Probability – Solutions
COR1-GB.1305 – Statistics and Data Analysis

Sample Points and Sample Spaces

1. In the following two experiments, what are the sample points and the sample space?
   (a) You flip a coin.

   Solution: The sample points are H, "the outcome is heads," and T, "the outcome is tails." The sample space is the set of all sample points: $\Omega = \{H, T\}$.

   (b) You roll a 6-sided die.

   Solution: The sample points are the possible outcomes of the die: 1, 2, 3, 4, 5, 6. The sample space is the set of all sample points: $\Omega = \{1, 2, 3, 4, 5, 6\}$.

2. Suppose that a customer visits a restaurant and leaves a review on Yelp with 1–5 stars. What are the sample points and the sample space for the customer's star rating?

   Solution: The sample points are the possible star ratings: 1, 2, 3, 4, 5. The sample space is the set of all sample points: $\Omega = \{1, 2, 3, 4, 5\}$.

3. Suppose that two customers visit a restaurant, and that they both leave Yelp reviews with 1–5 stars each. What are the sample points and the sample space for the pair of star ratings?

   Solution: Each sample point can be represented by an ordered pair (i, j), where i is the first customer's star rating, and j is second customer's star rating. The sample points are the elements of the following table.

	1	2	$\cdots$	5
1	(1,1)	(1,2)	$\cdots$	(1,5)
2	(2,1)	(2,2)	$\cdots$	(2,5)
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
5	(5,1)	(5,2)	$\cdots$	(5,5)

   The sample space is the set of all 25 sample points: $\Omega = \{(1, 1), (1, 2), \ldots, (5, 5)\}$.

4. Suppose you randomly pick a respondent from the class survey, then record their undergraduate major and gender. What are the sample points and the sample space? Assume that major is either "Business," "Humanities/Social Science," or "Science/Engineering."

Solution: Each sample point is a gender-major pair. The sample space is the set of all possible gender-major pairs: $\Omega = \{(\text{Bus.}, \text{Female}), (\text{Bus.}, \text{Male}), (\text{Hum./Soc. Sci.}, \text{Female}), (\text{Hum./Soc. Sci.}, \text{Male}), (\text{Sci./Eng.}, \text{Female}), (\text{Sci./Eng.}, \text{Male})\}$. Note that the sample space is all *possible* gender-major pairs, not all *observed* gender-major pairs.

Events

5. Suppose that a customer leaves a Yelp rating (1–5 stars) for a restaurant. Describe the event "the rating is odd (not even)."

Solution:

$$O = \{1, 3, 5\}.$$

6. Suppose you randomly pick a respondent from the class survey, then record their undergraduate major and gender. Assume that undergraduate major is listed as "Business", "Hum./Soc. Sci.", or "Sci./Eng.", and that gender is listed as "Male" or "Female".

   (a) List the sample points in the event "the major is Business."

   Solution: $\text{Business} = \{(\text{Business}, \text{Female}), (\text{Business}, \text{Male})\}$.

   (b) List the sample points in the event "the gender is Male."

   Solution: $\text{Male} = \big\{\big(\text{Business}, \text{Male}\big), \big(\text{Hum./Soc. Sci.}, \text{Male}\big), \big(\text{Sci./Eng.}, \text{Male}\big)\big\}$.

Probability

7. Suppose you randomly pick a respondent from the class survey and record their undergraduate major and gender.

   (a) Use the following table of recorded survey response frequencies to compute the probabilities of the sample points.

	Gender		
Undergrad Major	Female	Male	Total
Business	6	14	20
Hum./Soc. Sci.	15	7	22
Sci./Eng.	2	12	14
Total	23	33	56

Solution: To compute the probabilities for the 6 sample points corresponding to the cells of the table, we take the recorded frequency and divide by the total number of survey respondents. We have

$$\begin{aligned} \mathrm{P}\left(\text{Bus., Female}\right) &= \frac{6}{56} \approx .11, \\ \mathrm{P}\left(\text{Bus., Male}\right) &= \frac{14}{56} \approx .25, \\ \mathrm{P}\left(\text{Hum./Soc. Sci., Female}\right) &= \frac{15}{56} \approx .27, \\ \mathrm{P}\left(\text{Hum./Soc. Sci., Male}\right) &= \frac{7}{56} \approx .12, \\ \mathrm{P}\left(\text{Sci./Eng., Female}\right) &= \frac{2}{56} \approx .04, \\ \mathrm{P}\left(\text{Sci./Eng., Male}\right) &= \frac{12}{56} \approx .21. \end{aligned}$$

   (b) Find the probability that the undergraduate major is Business.

