ECN 5320/6320: Mathematical Methods in Economics and Finance II

HW6: Political Business Cycle & Exponential Memory Discounting

Consider the following economic problem: Time is continuous and indexed by t. From the perspective of being newly elected (or newly re-elected), it is assumed that at $t = 0$ an incumbent government desires to be re-elected at the next future election date $t = E$. As such, the government sets fiscal and monetary policy that influences (selects) the time path of the unemployment rate, $u(t)$, and the actual inflation rate, $\pi(t)$, in order to maximize its vote function. Given an instantaneous payoff (penalty) vote function $\nu(u, \pi)$ with $\partial\nu/\partial u < 0$ and $\partial\nu/\partial\pi < 0$, this problem can be stated formally as

$$\max \quad V = \int_0^E \nu(u(t), \pi(t)) M(E - t)\; dt, \tag{1}$$

where $M(x)$ is a memory discount function in general form for a retrospective discounting delay of x with $M(0) = 1$ and $dM/dx < 0$. The retrospective discount function governs how voters devalue memories about past pain from inflation and unemployment over the course of the incumbent government's term in office. Here, we assume that the memory discount function takes the exponential form: $M_e(x) = \exp[-\rho x]$ for a discounting delay of length x, where $\rho \in \mathbb{R}^+$ is the exponential memory discount rate.

The incumbent government is constrained by an expectations-augmented Phillips Curve relationship

$$\pi(t) = \varphi(u(t)) + \psi\varepsilon(t). \tag{2}$$

The parameter $\psi \in (0, 1]$ is the proportion by which expected inflation materializes into actual inflation.[1] Note that $d\varphi/du < 0$ and also note that the *expected* rate of inflation $\varepsilon(t)$ evolves adaptively in the vintage framework

$$\frac{d\varepsilon(t)}{dt} = \gamma[\pi(t) - \varepsilon(t)], \text{ for } t \in [0, E]\,, \tag{3}$$

$$\varepsilon(0) \textit{ given}, \tag{4}$$

$$\varepsilon(E) \textit{ free}. \tag{5}$$

Equation (3) represents the idea that revisions to the expected inflation rate are proportional to the forecast error by $\gamma > 0$. Therefore, γ can be considered the speed of adaptation or adjustment in inflation expectations.

The following functional forms are used to solve the model explicitly

$$\nu(u, \pi) = -u(t)^2 - \theta\pi(t), \tag{6}$$

$$\varphi(u) = \alpha - \zeta u(t), \tag{7}$$

where $\theta > 0$ governs the intensity with which inflation enters the penalty vote function and where $\zeta > 0$ is the slope of and $\alpha > 0$ affects the intercepts of the short-run and long-run Phillips Curves. Note that when $\varepsilon = \pi$ given $\psi = 1$, the long-run natural rate of unemployment is identified. This is equal to α/ζ for the functional forms specified in (2) and (7). Note also that $\partial\nu/\partial u < 0$ and $\partial\nu/\partial\pi < 0$ given (6). With the appropriate substitutions, the Hamiltonian is written as

$$\mathcal{H}(t) = \left[-u(t)^2 + \theta\zeta u(t) - \alpha\theta - \theta\psi\varepsilon(t)\right] M_e(E - t) + \lambda(t)\gamma\left[\alpha - \zeta u(t) - (1 - \psi)\varepsilon(t)\right]. \tag{8}$$

Assuming that $u(t) \in [0, 1]$ for all $t \in [0, E]$, application of the Maximum Principle for finite-horizon, free-endpoint control problems yields a system of optimality conditions

$$\frac{\partial\mathcal{H}(t)}{\partial u(t)} = \left[-2u(t) + \theta\zeta\right] M_e(E - t) - \gamma\zeta\lambda(t) \stackrel{\text{set}}{=} 0, \tag{9}$$

$$\frac{\partial\mathcal{H}(t)}{\partial\varepsilon(t)} = -\theta\psi M_e(E - t) - (1 - \psi)\gamma\lambda(t) \stackrel{\text{set}}{=} -\frac{d\lambda(t)}{dt}, \tag{10}$$

[1] The parameter ψ determines whether or not a tradeoff exists between inflation and unemployment in the long run, wherein $d\varepsilon/dt = 0$ and $\varepsilon = \pi$ by (3). In this case the Phillips Curve becomes $\pi = (1 - \psi)^{-1} \times \varphi(u)$ with slope $(1 - \psi)^{-1} \times d\varphi/du$. The Phillips Curve is vertical in the long run if $\psi = 1$. But if $\psi < 1$, then a tradeoff exists in the long run given that $d\varphi/du < 0$.

