ECN 5320/6320: Mathematical Methods in Economics and Finance II

HW9: LCPI Model of Consumption (Two-Stage Control Problem)

Time is a continuous variable indexed by t. The representative individual enters the workforce at $t = 0$, retires at $t = T$, and dies at $t = \bar{T}$. The dates of retirement and death are exogenous. There is no uncertainty of any kind in the model (i.e., no mortality risk, no wage income uncertainty, no asset return uncertainty, etc.). Wage income, $w(t)$, is earned for all $t \in [0, T]$. Private assets grow at a constant market rate of interest, r. The representative individual has perfect foresight over the entire life cycle, passes away with no assets, and constructs a consumption program that maximizes discounted lifetime utility. Combining the above information, the optimal control problem is

$$\max_{\{c(t)\}} \quad \int_0^{\bar{T}} e^{-\rho t} u[c(t)] \; dt, \tag{1}$$

where ρ is the rate of time preference and where $u[c(t)] = (c(t)^{1-\phi} - 1)/(1 - \phi)$ is the CIES instantaneous utility function. The representative individual is constrained by

$$\begin{aligned}
\frac{dk(t)}{dt} &= rk(t) + w(t) - c(t), \qquad \text{for } t \in [0, T], && (2)\\
\frac{dk(t)}{dt} &= rk(t) - c(t), \qquad \text{for } t \in \left[T, \bar{T}\right], && (3)\\
k(0) &= 0, && (4)\\
k(\bar{T}) &= 0. && (5)
\end{aligned}$$

Application of the *Maximum Principle* for a fixed-endpoint optimal problem yields the analytical solution for the *optimal control*,

$$c^*(t) = e^{gt} \left(\frac{\displaystyle\int_0^T e^{-rt} w(t) \; dt}{\displaystyle\int_0^{\bar{T}} e^{(g-r)t} \; dt} \right), \qquad \text{for } t \in \left[0, \bar{T}\right], \tag{6}$$

where $g \equiv \dfrac{r - \rho}{\phi}$.

Alternatively, let $W(t)$ be defined as this same individual's lifetime "wealth account" or "resource account" for $t \in \left[0, \bar{T}\right]$, where (7) is a differential equation that governs the evolution of this account over the entire lifetime of the individual,

$$\frac{dW(t)}{dt} = rW(t) - A(t) \qquad \forall t \in \left[0, \bar{T}\right]. \tag{7}$$

Let $A(t)$ be a time-dependent annuity that grows at the constant rate η, meaning $\dfrac{dA(t)/dt}{A(t)} = \eta$ which implies $A(t) = A(0)e^{\eta t}$ for $t \in \left[0, \bar{T}\right]$. The boundary values for the wealth account are

$$\begin{aligned}
W(0) &= \int_0^T e^{-rt} w(t) \; dt, && (8)\\
W(\bar{T}) &= 0, && (9)
\end{aligned}$$

in which (8) suggests that the individual is hypothetically endowed up-front with the present value of their total lifetime wealth or resources, and (9) suggests that $A(t)$ would exhaust the individual's lifetime wealth exactly at the date of death. Solving this boundary-value problem yields

$$A(t) = e^{\eta t} \left(\frac{\displaystyle\int_0^T e^{-rt} w(t) \; dt}{\displaystyle\int_0^{\bar{T}} e^{(\eta-r)t} \; dt} \right), \qquad \text{for } t \in \left[0, \bar{T}\right]. \tag{10}$$

It is straightforward to see that if η equals $g \equiv (r-\rho)/\phi$, then (6) is exactly equal to (10), which suggests that an individual who maximizes lifetime utility simply consumes the annuity value of their lifetime resources (where the annuity grows at rate $\eta = g$).

Exercises

1.) Derive equation (6) using two-stage optimal control techniques. Make sure to write legibly and to show all of your work in the derivations.

2.) Derive equation (10) which is the solution to the boundary-value problem given by equations (7), (8), and (9) above. Make sure to write legibly and to show all of your work in the derivations. Verify and discuss the intuition for the reason that (10) is the same as (6) if $\eta = g$.

3.) Using EXCEL simulate the time paths for $c^*(t)$ and for $k(t)$ for $t \in [0, \bar{T}]$. Remember that discretized-differential approximation of $k(t)$ can be employed as an alternative to solving for $k(t)$ analytically. Simulate the life cycle at a numerical step-size of $dt = 0.01$. Assume the following parameter values in numerical calculations: $\rho = 0.045$, $r = 0.035$, $\phi = 2$, and $w(t) = w = \$50000$ for all $t \in [0, T]$.

4.) Upload the derivations to Canvas by **Wednesday 11/30/2022 at 11:59pm**.