Patrick Neyland A02378992

ECN 5320/6320: Mathematical Methods in Economics and Finance II

HW3: 401(k) Asset Accumulation with Hump-Shaped Wages

Age is a continuous variable indexed by t. A representative individual enters the workforce at $t = 0$, retires at $t = T$, and dies at $t = \bar{T}$. The dates of retirement and death are exogenous. The individual receives a wage-income flow $w(t) = wq(t)$ for $t \in [0, T]$ where w is the constant market wage and $q(t)$ is a longitudinal age-efficiency profile that is used to model the hump-shaped pattern of wage income earnings over the working phase of the life cycle. The following functional form is used

$$q(t) = q_0 + q_1 t + q_2 t^2, \text{ for } t \in [0, T], \tag{1}$$

which is a second-order polynomial. In numerical work, the following parameter values can be used,

$$q_0 = 1, \tag{2}$$
$$q_1 = 0.0315, \tag{3}$$
$$q_2 = -0.00062, \tag{4}$$

which indicate that the wage income profile will peak at age 50 (which is $t = 25$ in the model since $t = 0$ corresponds to age 25) and that the ratio of peak wage income to initial income is 1.4, as evidenced in empirical data.[1]

With the above information, the individual's 401(k) asset account balance, $k(t)$, grows at the market rate of interest, r, and evolves according to the following system of differential equations and boundary conditions

$$\frac{dk(t)}{dt} = rk(t) + sw(t), \quad \forall t \in [0, T], \tag{5}$$
$$\frac{dk(t)}{dt} = rk(t) - A, \quad \forall t \in [T, \bar{T}], \tag{6}$$
$$k(0) = 0, \tag{7}$$
$$k(\bar{T}) = 0. \tag{8}$$

Solving this boundary-value problem can identify the individual's constant retirement annuity in (6), which is given as

$$A = \frac{\displaystyle\int_0^T sw(t)e^{-rt}dt}{\displaystyle\int_T^{\bar{T}} e^{-rt}dt}, \quad \forall t \in [T, \bar{T}], \tag{9}$$

and with substitution in for $w(t)$ this can be re-written as the following via integration

$$A = \frac{sw\left(q_0 r^2 + q_1 r + 2q_2 - \left(q_2 r^2 T^2 + \left(q_1 r^2 + 2q_2 r\right)T + q_0 r^2 + q_1 r + 2q_2\right)e^{-rT}\right)}{r^2\left(e^{-rT} - e^{-r\bar{T}}\right)}, \quad \forall t \in [T, \bar{T}]. \tag{9'}$$

Exercises Solve the ODEs in (5) & (6) and find P.S. at (7) & (8)

√**1.)** Given the mathematical setup, solve the boundary-value problem given by equations (5)–(8) in order to derive analytically the time path of $k(t)$ for $t \in [0, T]$ and also separately for $t \in [T, \bar{T}]$. Make sure to write legibly and to show all of your work in the derivations.

2.) Making use of the identity that $k(T) = k(T)$, derive equation (9). Make sure to write legibly and to show all of your work in the derivations.

3.) Use integration by parts to perform the integration in equation (9) in order to derive equation (9′). Make sure to write legibly and to show all of your work in the derivations.

4.) Using EXCEL simulate the 401(k) asset balance time path for $t \in [0, \bar{T}]$ of the representative individual, either using the derived analytical solutions to the boundary-value problem and/or using discretized-differential approximation at a numerical step-size of $dt = 0.01$. Assume the following parameter values in numerical calculations: $r = 0.035$, $w = \$50000$, $s = 0.1$, $T = 40$, $\bar{T} = 55$, along with the parameter values for the age-efficiency profile given by equations (2)–(4).

5.) Upload the derivations and the EXCEL simulation to Canvas by **Wednesday 10/12/2022 at 11:59pm.**

[1] Gourinchas, Pierre-Olivier, and Jonathan A. Parker (2002). Consumption Over the Life Cycle. *Econometrica* 70(1), 47-89.

1. $\frac{dk(t)}{dt} = rk(t) + sw(t)$, for $t \in [0, T]$ (5)

$\frac{dk(t)}{d(t)} = rk(t) - A$, for $t \in [T, \bar{T}]$ (6)

$k(0) = 0 \quad k(\bar{T}) = 0$ starting & ending values are zero.
-didn't receive any inheritance and not leaving any inheritance

General Solution for (5) - working years

$$e^{rt} \cdot \left\{ Z_1 + \int^{t} sw(v) e^{-rv} dv \right\}$$

$$k(0) = \cancel{e^{r(0)}} \cdot \left\{ Z_1 + \int^{0} sw(v) e^{-rv} dv \right\} = 0$$

$$Z_1 = -\int^{0} sw(v) e^{-rv} dv$$

$$k(t) = e^{rt} \cdot \left\{ -\int^{0} sw(v) e^{-rv} dv + \int^{t} sw(v) e^{-rv} dv \right\}$$

$$= e^{rt} \int_{0}^{t} sw(v) e^{-rv} dv$$ particular solution for $t \in [0, T]$

