

## Binomial Random Variables

1. A certain coin has a 25% of landing heads, and a 75% chance of landing tails.
  - (a) If you flip the coin 4 times, what is the chance of getting exactly 2 heads?

**Solution:** There are 6 outcomes with exactly 2 heads:

$$HHTT, HTHT, HTTH, THHT, THTH, TTHH.$$

By independence, each of these outcomes has probability  $(.25)^2(.75)^2$ . Thus,

$$P(\text{exactly 2 heads out of 4 flips}) = 6(.25)^2(.75)^2.$$

- (b) If you flip the coin 10 times, what is the chance of getting exactly 2 heads?

**Solution:** Rather than list all outcomes, we will use a counting rule. There are  $\binom{10}{2}$  ways of choosing the positions for the two heads; each of these outcomes has probability  $(.25)^2(.75)^8$ . Thus,

$$P(\text{exactly 2 heads out of 10 flips}) = \binom{10}{2}(.25)^2(.75)^8.$$

2. Suppose that you are rolling a die eight times. Find the probability that the face with two spots comes up exactly twice.

**Solution:** Let  $X$  be the number of times that we get the face with two spots. This is a binomial random variable with  $n = 8$  and  $p = \frac{1}{6}$ . We compute

$$\begin{aligned} P(X = 2) &= \binom{8}{2} p^2 (1 - p)^{8-2} \\ &= \binom{8}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^6 \\ &\approx 0.26. \end{aligned}$$

3. The probability is 0.04 that a person reached on a “cold call” by a telemarketer will make a purchase. If the telemarketer calls 40 people, what is the probability that at least one sale with result?

**Solution:** Let  $X$  be the number of sales. This is a binomial random variable with  $n = 40$  and  $p = 0.04$ . Thus,

$$\begin{aligned} P(X \geq 1) &= 1 - P(X < 1) \\ &= 1 - P(X = 0) \\ &= 1 - \binom{n}{0} p^0 (1 - p)^{n-0} \\ &= 1 - (0.96)^{40} \\ &\approx .805 \end{aligned}$$

4. A new restaurant opening in Greenwich village has a 30% chance of survival during their first year. If 16 restaurants open this year, find the probability that exactly 3 restaurants survive.

**Solution:** Let  $X$  be the number that survive. This is a binomial random variable with  $n = 16$  and  $p = 0.3$ . Therefore,

$$\begin{aligned} P(X = 3) &= \binom{16}{3} (0.3)^3 (1 - 0.3)^{(16 - 3)} \\ &= .146 \end{aligned}$$

5. The probability of winning at a certain game is 0.10. If you play the game 10 times, what is the probability that you win at most once?

**Solution:** Let  $X$  be the number of times that we win. This is a binomial random variable with  $n = 10$  and  $p = 0.10$ . We compute

$$\begin{aligned} P(X \leq 1) &= P(X = 0) + P(X = 1) \\ &= \binom{n}{0} p^0 (1 - p)^{n-0} + \binom{n}{1} p^1 (1 - p)^{n-1} \\ &= \binom{10}{0} (0.10)^0 (0.90)^{10} + \binom{10}{1} (0.10)^1 (0.90)^9 \\ &= (0.90)^{10} + 10 (0.10) (0.90)^9 \\ &\approx 0.736. \end{aligned}$$

6. The probability is 0.3 that an audit of a retail business will turn up irregularities in the collection of state sales tax. If 16 retail businesses are audited, find the probability that
- (a) fewer than 3 will have irregularities in the collection of state sales tax.

**Solution:** Let  $X$  be the number audited. This is a binomial random variable with  $n = 16$  and  $p = 0.3$ . Therefore,

$$\begin{aligned} P(X < 3) &= \binom{16}{0} (0.3)^0 (0.7)^{16} + \binom{16}{1} (0.3)^1 (0.7)^{15} + \binom{16}{2} (0.3)^2 (0.7)^{14} \\ &\approx .0994. \end{aligned}$$

- (b) more than 3 will have irregularities in the collection of state sales tax.

**Solution:**

$$\begin{aligned} P(X > 3) &= 1 - P(X \leq 3) \\ &= 1 - \left[ \binom{16}{0} (0.3)^0 (0.7)^{16} + \binom{16}{1} (0.3)^1 (0.7)^{15} + \binom{16}{2} (0.3)^2 (0.7)^{14} + \binom{16}{3} (0.3)^3 (0.7)^{13} \right] \\ &\approx .7541. \end{aligned}$$

## Poisson Random Variables

7. The number of calls arriving at the Swampside Police Station follows a Poisson distribution with rate 4.6/hour.

(a) What is the probability that exactly six calls will come between 8:00 p.m. and 9:00 p.m.?

