Revision of the paper

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1 Relax normality assumption in Proposition 2

From the original proof of Proposition 2, we can see the two actual properties that are used in the proof which are derived from the multivariate normal distribution are:

1. $E(Y|X) = \rho X$
2. Symmetric marginal distribution of X and Y

So we can relax the condition in Proposition 2 as following:
let $\{X, Y\}$ denotes a bivariate distribution with correlation ρ that can be decomposed as

- $X = Z_1$
- $Y = \rho Z_1 + \sqrt{1-\rho^2} Z_2$

where Z_1 and Z_2 are ***independent*** *random variables with mean zero, unit variance and symmetric distribution.*

One can see the above condition is sufficient to get the two properties used in the proof and it's more relax than multivariate normal. In fact, X and Y do not necessary to have the same distribution (just the same first and second moments). Actually, for any bivariate distribution $\{X, Y\}$ with mean zero, unit variance and correlation ρ, we can decompose $\{X, Y\}$ as:

- $X = Z_1$
- $Y = \rho Z_1 + \sqrt{1-\rho^2} Z_2$

where Z_1 and Z_2 are **uncorrelated** random variables through Gram-Schmidt procedure. However, we need stronger result than uncorrelated in order to get the first property used in the proof, i.e. $E(Y|X) = \rho X$.

2 Extend the result of Proposition 3 to higher dimension

Proof. There are two clusters G_1 and G_2, where observations from G_1 are distributed as

$$\mathcal{N}\left(\begin{pmatrix}\mu^X \\ \mu^Y\end{pmatrix}, \mathbf{I}\right)$$

and observations from G_2 are distributed as

$$\mathcal{N}\left(\begin{pmatrix}-\mu^X\\ -\mu^Y\end{pmatrix}, \mathbf{I}\right)$$

here μ^X and μ^Y are all vectors. Let G_i be the true cluster where observation i is generated from, by assumption

$$P(G_i = G_1) = P(G_i = G_2) = 1/2$$

To simplify the notation, let

$$\begin{pmatrix}X_1\\ Y_1\end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix}\mu^X\\ \mu^Y\end{pmatrix}, \mathbf{I}\right)$$

denote the observations from G_1 and

$$\begin{pmatrix}X_2\\ Y_2\end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix}-\mu^X\\ -\mu^Y\end{pmatrix}, \mathbf{I}\right)$$

denote the observations from G_2

Further, let's denote $\mu^X = \lambda_x e_x$ where λ_x is a scalar denotes the distance of μ^X from origin and e_x is the unit vector point at the same direction as μ^X; $\mu^Y = \lambda_y e_y$ has the same interpretation.

Apply K-mean on the Y-space, where the two clusters are $\mathcal{N}\left(\mu^Y, \mathbf{I}\right)$ and $\mathcal{N}\left(-\mu^Y, \mathbf{I}\right)$, the K-mean centroids are $\bar{\mu}_1^Y$ and $\bar{\mu}_2^Y$ with $\bar{\mu}_1^Y = -\bar{\mu}_2^Y$. Note that the boundary between the two clusters are $e_y^T Y > 0$. So

$$\begin{aligned}
\bar{\mu}_1^Y &= E(Y \mid Y > 0) && (1)\\
&= E(Y_1 \mid e_y^T Y_1 > 0) \cdot P(e_y^T Y_1 > 0) + E(Y_2 \mid e_y^T Y_2 > 0) \cdot P(e_y^T Y_2 > 0) && (2)
\end{aligned}$$

