Cycles of Length $2n + 1$ with Descent Set $[n]$

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1 Introduction

1.1 Definitions

The subsections Cycle structure, Quasi-symmetric function and Quasi-symmetric generating function are defined in [1]

1.1.1 Cycle Structure

Let $\sigma \in S_n$.
If σ has n_i cycles of length i, then the **cycle structure** of σ is the partition $1^{n_1}2^{n_2}...$

1.1.2 Descent set

The descent set of σ, denoted by $Des(\sigma)$ is:

$$\text{Des}(\sigma) = \{i | 1 \leq i \leq n \text{ and } \sigma_i > \sigma_{i+1}\} \tag{1}$$

Recall that subsets of $\{1, ..., n-1\}$ is given by the bijection with compositions of n given by:
$D \to C(D)$
$\{d_0 < d_1 < d_2 < ... < d_k\} \to (d_1 - d_0, d_2 - d_1, ..., d_{k+1} - d_k)$
where $d_0 = 0$ and $d_{k+1} = n$.
Let's denote $C(\sigma)$ to be the composition associated to the descent set of σ: $C(\sigma) = C(D(\sigma))$.

1.1.3 Quasi-symmetric function

Let X be an infinite totally ordered set of variables. A formal power series F in $\mathbb{Q}[[x]]$ is called quasi-symmetric if for any variables $x_1, ..., x_n$, $y_1, ..., y_n$ in X with $n_1 < ... < x_n$ and $y_1 < ... < y_n$ and for any positive interes $k_1, ..., k_n$ the coefficients of F of $x^{n_{k_1}}...x^{n_{k_2}}$ and $y^{n_{k_1}}...y^{n_{k_2}}$ are equal.

1.1.4 Quasi-symmetric generating function

Let A be a set of permutations. Lets define the quasi-symmetric generating function to be the quasi-symmetric function $\sum_{\pi \in A} F_{C(\pi)}$.

2 Cycles of Length $2n+1$ with Descent Set $[n]$ are Catalan Structures

Theorem: The number of cycles with descent $\{i\}$ is $(1/n)\sum_{d|gcd(n,i)} \mu(d)\binom{n/d}{i/d}$ for $n > 1$.
Proof: The quasi-symmetric generating function for n-cycles is $L_n(x)$.
We have $L_n(1) = 0$ for $n \geq 1$ and
$L_n(q, 1)$
$= (1/n) \sum_{d|n} \mu(d)(1 + q^d)^{n/d}$
$= (1/n) \sum_{d|gcd(n,i)} \sum_{j=0}^{n} \mu(d) \binom{n/d}{j} q^{d_j}$
$= (1/n) \sum_{i=1}^{n} q^i \sum_{d|n,d|i} \mu(d) \binom{n/d}{i/d}$
where $L_n(q, 1)$ has no constant term, $\mu(d)$ is the Mobius function and
$L_n = (1/n) \sum_{d|n} \mu(d) p_d^{n/d}$.
QED.

Theorem: Cycles of Length $2n+1$ with Descent Set $[n]$ are Catalan Structures
Proof: From the above theorem, if we set $n = 2i + 1$ then this is:
$\frac{1}{2i+1}\binom{2i+1}{i} = \frac{1}{i+1}\binom{2i}{i}$
QED.

References

[1] I.M Gessel and C. Reutenauer, *Counting permutations with given cycle structure and descent set*, Journal of Combinatoral Theory **64** (1993), 189 – 215.