UNIT – II

Fourier Series Representation of Periodic Signals :-

1) **Introduction :-**

→ Analysis of a signal is more convinent in frequency domain.

→ There are 3-types important classes of transformation methods available for Continous time signals.

ⓐ Fourier Series

ⓑ Fourier Transform

ⓒ Laplace Transform.

1.1) **Representation of Fourier series :-**

Fourier Series is a mathematical tool that allows the representation of any Periodic signal as the sum of harmonically related signals.

(or)

Representation of signals over a certain interval of time in terms of linear Combinations of orthogonal functions are called fourier series.

Purpose :-

→ used to analyse Periodic signals $(-\infty < t < \infty)$.

→ Harmonic Constant of the signal is analyzed with the help of fourier series.

→ It Can be developed for Continous time as well as discrete time signal.

Types of fourier Series :-

(i) Trignometric form

(ii) Cosine form

(iii) Exponential form.

(i) Trignometric form:-

Representing Periodic Signals in fourier Series which is a linear Combination of orthogonal functions is Called trigometric form.

(ii) Cosine form:-

Trignometric functions Containing sine & cosine terms of same frequency which is a linear combination of orthogonal functions is called as Cosine form.

(iii) Exponential form:-

Representation of Exponential functions over Certain time by linear Combinations of orthogonal functions is Called a Exponential form.

2) Existence of fourier series:-

① Conditions under which a Periodic signal can be represented by fourier series are known as Dirichlet's Conditions

② signal $x(t)$ should be represented in frequency domain.

③ $x(t)$ must be single valued function.

④ $x(t)$ should have a finite number of maxima and minima

⑤ function $x(t)$ has finite number of dis Continuities (Points 2,3,4 are Considered for Each Period).

⑥ The function $x(t)$ is absolutely integratable over Period i.e.,

$$\int_0^T x(t)\,dt < \infty$$

3) **Fourier Series Representation of Periodic Signals:-**

A periodic signal is one which repeat itself periodically over ~~ere~~ to $(-\infty < t < \infty)$

Eg:- A sinusoidal signal $x(t) = A \sin \omega_0 t$ is periodic signal with period $T = \frac{2\pi}{\omega_0}$

W.K.T the sum of two sinusoidal is periodic provided that their frequencies are integer multiples of fundamental frequency 'ω_0'.

Now:-

Let us consider $x(t)$ signal, a sum of sine & cosine function whose frequencies are integer multiple of 'ω_0'.

$$\therefore x(t) = a_0 + a_1 \cos(\omega_0 t) + a_2 \cos(2\omega_0 t) + a_3 \cos(3\omega_0 t) + \dots + a_K \cos(K\omega_0 t)$$

$$+ b_1 \sin(\omega_0 t) + b_2 \sin(2\omega_0 t) + b_3 \sin(3\omega_0 t) + \dots + b_K \sin(K\omega_0 t) \rightarrow (1)$$

$$\therefore x(t) = a_0 + \sum_{n=1}^{K} \left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right] \rightarrow (2)$$

where $a_0, a_1 \dots a_K$ & $b_0, b_1, b_2 \dots b_K$ are integer constants.

$\omega_0 \rightarrow$ fundamental frequency $= \frac{2\pi}{T}$ $\left(\because \omega_0 = 2\pi f \Rightarrow f = \frac{1}{T}, \ \omega_0 = \frac{2\pi}{T}\right)$

if $x(t)$ is a periodic signal then it has to satisfy condition $x(t+T) = x(t) \rightarrow (3)$

from eq (2)

$$x(t+T) = a_0 + \sum_{n=1}^{K} \left[a_n \cos n\omega_0(t+T) + b_n \sin \omega_0 (t+T) \right]$$

$$= a_0 + \sum_{n=1}^{K} \left[a_n \cos(n\omega_0 t + 2n\pi) + b_n \sin(n\omega_0 t + 2n\pi) \right] \quad \left(\because \omega_0 = \frac{2\pi}{T}\right)$$

$$x(t+T) = a_0 + \sum_{n=1}^{K} \left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right] \rightarrow (4)$$

$$x(t+T) = x(t) \rightarrow (5)$$

The Eq (2) Can be written as

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega_0 t)\right] + \sum_{n=1}^{\infty}\left[b_n \sin(n\omega_0 t)\right] \rightarrow (6)$$

The above Eq represents trigonometric fourier series.

where a_n, b_n are Constants.

Cofficient $a_0 \rightarrow$ is dc Component

$a_1 \cos\omega_0 t + b_1 \sin\omega_0 t$ are first harmonic

$a_2 \cos 2\omega_0 t + b_2 \sin 2\omega_0 t$ are second "

⋮

$a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$ " n^{th} " ".