Note that $e_y^T Y_1$ projects vector Y_1 on the direction of e_y. And because the e_y is the same direct as μ^Y, it goes through the center of the sphere $\mathcal{N}\left(\mu^Y, \mathbf{I}\right)$. Because the covariance matrix is $\mathbf{I}$, the sphere is symmetric around e_y. Therefore,

$$E(Y_1 \mid e_y^T Y_1 = a) = a e_y \qquad (3)$$

Also, $Y_1 \sim \mathcal{N}\left(\mu^Y, \mathbf{I}\right)$ so $e_y^T Y_1 \sim \mathcal{N}\left(\lambda_y, 1\right)$. We have

$$\begin{aligned}
E(Y_1 \mid e_y^T Y_1 > 0) &= E\left[E\left(Y_1 \mid e_y^T Y_1\right) \mid e_y^T Y_1 > 0\right] && (4)\\
\textit{from (3) above} &= E(e_y^T Y_1 e_y \mid e_y^T Y_1 > 0) && (5)\\
&= e_y E(e_y^T Y_1 \mid e_y^T Y_1 > 0) && (6)
\end{aligned}$$

Because $e_y^T Y_1 \sim \mathcal{N}(\lambda_y, 1) = \lambda_y + Z$, where Z is standard normal, by Lemma 3 from Appendix C we have

$$E(e_y^T Y_1 \mid e_y^T Y_1 > 0) = E(\lambda_y + Z \mid Z > -\lambda_y) \tag{7}$$
$$= \lambda_y + E(Z \mid Z > -\lambda_y) \tag{8}$$
$$= \lambda_y + \frac{\varphi(\lambda_y)}{\Phi(\lambda_y)} \tag{9}$$

where $\varphi()$ and $\Phi()$ are the standard normal probability and cumulative distribution function respectively. So, by (6) we have

$$E(Y_1 \mid e_y^T Y_1 > 0) = \left[\lambda_y + \frac{\varphi(\lambda_y)}{\Phi(\lambda_y)}\right] e_y \tag{10}$$

Similarly, we can have

$$E(Y_2 \mid e_y^T Y_2 > 0) = -E(Y_1 \mid e_y^T Y_1 < 0) \tag{11}$$
$$= -e_y E(e_y^T Y_1 \mid e_y^T Y_1 < 0) \tag{12}$$
$$= \left[\frac{\varphi(\lambda_y)}{1 - \Phi(\lambda_y)} - \lambda_y\right] e_y \tag{13}$$

Because $e_y^T Y_1 \sim \mathcal{N}(\lambda_y, 1) = \lambda_y + Z$, it's easy to get

$$P(e_y^T Y_1 > 0) = P(Z > -\lambda_y) \tag{14}$$
$$= \Phi(\lambda_y) \tag{15}$$

By symmetry, we can get

$$P(e_y^T Y_2 > 0) = 1 - \Phi(\lambda_y) \tag{16}$$

Put everything together, we have

$$\bar{\mu}_1^Y = E(Y_1 \mid e_y^T Y_1 > 0) \cdot P(e_y^T Y_1 > 0) + E(Y_2 \mid e_y^T Y_2 > 0) \cdot P(e_y^T Y_2 > 0) \tag{17}$$
$$= \left[\lambda_y + \frac{\varphi(\lambda_y)}{\Phi(\lambda_y)}\right] \cdot \Phi(\lambda_y) e_y + \left[\frac{\varphi(\lambda_y)}{1 - \Phi(\lambda_y)} - \lambda_y\right] \cdot (1 - \Phi(\lambda_y))\, e_y \tag{18}$$
$$= [2\lambda_y \Phi(\lambda_y) + 2\varphi(\lambda_y) - \lambda_y] e_y \tag{19}$$

After training the classifier, because of the identity covariance matrix, the classification boundary is $e_x^T X > 0$. So the Y center for observation with $e_x^T X > 0$ is

$$
\begin{aligned}
\hat{\mu}_1^Y &= E(Y_1 \mid e_x^T X_1 > 0) \cdot P(e_x^T X_1 > 0) + E(Y_2 \mid e_x^T X_2 > 0) \cdot P(e_x^T X_2 > 0) && (20)\\
\text{X independent of Y} &= E(Y_1) \cdot P(e_x^T X_1 > 0) + E(Y_2) \cdot P(e_x^T X_2 > 0) && (21)\\
&= \mu^Y \cdot P(e_x^T X_1 > 0) - \mu^Y \cdot P(e_x^T X_2 > 0) && (22)\\
&= \mu^Y \left(P(e_x^T X_1 > 0) - P(e_x^T X_2 > 0)\right) && (23)\\
&= \mu^Y \left[\Phi(\lambda_x) - (1 - \Phi(\lambda_x))\right] && (24)\\
&= (2\Phi(\lambda_x) - 1)\mu^Y && (25)\\
&= (2\Phi(\lambda_x) - 1)\lambda_y e_y && (26)
\end{aligned}
$$

