Improvement of Mathematical Teaching Through the Development of Virtual CAD Models and Physical Prototypes for Real Visualization Using 3D Printing

Adis J. Muminovic, Vahidin Hadziabdic, Sedin Musanovic, Nedim Pervan, Muamer Delic

University of Sarajevo – Faculty of Mechanical Engineering, Vilsonovo setaliste no.9, Sarajevo, Bosnia and Herzegovina

Abstract **– This article explores a possibility to improve mathematical teaching by using 3D printing technology. The question is whether it is possible to use low cost additive manufacturing technology to develop and manufacture real physical prototypes of complex mathematical surfaces and volumes and on that way improve mathematics education. Five mathematical problems were chosen as case studies. Visualization of this problems was done using professor hand drawing, using computer visualization and using development and manufacturing of real physical prototypes. To find out how much better is understanding of these problems, survey with 57 students is carried out. Results showed significant improvements of understanding and better visualization of selected mathematical problems.**

Keywords **– Mathematics, visualization, 3D printing, complex geometrical shapes.**

DOI: 10.18421/TEM121-02
https://doi.org/10.18421/TEM121-02

Corresponding author: Adis J. Muminovic,
University of Sarajevo – Faculty of Mechanical Engineering, Vilsonovo setaliste no.9, Sarajevo, Bosnia and Herzegovina.
Email: adis.muminovic@mef.unsa.ba

Received: 03 October 2022.
Revised: 12 December 2022.
Accepted: 15 December 2022.
Published: 27 February 2023.

1. Introduction

During the teachings in mathematics there is always a problem how to present, through visualization, complex mathematical problems whose goal is the calculation of areas and volumes.

Usually, visualization is carried out using hand drawings by the professor on the board. Understanding of the size and shape of the geometry drafted using this way is very limited by the students. In the most of the complex cases students do not understood what is the goal of mathematical problem. They solve the problem using knowledge from mathematics, but they basically do not understand what they calculated. Another way to visualize complex geometrical shape in 3D space is using computer software's like Mathematica, Matlab or Microsoft Excel. Usage of these software's can significantly increase the understanding of the shape and size of complex mathematical geometry but again students cannot feel the geometry in their hands. Low cost additive manufacturing technology (3D printing) can be used for easy and cheap manufacturing of real physical prototypes of complex mathematical shapes.

Low cost 3D printing technologies have great potential to be used in education purposes, especially at engineering courses. Research about enhancing the teaching of machine design by creating a basic hands-on environment with mechanical 'breadboards' was carried out in paper [1]. Also, similar research was carried out in paper [2]. Goal of the research was to improve the mechanical design education introducing hands-on experience with machine parts. Research about development of machine part exhibition and functional mock-ups to enrich design education was carried out in paper [3]. Similar research was carried out at machine design courses at the United States Air Force Academy and at the University of Texas in Austin in USA in paper [4].

In paper [5] three options for using the 3D-printing technology in the classroom is presented. First, there is the possibility to reproduce existing products. Another option is the development of products by the teacher in co-operation with students. Finally, the students can also develop 3D-printed objects on their own. In relation to inquiry-based learning. This third option is the most interesting application for education. General problems and methodologies for usage of 3D printers in classrooms are well described in paper [6].

For usage of 3D printers in mathematical education there is several papers published up to this date. Similar research (to the research carried out in this paper) was carried out in paper [7], but it is limited to calculus, in this paper using the example of 3D-printed graphs of functions, the use of 3D printing technology in calculus is discussed within the three approaches of Grundvorstellungen, Subjective Domains of Experience and Empirical Theories. Also similar research was carried out in paper [8], but it was limited to simple geometry and the goal of the research was to better understood the relations between changes in dimensions of the object and how that change reflect to the change of volume of the same object. Great example of real visualization is presented in another paper [9], again it is showed how different mathematical functions can be represented using 3D printing technology. Comparison of fused deposition modeling (FDM) and Color Jet 3D Printing (CJP) technologies for the printing of mathematical geometries is carried out in paper [10]. General research about importance of visualizations in mathematics teaching is carried out in paper [11].

Goal of this research is to analyze possibilities to use 3D printing for better visualization in mathematical education for complex mathematical problems for higher education of engineering students. With special focus on complex mathematical problems which needs to be solved using different forms of integrals.

2. Problem Definition

Visualization problems for complex mathematical shapes can be presented using following example.

