-T}^T \frac{1 + \cos 2t}{2} dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2} [T]_{-T}^T + \int_{-T}^T \cos 2t dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2} \cdot \frac{1}{2} [T + T] dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2} [RT] dt$$

$$E = \infty ; \text{ Not Finite}$$

$$P = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \cos^2 t dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \frac{(1 + \cos 2t)}{2} dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{4T} \int_{-T}^T (1 + \cos 2t) dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{4T} [T]_{-T}^T + 0$$

$$= \lim_{T \rightarrow \infty} \frac{1}{4T} [T + T]$$

$$= \frac{1}{4} \cdot \frac{1}{2} [2T]$$

$$P = \frac{1}{2}$$

State and Prove Schwartz Inequality :-

Proof :-
 $f(t), g(t)$ are two real signals then Schwartz inequality can be defined as

$$\int_a^b f^2(t) dt + \int_a^b g^2(t) dt \geq \left[\int_a^b f(t)g(t) dt \right]^2$$

W.K.T To prove the above eqn we require a condition having a value greater than zero.
we can write as

$$\int_a^b [f(t) + xg(t)]^2 dt > 0$$

$$\int_a^b f^2(t) dt + x^2 \int_a^b g^2(t) dt + 2x \int_a^b f(t)g(t) dt > 0$$

then W.K.T

$$a = \int_a^b f^2(t) dt \quad b = x^2 \int_a^b g^2(t) dt \quad c = 2x \int_a^b f(t)g(t) dt$$

$$a^2 + x^2c + b^2 > 0$$

$$a^2 + 2bx + b^2 > 0$$

$$cx^2 + 2bx + a > 0$$

This is a quadratic eqn of form having values
 $c < 0, b < 0, a < 0$

$$(2b)^2 - 4(c)(a) \leq 0 \leq 0$$

$$4b^2 - 4ac \leq 0$$

$$ax^2 + bx + c = 0$$

$$\text{condition } b^2 - 4ac \leq 0$$

$$4 \left[\int_a^b f^2(t) g^2(t) dt \right] \leq 4 \left[\int_a^b f^2(t) dt \cdot \int_a^b g^2(t) dt \right] \leq 0$$

$$\frac{4}{a} \int_a^b f^2(t) g^2(t) dt \leq \frac{4}{a} \int_a^b g^2(t) dt \int_a^b f^2(t) dt$$

$$\int_a^b f^2(t) g^2(t) dt \leq \int_a^b f^2(t) dt \int_a^b g^2(t) dt$$

② Explain distortion less transmission through systems:

for distortionless transmission of a signal system must :- 1. Attenuate all frequency components equally i.e. $H(\omega)$ should have constant magnitude for all frequencies.

③ The phase shift of each component must also satisfy certain relations which are as follows

④ for distortionless transmission the response must be an exact replica of the input signal.

This replica may have different magnitudes. They may also be some time delay associated with this replica.

⑤ we can say that the signal $f(t)$ is transmitted without distortion if the response $kf(t-t_0)$, the response is an exact replica of the input.

with magnitude k times the original signal and delayed by t_0 seconds.

④ if $f(t)$ is the input signal for the distortion less transmission, then the response $r(t)$ is given,

$$r(t) = k f(t - t_0) \quad \text{--- (1)}$$

⑤ Applying time shifting property to eqn. ① we get

$$R(\omega) = k f(\omega) e^{-j\omega t_0} \quad \text{--- (2)}$$

$$⑥ \omega \cdot k \cdot r R(\omega) = f(\omega) \cdot H(\omega) \quad \text{--- (3)}$$

Then we can write as

Comparing eqn ② & ③

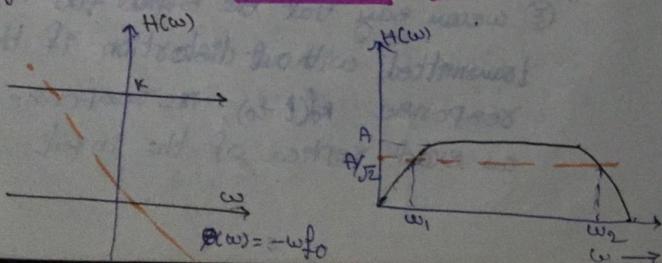
$$k f(\omega) e^{-j\omega t_0} = f(\omega) \cdot H(\omega)$$

$$H(\omega) = k \cdot e^{-j\omega t_0} \quad \text{--- (4)}$$

⑦ To achieve a distortion less transmission the signal must be in the form.

$$H(\omega) = k \cdot e^{-j\omega t_0}$$

Magnitude and phase characteristics of a system:-



⑧ It is evident that the magnitude of a transfer function is ' k ' and is constant for all frequencies.

⑨ The phase shift on the other hand is directly proportional to frequency.

$$\theta(\omega) = -\omega t_0 \quad \text{--- (5)}$$

⑩ The reason for this is that if two different frequency components are shifted by the same interval their corresponding phase changes are proportional to frequency.

for example:-

if signal $\cos \omega t$ is shifted by t_0 seconds
resulting signal can be expressed as

$$\cos(\omega(t + t_0)) = \cos(\omega t - \omega t_0) \quad \text{--- (6)}$$

⑪ It is clear that phase shift of a new signal $-\omega t_0$ is due to a frequency of ω . From eqn ⑤ we can write as

$$\theta(\omega) = n\pi - \omega t_0$$

where $n = \pm$, integers.

⑫ The addition of extra phase of $n\pi$ radians may almost change the sign of a signal hence the phase function of distortion less signal is in the form of

$$\theta(\omega) = n\pi - \omega t_0$$

(13) This eqn will clearly describe the characteristic of a distortion less system and its nature.

Explain causality and poly wiener criterion for physical realization

- i. Poly wiener criteria is a test b/w physically realizable characteristic form from an

Magnitude in realizable one.

M → magnitude = 0
A → amplitude $\int_{-\infty}^{\infty} |H(\omega)|^2 d\omega < \infty$
A physically realizable system shouldn't have a response before the driving function input function is applied. This is known as causality function.

S → smaller values can estimate causality function.
(2) A causality condition can also be expressed as "A unit impulse response $H(t)$ of a physically realizable system must be causal."

(4) A signal is said to be causal if it has time less than zero.

(5) Thus the impulse response $h(t)$ of a physically realizable system for $t < 0$

(6) This is a Time domain criteria of physical realizability. In frequency domain this criteria implies that a necessary and sufficient condition

for a magnitude function $H(\omega)$ - if it is physically realizable.

$$\int_{-\infty}^{\infty} \frac{|\ln|H(\omega)||}{1+\omega^2} d\omega < \infty \quad \text{--- (1)}$$

(7) Magnitude function $H(\omega)$ must however be square integrable before the Poly wiener criteria is valid.

$$\int_{-\infty}^{\infty} |H(\omega)|^2 d\omega < \infty \quad \text{--- (2)}$$

(8) A system whose magnitude function violates Poly wiener criteria as given by eqn (1) has a non causal impulse response. i.e. the response exist prior to application of driving function.

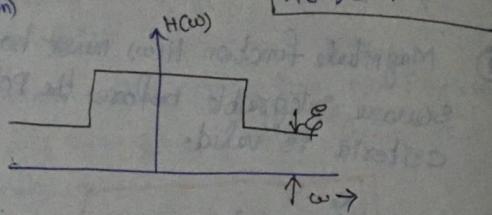
Significant conclusion drawn from poly wiener criteria

(1) The Magnitude function $H(\omega)$ may be zero at some discreet frequency but it can't be zero over a finite band of frequencies. since this will cause the integral in eqn (1) to become infinity.

(2) we can conclude from (1) that the amplitude function can't fall off to zero faster than a function of exponential order

$$H(\omega) = k e^{-\alpha |\omega|} \text{ is permissible.}$$

- (1) The realizable magnitude characteristics can't have high total attenuation.
- (2) The physically realizable low pass filter can be realized for arbitrary small values of α (ϵ psilon)

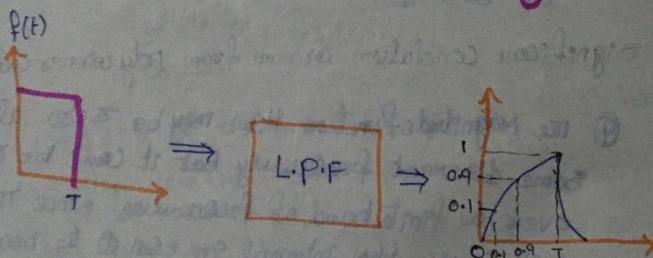


Relationship between Rise time and Band width

1. The time taken for a pulse to rise from 10% of initial value to 90% of its final value is called

Rise time

$$y(t) = (1 - e^{-t_0/Rc})$$



The output of the low pass filter is defined as

$$y(t) = 1 - e^{-t_0/Rc}$$

Then we get

$$y(t) = 1 - e^{t_0/Rc} \text{ at } 0.1 \text{ time}$$

Then

$$0.1 = 1 - e^{-t_0/Rc}$$

$$e^{-t_0/Rc} = 1 - 0.1$$

$$e^{-t_0/Rc} = 0.9$$

Applying \log_e on B.S.

$$\log e^{-t_0/Rc} = \log 0.9$$

$$-\frac{t_0}{Rc} = \log 0.9$$

$$-t_0 = R_c \log 0.9$$

$$t_0 = -R_c \log \frac{0.9}{e}$$

$$y(t) = 1 - e^{-t_0/Rc}$$

$$0.9 = 1 - e^{-t_0/Rc}$$

$$e^{-t_0/Rc} = 1 - 0.9$$

$$e^{-t_0/Rc} = 0.1$$

Apply \log on B.S.

$$\log e^{-t_0/Rc} = \log \frac{0.1}{e}$$

$$-\frac{t_0}{Rc} = \log \frac{0.1}{e}$$

$$t_{0.9} = -Rc \log(0.1) \quad \text{--- (3)}$$

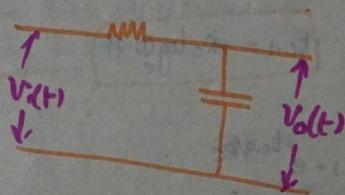
$$t_0 = t_{0.9} - t_{0.1}$$

$$= -Rc \log(0.1) + Rc \log(0.9)$$

Half-time $t_{0.5}$ of an Rc low pass filter is $\frac{0.25}{f_{3dB}}$

where $f_{3dB} = \frac{1}{2\pi R_c}$ which is $3dB$ Band width of ω filter

Low pass filter can be drawn as



$$y(t) = h(t) * x(t)$$

Unit step response of ω low pass filter is obtained by convolution integral $y(t) = h(t) * x(t)$. We write as $h(t)$ is unit impulse response $h(t) = \frac{1}{Rc} \{ e^{-t/Rc} \}$

$$\begin{aligned} h(t) &= \int_{-\infty}^t \frac{1}{Rc} e^{-t/Rc} dt \\ &= \int_0^t \frac{1}{Rc} e^{-t/Rc} dt \\ &= \frac{1}{Rc} \int_0^t e^{-t/Rc} dt \end{aligned}$$

$$= \frac{1}{Rc} \left[\frac{e^{-t/Rc}}{-1/Rc} \right]_0^t$$

$$= \frac{1}{Rc} \times -Rc \left[e^{-t/Rc} \right]_0^t$$

$$= - \left[e^{-t/Rc} - e^0 \right]$$

$$= - \left[e^{-t/Rc} - 1 \right]$$

$$y(t) = 1 - e^{-t/Rc}$$

$$0.1 = 1 - e^{-t_{0.1}/Rc}$$

$$e^{-t_{0.1}/Rc} = 1 - 0.1$$

$$e^{-t_{0.1}/Rc} = 0.9$$

Apply log on B.S

$$\frac{-t_{0.1}}{Rc} = \log(0.9)$$

$$0.9 = 1 - e^{-t_{0.9}/Rc}$$

$$e^{-t_{0.9}/Rc} = 1 - 0.9$$

$$e^{-t_{0.9}/Rc} = 0.1$$

Apply log on B.S

$$\frac{-t_{0.9}}{Rc} = \log(0.1)$$

$$t_{0.9} = -Rc \log(0.1)$$

$$t_{0.1} = -Rc \log(0.9)$$

$$\textcircled{1} \quad \frac{-t_{0.1}}{Rc} = \frac{e^{-\frac{t_{0.1}}{Rc}}}{e^{-\frac{t_{0.9}}{Rc}}} = \frac{0.9}{0.1} = 9$$

$$\therefore e^{-\frac{t_{0.1}}{Rc}} + \frac{t_{0.9}}{Rc} = 9$$

$$= e^{\left[\frac{t_{0.9}}{Rc} - \frac{t_{0.1}}{Rc} \right]} = 9$$

$$= e^{\frac{1}{Rc} [t_{0.9} - t_{0.1}]} = 9$$

Apply log on B.S

$$\textcircled{2} \quad \frac{1}{Rc} [t_{0.9} - t_{0.1}] = \log 9$$

$$t_{0.9} - t_{0.1} = R_c \log 9$$

$$t_0 = R_c \log(9)$$

$$t_0 = R_c 2.197$$

$$\text{W.K.T} \quad t_0 = R_c \times \frac{1}{2\pi f_{3dB}}$$

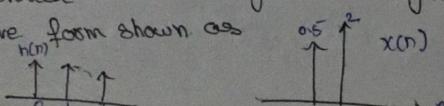
$$t_0 = \frac{0.35}{f_{3dB}}$$

Consider an LTI system with impulse response $h(n) \leq x(n)$

a) find the response (or) ~~for~~ non-zero values of input

b) the overall response of $y(n)$, $h(n)$ is given by the

c) wave form shown as



$$y(n) = x(n) * h(n)$$

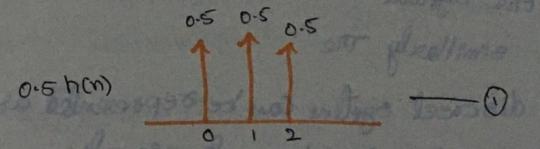
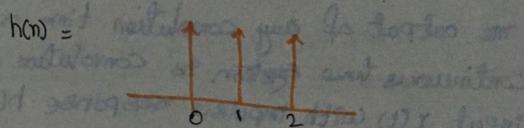
$$y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

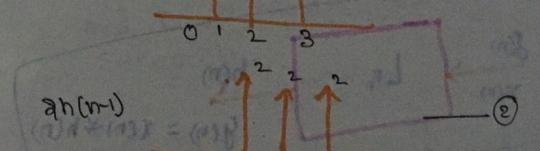
$$= \sum_{k=0}^1 x(k) h(n-k)$$

$$= x(0) h(n-0) + x(1) h(n-1)$$

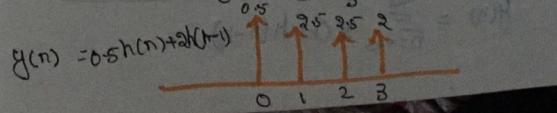
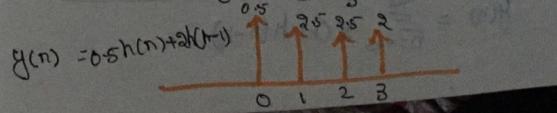
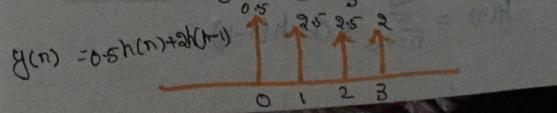
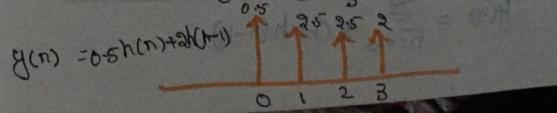
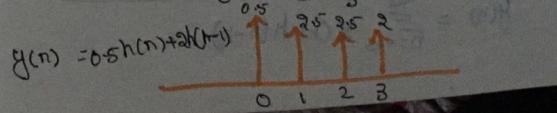
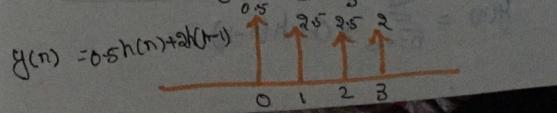
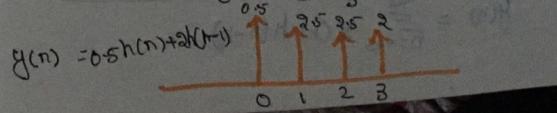
$$= 0.5 h(n) + 2 h(n-1)$$



$$h(n-1)$$

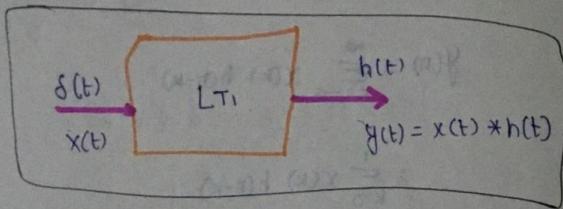


$$2h(n-1)$$



Convolution and correlation of signals

concept of convolution in time domain and frequency domain represented as the block diagram

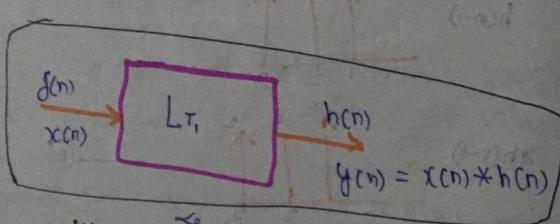


$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \quad \text{--- (1)}$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \quad \text{--- (2)}$$

The output of any convolution time or continuous time system is convolution of input $x(t)$ with impulse response $h(t)$ of the system.

Similarly the discrete system can be represented as convolution for discrete signals.



$$y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

out of any discrete time LTI system is convolution of input $x(n)$ with impulse response we can obtain

$$x_1(t) * x_2(t) \longleftrightarrow X_1(j\omega) \cdot X_2(j\omega)$$

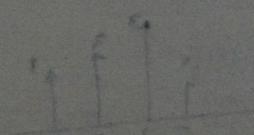
The Laplace LT play important role in convolution the linear time invariant system can be represented as $X_1(t) * X_2(t)$

$$X_1(t) * X_2(t) \longleftrightarrow \frac{1}{2\pi} |X_1(j\omega)| |X_2(j\omega)|$$

If the multiplication property and is often referred as the frequency convolution the multiplication in time domain will become convolution in frequency domain.