Because of symmetry and $P(G_i = G_1) = P(G_i = G_2) = 1/2$, it's sufficient to show that for observations with $e_x^T X > 0$, if $CV(2) < CV(1)$ then the Gabriel CV method correctly picks $k = 2$ over $k = 1$.

Similar as in the proof of Proposition 4, by the variance and bias decomposition of MSE, the variance is the same, so only the bias influences the result. Note the predicted center is grand 0 for $CV(1)$, so to see if $CV(2) < CV(1)$ one only need to see if $||\bar{\mu}_1^Y - \hat{\mu}_1^Y||^2 < ||\hat{\mu}_1^Y - 0||^2$, which is true if

$$2\Phi(\lambda_y) + 2\frac{\varphi(\lambda_y)}{\lambda_y} < 4\Phi(\lambda_x) - 1$$

such result reduce to the original result of Proposition 3 if one set $\lambda_x = \mu^X$ and $\lambda_y = \mu^Y$

□

3 Correct description of simulation and reply to comments

1. **Comment** 2 **of referee** 1
   It is true that $\hat{k}_0$ tends to overestimate k when correlation is high, and the covariance matrices for the $\hat{k}_0$ clusters is not the same as the true underlying covariance matrices for the k clusters. The point is that we only need the estimated noise covariance matrix to be "close" to the actual covariance matrix such that after adjusting for correlation, the (transformed) data become weakly correlated (not necessary zero-correlated). We have demonstrate that the proposed method works well when the correlation is weak in both theoretical results and simulation results.

2. **Comment** 3.1 **of referee** 1
   The previous description of how the clusters are generated in the simulation and the distance between them are incorrect (last paragraph in page 20 of previous draft). The correct description should be "any simulation with clusters having minimum distance less than 1.0

units between them was discarded; ... the settings are such that about half of the random realizations were discarded" **?**. This idea was directly borrowed from **?**.

The idea is that any observation i which is generated from cluster G_i will has the distance between i and μ_{G_i} (its true cluster center) less than the distance between i and $\mu_{G_j} (i \neq j)$ by more than 1.0 unit, i.e. $||i - \mu_{G_i}|| + 1 \leq ||i - \mu_{G_j}||$. Such procedure ensures that the generated clusters have no overlap, so it's clear what the "true" number of cluster is.

3. **Comment** 3.2 **of referee** 1

   All our simulation setting are similar to the (c) and (d) settings in section 6 of **?**, given how we generated the clusters (see previous reply).

4 Simulation results in tables

4.1 Setting 1

	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0
6	97	84	28	3	1	0	0	0	0	0
7	3	7	15	1	1	1	0	0	1	2
8	0	4	16	7	4	0	0	0	0	0
9	0	4	24	27	16	11	7	4	4	2
10	0	1	17	62	78	88	93	96	95	96
NA	0	0	0	0	0	0	0	0	0	0

Table 1: gabriel-nearest-5x2

	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0
6	96	97	91	89	89	95	95	95	100	97
7	1	2	6	6	4	2	2	3	0	0
8	1	1	2	2	3	2	3	2	0	1
9	2	0	1	3	3	0	0	0	0	1
10	0	0	0	0	1	1	0	0	0	1
NA	0	0	0	0	0	0	0	0	0	0

Table 2: gabriel-corr-correct2

	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	1	1	2	2	2	1	2	0
4	4	3	5	5	9	4	4	8	8	4
5	15	16	15	14	15	16	18	20	17	17
6	74	72	69	68	58	65	62	57	58	48
7	4	6	7	9	11	7	6	8	9	9
8	3	3	3	3	3	2	5	2	4	6
9	0	0	0	0	1	1	1	1	0	9
10	0	0	0	0	1	3	2	3	2	7
NA	0	0	0	0	0	0	0	0	0	0