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4) Trigonometric fourier series (quadrature fourier series):-

it is expressed as

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega_0 t)\right] + \sum_{n=1}^{\infty}\left[b_n \sin(n\omega_0 t)\right] \rightarrow (1)$$

where a_0, a_n, b_n are the trigonometric fourier series Co-efficients

the Co-efficients may be obtained from $x(t)$ using

$$a_0 = \frac{1}{T}\int_0^T x(t).dt \rightarrow (2)$$

$$a_n = \frac{2}{T}\int_0^T x(t).\cos(n\omega_0 t).dt \rightarrow (3)$$

$$b_n = \frac{2}{T}\int_0^T x(t).\sin(n\omega_0 t).dt \rightarrow (4)$$

where $\int$ indicates integration over time period

$\omega_0 = \frac{2\pi}{T}$; $T \Rightarrow$ time Period of signal $x(t)$.

Note :-

1

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega_0 t)\right] + \sum_{n=1}^{\infty}\left[b_n \sin(n\omega_0 t)\right] \rightarrow (1)$$

2

$$\left.\begin{aligned}&\int_0^T \sin(m\omega_0 t)\cdot dt = 0 \quad \forall\ m\\ &\int_0^T \cos(n\omega_0 t)\cdot dt = 0 \quad \forall\ n \neq 0\end{aligned}\right\} \rightarrow (2)$$

because avg value of a sinusoidal over m and n complex cycles in the period T is zero.

3

$$\int_0^T \sin(m\omega_0 t)\cdot\cos(n\omega_0 t)\cdot dt = 0 \quad \forall\ m, n \rightarrow (3)$$

$$\int_0^T \sin(m\omega_0 t)\cdot\sin(n\omega_0 t)\cdot dt = \begin{cases} 0 & ; m \neq n \\ \frac{T}{2} & ; m = n \end{cases} \rightarrow (4)$$

$$\int_0^T \cos(m\omega_0 t)\cdot\cos(n\omega_0 t)\cdot dt = \begin{cases} 0 & , m \neq n \\ \frac{T}{2} & , m = n \end{cases} \rightarrow (5)$$

Proof to get Co-efficients:-

Proof (i) $a_0 = \frac{1}{T}\int_0^T x(t).dt \rightarrow$ from above Eq 4.2 Eq

from Eq 1. WKT

$$x(t) = a_0 + [a_1 \cos(\omega_0 t) + b_1 \sin(\omega_0 t)] + [a_2 \cos(2\omega_0 t) + b_2 \sin(2\omega_0 t)] + \dots$$
$$\dots + [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)] + \dots \rightarrow (b)$$

Integrating above Eq over the interval of time 0 to T gives

$$\int_0^T x(t).dt = \int_0^T a_0 .dt + \int_0^T a_1 \cos(\omega_0 t) dt + \int_0^T b_1 \sin(\omega_0 t) dt + \int_0^T a_2 \cos 2\omega_0 t . dt +$$
$$+ \int_0^T b_2 \sin 2\omega_0 t . dt + \dots + \int_0^T a_n \cos(n\omega_0 t).dt + \int_0^T b_n \sin(n\omega_0 t).dt + \dots \rightarrow (c)$$

using Eq 2 we know that all terms on RHS of Eq above are found to have zero value except the first term

$$\int_0^T x(t).dt = \int_0^T a_0 \, dt + 0 + 0 + \dots$$

$$\Rightarrow \int_0^T x(t).dt = a_0 T \Rightarrow \boxed{a_0 = \frac{1}{T}\int_0^T x(t).dt}$$

$\rightarrow$ avg value of $x(t)$ over a Period (or) dc value of the signal.

Proof (ii) $a_n = \frac{2}{T}\int_0^T x(t) \cdot \cos(n\omega_0 t) \cdot dt$

from Eq 1 WKT

$$x(t) = a_0 + [a_1 \cos(\omega_0 t) + b_1 \sin(\omega_0 t)] + a_2[\cos(2\omega_0 t) + b_2(\sin(2\omega_0 t))] + \dots$$
$$+ a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)].$$

By multiplying B.S by $\cos n\omega_0 t$ and integrating over the interval of time 0 to T we get

$$\int_0^T x(t) . \cos(n\omega_0 t).dt = \int_0^T a_0 \cos(n\omega_0 t).dt + \int_0^T a_1 \cos\omega_0 t \cos(n\omega_0 t).dt +$$
$$\int_0^T b_1 \sin(\omega_0 t) . \cos(n\omega_0 t) . dt + \int_0^T a_2 \cos(2\omega_0 t) \cos(n\omega_0 t) dt + \dots$$

$+ \int_0^T a_n \cos^2(n\omega_0 t).dt + \int_0^T b_n \sin(n\omega_0 t).\cos(n\omega_0 t).dt + \ldots$

now using note eq all the terms on RHS is having zero value except the integration of $\cos^2(n\omega_0).dt$ which has value $\frac{T}{2}$.

