ECN 5320/6320: Mathematical Methods in Economics and Finance II

HW8: LCPI Model of Consumption (One-Stage Control Problem)

Time is a continuous variable indexed by t. The representative individual enters the workforce at $t = 0$, retires at $t = T$, and dies at $t = \bar{T}$. The dates of retirement and death are exogenous. There is no uncertainty of any kind in the model (i.e., no mortality risk, no wage income uncertainty, no asset return uncertainty, etc.). Wage income, $w(t)$, is earned for all $t \in [0, T]$. Private assets grow at a constant market rate of interest, r. The representative individual has perfect foresight over the entire life cycle, passes away with no assets, and constructs a consumption program that maximizes discounted lifetime utility. Combining the above information, the optimal control problem is

$$\max_{\{c(t)\}} \quad \int_0^{\bar{T}} e^{-\rho t} u[c(t)] \; dt \tag{1}$$

where ρ is the rate of time preference and where $u[c(t)] = (c(t)^{1-\phi} - 1)/(1 - \phi)$ is the CIES instantaneous utility function.[1] The representative individual is constrained by

$$\begin{aligned} \frac{dK(t)}{dt} &= rK(t) - c(t), && (2) \\ K(0) &= \int_0^T e^{-rt} w(t) \; dt, && (3) \\ K(\bar{T}) &= 0. && (4) \end{aligned}$$

where $K(0)$ is the present value lifetime resources. Application of the *Maximum Principle* for a fixed-endpoint optimal problem yields analytical solutions for the *optimal state path* and the *optimal control*. These are given as (5) and (6) respectively.

$$\begin{aligned} K(t) &= \frac{K(0)\left[e^{(g-r)\bar{T}+rt} - e^{gt}\right]}{e^{(g-r)\bar{T}} - 1} && (5) \\ c^*(t) &= \frac{K(0)e^{gt}}{\displaystyle\int_0^{\bar{T}} e^{(g-r)t} \; dt} = \frac{K(0)e^{r\bar{T}+gt}\,(g-r)}{e^{g\bar{T}} - e^{r\bar{T}}} && (6) \end{aligned}$$

where $g \equiv \dfrac{r-\rho}{\phi}$. As an alternative to deriving (5) analytically, the optimal path for the lifetime resource account, $K(t)$, can be approximated for $t \in [0, \bar{T}]$. First, approximate the actual path of the lifetime resource account, $K(t)$, by transforming equation (2) into differential form such that exact or true changes in the resource account, $\Delta K(t)$, will approximately follow

$$dK(t) = (rK(t) - c^*(t))\, dt \tag{7}$$

given a discrete change in time of $dt = \Delta t$ in advancing from one period during the life cycle to the next. With $K(0)$ given by (3) this implies that the path of the lifetime resource account will follow

$$\begin{aligned} K(t + dt) &= K(t) + \Delta K(t) \\ &\approx K(t) + dK(t) \\ &= K(t) + (rK(t) - c^*(t))\, dt && (8) \end{aligned}$$

for all $t \in [0, \bar{T}]$, remembering that the approximation becomes more precise as $dt \to 0$.

Exercises

1.) Derive equation (6). Make sure to write legibly and to show all of your work in the derivations.

2.) Using EXCEL simulate the time paths for $c^*(t)$ and for $K(t)$ for $t \in [0, \bar{T}]$. Remember that discretized-differential approximation of $K(t)$ can be employed as an alternative to solving for (5) analytically. Simulate the life cycle at a numerical step-size of $dt = 0.01$. Assume the following parameter values in numerical calculations: $\rho = 0.045$, $r = 0.035$, $\phi = 2$, and $w(t) = w = \$50000$ for all $t \in [0, T]$.

3.) Upload the derivations and the EXCEL simulation to Canvas by **Wednesday 11/23/2022 at 11:59pm**.

[1] Note that $u[c(t)] = \ln[c(t)]$ is a special case of the CIES utility function as $\phi \to 1$.This limiting result can be derived using *L'Hôpital's Rule*.