Table 3: wold

	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0
6	100	100	62	9	1	0	0	0	0	0
7	0	0	8	5	0	0	0	0	0	0
8	0	0	17	16	2	0	0	0	0	0
9	0	0	8	18	11	3	0	0	0	0
10	0	0	5	52	86	97	100	100	100	100
NA	0	0	0	0	0	0	0	0	0	0

Table 4: gap

	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0
6	100	92	98	100	99	99	98	99	99	94
7	0	7	1	0	0	0	2	0	0	3
8	0	0	1	0	0	0	0	0	1	2
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	1
NA	0	1	0	0	1	1	0	1	0	0

Table 5: BIC

	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	0	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0
3	3	1	2	0	1	1	1	1	1	0
4	6	3	5	3	3	3	3	3	3	2
5	15	17	10	7	10	6	7	6	4	5
6	38	35	27	32	35	27	26	25	23	23
7	23	18	26	27	19	33	26	23	14	10
8	6	15	11	18	15	12	15	13	18	15
9	4	4	15	7	13	13	13	17	16	20
10	4	6	4	6	4	5	9	12	21	25
NA	0	0	0	0	0	0	0	0	0	0

Table 6: CH

	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	3	8	7	7	7	5	5	4	3	8
4	9	11	11	7	8	11	9	6	6	13
5	17	21	13	12	11	9	12	7	10	10
6	45	34	31	39	36	31	34	34	32	29
7	14	10	15	18	17	24	16	22	14	17
8	4	7	15	9	13	9	8	11	18	8
9	7	7	7	4	7	7	9	12	12	9
10	1	2	1	4	1	4	7	4	5	6
NA	0	0	0	0	0	0	0	0	0	0

Table 7: Hartigan

	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0
6	100	100	100	96	22	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0
8	0	0	0	1	3	1	0	0	0	0
9	0	0	0	0	13	15	9	5	3	1
10	0	0	0	3	62	84	91	95	97	99
NA	0	0	0	0	0	0	0	0	0	0

Table 8: Jump

	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	89	95	90	95	88	89	91	92	91	93
2	10	4	10	5	10	11	8	8	8	6
3	1	1	0	0	2	0	1	0	1	1
4	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0
NA	0	0	0	0	0	0	0	0	0	0

Table 9: PS

	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	0	0	0	0	0	0	0	0	0	0
2	5	2	2	0	1	1	1	1	1	1
3	0	0	0	0	1	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0
5	1	1	0	0	0	0	0	0	1	0
6	5	7	7	1	2	0	0	0	1	3
7	14	19	8	4	5	7	3	3	3	2
8	30	22	27	17	21	13	15	9	11	13
9	25	23	24	31	38	25	26	26	27	27
10	20	26	32	47	32	54	55	61	56	54
NA	0	0	0	0	0	0	0	0	0	0

Table 10: Stab

4.2 Setting 2

	0	6	12	18	24	30	36	42	48	54
1	0	3	9	3	19	0	12	6	0	1
2	0	5	7	1	6	0	12	6	1	1
3	99	88	82	92	74	99	73	84	97	97
4	0	2	0	2	0	0	3	3	0	0
5	1	0	2	1	1	1	0	1	2	1
6	0	0	0	0	0	0	0	0	0	0
7	0	1	0	1	0	0	0	0	0	0
8	0	1	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0
NA	0	0	0	0	0	0	0	0	0	0

Table 11: gabriel-nearest-5x2

	0	6	12	18	24	30	36	42	48	54
1	0	3	9	3	19	0	12	6	0	1
2	1	5	6	1	5	0	11	6	1	1
3	98	92	85	96	76	100	77	88	99	98
4	0	0	0	0	0	0	0	0	0	0
5	1	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0
NA	0	0	0	0	0	0	0	0	0	0