The Correlation function:-

- ① It gives the similarity b/w two signals.
- ② The correlation functions for energy signals and power signals can be represented separately.



Problem :-
convolute $x(n), h(n)$ find $y(n)$ where $x(n) = \{1, 4, 3, 2\}$

$$h(n) = \{1, 3, 2, 1\}$$

Step 1 :-

$n_1 \rightarrow$ starting time of $x(n) = -1$
 $n_2 \rightarrow$ starting time of $h(n) = 0$

$$n = n_1 + n_2$$

$$n = -1 + 0 = -1$$

$$n = -1$$

Step 2 :-

$h(n) = \{1, 3, 2, 1\}$ represent in k terms

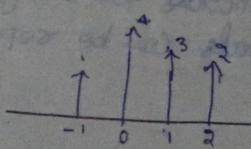
$$h(k) = \{1, 3, 2, 1\}$$

Step 3 :- $h(-k)$

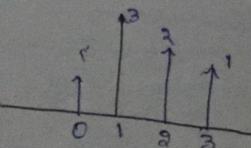
Step 4 :- $h(-k-1)$

steps :-

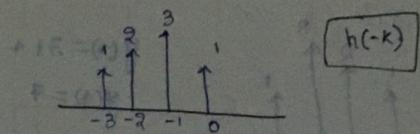
$$y(n) = x(n) * h(n)$$



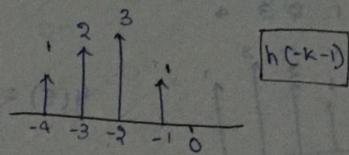
$x(n)$



$h(x)$

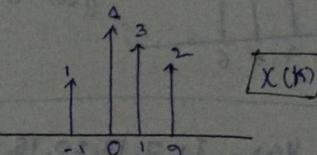


$h(-k)$

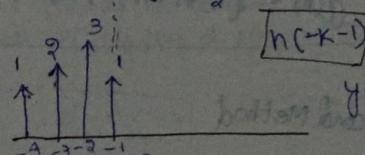


$h(-k-1)$

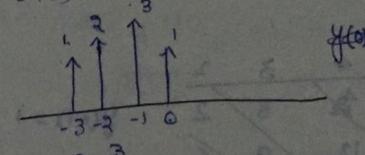
(or) directly we can write as



$x(k)$



$h(-k)$



$h(-k-1)$

$$y(-1) = 1$$

$$y(0) = 4 + 3 = 7$$

$$y(0) = 3 + 6 + 2$$

$$y(0) = 7$$

$$y(1) = 12 + 3 + 2$$

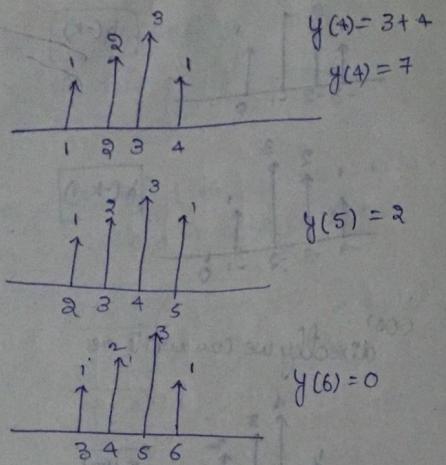
$$y(1) = 17$$

$$y(2) = 8 + 9 + 2 + 1$$

$$y(2) = 20$$

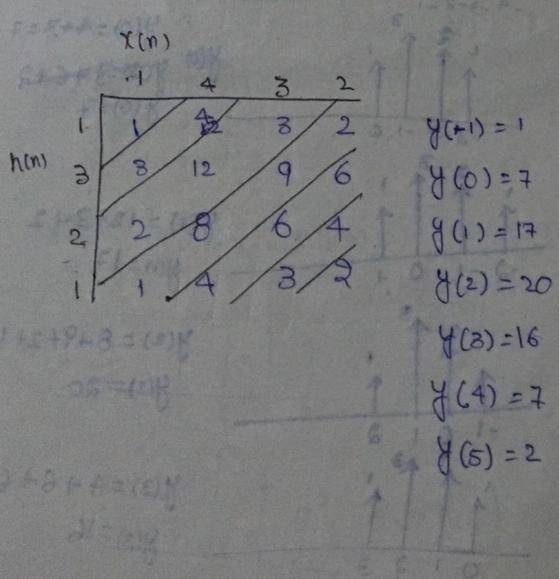
$$y(3) = 4 + 6 + 6$$

$$y(3) = 16$$

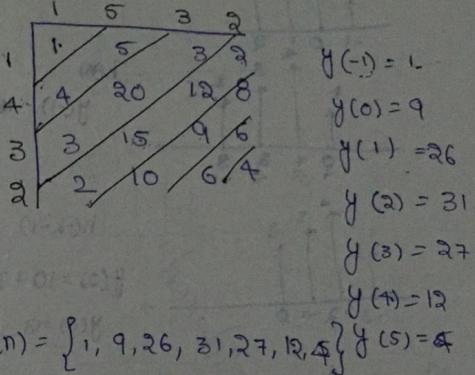


Output $y(n) = \{1, 7, 17, 20, 16, 7, 2\}$

second method



$X(n) = \{1, 5, 3, 2\}$ $h(n) = \{1, 4, 3, 2\}$



step 1:-

n_1 :- starting time of $X(n)$
 n_2 :- starting time of $h(n)$

$n = n_1 + n_2$

$n = -1 + 0 = -1$

$n = -1$

step 2:-

$h(n) = \{1, 4, 3, 2\}$ represent in k teams.

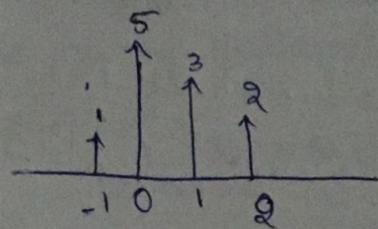
$h(k) = \{1, 4, 3, 2\}$

step 3:- $h(k)$

step 4:- $h(k-1)$

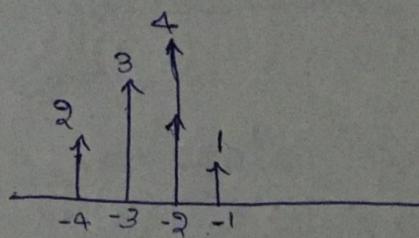
step 5:- $y(n) = x(n)*h(n)$

$$x(n) = \{1, 5, 3, 2\}$$



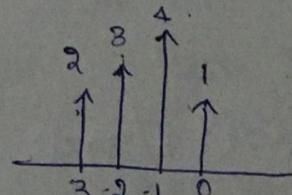
$x(k)$

$$h(-k-1)$$

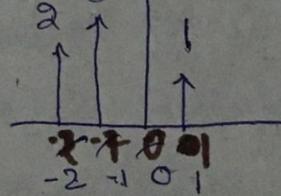


$$y(-1) = 1$$

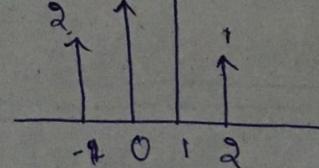
$$y(0) = 2 + 5 + 4 \\ = 9$$



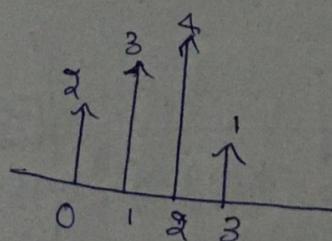
$$y(1) = 3 + 20 + 3 \\ = 26$$



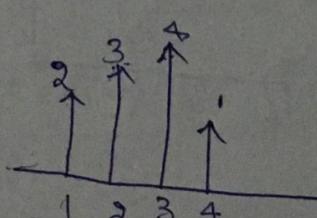
$$y(2) = 2 + 15 + 12 + 2 \\ = 31$$



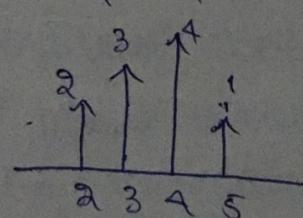
$$y(3) = 10 + 9 + 8 \\ = 27$$



$$y(4) = 6 + 6 \\ = 12$$



$$y(5) = 4$$



$$y(6) = 0$$

Energy signals-

let us now define correlation for energy signal.
it is given by the following expression.

$$R_{12}[\tau] = \int_{-\infty}^{\infty} x_1(t) * x_2(t - \tau) dt$$

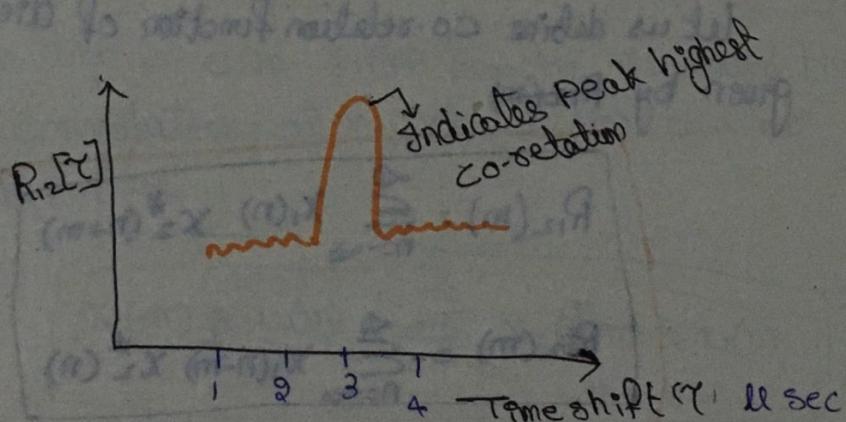
$x_1(t), x_2(t)$ are complex valued signal of finite energy.

$R_{12}[\tau]$ is co-correlation function of $x_1(t)$ & $x_2(t)$.
in the above eqn we have defined a co-correlation function. Here note that co-correlation function is a function of τ . and τ is time delay or time shift of one of the two signals.
i.e. one of the time signal is time shifted

$R_{12}[\tau]$ is calculated for various values of τ .

This co-correlation function has many values.

Explanation of co-correlation function:-



In the above figure observe that co-correlation function has highest values at $\tau = 3$ msec and the rest of the other values of τ .

The value of the co-correlation function is near zero or minimum. i.e. Two signals have highest co-correlation

for delay $\tau = 3$ sec for other delay's the two signals are not much correlated.

② if the two signals are real then the correlation function $R_{12}[\tau]$ then

$$R_{12}[\tau] = \int_{-\infty}^{\infty} x_1(t) x_2(t-\tau) dt$$

$$R_{12}[\tau] = \int_{-\infty}^{\infty} x_1(t) x_2^*(t+\tau) dt$$

$$R_{12}[\tau] = \int_{-\infty}^{\infty} x_1(t-\tau) x_2(t) dt$$

$$R_{12}[\tau] = \int_{-\infty}^{\infty} x_1(t+\tau) x_2^*(t) dt$$

The above eqn's produce similar results.

co-relation function of discrete time energy signals:

let us define co-relation function of discrete times given by $R_{12}(m)$

$$R_{12}(m) = \sum_{n=-\infty}^{\infty} x_1(n) x_2^*(n+m)$$

$$R_{12}(m) = \sum_{n=-\infty}^{\infty} x_1(n-m) x_2^*(n)$$

In the above eqn's one signal is delayed or advanced by m samples. The co-relation function $R_{12}[m]$ is the samples of m .

Relationship b/w co-relation and convolution :-
consider the convolution of two sequences $x_1(n) \otimes x_2(n)$

$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(n-k) x_2(k)$$

$$x_1(m) * x_2$$

In the above eqn n is just index.
hence it can be replaced by any other variable
let replace n by m then we get

$$x_1(m) * x_2(m) = \sum_{k=-\infty}^{\infty} x_1(m-k) x_2(k)$$

In the above eqn k is just an index. Replace it with any other index ' n ' then the eqn becomes.

$$x_1(m) * x_2(m) = \sum_{n=-\infty}^{\infty} x_1(m-n) x_2(n)$$

In the above eqn $x_1(m)$ be inverted time then convolution of $x_1(-m)$, $x_2(m)$ can be written as.

$$x_1(-m) * x_2(m) = \sum_{n=-\infty}^{\infty} x_1(n-m) * x_2(n)$$

after stdn

$$x_1(-m) * x_2(m) = \sum_{n=-\infty}^{\infty} x_1(-m-n) x_2(n)$$

$$x_1(-m) * x_2(m) = \sum_{n=-\infty}^{\infty} x_1(-(m+n)) x_2(n)$$

The above eqn (Comparing R.H.S)

$$x_1(-m) * x_2(m) = \sum_{n=-\infty}^{\infty} x_1(n-m) * x_2(n) \quad -④$$

Thus the co-relation of two signals is obtained by convolution of one signal with time folded version of another signal. This similar principle is true for continuous time signals can be stated as.

$$x_1(-m) * x_2(m) = R_{12}(m) \quad -⑤$$

$$R_{12}(m) = x_1(-m) * x_2(m)$$

$$R_{12}(\tau) = x_1(-\tau) * x_2(\tau) \quad -⑥$$

Properties of Fourier transform

$$① x^*(t) \rightarrow X^*(\omega)$$

$$X(t) \rightarrow x(\omega)$$

$$R_{12}(m) \rightarrow x_1(-\omega) \cdot x_2(\omega)$$

$$R_{12}(\tau) \rightarrow x_1^*(-\omega) \cdot x_2(\omega)$$

Thus co-relation can be calculated by multiplying fourier transform of one function and complex conjugate of fourier transform of another function. Inverse fourier transform of product gives co-relation function.

Power Signals

⑤ The Co-relation function of Periodic power signals is given by $R_{12}[\tau] = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{T} x_1(t) x_2^*(t-\tau) dt \quad -⑦$

If we compare above eqn with the eqn of energy signals is given as

$$R_{12}[T] = \int_{-\infty}^{\infty} x_1(t) x_2^*(t-\tau) dt \quad -⑧$$

If we compare above eqn with correlation function of energy signals given by above eqn we find that integration result is divided by $\frac{1}{2T}$ and limiting condition as $T \rightarrow \infty$ when x_1, x_2 both are real, we can write the eqns as

$$R_{12}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-\tau}^{\tau} x_1(t) x_2(t-\tau) dt \quad -⑨$$

$$R_{12}(\tau) = \lim_{T \rightarrow \infty} \int_{-\tau_2}^{\tau_2} x_1(t) x_2^*(t+\tau) dt \quad -⑩$$

$$R_{12}(\tau) = \lim_{T \rightarrow \infty} \int_{-\tau_2}^{\tau_2} x_1(t-\tau) x_2^*(t) dt \quad -⑪$$

for discrete values we can write as

$$R_{12}[m] = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=-N}^{N} x_1(n) x_2^*(n+m) \quad -⑫$$

$$R_{12}(m) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=-N}^{N} x_1(n-m) x_2^*(m)$$

⑦

Auto OR NO CO-relation:-

- ① when we calculate co-relation of a signal with it self then it is called auto -> Auto Co-relation
- ② Thus if $x_1(t) = x_2(t)$ then Auto co-relation function can be written as [Auto correlation function for energy signals]

is

$$R[\tau] = \int_{-\infty}^{\infty} x(t) x^*(t-\tau) dt \quad \text{--- ①}$$

In the above case if the complex variables is delayed in other words the real valued signal is delayed then the above function can be written as

$$R[\tau] = \int_{-\infty}^{\infty} x(t+\tau) x^*(t) dt$$

Similarly Auto correlation of a discrete time signal is given by

REFT:

$$R[m] = \sum_{n=-\infty}^{\infty} x(n) x^*(n+m)$$

$$= \sum_{n=-\infty}^{\infty} x(n+m) x^*(n)$$

Auto correlation of power signals:-

It can be represented as

$$R(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) x^*(t-\tau) dt \quad \text{--- ②}$$

τ is ∞ in negative direction then we can write

as

$$R[\tau] = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t+\tau) x^*(t) dt \quad \text{--- ③}$$

for any values of t we can write as

$$R[\tau] = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) x^*(t-\tau) dt \quad \text{--- ④}$$

Auto correlation function for discrete

$$R[m] = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=-N}^{N} x(n) x^*(n+m)$$

explain the

Relation to signal energy and signal power
(or)

explain the relationship b/w ^{signal} Energy and signal Power

Proof:-

we define Auto correlation function of energy signals and power signals now let's us relate Auto correlation to energy and power of a signal

Relation of Auto Correlation with energy:-

consider the Auto Correlation of energy signal as

$$R[x] = \int_{-\infty}^{\infty} x(t) x^*(t-\tau) dt$$

$$= \int_{-\infty}^{\infty} x(t) x^*(t) dt$$

$$\because \tau = 0$$

$$= \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$R(\tau) = E$$

Relation of Auto correlation with Power

$$R(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) x(t-\tau) dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) y(t) dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

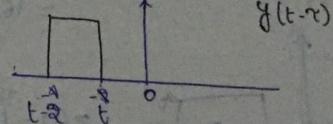
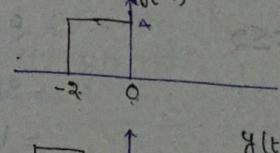
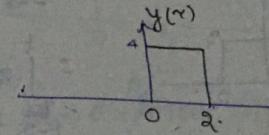
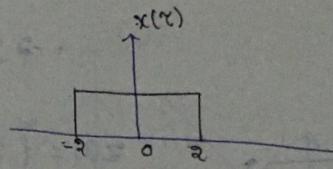
$$R(\tau) = P$$

problem:-
Given $x(t)$ convolved with $y(t)$ and sketch the results.

