# Homework 2 - ACT3230

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## 1 Question 1

**A) Show that $r_t = \theta + (r_0 - \theta)e^{-kt}$ satisfies the differential equation $dr_t = k(\theta - r_t)dt$.**

We begin by computing the derivative of $r_t$ with respect to time $t$. Since $r_t$ evolves deterministically, its differential is also deterministic. We have that $r_t - \theta = (r_0 - \theta)e^{-kt}$.

$$\frac{dr_t}{dt} = -k(r_0 - \theta)e^{-kt} = -k(r_t - \theta) = k(\theta - r_t)$$

Therefore,

$$dr_t = k(\theta - r_t)dt$$

Thus, $r_t = \theta + (r_0 - \theta)e^{-kt}$ satisfies the differential equation $dr_t = k(\theta - r_t)dt$.

**B) Show that $B_t = \exp\left(\int_0^t r_s ds\right)$ satisfies the differential equation $dB_t = r_t B_t dt$.**

We start by computing $\frac{dB_t}{dt}$. Using the fundamental theorem of calculus, the derivative with respect to time is

$$\frac{dB_t}{dt} = \frac{d}{dt}\exp\left(\int_0^t r_s ds\right) = \exp\left(\int_0^t r_s ds\right)\cdot\frac{d}{dt}\left(\int_0^t r_s ds\right) = \exp\left(\int_0^t r_s ds\right)\cdot r_t$$

$$\frac{dB_t}{dt} = \exp\left(\int_0^t r_s ds\right)\cdot r_t = B_t \cdot r_t \iff dB_t = r_t B_t dt$$

**C) Determine the stochastic differential equation followed by the process $\tilde{S}_t$, where $\tilde{S}_t = \frac{S_t}{B_t}$.**

We are given the SDE for $S_t$,

$$dS_t = \mu S_t dt + \sigma S_t dW_t^P,$$

and we want to determine the SDE for

$$\tilde{S}_t = \frac{S_t}{B_t},$$

where $B_t = \exp\left(\int_0^t r_s ds\right)$ satisfies the differential equation $dB_t = r_t B_t dt$.

To determine the differential of $\tilde{S}_t$, we apply Itô's formula. We express $\tilde{S}_t = \frac{S_t}{B_t}$ as the ratio of two processes $X_t = S_t$ and $Y_t = B_t$. The formula for $\tilde{S}_t = \frac{X_t}{Y_t}$ is:

$$d\tilde{S}_t = \frac{1}{Y_t} dX_t - \frac{X_t}{Y_t^2} dY_t + d\langle X, Y\rangle_t,$$

where $d\langle X, Y\rangle_t$ is the quadratic covariation between $X_t$ and $Y_t$.

Substituting $X_t = S_t$ and $Y_t = B_t$, we have:

$$dS_t = \mu S_t dt + \sigma S_t dW_t^P,$$
$$dB_t = r_t B_t dt.$$

The quadratic covariation term satisfies $d\langle S, B\rangle_t = 0$ because $B_t$ is deterministic while $W_t^P$ is random.

Substituting into the expression:

$$d\tilde{S}_t = \frac{1}{B_t} dS_t - \frac{S_t}{B_t^2} dB_t = \frac{1}{B_t}\left(\mu S_t dt + \sigma S_t dW_t^P\right) - \frac{S_t}{B_t^2} \cdot r_t B_t dt$$

Thus, the differential form becomes:

$$d\tilde{S}_t = \tilde{S}_t \left((\mu - r_t)dt + \sigma dW_t^P\right)$$

**D) Use Girsanov's theorem to show how to change the probability measure on $(\Omega, F)$ to obtain the evolution of the risky asset price under the risk-neutral measure $\mathbb{Q}$. Also give the stochastic differential equation satisfied by the asset price $\{S_t\}_{t\geq 0}$ under $\mathbb{Q}$.**

We begin by showing how to change the probability measure using Girsanov's theorem. Under the physical measure $\mathbb{P}$, the evolution of the risky asset price $\{S_t\}_{t\geq 0}$ is:

$$dS_t = \mu S_t dt + \sigma S_t dW_t^P.$$

We want to transform to the risk-neutral measure $\mathbb{Q}$, where the expected return of $S_t$ equals the risk-free rate $r$ instead of $\mu$. Under $\mathbb{Q}$, the process satisfies:

$$dS_t = rS_t dt + \sigma S_t dW_t^Q,$$

where the Brownian motion is now under $\mathbb{Q}$.