Solution:

$$\begin{aligned} \mathrm{P}(\text{Business}) &= \tfrac{6}{56} + \tfrac{14}{56} \\ &= \tfrac{20}{56} \\ &\approx 36\%. \end{aligned}$$

(c) Find the probability that the gender is Male.

Solution:

$$\begin{aligned} \text{P(Male)} &= \tfrac{14}{56} + \tfrac{7}{56} + \tfrac{12}{56} \\ &= \tfrac{33}{56} \\ &\approx 59\%. \end{aligned}$$

(d) Find the probability the undergraduate major is Humanities/Social Science.

Solution:

$$\begin{aligned} \text{P(Hum./Soc. Sci.)} &= \tfrac{15}{56} + \tfrac{7}{56} \\ &= \tfrac{22}{56} \\ &\approx 39\%. \end{aligned}$$

8. Suppose that a customer's Yelp rating is random, and that the probabilities for the possible star ratings are $p_1 = 10\%$, $p_2 = 30\%$, $p_3 = 25\%$, $p_4 = 20\%$, $p_5 = 15\%$. Find the probability that the rating is odd.

Solution: We add up the probabilities of the sample points in the event:

$$\begin{aligned} \mathrm{P}(\{1,3,5\}) &= p_1 + p_3 + p_5 \\ &= 10\% + 25\% + 15\% \\ &= 50\%. \end{aligned}$$

Compound Events and the Additive Rule

9. Suppose you pick a random survey respondent and record their undergraduate major and gender.

   (a) List the sample points in the event "the major is Business or the gender is Male."

   Solution: Denote the event by A. Then,

   $$A = \{(\text{Business}, \text{Female}), (\text{Business}, \text{Male}), (\text{Hum./Soc. Sci.}, \text{Male}), (\text{Sci./Eng.}, \text{Male})\}.$$

   (b) Compute the probability of the event in part (a) by adding the probabilities of all of the sample points in the event.

   Solution:

   $$\begin{aligned} \mathrm{P}(A) &= \tfrac{6}{56} + \tfrac{14}{56} + \tfrac{7}{56} + \tfrac{12}{56} \\ &= \tfrac{39}{56} \\ &\approx 70\%. \end{aligned}$$

   (c) Express the event "the major is Business or the gender is Male" as a union of two other events.

   Solution:

   $$\begin{aligned} A &= \{\text{major is Business or gender is Male}\} \\ &= \text{Business} \cup \text{Male}. \end{aligned}$$

   (d) Compute the probability of the event using the additive rule.

   Solution:

   $$\begin{aligned} \mathrm{P}(A) &= \mathrm{P}(\text{Business} \cup \text{Male}) \\ &= \mathrm{P}(\text{Business}) + \mathrm{P}(\text{Male}) - \mathrm{P}(\text{Business} \cap \text{Male}) \\ &= \tfrac{20}{56} + \tfrac{33}{56} - \tfrac{14}{56} \\ &= \tfrac{39}{56} \\ &\approx 70\%. \end{aligned}$$

10. Suppose that two customers give ratings (1–5 stars) to the same restaurant on Yelp.

(a) Express the event "at least one customer gives a 1 star rating" as a union of two other events.

> **Solution:**
>
> $$A = \{ \text{ the first customer gives a 1 star rating } \} \\ \cup \{ \text{ the second customer gives a 1 star rating } \}.$$

(b) Suppose that both customers randomly assign their ratings, giving equal probabilities to all possible star ratings. In this case, all 25 sample points have equal probability. Compute the probability of the event in part (a).

