Triangulations of an n-gon

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1 Introduction

In a letter from Euler to Christian Goldbach in 1751 Euler described the following problem:
How many ways may a convex polygon of $n+2$ edges may be triangulated by $n-1$ nonintersecting diagonals.
This can be defined in less formal terms as:
Find the number of ways that the interior of a convex polygon can be divided into triangles by drawing nonintersecting diagonals where $n \geq 3$.
This is the exact same problem as the well known puzzle of if there are $2n$ friends sitting at a round table, how many ways can they shake hands without crossing handshakes.

1.1 Convex Polygon

A convex polygon is a polygon whose interior is a convex set. A convex set is defined for every pair of points within the topological object where every point on the straight line segment which joins them is also in the object. For example you have two points x and y within a polygon and there is a straight line, l joining x and y. If l lies within the polygon then it is in the convex set. If any part of l lies outwith the boundries of the polygon then it is **not** in the convex set.
A convex polygon also holds the following properties:

- Every internal angle is less than or equal to 180 degrees.
- Every line segment between two vertices remains inside or on the boundary of the polygon.

2 Proof that the amount of triangulations is C_n

Theorem: The number of triangulations of a convex polygon with $n+2$ vertices is the Catlan number, $C_n = \frac{1}{n+1}\binom{2n}{n}$.
Proof: Let P_{n+2} be a convex polygon with vertices labelled from 1 to n+2.
Let τ be the set of triangulations of P_{n+2} where τ has two elements.

We will show that t_{n+2} is the Catalan number, C_n.
Let ϕ be a map from τ_{n+2} to τ_{n+1} given by contracting the edge $\{1, n+2\}$ of P_{n+2}. Let T be an element of τ_{n+1}. It is important to note that the number of triangulations of τ_{n+2} that map to T equals the degree of vertex 1 in T.
Let's define $deg(i, T)$ to be the degree of vertex i of T.
It follows that, $t_{n+1} = \sum\limits_{T \in \tau_{n+1}} deg(1, T)$.
Since this polygon is convex the above formula holds for all vertices of T.
$\therefore (n+1).t_{n+2} = \sum\limits_{i=1}^{n+1} \sum\limits_{T \in \tau_{n+1}} deg(i, T)$
$= \sum\limits_{T \in \tau_{n+1}} \sum\limits_{i=1}^{n+1} deg(i, T)$
$= 2(2n-1).t_{n+1}$
The above line follows as the sum of degrees of all verticies of T double counts the number of edges of T and the number of diagonals of T.
Since we need $n-2$ diagonals lets solve for t_{n+2}:
$(n+1)t_{n+2} = 2(2n-1)t_{n+1}$
$=> t_{n+2} = \frac{2(2n-1)}{n+1} t_{n+1} = 2^n . \frac{2n-1}{n+1} . \frac{2n-3}{n} ... \frac{3}{3} . \frac{1}{2}$
$= \frac{(2n!)}{(n+1)!n!}$
$= \frac{1}{n+1}\binom{2n}{n} \square$

References