The change of measure in Girsanov's theorem uses the Radon–Nikodym density $Z_t$. We assert that there exists a measure $\mathbb{Q} \sim \mathbb{P}$ such that $W_t^{\mathbb{Q}} = \{W_t^{\mathbb{Q}}\}_{t\in[0,T]}$. Under $\mathbb{Q}$, the Brownian motion is defined as:

$$W_t^{\mathbb{Q}} = W_t^{\mathbb{P}} + \int_0^t \lambda_s ds.$$

Thus, the Radon–Nikodym density is:

$$Z_t = \frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^t \lambda_s dW_s^{\mathbb{P}} - \frac{1}{2}\int_0^t \lambda_s^2 ds\right),$$

where $\lambda_s$ is the drift-adjustment process.

We now determine $\lambda_t$. Under $\mathbb{P}$,

$$dS_t = \mu S_t dt + \sigma S_t dW_t^{\mathbb{P}},$$

and under $\mathbb{Q}$,

$$dS_t = rS_t dt + \sigma S_t dW_t^{\mathbb{Q}}.$$

Using the relation $W_t^{\mathbb{Q}} = W_t^{\mathbb{P}} + \int_0^t \lambda_s ds$, we substitute $dW_t^{\mathbb{P}} = dW_t^{\mathbb{Q}} - \lambda_t dt$ into the SDE:

$$dS_t = \mu S_t dt + \sigma S_t(dW_t^{\mathbb{Q}} - \lambda_t dt) = (\mu - \sigma\lambda_t)S_t dt + \sigma S_t dW_t^{\mathbb{Q}}$$

For the drift under $\mathbb{Q}$ to equal $r$, we must have:

$$\mu - \sigma\lambda_t = r \iff \lambda_t = \frac{\mu - r}{\sigma}.$$

Therefore, under the risk-neutral measure $\mathbb{Q}$, the asset price satisfies:

$$dS_t = rS_t dt + \sigma S_t dW_t^{\mathbb{Q}}.$$

## 2 Question 2

**This exercise concerns the Black–Scholes formula for the price of a European vanilla option and the computation of implied volatility.**

**A) Create a function `BSOptionPrice` that outputs the price of a European vanilla option under the Black–Scholes model (either a call option or a put option).**
See R code.

**B) Evaluate your function for a call option and a put option with the following parameters:** $S_0 = 100$, $K = 105$, $r = 0.02$, $T = 0.5$, $\sigma = 0.2$.
The price of the call option is: 3.985574.
The price of the put option is: 7.940807.

**C) Create a function `BSImplicitVol` that outputs the implied volatility of the given option.**
See R code.

**D) Use your function to compute the implied volatility of a call option with a price of 2.7852 under the assumptions:** $S_0 = 100$, $K = 105$, $r = 0.02$, $T = 0.25$.
The resulting implied volatility for the call option is: 0.2300165.

**E) Use your function to compute the implied volatility of a put option with a price of 6.8249 under the assumptions:** $S_0 = 100$, $K = 105$, $r = 0.02$, $T = 0.75$.
The resulting implied volatility for the put option is: 0.1400008.

