Introduction to Doubly Alternating Baxter Permutations

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1 Introduction

A Baxter permutation $\sigma \in S_n$ is a permutation of length n which satisfies the following properties: there are no indicies $i < j < k$ such that $\sigma_{j+1} < \sigma_i < \sigma_k < \sigma_j$ **or** $\sigma_j < \sigma_k < \sigma_i < \sigma_{j+1}$
For Baxter permutations of length n:

- mirror: $\sigma_i^* = \sigma_{n+1-i}$
- complement: $\sigma_i^c = n + 1 + \sigma_i$
- inverse: $\sigma_i^{-1} = \sigma_j \Leftrightarrow \sigma_j = i$

A permutation is *doubly alternating* if it is alternating **and** its inverse is alternating.

2 Relation to Catalan numbers

We wish to prove that the number of doubly alternating Baxter permutations of length $2n + \epsilon$ where $\epsilon = 0$ or $\epsilon = 1$ is the Catalan number:

$$\mathrm{C}_n = \tfrac{1}{n+1}\tbinom{2n}{n}$$

The length $2n + \epsilon$ is denoted by $\delta_{2n+\epsilon}$. By considering $\sigma = \emptyset$, where $\emptyset$ denotes the empty permutation, we have $\delta_0 = 1$.