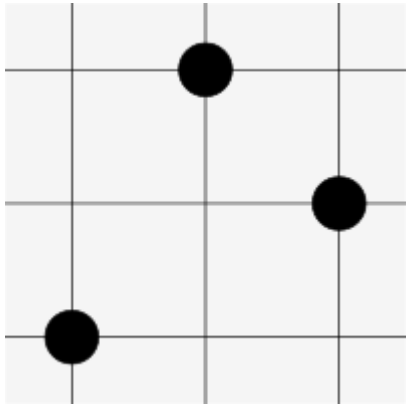
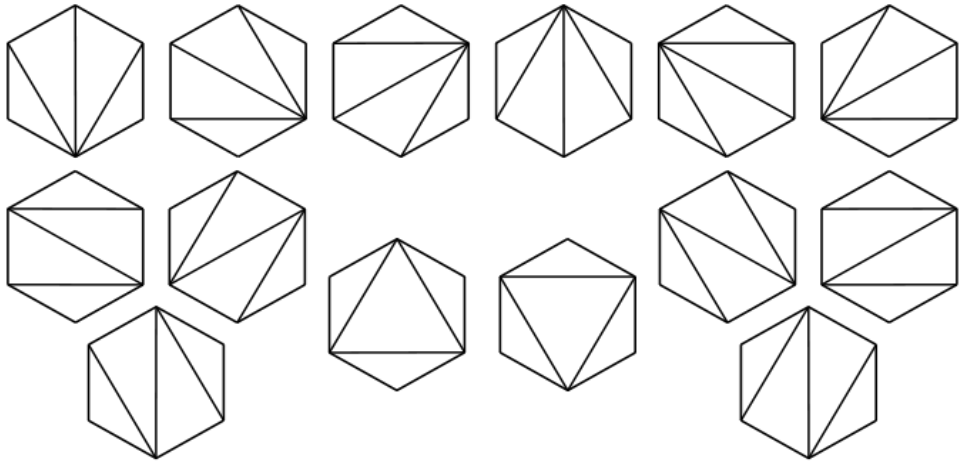
Catalan Structures and Bijections





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Repository: https://www.github.com/patons02/catalan-structures

Aims and Objectives

- Explore various Catalan Structures.
  - Choose between 5 and 10 structures to explore.
- Create bijections between chosen Catalan Structures.
  - Research bijections.
  - Create bijections to correspond to the chosen structures.
- Create application to model Catalan Structures and their bijections.
  - Written in Haskell
  - Each structure is its own module.
  - One module for bijections.
  - Represent them on screen using graphics packages.

Background Study

- Learn basics of combinatorics.
  - Binomial theorem
  - Permutations
  - Generating functions
- Catalan structures
  - What are they?
  - Relation to Catalan numbers
  - Proof?
- The following Catalan structures:
  - Dyck Paths
  - Permutations which avoid any fixed pattern in S_3.
  - Young Tablaeux
  - Binary Trees
  - Triangulations of an n-gon

Specification and Requirements

- Model Dyck Paths
- Model permutations that are in the following permutation classes:
  - Av(123), Av(213), Av(321), Av(231), Av(312), Av(132)
- Model Young Tableaux.
- Model Triangulations of an n-gon.
- Create bijections between all structures.
- Visualise the above structures.

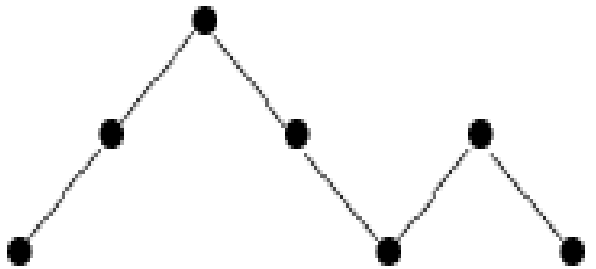
Project Design

- Structures modelled by recursive formula
  - Cons – Construct structure
  - Decons – Deconstruct structure

- Each structure is an instance of Catalan:

```
class Catalan where
    empty :: a
    cons :: a → a → a
    decons :: a → Maybe (a,a)
```

- Dyck paths:
  - Empty: Empty list of steps
  - Cons: Put into indecomposable form
    - D = uDdD where u = an up-step and d = a down-step
  - Decons: decompose into list of Dyck paths then produce alpha and beta.