## 3 Question 3

**This question concerns the use of the binomial model to approximate option prices under the Black–Scholes model. Consider the binomial tree parameterized as follows:**

$$u = e^{\mu h + \sigma\sqrt{h}} \tag{1}$$

$$d = e^{\mu h - \sigma\sqrt{h}} \tag{2}$$

$$h = \frac{1}{n} \tag{3}$$

**a) Create a function `BinOptionPrice`.**
See R code.

**b) Evaluate your function for a call option and a put option with the following parameters:** $S_0 = 100$, $K = 105$, $r = 0.02$, $T = 0.5$, $\sigma = 0.2$, $\mu = r - \sigma^2/2$, $n = 20$**.**
The price obtained for the call option is: 4.011795.
The price obtained for the put option is: 7.967028.

**c) Using your functions `BSOptionPrice` and `BinOptionPrice`, produce numerical results illustrating, for different values of** $\mu$**, the convergence of European vanilla option prices as a function of** $n$ **toward the price given by the Black–Scholes formula. Consider in particular the Cox–Ross–Rubinstein tree and the lognormal tree. Use the following assumptions:** $S_0 = 100$, $K = 105$, $r = 0.02$, $T = 0.5$, $\sigma = 0.2$**.**

First, we define the variables of interest in the function. We begin by computing the theoretical price using the `BSOptionPrice` function (Black–Scholes method) defined in Exercise 2 of the assignment. We obtain:
Black–Scholes price (Call): 3.985574
Black–Scholes price (Put): 7.940807

We then create a vector of values for $n$ that increases, in order to show that as $n$ becomes large, the price for different values of $\mu$ approaches the price from the Black–Scholes formula for both call and put options. For the values of $n$, we choose values up to $n = 200$.

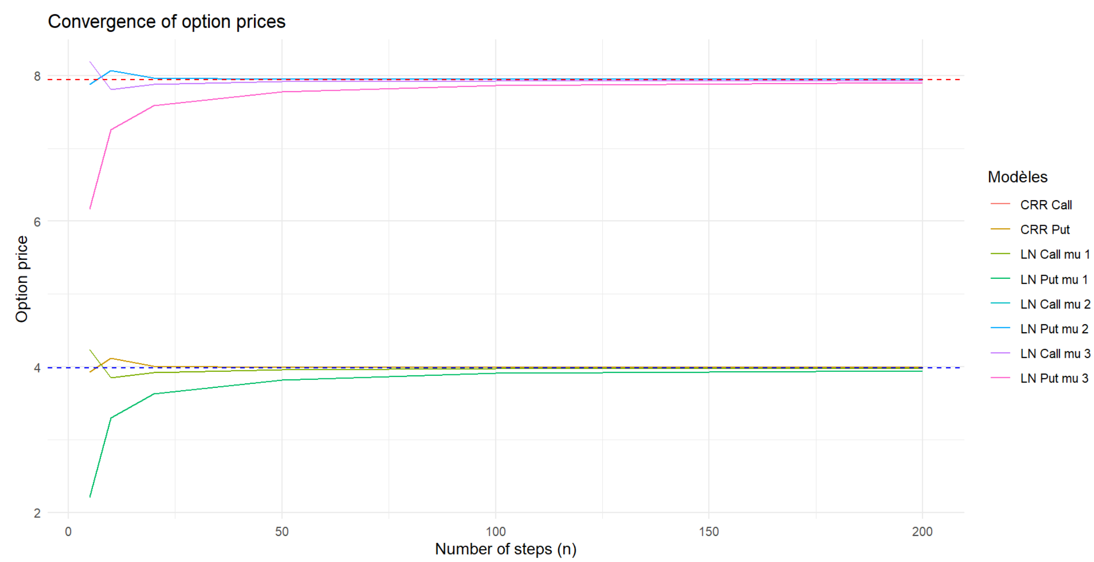
We define the $\mu$ parameters for the Cox–Ross–Rubinstein model and for the lognormal model using their definitions from class and the given assumptions. Thus,

$$\mu_{CRR} = 0$$
$$\mu_{LN} = r - \frac{\sigma^2}{2} = 0.02 - \frac{0.2^2}{2} = 0$$

