Probability – Solutions
COR1-GB.1305 – Statistics and Data Analysis

Sample Points and Sample Spaces

1. In the following two experiments, what are the sample points and the sample space?
   (a) You flip a coin.

   Solution: The sample points are H, "the outcome is heads," and T, "the outcome is tails." The sample space is the set of all sample points: $\Omega = \{H, T\}$.

   (b) You roll a 6-sided die.

   Solution: The sample points are the possible outcomes of the die: 1, 2, 3, 4, 5, 6. The sample space is the set of all sample points: $\Omega = \{1, 2, 3, 4, 5, 6\}$.

2. Suppose that a customer visits a restaurant and leaves a review on Yelp with 1–5 stars. What are the sample points and the sample space for the customer's star rating?

   Solution: The sample points are the possible star ratings: 1, 2, 3, 4, 5. The sample space is the set of all sample points: $\Omega = \{1, 2, 3, 4, 5\}$.

3. Suppose that two customers visit a restaurant, and that they both leave Yelp reviews with 1–5 stars each. What are the sample points and the sample space for the pair of star ratings?

   Solution: Each sample point can be represented by an ordered pair (i, j), where i is the first customer's star rating, and j is second customer's star rating. The sample points are the elements of the following table.

	1	2	$\cdots$	5
1	(1,1)	(1,2)	$\cdots$	(1,5)
2	(2,1)	(2,2)	$\cdots$	(2,5)
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
5	(5,1)	(5,2)	$\cdots$	(5,5)

   The sample space is the set of all 25 sample points: $\Omega = \{(1, 1), (1, 2), \ldots, (5, 5)\}$.

4. Suppose you randomly pick a respondent from the class survey, then record their industry and gender. What are the sample points and the sample space? Assume that industry is either "Finance," "Manufacturing," or "Other."

Solution: Each sample point is a gender-industry pair. The sample space is the set of all possible gender-industry pairs: $\Omega = \{(\text{Finance}, \text{Female}), (\text{Finance}, \text{Male}), (\text{Manufacturing}, \text{Female}), (\text{Manufacturing}, \text{Male}), (\text{Other}, \text{Female}), (\text{Other}, \text{Male})\}$. Note that the sample space is all *possible* gender-industry pairs, not all *observed* major-industry pairs.

Events

5. Suppose that a customer leaves a Yelp rating (1–5 stars) for a restaurant. Describe the event "the rating is odd (not even)."

Solution:

$$O = \{1, 3, 5\}.$$

6. Suppose you randomly pick a respondent from the class survey, then record their industry and gender. Assume that industry is listed as Finance, Manufacturing, or Other, and that gender is listed as Male or Female.

   (a) List the sample points in the event "the industry is Finance."

   Solution: $\text{Finance} = \{(\text{Finance}, \text{Female}), (\text{Finance}, \text{Male})\}$.

   (b) List the sample points in the event "the gender is Male."

   Solution: $\text{Male} = \big\{\big(\text{Finance}, \text{Male}\big), \big(\text{Manufacturing}, \text{Male}\big), \big(\text{Other}, \text{Male}\big)\big\}$.

Probability

7. Suppose you randomly pick a respondent from the class survey and record their industry and gender.

   (a) Use the following table of recorded survey response frequencies to compute the probabilities of the sample points.

	Gender		
Industry	Female	Male	Total
Finance	3	9	12
Manufacturing	2	4	6
Other	8	10	18
Total	13	23	36

Solution: To compute the probabilities for the 6 sample points corresponding to the cells of the table, we take the recorded frequency and divide by the total number of survey respondents. We have

$$\begin{aligned}
\mathrm{P}\left(\text{Finance}, \text{Female}\right) &= \frac{3}{36} \approx .08, \\
\mathrm{P}\left(\text{Finance}, \text{Male}\right) &= \frac{9}{36} \approx .25, \\
\mathrm{P}\left(\text{Manufacturing}, \text{Female}\right) &= \frac{2}{36} \approx .06, \\
\mathrm{P}\left(\text{Manufacturing}, \text{Male}\right) &= \frac{4}{36} \approx .11, \\
\mathrm{P}\left(\text{Other}, \text{Female}\right) &= \frac{8}{36} \approx .22, \\
\mathrm{P}\left(\text{Other}, \text{Male}\right) &= \frac{10}{36} \approx .28.
\end{aligned}$$

   (b) Find the probability that the industry is Finance.