General Solution for (6) - retirement

$$k(\bar{T}) = e^{r\bar{T}} \cdot \left\{ Z_2 - \int^{\bar{T}} A e^{-rv} dv \right\} = 0$$

$$k(\bar{T}) = \cancel{e^{r0}} \cdot \left\{ Z_2 - \int^{t} A e^{-rv} dv \right\} =$$

$$Z_2 = A \int^{\bar{T}} e^{-rv} dv$$

$$k(\bar{T}) = e^{rt} \cdot \left\{ A \int^{\bar{T}} e^{-rv} dv - A \int^{t} e^{-rv} dv \right\}$$

$$k(\bar{T}) = e^{rt} \cdot A \cdot \int_{t}^{\bar{T}} e^{-rv} dv$$ particular solution for $t \in [T, \bar{T}]$

2. Add the boundary T to both of the particular solutions in 1. which makes them both $k(T)$. Because $k(T) = k(T)$, we can set these solutions equal to each other and solve for A

$$e^{rt}\int_0^T sw(v)e^{-rv}\,dv = e^{rt}A\cdot\int_T^{\bar{T}} e^{-rv}\,dv$$

divide both sides by $e^{rt}\cdot\int_T^{\bar{T}} e^{-rv}\,dv$ to isolate A

$$A = \frac{\cancel{e^{rt}}\int_0^T sw(v)e^{-rv}\,dv}{\cancel{e^{rt}}\int_T^{\bar{T}} e^{-rv}\,dv}$$

$$A = \frac{\int_0^T sw(v)e^{-rv}\,dv}{\int_T^{\bar{T}} e^{-rv}\,dv}, \quad [T, \bar{T}$$

Prep for exercise 3

Consider variable nature of $w(v)$. $w(v) = w\cdot q(v)$

$$q(t) = q_0 + q_1 t + q_2 t^2 \text{ for } t \in [0, T]$$

3. Start by integrating numerator

$$\int_0^T sw(t)e^{-rt}\,dt \qquad w(t)=wq(t)=w(q_0+q_1t+q_2t^2)$$

$$sw\int_0^T (q_0+q_1t+q_2t^2)e^{-rt}\,dt$$

$f=q_0+q_1t+q_2t^2 \qquad g'=e^{-rt}$

$f'=2q_2+q_1 \qquad g=-\dfrac{e^{-rt}}{r}$

$$-\frac{e^{-rt}(q_0+q_1t+q_2t^2)}{r} - \int -\frac{(2q_2+q_1)\cdot e^{-rt}}{r}\,dt$$

factor out $-\frac{1}{r}$

$$-\frac{1}{r}\int(2q_2+q_1)e^{-rt}\,dt$$

$f=2q_2+q_1 \qquad g'=e^{-rt}$

$f'=2 \qquad g=-\dfrac{e^{-rt}}{r}$

$$-\frac{e^{-rt}(2q_2+q_1)}{r} - \int -\frac{2e^{-rt}}{r}$$

$u=-rt \qquad dt=-\frac{1}{r}\,du$

$$\frac{2q_2}{r^2}\int e^u\,du$$

$$=\frac{2q_2e^u}{r^2}$$

$$=\frac{2q_2e^{-rt}}{r^2}$$

Combin all Solved integrals

(on next page)

3. continued

$$-\frac{sw\left(q_2\left(rt\left(rt+2\right)+2\right)+r\left(q_1\left(rt+1\right)+rq_0\right)\right)e^{-rt}}{r^3}$$

Sub in T and simplify

$$\frac{sw\left(q_0r^2+q_1r+2q_2-\left(q_2r^2T^2+\left(q_1r^2+2q_2r\right)T+q_0r^2+q_1r+2q_2\right)e^{-rT}\right)}{r^3}$$

now the denominator

$$\int_T^{\bar{T}} e^{-rt}\,dt = \frac{e^{-rT}-e^{-r\bar{T}}}{r}$$

now to divide the top integral by the bottom.

$$\frac{sw\left(\ldots e^{-rt}\right)}{r^{\cancel{3}2}} \cdot \frac{\cancel{r}}{e^{-rT}-e^{-r\bar{T}}} =$$

$$A = \frac{sw\left(q_0r^2+q_1r+2q_2-\left(q_2r^2T^2+\left(q_1r^2+2q_2r\right)T+q_0r^2+q_1r+2q_2\right)e^{-rT}\right)}{r^2\left(e^{-rT}-e^{-r\bar{T}}\right)}$$