$$\int_0^T x(t).\cos(n\omega_0 t).dt = a_n.\frac{T}{2}$$

$$\Rightarrow \boxed{a_n = \frac{2}{T}\int_0^T x(t).\cos(n\omega_0 t).dt}$$

Similar to above derivation b_n can be obtained:-

$$\Rightarrow \boxed{b_n = \frac{2}{T}\int_0^T x(t).\sin(n\omega_0 t).dt}$$

5) Evaluation of a_0, a_n, b_n:-

i) for a_0:-

W.K.T $x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$.

$$\rightarrow \int_0^T x(t).dt = \int_0^T a_0\,dt + \int_0^T \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t).dt + \int_0^T \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t).dt$$

$$\cancel{= a_0[t]_0^T + \sum_{n=1}^{\infty} a_n \int_0^T \cos(n\omega_0 t).dt + \sum_{n=1}^{\infty} b_n \int_0^T \sin}$$

$$= a_0[t]_0^T + \sum_{n=1}^{\infty} a_n\left[\frac{\sin(n\omega_0 t)}{n\omega_0}\right]_0^T + \sum_{n=1}^{\infty} b_n\left[\frac{-\cos(n\omega_0 t)}{n\omega_0}\right]_0^T$$

$$= a_0[T] + \sum_{n=1}^{\infty} a_n\left[\frac{\sin(n\omega_0 T)}{n\omega_0}\right] + \sum_{n=1}^{\infty} b_n\left[-\frac{\cos n\omega_0 T}{n\omega_0} + \frac{\cos 0}{n\omega_0}\right]$$

$$\Rightarrow a_0 T + \sum_{n=1}^{\infty} a_n\left[\frac{\sin n.\frac{2\pi}{T}.T}{n.\frac{2\pi}{T}}\right] + \sum_{n=1}^{\infty} b_n\left[\frac{-\cos n.\frac{2\pi}{T}.T}{n.\frac{2\pi}{T}} + \frac{1}{n.\frac{2\pi}{T}}\right]$$

$$= a_0 T + \sum_{n=1}^{\infty} a_n T \left[\frac{\sin n2\pi}{n2\pi} \right] + \sum_{n=1}^{\infty} b_n T \left[\frac{-\cos n2\pi}{n2\pi} + \frac{1}{n2\pi} \right]$$

$$= a_0 T + \sum_{n=1}^{\infty} a_n T (0) + \sum_{n=1}^{\infty} b_n T \left[\frac{-1}{n2\pi} + \frac{1}{n2\pi} \right]$$

$(\because \sin n2\pi = 0 ;\ \cos n2\pi = 1$ for integer $k)$

$$\int_0^T x(t)\,dt = a_0 T$$

$$\therefore a_0 = \frac{1}{T} \int_0^T x(t)\,dt$$

(ii) <u>for an a_n</u> :-

where $a_n = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \cdot \cos(n\omega_0 t)\,dt$ (or)

$$a_n = \frac{2}{T} \int_0^T x(t) \cdot \cos(n\omega_0 t)\,dt$$

<u>Proof</u>:-

WKT $x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$.

$= a_0 + a_1 \cos\omega_0 t + a_2 \cos 2\omega_0 t + \dots + a_n \cos(n\omega_n t) + \dots$
$+ b_1 \sin\omega_0 t + b_2 \sin(2\omega_0 t) + \dots + b_n \sin(n\omega_0 t) + \dots$

Let us multiply the above eq by $\cos n\omega_0 t$

$\therefore x(t) \cdot \cos n\omega_0 t = a_0 \cos n\omega_0 t + a_1 \cos\omega_0 t \cdot \cos n\omega_0 t + a_2 \cos 2\omega_0 t \cdot \cos n\omega_0 t + \dots$
$\dots + \cancel{b_1 \sin\omega_0 t} + \cancel{b_2 \sin 2\omega_0 t} + \dots + \cancel{b_n \sin n\omega_0 t} + \dots$
$\dots + a_n \cos^2 n\omega_0 t + \dots + b_1 \sin\omega_0 t \cdot \cos n\omega_0 t + b_2 \sin 2\omega_0 t \cdot \cos n\omega_0 t + \dots$
$+ b_n \sin n\omega_0 t \cdot \cos n\omega_0 t + \dots$

Let us integrate the above eq b/w limits 0 to T..

$$\therefore \int_0^T x(t).\cos n\omega_0 t.dt = \int_0^T a_0 \cos n\omega_0 t.dt + \int_0^T a_1 \cos \omega_0 t.\cos n\omega_0 t.dt + \ldots$$

{all definite integral will be zero}.

