ECN 5320/6320: Mathematical Methods in Economics and Finance II

HW5: Save More Tomorrow (SMarT Program)

Age is a continuous variable indexed by t. A representative individual enters the workforce at $t = 0$, retires at $t = T$, and dies at $t = \bar{T}$. The dates of retirement and death are exogenous. The individual receives a wage-income flow $w(t) = wq(t)$ for $t \in [0, T]$ where w is the constant market wage and $q(t)$ is a longitudinal age-efficiency profile that is used to model the hump-shaped pattern of wage-income earnings over the working phase of the life cycle. The following functional form is used

$$q(t) = q_0 + q_1 t + q_2 t^2, \text{ for } t \in [0, T], \tag{1}$$

which is a second-order polynomial. In numerical work, the following parameter values can be used,

$$q_0 = 1, \tag{2}$$
$$q_1 = 0.0315, \tag{3}$$
$$q_2 = -0.00062, \tag{4}$$

which indicate that the wage-income profile will peak at age 50 (which is $t = 25$ in the model since $t = 0$ corresponds to age 25 by assumption) and that the ratio of peak wage income to initial wage income is 1.4, as evidenced in empirical data.

We assume that the representative individual is automatically enrolled into participation in the SMarT program upon entry into the labor force. The individual saves a constant fraction, s, of their entry-level wage earnings. In addition, the individual can also save at a higher rate γ on wage raises, such that $s \leq \gamma \leq 1$. Note that if $\gamma = s$, then no SMarT saving is in force. The individual's asset account balance, $k(t)$, grows at the market rate of interest, r. With this information, the asset account balance evolves according to the following system of differential equations and boundary conditions

$$\frac{dk(t)}{dt} = rk(t) + sw(0) + \gamma\left[w(t) - w(0)\right], \quad \forall t \in [0, T], \tag{5}$$
$$\frac{dk(t)}{dt} = rk(t) - A, \quad \forall t \in \left[T, \bar{T}\right], \tag{6}$$
$$k(0) = 0, \tag{7}$$
$$k(\bar{T}) = 0. \tag{8}$$

Solving this boundary-value problem can identify the individual's constant retirement annuity in (6), which is given as

$$A = \frac{\displaystyle\int_0^T \left[sw(0) + \gamma\left(w(t) - w(0)\right)\right] e^{-rt} dt}{\displaystyle\int_T^{\bar{T}} e^{-rt} dt}, \quad \forall t \in [T, \bar{T}] \;. \tag{9}$$

Note that the individual's rule-of-thumb consumption path is given as

$$c(t) = w(t) - sw(0) - \gamma\left[w(t) - w(0)\right], \quad \forall t \in [0, T], \tag{10}$$
$$c(t) = A, \quad \forall t \in \left[T, \bar{T}\right], \tag{11}$$

which is implied by (5)–(8). For purposes of normative welfare calculations, assume that instantaneous utility is measured via the functional form,

$$u(c(t)) = \frac{c(t)^{1-\phi} - 1}{1 - \phi} \tag{12}$$

in which ϕ is the inverse elasticity of intertemporal substitution. Lastly, assume that experienced lifetime utility is measured as

$$U = \int_0^{\bar{T}} e^{-\rho t} u(c(t))\; dt \tag{13}$$

where ρ is the rate of time preference (discount rate) used in measuring lifetime well-being.

Exercises

1.) Given the mathematical setup, solve the boundary-value problem given by equations (5)–(8) in order to derive analytically the time path of $k(t)$ for $t \in [0, T]$ and also separately for $t \in [T, \bar{T}]$. Make sure to write legibly and to show all of your work in the derivations.

2.) Making use of the identity that $k(T) = k(T)$, derive equation (9). Make sure to write legibly and to show all of your work in the derivations.

3.) Using EXCEL simulate the asset balance time path, $k(t)$, for $t \in [0, \bar{T}]$ of the representative individual, either using the derived analytical solutions to the boundary-value problem and/or using discretized-differential approximation at a numerical step-size of $dt = 0.01$. Assume the following parameter values in numerical calculations: $r = 0.035$, $w = \$50000$, $s = 0.1$, $\gamma = 0.3$, $T = 40$, $\bar{T} = 55$, along with the parameter values for the age-efficiency profile given by equations (2)–(4).

4.) Given $\rho = 0.035$ and holding all other parameters at their values stated above, allow the SMarT saving rate to be free and find the particular value of the SMarT saving rate that maximizes lifetime well-being, $\gamma^* = \underset{[s,1]}{\arg\max}\,\{U\}$, for the three cases of $\phi = 0.5$, $\phi = 1$, and $\phi = 2$.

5.) Upload the derivations and the EXCEL simulation to Canvas by **Monday 10/31/2022 at 11:59pm**.