Solution:

$$\begin{aligned}
\mathrm{P}(\text{Finance}) &= \tfrac{3}{36} + \tfrac{9}{36} \\
&= \tfrac{12}{36} \\
&\approx 33\%.
\end{aligned}$$

(c) Find the probability that the gender is Male.

Solution:

$$\begin{aligned} \mathrm{P}(\text{Male}) &= \tfrac{9}{36} + \tfrac{4}{36} + \tfrac{10}{36} \\ &= \tfrac{23}{36} \\ &\approx 64\%. \end{aligned}$$

(d) Find the probability the industry is Manufacturing.

Solution:

$$\begin{aligned} \mathrm{P}(\text{Manufacturing}) &= \tfrac{2}{36} + \tfrac{4}{36} \\ &= \tfrac{6}{36} \\ &\approx 17\%. \end{aligned}$$

8. Suppose that a customer's Yelp rating is random, and that the probabilities for the possible star ratings are $p_1 = 10\%$, $p_2 = 30\%$, $p_3 = 25\%$, $p_4 = 20\%$, $p_5 = 15\%$. Find the probability that the rating is odd.

Solution: We add up the probabilities of the sample points in the event:

$$\begin{aligned} \mathrm{P}(\{1,3,5\}) &= p_1 + p_3 + p_5 \\ &= 10\% + 25\% + 15\% \\ &= 50\%. \end{aligned}$$

Compound Events and the Additive Rule

9. Suppose you pick a random survey respondent and record their industry and gender.

   (a) List the sample points in the event "the industry is Finance or the gender is Male."

   Solution: Denote the event by A. Then,

   $$A = \{(\text{Finance}, \text{Female}), (\text{Finance}, \text{Male}), (\text{Manufacturing}, \text{Male}), (\text{Other}, \text{Male})\}.$$

   (b) Compute the probability of the event in part (a) by adding the probabilities of all of the sample points in the event.

   Solution:

   $$\begin{aligned} \text{P}(A) &= \tfrac{3}{36} + \tfrac{9}{36} + \tfrac{4}{36} + \tfrac{10}{36} \\ &= \tfrac{26}{36} \\ &\approx 72\%. \end{aligned}$$

   (c) Express the event "the industry is Finance or the gender is Male" as a union of two other events.

   Solution:

   $$\begin{aligned} A &= \{\text{industry is Finance or gender is Male}\} \\ &= \text{Finance} \cup \text{Male}. \end{aligned}$$

   (d) Compute the probability of the event using the additive rule.

   Solution:

   $$\begin{aligned} \text{P}(A) &= \text{P}(\text{Finance} \cup \text{Male}) \\ &= \text{P}(\text{Finance}) + \text{P}(\text{Male}) - \text{P}(\text{Finance} \cap \text{Male}) \\ &= \tfrac{12}{36} + \tfrac{23}{36} - \tfrac{9}{36} \\ &= \tfrac{26}{36} \\ &\approx 72\%. \end{aligned}$$

10. Suppose that two customers give ratings (1–5 stars) to the same restaurant on Yelp.

(a) Express the event "at least one customer gives a 1 star rating" as a union of two other events.

Solution:

$$A = \{ \text{ the first customer gives a 1 star rating } \} \cup \{ \text{ the second customer gives a 1 star rating } \}.$$

(b) Suppose that both customers randomly assign their ratings, giving equal probabilities to all possible star ratings. In this case, all 25 sample points have equal probability. Compute the probability of the event in part (a).