> **Solution:** Using the additive rule,
>
> $$\begin{aligned} \mathrm{P}(A) &= \mathrm{P}(1 \text{ from first customer}) + \mathrm{P}(1 \text{ from second customer}) \\ &\quad - \mathrm{P}(1 \text{ from the first customer } \textit{and } 1 \text{ from second customer}) \\ &= \tfrac{1}{5} + \tfrac{1}{5} - \tfrac{1}{25} \\ &= \tfrac{9}{25} \\ &= 36\%. \end{aligned}$$

11. Suppose that two customers give ratings to the same restaurant on Yelp.

(a) Express the event "the average of their ratings is 3.5 or 4" as a union of two other events. *Hint: this is the same event as "the sum of their ratings is 7 or 8."*

> **Solution:** Define two events:
>
> $$\begin{aligned} S_7 &= \{ \text{ the sum of their ratings is 7 } \} \\ &= \{(2,5),(3,4),(4,3),(5,2)\}, \\ S_8 &= \{ \text{ the sum of their ratings is 8 } \} \\ &= \{(3,5),(4,4),(5,3)\}. \end{aligned}$$
>
> Then, the event we care about is $A = S_7 \cup S_8$.

(b) As in problem 10(b), suppose that both customers randomly assign their ratings with equal probability for all possible star ratings, so that all 25 sample points have equal probability. Compute the probability of the event in part (a).

> **Solution:** Using the additive rule,
>
> $$\begin{aligned} \mathrm{P}(A) &= \mathrm{P}(S_7 \cup S_8) \\ &= \mathrm{P}(S_7) + \mathrm{P}(S_8) - \mathrm{P}(S_7 \cap S_8). \end{aligned}$$

We note that the sum can't be 7 and 8 simultaneously, so S_7 and S_8 are mutually exclusive events, i.e. $S_7 \cap S_8 = \emptyset$. Thus,

$$\begin{aligned} \mathrm{P}(A) &= \mathrm{P}(S_7) + \mathrm{P}(S_8) \\ &= \tfrac{4}{25} + \tfrac{3}{25} \\ &= \tfrac{7}{25} \\ &= 28\%. \end{aligned}$$

Complementary Events and the Complement Rule

12. Here are the tabulated undergraduate major and gender frequencies from the class survey.

	Gender		
Undergrad Major	Female	Male	Total
Business	6	14	20
Hum./Soc. Sci.	15	7	22
Sci./Eng.	2	12	14
Total	23	33	56

Use the data and the complement rule to answer the following questions:

(a) If you pick a random survey respondent, what is the probability that the undergraduate major will not be Business?

Solution: Let

$$A = \{ \text{ the randomly picked student's major } \textit{is not} \text{ Business } \}.$$

Then, the complement of this event is

$$A^c = \{ \text{ the randomly picked student's major } \textit{is} \text{ Business } \}.$$

By the complement rule,

$$\begin{aligned} \mathrm{P}(A) &= 1 - \mathrm{P}(A^c) \\ &= 1 - \tfrac{20}{56} \\ &= \tfrac{36}{56} \\ &\approx 64\%. \end{aligned}$$

(b) What proportion of survey respondents have an undergraduate major that is not listed as "Sci./Eng."?

Solution: Again, using the complement rule,

$$1 - \tfrac{14}{56} = \tfrac{42}{56} = 75\%.$$

13. Suppose you flip five coins. What is the probability of getting at least one head?
Hint: what is the complement of this event?

Solution: The sample space, Ω, is the set of all possible outcomes for the five flips. Since there are 5 independent flips, and each has 2 possible outcomes, we have that $|\Omega| = 2^5 = 32$.

Let

$$A = \{ \text{ you get at least one head } \}.$$

Then,

$$\begin{aligned} A^c &= \{ \text{ you don't get any heads } \} \\ &= \{(T, T, T, T, T)\}. \end{aligned}$$

Thus, by the complement rule,

$$\begin{aligned} \mathrm{P}(A) &= 1 - \mathrm{P}(A^c) \\ &= 1 - \tfrac{1}{32} \\ &= \tfrac{31}{32}. \end{aligned}$$