Introduction to Dyck Paths

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1 Introduction

A Dyck path of length $2n$ is a lattice path from $(0,0)$ to $(2n,0)$ with steps:

$$\begin{aligned} u &= (1,1) \\ d &= (1,-1) \end{aligned} \tag{1}$$

that never go below the x-axis.

We can see that if D denotes the set of all Dyck paths then one has the following relation for D:

$$\text{D} = 1 + \text{udD} + \text{uDdD} \tag{2}$$

but since the first path does not have to have its first pattern as ud then we can generalise to

$$D = \epsilon + \text{uDdD} \tag{3}$$

by letting $1 = \epsilon$ (empty set).[**?**]

2 Proof of composition and generalisation

Keep encodings as above with ϵ being the empty set.
Pictorially:

$$\begin{aligned} D &= ... \\ &= \epsilon + \text{ud} + \text{udud} + \text{uudd} + \text{ududud} + \text{uduudd} + ... \\ &= \epsilon + \text{uDdD} \end{aligned} \tag{4}$$

From this we can get the Catalan number by looking at formal power series:

$$\psi : \mathbb{Q} << u, d >> \rightarrow \mathbb{Q}[[x]] \tag{5}$$

and by letting $u \to x$ and $d \to 1$ for ψ.
To show the relation let's look at the Catalan numbers generation function, c:

$$\text{c} = \psi(D) \text{ and } c_{coeff} = \sum_{n\geq 0} c_n x^n \tag{6}$$

so from our $u \to x$ and $d \to 1$ propositions we get:

$$\begin{aligned} &c = \psi(D) \\ &= \psi(\epsilon + uDdD) \\ &= \psi(\epsilon + xDD) \\ &let\ \epsilon = 1 \text{ also, so} \\ &\therefore c = \psi(1 + xD.D) \\ &= \psi(1 + xD^2) \end{aligned}$$

so for Catalan numbers, the recurrence is

$$c = 1\text{+xc}^2 \tag{8}$$

which is analogous to $1 + xD^2$, so the formal power series of $c = 1 + xc^2$ is the same as the formal power series of $D = 1 + xD^2$ which is $1, 1, 2, 5, 14, 42, 132, ...$ and this is the Catalan numbers.
This is given by:

$$\begin{aligned} &c(\text{x}) = \sum 1 + xc(x)^2 \\ &= \sum \tfrac{1-\sqrt{1-4x}}{2x} \\ &= 1 + \text{x} + 2\text{x} + 5\text{x}^2 + 14x^3 + 42^4 + 132^5 + ... \end{aligned} \tag{9}$$

$\square$

References

[1] Sergey Kitaev, *Patterns in permutations and words*, Springer, 2011.