Solution: Using the additive rule,

$$\begin{aligned} \mathrm{P}(A) &= \mathrm{P}(\text{1 from first customer}) + \mathrm{P}(\text{1 from second customer}) \\ &\quad - \mathrm{P}(\text{1 from the first customer } \textit{and} \text{ 1 from second customer}) \\ &= \tfrac{1}{5} + \tfrac{1}{5} - \tfrac{1}{25} \\ &= \tfrac{9}{25} \\ &= 36\%. \end{aligned}$$

11. Suppose that two customers give ratings to the same restaurant on Yelp.

(a) Express the event "the average of their ratings is 3.5 or 4" as a union of two other events. *Hint: this is the same event as "the sum of their ratings is 7 or 8."*

Solution: Define two events:

$$\begin{aligned} S_7 &= \{ \text{ the sum of their ratings is 7 } \} \\ &= \{(2,5),(3,4),(4,3),(5,2)\}, \\ S_8 &= \{ \text{ the sum of their ratings is 8 } \} \\ &= \{(3,5),(4,4),(5,3)\}. \end{aligned}$$

Then, the event we care about is $A = S_7 \cup S_8$.

(b) As in problem 10(b), suppose that both customers randomly assign their ratings with equal probability for all possible star ratings, so that all 25 sample points have equal probability. Compute the probability of the event in part (a).

Solution: Using the additive rule,

$$\begin{aligned} \mathrm{P}(A) &= \mathrm{P}(S_7 \cup S_8) \\ &= \mathrm{P}(S_7) + \mathrm{P}(S_8) - \mathrm{P}(S_7 \cap S_8). \end{aligned}$$

We note that the sum can't be 7 and 8 simultaneously, so S_7 and S_8 are mutually exclusive events, i.e. $S_7 \cap S_8 = \emptyset$. Thus,

$$\begin{aligned} \mathrm{P}(A) &= \mathrm{P}(S_7) + \mathrm{P}(S_8) \\ &= \tfrac{4}{25} + \tfrac{3}{25} \\ &= \tfrac{7}{25} \\ &= 28\%. \end{aligned}$$

Complementary Events and the Complement Rule

12. Here are the tabulated industry and gender frequencies from the class survey.

	Gender		
Industry	Female	Male	Total
Finance	3	9	12
Manufacturing	2	4	6
Other	8	10	18
Total	13	23	36

Use the data and the complement rule to answer the following questions:

(a) If you pick a random survey respondent, what is the probability that the industry will not be Finance?

Solution: Let

$$A = \{ \text{ the randomly picked student's industry } \textit{is not} \text{ Finance } \}.$$

Then, the complement of this event is

$$A^c = \{ \text{ the randomly picked student's industry } \textit{is} \text{ Finance } \}.$$

By the complement rule,

$$\begin{aligned} \mathrm{P}(A) &= 1 - \mathrm{P}(A^c) \\ &= 1 - \tfrac{12}{36} \\ &= \tfrac{24}{36} \\ &\approx 67\%. \end{aligned}$$

(b) What proportion of survey respondents have an industry that is not listed as Other?

Solution: Again, using the complement rule,

$$1 - \tfrac{18}{36} = \tfrac{18}{36} = 50\%.$$

13. Suppose you flip five coins. What is the probability of getting at least one head?
Hint: what is the complement of this event?

Solution: The sample space, Ω, is the set of all possible outcomes for the five flips. Since there are 5 independent flips, and each has 2 possible outcomes, we have that $|\Omega| = 2^5 = 32$.

Let

$$A = \{ \text{ you get at least one head } \}.$$

Then,

$$\begin{aligned} A^c &= \{ \text{ you don't get any heads } \} \\ &= \{(T, T, T, T, T)\}. \end{aligned}$$

Thus, by the complement rule,

$$\begin{aligned} \mathrm{P}(A) &= 1 - \mathrm{P}(A^c) \\ &= 1 - \tfrac{1}{32} \\ &= \tfrac{31}{32}. \end{aligned}$$