Project Design (2)

- Stack sortable permutations (132-avoiding permutations)
  - Empty: empty list
  - Cons: In form: αnβ
  - Decons: Returns Just (α, β)
- Make similar recursive formulae for other structures.
- Get bijections for free!

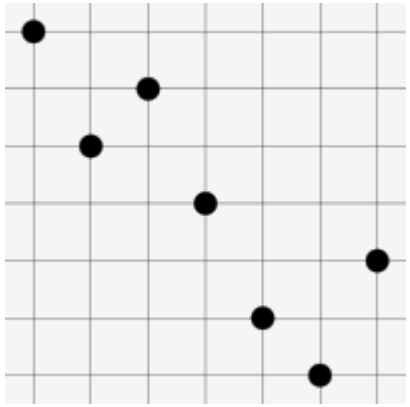
```
bijection :: (Catalan a, Catalan b) => a -> b
bijection w = case decons w of
        Nothing -> empty
        Just (u,v) -> cons (bijection u, bijection v)
```

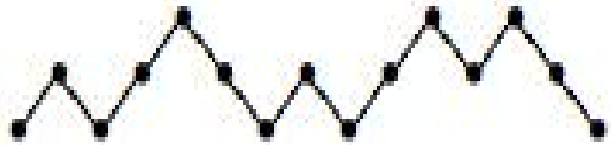
Progress Status

- Initial research complete
  - Proofs that structures are Catalan structures
  - Bijections between them:
    - Knuth's bijection
    - Knuth-Rotem's bijection
    - Simon-Schmidt's bijection
- Dyck Paths
  - Modelled
  - About to be visualised
- Stack sortable permutations
  - Also known as 132-avoiding permutations
  - Modelled
  - About to be visualised

Progress Status (2)

- Standard bijection
  - Converts stack sortable permutation to Dyck path
  - Recursive formula: $f(\pi) = uf(\pi'_L)df(\pi_R)$ *and* $f(\epsilon) = \epsilon$
  - π'_L = permutation obtained by subtracting $|\pi_R|$ from each of its original letters
  - Modelled
- Next steps:
  - Model Young Tableaux
  - Implement Knuth's Bijection [1].





Evaluation (1)

- Analyse statistics for each bijection
- List of statistics will be made for each structure
- Statistics for each bijection will be recorded and compared against other bijections.
- Process:
  - F: C → D
  - C and D have their own statistics
  - Statistics will be such that $stat_1(\pi) = stat_2(f(\pi))$
  - Finally check which statistics are respected by the bijection

Evaluation (2)

- Process ctd:
  - Start with the standard bijection and find statistics.
  - Repeat for all bijections.
  - Complete analysis and present best bijections.
- Presented via evaluation report
- List of all statistics respected included also

Project Plan

Revised time scale is as follows:

- 11th Jan 2013 – Literature review complete
- 18th Jan 2013 – Analysis of initial structures complete
- 4th Jan 2013 – Model of initial structures complete
- 5th Jan 2013 – Poster presentation complete
- 13th Feb 2013 – Analysis of remaining structures complete
- 20th Feb 2013 – Remaining structures added to program
- 4th March 2013 – Visualisations added to program
- 4th March 2013 – Write up started
- 9th March 2013 – Evaluation complete
- 13th March 2013 – Project submitted to supervisor
- 27th March 2013 – Project submitted for binding
- 29th March 2013 – Project submitted for marking

References

[1] – Anders Claesson and Sergey Kitaev, Classification of bijections between 321- and 132-avoiding permutations, 2008.