HW 4 Patrick Neyland

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1. Given: $\frac{dk(t)}{dt} = r(t)\,k(t) + sw(t), \ \forall t \in [0,T]$ (5)

$$\frac{dk(t)}{dt} = r(t)\,k(t) - A, \ \forall t \in [T, \bar{T}] \quad (6)$$

$$k(0) = 0 \qquad k(\bar{T}) = 0$$

G.S. for (5): $k(t) = e^{\int^t r(v)\,dv} \cdot \left\{ z_1 + \int^t sw(v)\, e^{-\int^v r(j)\,dj} \, dv \right\}$

Now move to the particular solution

$$k(0) = \frac{\cancel{e^{\int^0 r(v)\,dv}}}{\cancel{e^{\int^0 r(v)\,dv}}} \cdot \left\{ z_1 + \int^0 sw(v)\, e^{-\int^v r(j)\,di} \, dv \right\} = \frac{\cancel{0}}{\cancel{e^{\int^0 r(v)\,dv}}}$$

$$z_1 + \int^0 sw(v)\, e^{-\int^v r(j)\,di} \, dv = 0$$

$$z_1 = -\int^0 sw(v)\, e^{-\int^v r(j)\,di} \, dv$$

Now plug z_1 back into G.S.

$$e^{\int^t r(v)\,dv} \cdot \left\{ -\int^t sw(v)\, e^{-\int^v r(j)\,di} \, dv + \int^0 sw(v)\, e^{-\int^v r(j)\,di} \, dv \right\}$$

Combine integral terms

P.S. for (5): $k(t) = e^{\int^t r(v)\,dv} \cdot \left\{ -\int_0^t sw(v)\, e^{-\int^v r(j)\,di} \, dv \right\}$
$t \in [0,T]$

This is the time path for the balance of the retirement account for the time period starting at day 1 of working and ending at day 1 of retirement

Repeat for (6)

G.S. for (6): $k(t) = e^{\int^t r(v)\,dv} \cdot \left\{ z_2 - \int^t A e^{-\int r(j)\,dj} \, dv \right\}$

$$k(T) = e^{\int^T r(v)\,dv} \cdot \left\{ z_2 - \int^T A e^{-\int^v r(j)\,dj} \, dv \right\}$$

$$z_2 = k(T) \cdot e^{-\int^T r(v)\,dv} + \int^T A e^{-\int^v r(j)\,dj} \, dv$$

$$k(t) = e^{\int^t r(v)\,dv} \cdot \left\{ k(T) \cdot e^{-\int^T r(v)\,dv} + \int^T A e^{-\int^v r(j)\,dj} \, dv - \int^t A e^{-\int^v r(j)\,dj} \, dv \right\}$$

Simplify

P.S. for (6): $k(t) = e^{\int^t r(v)\,dv} \cdot \left\{ k(T) \cdot e^{-\int^T r(v)\,dv} - \int_T^t A e^{-\int^v r(j)\,dj} \, dv \right\}$

Time path from retirement until death

Both particular solutions match what is given by equations (14) & (15) in the HW. Therefore, we can move on to excel to do the numerical integration

$$r(t) = r_0 + r_1 t + r_2 t^2 + r_3 t^3$$

$$\int r_0 + r_1 t + r_2 t^2 + r_3 t^3 \, dt$$

$$= r_0 t + \frac{r_1}{2} t^2 + \frac{r_2}{3} t^3 + \frac{r_3}{4} t^4$$