We also want to test price convergence when $\mu$ varies, so we keep $\mu = 0$ for the Cox–Ross–Rubinstein model (this is one of the model assumptions). For the lognormal model, we vary $\mu$ as follows: $\mu_1 = 0$, $\mu_2 = 0.1$, and $\mu_3 = 0.5$. The following table shows the call and put prices for each value of $n$ and each value of $\mu$, computed using the binomial tree under the respective assumptions of both models. The table therefore contains 8 columns of interest and one column for $n$.

| n | CRR Call | L Call mu 1 | L Call mu 2 | L Call mu 3 | CRR Put | L Put mu 1 | L Put mu 2 | L Put mu 3 |
|---|---|---|---|---|---|---|---|---|
| 5 | 3.927930 | 3.927930 | 4.240867 | 2.213751 | 7.883163 | 7.883163 | 8.196100 | 6.168983 |
| 10 | 4.115990 | 4.115990 | 3.853222 | 3.298018 | 8.071223 | 8.071223 | 7.808455 | 7.253250 |
| 20 | 4.011795 | 4.011795 | 3.921892 | 3.630844 | 7.967028 | 7.967028 | 7.877124 | 7.586077 |
| 50 | 3.993589 | 3.993589 | 3.962330 | 3.819251 | 7.948822 | 7.948822 | 7.917562 | 7.774484 |
| 100 | 3.993118 | 3.993118 | 3.975218 | 3.911124 | 7.948351 | 7.948351 | 7.930450 | 7.866356 |
| 200 | 3.991831 | 3.991831 | 3.981269 | 3.946003 | 7.947063 | 7.947063 | 7.936501 | 7.901236 |

We observe from the graph below that the CRR and lognormal models both converge properly to the Black–Scholes prices (Call and Put) when $n$ is large (as in our case where $n = 200$).




**d) In Question c, discuss the impact of the choice of the parameter $\mu$. Is there a value of $\mu$ for which you observe faster convergence?**

First, using the assumptions from the problem statement, both trees (Cox–Ross–Rubinstein and Lognormal) were parameterized the same way because they had $\mu = 0$. Thus, we study convergence for other values of $\mu$ as well.

For the Cox–Ross–Rubinstein tree, we have $\mu = 0$ by definition. We keep this value fixed, and modify the assumptions for the lognormal tree to vary its $\mu$.

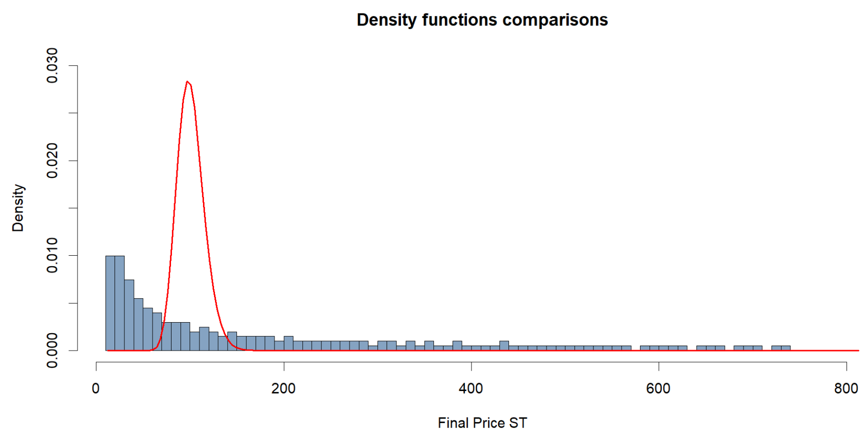
For the lognormal tree, the parameter $\mu = r - \sigma^2/2$ includes an adjustment that accounts for the true drift of the simulated lognormal process.

For large $n$, the CRR model ($\mu = 0$) offers more accurate convergence, as seen in the graph for Question c).

