Bijective proof between 132-avoiding permutations and Dyck Paths

Stuart Paton

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1 Introduction

Given the Catalan structures Dyck paths and 132-avoiding permutations commonly known as the One Stack Sortable Permutations can we formulate a bijective proof to show the equivilance and hence convert between Dyck Paths and 132-avoiding permutations? In this paper I will show the proof known as Knuth's bijection[1] in order to answer this question.

2 132-avoiding permutation to Dyck path

Knuth[2] gives a bijection from 312-avoiding permutations. By taking the complement of 312, it is a bijection form 132-avoiding permutations to Dyck paths. We are by describing the bijection from 132-avoiding permutations to Dyck Paths. Henceforth this will be described as the bijection between Dyck paths and 132-avoiding permutations.
Let $\sigma = \alpha n \beta$ be a 132-avoiding permutation of length n. As we saw in the proof of One Stack Sortable Permutations, every element of α is larger than every element of β or else a 132 pattern would be formed by the permutation.
When converting between 132-avoiding permutations and Dyck paths we use the encodings that we formulated when defining Dyck paths. These are up $\to u$ and down $\to d$. We define the bijection between Dyck paths and 132-avoiding permutations recursively by:

$$\mathrm{f}(\sigma) = uf(\alpha)df(\beta) \text{ and } f(\epsilon) = \epsilon \tag{1}$$

where ϵ is the empty word, or permutation. Therefore by the bijection between Dyck paths and 132-avoiding permutations, the position of the largest element in a 132-avoiding permutation detemines the first to return to the x-axis and vice versa.

2.1 Examples of 132-avoiding permutation to Dyck path

2.2 Example 1

Let $\sigma = 2134$.
Now we must calculate $f(\sigma)$.

$$\begin{aligned} &\text{f}(\sigma) = f(2134) \\ &= \text{u f(213) d} \\ &= \text{uu f(21) dd} \\ &= \text{uuududdd} \end{aligned} \tag{2}$$

Hence the corresponding Dyck path for the permutation 2134 has the encoding *uuududdd*.

2.3 Example 2

Let $\pi = 7564213$.
Now we must calculate $f(\pi)$.

$$\begin{aligned} &\text{f}(\pi) = f(7654213) \\ &= \text{ud f(564213)} \\ &= \text{udu f(5) d f(4213)} \\ &= \text{uduuddud f(213)} \\ &= \text{uduuddudu f(21) d} \\ &= \text{uduuddudu udd} \end{aligned} \tag{3}$$

Hence the corresponding Dyck path for the permutation 7654213 has the encoding *uduudduduududd*.

3 Dyck path to 132-avoiding permutation

To solve the problem of converting a Dyck Path to a 132-avoiding permutation we will use the Robinson-Schensted-Knuth correspondence.
Given a 132-avoiding permutation we will start by applying the Robinson-Schensted-Knuth correspondence (RSK-correspondence) to the permutation. As is known the RSK-correspondence gives a bijection between a permutation σ of length n and pairs (P, Q) of *standard Young tableaux* of the same shape $\lambda \vdash n$, hence for 132-avoiding permutations Young tableau has at most two rows. The *insertion tableau*, P, is obtained by reading in the permutation $\sigma = a_1 a_2 ... a_n$ left to right and inserting the element of the permutation into the partial tableau that has already been obtained by using Schensted's insertion algorithm. Assume that $a_1 a_2 ... a_{i-1}$ have already been inserted. If a_i is larger than all the elements of the first row of the tableu, place a_i at the end of the first row of the tableau. If it is not, then let m be the leftmost element in the first row that is larger than a_i, then place a_i in the cell that is currently occupied by m and

move m to the end of the second row.
The *recording tableau*, Q, is obtained by placing i, where i is from 1 to n, in the position of the cell that in the construction of P was inserted at step i (that is, the stage where a_i was inserted).
<insert example>
Finally, to turn the pair of tableaux, (P, Q), into a Dyck path, D, we do it in two stages. Firstly, the first half, X we get by recording, for i from 1 to n. If i is in the first row of P we record an up-step, u, and a down-step, d, if i is in the second row of P. Let Y be the word obtained by replacing all the u's in A with a d, and all the d's in A with a u, then $D = XY^r$ where Y^r is the reverse of Y.
<insert example>

References

[1] Anders Claesson and Sergey Kitaev, *Classification of bijections between 321- and 132-avoiding permutations*, 2008.

[2] Donald E. Knuth, *The art of computer programming*, vol. 1, Addison-Wesley, Reading, Massachusetts, 1973.