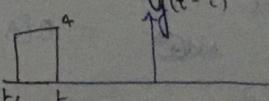
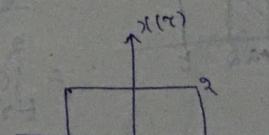
$$x(t) = 2 \quad \text{for } -2 \leq t \leq 2 \\ = 0 \quad \text{elsewhere}$$

$$y(t) = 4 \quad \text{for } 0 \leq t \leq 2 \\ = 0 \quad \text{elsewhere}$$

$$z(t) = x(t) * y(t)$$

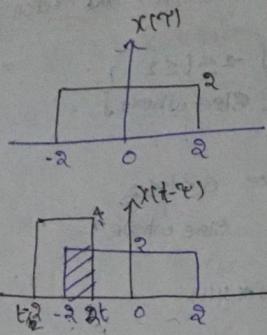


step 1 :-



$$z(t) = \int_{-\infty}^{\infty} x(\tau) y(t-\tau) d\tau \\ = \int_{-\infty}^{t_2} 0 + \int_{t_2}^0 0 + \int_0^t 2 \cdot 4 d\tau \\ = \int_0^t 8 d\tau \\ = 8t$$

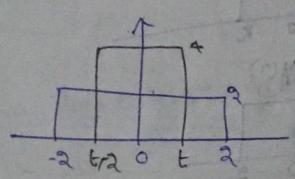
Step 2



$$\begin{aligned} z(t) &= \int_{-\infty}^{\infty} x(r) y(t-r) dr \\ &= \int_{-2}^{t+2} (1)(4) dr \\ &= 4r \Big|_{-2}^{t+2} \\ &= 8[t+2] \end{aligned}$$

$-2 \leq t \leq 0$

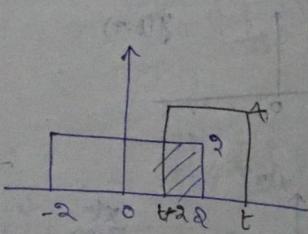
Step 3



$0 \leq t \leq 2$

$$\begin{aligned} z(t) &= \int_{-\infty}^{\infty} x(r) y(t-r) dr \\ &= \int_{-2}^{t-2} (1)(4) dr \\ &= 8[t-(t-2)] \\ &= 8[2-t+2] \\ &= 16 \end{aligned}$$

Step 4



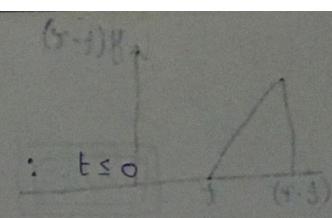
$t-2 \leq 2 < t$

$$\begin{aligned} t-2 &\leq 2 < t \\ t-2 &\leq -2 < t \leq t \\ -2 &\leq 2-t \leq 0 \quad \text{Add } (-2) \\ -4 &\leq 2-t \leq -2 \end{aligned}$$

$$\begin{aligned} z(t) &= \int_{-\infty}^{\infty} x(r) y(t-r) dr \\ &= \int_{t-2}^{2} (1)(4) dr \\ &= 8[2-t+2] \\ &= 8[4-t] \end{aligned}$$

$2 \leq t \leq 4$

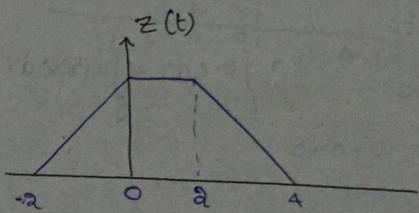
$$z(t) = 0 : t \leq 0$$



$$8t + 16 : -2 \leq t \leq 0$$

$$16 : 0 \leq t \leq 2$$

$$-8t + 32 : 2 \leq t \leq 4$$



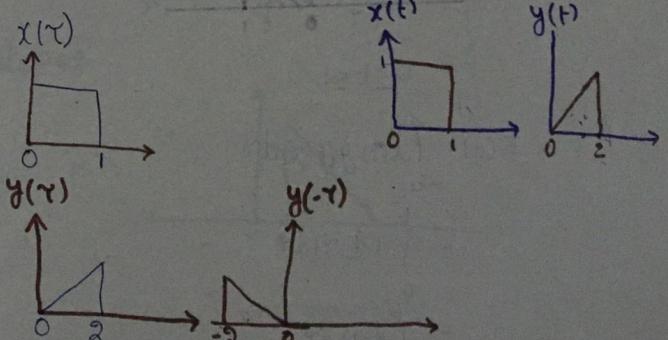
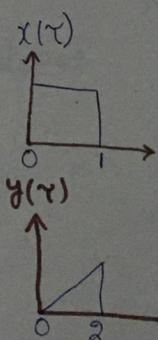
* Convolute Two different signals $y(t) = t : 0 < t < 2$

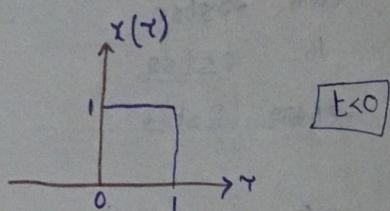
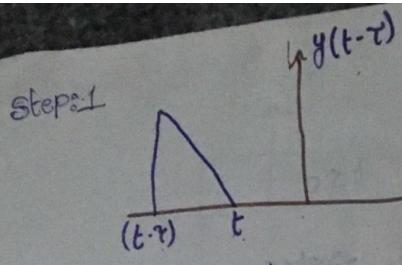
$$= 0 ; \text{elsewhere}$$

$$z(t) = \int_{-\infty}^{\infty} x(r) y(t-r) dr$$

$$x(t) = 1 : 0 < t < 1$$

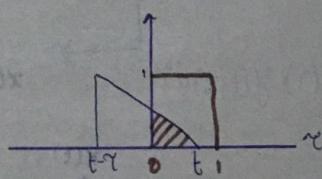
$$= 0 ; \text{elsewhere}$$





$$\begin{aligned} Z(t) &= \int_{-\infty}^t y(t-x) dx + \int_t^0 0 dx + \int_0^t x(t-x) dx \\ &= 0 + 0 + 0 \\ &= 0 \end{aligned}$$

Step 2 :-



$$0 \leq t \leq 1$$

$$\begin{aligned} Z(t) &= \int_{-\infty}^t x(\tau) y(t-\tau) d\tau \\ &= \int_0^t 1(t-\tau) d\tau \\ &\stackrel{\text{def}}{=} \int_0^t (t-\tau) d\tau \\ &= t \left(\tau \right)_0^t - \left(\frac{\tau^2}{2} \right)_0^t \end{aligned}$$

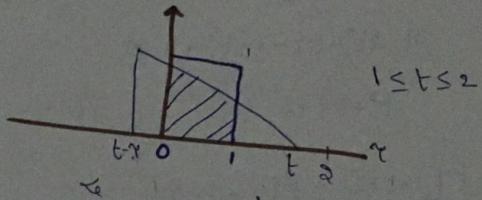
$$t(t) = \frac{t^2}{2}$$

$$\frac{2t^2 - t^2}{2}$$

$$\frac{t^2}{2}$$

$$0 \leq t \leq 1$$

Step 3 :-



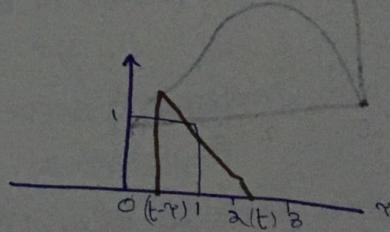
$$\begin{aligned} Z(t) &= \int_{-\infty}^t x(\tau) y(t-\tau) d\tau \\ &= \int_0^t 1(t-\tau) d\tau \end{aligned}$$

$$= t(t)_0^t - \left(\frac{\tau^2}{2} \right)_0^t$$

$$= t - \frac{t^2}{2}$$

$$= \frac{2t - t^2}{2} \Rightarrow 1 \leq t \leq 2$$

Step 4 :-



$$\begin{aligned} Z(t) &= \int_{-\infty}^t x(\tau) y(t-\tau) d\tau \\ &= \int_{t-2}^t (t-\tau) d\tau \end{aligned}$$

$$t \left[\gamma \right] - \left[\frac{\gamma^2}{2} \right]$$

$$t \left[1 - (t-\gamma) \right] - \frac{1}{2} \left[t^2 - (t-\gamma)^2 \right]$$

$$t - t^2 + t\gamma - \frac{1}{2} [1 - t^2 + 2t - \gamma^2]$$

$$\cancel{t^2 - t^2 + t\gamma + \cancel{t^2}} - \cancel{\frac{1}{2}} [1 + t^2 + 1 - 2t - \gamma^2]$$

$$2t - 2t^2 + t\gamma - 1 + t^2 + 1 - 2t - \gamma^2$$

$$-t^2 + t\gamma + 3$$

$$t \left[1 - t + \frac{3}{2} \right] - \frac{1}{2} \left[1 - (t-2)^2 \right]$$

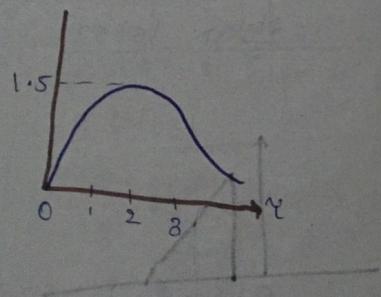
$$t \left[1 - t \right] - \frac{1}{2} \left[-t^2 - 4 + 2t - 1 \right]$$

$$2t - t^2 - \frac{1}{2} \left[-t^2 + 2t - 5 \right]$$

$$3t - t^2 - \frac{1}{2} + \frac{15}{2} + 2 - 2t$$

$$2 \leq t \leq 3$$

4M Energy spectral density :-



Energy

4M write a note on Energy spectral density.
The spectral density function gives the distribution of energy of a signal in a frequency domain. It can be represented as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \quad \text{--- (1)}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |x(\omega)|^2 d\omega \quad \text{--- (2)}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |x(2\pi f)|^2 2\pi \cdot df \quad \text{--- (3)}$$

$$E = \int_{-\infty}^{\infty} |x(f)|^2 df$$

$$4(f) = |x(f)|^2$$

$$E = \int_{-\infty}^{\infty} 4(f) df \quad \text{--- (4)}$$

This EQN shows that Total energy of a signal is given by the total area under the curve $4(f)$.

(2) $4(f)$ represents energy density of a signal.

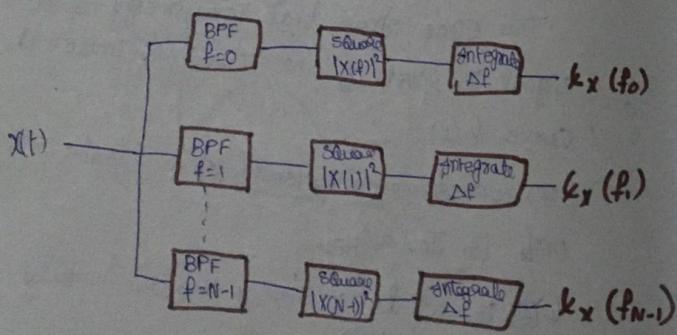
units is Joules/Hz.

(3) $4(f)$ is called as energy spectral density of a signal $x(t)$.

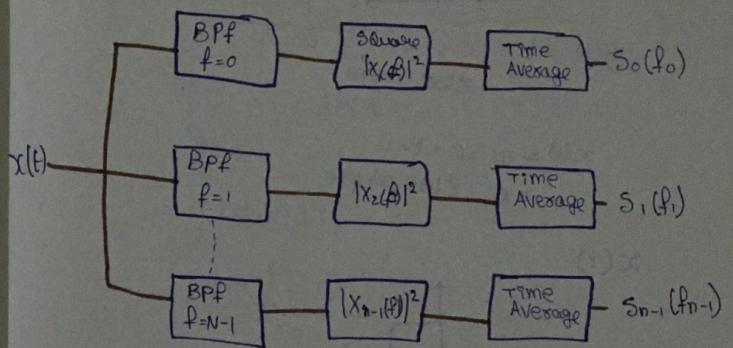
Explain conceptional block diagram of energy spectral density?

- ① Normally ESD spectrum is plotted for positive as well as -ve frequencies.
- ② The Single sided ESD of signal is twice the Double side ESD.
- ③ ESD is symmetric around $f=0$.
- ④ $X(t)$ is applied to various Band width filters having the same Pass Band of Δf . and centred frequencies $f_0, f_1, f_2, \dots, f_{N-1}$.
- ⑤ Then signal of each section is squared and then integrated over Δf . which gives the spectral density $k_x(f)$ for different frequencies $f_0, f_1, f_2, \dots, f_{N-1}$

Block diagram of Conceptional ESD.



Draw the conceptional block diagram of power spectrum density?



Explain the differences b/w ESD and PSD?

ESD

PSD

1. It gives the distribution of energy of a signal in frequency domain
2. It gives the distribution of power of signal in frequency domain

2. ESD is defined as

$$k_x(f) = |X(f)|^2$$

$$E = \int_{-\infty}^{\infty} |X(f)|^2 df$$

$$= \int k_x(f) df$$

3. Auto correlation function for energy signal and ESD form a Fourier transform pair

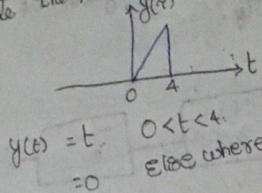
② PSD is defined as

$$S(f) = \frac{1}{T^2} \sum_{k=-\infty}^{\infty} |X(kf_0)|^2 \delta(f+kf_0)$$

$$S(f) \rightarrow \int_{-\infty}^{\infty} S(f') df'$$

- ③ Auto correlation of Power signal and PSD from a Fourier transform pair

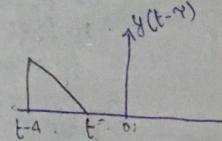
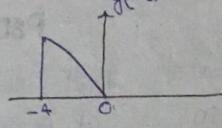
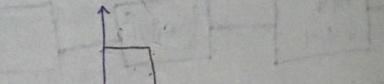
convolute the following signal and sketch the results



$$y(t) = \begin{cases} t & 0 < t < 4 \\ 0 & \text{elsewhere} \end{cases}$$

$$x(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$x(\tau)$



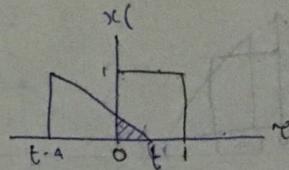
$$z(t) = \int_{-\infty}^{\infty} x(\tau) y(t-\tau) d\tau$$

Step 1:-

$$= \int_{t-4}^t y(t-\tau) \cdot 0 + \int_t^0 0 \cdot 0 + \int_0^1 x(\tau) \cdot 0 + \int_1^t 0 \cdot 0$$

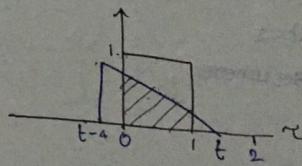
= 0

Step 2:-



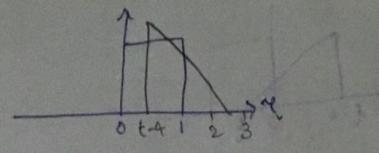
$$\begin{aligned} z(t) &= \int_{-\infty}^t x(\tau) y(t-\tau) d\tau & 0 \leq t \leq 1 \\ &= \int_0^t 1 \cdot (t-\tau) d\tau \\ &= t \left[\tau \right]_0^t - \left[\frac{\tau^2}{2} \right]_0^t \\ &= t^2 - \frac{t^2}{2} \\ &= t^2 \left[1 - \frac{1}{2} \right] \\ &= \frac{t^2}{2} \end{aligned}$$

Step 3:-



$$\begin{aligned} z(t) &= \int_{-\infty}^t x(\tau) y(t-\tau) d\tau \\ &= \int_0^1 1 \cdot (t-\tau) d\tau \\ &= t \left[\tau \right]_0^1 - \left[\frac{\tau^2}{2} \right]_0^1 \\ &= t \cdot [1] - \frac{1}{2} \\ &= \frac{2t-1}{2} & 1 \leq t \leq 2 \end{aligned}$$

Step 4:-

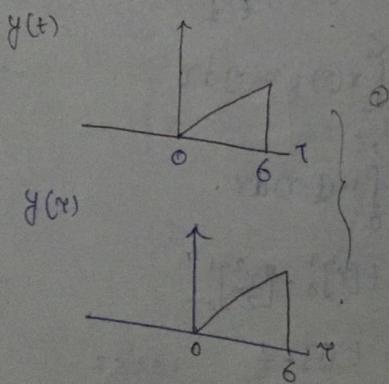


$$\begin{aligned}
 z(t) &= \int_{-\infty}^t x(\tau) y(t-\tau) d\tau \\
 &= \int_{t-4}^t (\tau - t) d\tau \\
 &= t[\tau]_{t-4}^t - \left[\frac{\tau^2}{2} \right]_{t-4}^t \\
 &= t[0 - t + 4] - \frac{1}{2}[1 - (t^2 + 16 - 8t)] \\
 &= t - t^2 + 4t - \frac{1}{2}[-t^2 + 8t - 15]
 \end{aligned}$$

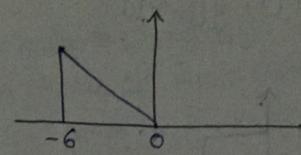
Problem 5:-

$$\begin{cases} y(t) = t & 0 < t < 6 \\ = 0 & \text{elsewhere} \end{cases}$$

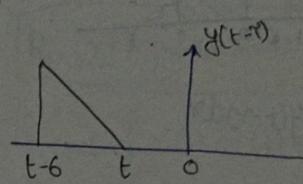
$$\begin{cases} x(t) = 21 & 0 < t < 2 \\ = 0 & \text{elsewhere} \end{cases}$$



Step 1:-



\$y(t-\tau)\$



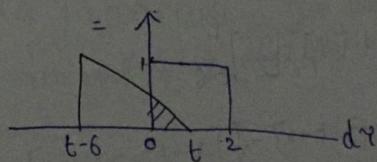
Step 1:-

$$\begin{aligned}
 z(t) &= \int_{-\infty}^t x(\tau) y(t-\tau) d\tau \\
 &= \int_{t-6}^t (21)(t-\tau) d\tau + \int_t^6 0 \cdot 0 d\tau + \int_6^6 1 \cdot 0 d\tau
 \end{aligned}$$

\$= 0\$

Step 2:-

$$z(t) = \int_{-\infty}^t x(\tau) y(t-\tau) d\tau$$



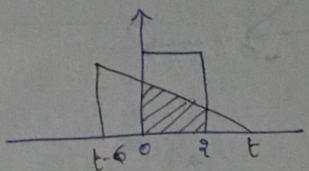
$$= \int_0^t 21 \cdot (t-\tau) d\tau$$

$$= t[\tau]_0^t - \left[\frac{\tau^2}{2} \right]_0^t$$

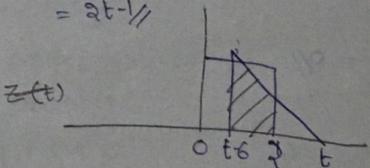
$$= t[t] - \frac{1}{2}[t^2 - 0] = t^2 - \frac{t^2}{2} = \frac{t^2}{2}$$

Step 3 :-

$$z(t) = \int_{-\infty}^t x(\tau) y(t-\tau) d\tau$$



$$\begin{aligned} z(t) &= \int_0^t x(\tau) y(t-\tau) d\tau \\ &= \int_0^t 1(t-\tau) d\tau \\ &= t[t^2]_0 - [\frac{\tau^2}{2}]_0 \\ &= t[2] - [1] \\ &= 2t - 1 \end{aligned}$$



$$\begin{aligned} z(t) &= \int_{-6}^t (t-\tau) d\tau \\ &= t[\tau]^2 - [\frac{\tau^2}{2}]_{-6}^t \\ &= t[2-t+6] - \frac{1}{2}[4-t^2-36+12t] \\ &= 8t - t^2 + 6t - \frac{1}{2}[-t^2 + 12t - 32] \end{aligned}$$

Cross correlation of energy signals and properties:-

1. Two signals $x_1(t)$ & $x_2(t)$ is a pair of complex valued signals of finite energy the cross correlation of two signals is given by $R_{12}[\tau] = \int_{-\infty}^{\infty} x_1(t) x_2^*(t-\tau) dt$

$$R_{12}[\tau] = \int_{-\infty}^{\infty} x_1(t) x_2^*(t-\tau) dt \quad \text{--- (1)}$$

If $\int_{-\infty}^{\infty} x_1(t) x_2^*(t) dt = 0$ then $R_{12} = 0$

$x_1(t), x_2^*(t)$ are orthogonal signals.

