CSCE A405/A605 (Adv) Artificial Intelligence

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1. Fill in the following probability table:

X	**Y**	$\mathbf{P(X \cap Y)}$
1	1	$\frac{4}{10}$
1	0	$\frac{1}{10}$
0	1	
0	0	

Answer:

The easiest way to solve this one is to note some facts that we get from the table. First, note that X and Y are independent. How can we tell? Because they can both occur at the same time, and don't have a probability of 0. That is to say, learning something about X doesn't necessarily tell us anything about Y.

From that we can deduce the following facts:

$$P(X)P(Y) = P(X \cap Y) = \frac{4}{10}$$

Further, because the complements are also dependent we get

$$P(X)P(Y^c) = P(X \cap Y^c) = \frac{1}{10}$$

This means that each variable's complement is also independent and the combinations of complements and original variables are independent[1], so:

$$P(X^c)P(Y) = P(X^c \cap Y)$$

and

$$P(X^c)P(Y^c) = P(X^c \cap Y^c)$$

From here we need to do a little algebra and solve for $P(X)$.

$$\begin{aligned} P(X)P(Y^c) &= P(X \cap Y^c) \\ P(X)\big[1 - P(Y)\big] &= P(X \cap Y^c) \\ P(X) - P(X)P(Y) &= P(X \cap Y^c) \\ P(X) - P(X \cap Y) &= P(X \cap Y^c) \end{aligned}$$

[1]Theorem 1.3.1 in [1] - text available on request.

$$P(X) = P(X \cap Y^c) + P(X \cap Y)$$
$$P(X) = \frac{1}{10} + \frac{4}{10} = \frac{1}{2}$$

Now we can simply solve for $P(Y)$

$$P(X)P(Y) = P(X \cap Y)$$

$$P(Y) = \frac{P(X \cap Y)}{P(X)} \frac{4}{10} \div \frac{1}{2} = \frac{4}{5}$$

Consequently,

$$P(X^c) = \frac{1}{2}$$

and

$$P(Y^c) = \frac{1}{5}$$

so we can substitute these values in on demand:

$$P(X^c)P(Y^c) = P(X^c \cap Y^c) = \frac{1}{2} \cdot \frac{1}{5} = \frac{1}{10}$$

$$P(X^c)P(Y) = P(X^c \cap Y) = \frac{1}{2} \cdot \frac{4}{5} = \frac{4}{10} = \frac{2}{5}$$

hence:

X	**Y**	$\mathbf{P(X \cap Y)}$
1	1	$\frac{4}{10}$
1	0	$\frac{1}{10}$
0	1	$\frac{2}{5}$
0	0	$\frac{1}{10}$

2. Naive Bayes models are widely used for spam filtering. Assume you have trained a spam classifier using a dataset and have estimated the following word probabilities

W	**Free**	**Meet**	**Membership**	**Order**	**Now**
$P(W\|Y = spam)$	$\frac{1}{4}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{1}{2}$
$P(W\|Y = ham)$	$\frac{1}{5}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{10}$

You have received a new email which contains "Order Now." For what values of $P(Y = spam)$ will the email be classified as spam? Note: The answer is in the form of $P(Y = spam) > ...$

Answer:

I approached this by looking at the whole problem as a big naive Bayes classifier.

We're looking for the case where

$$P(Spam|matching\ words) > P(Ham|matching\ words)$$

So, we can evaluate this as:

$$P(Spam|order, now) > P(\sim Spam|order, now)$$

or, more generally,

$$\frac{P(Spam)\prod_{w_i \in email} P(w_i|Spam)}{P(w_1 \cap w_2 \cap ... \cap w_n)} > \frac{P(\sim Spam)\prod_{w_i \in email} P(w_i| \sim Spam)}{P(w_1 \cap w_2 \cap ... \cap w_n)}$$

