Conditional Probability 1 – Solutions
COR1-GB.1305 – Statistics and Data Analysis

Compound Events and the Additive Rule

1. Suppose that two customers give ratings (1–5 stars) to the same restaurant on Yelp.
   (a) Express the event "at least one customer gives a 1 star rating" as a union of two other events.

   Solution:

   $$A = \{ \text{ the first customer gives a 1 star rating } \} \cup \{ \text{ the second customer gives a 1 star rating } \}.$$

   (b) Suppose that both customers randomly assign their ratings, giving equal probabilities to all possible star ratings. In this case, all 25 sample points have equal probability. Compute the probability of the event in part (a).

   Solution: Using the additive rule,

   $$\begin{aligned} \mathrm{P}(A) &= \mathrm{P}(1 \text{ from first customer}) + \mathrm{P}(1 \text{ from second customer}) \\ &\quad - \mathrm{P}(1 \text{ from the first customer } \textit{and } 1 \text{ from second customer}) \\ &= \tfrac{1}{5} + \tfrac{1}{5} - \tfrac{1}{25} \\ &= \tfrac{9}{25} \\ &= 36\%. \end{aligned}$$

2. Suppose that two customers give ratings to the same restaurant on Yelp.
   (a) Express the event "the average of their ratings is 3.5 or 4" as a union of two other events. *Hint: this is the same event as "the sum of their ratings is 7 or 8."*

   Solution: Define two events:

   $$\begin{aligned} S_7 &= \{ \text{ the sum of their ratings is 7 } \} \\ &= \{(2,5),(3,4),(4,3),(5,2)\}, \\ S_8 &= \{ \text{ the sum of their ratings is 8 } \} \\ &= \{(3,5),(4,4),(5,3)\}. \end{aligned}$$

   Then, the event we care about is $A = S_7 \cup S_8$.

(b) As in problem 1(b), suppose that both customers randomly assign their ratings with equal probability for all possible star ratings, so that all 25 sample points have equal probability. Compute the probability of the event in part (a).

Solution: Using the additive rule,

$$\begin{aligned} \mathrm{P}(A) &= \mathrm{P}(S_7 \cup S_8) \\ &= \mathrm{P}(S_7) + \mathrm{P}(S_8) - \mathrm{P}(S_7 \cap S_8). \end{aligned}$$

We note that the sum can't be 7 and 8 simultaneously, so S_7 and S_8 are mutually exclusive events, i.e. $S_7 \cap S_8 = \emptyset$. Thus,

$$\begin{aligned} \mathrm{P}(A) &= \mathrm{P}(S_7) + \mathrm{P}(S_8) \\ &= \tfrac{4}{25} + \tfrac{3}{25} \\ &= \tfrac{7}{25} \\ &= 28\%. \end{aligned}$$

Complementary Events and the Complement Rule

3. Here are the tabulated undergraduate major and gender frequencies from the class survey.

	Gender		
Undergrad Major	Female	Male	Total
Business	5	5	10
Hum./Soc. Sci.	7	16	23
Sci./Eng.	4	8	12
Total	16	29	45

Use the data and the complement rule to answer the following questions:

(a) If you pick a random survey respondent, what is the probability that the undergraduate major will not be Business?

Solution: Let

$$A = \{ \text{ the randomly picked student's major } \textit{is not} \text{ Business } \}.$$

Then, the complement of this event is

$$A^c = \{ \text{ the randomly picked student's major } \textit{is} \text{ Business } \}.$$

By the complement rule,

$$\begin{aligned} \mathrm{P}(A) &= 1 - \mathrm{P}(A^c) \\ &= 1 - \tfrac{10}{45} \\ &= \tfrac{35}{45} \\ &\approx 78\%. \end{aligned}$$

(b) What proportion of survey respondents have an undergraduate major that is not listed as "Sci./Eng."?

Solution: Again, using the complement rule,

$$1 - \tfrac{12}{45} = \tfrac{33}{45} = 73\%.$$

4. Suppose you flip five coins. What is the probability of getting at least one head?
Hint: what is the complement of this event?

Solution: The sample space, Ω, is the set of all possible outcomes for the five flips. Since there are 5 independent flips, and each has 2 possible outcomes, we have that $|\Omega| = 2^5 = 32$.

Let

$$A = \{ \text{ you get at least one head } \}.$$

Then,

$$\begin{aligned} A^c &= \{ \text{ you don't get any heads } \} \\ &= \{(T, T, T, T, T)\}. \end{aligned}$$

Thus, by the complement rule,

$$\begin{aligned} \mathrm{P}(A) &= 1 - \mathrm{P}(A^c) \\ &= 1 - \tfrac{1}{32} \\ &= \tfrac{31}{32}. \end{aligned}$$

Conditional Probability

5. Here is a table of the tabulated frequencies for the political parties and presidential votes of the respondents to the class survey.

	Vote			
Party	Clinton	Trump	Other	Total
Democrat	34	1	1	36
Republican	1	2	2	5
Other	2	1	2	5
Total	37	4	5	46

(a) Express the following statements as conditional probabilities:

- $\frac{34}{36} \approx 94\%$ of the Democrats vote Clinton.
- $\frac{1}{4} \approx 25\%$ of the Trump votes come from Democrats.

Solution:

$$P(\text{Clinton} \mid \text{Democrat}) = \frac{34}{36},$$
$$P(\text{Democrat} \mid \text{Trump}) = \frac{1}{4}.$$

(b) Compute $P(\text{Trump} \mid \text{Republican})$ and $P(\text{Republican} \mid \text{Trump})$. Explain the difference between these two quantities.