Properties :-

1. The cross correlation function exhibits symmetry

$$R_{12}[\tau] = R_{21}^*[\tau] \quad \text{--- (1)}$$

Cross correlation is not a general commutative

$$\text{i.e. } R_{12}[\tau] \neq R_{21}[\tau] \quad \text{--- (2)}$$

② Property 2 :-

If $R_{12}[\phi] = 0$ then the signals can be represented

i.e. $\int_{-\infty}^{\infty} x_1(t) x_2^*(t) dt = 0$ then the signals are orthogonal signals over the entire interval.

Property 3 :-

The cross correlation of two energy signals corresponds to multiplication of Fourier transform of one signal by complex conjugate of Fourier transform of second signal.

$$R_{12}[\tau] \rightarrow x_1(f) x_2^*(f)$$

This is known as co-relation theorem.

LAPLACE TRANSFORM

Laplace transform is a function of $f(t)$ is defined for $t > 0$ and can be represented as

$$f(s) = L[f(t)]$$

$$F(s) = L[f(t)] = \int_0^\infty f(t) e^{-st} dt$$

problem:- find the Laplace transform of a standard function K ?

$$= \int_0^\infty f(t) e^{-st} dt$$

$$= \int_0^\infty k e^{-st} dt$$

$$= k \int_0^\infty e^{-st} dt$$

$$= -k \left[\frac{e^{-st}}{-s} \right]_0^\infty$$

$$= k \left[0 - \left(\frac{1}{s} \right) \right]$$

$$= \frac{k}{s}$$

$$L[f(t)] = k = \frac{k}{s}$$

find the Laplace transform of e^{at} ?

$$\int_0^\infty e^{at} e^{-st} dt$$

$$\int_0^\infty e^{-t(s-a)} dt$$

$$\left\{ \frac{e^{-t(s-a)}}{-(s-a)} \right\}_0^\infty$$

$$= \frac{e^0}{-(s-a)} - \left\{ \frac{e^{0(s-a)}}{-(s-a)} \right\}$$

$$= \frac{1}{s-a}$$

$$\therefore L[e^{at}] = \frac{1}{s-a}$$

$s-a > 0$

find the Laplace transform of e^{-at} ?

$$\int_0^\infty f(t) e^{-st} dt = \int_0^\infty e^{-at} e^{-st} dt$$

$$= \int_0^\infty e^{-t(s+a)} dt$$

$$= \left\{ \frac{e^{-t(s+a)}}{-t(s+a)} \right\}_0^\infty$$

$$= \frac{e^0}{-0(s+a)} - \left[\frac{e^{0(s+a)}}{-0(s+a)} \right]$$

$$= \frac{1}{s+a}$$

Find $L[\sin at]$

$$\int_0^\infty f(t) e^{-st} dt$$

$$\int_0^\infty \sin at e^{-st} dt$$

$$\int_0^\infty \frac{e^{iat} - e^{-iat}}{2i} e^{-st} dt$$

$$\frac{1}{2i} \left[\int_0^\infty e^{iat} e^{-st} dt - \int_0^\infty e^{-iat} e^{-st} dt \right]$$

$$\frac{1}{2i} L[e^{iat}] - L[e^{-iat}]$$

$$\frac{1}{2i} \left[\frac{1}{(s-i\alpha)} - \frac{1}{s+i\alpha} \right]$$

$$(s-i\alpha)(s+i\alpha)$$

$$s^2 + s\alpha - s\alpha + \alpha^2$$

$$L[\sin at] = \frac{\alpha}{s^2 + \alpha^2}$$

Find $L[\cos at]$

$$\int_0^\infty f(t) e^{-st} dt$$

$$\int_0^\infty \frac{e^{iat} + e^{-iat}}{2i} e^{-st} dt$$

$$\frac{1}{2} \left[\int_0^\infty e^{iat} e^{-st} dt + \int_0^\infty e^{-iat} e^{-st} dt \right]$$

$$\frac{1}{2} \left[\int_0^\infty e^{-t(s-i\alpha)} dt + \int_0^\infty e^{-t(s+i\alpha)} dt \right]$$

$$\cos at = \frac{e^{iat} + e^{-iat}}{2i}$$

$$e^{iat}$$

$$e^{-iat}$$

$$\frac{1}{2} \left[\frac{e^{-t(s-i\alpha)}}{-(s-i\alpha)} \right]_0^\infty$$

$$\frac{1}{2} [L[e^{iat}]$$

$$[e^{ia0}] - [e^{ia\infty}] \frac{1}{2}$$

$$\left[\frac{1}{s-i\alpha} - \frac{1}{s+i\alpha} \right] \frac{1}{2}$$

$$\left[\frac{2s\alpha}{s^2 + \alpha^2} \right] \frac{1}{2}$$

$$L[\cos at] = \frac{s}{s^2 + \alpha^2}$$

$L[\cosh at]$

$$\int_0^\infty f(t) e^{-st} dt$$

$$\frac{1}{2} \int_0^\infty \frac{e^{at} + e^{-at}}{2} e^{-st} dt$$

$$\frac{1}{2} \left[\int_0^\infty e^{at} e^{-st} dt + \int_0^\infty e^{-at} e^{-st} dt \right]$$

$$\frac{1}{2} [L[e^{at}] + L[e^{-at}]]$$

$$\frac{1}{2} \left[\frac{1}{s-a} + \frac{1}{s+a} \right]$$

$$\frac{1}{2} \left[\frac{2s}{s^2 - a^2} \right]$$

$$L[\cosh at] = \frac{s}{s^2 - a^2}$$

$L[\sinh at]$

$$\int_0^\infty f(t) e^{-st} dt$$

$$\frac{1}{2} \int_0^\infty \frac{e^{at} - e^{-at}}{2} e^{-st} dt$$

$$\frac{1}{s} \left[\int_0^{\infty} e^{at} e^{-st} dt - \int_0^{\infty} e^{-at} e^{-st} dt \right]$$

$$\frac{1}{s} \left[L[e^{at}] - L[e^{-at}] \right]$$

$$\frac{1}{s} \left[\frac{1}{s-a} - \frac{1}{s+a} \right]$$

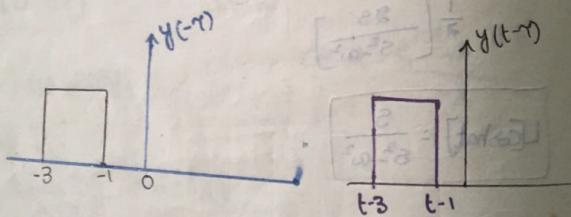
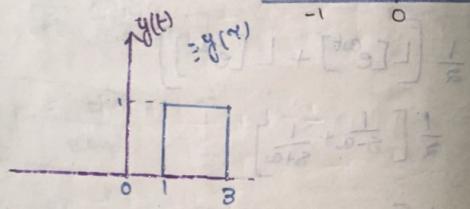
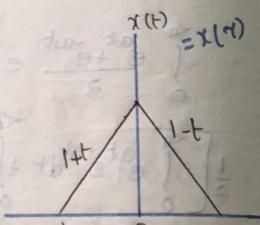
$$\frac{1}{s} \left[\frac{s\omega}{s^2 + \omega^2} \right]$$

$$L[\sinh at] = \frac{a}{s^2 - a^2}$$

Convolute the following functions

$$y(t) = 1 \quad 1 < t < 3 \\ = 0 \quad \text{elsewhere}$$

$$x(t) = 1+t \quad -1 \leq t \leq 0 \\ = 1-t \quad 0 \leq t \leq 1$$



Step 1:-

$$\begin{aligned} z(t) &= \int_{-\infty}^{\infty} x(\tau) y(t-\tau) d\tau \\ &= \int_{-1}^{t-1} (1+\tau) 1 d\tau \\ &= \left[\tau + \frac{\tau^2}{2} \right]_{-1}^{t-1} \\ &= \left[t + \tau + \frac{\tau^2}{2} \right]_{-1}^{t-1} \\ &= t + \left(\frac{(t-1)^2}{2} + \frac{1}{2} \right) \\ &= t + \frac{(t-1)^2 + 1}{2} \end{aligned}$$

Step 2:-

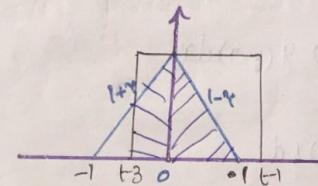
$$\begin{aligned} z(t) &= \int_{-\infty}^{\infty} x(\tau) y(t-\tau) d\tau \\ &= \int_{-1}^0 (1+\tau) 1 d\tau + \int_0^{t-1} (1-\tau) 1 d\tau \\ &= \left[\tau + \frac{\tau^2}{2} \right]_{-1}^0 + \left[\tau - \frac{\tau^2}{2} \right]_0^{t-1} \end{aligned}$$

$$t + \frac{1}{2} - \frac{(t-1)^2}{2}$$

$$0 < t-1 < 1$$

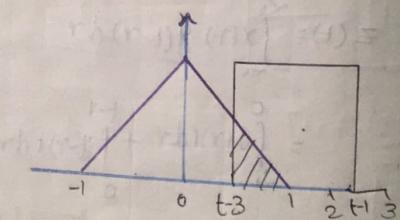
add ① on B.G.

$$1 < t < 2$$



$$\begin{aligned} Z(t) &= \int_{-1}^t x(r) y(t-r) dr \\ &= \int_{-1}^0 (t+r) dr + \int_0^1 (t-r) dr \\ &= \left[\frac{r^2}{2} \right]_{-1}^0 + \left[\frac{r^2}{2} \right]_0^1 \\ &= -t+3 + \frac{(t-3)^2}{2} + t - \frac{1}{2} \\ &= -t + \frac{7}{2} + \frac{(t-3)^2}{2} // \end{aligned}$$

step 48-

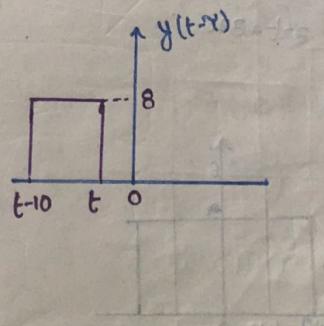
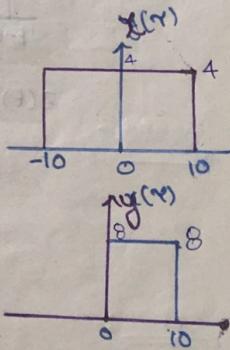


$$\begin{aligned} Z(t) &= \int_{-3}^t x(r) y(t-r) dr \\ &= \int_{-3}^1 (t-r) dr \\ &= \left[r \right]_{-3}^1 - \left[\frac{r^2}{2} \right]_{-3}^1 \\ &= [1-t+3] - \left[\frac{1}{2} - \frac{(t-3)^2}{2} \right] \\ &= 4-t - \frac{1}{2} [t-2] \\ &= \frac{8-2t-t^2}{2} \\ &= \frac{6-3t}{2} // \end{aligned}$$

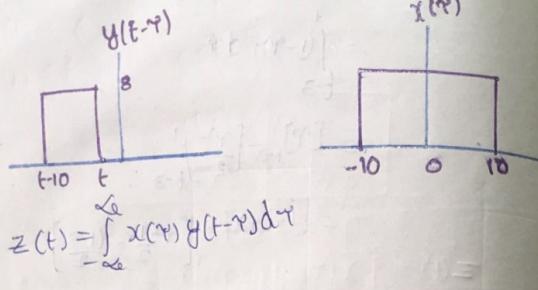
find the convolution of

$$x(t) = 4 \quad -10 \leq t \leq 10$$

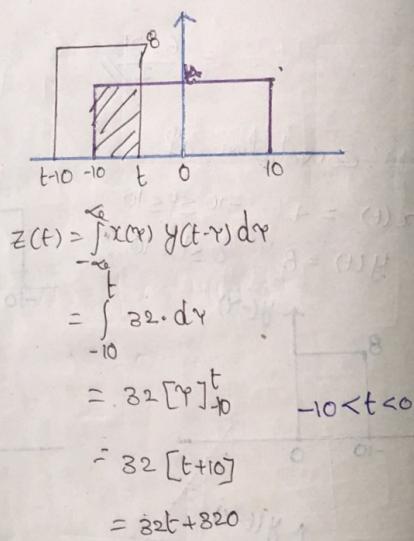
$$y(t) = 8 \quad 0 \leq t \leq 10$$



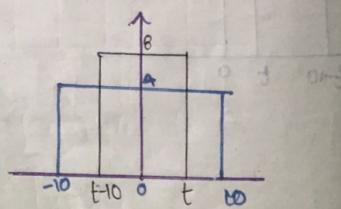
Step 1 8-



Step 28-



Step 38-



$$z(t) = \int_{-\infty}^t x(\tau) y(t-\tau) d\tau$$

$$= \int_{-10}^0 82 d\tau + \int_0^{t-10} 82 d\tau$$

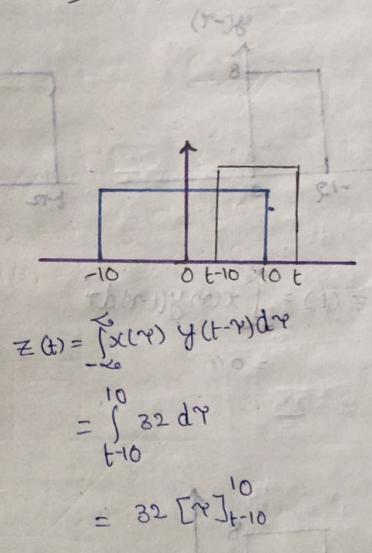
$$= 82[\tau]_0^{t-10} + 82[\tau]_0^t$$

$$= 82[-t+10] + 82[t]$$

$$= -82t + 820 + 82t$$

$$\therefore z(11) = 820$$

Step 48-

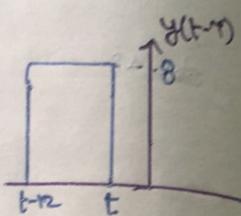
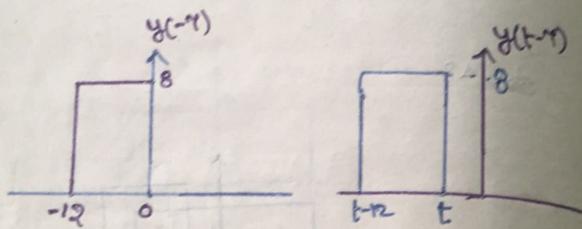
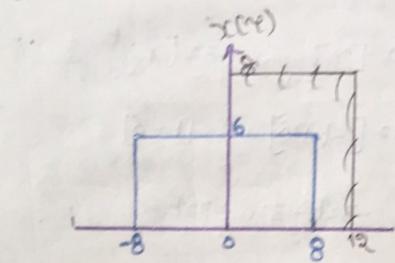


$$= 32[10-t+10]$$
 ~~$= 32[20-t]$~~

~~$= 640 - 32t$~~

$$x(t) = 6$$

$$\begin{aligned} -8 \leq t &\leq 8 \\ 0 \leq t &\leq 12 \end{aligned}$$



$$= 48 \int_{-8}^t$$

$$= 48 [t+8]$$

$$= 48t + 384$$

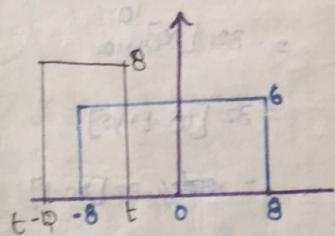
Step 1 :-

$$z(t) = \int_{-\infty}^t x(\tau) y(t-\tau) d\tau$$

$$= 0.5 \int_{-8}^{t-12} 6 \cdot 8 d\tau$$

$$= 0.11$$

Step 2 :-



$$z(t) = \int_{-\infty}^t x(\tau) y(t-\tau) d\tau$$

$$= \int_{-8}^t 48 d\tau$$

Find the Laplace transform for the following function:

$$L[t^{\frac{3}{2}} + \frac{1}{s}]$$

$$L[t^n] = \frac{n!}{s^{n+1}} \quad n \text{ is integer}$$

$$\checkmark L[t^n] = \frac{\sqrt{n+1}}{s^{n+1}} \quad n \text{ is fraction.}$$

$$L[t^{\frac{3}{2}}] + L[t^{-\frac{1}{2}}]$$

$$\frac{\sqrt{\frac{3}{2}+1}}{s^{\frac{3}{2}+1}} + \frac{\sqrt{-\frac{1}{2}+1}}{s^{-\frac{1}{2}+1}}$$

$$\frac{\sqrt{\frac{5}{2}}}{s^{\frac{5}{2}}} + \frac{\sqrt{\frac{1}{2}}}{s^{\frac{1}{2}}} \quad \sqrt{n+1} = \sqrt{n}\sqrt{n+1}$$

$$\frac{\frac{3}{2}\sqrt{\frac{3}{2}+1}}{s^{\frac{5}{2}}} + \frac{\frac{1}{2}\sqrt{\frac{1}{2}}}{s^{\frac{1}{2}}}$$

$$\frac{\frac{3}{2}\sqrt{\frac{1}{2}+1}}{s^{\frac{5}{2}}} + \frac{\sqrt{\pi}}{s^{\frac{1}{2}}}$$

$$\frac{\frac{3}{2}\times\frac{1}{2}\sqrt{\frac{1}{2}}}{s^{\frac{5}{2}}} + \frac{\sqrt{\pi}}{s^{\frac{1}{2}}}$$

$$\boxed{\frac{\frac{3}{4}\sqrt{\pi}}{s^{\frac{5}{2}}} + \frac{\sqrt{\pi}}{s^{\frac{1}{2}}}}$$