**Solution:** Let  $X$  be the number of calls that arrive between 8:00 p.m. and 9:00 p.m. This is a Poisson random variable with mean

$$\lambda = E(X) = (4.6 \text{ calls/hour})(1 \text{ hour}) = 4.6 \text{ calls.}$$

Thus,

$$P(X = 6) = \frac{\lambda^6}{6!} e^{-\lambda} = \frac{(4.6)^6}{6!} e^{-4.6}.$$

(b) Find the probability that exactly 7 calls will come between 9:00 p.m. and 10:30 p.m.

**Solution:** Let  $X$  be the number of calls that arrive between 9:00 p.m. and 10:30 p.m. This is a Poisson random variable with mean

$$\lambda = E(X) = (4.6 \text{ calls/hour})(1.5 \text{ hours}) = 6.9 \text{ calls.}$$

Thus,

$$P(X = 7) = \frac{\lambda^7}{7!} e^{-\lambda} = \frac{(6.9)^7}{7!} e^{-6.9}.$$

8. Car accidents occur at a particular intersection in the city at a rate of about 2/year.

(a) Estimate the probability of no accidents occurring in a 6-month period.

**Solution:** Let  $X$  be the number of car accidents. This is Poisson random variable with mean

$$\lambda = E(X) = (2 \text{ accidents/year})(0.5 \text{ years}) = 1 \text{ accident.}$$

Thus,

$$P(X = 0) = \frac{\lambda^0}{0!} e^{-\lambda} = e^{-1} \approx .368.$$

(b) Estimate the probability of two or more accidents occurring in a year.

**Solution:** Let  $X$  be the number of car accidents. This is Poisson random variable with mean

$$\lambda = E(X) = (2 \text{ accidents/year})(1.0 \text{ years}) = 2 \text{ accident.}$$

Thus,

$$\begin{aligned} P(X \geq 2) &= 1 - P(X < 2) \\ &= 1 - \left[ \frac{\lambda^0}{0!} e^{-\lambda} + \frac{\lambda^1}{1!} e^{-\lambda} \right] \\ &\approx .594. \end{aligned}$$

## Empirical Rule with Binomial and Poisson Random Variables

9. If you flip a fair coin 100 times, would it be unusual to get 42 heads and 58 tails?

**Solution:** Let  $X$  be the number of heads. Then,  $X$  is binomial with  $n = 100$  and  $p = 0.5$ . Thus, its expectation and standard deviation are

$$\mu = np = (100)(0.5) = 50,$$

and

$$\sigma = \sqrt{np(1-p)} = \sqrt{(100)(0.5)(1-0.5)} = 5.$$

Since  $np \geq 15$  and  $n(1-p) \geq 15$ , we can use the empirical rule to approximate the distribution of  $X$ . Thus, approximately 95% of the time,  $X$  will be in the range  $\mu \pm 2\sigma$ , or  $(40, 60)$ . So, it would not be unusual to observe  $X = 42$ .

10. If  $X$  is a Poisson random variable with  $\lambda = 225$ , would it be unusual to get a value of  $X$  which is less than 190?

**Solution:** Set

$$\mu = E(X) = \lambda = 225,$$

$$\sigma = \text{sd}(X) = \sqrt{\lambda} = 15.$$

Define  $z$  to be the number of standard deviations above the mean that 190 is, i.e.

$$190 = \mu + \sigma z.$$

Then,

$$z = \frac{190 - \mu}{\sigma} = \frac{-35}{15} \approx -2.33.$$

A value of  $X$  which is below 190 is more than 2.33 standard deviations below the mean of  $X$ . The empirical rule tells us that observations more than 2 standard deviations away from the mean are unusual (they occur less than 95% of the time). Therefore, values of  $X$  below 190 are unusual.

11. The probability is 0.10 that a person reached on a “cold call” by a telemarketer will make a purchase. If the telemarketer calls 200 people, would it be unusual for them to get 30 purchases?

**Solution:** Let  $X$  be the number of purchases. This is a Binomial random variable with size  $n = 200$  and success probability  $p = 0.10$ . Thus, the expectation and standard deviation of  $X$  are

$$\begin{aligned}\mu &= np = (200)(.10) = 20 \\ \sigma &= \sqrt{np(1-p)} = \sqrt{(200)(.10)(.90)} = \sqrt{18} \approx 4.2\end{aligned}$$

Since  $np \geq 15$  and  $np(1-p) \geq 15$ , the distribution of  $X$  can be approximated by the empirical rule. Using the empirical rule approximation, 95% of the time,  $X$  will be in the range  $\mu \pm 2\sigma$ , or (11.6, 28.4), and 99.7% of the time,  $X$  will be in the range  $\mu \pm 3\sigma$ , or (7.4, 32.6). We would see  $X \geq 30$  less than 5% of the time. It would be unusual to see  $X = 30$ , but not highly unusual.