The lognormal model ($\mu = r - \sigma^2/2$), however, exhibits faster convergence toward the Black–Scholes price for small values of $n$. For example, based on the graph in Question c), convergence is faster when $\mu_3 = 0.5$ for both call and put options under the lognormal model, and similarly for $\mu = 0.1$.

We conclude that the CRR model ($\mu = 0$) is more robust than the lognormal model. It guarantees the preservation of risk-neutrality and asymptotic convergence. The lognormal model, however, can be useful to study variations in convergence for small tree sizes, as it converges more sharply.

**e) Produce numerical results showing that the random variable $S_T$ in the risk-neutral binomial tree is approximately lognormally distributed, i.e., LN$(\log S_0 + (r - \sigma^2/2)T,\ \sigma^2 T)$, when $n$ is large. Use the following assumptions: $S_0 = 100$, $K = 105$, $r = 0.02$, $T = 0.5$, $\sigma = 0.2$.**




Thus, when $n$ is large (in our case $n = 200$), the density of terminal prices $S_T$ in the binomial tree approaches the density of a lognormal distribution.

**f) Explain why using the binomial model to approximate the price of exotic options under the Black–Scholes model is generally not numerically appropriate.**

First, by definition, exotic options depend on the entire path of the underlying asset, since their payoff depends on the trajectory taken. In a binomial tree, the number of paths increases extremely quickly as the number of steps $n$ grows. For very large $n$, such as in Question 3c ($n = 200$), evaluating all possible paths to price an exotic option becomes computationally expensive.

Another important point is that some exotic options, such as Asian options, depend on the average of the asset prices over time. The binomial tree only stores terminal prices, not intermediate ones. Storing all intermediate prices between the initial and terminal times adds considerable computational complexity.

Monte Carlo simulations are more efficient for this type of computation, since we can directly compute average payoffs after simulating all possible trajectories of the underlying asset without needing to store intermediate prices explicitly.

# 4 Question 4

**a) Create a function `SimulateStockPaths`.**

See R code.

**b) By adapting the function `SimulateStockPaths`, create a new function `SimulateStockPathsAntithetic` that also outputs a matrix of dimension $M \times (N+1)$ representing $M$ simulated price paths, but this time using the variance reduction method based on antithetic variables.**

See R code.

**c) Suppose that: $S_0 = 100$, $\mu = 0.07$, $r = 0.02$, $\sigma = 0.2$, $T = 0.5$, $\Delta t = 1/52$. Use your functions `SimulateStockPaths` and `SimulateStockPathsAntithetic` to simulate $M = 10\,000$ trajectories under $\mathbb{P}$ and under $\mathbb{Q}$.**

See R code.

Note: We also verify that the output matrix for the function `SimulateStockPathsAntithetic` has dimension $M \times (N+1)$, that is $10\,000 \times 27$. We consider $N = 26$ from the formula given in the problem statement $T = n\Delta t$, since $T = 0.50$ and $\Delta t = \frac{1}{52}$.

**d) Using the trajectories you simulated, estimate the expectations of the terminal prices under $\mathbb{P}$ and under $\mathbb{Q}$, then construct a 0.95 confidence interval for your Monte Carlo estimators. Compare your results with the exact values of these expectations.**

By definition, the expectation under the real-world measure $\mathbb{P}$ is $\mathbb{E}^{\mathbb{P}}[S_T] = S_0 e^{\mu T}$, where $\mu = 7\%$ and $T = 0.5$. Thus, the exact expectation under $\mathbb{P}$ is:

$$\mathbb{E}^{\mathbb{P}}[S_T] = S_0 e^{\mu T} = 100 e^{0.07 \cdot 0.5} = 103.562$$