We can simplify this a little bit

$$P(Spam) \prod_{w_i \in email} P(w_i|Spam) > P(\sim Spam) \prod_{w_i \in email} P(w_i| \sim Spam)$$

now let's make this less messy to look at by declaring a variables to contain those massive products:

$$\pi_1 = \prod_{w_i \in email} P(w_i|Spam)$$

$$\pi_2 = \prod_{w_i \in email} P(w_i| \sim Spam)$$

$$P(Spam)\pi_1 > P(\sim Spam)\pi_2$$

$$P(Spam)\pi_1 > \big[1 - P(Spam)\big]\pi_2$$

$$P(Spam)\pi_1 > \pi_2 - \pi_2 P(Spam)$$

$$P(Spam)\pi_1 + \pi_2 P(Spam) > \pi_2$$

$$P(Spam)\big(\pi_1 + \pi_2\big) > \pi_2$$

$$P(Spam) > \frac{\pi_2}{\pi_1 + \pi_2}$$

Now let's think about this a minute - does it make sense for $\pi_1 + \pi_2 = 0$? No, that will never sum to zero unless none of the words in the table are found in an email that we're trying to classify - for our purposes, in this example these are safe values. So:

$$P(Spam) > \frac{\prod_{w_i \in email} P(w_i| \sim Spam)}{\prod_{w_i \in email} P(w_i|Spam) + \prod_{w_i \in email} P(w_i| \sim Spam)}$$

$$P(Spam) > \frac{\frac{1}{4} \cdot \frac{1}{10}}{\frac{1}{3} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{10}}$$

$$P(Spam) > \frac{6}{46}$$

$$P(Spam) > \frac{3}{23}$$

$$P(Spam) > 0.13043...$$

Note:

I would say that this is probably a pretty good tiny subset of a data set for a classifier then, no? Because at least anecdotally, a lot larger number than 13% of my email appears to be spam.

3. Sally is new to the area nad listens to some friends discussing about another firend. Sally knows that they are talking about either Alice or Bella but doesn't know which. From previous conversations Sally knows some independent pieces of information: She's 90% sure that Alice has a white car, but doesn't know if Bella's car is white or black. Similarly, she's 90% sure that Bella likes sushi, but doesn't know if Alice likes sushi.

   Sally hears from the conversation that the person being discussed hates suchi and drives a white car.

   What is the probability hat the friends are talking about Alice? Assume maximum uncertainty in the absence of knowledge of the probabilities.

   Answer:

   At it's heart, this is a Bayes problem. We'll need to apply Bayes' Rule to solve it.

   We should start by listing what we know.

   Let W refer to "drives a white car, A refer to "Alice calls, S refer to "likes sushi" and B refer to "Bella calls." Out of the gate then, we have the following knowns and unknowns:

$$P(W|A) = 0.9 \text{ (this is given in the problem)}$$
$$P(S|B) = 0.9 \text{ (this is given in the problem)}$$
$$P(W|B) = 0.5 \text{ (this is an unknown)}$$
$$P(S^c|A) = 0.5 \text{ (this is an unknown)}$$
$$P(A) = 0.5 \text{ (this is an unknown)}$$
$$P(B) = 0.5 \text{ (this is an unknown)}$$

   With the information we're given, we're looking to find this:

$$P(A|W \cap S^c)$$

   That is, the probability that it's Alice calling given the caller drives a white car, and doesn't like (hates) sushi. Technically, there's a lot of room between "likes sushi" and "hates sushi" - but let's think of it as a binary - you either love it our hate it.

   So, applying Bayes' rule:

$$P(A|W \cap S^c) = \frac{P(S^c \cap W|A)P(A)}{P(W \cap S^c)}$$

   then apply the chain rule for probability

$$P(A|W \cap S^c) = \frac{P(W|A) \cdot P(S^c|A)P(A)}{P(W \cap S^c)}$$

   now let's consider the denominator. We need to think carefully about what $P(W \cap S^c)$ is. In words, $P(W \cap S^c)$ is the probability that the caller drives a white car and doesn't like sushi. Given that there's a chance either one of them could drive a white car, and either one of them could not like sushi, we're going to have to sum the probabilities for both of them:

$$P(W \cap S^c) = P(W \cap S^c|A)P(A) + P(W \cap S^c)P(B)$$

Let's pause and consider this a moment - we're looking at the "probability that a white car is driven and the driver doesn't like sushi given Alice multiplied by the probability of Alice in general" on the left, and the same thing for Bella on the right. This covers all of the possibilities.

