Symmetric Groups, Young Tableau and the Robinson-Schensted-Knuth Correspondence

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1 Symmetric Groups

A symmetric group, represented by $\sum_x$, S_x, $Sym(x)$ and $\mathfrak{S}_X$, on the finate set X is the group whose elements are all bijective functions from X to X and whose group operation is that of function composition.
For finate sets, permutation and bijective functions both refer to the operation of arrangement.

1.1 Permutation Groups

The group of **all** permutations of a set is the symmetric group.
$\mathfrak{S}_x \subseteq$ all other permutation groups.

2 Young Tableau

A Young Tableau is a combinatorial object which provides a convient way to describe the group representations of Symmetric and general linear groups and to study their properties.

2.1 Young Diagram

A Young diagram is a finite collection of cells arranged in left-justified rows, with the row lengths weakly decreasing.
Listing the number of boxes in each row gives a partition λ of a non-negative integer, n, the total number of boxes in the diagram.
The diagram is said to be of shape λ, and it carries the same information as that partition. If we list the number of boxes of a Young diagram in each column gives another partition: the **conjugate** or *transpose* partition of λ; we obtain a Young diagram of that shape by reflecting the original diagram along its main diagonal.

2.2 Young Tableaux

A Young tableaux is created by filling in the cells of the Young diagram with symbols taken from teh same alphabet, which is usually a totally ordered set. Young tableaux have n distinct entries arbitarily assigned to cells of the diagram.
A tableau is called **standard** if the entries in each row and each column are increasing. The number of distinct standard Young tableaux on n entries is given by the telephone numbers:

$$1, 1, 2, 4, 10, 26, 76, 232, 764, 2620, 9496, ...[1] \tag{1}$$

A tableau is called **semistandard**, or *column-strict*, if the entries weakly increase along each row and strictly increase down each column.

The weight of a tableau is the sequence of the number of times each number appears in a tableau. For example, the standard Young tableau are the semistandard tableaux of weight (1, 1, ..., 1) which requires every integer up to n to occur exactly once.

3 Robinson-Schensted-Knuth correspondence

The Robinson-Schensted-Knuth correspondence (RSK-correspondence) is a bijective mapping between permutations and pairs of standard Young tableaux, both having the same shape.
The bijection is constructed using the Schensted insertion algorithm, starting with an empty tableau and successively inserting the values $\sigma_1, ..., \sigma_2$ of the permutation σ at index $1, 2, ..., n$; these are for the second line when σ is in two-line notation.

$$\sigma = \begin{pmatrix} 1 & 2 & ... & n \\ \sigma_1 & \sigma_2 & ... & n \end{pmatrix}$$

The resulting standard tableau P is the first of the pair corresponding to σ; the other standard tabeau Q records the successive shapes of the intermediate tableaux during the construction of P.

3.1 Two-line arrays

The two-line array w_A corresponding to a matrix A is defined as:

$$w_A = \begin{pmatrix} i_1 & i_2 & ... & i_m \\ j_1 & j_2 & ... & j_m \end{pmatrix}$$

in which for any pair (i, j) that indexes an entry $A_{i,j}$ of A there are $A_{i,j}$ columns equal to $\binom{i}{j}$ and all columns are in lexicographic ordering, so:

$$\begin{aligned}&1)\ \mathrm{i}_1 \le i_2 \le i_3 \le ... \le i_m \text{ and,}\\&2) if\ \mathrm{i}_r = i_s \text{ and } r \le s \text{ then } j_r \le j_s\end{aligned} \tag{2}$$

3.2 Schensted Insertion Algorithm

The Schensted Insertion Algorithm [**?**] is a procedure to insert each σ_i into an initially empty Young tableaux, T. Each σ_i will be represented by the value x, and each cell will be represented by c.
The algorithm is as follows:
1) Set $i = 1$ and j to be one more than the length of the first row of T.
2) While $j > 1$ and $x < T_{j,i-1}$ decreast j by 1. (Now $(i,\, j)$ is the first cell in row i with either an entry larger than x in t or no entry at all).
3) If the cell $(i,\, j)$ is empty in T, terminate after adding x to T in cell $(i,\, j)$ and setting $c = (i,\, j)$.
4) Swap the values x and $T_{i,j}$. (This inserts the old x into row i and saves the value it replaces for insertion into the next row).
5) Increase i by 1 and go to step 2.
The result of the algorithm is that the shape of T grows by exactly one cell, namely c.

3.3 Definition of RSK-correspondence

By applying the Schensted insertion algorithm to the bottom line of the two-line array, we get a pair $(P,\, Q_0)$ which consists of a semistandard tableau, P, and a standard tableau, Q_0, where Q_0 can be turned into a semistandard tableau, Q, by replacing each entry, b, of Q_0 by the b-th entry of the top line of w_A.
Now we obtain a bijection from matrix A to ordered pairs $(P,\, Q)$ of semistandard Young tableaux of the same shape, in which the set of entries of P is that of the second line of q_A, and the set of entries of Q is that of the first line of w_A.
The number of entries j in p is therefore equal to the sum of the entries on column j of A, and the number of entries i in Q is equal to the sum of entries in row i in A.
See sheet: board 4 17/01/2013 for example.

References

[1] The On-Line Encyclopedia of Integer Sequences, *Sequence a000085*, 2012.

[2] C. Schensted, *Longest increasing and decreasing subsequences*, Canadian Journal of Mathematics **13** (1961), no. 1, 179–191.