COMP8760

Lecture 3

Worksheet for Practice

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1. Consider the set $\mathbb{Z}_{11} = \{0, 1, 2, \ldots, 10\}$. How many elements are there in the set $\mathbb{Z}_{11}$?
   Note: The number of elements in $\mathbb{Z}_{11}$ is called its cardinality and is denoted as $|\mathbb{Z}_{11}|$.

2. Consider the set $\mathbb{Z}_{11}$ and the operation $+ \pmod{11}$ on its elements. Create a table for all operations $x + y \pmod{11}$ where $x, y \in \mathbb{Z}_{11}$. We will call it the "addition table of $\mathbb{Z}_{11}$".

3. From the table in question 2, find the element $\bar{x}$ for each $x \in \mathbb{Z}_{11}$ such that

   $$x + \bar{x} = 0 \pmod{11}.$$

   Hint: In the row corresponding to the number x, find the column $\bar{x}$ with the entry 0. Such a pair $x, \bar{x}$ are additive inverses of each other with respect to $+ \pmod{11}$.

4. Prove that $(\mathbb{Z}_{11}, + \pmod{11})$ is a group.
   Hint: Show that all four properties (closure, associativity, identity and inverse) hold.
   Note: This proof is for advanced learning only. You may skip this question.

5. Find all numbers $1 \leq x \leq 20$ that are mutually prime to 20. We will denote this set of numbers as $\mathbb{Z}_{20}^{\star}$.

6. Find all numbers $1 \leq x \leq 11$ that are mutually prime to 11. We will denote this set of numbers as $\mathbb{Z}_{11}^{\star}$.

7. What is the value of $|\mathbb{Z}_{11}^{\star}|$?

8. What is the value of $\phi(11)$?

9. What is the relationship between $\phi(11)$ and $\mathbb{Z}_{11}^{\star}$?

10. Consider the set $\mathbb{Z}_{11}^{\star} = \{1, 2, \ldots, 10\}$ of all $1 \leq x \leq 11$ that are mutually prime to 11 and the operation $\cdot \pmod{11}$ on its elements. Create a table for all operations $x \cdot y \pmod{11}$ where $x, y \in \mathbb{Z}_{11}^{\star}$. We will call it the "multiplication table of $\mathbb{Z}_{11}^{\star}$".

11. From the table in question 10, find the element $\bar{x}$ for each $x \in \mathbb{Z}_{11}^{\star}$ such that

    $$x \cdot \bar{x} = 1 \pmod{11}.$$

    Hint: In the row corresponding to the number x, find the column $\bar{x}$ with the entry 1. Such a pair $x, \bar{x}$ are multiplicative inverses of each other with respect to $\cdot \pmod{11}$.

12. Prove that $(\mathbb{Z}_{11}^{\star}, \cdot \pmod{11})$ is a group.
    Hint: Show that all four properties (closure, associativity, identity and inverse) hold.
    Note: This proof is for advanced learning only. You may skip this question.

13. What is the value of 3^{10} where $3 \in \mathbb{Z}_{11}^{\star}$?
In other words, find the value of $3^{10} = \underbrace{3 \times 3 \times \cdots \times 3}_{10 \text{ times}} \pmod{11}$.

14. For any $x \in \mathbb{Z}_{11}^{\star}$, let us define $x^{10} = \underbrace{x \times x \times \cdots \times x}_{10 \text{ times}} \pmod{11}$. Find the values of x^{10} for all $x \in \mathbb{Z}_{11}^{\star}$.