# COMP8760

Class Worksheet - Lattice

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This worksheet contains programming tasks on lattices. You may use any programming language of your choice.

1. **Length of a vector (its $L_2$ norm).** You are given a file containing $m \times n$ integers in $m$ rows and $n$ columns. Each column represents a vector $\mathbf{b} = (b_1, b_2, \ldots, b_m)$, where each $b_i \in \mathbb{Z}$ is a 32-bit integer. There are $n$ such column vectors. The square of the $L_2$ norm of a vector $\mathbf{b}$ is defined as

$$\|\mathbf{b}\|^2 = \sum_{i=1}^{m} b_i^2 = b_1^2 + b_2^2 + \cdots + b_m^2.$$

   Write a program that will read the file for all the vectors $(\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ and compute the square of the $L_2$ norm for each.

2. **Inner product of vectors.** Write a program that will read the vectors from a file as above and compute the inner product for each pair $\mathbf{b}_i, \mathbf{b}_j$ where $1 \leq i, j \leq n$. Let $\mathbf{b}_i = (b_{i,1}, b_{i,2}, \ldots, b_{i,m})$ and $\mathbf{b}_j = (b_{j,1}, b_{j,2}, \ldots, b_{j,m})$. Then the inner product is defined as follows:

$$\begin{aligned}\langle \mathbf{b}_i, \mathbf{b}_j \rangle &= \sum_{\ell=1}^{m} b_{i,\ell} \times b_{j,\ell} \\ &= b_{i,1} \times b_{j,1} + b_{i,2} \times b_{j,2} + \cdots + b_{i,m} \times b_{j,m}.\end{aligned}$$

3. **Gram-Schmidt orthogonalisation (GSO) vectors.** Write a program that will read the vectors from a file as above. These vectors form an ordered set $\mathbf{B}$ denoted as follows:

$$\mathbf{B} = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n).$$

   We define the GSO vector $\mathbf{b}_i^*$ incrementally with respect to all its predecessors $\mathbf{b}_1, \ldots, \mathbf{b}_{i-1}$. The first GSO vector is defined as $\mathbf{b}_1^* = \mathbf{b}_1$. The other vectors $\mathbf{b}_i^*, 2 \leq i \leq n$ are defined iteratively as the projection of $\mathbf{b}_i$ in a direction orthogonal to $\mathtt{span}(\mathbf{b}_1, \ldots, \mathbf{b}_{i-1})$, i.e.

$$\mathbf{b}_i^* = \pi_i(\mathbf{b}_i) = \mathbf{b}_i - \sum_{j=1}^{i-1} \mu_{i,j} \mathbf{b}_j^*$$

   where

$$\mu_{k,\ell} = \frac{\langle \mathbf{b}_k, \mathbf{b}_\ell^* \rangle}{\langle \mathbf{b}_\ell^*, \mathbf{b}_\ell^* \rangle} = \frac{\langle \mathbf{b}_k, \mathbf{b}_\ell^* \rangle}{\|\mathbf{b}_\ell^*\|^2}.$$

   These vectors form the GSO basis $\mathbf{B}^* = (\mathbf{b}_1^*, \mathbf{b}_2^*, \ldots, \mathbf{b}_n^*)$. Your program should compute and output the GSO basis $\mathbf{B}^*$.

4. **General projection of vectors.** Write a program that will read the vectors $\mathbf{B} = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ from a file as above. We define the projection of a vector $\mathbf{b}_k$ with respect to its predecessors $\mathbf{b}_1, \ldots, \mathbf{b}_{i-1}$ (where $i < k$) as:

$$\begin{aligned}\pi_i(\mathbf{b}_k) &= \mathbf{b}_k^* + \sum_{\ell=i}^{k-1} \mu_{k,\ell} \mathbf{b}_\ell^* \\ &= \mathbf{b}_k - \sum_{\ell=1}^{i-1} \mu_{k,\ell} \mathbf{b}_\ell^*.\end{aligned}$$

   For each $1 \leq i < k \leq n$, compute and output $\pi_i(\mathbf{b}_k)$.