$$\therefore \int_0^T x(t).\cos n\omega_0 t.dt = \int_0^T a_n \cos^2 n\omega_0 t$$

$$= a_n \int_0^T \left(\frac{1+\cos 2n\omega_0 t}{2}\right) dt$$

$$= \frac{a_n}{2} \int_0^T (1+\cos 2n\omega_0 t).dt$$

$$= \frac{a_n}{2} \left[t + \frac{\sin 2n\omega_0 t}{2\omega_0 n} \right]_0^T$$

$$= \frac{a_n}{2} \left[T + \frac{\sin 2n\omega_0 T}{2\omega_0 n} - 0 - \frac{\sin 0}{2\omega_0 n} \right] = \frac{a_n}{2} \left[T + \frac{\sin .2n.\frac{2\pi}{T}.T}{2.\frac{2\pi}{T}.n} \right]$$

$$= \frac{a_n}{2} [T]$$

$$\Rightarrow a_n = \frac{2}{T} \int_0^T x(t).\cos n\omega_0 t.dt \,//.$$ {gives the n^{th} co.eff of a_n}

(iii) for b_n:-

where $b_n = \frac{2}{T} \int_{-T/2}^{T/2} x(t).\sin n\omega_0 t.dt$ (or)

$$b_n = \frac{2}{T} \int_0^T x(t).\sin n\omega_0 t.dt$$

Proof:- $x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$

$$x(t) = a_0 + a_1 \cos \omega_0 t + a_2 \cos 2\omega_0 t + \ldots + a_n \cos n\omega_0 t + \ldots$$
$$b_1 \sin \omega_0 t + b_2 \sin 2\omega_0 t + \ldots b_n \sin n\omega_0 t + \ldots$$

multiply above eq with $\sin(n\omega_0 t)$ on B.S then

$$x(t)\cdot\sin n\omega_0 t = a_0 \sin n\omega_0 t + a_1 \cos\omega_0 t\cdot\sin n\omega_0 t + a_2 \sin n\omega_0 t\cdot\cos 2\omega_0 t + \dots + a_n \cos n\omega_0 t\cdot\sin n\omega_0 t + \dots + b_1 \sin\omega_0 t\cdot\sin n\omega_0 t + b_2 \sin 2\omega_0 t\cdot\sin n\omega_0 t + \dots + b_n \sin^2 n\omega_0 t + \dots$$

integration the above eq b/w limits 0 to T.

$$\int_0^T x(t)\cdot\sin n\omega_0 t = \int_0^T a_0 \sin n\omega_0 t + \int_0^T a_1 \cos\omega_0 t\cdot\sin n\omega_0 t + \dots$$

$$\int_0^T x(t)\cdot\sin n\omega_0 t = \int_0^T b_n \sin^2 n\omega_0 t$$

$$= \int_0^T b_n\cdot\left(\frac{1-\cos 2n\omega_0 t}{2}\right)\cdot dt$$

$$= \frac{b_n}{2}\int_0^T [1-\cos 2n\omega_0 t]\cdot dt$$

$$= \frac{b_n}{2}\left[\int_0^T t\cdot dt - \int_0^T \cos 2n\omega_0 t\cdot dt\right]$$

$$= \frac{b_n}{2}\left\{[t]_0^T - \left[\frac{\sin 2n\omega_0 t}{2n\omega_0}\right]_0^T\right]$$

$$= \frac{b_n}{2}\left[T - \frac{\sin 2n\omega_0 T}{2n\omega_0}\right]$$

$$= \frac{b_n}{2}\left[T - \frac{\sin 2n\left(\frac{2\pi}{T}\right)T}{2n(\omega_0)}\right] \quad \left(\because \omega_0 = \frac{2\pi}{T}\right)$$

$$= \frac{b_n}{2}\left[T - \frac{\sin 2n(2\pi)}{2n\omega_0}\right] = \frac{T}{2}b_n.$$

$$\therefore b_n = \frac{2}{T}\int_0^T x(t)\sin n\omega_0 t\cdot dt$$

gives n^{th} coefficient b_n. Here the n^{th} coeff b_n is given

$$\boxed{b_n = \frac{2}{T}\int_0^T x(t)\cdot\sin n\omega_0 t\cdot dt}$$

6) Cosine Representation :-

A trignometric fourier series of $x(t)$ contains sine & cosine terms of same frequency.

w.k.t $$x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \right] \rightarrow (1)$$

To obtain cosine representation i.e $x(t)$ will be in terms cosine form.