Find $L[\sin^2 3t]$

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$$

$$L\left[\frac{1 - \cos 2(3t)}{2}\right]$$

$$= \frac{1}{2} [L[1] - L[\cos 6t]]$$

$$= \frac{1}{2} \left[\frac{1}{s} - \frac{s}{s^2 + 36} \right] = \frac{1}{2} \left[\frac{1}{s} - \frac{3}{s^2 + 36} \right]$$

$$\text{Find } L[t^n] = \frac{n!}{s^{n+1}} \mid \frac{\sqrt{n+1}}{s^{n+1}}$$

$$\int_0^\infty t^n e^{-st} dt \\ st = x \\ dt = \frac{dx}{s}$$

$$\int_0^\infty \frac{x^n}{s^n} e^{-x} dx$$

$$\frac{1}{s^{n+1}} \int_0^\infty x^n e^{-x} dx$$

$$\frac{1}{s^{n+1}} \int_0^\infty x^n e^{-x} dx$$

$$\boxed{\sqrt{n} = \int_0^\infty x^{n+1} e^{-x} dx}$$

$$= \frac{-1}{s^{n+1}} \int_0^\infty x^{n+1} \cdot -e^{-x} dx$$

$$= \frac{\sqrt{n+1}}{s^{n+1}} = \frac{n!}{s^{n+1}}$$

$n = \text{integer}$

find $L[\cos^3 st]$

$$\cos 3\theta = 4\cos^3 \theta - 3\cos \theta$$

$$\cos^3 \theta = \frac{\cos 3\theta + 2\cos \theta}{4}$$

$$\frac{1}{4} L[\cos 3(st) + 3\cos(st)]$$

$$\frac{1}{4} [L[\cos 15t] + 3L[\cos 5t]]$$

$$\boxed{\frac{1}{4} \left[\frac{5}{s^2+15^2} + 3 \left[\frac{5}{s^2+5^2} \right] \right]}$$

find $L[\sin t \cos 2t]$

$$\sin A \cos B = \frac{\sin(A+B) + \sin(A-B)}{2}$$

$$L\left[\frac{\sin(t+2t) + \sin(t-2t)}{2} \right]$$

$$\frac{1}{2} [L[\sin 3t] + L[\sin(-t)]]$$

$$\frac{1}{2} [L[\sin 3t] - L[\sin(t)]]$$

$$\boxed{\frac{1}{2} \left[\frac{3}{s^2+3^2} - \frac{1}{s^2+1^2} \right]}$$

find $L[\sin t \cdot \sin at \cdot \sin bt]$

$$\frac{1}{2} [\cos(at+bt) - \cos(t+at)] \sin bt$$

$$\frac{1}{2} [\cos(bt) - \cos(3t)] \sin at$$

$$\frac{1}{2} [\cos t \sin at - \cos 3t \sin at]$$

$$\frac{1}{2} \left[\frac{\sin(t+3t) - \sin(t-3t)}{2} \right] - \frac{\sin 2(3t)}{2}$$

$$\cos 3\theta = 4\cos^3 \theta - 3\cos \theta$$

$$\cos^3 \theta = \frac{\cos 3\theta + 2\cos \theta}{4}$$

$$\frac{1}{2} \left[\sin 4t + \sin 2t \right] - \sin 6t$$

$$\frac{1}{4} L[\sin 4t] + L[\sin 2t] - L[\sin 6t]$$

$$\frac{1}{4} \left[\frac{4}{s^2+4^2} + \frac{2}{s^2+2^2} - \frac{6}{s^2+6^2} \right]$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$2 \cos A \sin B = \sin(A+B) - \sin(A-B)$$

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

$L[\sin at \sin bt \cos ct]$

$$L\left[\frac{\cos(2t-4t) - \cos(2t+4t)}{2} \times \cos ct \right]$$

$$\frac{1}{2} \left[\cos t - \cos 7t \right] \cos 6t$$

$$\frac{1}{2} L[\cos t + \cos 6t - \cos 7t \cos 6t]$$

$$\frac{1}{2} L\left[\frac{\cos(t+6t) + \cos(t-6t)}{2} \right] - \frac{1}{2} \left[\cos(7t+6t) + \cos(7t-6t) \right]$$

$$\frac{1}{4} \left[L[\cos 7t] + L[\cos t] - L[\cos 13t] - L[\cos 11t] \right]$$

$$\frac{1}{4} \left[\frac{3}{s^2+7^2} + \frac{3}{s^2+1^2} - \frac{3}{s^2+13^2} - \frac{3}{s^2+11^2} \right]$$

Find $L[\cos(At+B)]$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$L[\cos At \cos Bt - \sin At \sin Bt]$

$$\cos B L[\cos At] - \sin B L[\sin At]$$

$$\cos B \cdot \frac{s}{s^2+a^2} - \sin B \cdot \frac{a}{s^2+a^2}$$

Find the Laplace transform for the following functions.

$$e^{-at} \sin bt, e^{-at} \cos bt, e^{At} t^n, e^{-at} \sin t, e^{2t} \cos^2 t$$

1) $L[e^{-at} \sin bt]$

here, e^{-at} multiply with some function gives solution
which is one of the property of Laplace.

$L[\sin bt] \xrightarrow{s \rightarrow s+a}$

$$\frac{b}{s^2+b^2} = \frac{\cancel{s+a}}{\cancel{s+a}} \frac{b}{(s+a)^2+b^2}$$

2) $L[e^{-at} \cos bt]$

$L[\cos bt] \xrightarrow{s \rightarrow s+a}$

$$\frac{s+a}{s^2+b^2}$$

3) $L[e^{at} t^n]$

$$L[t^n] \xrightarrow{s \rightarrow s-a}$$

$$\left[\frac{n!}{s^{n+1}} \right] \xrightarrow{s \rightarrow s-a}$$

$$\frac{n!}{(s-a)^{n+1}} \text{ similarly } \frac{1}{(s-a)^{n+1}} \text{ if } n \text{ is fraction}$$

4) $L[e^{-2t} \cos^2 t]$

$$\cos^2 t = \frac{1 + \cos 2t}{2}$$

$$L[\cos^2 t] \xrightarrow{s \rightarrow s+2}$$

$$L\left[\frac{1 + \cos 2t}{2}\right]$$

$$\frac{1}{2} \left[\frac{1}{s} + \frac{2s}{s^2+4} \right] \xrightarrow{s \rightarrow s+2}$$

$$\frac{1}{2} \left[\frac{1}{s+2} + \frac{s+2}{(s+2)^2+4} \right]$$

Properties of Laplace transformations:-

1. Linearity property :-

$$\text{It can be stated as } L[a f(t) + b g(t)] = a L[f(t)] + b L[g(t)]$$

$$L[a \cdot f(t) + b \cdot g(t)] = a \cdot L[f(t)] + b \cdot L[g(t)]$$

$$\int_0^\infty f(t) e^{-st} dt$$

$$\int_0^\infty [a f(t) + b g(t)] e^{-st} dt$$

$$a \int_0^\infty f(t) e^{-st} dt + b \int_0^\infty g(t) e^{-st} dt$$

$$a \cdot L[f(t)] + b \cdot L[g(t)]$$

2. scaling property :-

$$\text{if Laplace of } f(t) = F$$

$$\text{if } L[f(t)] = F(s) \text{ Then } L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$$

$$= \int_0^\infty f(at) e^{-st} dt$$

$$\begin{aligned} &\text{Put } at=x \\ &t=\frac{x}{a} \\ &dt=\frac{dx}{a} \end{aligned}$$

$$\int_0^\infty f(x) e^{-sx/a} \frac{dx}{a}$$

$$\frac{1}{a} \int_0^\infty f(x) e^{-sx/a} dx$$

$$= \frac{1}{a} F\left(\frac{s}{a}\right)$$

3. Differentiation of Transforms :-

$$L[f(t)] = F(s) \text{ Then } L[t f(t)] = -\frac{d}{ds} F(s)$$

$$F(s) = L[f(t)] \quad \dots$$

$$F(s) = \int_0^\infty f(t) e^{-st} dt$$

D. w.r.t s.

$$\frac{d}{ds} F(s) = \int_0^\infty f(t) \frac{d}{ds} e^{-st} dt$$

$$\frac{d}{ds} F(s) = \int_0^\infty f(t) - t e^{-st} dt$$

$$\frac{d}{ds} F(s) = -t \int_0^\infty f(t) e^{-st} dt$$

$$-\frac{d}{ds} [F(s)] = t \int_0^\infty f(t) e^{-st} dt$$

$$L[t f(t)] = -\frac{d}{ds} F(s)$$

$$\text{Note:- } -\frac{d}{ds} F(s) = L[t f(t)]$$

$$L[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2} F(s)$$

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$$

Integration of Transforms :-

If Laplace transformation of a function exists then the integral of the corresponding transform w.r.t s area in the complex frequency domain is equal to its division by in the time domain

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(u) du \quad \text{or} \quad \int_s^\infty F(u) ds$$

$$F(s) = \int_0^\infty f(t) e^{-st} dt$$

Replace s $\rightarrow u$

$$f(u) = \int_0^\infty f(t) e^{-ut} dt$$

Apply \int_s^∞ on both sides

$$\int_s^\infty F(u) du = \int_s^\infty \int_0^\infty f(t) e^{-ut} dt du$$

$$\int_s^\infty F(u) du = \int_0^\infty f(t) \left[\int_s^\infty e^{-ut} du \right] dt$$

$$= \int_0^\infty f(t) \left[\frac{e^{-ut}}{-t} \right]_s^\infty dt$$

$$= \int_0^\infty f(t) \left[0 - \frac{e^{-st}}{-t} \right] dt$$

$$\int_0^{\infty} f(u)du = \int_0^{\infty} \frac{f(t)}{t} e^{-st} dt$$

$$\Rightarrow \int_0^{\infty} f(u)du = L\left[\frac{f(t)}{t}\right]$$

ROC of Laplace Transformations :-

(Region of Convergence)

steps to be followed

① ROC consists of strips of area parallel to $j\omega$ axis in s-Plane.

Proof:-

① ROC $[X(s)]$ consists of values of s for which Fourier transform $\int_{-\infty}^{\infty} x(t) e^{-pt} dt$ is possible when $x(t) e^{-pt}$ is fully integrable condition depends on p . ROC is strips which is only in the terms of poles of $X(s)$.

Step 2:-
ROC doesn't contain any poles. Values of $X(s)$ is infinite and $X(s)$ has finite values in ROC at one. hence ROC shouldn't contain any poles.

② ROC is entire s-plane for absolutely integrable and finite duration signals.

Proof:- $x(t)$ lies in interval $\int_{t_1}^{t_2} x(t) e^{-pt} dt < \infty$

This integration is finite for any value of P

ROC is entire s-plane.

Integration Property :-

$$L\left\{\int_0^t f(t) dt\right\} = \frac{F(s)}{s}$$

The function $f(t)$ is continuous than the

$$L\left\{\int_0^t f(t) dt\right\} = \frac{F(s)}{s}$$

$$\int_0^t f(t) e^{-st} dt$$

$$u \cdot v = u \int v dt - \int u' v dt$$

$$\int_0^t [f(t) dt] e^{-st} dt$$

$$\int_0^t f(t) dt \int_0^t e^{-st} dt - \int_0^t \frac{d}{dt} (\int_0^t f(t) dt) e^{-st} dt$$

$$\int_0^t f(t) dt \left[\frac{e^{-st}}{-s} \right]_0^\infty - \int_0^t f(t) dt \left[\frac{e^{-st}}{-s} \right]_0^\infty dt$$

$$0 + \frac{1}{s} \int_0^\infty f(t) e^{-st} dt$$

$$\frac{F(s)}{s}$$

Differentiation Property :-

If a function $f(t)$ is piecewise continuous then the Laplace transform of the derivative $\frac{d}{dt} f(t)$ is given by

$$L\left\{f'(t)\right\} = sF(s) - f(0)$$

$$\int_0^\infty f(t) e^{-st} dt$$

$$f'(t) = \frac{d}{dt} f(t)$$

$$\int_0^\infty \frac{d}{dt} f(t) e^{-st} dt$$

$$\int_0^\infty e^{-st} \frac{d}{dt} f(t) dt$$

$$e^{-st} \int_0^t f(t) dt - \int_0^t \frac{d}{dt} e^{-st} \frac{d}{dt} f(t) dt$$

$$\int_0^t e^{-st} f(t) dt - \int_0^t (-s) e^{-st} dt \cdot f(t) dt$$

$$f(0) + s \int_0^t f(t) e^{-st} dt$$

$$= SF(s) - f(0)$$

Note :-

$$L\{f'(t)\} = SF(s) - f(0)$$

$$L\{f''(t)\} = S^2 F(s) - sf(0) - f'(0)$$

$$L\{f'''(t)\} = S^3 F(s) - S^2 f(0) - Sf'(0) - f''(0)$$

$$L\{f^n(t)\} = S^n F(s) - S^{n-1} f(0) - S^{n-2} f'(0) - \dots - f^{(n)}(0)$$

Imp first shifting theorem :- If the function $f(t)$ has transformed $F(s)$ then the Laplace transform of $e^{-at} f(t) \rightarrow F(s+a)$

$$\text{Proof :- } F(s) = \int_0^\infty f(t) e^{-st} dt$$

$$F(s+a) = \int_0^\infty f(t) e^{-(s+a)t} dt$$

$$= \int_0^\infty f(t) e^{-st} e^{-at} dt$$

$$= \int_0^\infty e^{-at} \int_0^t f(t) e^{-st} dt dt$$

$$= e^{-at} \int_0^\infty [f(t)] e^{-st} dt$$

$$= e^{-at} f(t)$$

similarly

$$F(s-a) = L[e^{at} f(t)]$$

second shifting theorem :-

If the function $f(t)$ has the function $f(s)$ Then

Laplace Transform $f(t-a) u(t-a)$ is $e^{-as} f(s)$

Proof :-

$$L\{f(t-a) u(t-a)\} = e^{-as} F(s)$$

$$\int_0^\infty f(t) e^{-st} dt$$

$$\int_0^\infty f(t-a) u(t-a) e^{-st} dt$$

$$f(t-a) u(t-a) < 0$$

$$t-a = \gamma$$

$$t = \gamma + a$$

$$\int_0^\infty f(t-a) e^{-s(t-a)} dt$$

$$\int_0^\infty f(t-a) e^{-s(t-a)} dt$$

$$t-a = \gamma$$

$$\int_0^\infty f(\gamma) e^{-s(\gamma+a)} d\gamma$$

$$\int_0^\infty f(\gamma) e^{-s\gamma} e^{-sa} d\gamma$$

$$e^{-as} \int_0^\infty f(\gamma) e^{-s\gamma} d\gamma$$

$$e^{-as} F(s)$$

initial value theorems :-

If the function $f(t)$ and its derivative are Laplace transformable then we can write as

$$\lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} s F(s)$$

Proof :-

$$\text{w.r.t } L[f'(t)] = s [L(f'(t))] - f(0)$$

$$\int_0^t f'(t) e^{-st} dt = s(F(s)) - f(0)$$

Taking limits as \int_0^∞

$$\lim_{s \rightarrow \infty} \int_0^\infty f'(t) e^{-st} dt = \lim_{s \rightarrow \infty} [sF(s) - f(0)]$$

$$LHS \lim_{s \rightarrow \infty} \int_0^\infty f'(t) e^{-st} dt = 0$$

$$0 = \lim_{s \rightarrow \infty} [sF(s) - f(0)]$$

$$0 = \lim_{s \rightarrow \infty} sF(s) - f(0)$$

$$\lim_{s \rightarrow \infty} sF(s) = f(0)$$

$$\lim_{s \rightarrow \infty} s.F(s) = \lim_{t \rightarrow \infty} f(t)$$

JMP

Final value theorem :-

If $f(t)$ and $f'(t)$ are Laplace transformable then

$$\boxed{\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)}$$

Proof :-

$$L[f'(t)] = sF(s) - f(0)$$

Applying

$$\int_0^\infty f'(t) e^{-st} dt = sF(s) - f(0)$$

Taking $\lim_{s \rightarrow 0}$ on L.H.S

we have

$$\lim_{s \rightarrow 0} \int_0^\infty f'(t) e^{-st} dt = \lim_{s \rightarrow 0} [sF(s) - f(0)]$$

here $\lim_{s \rightarrow 0}$ and $\frac{d}{dt}$ canceled out each other

$$\therefore \int_0^\infty f'(t) dt = \lim_{s \rightarrow 0} [sF(s)] - \lim_{s \rightarrow 0} f(0)$$

$$[f'(t)]_0^\infty = \lim_{s \rightarrow 0} [sF(s)] - \lim_{s \rightarrow 0} f(0)$$

$$\lim_{t \rightarrow \infty} [f(\infty) - f(0)] = \lim_{s \rightarrow 0} sF(s) - \lim_{s \rightarrow 0} f(0)$$

$$\lim_{t \rightarrow \infty} f(t) - \lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow 0} sF(s) - \lim_{s \rightarrow 0} f(t)$$

$$\boxed{\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)}$$

Properties of ROC :-

If $x(t)$ is right sided signals and real value of $s = p_0$ is in ROC then all values of 's' for which real value of s greater than p_0 will also be in ROC.

