Exponencial na unha!

Tópicos em controle

IME USP

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A Matriz

Queremos calcular $\exp(tA)$ onde

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{bmatrix}$$

A minha estratégia será usar a definição

$$\exp(tA) = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$$

e tentar inferir as somas dos termos $a_{ij}(t)$ de $\exp(tA)$.

Contas

		$A^{(1,3,5,7)}$	$\rightarrow$	0	1	0	0
				$3\omega^2$	0	0	2ω
$A^{(2,4,6)}$				0	0	0	1
$\downarrow$				0	-2ω	0	0
$3\omega^2$	0	0	2ω	0	$-\omega^2$	0	0
0	$-\omega^2$	0	0	$-3\omega^4$	0	0	$-2\omega^3$
0	-2ω	0	0	$-6\omega^3$	0	0	$-4\omega^2$
$-6\omega^3$	0	0	$-4\omega^2$	0	$2\omega^3$	0	0
$-3\omega^4$	0	0	$-2\omega^3$	0	ω^4	0	0
0	ω^4	0	0	$3\omega^6$	0	0	$2\omega^5$
0	$2\omega^3$	0	0	$6\omega^5$	0	0	$4\omega^4$
$6\omega^5$	0	0	$4\omega^4$	0	$-2\omega^5$	0	0
$3\omega^6$	0	0	$2\omega^5$	0	$-\omega^6$	0	0
0	$-\omega^6$	0	0	$-3\omega^8$	0	0	$-2\omega^7$
0	$-2\omega^5$	0	0	$-6\omega^7$	0	0	$-4\omega^6$
$-6\omega^7$	0	0	$-4\omega^6$	0	$2\omega^7$	0	0

Tabela dos coeficientes - Linhas 1 e 2

0	1	2	3	4	5	6	7
a_{11}	0	$3\omega^2$	0	$-3\omega^4$	0	$3\omega^6$	0
a_{12}	1	0	$-\omega^2$	0	ω^4	0	$-\omega^6$
a_{13}	0	0	0	0	0	0	0
a_{14}	0	2ω	0	$-2\omega^3$	0	$2\omega^5$	0
$*$	—	—	—	—	—	—	—
a_{21}	$3\omega^2$	0	$-3\omega^4$	0	$3\omega^6$	0	$-3\omega^8$
a_{22}	0	$-\omega^2$	0	ω^4	0	$-\omega^6$	0
a_{23}	0	0	0	0	0	0	0
a_{24}	2ω	0	$-2\omega^3$	0	$2\omega^5$	0	$-2\omega^7$
$*$	t	$t^2/2!$	$t^3/3!$	$t^4/4!$	$t^5/5!$	$t^6/6!$	$t^7/7!$

0	1	2	3	4	5	6	7
a_{31}	0	0	$-6\omega^3$	0	$6\omega^5$	0	$-6\omega^7$
a_{32}	0	-2ω	0	$2\omega^3$	0	$-2\omega^5$	0
a_{33}	0	0	0	0	0	0	0
a_{34}	1	0	$-4\omega^2$	0	$4\omega^4$	0	$-4\omega^6$
$*$	—	—	—	—	—	—	—
a_{41}	0	$-6\omega^3$	0	$6\omega^5$	0	$-6\omega^7$	0
a_{42}	-2ω	0	$2\omega^3$	0	$-2\omega^5$	0	$2\omega^7$
a_{43}	0	0	0	0	0	0	0
a_{44}	0	$-4\omega^2$	0	$4\omega^4$	0	$-4\omega^6$	0
$*$	t	$t^2/2!$	$t^3/3!$	$t^4/4!$	$t^5/5!$	$t^6/6!$	$t^7/7!$

Cálculo do a_{11}

Agora $\exp(tA) = I + \sum_{n=1}^{\infty} \frac{t^n A^n}{n!}$ então para a_{11} temos:

$$a_{11} = 1 + 3(\frac{(\omega t)^2}{2!} - \frac{(\omega t)^4}{4!} + \frac{(\omega t)^6}{6!}) =$$

$$= 1 - 3(\sum_{k=1}^{\infty} \frac{(i\omega t)^{2k}}{(2k)!})$$

$$s_1(t) = e^{i\omega t} = \sum \frac{(i\omega t)^n}{n!}$$

$$s_2(t) = e^{-i\omega t} = \sum \frac{(-i\omega t)^n}{n!}$$

$$\sum_{k=1}^{\infty} \frac{(i\omega t)^{2k}}{(2k)!} = \frac{s_1(t) + s_2(t)}{2} - 1 = \cos(\omega t) - 1$$

$$a_{11}(t) = 1 - 3(\cos(\omega t) - 1) = 4 - 3\cos(\omega t)$$

Os outros coeficientes ficam como exercício!