→ Let $a_0 = A_0$; $a_n = \cos\theta_n A_n$ & $b_n = -\sin\theta_n A_n$.

and $A_n = \sqrt{a_n^2 + b_n^2}$; $\theta_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$. $\rightarrow (2)$

→ We can obtain cosine representation from trignometric fourier series

$$x(t) = A_0 + \sum_{n=1}^{\infty} \left[\cos\theta_n A_n \cdot \cos n\omega_0 t - \sin\theta_n \cdot A_n \sin n\omega_0 t \right] \rightarrow (3)$$

$$= A_0 + \sum_{n=1}^{\infty} A_n \left[\cos\theta_n \cdot \cos n\omega_0 t - \sin\theta_n \cdot \sin n\omega_0 t \right] \rightarrow (4)$$

$$\boxed{x(t) = A_0 + \sum_{n=1}^{\infty} A_n \left[\cos\left(\theta_n + n\omega_0 t\right) \right]} \rightarrow (5)$$

$(\because \cos a \cdot \cos b - \sin a \cdot \sin b = \cos(a+b))$

∴ Above Eq (5) is cosine representation of $x(t)$ which contain sinusoidal of freq $\omega_0, 2\omega_0, 3\omega_0, 4\omega_0 \ldots$ and $a_n \cos(\theta_n + n\omega_0 t)$ is n^{th} harmonic

→ A_0 is dc-component and

→ θ_n → phase angle (or) spectrum phase of fourier series.

→ A_n → Amplitude co-efficient (or) spectral amplitudes of fourier series //.

7) Exponential fourier Series :-

→ " " " is widely used in fourier series

→ " " " $x(t)$ expressed as weighted sum of Complex exponential functions.

$$\rightarrow x(t) = A_0 + \sum_{n=1}^{\infty}\left[A_n \cos(n\omega_0 t + \theta_n)\right] \rightarrow (1)$$

where $\omega_0 \rightarrow$ fundamental freq so $(\cos n\omega_0 t + \theta_n)$ is expressed as

$$\cos(n\omega_0 t + \theta_n) = \frac{e^{j(n\omega_0 t + \theta_n)} + e^{-j(n\omega_0 t + \theta_n)}}{2} \rightarrow (2)$$

Sub (2) in eq (1)

$$\therefore x(t) = A_0 + \sum_{n=1}^{\infty}\left(A_n\left(\frac{e^{j(n\omega_0 t + \theta_n)} + e^{-j(n\omega_0 t + \theta_n)}}{2}\right)\right)$$

$$\Rightarrow x(t) = A_0 + \underbrace{\sum_{n=1}^{\infty}\frac{A_n}{2}\left(e^{j(n\omega_0 t + \theta_n)}\right)}_{(1)} + \underbrace{\sum_{n=1}^{-\infty}\left(\frac{A_n}{2}\left(e^{-j(n\omega_0 t + \theta_n)}\right)\right)}_{(2)} \rightarrow (3)$$

→ in 2nd summation let $n = -k$ then

$$x(t) = A_0 + \sum_{n=1}^{\infty}\frac{A_n}{2}\left(\left(e^{jn\omega_0 t} \cdot e^{j\theta_n}\right)\right) + \sum_{k=1}^{-\infty}\left(\frac{A_{-k}}{2}\right) \cdot e^{jk\omega_0 t} \cdot e^{-j\theta_{-k}} \rightarrow (4)$$

Comparing eq (3) & (4)

$$A_n = A_{-k} \quad \& \quad -\theta_n = -\theta_{-k}$$

replacing $A_0 = C_0$

$$x(t) = C_0 + \underbrace{\sum_{n=1}^{\infty}\left(\frac{A_n}{2}\right) \cdot e^{jn\omega_0 t} \cdot e^{j\theta_n}}_{(1)} + \underbrace{\sum_{k=1}^{-\infty}\left(\frac{A_{-k}}{2}\right) \cdot e^{jk\omega_0 t} \cdot e^{-j\theta_{-k}}}_{(2)} \rightarrow (5)$$

from above eq (5) 2nd summation $n = k$ then

$$x(t) = C_0 + \sum_{n=1}^{\infty}\left(\frac{A_n}{2}\right) \cdot e^{jn\omega_0 t} \cdot e^{j\theta_n} + \sum_{n=-1}^{-\infty}\left(\frac{A_{-n}}{2}\right) \cdot e^{jn\omega_0 t} \cdot e^{-j\theta_{-n}} \rightarrow (6)$$

now let $A_{-n} = A_n$ & $\theta_{-n} = \theta_n$ then above Eq ⑥ is

$$x(t) = C_0 + \sum_{n=1}^{\infty}\left(\frac{A_n}{2}\right).e^{jn\omega_0 t}.e^{j\theta_n} + \sum_{n=-1}^{-\infty}\left(\frac{A_n}{2}\right).e^{-jn\omega_0 t}.e^{j\theta_n}$$

$$x(t) = C_0 + \underbrace{\sum_{n=-\infty}^{\infty}\left(\frac{A_n}{2}\right).e^{jn\omega_0 t}.e^{j\theta_n}}_{\text{Except } n=0} \rightarrow ⑦$$

from Eq ⑦ let $\left(\frac{A}{2}\right)e^{j\theta_n} = C_n$ then

$$x(t) = C_0 + \sum_{n=-\infty}^{\infty} C_n.e^{jn\omega_0 t} \rightarrow ⑧$$ which represents Exponential Fourier Series where C_0 & C_n are Co-efficients an there values are

$$C_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t).dt \rightarrow ⑨$$

$$C_n = \frac{1}{T}\int_{0}^{t_0+T} x(t).e^{-jn\omega_0 t}.dt \rightarrow ⑩ //.$$