Find the Laplace transform of unit step function :- draw its ROC

$$U(t) = 1 \quad t > 0 \\ = 0 \quad t < 0$$

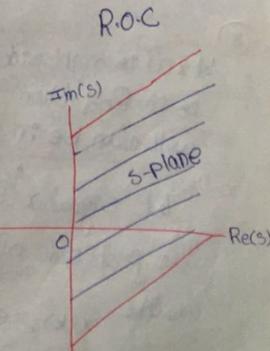
$$(0 \cdot K \cdot T) U(t) = \int_0^\infty f(t) e^{-st} dt$$

$$= \int_0^\infty 1 e^{-st} dt$$

$$= \left[\frac{e^{-st}}{-s} \right]_0^\infty$$

$$= \left[0 - \left(\frac{1}{s} \right) \right]$$

$$= \frac{1}{s}$$



⑤ Laplace transform of unit Ramp function draw its Roc.

w.r.t. Ramp function can be defined as

$$x(t) = t \quad t > 0 \\ = 0 \quad t < 0$$

$$\int_0^t f(t) e^{-st} dt$$

$$\int_0^t t e^{-st} dt$$

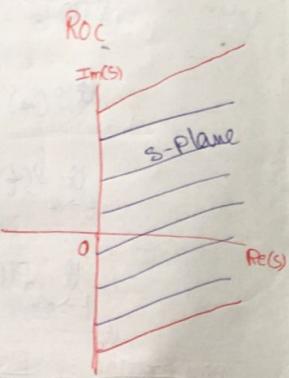
$$\int_0^\infty t e^{at} = \frac{e^{at}}{a^2} (at - 1)$$

$$\left[\frac{e^{-st}}{s^2} (-st - 1) \right]_0^\infty$$

Multiply with $-$

$$\left[\frac{-e^{-st}}{s^2} (st + 1) \right]_0^\infty$$

$$-\frac{1}{s^2} \left[e^{-st} (st + 1) \right]_0^\infty \Rightarrow -\frac{1}{s^2} [0 - 1] = \frac{1}{s^2}$$



⑤ If $x(t)$ is right sided signal and line real value of $s = p_0$ is in Roc then all values of s for which real values $> p_0$ will also be in Roc.

Proof :- Let signal $x(t)$ is right sided then its Roc is also right side of some line parallel to jw axis.

example $x(t) = e^{at} u(t)$

is a right sided signal then its Laplace transform is $S(x) = \frac{1}{s-a}$ with Roc real value of $s > -a$ which is right sided hence if the line at real value of $s = p_0$ then this function

falls in Roc for all the values of 's', real value of $s > p_0$ will also present in Roc.

⑥ let $x(t)$ be left sided signal and the line real value of $s = p_0$ is in Roc then all values of s' for which real values of $s < p_0$ will also in Roc.

Proof :- for Left hand sided signal the Roc is also left side of some line parallel to jw axis hence if real value of $s = p_0$ falls in Roc then all the values of 's' for which real values of $s < p_0$ will lie in Roc.

⑦ if $x(t)$ is Two sided signal and real value of $s = p_0$ is in Roc then Roc will be strip in s-plane that includes line real value of $s = p_0$

Proof :- for both sided sequences Roc lies in the region $\sigma_1 < \text{Re}(s) < \sigma_2$. This Roc is the strip parallel to jw axis in s-plane.

⑧ The Roc is bounded by poles (or) extends to infinity

Proof :- if the function has two poles then Roc the area b/w these two poles parallel to jw axis for Two sided sequence, for single sided sequence the area extends from one pole to infinity but it doesn't include any pole.

⑨ if $x(t)$ is right sided then Roc is region to right of right most pole and if $x(t)$ left side then Roc is region in s-plane to left side of left most pole.

Proof :- It is clear from fact Roc should not any pole left most pole has all the

Poles on its right side and ROC is on left side of left most pole for left sided signal.

⑩ List out all the properties of Laplace transform?

The ROC of $f(s)$ includes the strong lines parallel to imaginary axis in the s -plane. This property has the validity for the ROC of $f(s)$ that consist of values of both real and imaginary parts i.e. $s = \sigma + j\omega$

where σ = real part

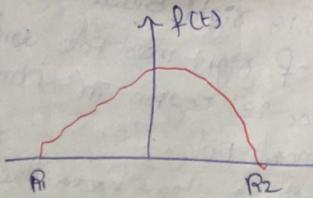
$j\omega$ = imaginary part

and it can be represented as

$$\int_{-\infty}^{\infty} |f(t)| e^{-\sigma t} dt < \infty$$

⑪ The ROC doesn't contain any poles for the values of rational Laplace transform. This is because $f(s)$ is infinite at poles and integral can't converge at this point hence ROC can't contain the values of 's' that are poles.

⑫ if $f(t)$ is absolutely integrable and of finite duration then the ROC is the entire s -plane. The logic behind this property is that zero outside an interval of finite duration as shown in figure.



If $f(t)$ is right sided and if ROC of Real value of $s = \sigma_0$ then all the values of complex variable for which real value of s will also be in ROC

⑬ This property will obtain from the two fundamental points for a right hand side signal the ROC will lie in right half of the s -plane.

⑭ The ROC doesn't contain any poles.

⑮ If $f(t)$ is left sided and if ROC of real value of $s = \sigma_0$ then all values of complex variable s for which real value of $s < \sigma_0$ will also be in ROC

⑯ If the function $f(t)$ is rational then its ROC extends to infinity in addition no poles of $f(s)$ are involved in ROC region

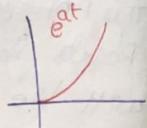
problem 3-

Find the Laplace transform of derivative of signal $x(t)$ invert the transform to find signal derivative

Sol :- Differentiation property proof.

- ② Laplace transform of growing exponential function
 (or)
 find the Laplace transform for the signal shown.

$$= \int_0^{\infty} f(t) e^{-st} dt$$



$$= \int_0^{\infty} e^{at} e^{-st} dt$$

$$= \int_0^{\infty} e^{-(s-a)t} dt$$

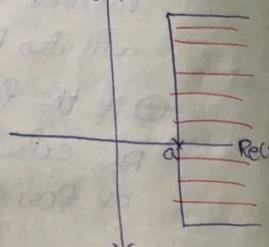
$$= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^{\infty}$$

$$= 0 - \frac{1}{-(s-a)}$$

$$= 0 + \frac{1}{s-a} = \frac{1}{s-a}$$

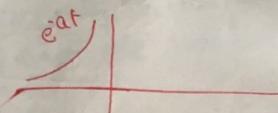
ROC:

$$\begin{matrix} s-a > 0 \\ s > a \\ \text{Im}(s) \end{matrix}$$



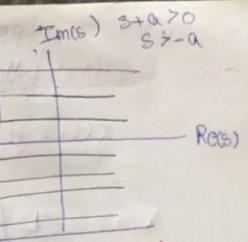
Problem 3: find Laplace transform of decaying function

$$\begin{aligned} &= \int_0^{\infty} f(t) e^{-st} dt \\ &= \int_0^{\infty} e^{-at} e^{-st} dt \\ &= \int_0^{\infty} e^{-t(s+a)} dt \\ &= \left[\frac{e^{-t(s+a)}}{-s-a} \right]_0^{\infty} \end{aligned}$$



$$= 0 + \frac{1}{s+a}$$

$$= \frac{1}{s+a}$$



if $x(t)$ is even function, Prove that $X(s) = X(-s)$
 and if $x(t)$ is odd prove that $X(s) = -X(-s)$

(OR)
 Prove that L [even & odd functions] is even and odd functions respectively.

Sol:-

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$X(-s) = \int_{-\infty}^{\infty} x(t) e^{st} dt$$

$$t = -P$$

$$dt = -dP$$

$$= \int_{\infty}^{-\infty} x(-P) e^{s(-P)} (-dP)$$

$$= - \int_{\infty}^{-\infty} x(-P) e^{-sP} dP$$

$$= -X(-s)$$

Conditions:
 \uparrow
 $E-10 = -X(s)$

1. If $x(t) = x(-t)$

$$X(-s) = \int_{-\infty}^{\infty} x(t) e^{st} dt = X(s)$$

2. If $x(t) = -x(-t)$, $X(-s) = - \int_{-\infty}^{\infty} x(t) e^{st} dt = -X(s)$

Then we can write as

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

Determine the Laplace transform, ROC, pole and zero for the following function.

$$x(t) = e^{-2t} u(t) + e^{-3t} u(t)$$

$$= L[e^{-2t} u(t) + e^{-3t} u(t)]$$

$$= L[e^{-2t} u(t)] + L[e^{-3t} u(t)]$$

$$= \int_0^{\infty} e^{-2t} e^{-st} dt + \int_0^{\infty} e^{-3t} e^{-st} dt$$

$$= \int_0^{\infty} e^{-(s+2)t} dt + \int_0^{\infty} e^{-(s+3)t} dt$$

$$= \left\{ \frac{e^{-(s+2)t}}{-(s+2)} \right\}_0^{\infty} + \left\{ \frac{e^{-(s+3)t}}{-(s+3)} \right\}_0^{\infty}$$

$$= \left\{ 0 - \left(\frac{1}{-(s+2)} \right) \right\} + \left\{ 0 - \left(\frac{1}{-(s+3)} \right) \right\}$$

$$= \frac{1}{s+2} + \frac{1}{s+3}$$

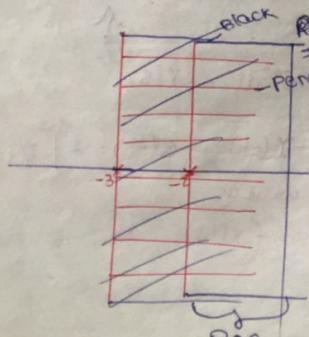
$$\therefore s+2 > 0$$

$$\therefore s > -2$$

$$s+3 > 0$$

$$\therefore s > -3$$

$$s > -2 \\ \text{is ROC}$$



Find the Laplace transform, ROC, poles for the function

$$x(t) = e^{6t} u(t) + e^{9t} u(t)$$

$$= L[e^{6t} u(t) + e^{9t} u(t)]$$

$$= L[e^{6t} u(t)] + L[e^{9t} u(t)]$$

$$= \int_0^{\infty} e^{6t} e^{-st} dt + \int_0^{\infty} e^{9t} e^{-st} dt$$

$$= \int_0^{\infty} e^{-(s-6)t} dt + \int_0^{\infty} e^{-(s-9)t} dt$$

$$= \left\{ \frac{e^{-(s-6)t}}{-(s-6)} \right\}_0^{\infty} + \left\{ \frac{e^{-(s-9)t}}{-(s-9)} \right\}_0^{\infty}$$

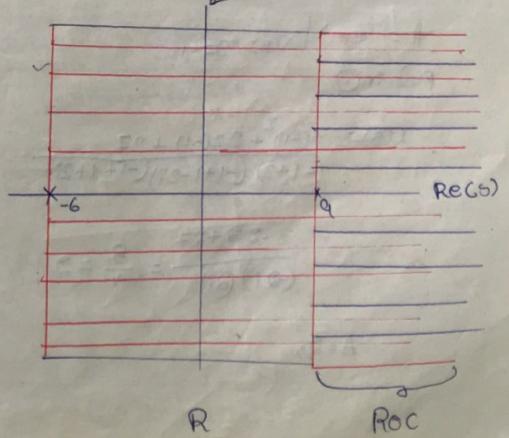
$$= \left\{ 0 - \left[\frac{1}{-(s-6)} \right] \right\} + \left\{ 0 - \left[\frac{1}{-(s-9)} \right] \right\}$$

$$= \frac{1}{s+6} + \frac{1}{s+9}$$

$$\therefore s+6 > 0 \quad s-9 > 0$$

$$s > -6 \quad s > 9$$

$$\therefore s > 9$$



a find signal $x(t)$ that a Laplace given

$$X(s) = \frac{3s^2 + 22s + 27}{(s^2 + 2s + 2)(s^2 + 2s + 5)}$$

$$X(s) = \frac{3s^2 + 22s + 27}{(s+2)(s+1)(s^2 + 2s + 5)}$$

$$s^2 + 2s + 5 = \alpha x^2 + bx + c$$

$$-b \pm \sqrt{b^2 - 4ac}$$

$$= \frac{-2 \pm \sqrt{4 - 20}}{2}$$

$$= \frac{-2 \pm \sqrt{16}}{2}$$

$$= \frac{-2 \pm 4i}{2}$$

$$= -1 \pm 2i$$

$$s = -1 + 2i, -1 - 2i$$

$$X(s) = \frac{3s^2 + 22s + 27}{(s+1)(s+2)(s+1-2i)(s+1+2i)}$$

$$= \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+1-2i} + \frac{D}{s+1+2i}$$

$$A = (s+1) | X(s), s = -1$$

Put in ①

$$X(s) = \frac{3(-1)^2 + 22(-1) + 27}{(-1+2)(-1+1-2i)(-1+1+2i)}$$

$$= \frac{3 - 2i + 27}{(-2i)(2i)} = \frac{8}{4} = 2$$

$$A = 2$$

$$\begin{aligned} & s^2 + 2s + 2 \\ & s^2 + 2s + 8 + 2 \\ & s(s+2) + (Cs+2) \\ & (s+1)(s+2) \end{aligned}$$

$$\text{Now } B = (s+2) | X(s); s = -2$$

Put $s = -2$ in eqn ①

$$\frac{3(-2)^2 + 22(-2) + 27}{(-2+1)(-2+1-2i)(-2+1+2i)}$$

$$= \frac{3(4) - 44 + 27}{(-1)(-1-2i)(-1+2i)}$$

$$= \frac{12 - 44 + 27}{(-1)(-1-4i^2)}$$

$$= \frac{-5}{-5} = 1$$

$$\therefore B = 1$$

$$\text{Now } C = (s+1-2i) | s = -1+2i$$

Put in eqn ①

$$\frac{3(-1+2i)^2 + 22(-1+2i) + 27}{(-1+2i+1)(-1+2i+2)(-1+2i+1+2i)}$$

$$\frac{3(1+4i^2 - 4i) - 22 + 44i + 27}{2i(2i+1)(4i)}$$

$$\frac{3 - 12 - 12i - 22 + 44i + 27}{(-4+2i)(4i)}$$

$$\frac{4+82i}{-16i+8i^2} = \frac{4+82i}{-16i-8}$$

$$\text{Now } D = (s+1+2i) | X(s); s = -1-2i$$

Put in eqn ①

$$\frac{3(-1-2i)^2 + 22(-1-2i) + 27}{(-1-2i+1)(-1-2i+2)(-1-2i+1-2i)}$$

$$= \frac{3(1-4i) - 22-44i+27}{(-2i)(-2i+1)(-4i)}$$

$$= \frac{3(-3-4i) - 22-44i+27}{-8(-2i+1)}$$

$$= \frac{3(-7) - 44i+5}{-8(-2i+1)}$$

$$= \frac{-21-44i+5}{-16i-8}$$

Q. find the inverse Laplace of $X(s) = \frac{1+e^{-2s}}{s^2+2s+5}$

$$\frac{1}{s(s+2)} = \frac{A}{s} + \frac{B}{(s+2)}$$

$$1 = A(s+2) + B(s)$$

$$s=0$$

$$1 = 2A$$

$$A = \frac{1}{2}$$

Calculate constant co-eff. of s.

$$0 = 2A + B$$

$$0 = 3(\frac{1}{2}) + B$$

$$B = -\frac{3}{2}$$

$$\frac{Y_2}{s} + \frac{-\frac{3}{2}}{(s+2)} + e^{-2s} \frac{\frac{1}{2}}{s} + e^{-2s} \frac{-\frac{3}{2}}{(s+2)}$$

$$\frac{\frac{1}{2}}{s} + \frac{-\frac{3}{2}}{s(s+2)} + e^{-2s} \frac{Y_2}{s} + e^{-2s} \frac{-\frac{3}{2}}{s(s+2)}$$

Inverse bilateral Transform

$$\left\{ \frac{1}{2} u(t) - \frac{1}{2} e^{-2s} t \right\} + e^{-2s} u(t-2) + (-\frac{3}{2}) e^{-2s} t$$

⑥ The inverse Laplace transform of a signal is given by $X(s) = \left\{ \frac{w_n^2}{s^2 + 2s w_n + w_n^2} \right\}$

where w_n or z is natural frequency and damping factor for what value of $z = -1 < 2 < 1$ can be Fourier transformed to completed Laplace Transform.

$X(s)$ can be solved as

$$X(s) = \frac{ce^2}{(s-p_1)(s-p_2)}$$

where p_1, p_2 are the two poles and w_n = natural frequency and ξ = damping factor.