$$\frac{\partial \mathcal{H}(t)}{\partial \lambda(t)} = \gamma\left[\alpha - \zeta u(t) - (1-\psi)\varepsilon(t)\right] \overset{\text{set}}{=} \frac{d\varepsilon(t)}{dt}, \tag{11}$$

$$\varepsilon(0) \textit{ given}, \tag{12}$$

$$\lambda(E) = 0, \tag{13}$$

where $u(t)$ is the control variable, $\varepsilon(t)$ is the state variable, and $\lambda(t)$ is a multiplier function. Solving this system of equations yields the optimal path of the unemployment rate and the inflation rate

$$u^*(t) = \frac{\theta\zeta}{2}\left[1 + \frac{\gamma\psi \exp[(1-\psi)\gamma t]}{M_e(E-t)} \int_t^E M_e(E-s)\exp[-(1-\psi)\gamma s]\, ds\right], \text{ for } t \in [0,E], \tag{14}$$

upon making the substitution for the memory discount function, (14) can be rewritten as

$$u^*(t) = \frac{\theta\zeta}{2}\left[1 + \frac{\gamma\psi \exp[(1-\psi)\gamma t - \rho E]}{\exp[-\rho(E-t)]\,(\rho-(1-\psi)\gamma)}\left[\exp[(\rho-(1-\psi)\gamma)\,E] - \exp[(\rho-(1-\psi)\gamma)\,t]\right]\right], \text{ for } t \in [0,E], \tag{14$'$}$$

where s is a dummy variable of integration. Following equation (14) (or (14$'$)) maximizes the hope of re-election for an opportunistic incumbent government. It is important to recognize that

$$\lim_{t\to E}\{u^*(t)\} = \frac{\theta\zeta}{2}$$

which can be used to definitize the unobservable preference parameter, θ, given numerical estimates of the slope of the *Phillips Curve* and the unemployment rate that immediately precedes an election.

The optimal path for the inflation rate, $\pi^*(t)$, can be approximated alternatively for $t \in [0,E]$. First, approximate the actual path of the expected inflation rate, $\varepsilon(t)$, by transforming equation (3) into differential form such that exact or true changes in the expected inflation rate, $\Delta\varepsilon(t)$, will approximately follow

$$d\varepsilon(t) = (\gamma[\pi(t) - \varepsilon(t)])\, dt \tag{15}$$

given a discrete change in time of $dt = \Delta t$ in advancing from one period during the term in office to the next. With $\varepsilon(0)$ *given* this implies that the path of the expected inflation rate will follow

$$\begin{aligned} \varepsilon(t+dt) &= \varepsilon(t) + \Delta\varepsilon(t) \\ &\approx \varepsilon(t) + d\varepsilon(t) \\ &= \varepsilon(t) + (\gamma[\pi(t) - \varepsilon(t)])\, dt \end{aligned} \tag{16}$$

for all $t \in [0,E]$, remembering that the approximation becomes more precise as $dt \to 0$. Lastly, insert (14$'$) and (16) into (2) to approximate $\pi^*(t)$.

Exercises

1.) Derive equation (14$'$). Make sure to write legibly and to show all of your work in the derivations.

2.) Using EXCEL simulate the time path of the exponential memory discount function $M_e(E-t) = \exp[-\rho(E-t)]$ for $t \in [0,E]$ at a numerical step-size of $dt = 0.01$. Assume the following values for ρ, the memory discount rate: $\rho = 0.50$, $\rho = 0.95$, and $\rho = 1.35$, which reflects "low", "moderate", and "high" rates of memory decay in voters. Assume a parameter value of $E = 4$.

3.) Using EXCEL simulate the time path for $u^*(t)$ and for $\pi^*(t)$ for $t \in [0,E]$, for each of the three numerical values for the memory discount rate. Use discretized-differential approximation of $\varepsilon(t)$ at the same numerical step-size of $dt = 0.01$. Assume an initial condition on inflation expectations of $\varepsilon(0) = 0.041660142$. Also assume the following parameter values in numerical calculations: $\gamma = 0.14$, $\psi = 1$, $\zeta = 0.6$, $\theta = 0.133$, $\alpha = 0.03$.

4.) Upload the derivations and the EXCEL simulation to Canvas by **Wednesday 11/9/2022 at 11:59pm**.