— x —

8) <u>Complex Fourier Exponential Series : (Alternate) :-</u>

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos n\omega_0 t + b_n \sin \omega_0 t\right] \rightarrow ①$$

We know Euler's identity

$$e^{j\theta} = \cos\theta + j\sin\theta \rightarrow ②, \quad e^{-j\theta} = \cos\theta - j\sin\theta \rightarrow ③$$

Adding ② & ③

$$e^{j\theta} + e^{-j\theta} = 2\cos\theta \Rightarrow \cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2} \rightarrow ④$$

Subtracting ② & ③

$$e^{j\theta} - e^{-j\theta} = 2.j\sin\theta$$

$$\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j} \rightarrow ⑤$$

so.
$$\cos(n\omega_0 t) = \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2} \;;\; \sin(n\omega_0 t) = \frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j}$$

substituting the values of $\cos n\omega_0 t$ & $\sin n\omega_0 t$ in eq (1)

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \left[\frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2} \right] + b_n \left[\frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j} \right] \right].$$

$$\cancel{x(t) = a_0 + \sum_{n=1}^{\infty} \left[\frac{(a_n - jb_n)}{2} \cdot e^{jn\omega_0 t} + \frac{(a_n + jb_n)}{2} \cdot e^{-jn\omega_0 t} \right]}$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[\left(\frac{a_n}{2} + \frac{b_n}{2j} \right) \cdot e^{jn\omega_0 t} + \left(\frac{a_n}{2} - \frac{b_n}{2j} \right) \cdot e^{-jn\omega_0 t} \right]$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} \frac{1}{2} (a_n - jb_n) \cdot e^{jn\omega_0 t} + \sum_{n=1}^{\infty} \frac{1}{2} (a_n - (-j) b_n) \cdot e^{-jn\omega_0 t}$$

$$\therefore \text{Let } C_n = \frac{a_n - jb_n}{2} \;;\; C_n^* = \frac{a_n + jb_n}{2} \;;\; C_0 = a_0$$

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cdot e^{jn\omega_0 t} + \sum_{n=1}^{\infty} C_n^* e^{-jn\omega_0 t} \qquad [C_n^* = C_{-n}]$$

$$= C_0 + \sum_{n=1}^{\infty} C_n \cdot e^{jn\omega_0 t} + \sum_{n=1}^{\infty} C_{-n} \cdot e^{-jn\omega_0 t}$$

$$= C_0 + \sum_{n=1}^{\infty} C_n \cdot e^{jn\omega_0 t} + \sum_{n=-\infty}^{-1} C_n \cdot e^{+jn\omega_0 t}$$

$$x(t) = \sum_{n=-\infty}^{\infty} C_n \cdot e^{jn\omega_0 t}$$

$$\therefore C_n = \frac{1}{2} (a_n - jb_n) = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \left[\cos n\omega_0 t - j \sin n\omega_0 t \right] . dt$$

(or)

$$C_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \cdot e^{-jn\omega_0 t} . dt$$

similly. $$C_{-n} = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \cdot e^{jn\omega_0 t} . dt \;\text{ ||.}$$

Fourier Spectrum:-

→ Fourier Spectrum of Periodic signal $x(t)$ is a plot of its Fourier Co-efficient and frequency 'ω'.

→ Fourier Spectrum is divided in two Parts
(i) Amplitude Spectrum &
(ii) Phase Spectrum.

(i) **Amplitude Spectrum**:- $(A_n$ & $\omega_0)$ (or) $(C_n$ & $\omega_0)$
The spectrum (or) plot drawn for amplitude of Fourier Co-efficients versus frequency is called as amplitude spectrum.

(ii) **Phase spectrum**:- $(\theta_n$ & $\omega_0)$
The Spectrum (or) Plot drawn for Phase of Fourier Co-efficients versus frequency is known as Phase Spectrum.

(*) If these two Spectrums are represented together then it is called as frequency domain representation.

(*) Trignometric representation of Periodic signal contain both sine & Cosine terms with Positive and negative amplitude Co-efficients which doesn't Contain Phase angle.

(*) The Cosine representation of a periodic signal contains only Positive amplitudes Co-efficients with phase angle θ_n, we can Plot amplitude spectrum (A_n Vs ω_0) and Phase spectrum (θ_n Vs ω_0)

(*) The Exponential representation of periodic signal $x(t)$ contain amplitude Co-efficient 'C_n' which is Complex.

(*) For trignometric, exponential fourier series we can only draw amplitude spectrum.

→ If fourier spectrum exits only for positive frequency
This spectrum is called single side spectrum.

