ECN 5320/6320: Mathematical Methods in Economics and Finance II

HW4: 401(k) Asset Accumulation with Time-Dependent Interest Rate

Age is a continuous variable indexed by t. A representative individual enters the workforce at $t = 0$, retires at $t = T$, and dies at $t = \bar{T}$. The dates of retirement and death are exogenous. The individual receives a wage-income flow $w(t) = wq(t)$ for $t \in [0, T]$ where w is the constant market wage and $q(t)$ is a longitudinal age-efficiency profile that is used to model the hump-shaped pattern of wage income earnings over the working phase of the life cycle. The following functional form is used

$$q(t) = q_0 + q_1 t + q_2 t^2, \text{ for } t \in [0, T], \tag{1}$$

which is a second-order polynomial. In numerical work, the following parameter values can be used,

$$\begin{aligned} q_0 &= 1, && (2)\\ q_1 &= 0.0315, && (3)\\ q_2 &= -0.00062, && (4) \end{aligned}$$

which indicate that the wage income profile will peak at age 50 (which is $t = 25$ in the model since $t = 0$ corresponds to age 25) and that the ratio of peak wage income to initial income is 1.4, as evidenced in empirical data.[1]

With the above information, the individual's 401(k) asset account balance, $k(t)$, grows at a time-dependent rate of interest, $r(t)$, and evolves according to the following system of differential equations and boundary conditions

$$\begin{aligned} \frac{dk(t)}{dt} &= r(t)k(t) + sw(t), \quad \forall t \in [0, T], && (5)\\ \frac{dk(t)}{dt} &= r(t)k(t) - A, \quad \forall t \in \left[T, \bar{T}\right], && (6)\\ k(0) &= 0, && (7)\\ k(\bar{T}) &= 0. && (8) \end{aligned}$$

where s is a constant contribution rate into the 401(k) asset account that gets annuitized at the date of retirement at rate A such that equation (8) is obeyed. For purposes of numerical work, assume that the interest rate takes the following form,

$$r(t) = r_0 + r_1 t + r_2 t^2 + r_3 t^3, \quad \forall t \in [0, \bar{T}] \quad , \tag{9}$$

with

$$\begin{aligned} r_0 &= 0.022248, && (10)\\ r_1 &= 0.008369, && (11)\\ r_2 &= -0.000346, && (12)\\ r_3 &= 0.000003, && (13) \end{aligned}$$

which matches the time-series of the 3-Month T-Bill over a 55-year period from 1965 to 2020. The particular/definite solutions to this boundary-value problem are given as

$$\begin{aligned} k(t) &= e^{\int^t r(v)dv} \int_0^t sw(v) e^{-\int^v r(j)dj} \, dv, \quad \forall t \in [0, T], && (14)\\ k(t) &= e^{\int^t r(v)dv} \left(k(T) e^{-\int^T r(v)dv} - \int_T^t A e^{-\int^v r(j)dj} \, dv \right), \quad \forall t \in \left[T, \bar{T}\right], && (15) \end{aligned}$$

where $k(T)$ is the accumulated balance of the asset account at the date of retirement.

[1] Gourinchas, Pierre-Olivier, and Jonathan A. Parker (2002). Consumption Over the Life Cycle. *Econometrica* 70(1), 47-89.

Exercises

1.) Given the mathematical setup, solve the boundary-value problem given by equations (5)–(8) in order to derive analytically the time path of $k(t)$ for $t \in [0, T]$ and also separately for $t \in [T, \bar{T}]$, given by equations (14) and (15). Make sure to write legibly and to show all of your work in the derivations.

2.) Using EXCEL simulate the 401(k) asset balance time path for $t \in [0, \bar{T}]$, using the derived analytical solutions to the boundary-value problem (coupled with numerical integration techniques at a numerical step-size of $dv = 0.01$) AND separately using discretized-differential approximation at a numerical step-size of $dt = 0.01$. Assume the following parameter values in numerical calculations: $w = \$50000$, $s = 0.1$, $T = 40$, $\bar{T} = 55$, along with the parameter values for the age-efficiency profile given by equations (2)–(4) and the paramter values for the interest rate given by equations (10)–(13). Make sure to perform a numerical grid search in order to identify the value of A that allows equation (8) to be obeyed.

3.) Upload the derivations and the EXCEL simulation to Canvas by **Monday 10/24/2022 at 11:59pm**.