Next, the expectation under the risk-neutral measure $\mathbb{Q}$ is $\mathbb{E}^{\mathbb{Q}}[S_T] = S_0 e^{rT}$, where $r = 2\%$ and $T = 0.5$. Thus, the exact expectation under $\mathbb{Q}$ is:

$$\mathbb{E}^{\mathbb{Q}}[S_T] = S_0 e^{r \cdot T} = 100 e^{0.02 \cdot 0.5} = 101.005$$

For the Monte Carlo confidence interval, it is given by the formula:

$$IC = \hat{\mu}^{(MC)} \pm z_{0.975} \cdot \frac{\hat{\sigma}^{\mathbb{Q}}}{\sqrt{M}}$$

where $z_{0.975} = 1.96$ for a 95% confidence interval, $\hat{\mu}^{(MC)}$ is the Monte Carlo estimate of the expectation, and $\hat{\sigma}^{\mathbb{Q}}$ is the standard deviation under the risk-neutral measure.

See R code. We obtain the Monte Carlo expectation under $\mathbb{P}$: 104.7547.
Thus, the 95% confidence interval under $\mathbb{P}$ is:

$$[104.4649;\ 105.0445].$$

Similarly, we obtain the Monte Carlo expectation under $\mathbb{Q}$: 101.1517.
The 95% confidence interval under $\mathbb{Q}$ is:

$$[100.8718;\ 101.4315].$$

To compare our results with the exact expectations, we compute the relative error under $\mathbb{P}$ and under $\mathbb{Q}$. See R code.

We obtain a relative error under $\mathbb{P}$ of 1.152% and a relative error under $\mathbb{Q}$ of 0.145%.

# 5 Question 5

**This question concerns pricing, under the Black–Scholes model, of an Asian option with payoff at time $T$.**

**a) Use the simulated paths `StockPathsQ` and `StockPathsQantithetic` from Exercise 4 to obtain a Monte Carlo estimate of the price of the Asian option, as well as a 0.95 confidence interval.**

To obtain a 95% confidence interval for the Monte Carlo price estimate, we apply the following formula:

$$\left[\hat{\Pi}_0^{(MC)} - z_{\frac{\alpha}{2}} \cdot \frac{\hat{Std}^{\mathbb{Q}}(\tilde{H})}{\sqrt{M}}\ ;\ \hat{\Pi}_0^{(MC)} + z_{\frac{\alpha}{2}} \cdot \frac{\hat{Std}^{\mathbb{Q}}(\tilde{H})}{\sqrt{M}}\right].$$

See R code. We obtain a Monte Carlo price (using the function **StockPathsQ**) of 3.625952$. The associated 95% confidence interval (StockPathsQ) is:

$$[3.520199;\ 3.731705].$$

Similarly, the Monte Carlo price (using the function **StockPathsQantithetic**) is 3.570545$. The associated 95% confidence interval (StockPathsQantithetic) is:

$$[3.465242;\ 3.675848].$$

**b) Show that**

$$E^{\mathbb{Q}}[A_T^{(ari)}] = \frac{S_0 \cdot e^{r\Delta_t}}{n} \cdot \frac{1 - e^{rn\Delta_t}}{1 - e^{r\Delta_t}}. \tag{4}$$

We begin by recalling the definition of $A_T^{(ari)}$:

$$A_T^{(ari)} = \frac{1}{n} \sum_{i=1}^{n} S_{t_i}.$$

Using this, we substitute directly into the expression for $E^{\mathbb{Q}}[A_T^{(ari)}]$. By linearity of expectation (constants and sums factor out), we obtain:

$$\mathbb{E}^{\mathbb{Q}}\left[A_T^{(ari)}\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{n}\sum_{i=1}^{n} S_{t_i}\right] = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}^{\mathbb{Q}}[S_{t_i}].$$

We now compute $\mathbb{E}^{\mathbb{Q}}[S_{t_i}]$. Under the risk-neutral measure $\mathbb{Q}$, we know that $\mathbb{E}^{\mathbb{Q}}[S_{t_i}] = S_0 e^{rt_i}$.