Apply the chain rule again

$$P(W \cap S^c) = P(W|A)P(S^c|A)P(A) + P(W|B)P(S^c|B)P(B)$$

$$= P(W|A)P(S^c|A)P(A) + P(W|B)\big[1 - P(S|B)\big]P(B)$$

Now we have all the numbers, we can throw them into this monstrosity.

$$P(A|W \cap S^c) = \frac{P(W|A) \cdot P(S^c|A)P(A)}{P(W|A)P(S^c|A)P(A) + P(W|B)\big[1 - P(S|B)\big]P(B)}$$

$$P(A|W \cap S^c) = \frac{(0.9) \cdot (0.5) \cdot (0.5)}{(0.9) \cdot (0.5) \cdot (0.5) + (0.5)\big[1 - (0.9)\big](0.5)}$$

$$P(A|W \cap S^c) = \frac{(0.9) \cdot (0.5) \cdot (0.5)}{(0.9) \cdot (0.5) \cdot (0.5) + (0.5) \cdot (0.1) \cdot (0.5)}$$

$$P(A|W \cap S^c) = \frac{0.225}{0.225 + 0.025}$$

$$P(A|W \cap S^c) = \frac{0.225}{0.25} = 0.9$$

Sally can be 90% sure that the call is about Alice.

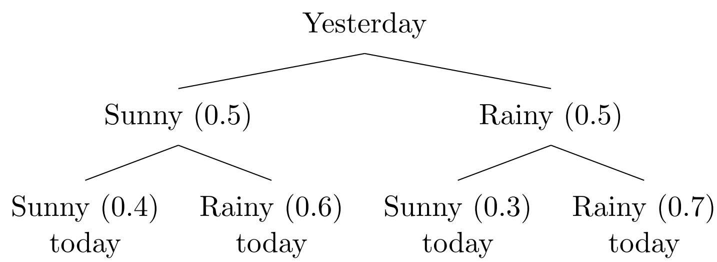
Note: At it's heart, this should intuitively make sense because the biggest clue is the white car, and the other things only change things a little bit, and in this case it works out such that they don't change anything.

4. The weather in London can be summarized as: if it rains one day, there's a 70% chance oit will rain the following day; if it's sunny one day, there's a 40% chance it will be sunny the following day.

   Assuming that the prior probability it rained yesterday is 0.5 what is the probability that it was raining yesterday given that it's sunny today.

   Answer:

   I used a tree at first for this problem, then justified the answer to myself with Bayes' Rule.



   Let's try to put this into an equation. Let's let $Rainy_y$ be "rainy yesterday", and $Rainy_t$ be "rainy today". Similarly, let's let $Sunny_t$ and $Sunny_y$ have the meaning "sunny today" and "sunny yesterday."

$$P(Rainy_y|Sunny_t) = \frac{P(Sunny_t|Rainy_y)P(Rainy_y)}{P(Sunny_t)}$$

   Looking at the tree, the probability of $P(Sunny_t$ has 2 different pathways that can get to it, so we have to add those together to get the complete probability.

$$P(Sunny_t) = P(Sunny_y)P(Sunny_t|Sunny_y) + P(Rainy_y)P(Sunny_t|Rainy_y)$$

$$P(Sunny_t) = (0.4) \cdot (0.5) + (0.5) \cdot (0.3) = 0.35$$

   Then to get the numerator, we need to go down the right branch of the tree, then the left sub-branch, yielding:

$$P(Sunny_t|Rainy_y)P(Rainy_y) = (0.3) \cdot (0.5) = 0.15$$

$$P(Rainy_y|Sunny_t) = \frac{P(Sunny_t|Rainy_y)P(Rainy_y)}{P(Sunny_t)} = \frac{0.15}{0.35} = 0.4287..$$