Table 12: gabriel-corr-correct2

	0	6	12	18	24	30	36	42	48	54
1	6	6	6	6	6	6	6	7	6	6
2	25	24	23	23	24	24	23	24	26	25
3	47	49	52	52	51	51	51	50	48	49
4	8	14	12	13	14	15	11	14	15	15
5	3	1	2	2	2	1	5	0	1	3
6	1	1	3	2	2	2	4	3	3	1
7	0	1	0	2	0	0	0	0	1	0
8	4	2	2	0	1	1	0	1	0	1
9	2	1	0	0	0	0	0	1	0	0
10	4	1	0	0	0	0	0	0	0	0
NA	0	0	0	0	0	0	0	0	0	0

Table 13: wold

	0	6	12	18	24	30	36	42	48	54
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	100	100	100	100	100	100	100	100	100	100
4	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0
NA	0	0	0	0	0	0	0	0	0	0

Table 14: gap

	0	6	12	18	24	30	36	42	48	54
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	100	99	100	100	100	100	100	100	100	100
4	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0
7	0	1	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0
NA	0	0	0	0	0	0	0	0	0	0

Table 15: BIC

	0	6	12	18	24	30	36	42	48	54
1	0	0	0	0	0	0	0	0	0	0
2	1	2	4	4	5	5	5	7	7	7
3	80	80	80	80	80	80	80	79	79	79
4	12	12	10	11	11	10	11	11	10	11
5	5	5	5	4	3	4	3	2	3	2
6	2	1	1	1	1	1	1	1	1	1
7	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0
NA	0	0	0	0	0	0	0	0	0	0

Table 16: CH

	0	6	12	18	24	30	36	42	48	54
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	89	89	88	89	89	88	89	89	88	89
4	7	7	8	7	7	8	7	7	8	7
5	2	2	2	2	2	2	2	2	2	2
6	0	0	0	0	0	0	0	0	0	0
7	2	2	2	2	2	2	2	2	2	2
8	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0
NA	0	0	0	0	0	0	0	0	0	0

Table 17: Hartigan

	0	6	12	18	24	30	36	42	48	54
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	100	100	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0
9	0	0	8	3	1	0	0	0	0	0
10	0	0	92	97	99	100	100	100	100	100
NA	0	0	0	0	0	0	0	0	0	0

Table 18: Jump

	0	6	12	18	24	30	36	42	48	54
1	30	28	30	34	31	31	32	26	27	31
2	30	29	26	27	30	28	32	29	28	27
3	40	43	44	39	39	41	36	45	45	42
4	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0
NA	0	0	0	0	0	0	0	0	0	0

Table 19: PS

	0	6	12	18	24	30	36	42	48	54
1	0	0	0	0	0	0	0	0	0	0
2	23	23	22	23	24	24	23	23	23	22
3	55	62	59	62	60	56	56	59	57	57
4	13	8	11	7	9	13	13	10	12	14
5	0	2	0	0	2	1	0	1	2	3
6	0	0	0	1	0	0	0	0	0	0
7	0	0	0	0	1	1	0	1	0	0
8	0	0	1	0	1	0	1	0	0	0
9	0	1	2	2	1	0	2	0	3	0
10	9	4	5	5	2	5	5	6	3	4
NA	0	0	0	0	0	0	0	0	0	0

Table 20: Stab

4.3 Setting 3

	10	20	30	40	50	60	70	80	90	100
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0
8	91	98	100	100	100	100	100	100	100	100
9	6	2	0	0	0	0	0	0	0	0
10	3	0	0	0	0	0	0	0	0	0
NA	0	0	0	0	0	0	0	0	0	0

Table 21: gabriel-nearest-5x2

	10	20	30	40	50	60	70	80	90	100
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0
8	90	100	100	100	100	100	100	100	100	100
9	7	0	0	0	0	0	0	0	0	0
10	3	0	0	0	0	0	0	0	0	0
NA	0	0	0	0	0	0	0	0	0	0