From the problem statement, $t_i = i\Delta_t$. Substituting:

$$\mathbb{E}^{\mathbb{Q}}[A_T^{(ari)}] = \frac{1}{n}\sum_{i=1}^{n} S_0 e^{rt_i} = \frac{1}{n}\sum_{i=1}^{n} S_0 e^{ri\Delta_t} = \frac{1}{n} S_0 \sum_{i=1}^{n} e^{r\Delta_t \cdot i}.$$

The sum above is a geometric series with ratio $e^{r\Delta_t}$. We use the known identity $\sum_{i=1}^{n} x^i = x\frac{1-x^n}{1-x}$. Setting $x = e^{r\Delta_t}$, we obtain:

$$\sum_{i=1}^{n} e^{ri\Delta_t} = e^{r\Delta_t} \cdot \frac{1 - e^{(r\Delta_t)^n}}{1 - e^{r\Delta_t}} = e^{r\Delta_t} \cdot \frac{1 - e^{(rn\Delta_t)}}{1 - e^{r\Delta_t}}.$$

Therefore,

$$\mathbb{E}^{\mathbb{Q}}[A_T^{(ari)}] = \frac{1}{n} \cdot S_0 \cdot e^{r\Delta_t} \cdot \frac{1 - e^{rn\Delta_t}}{1 - e^{r\Delta_t}} = \frac{S_0 \cdot e^{r\Delta_t}}{n} \cdot \frac{1 - e^{rn\Delta_t}}{1 - e^{r\Delta_t}}.$$

**c) Use the simulated paths `StockPathsQ` and `StockPathsQantithetic` to obtain a Monte Carlo estimate of the Asian option price based on the control variate $A_T^{(ari)}$. Also obtain a 0.95 confidence interval for your control variate estimator. State the computed value of $c^*$.**

The Asian option price is estimated as:

$$\Pi_0^{(ari)} \approx \frac{1}{M}\sum_{j=1}^{M}\left[e^{-rT} H_j^{(ari)} + c^*\left(A_T^{(ari),j} - \mathbb{E}^{\mathbb{Q}}[A_T^{(ari)}]\right)\right],$$

where:

- $H_j^{(ari)} = \max(0, A_T^{(ari),j} - K)$ is the simulated payoff for the $j$-th path,
- $c^*$ is the optimal control variate coefficient,

$$c^* = -\frac{\mathrm{Cov}^{\mathbb{Q}}\left(H^{(ari)}, A_T^{(ari)}\right)}{\mathrm{Var}^{\mathbb{Q}}\left(A_T^{(ari)}\right)}.$$

From part 5b), the theoretical expectation of $A_T^{(ari)}$ under $\mathbb{Q}$ is:

$$\mathbb{E}^{\mathbb{Q}}[A_T^{(ari)}] = S_0 e^{r\Delta t} \cdot \frac{1 - e^{rn\Delta t}}{n(1 - e^{r\Delta t})} = 100 \cdot e^{0.02 \cdot \frac{1}{52}} \cdot \frac{1 - e^{0.02 \cdot 26 \cdot \frac{1}{52}}}{26(1 - e^{0.02 \cdot \frac{1}{52}})} = 100.521.$$

From the course notes, to compute $c^*$ we must simulate the quantities involved using $A_T^{(ari)}$ and $H^{(ari)}$ obtained from **StockPathsQ** and **StockPathsQantithetic**. We use the parameters:

$$S_0 = 100, \quad r = 2\%, \quad \sigma = 20\%, \quad T = 0.5, \quad \Delta t = \frac{1}{52}, \quad n = 26,$$

with $M = 10{,}000$ simulated paths.

Using R, we obtain the value:

$$c^* = -0.5629245.$$

We then estimate $\Pi_0^{(ari)}$ using the control variate method and obtain the estimated Asian option price (control):

$$\Pi_0^{(ari)} \approx 3.564768\$$$

The associated 95% confidence interval is:

$$[3.51628;\ 3.613255].$$