→ The spectrum which can be plotted for both positive &
-ve frequencies called double sided spectrum.

→ Amplitude spectrum is also called as "magnitude spectrum".

Fourier Transforms & Sampling:-

1) **Introduction of F.T:-** (Time domain to frequency domain)
(T_d to W_d)

i) → In fourier series any continuous time periodic signal $x(t)$ can be represented as linear combinations of complex exponential and fourier co-efficients are discrete.

ii) → F.S represents a periodic signals as sum of sinusoidals (or) complex sinusoidal $e^{jn\omega_0 t}$

iii) → because F.S uses harmonically related frequencies $n\omega_0$ it can only create periodic signals.

iv) The F.S. can be applied to periodic signals only but F.T can also be applied to non-periodic functions like rectangular, step func, ramp, ...

v) the F.T can be developed by finding fourier series of periodic function and then tending 'T' to infinity (∞).

(vi) If T tends to infinity then f(t) becomes aperiodic signal.

2) **Deriving fourier Transform from fourier series:-**

Let $x(t)$ be non-periodic function and $x_T(t)$ be periodic function with period 'T'.

→ their relation is given by $x(t) = \underset{T\to\infty}{Lt}\ x_T(t)$ → ①

→ W.K.T fourier series of periodic $x_T(t)$ is

$$x_T(t) = C_0 + \sum_{n=-\infty}^{\infty} C_n . e^{jn\omega_0 t} \rightarrow ②$$

Co-efficient is

$$\therefore C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x_T(t) . e^{-jn\omega_0 t} . dt \rightarrow ③$$

$$\Rightarrow C_n . T = X(jn\omega_0) \rightarrow ④ \quad \left(\because \text{where} \int_{t_0}^{t_0+T} x_T(t) . e^{-jn\omega_0 t} . dt = X(jn\omega_0)\right)$$

w.k.T $T = \frac{2\pi}{\omega_0}$ then $\left[\frac{1}{T} = \frac{\omega_0}{2\pi}\right]$

from Eq (2) Co-efficient 'C_0' is neglected then.

$$x_T(t) = \sum_{n=-\infty}^{\infty} C_n . e^{jn\omega_0 t}$$

Multiply & divided by 'T' on R.H.S

$$x_T(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} T . C_n . e^{jn\omega_0 t}$$

$$\Rightarrow x_T(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} T . C_n . e^{jn\omega_0 t} \rightarrow (5)$$

Substitute Eq (4) in Eq (5) then $(\because T.C_n = X(jn\omega_0))$

$$x_T(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(jn\omega_0) . e^{jn\omega_0 t} \rightarrow (6)$$

→ when $T \to \infty$ fundamental freq $\omega_0 \doteq 0$ and the harmonics of frequency i.e $n\omega_0 \to \omega$ $(\because \omega_0 = \frac{2\pi}{T}$ if $T = 0$ then $\omega_0 = 0)$

Substitue $n\omega_0 = \omega$ in Eq (6) then

$$x_T(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(j\omega) . e^{j\omega t} \rightarrow (7)$$

→ when $T \to \infty$ for a Periodic signal summation becomes integration.

$$x_T(t) = \frac{\omega_0}{2\pi} \int_{-\infty}^{\infty} X(j\omega) . e^{j\omega t} . dt \rightarrow (8)$$

Continous fourier series.

∴ From above Equation (8) states the discrete fourier series becomes Continous fourier series.

∵ w.k.T $T = \frac{2\pi}{\omega}$ then.

from Eq (7) Can be written as

$$x_T(t) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} X(j\omega) . e^{j\omega t} . \omega \rightarrow (9)$$

∴ Equation (9) is transformed to → Inverse F.T

$$\boxed{x_T(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) . e^{j\omega t} . d\omega} \rightarrow (10)$$ inverse. F.T

In above Eq (10) 'ω' changed to '$d\omega$' which is nothing but change in frequency. (3) (23)

$\therefore$ Eq (10) is represented for Inverse fourier Transform.

$\therefore$ from Eq (10) we can write.

$$\boxed{X(j\omega) = \int_{-\infty}^{\infty} x_T(t) \cdot e^{-j\omega t} \cdot dt} \rightarrow (11) \rightarrow \text{F.T}$$

$\therefore$ Eq (10) is F.T & Eq (11) is inverse fourier Transform.

Note:-

$\boxed{\text{frequency}} = \boxed{\text{Time domain}}$ (transformed)

fourier Transform.

$\boxed{\text{Time}} = \boxed{\text{frequency domain}}$ (transformed)

Inverse fourier Transform.