Table 22: gabriel-corr-correct2

	10	20	30	40	50	60	70	80	90	100
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0
4	1	0	0	0	0	0	0	0	0	0
5	3	0	0	0	0	0	0	0	0	0
6	5	0	0	0	0	0	0	0	0	0
7	20	0	0	0	0	0	0	0	0	0
8	62	98	100	100	100	100	100	100	100	100
9	7	2	0	0	0	0	0	0	0	0
10	2	0	0	0	0	0	0	0	0	0
NA	0	0	0	0	0	0	0	0	0	0

Table 23: wold

	10	20	30	40	50	60	70	80	90	100
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0
8	100	100	100	100	100	100	100	100	100	100
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0
NA	0	0	0	0	0	0	0	0	0	0

Table 24: gap

	10	20	30	40	50	60	70	80	90	100
1	0	0	0	0	0	0	2	9	33	76
2	0	0	0	0	0	4	4	6	8	6
3	0	0	0	0	0	4	8	9	5	5
4	0	0	0	0	0	4	8	7	9	6
5	0	0	0	0	1	9	15	12	5	1
6	0	0	0	0	4	11	8	8	9	1
7	0	0	0	0	14	14	12	10	8	1
8	100	100	100	100	73	21	18	17	6	0
9	0	0	0	0	3	14	9	8	8	1
10	0	0	0	0	5	19	16	14	9	3
NA	0	0	0	0	0	0	0	0	0	0

Table 25: BIC

	10	20	30	40	50	60	70	80	90	100
1	0	0	0	0	0	0	0	0	0	0
2	1	6	14	22	24	26	35	28	27	29
3	0	3	2	11	9	10	14	19	22	23
4	2	2	8	9	12	11	9	14	14	12
5	3	6	11	8	11	10	11	9	10	13
6	12	14	13	16	11	11	10	10	8	11
7	19	24	23	15	13	19	12	14	12	5
8	23	23	21	13	20	13	9	4	7	7
9	20	14	5	6	0	0	0	2	0	0
10	20	8	3	0	0	0	0	0	0	0
NA	0	0	0	0	0	0	0	0	0	0

Table 26: CH

	10	20	30	40	50	60	70	80	90	100
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	7	7	3	4	1	4	4	3	3	2
4	4	1	3	5	4	7	5	6	3	6
5	5	3	5	7	13	3	7	7	5	8
6	14	13	17	17	13	15	12	8	9	19
7	19	24	23	14	19	17	25	28	26	14
8	18	25	20	27	35	33	30	32	27	32
9	17	12	12	18	9	9	14	11	17	10
10	16	15	17	8	6	12	3	5	10	9
NA	0	0	0	0	0	0	0	0	0	0

Table 27: Hartigan

	10	20	30	40	50	60	70	80	90	100
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0
8	100	100	100	100	100	100	100	100	100	100
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0
NA	0	0	0	0	0	0	0	0	0	0

Table 28: Jump

	10	20	30	40	50	60	70	80	90	100
1	90	94	98	97	100	100	100	100	100	100
2	9	6	2	3	0	0	0	0	0	0
3	1	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0
NA	0	0	0	0	0	0	0	0	0	0

Table 29: PS

	10	20	30	40	50	60	70	80	90	100
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0
7	1	1	0	0	0	0	0	0	0	0
8	9	6	8	12	7	2	2	2	3	1
9	27	31	32	26	27	36	32	37	28	28
10	63	62	60	62	66	62	66	61	69	71
NA	0	0	0	0	0	0	0	0	0	0

Table 30: Stab

4.4 Setting 4

	1	5	10	15	20	25	30	35	40	45
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	100	100	99	99	99	99	99	99	100	100
4	0	0	1	1	1	1	1	1	0	0
5	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0
NA	0	0	0	0	0	0	0	0	0	0

Table 31: gabriel-nearest-5x2

	1	5	10	15	20	25	30	35	40	45
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	99	100	99	100	99	99	99	99	100	100
4	1	0	1	0	1	1	1	1	0	0
5	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0
NA	0	0	0	0	0	0	0	0	0	0