(i) both poles p_1 & p_2 lies on the real axis for damping factor $\xi > 1$ This factor is essentially a product of first two order terms as a result in which $X(s)$ decreases monotonically and magnitude of ω increases. Thus $X(s)$ varies from ∞ to 0 .

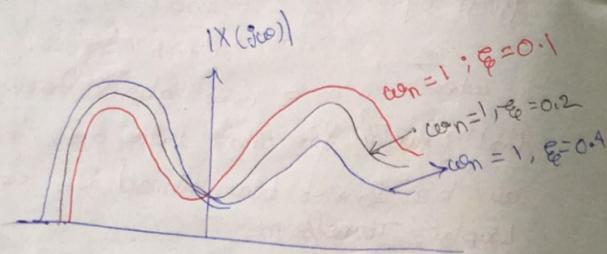
(ii) $s=j\omega$ increases monotonically as ω increases from 0 to ∞ and the angle of these vectors also increases from 0 to $\pi/2$ By absorbing the length of vectors from each of the two poles.

(iii) consider the pole vector for damping ratio $\xi > 1$ Then the length and angle for the pole from the

imaginary axis and less effective to the changes in ω . Then the length and angle of the vector for the pole close to imaginary axis.

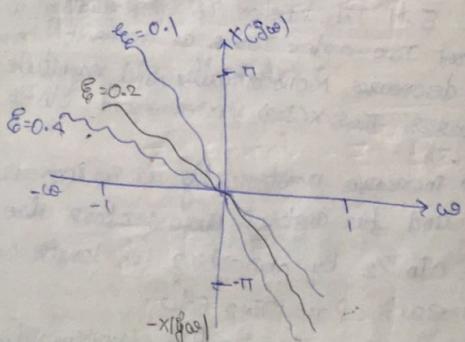
(iv) consider P_1, P_2 are complex for damping factor $\xi < 1$, as a result the step response and impulse response of this function have oscillating parts.

(v) the frequency response is drawn for several values of damping factor ξ and natural frequency $\omega_n = 1$



Magnitude and phase of frequency response for second order system with damping factor

$$0 < \xi < 1 \text{ the range of the frequency is } \omega_n \sqrt{1 - \xi^2} < \omega < \omega_n \sqrt{1 - \xi^2 + \xi \omega_n}$$



Z-Transforms- Unit VIII

There are two types of signals.

i. Continuous signal $f(t)$

ii. Discrete signal $f(n)$

where $n=0, 1, 2, 3$

Definition:

Let $f(n)$ be a sequence defined for $n=-\infty, 0, 1, 2, 3, \dots$

Then the Z-transform of $f(n)$ is defined as

$$\sum_{n=-\infty}^{\infty} f(n) z^{-n}$$

and is known as bilateral or two-sided Z-transform and is denoted by $Z\{f(n)\}$ or $F(z)$, where 'z' is a complex variable.

If the $f(n)$ is a causal sequence for which $f(n)=0$ when $n < 0$, the Z-transform of $f(n)$ is known as unilateral or one-sided Z-transform.

$$\therefore Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n}$$

Z-transform of a sequence is defined only for certain values of 'z' and the region in Z-plane in which Z-transform is defined is known as region of convergence.

For example

$$1 + \frac{1}{z} + \frac{1}{z^2} + \dots$$

for example

$$\left| \frac{1}{z} \right| < 1$$

$$|z| > 1$$

Properties of Z-transforms-

Linearity Property:-

Z-transform is linear or prove

prove the Z-transform property

$$Z\{a \cdot f(n) + b \cdot g(n)\} = a \cdot Z\{f(n)\} + b \cdot Z\{g(n)\}$$

$$\text{RHS} \Rightarrow \sum_{n=-\infty}^{\infty} f(n) z^{-n}$$

$$\sum_{n=-\infty}^{\infty} [a \cdot f(n) + b \cdot g(n)] z^{-n}$$

$$\sum_{n=-\infty}^{\infty} [a \cdot f(n) z^{-n} + b \cdot g(n) z^{-n}]$$

$$a \left[\sum_{n=-\infty}^{\infty} f(n) z^{-n} \right] + b \left[\sum_{n=-\infty}^{\infty} g(n) z^{-n} \right]$$

$$a \cdot Z\{f(n)\} + b \cdot Z\{g(n)\}$$

Time shifting property:-

$$Z\{f(n-n_0)\} = z^{-n_0} F(z)$$

it can be written as

$$Z\{f(n-n_0)\} = z^{-n_0} F(z)$$

$$\sum_{n=0}^{\infty} f(n-n_0) z^{-n}$$

$$\begin{aligned} & \text{assume } m = n - n_0 \\ & m = m + n_0 \end{aligned}$$

$$\sum_{m=n-n_0}^{\infty} f(m) z^{-(m+n_0)}$$

$$\sum_{m=n-n_0}^{\infty} f(m) z^{m-n_0}$$

$$\left[\sum_{m=0}^{\infty} f(m) z^m \right] z^{-n_0}$$

$$Z\{F(z)\} z^{-n_0}$$

$$= \bar{F}(z) z^{-n_0}$$

Hence proved,

frequency shifting property:-

$$Z\{a^n f(n)\} = \bar{F}(z/a)$$

$$Z\{a^n f(t)\} = \bar{F}(z/a)$$

$$Z\{e^{at} f(t)\} = \bar{F}(z \cdot e^{at})$$

$$\sum_{n=0}^{\infty} a^n f(n) z^{-n} = \{a^n f(n)\} = \bar{F}(z/a) \quad \text{--- ①}$$

$$\left[\sum_{n=0}^{\infty} a^n f(n) \left(\frac{z}{a} \right)^{-n} \right]$$

$$\bar{F}(z/a) - ①$$

$$Z\{a^n f(t)\} = \bar{F}(z/a)$$

$$\sum_{n=0}^{\infty} a^n f(nt) z^{-n}$$

$$\sum_{n=0}^{\infty} f(nt) z^{-n} = \bar{F}(z/a)$$

$$z \{ e^{-at} f(t) \} = \bar{f}(z \cdot e^{at})$$

$$\sum_{n=0}^{\infty} e^{-at} f(n) z^{-n}$$

$$\sum_{n=0}^{\infty} f(n) (z \cdot e^{at})^{-n}$$

$$\boxed{z \{ e^{-at} f(t) \}} = \bar{f}(z \cdot e^{at}) \quad \text{--- (3)}$$

Prooved

Time reversed property for Bilateral transform.

$$z \{ f(-n) \} = \bar{f}(\frac{1}{z})$$

$$\sum_{n=-\infty}^{\infty} f(-n) z^{-n}$$

$$p+q+m = n$$

$$\sum_{m=-n}^{\infty} f(m) z^m$$

$$\sum_{m=-n}^{\infty} f(m) (\frac{1}{z})^m$$

$$\boxed{z \{ f(-n) \} = \bar{f}(\frac{1}{z})}$$

Hence Prooved

Differentiation property

$$\boxed{z \{ n \cdot f(n) \} = -z \frac{d}{dz} \bar{f}(z)}$$

$$\boxed{z \{ n \cdot f(t) \} = -z \frac{d}{dz} \bar{f}(z)}$$

$$\bar{f}(z) = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$\bar{f}(z) = \sum_{n=0}^{\infty} f(n) z^{-n} \quad \text{--- (1)}$$

Do w.r.t. z on B.S

$$\frac{d}{dz} \bar{f}(z) = \sum_{n=0}^{\infty} \frac{d}{dz} f(n) z^{-n}$$

$$\frac{d}{dz} (\bar{f}(z)) = \sum_{n=0}^{\infty} f(n) -n z^{-n-1}$$

$$\frac{d}{dz} (\bar{f}(z)) = \sum_{n=0}^{\infty} f(n) (-n) z^{-n-1}$$

$$\frac{d}{dz} (\bar{f}(z)) = -\frac{1}{z} \sum_{n=0}^{\infty} f(n) (n) z^{-n}$$

Multiply with -1

$$-z \frac{d}{dz} (\bar{f}(z)) = \sum_{n=0}^{\infty} n f(n) z^{-n}$$

$$\boxed{-z \frac{d}{dz} \bar{f}(z) = z \{ n \cdot f(n) \}}$$

Hence Prooved.

$$z \{ n \cdot f(t) \} = -z \frac{d}{dz} \bar{f}(z)$$

$$\bar{f}(z) = \sum_{n=0}^{\infty} f(n) z^{-n} \quad \text{--- (1)}$$

Do w.r.t. z on B.S

$$\frac{d}{dz} \bar{f}(z) = \sum_{n=0}^{\infty} f(n) \frac{d}{dz} z^{-n}$$

$$= -\sum_{n=0}^{\infty} f(n) (-n) z^{-n-1}$$

$$\frac{d}{dz} \bar{f}(z) = -\frac{1}{z} \sum_{n=0}^{\infty} f(n) (-n) z^{-n}$$

Multiply with -1 on B.S

$$-\sum_{n=0}^{\infty} n \cdot f(n) = -z \frac{d}{dz} \bar{f}(z) = z \{ n \cdot f(n) \}$$

Hence Prooved

Proove the initial value Theorem :- in Z-transform
it can be written as

$$z \{ f(n) \} = \bar{f}(z); \text{ Then } f(0) = \lim_{z \rightarrow \infty} \bar{f}(z)$$

$$\sum_{n=0}^{\infty} f(n) z^{-n}$$

$$\sum_{n=0}^{\infty} f(n) \left(\frac{1}{z}\right)^n$$

$$= f(0)$$

$$\lim_{n \rightarrow \infty} \bar{f}(z) = f(0)$$

Final value Theorem of Z-transform if $\bar{f}(z) = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$\text{if } \bar{f}(z) = \bar{f}(z), \text{ then } \lim_{z \rightarrow 1} (z-1) \bar{f}(z) = \lim_{n \rightarrow \infty} f(n)$$

$$z \{ f(n) \} = \bar{f}(z) \text{ then } \lim_{z \rightarrow 1} (z-1) \bar{f}(z) = \lim_{n \rightarrow \infty} f(n)$$

Assume

$$\sum_{n=0}^{\infty} f(n+1) z^{-n}$$

$$m=n+1$$

$$n=m-1$$

$$= \sum_{m=1}^{\infty} f(m) \frac{z^{-(m-1)}}{z}$$

$$= \sum_{m=1}^{\infty} f(m) z^{-m} z$$

$$z \{ f(n+1) \} = z \sum_{m=1}^{\infty} f(m) z^{-m} + f(0) z^0 - f(0) z^0$$

$$z \{ f(n+1) \} = z \{ f(z) \} - f(0) \quad \dots \textcircled{1}$$

add $-f(0)$ on L.S

$$z \{ f(n+1) \} - \bar{f}(z) = z \{ f(z) \} - \bar{f}(z) - f(0)$$

$$\bar{f}(z) = z \{ f(z) \}$$

$$z \{ f(n+1) \} - z \{ f(z) \} = z \{ \bar{f}(z) \} - f(0) - \bar{f}(z)$$

$$z \{ f(n+1) - f(n) \} = \bar{f}(z) [z-1] - f(0)$$

Taking $\lim_{z \rightarrow 1}$ on R.S we get

$$\lim_{z \rightarrow 1} z \{ f(n+1) - f(n) \} = \lim_{z \rightarrow 1} \{ \bar{f}(z) [z-1] - f(0) \}$$

$$\lim_{z \rightarrow 1} \bar{f}(z) (z-1) - f(0) = \lim_{N \rightarrow \infty} \sum_{n=0}^{\infty} \{ f(n+1) - f(n) \}$$

$$\lim_{z \rightarrow 1} \bar{f}(z) (z-1) \bar{f}(z) = \lim_{N \rightarrow \infty} f(n)$$

Hence Prooved

Proove the convolution theorem in Z-transform :-

It states that if $\bar{f}(z) = \sum_{n=0}^{\infty} f(n) z^{-n}$

Then

$$z \{ f(n) * g(n) \} = \bar{f}(z) \bar{g}(z)$$

$$f(n) * g(n) = \sum_{\sigma=-\infty}^{\infty} f(\sigma) g(n-\sigma) \quad \text{--- (1)}$$

$$\sum_{n=-\infty}^{\infty} f(n) z^{-n}$$

$$\sum_{n=-\infty}^{\infty} f(n) * g(n) z^{-n}$$

$$\sum_{n=-\infty}^{\infty} \left\{ \sum_{\sigma=-\infty}^{\infty} f(\sigma) g(n-\sigma) \right\} z^{-n}$$

Interchanging the integral

$$\sum_{\sigma=-\infty}^{\infty} f(\sigma) \left\{ \sum_{n=-\infty}^{\infty} g(n-\sigma) \right\} z^{-n}$$

$$\sum_{\sigma=-\infty}^{\infty} f(\sigma) \left\{ \sum_{n=\sigma}^{\infty} g(n-\sigma) \right\} z^{-n}$$

$$m = n - \sigma \quad n = m + \sigma$$

$$\sum_{\sigma=-\infty}^{\infty} f(\sigma) \left\{ \sum_{m=\sigma}^{\infty} g(m) \right\} z^{-(m+\sigma)}$$

$$\sum_{\sigma=-\infty}^{\infty} f(\sigma) z^{-\sigma} \quad \sum_{m=\sigma}^{\infty} g(m) z^{-m}$$

$$f(z) * g(z)$$

$$z \{ f(n) * g(n) \} = f(z) * g(z)$$

Hence Proved

* write the differences between L, Fourier, Z-Transforms

1. Laplace Transformation is an extension of continuous time Fourier transform.
2. The broader broad class of signals, there are many signals for which Fourier transform doesn't converge but Laplace Transform does converge.

3. The Laplace transform allow us to perform transform analysis of unstable systems and to develop additional insight and tool for LTI system Analysis

4. Z-Transform is discrete time counterpart of Laplace transform.

5. Laplace Transform is meant for continuous time signals and Z-transform is meant for discrete time signal.

6. To correlate Z-transform and Laplace Transform Sampling Theorem is used as a bridge b/w continuous and discrete signals.

7. In continuous time system Laplace Transformation is considered as a generalization of F.T similarly it is possible to generate F.T for discrete time signals and systems

8. Z-transform play a vital role in Analysis of representation of discrete time LTI system

9. Discrete time F.T can be applied only to stable system since it exists only in the impulse response is absolutely summable whereas Z-transform of impulse exist for unstable system Thus Z-transform can be used to study much larger class of systems and signals.

10. As in continuous time the discrete time complex exponentials are eigen functions of LTI system which provide the Z-transform with powerful set of properties for Analysis of signals and systems

11. The primary role of Z-transform in engg. Study of system characteristics and derivation

of computational structures for implementing
discrete-time signal applications

12. unilateral z-transform is used to solve d.e
subject to initial conditions.

13. limits are one side in L.T. but two sided
in f.t.

14. L.T of any function $f(t)$ equals to $\mathcal{L}[f(t)]$
multiplied by convergence factor e^{-st} , if
 $f(t) = 0$ where $t < 0$ then we can write all the
eqn's as

$$\mathcal{L}[f(t)] = \int_0^\infty f(t) e^{-st} dt \quad \text{if } f(t) = 0; t < 0$$

$$F.T F(j\omega) = \int_{-\infty}^0 f(t) e^{j\omega t} dt$$

$$L.T F(s) = \int_{-\infty}^s f(t) e^{-st} dt$$

$$z \{f(z)\} = \sum_{n=0}^{\infty} f(n) z^{-n}$$

similarly.

Inverse F.T

$$f(t) = \frac{1}{2\pi j} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega$$

Inverse L.T

$$f(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s) e^{st} ds$$

Inverse integral of Z.T

$$z^{-1} \{x(z)\} = x(k) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} x(z) z^{-k-1} dz$$

sequences, Their ROC in z-transforms

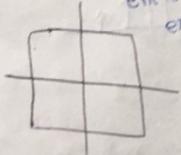
Sequences

causal

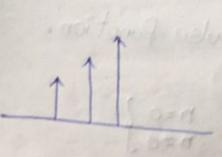


ROC

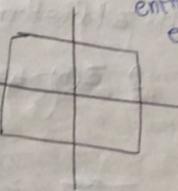
entire z-plane
except $z=0$



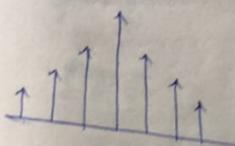
non causal:



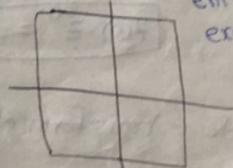
entire z-plane
except $z=\infty$



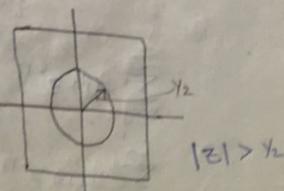
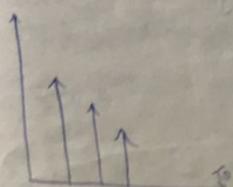
Two-sided signal

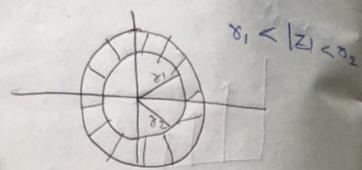
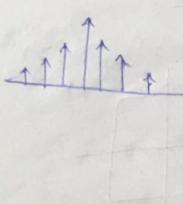
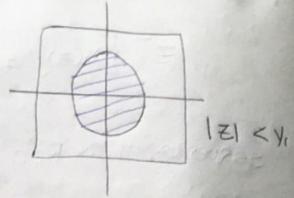


entire z-plane
except
 $z=0$
 $z=\infty$



Infinite eqns.