3) Fourier Transform & Inverse fourier transform pairs:-

from above concept Eq (10) & Eq (11) are

$$x_T(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) \cdot e^{j\omega t} \cdot d\omega \rightarrow \text{Inverse F.T}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x_T(t) \cdot e^{-j\omega t} \cdot dt \rightarrow \text{F.T}$$

above Eq's can be written as

$$\left.\begin{aligned} X(j\omega) &= F[x(t)] \rightarrow (12) \\ x_T(t) &= F^{-1}(X(j\omega)) \rightarrow (13) \end{aligned}\right\} \text{Pairs}$$

which is denoted as

$$\underset{T_d}{\boxed{x_T(t)}} \underset{\text{IFT}}{\overset{}{\rightleftarrows}} \underset{F_d}{\boxed{X(j\omega)}}$$

In general $X(j\omega)$ is a complex valued function of 'ω'. (4)

$$\therefore \quad X(j\omega) = \underbrace{X_R(j\omega)}_{\text{Real Part}} + j\,\underbrace{X_i(j\omega)}_{\text{imaginary Part}} \rightarrow (14)$$

The magnitude of $X(j\omega)$ is given by

$$|X(j\omega)| = \sqrt{\left[X_R(j\omega)\right]^2 + \left[X_i(j\omega)\right]^2} \rightarrow (15)$$

$\therefore$ The Phase of $X(j\omega)$ is given by

$$\angle X(j\omega) = \tan^{-1}\left(\frac{X_i(j\omega)}{X_R(j\omega)}\right) \rightarrow (16)$$

→ If a graph is drawn b/w magnitude of $X(j\omega)$ and freq (ω) This plot is called amplitude spectrum (or) magnitude spectrum.

→ A graph drawn b/w Phase of $X(j\omega)$ & freq (ω).

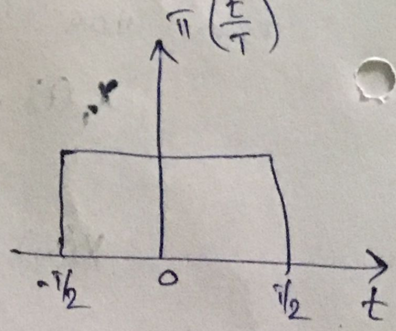
→ This plot is called Phase spectrum.

→ The Combination of both Amplitude spectrum & phase spectrum is called freq spectrum. //.

4) **Fourier transform of some Standard signals**:-

(i) Rectangular Pulse:-

Consider a rectangular Pulse which is unit gate function defined.



$$\Pi\left(\frac{t}{T}\right) = 1 \quad \text{for } |t| \leq \frac{T}{2}$$

$$\Pi\left(\frac{t}{T}\right) = 0 \quad \text{others}.$$

now Let $x(t) = \Pi\left(\frac{t}{T}\right)$ then

$$X(j\omega) = F\left(\Pi\left(\frac{t}{T}\right)\right) = \int_{-\infty}^{\infty} \Pi\left(\frac{t}{T}\right) \cdot e^{-j\omega t} \cdot dt$$

$$= \cancel{\int_{-\infty}^{\infty} e^{-j\omega t}\, dt}$$

$$X(j\omega) = \int_{-T/2}^{T/2} e^{-j\omega t}\,.dt = \frac{1}{-j\omega} e^{-j\omega t}\Big|_{-T/2}^{T/2}$$

$$X(j\omega) = \frac{1}{-j\omega}\left[e^{-j\omega T/2} - e^{-j\omega T/2} \right] = \frac{e^{j\omega T} - e^{-j\omega T}}{j\omega}$$

$$X(j\omega) = \frac{2 \sin \frac{\omega T}{2}}{j\omega} \quad \left(\because \frac{e^{j\theta} - e^{-j\theta}}{2} = \sin\theta \right)$$

$$X(j\omega) = 2T \frac{\sin \frac{\omega T}{2}}{\omega T}$$

$$X(j\omega) = T \frac{\sin\left(\frac{\omega T}{2}\right)}{\frac{\omega T}{2}}$$

$$X(j\omega) \cancel{\Pi(\tfrac{t}{T})} = T \operatorname{sinc}\left(\frac{\omega T}{2}\right)$$

$$\Pi\left(\frac{t}{T}\right) \xleftrightarrow{f.T} T \operatorname{sinc}\left(\frac{\omega T}{2}\right)$$

The amplitude spectrum is at $\omega = 0$; $\operatorname{sinc}\left(\frac{\omega}{2}\right) = T$

$\therefore$ $X(j\omega)$ at $\omega = 0$ is equal to T

at $\frac{\omega T}{2} = \pm n\pi$; $\operatorname{sinc}\left(\frac{\omega T}{2}\right) = 0$; i.e

$X(j\omega) = 0$ at $\omega = \pm \frac{2n\pi}{T}$; $n = 1, 2 \ldots$

The phase spectrum is

$\angle X(j\omega) = 0$ if $\operatorname{sinc}\left(\frac{\omega T}{2}\right) > 0$

$= \pm\pi$ if $\operatorname{sinc}\left(\frac{\omega T}{2}\right) < 0$,,