   So the probability that it was rainy yesterday given that it was sunny today is 42.87%

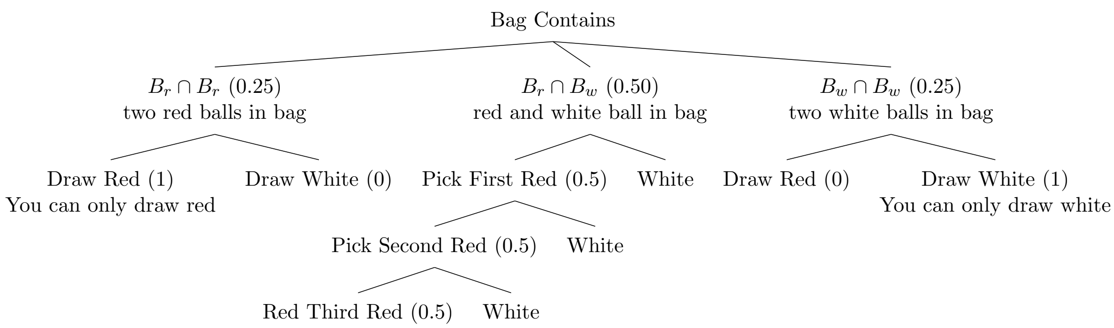
5. Two balls are placed in a box as follows: A fair coin is tossed and a white ball is placed in the box if a head occurs, otherwise a red ball is placed in the box. The coin is tossed again and a red ball is placed in the box if a tail occurs, otherwise a white ball is placed in the box. Balls are drawn from the box three times in succession (always with replacing the drawn ball back in the box). It is found that on all three occasions a red ball is drawn. What is the probability that both balls in the box are red?

   Answer:

   This is another Bayes' Rule problem. In essence if W is "a white ball is drawn" and R is "a red ball is drawn" and B_r is "the ball that goes into the bag is red" and B_w is "the ball that goes into the bag is white," then we're looking to solve the following problem:

$$P(B_r \cap B_r|RRR) = \frac{P(RRR|B_r \cap B_r)P(B_r \cap B_r)}{P(RRR)}$$

   Personally, I needed a tree to visualize this one initially:



   So $P(RRR)$ is found by adding up the results that give us red on the left branch, and the ones that give us red on the center branch.

   This is just a visual depiction of the chain rule! If we apply it to the original equation. Cool!

$$P(B_r \cap B_r|RRR) = \frac{P(RRR|B_r \cap B_r)P(B_r \cap B_r)}{P(RRR)}$$

   This is horrifying when typed out, but

$$P(B_r \cap B_r|RRR) = \frac{P(R|B_r \cap B_r)P(B_r \cap B_r)}{P(R|B_r \cap B_r)P(B_r \cap B_r) + P(R|RR \cap B_r \cap B_r) \cdot P(R|R \cap B_r \cap B_r) \cdot P(R| \cap B_r \cap B_r) \cdot P(B_r \cap B_r)}$$

   On the bright side, we do have all of these numbers now, so we can actually solve this.

$$P(B_r \cap B_r|RRR) = \frac{(1.0) \cdot (0.25)}{(1.0) \cdot (0.25) + (0.5) \cdot (0.5) \cdot (0.5) \cdot (0.5)}$$

$$P(B_r \cap B_r|RRR) = \frac{(0.25)}{(0.25) + (0.5)^4} = 0.80$$

So, if you pull out three red balls in a row, you can be 80% certain that the bag had 2 red balls in it from the start.

This makes sense, as the 50 percent probability of there being a white and red ball in the bag only moves the needle a little bit off of the left branch where you can only get red balls out of the bag.

References

[1] George Casella, Roger L. Berger (1990) *Statistical Inference*, Duxberry Press.