Solution:

$$P(\text{Trump} \mid \text{Republican}) = \frac{2}{5} \approx 40\%,$$
$$P(\text{Republican} \mid \text{Trump}) = \frac{2}{4} \approx 50\%.$$

The quantity $P(\text{Trump} \mid \text{Republican})$ is the proportion of Republicans who vote for Trump; the quantity $P(\text{Republican} \mid \text{Trump})$ is the proportion of Trump votes that come from Republicans.

6. The following table lists the pick-up and drop-off locations of approximately 170 million yellow cab taxi trips made in New York City in 2013. Numbers are reported in thousands.

Pick-up	Drop-off Bronx	Brooklyn	Manhattan	Queens	Staten Is.	Total
Bronx	53	1	37	4	0	95
Brooklyn	8	2,707	1,598	273	2	4,588
Manhattan	638	5,458	143,656	5,906	22	155,680
Queens	122	1,022	5,058	2,281	8	8,491
Staten Is.	0	0	0	0	3	3
Total	821	9,188	150,349	8,464	35	168,857

(a) Find $P(\text{drop-off Brooklyn} \mid \text{pick-up Manhattan})$ and $P(\text{pick-up Manhattan} \mid \text{drop-off Brooklyn})$. Explain the difference between these two quantities.

Solution:

$$P(\text{drop-off Brooklyn} \mid \text{pick-up Manhattan}) = \frac{5458}{155680} \approx 3.5\%,$$
$$P(\text{pick-up Manhattan} \mid \text{drop-off Brooklyn}) = \frac{5458}{9188} \approx 59.4\%.$$

3.5% of the rides that pick up in Manhattan drop off in Brooklyn; 59.4% of the rides that drop off in Brooklyn originate in Manhattan.

(b) Express the following statement as a conditional probability: "29% of the trips with drop-off locations in Brooklyn originated in the same borough."

Solution:

$$P(\text{pick-up Brooklyn} \mid \text{drop-off Brooklyn}) = \frac{2707}{9188} = 29\%.$$

Note:

$$P(\text{drop-off Brooklyn} \mid \text{pick-up Brooklyn}) = \frac{2707}{4588} = 59\%.$$

The Multiplicative Rule

7. Out of the 51 students enrolled in the class, 18 are female (35%) and 33 are male (65%). Suppose that we randomly select two different students.

   (a) What is the probability that both students are male?

   Solution: Define the two events

   $$\begin{aligned} A &= \text{the first student picked is male} \\ B &= \text{the second student picked is male.} \end{aligned}$$

   Then, $P(A) = \frac{33}{51}$, and $P(B \mid A) = \frac{32}{50}$. Thus, the probability that both will be male is

   $$\begin{aligned} P(A \cap B) &= P(A)P(B \mid A) \\ &= \frac{33}{51} \cdot \frac{32}{50} \\ &= \frac{1056}{2550} \\ &\approx 41\%. \end{aligned}$$

   (b) What is the probability that both students are female?

   Solution: Using the events A and B defined in the previous part, $P(A^c) = \frac{18}{51}$ and $P(B^c \mid A^c) = \frac{17}{50}$. Thus, the probability that both will be female is

   $$\begin{aligned} P(A^c \cap B^c) &= P(A^c)P(B^c \mid A^c) \\ &= \frac{18}{51} \cdot \frac{17}{50} \\ &= \frac{306}{2550} \\ &\approx 12\%. \end{aligned}$$

   (c) What is the probability that one of the students is male and one of the students is female?

   Solution: The event "one student is male and the other is female" is equivalent to the compound event $(A \cap B^c) \cup (A^c \cap B)$; that is, either the first is male and the second is female, or the first is female and the second is male. Since $A \cap B^c$ and $A^c \cap B$ are mutually exclusive, it follows that

   $$P(\text{one male and one female}) = P(A \cap B^c) + P(A^c \cap B).$$

Using the multiplicative rule,

$$\begin{aligned} P(A \cap B^c) &= P(A)P(B^c \mid A) \\ &= \frac{33}{51}\frac{18}{50} \\ &= \frac{594}{2550} \end{aligned}$$

$$\begin{aligned} P(A^c \cap B) &= P(A^c)P(B \mid A^c) \\ &= \frac{18}{51}\frac{33}{50} \\ &= \frac{594}{2550}. \end{aligned}$$

Thus,

$$\begin{aligned} P(\text{one male and one female}) &= \frac{594}{2550} + \frac{594}{2550} \\ &= \frac{1188}{2550} \\ &\approx 47\%. \end{aligned}$$

8. Of the 47 students who filled out the survey, 36 indicated that they drink at least one cup of coffee per day, while 11 indicated that they do not drink coffee on a typical day. Suppose that we randomly select two different survey respondents.

(a) What is the probability that both students regularly drink coffee?

Solution:

$$\frac{36}{47} \cdot \frac{35}{46} = \frac{1260}{2162} \approx 58\%.$$

(b) What is the probability that neither student regularly drinks coffee?

Solution:

$$\frac{11}{47} \cdot \frac{10}{46} = \frac{110}{2162} \approx 5\%.$$

(c) What is the probability that exactly one student regularly drinks coffee?

Solution:

$$\frac{36}{47} \cdot \frac{11}{46} + \frac{11}{47} \cdot \frac{36}{46} = \frac{792}{2162} \approx 37\%.$$