Table 32: gabriel-corr-correct2

	1	5	10	15	20	25	30	35	40	45
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	100	96	97	97	95	96	94	93	94	94
4	0	4	3	3	5	4	6	7	6	5
5	0	0	0	0	0	0	0	0	0	1
6	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0
NA	0	0	0	0	0	0	0	0	0	0

Table 33: wold

	1	5	10	15	20	25	30	35	40	45
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	100	100	100	96	85	75	62	54	49	41
4	0	0	0	1	4	6	7	5	3	2
5	0	0	0	1	1	1	0	0	0	0
6	0	0	0	0	0	0	0	0	1	1
7	0	0	0	0	0	1	0	1	1	1
8	0	0	0	0	1	3	3	5	5	4
9	0	0	0	0	2	5	6	6	7	10
10	0	0	0	2	7	9	22	29	34	41
NA	0	0	0	0	0	0	0	0	0	0

Table 34: gap

	1	5	10	15	20	25	30	35	40	45
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	100	100	100	97	97	82	76	69	61	65
4	0	0	0	1	2	16	20	28	28	17
5	0	0	0	0	0	1	4	3	10	15
6	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0
NA	0	0	0	2	1	1	0	0	1	3

Table 35: BIC

	1	5	10	15	20	25	30	35	40	45
1	0	0	0	0	0	0	0	0	0	0
2	14	11	10	8	9	10	11	10	10	14
3	83	80	80	81	81	81	79	81	82	80
4	3	9	10	11	10	9	10	9	8	6
5	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0
NA	0	0	0	0	0	0	0	0	0	0

Table 36: CH

	1	5	10	15	20	25	30	35	40	45
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	96	91	90	91	90	91	90	90	91	90
4	4	7	9	8	8	7	7	7	7	7
5	0	2	1	1	2	2	3	3	2	3
6	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0
NA	0	0	0	0	0	0	0	0	0	0

Table 37: Hartigan

	1	5	10	15	20	25	30	35	40	45
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	3	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0
8	0	4	3	2	1	2	1	2	2	2
9	10	31	30	24	19	24	21	18	19	20
10	87	65	67	74	80	74	78	80	79	78
NA	0	0	0	0	0	0	0	0	0	0

Table 38: Jump

	1	5	10	15	20	25	30	35	40	45
1	5	13	29	28	28	28	28	30	30	34
2	1	3	1	2	2	2	1	1	1	2
3	94	84	70	70	70	70	71	69	69	64
4	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0
NA	0	0	0	0	0	0	0	0	0	0

Table 39: PS

	1	5	10	15	20	25	30	35	40	45
1	0	0	0	0	0	0	0	0	0	0
2	2	2	0	0	0	0	0	0	0	1
3	73	55	57	61	61	58	58	58	57	52
4	24	34	32	26	27	30	31	30	28	32
5	1	9	11	11	10	10	8	11	12	13
6	0	0	0	1	2	2	3	1	3	2
7	0	0	0	1	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0
NA	0	0	0	0	0	0	0	0	0	0

Table 40: Stab

4.5 Setting 5

	2	3	4	5	6	7	8	9	10	11
1	0	0	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0	0	0
3	1	0	0	0	0	0	0	0	0	0
4	5	2	0	0	0	0	0	0	0	0
5	91	97	99	100	99	99	100	99	99	99
6	2	1	1	0	1	1	0	1	1	1
7	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0
NA	0	0	0	0	0	0	0	0	0	0

Table 41: gabriel-nearest-5x2

	2	3	4	5	6	7	8	9	10	11
1	0	0	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0	0	0
3	1	0	0	0	0	0	0	0	0	0
4	2	2	0	0	0	0	0	0	0	0
5	73	96	98	100	100	100	99	100	100	100
6	20	2	2	0	0	0	0	0	0	0
7	3	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	1	0	0	0
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0
NA	0	0	0	0	0	0	0	0	0	0