Find the Z-transform of unit impulse function.

$$f(n) = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases}$$

$$\sum_{n=0}^{\infty} f(n) z^{-n}$$

$$\sum_{n=0}^{\infty} \delta(n) z^{-n}$$

$$f(0) \cdot z^0 = 1$$

Find the Z-transform of unit step function

$$u(n) = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$$

$$\sum_{n=0}^{\infty} f(n) z^{-n}$$

$$\sum_{n=0}^{\infty} u(n) z^{-n}$$



$$\begin{aligned} \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n &= 1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \dots \\ &= \frac{1}{1 - \left(\frac{1}{z}\right)} = \frac{z}{z-1} \quad |z| > 1 \end{aligned}$$

Find the Z-transform of constant function.

$$\sum_{n=0}^{\infty} f(n) z^{-n}$$

$$\sum_{n=0}^{\infty} k z^{-n}$$

$$k \sum_{n=0}^{\infty} z^{-n}$$

$$k \left\{ 1 + \frac{1}{z} + \frac{1}{z^2} + \dots \right\}$$

$$k \left\{ \frac{z}{z-1} \right\}$$

Find the Z-transform of an

$$\sum_{n=0}^{\infty} f(n) z^{-n}$$

$$\sum_{n=0}^{\infty} a^n z^{-n}$$

$$\sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

$$1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \dots$$

$$= \frac{1}{1 - \left(\frac{a}{z}\right)} = \left| \frac{z}{za} \right| ; |z| > |a|$$

Find the Z-transform of $z^k (-1)^n$

$$\sum_{n=0}^{\infty} f(n) z^{-n}$$

$$\sum_{n=0}^{\infty} (-1)^n z^{-n}$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n$$

$$-1 + \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \dots = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots$$

$$\frac{1}{1 - \left(\frac{1}{z}\right)} = \left|\frac{z}{z+1}\right| = |z|$$

$$z\{e^{at}\}$$

$$\sum_{n=0}^{\infty} f(n) z^n$$

$$\sum_{n=0}^{\infty} e^{at} z^{-n}$$

$$\sum_{n=0}^{\infty} \frac{e^{at}}{z} z^{-n}$$

$$\sum_{n=0}^{\infty} \left(\frac{e^{at}}{z}\right)^n$$

$$1 + \frac{e^{at}}{z} + \left(\frac{e^{at}}{z}\right)^2 + \dots$$

$$\frac{1}{1 - \frac{e^{at}}{z}} = \left|\frac{z}{z - e^{at}}\right|$$

$$z\{a^{n-1}\}$$

$$\sum_{n=0}^{\infty} f(n) z^n$$

$$\sum_{n=1}^{\infty} a^{n-1} z^{-n}$$

$$\sum_{n=1}^{\infty} \frac{a^{n-1}}{z} z^{-n}$$

$$\sum_{n=1}^{\infty} a^{n-1} z^{-n}$$

$$a^0 z^0 + a^1 z^{-1} + a^2 z^{-2} + a^3 z^{-3} + \dots$$

$$\frac{a^0 + \frac{a^1}{z}}{1 - \frac{a^1}{z}} = \frac{1}{z} + \frac{a^0}{z^2} + \frac{a^2}{z^3} + \dots$$

$$\frac{1}{z} \left\{ 1 + \frac{a_0}{z} + \frac{a_1^2}{z^2} + \dots \right\}$$

$$\frac{1}{z} \left\{ \frac{1}{1 - \frac{a_1}{z}} \right\} = \frac{1}{z} \left\{ \frac{z}{z - a_1} \right\} = \frac{1}{z - a_1} \quad |z| > a_1$$

$$\text{Find } z\{n\}$$

$$\sum_{n=0}^{\infty} f(n) z^n$$

$$\sum_{n=0}^{\infty} n a z^{-n}$$

$$\sum_{n=0}^{\infty} n z^{-n}$$

$$\frac{1}{z} + \frac{3}{z^2} + \frac{3}{z^3} + \dots$$

$$\frac{1}{z} \left\{ 1 + \frac{3}{z} + \frac{3}{z^2} + \dots \right\}$$

$$\frac{1}{z} \left\{ 1 - \frac{1}{z} \right\}^{-2}$$

$$\frac{1}{z} \left\{ z-1 \right\}$$

$$\frac{1}{z} \left\{ 1 - \frac{1}{z} \right\}^{-2}$$

$$\frac{1}{z} \left\{ \frac{1}{(1 - \frac{1}{z})^2} \right\}$$

$$\frac{1}{z} \left\{ \frac{z^2}{(z-1)^2} \right\} = \frac{z}{(z-1)^2}$$

$|z| > 1$

find $z \{ n^2 \}$

$$\sum_{n=0}^{\infty} f(n) z^{-n}$$

$$(z - 1)^{-1} z \{ n^2 \} = -z \frac{d}{dz} \left(\frac{z}{z-1} \right)^2 \left\{ \frac{z}{z-1} \right\}$$

$$= -z \left\{ \frac{(z-1)^2 (1) - z(z)(z-1)}{(z-1)^4} \right\}$$

$$= -z \left\{ \frac{(z-1)(z-1) - 2z^2}{(z-1)^3} \right\}$$

$$= -z \left\{ \frac{(z-1) - 2z^2}{(z-1)^3} \right\}$$

$$= -z \left\{ \frac{z-1}{(z-1)^3} \right\}$$

$$= -z \left\{ \frac{1}{(z-1)^2} \right\}$$

$$= \frac{z}{(z-1)^2}$$

$$z \{ n^2 \} = z \left\{ \frac{z+1}{(z-1)^3} \right\}$$

* find the $z \{ \cos ne \}$

$$\text{Consider } z \{ (re^{i\theta})^n \} = \left[\frac{z}{z-re^{i\theta}} \right]$$

$$z \{ r^n (\cos \theta + i \sin \theta)^n \} = \frac{z}{z-r(\cos \theta + i \sin \theta)}$$

$$z \{ r^n (\cos \theta + i \sin \theta)^n \} = \frac{z}{(z-r(\cos \theta + i \sin \theta)) - ri \sin \theta}$$

Denominator on R.H.S

$$= \frac{z}{(z-r(\cos \theta + i \sin \theta)) - ri \sin \theta} \times \frac{(z-r(\cos \theta + i \sin \theta)) + ri \sin \theta}{(z-r(\cos \theta + i \sin \theta)) + ri \sin \theta}$$

$$= z \frac{[(z-r(\cos \theta + i \sin \theta)) + ri \sin \theta]}{(z-r(\cos \theta + i \sin \theta))^2 + r^2 \sin^2 \theta}$$

$$= z \frac{[z-r(\cos \theta + i \sin \theta)] + z(i \sin \theta)}{z^2 + r^2 \cos^2 \theta - 2zr \cos \theta + r^2 \sin^2 \theta}$$

$$= \frac{z [z-r(\cos \theta) + z(i \sin \theta) \sin \theta]}{z^2 - 2zr \cos \theta + r^2}$$

$$z \{ r^n \cos n\theta + i \sin n\theta \} = \frac{z [z-r(\cos \theta) + z(i \sin \theta) \sin \theta]}{z^2 - 2zr \cos \theta + r^2}$$

Composite real and imaginary parts

$$z \{ r^n \cos n\theta \} = \frac{z [z-r(\cos \theta)]}{z^2 - 2zr \cos \theta + r^2} = \frac{z [z-r(\cos \theta)]}{z^2 - 2zr \cos \theta + r^2}$$

$$z \{ r^n \sin n\theta \} = \frac{z i \sin \theta}{z^2 - 2zr \cos \theta + r^2} = \frac{z i \sin \theta}{z^2 - 2zr \cos \theta + r^2}$$

Put $\theta = \pi/2$

$$z \{ r^n \cos n\theta \} = \frac{z [z - r \cos(\pi/2)]}{z^2 - 2zr \cos(\pi/2) + r^2} = \frac{z^2}{z^2 + 1}$$

$$z \{ r^n \sin n\theta \} = \frac{i z \sin(\pi/2)}{z^2 - 2zr \cos(\pi/2) + r^2} = \frac{z}{z^2 + 1}$$

$$f_{\text{find}} = \{e^{at} \cos bt\}$$

$$z\{f(t)e^{-at}\} = f(z e^{at})$$

$$z\{\cos bt\} = \left| \frac{z[z - \cos bt]}{z^2 - 2z \cos bt + 1} \right| \quad z \rightarrow z e^{at}$$

$$\frac{ze^{at}(z \cdot e^{at} - \cos bt)}{(ze^{at})^2 - 2ze^{at} \cos bt + 1}$$

Find Z-transform, sketch ROC for the sequence

$$x(n) = u(n)$$

$$\sum_{n=0}^{\infty} u(n) z^{-n}$$

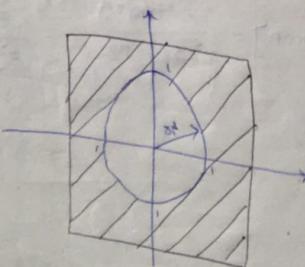
$$\sum_{n=0}^{\infty} 1 z^{-n}$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n$$

$$1 + \frac{1}{z} + \frac{1}{z^2} + \dots$$

$$\frac{1}{1 - \left(\frac{1}{z}\right)} = \frac{|z|}{|z| - 1} = |z| > 1$$

ROC :-



Inverse Z-transform :-

If $\tilde{F}(f(z)) = f(z)$ Then $\tilde{z}^{-1}[f(z)]$ can be found out by the following methods.

These are four methods to find \tilde{z}^{-1}

1. Expansion method

2. Long division method

3. Partial fraction method

4. Residue theorem or contour integral method

Find \tilde{z}^{-1} for the following using expansion method:-

$$\textcircled{1} \quad \tilde{z}^{-1} \left\{ \frac{1}{z+2} \right\}$$

In expansion method we need to have the problem in terms of specific format then only we can analyse.

$$\left(\frac{1}{z+2} \right)$$

$$\frac{1}{z(1+\frac{2}{z})}$$

$$\frac{1}{z} \left(1 + \frac{2}{z} \right)^{-1} \quad (1+x)^{-1} = 1 - x + x^2 - x^3 + x^4$$

$$\frac{1}{z} \left[1 - \left(-\frac{2}{z} \right) + \left(\frac{2}{z} \right)^2 - \left(\frac{2}{z} \right)^3 + \dots - (-1)^{n-1} \left(\frac{2}{z} \right)^{n-1} + (-1)^n \left(\frac{2}{z} \right)^n \right]$$

$$\frac{1}{z} (-1)^{n-1} \left(\frac{2}{z} \right)^{n-1}$$

$$(-1)^{n-1} \cdot \frac{2^{n-1}}{z^{n-1}} \times \frac{1}{z} = \frac{(-1)^{n-1} 2^{n-1}}{z^n}$$

$$= (-1)^{n-1} 2^{n-1} z^{-n}$$

Co-

$$\text{Coefficient of } z^{-n} = (-1)^{n-1} 2^{n-1}$$

$$\text{find } z^{-1} \left\{ \frac{1+2z^{-1}}{1-z^{-1}} \right\} \quad \text{by Long division method}$$

$$\frac{1 - z^{-1}}{1 + 3z^{-1} + 3z^{-2} + z^{-3}}$$

$$\text{Find } Z^{-1} \left\{ \frac{3z^2 - 18z + 26}{(Z-2)(Z-3)(Z-4)} \right\} \text{ using partial fraction method}$$

$$3z^2 - 18z + 26 = \frac{A}{z-2} + \frac{B}{z-8} + \frac{C}{z-4}$$

$$3z^2 - 18z + 26 = A(z-3)(z-4) + B(z-2)(z-4) + C(z-2)(z-3)$$

put $z = 2$

$$12 - 36 + 26 = A(-1)(-2) + 0 + 0$$

$$2 = 2A$$

Pat z = 3

$$27 - 54 + 26 = B(1)(-1)$$

$$-1 = -B$$

$$\text{put } z = 4$$

$$48 - 72 + 26 = C(2)(1)$$

$$2c = 2$$

$$C = 1$$

$$= \bar{z} \left[\frac{1}{(z-2)} + \frac{1}{(z-3)} + \frac{1}{(z-4)} \right]$$

$$z \{a^{n-1}\} = \frac{1}{z-a}$$

$$= 2^{n-1} + 3^{n-1} + 4^{n-1}$$

Cauchy Residue Theorem-

It states that

$$\oint_C f(z) \cdot dz = 2\pi i (\text{sum of Residues of poles lying inside})$$

These are two types.

Residues:-

① If $z = a$ is a simple pole i.e. a pole of order 1
 Then residue of $f(z)$ at $z = a$ is given by

osdel 1

$$\text{Res} \left[f(z) \right]_{z=\infty} = \lim_{z \rightarrow \infty} (z-a) f(z) z^{n-1}$$

② If $\infty = a$ is a pole of order n then

ocean

$$\lim_{z \rightarrow a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z) z^{n-1}$$

Find z^{-1} using residue theorem

$$z^{-1} \left\{ \frac{z^2}{(z-a)(z-b)} \right\}$$

$$\text{Res}[f(z)]_{z=a} = \lim_{z \rightarrow a} (z-a) \frac{z^2}{(z-a)(z-b)} z^{n-1}$$

$$= \lim_{z \rightarrow a} \frac{z^{n+1}}{(z-b)}$$

$$= \frac{a^{n+1}}{(a-b)} \quad \text{--- ①}$$

$$\text{Res}[f(z)]_{z=b} = \lim_{z \rightarrow b} (z-b) \frac{z^2}{(z-a)(z-b)} z^{n-1}$$

$$= \lim_{z \rightarrow b} \frac{z^{n+1}}{(z-a)} \quad \text{--- ②}$$

$$= \frac{b^{n+1}}{b-a}$$

$$\int_C f(z) dz = 2\pi i \left[\frac{a^{n+1}}{a-b} + \frac{b^{n+1}}{b-a} \right]$$

Find z^{-1} using contour integral method.

$$z^{-1} \left\{ \frac{2z^2+4z}{(z-2)^3} \right\}$$

$m=3$

$$\text{Res}[f(z)]_{z=2} = \lim_{z \rightarrow 2} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (z-2)^m f(z) z^{n-1}$$

$$= \lim_{z \rightarrow 2} \frac{1}{(3-1)!} \frac{d^2}{dz^2} (z-2)^3 \left[\frac{2z^2+4z}{(z-2)^3} \right] z^{n-1}$$

$$= \lim_{z \rightarrow 2} \frac{1}{2!} \frac{d^2}{dz^2} [2z^2+4z] z^{n-1}$$

$$\text{If } z \rightarrow 2 \frac{1}{2!} \frac{d^2}{dz^2} [2z^{n+1} + 4z^n]$$

$$\text{If } z \rightarrow 2 \frac{1}{2!} \frac{d}{dz} [2(n+1)z^n + 4z^{n-1}]$$

$$\text{If } z \rightarrow 2 \frac{1}{2!} \left\{ 2(n+1)n \cdot z^{n-1} + 4 \cdot n \cdot (n+1) z^{n-2} \right\}$$

$$\frac{1}{2!} \left[2(n+1)n \cdot 2^{n-1} + 4 \cdot n \cdot (n+1) 2^{n-2} \right]$$

$$\frac{1}{2!} \left[2(n+1)n \cdot \frac{2^n}{2} + 4 \cdot n(n+1) \frac{2^{n-1}}{2} \right]$$

$$\frac{1}{2!} \left[n(n+1)2^n + n(n+1)2^n \right]$$

$$\frac{1}{2!} n2^n [n+x+n-x]$$

$$\frac{2^n n (2^n)}{2}$$

$$= 2^n \cdot n^2 //$$

Find the z^{-1} for the following

partial fraction

$$\frac{4z^2 - 24z + 28}{(z-4)(z-6)(z-8)}$$

$$\frac{4z^2 - 24z + 28}{(z-4)(z-6)(z-8)} = \frac{A}{(z-4)} + \frac{B}{(z-6)} + \frac{C}{(z-8)}$$

$$4z^2 - 24z + 28 = A(z-6)(z-8) + B(z-4)(z-8) + C(z-4)(z-6)$$

$$\text{Put } z=4$$

$$64 - 96 + 28 = A(-2)(-4)$$

$$-4 = 8A$$

$$A = -\frac{1}{2}$$

put $z=6$
 $4z^2 - 24z + 28 = B(z-4)(z-8)$

$$144 - 144 + 28 = B(2)(-2)$$

$$-4B = 28$$

$$B = -7$$

put $z=8$

$$4z^2 - 24z + 28 = C(z-4)(z-16)$$

$$512 - 192 + 28 = C(4)(2)$$

$$92 = 8C$$

$$C = \frac{23}{8}$$

Find $z^{-1} \left\{ \frac{4z}{(z-1)^3} \right\}$ long division method

$$= \frac{4z}{(z-1)^3}$$

$$= \frac{4z}{z^3(1-\frac{1}{z})^3}$$

$$= \frac{4z}{z^3(1-\frac{1}{z})^3}$$

$$= \frac{4z^{-2}}{(1-z^{-1})^3}$$

$$(a-b)^3 = a^3 - 3a^2b^2 + 3ab^2 - b^3$$

$$= \frac{4z^{-2}}{1-3z^{-2}+3z^{-1}-z^{-3}}$$

$$\begin{array}{r} 4z^2 + 12z^4 \\ \hline 1-3z^2 + 3z^{-1}z^3 \\ 4z^2 - 12z^4 \\ \hline 4z^2 + 12z^3 + 24z^4 + \\ 4z^2 + 12z^3 - 12z^4 - 4z^5 \\ \hline 12z^3 + 12z^4 + 4z^5 \\ 12z^3 + 36z^4 - 36z^5 - 12z^6 \\ \hline (-) (-) (+) (+) \end{array}$$

$$\text{if } z=1 \quad \frac{1}{2!} \frac{d^2}{dz^2} \frac{4z}{(z-1)^3} z^{n-1}$$

$$\frac{1}{2!} \frac{d^2}{dz^2} 4z^n$$

$$\frac{1}{2!} \frac{d}{dz} n z^{n-1}$$

$$\text{if } z=1 \quad \frac{1}{2!} \stackrel{\oplus}{=} n(n-1) z^{n-2}$$

$$\frac{1}{2!} n(n-1) z^{n-2}$$

$$\begin{matrix} n \neq 0 \\ n=0 \\ 0 \end{matrix}$$