Table 42: gabriel-corr-correct2

	2	3	4	5	6	7	8	9	10	11
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0
4	5	3	0	0	1	0	1	0	0	1
5	87	95	94	96	97	98	97	97	93	94
6	8	2	5	4	2	2	2	3	7	4
7	0	0	1	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	1
NA	0	0	0	0	0	0	0	0	0	0

Table 43: wold

	2	3	4	5	6	7	8	9	10	11
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0
5	50	98	99	100	100	100	100	100	100	100
6	21	2	1	0	0	0	0	0	0	0
7	9	0	0	0	0	0	0	0	0	0
8	4	0	0	0	0	0	0	0	0	0
9	6	0	0	0	0	0	0	0	0	0
10	10	0	0	0	0	0	0	0	0	0
NA	0	0	0	0	0	0	0	0	0	0

Table 44: gap

	2	3	4	5	6	7	8	9	10	11
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0
5	0	0	31	64	86	97	96	100	100	100
6	3	29	53	32	12	3	4	0	0	0
7	0	24	15	0	0	0	0	0	0	0
8	1	22	0	0	0	0	0	0	0	0
9	5	12	0	0	0	0	0	0	0	0
10	91	10	0	0	0	0	0	0	0	0
NA	0	3	1	4	2	0	0	0	0	0

Table 45: BIC

	2	3	4	5	6	7	8	9	10	11
1	0	0	0	0	0	0	0	0	0	0
2	0	0	1	1	0	1	2	3	3	2
3	0	1	0	3	1	2	3	2	4	1
4	5	11	8	14	16	10	17	20	10	17
5	37	32	51	46	53	58	50	49	50	45
6	23	33	25	29	22	20	22	20	29	29
7	16	18	10	3	5	8	4	4	2	3
8	13	1	2	3	3	1	2	2	1	2
9	4	3	2	1	0	0	0	0	1	1
10	2	1	1	0	0	0	0	0	0	0
NA	0	0	0	0	0	0	0	0	0	0

Table 46: CH

	2	3	4	5	6	7	8	9	10	11
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	1	5	2	5	5	2	4	1	3	3
4	2	15	10	15	13	9	13	17	10	15
5	45	40	59	59	54	66	60	54	58	52
6	24	26	19	11	13	13	14	18	17	23
7	15	9	4	4	10	7	4	7	7	4
8	7	2	3	3	3	3	2	2	4	2
9	2	2	1	2	2	0	1	1	1	0
10	4	1	2	1	0	0	2	0	0	1
NA	0	0	0	0	0	0	0	0	0	0

Table 47: Hartigan

	2	3	4	5	6	7	8	9	10	11
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0
5	13	92	98	100	100	100	100	100	100	100
6	8	3	2	0	0	0	0	0	0	0
7	9	3	0	0	0	0	0	0	0	0
8	7	0	0	0	0	0	0	0	0	0
9	15	1	0	0	0	0	0	0	0	0
10	48	1	0	0	0	0	0	0	0	0
NA	0	0	0	0	0	0	0	0	0	0

Table 48: Jump

	2	3	4	5	6	7	8	9	10	11
1	88	99	96	93	94	94	96	91	94	94
2	2	1	4	7	6	5	4	9	6	6
3	0	0	0	0	0	1	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0
6	4	0	0	0	0	0	0	0	0	0
7	2	0	0	0	0	0	0	0	0	0
8	4	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0
NA	0	0	0	0	0	0	0	0	0	0

Table 49: PS

	2	3	4	5	6	7	8	9	10	11
1	0	0	0	0	0	0	0	0	0	0
2	0	0	1	2	1	1	1	1	1	2
3	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0
5	0	0	1	0	2	0	3	5	2	2
6	0	16	21	23	25	15	35	35	43	37
7	10	31	33	40	44	48	45	37	34	36
8	26	37	27	24	21	25	13	16	19	18
9	32	10	15	9	6	6	3	5	1	2
10	27	5	2	2	1	5	0	1	0	3
NA	5	1	0	0	0	0	0	0	0	0

Table 50: Stab