Definition of problem: It is necessary to calculate the volume of a shape which is limited by cylinders $x^2 + y^2 = 2y$ and $x^2 + y^2 = 4y$, and by paraboloids $z = x^2 + y^2$ and $z = 5(x^2 + y^2)$.

Mathematical solution: First step is to solve following equation system:

$$\begin{gathered} x^2 + y^2 = 2y \\ x^2 + y^2 = 4y \end{gathered} \tag{1}$$

Cylindrical coordinates need to be introduced:

$$\begin{gathered} x = \rho \cos\varphi \\ y = \rho \sin\varphi \\ z = z \\ |J| = \rho \end{gathered} \tag{2}$$

Now, the system will be:

$$\begin{gathered} \rho^2 = 2\rho \sin\varphi \\ \rho^2 = 4\rho \sin\varphi \end{gathered} \tag{3}$$

Equation (3) is divided by ρ and limits of integral are obtained in a form: $2\sin\varphi \leq \rho \leq 4\sin\varphi$. Also, it will be assumed that $\rho = 0$ so additional limits can be obtained in a form: $\pi \leq \varphi \leq 2\pi$.

Limits for z are obtained by solving the system and introducing cylindrical coordinates:

$$\begin{matrix} z = x^2 + y^2 \\ z = 5(x^2 + y^2) \end{matrix} \Rightarrow \begin{matrix} z = \rho^2 \\ z = 5\rho^2 \end{matrix} \tag{4}$$

Now volume of the shape can be calculated:

$$V = \iiint\limits_D dxdydz = \int\limits_{\pi}^{2\pi} d\varphi \int\limits_{2\sin\varphi}^{4\sin\varphi} d\rho \int\limits_{\rho^2}^{5\rho^2} \rho dz = 90\pi \tag{5}$$

Solution for visualization:

For solution of this problem first step which professors usually do is to try to visualize the shape

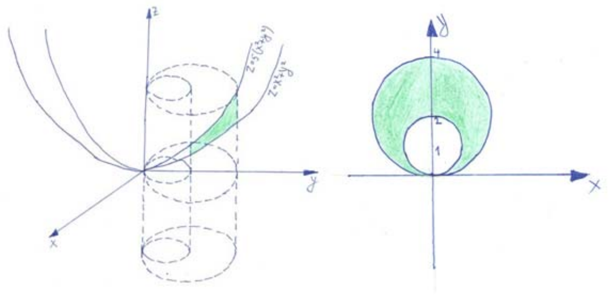

Figure 1. Visualization done by professor hand drawing, 3D view left, top view right

of the volume, usually they do it by one of the two ways, by hand drawings or by a computer software. Figure 1. shows professor hand drawing for this problem. Professor tries to do it through two views.

From Figure 1. it can be noticed that it is very hard to visually identify the shape which needs to be calculated. It is almost impossible to draw it on the way that students can understand the shape of it. Second way to make a visualization in this case is to use computer software.

Problem with computer software is in the fact that needed complex shape will be hidden inside two paraboloids and two cylinders and because of that it will not be visible (Figure 2.).

In addition, mathematical computer software's do not allow to cut one surface using another one, so the final complex volume is still hard to see. Looking to the figures 1 and 2 it can be noticed that visualization is improved, but still it is not perfectly clear from the figures what volume needs to be calculated.

Professor needs to make a significant effort to try to explain students which surfaces are intersecting with each other. It is clear that better visualization is needed for this case. Authors of this paper proposed a development of computer aided design (CAD) model and manufacturing of real physical model for this problem.

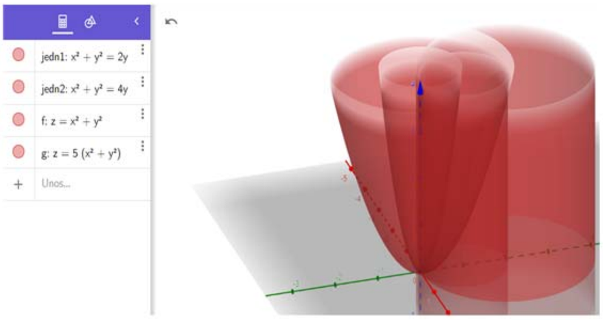

Figure 2. Visualization of defined problem carried out using GeoGebra mathematical software

3. Development and Manufacturing of Real Physical Models

In this part of the paper methodology for development of real physical model, for problem presented in section 2, is presented. This methodology includes development of solid CAD model and manufacturing of a model using low cost FDM (Fused Deposition Modelling) 3D printing technology.

A. Development of Solid CAD Model

Computer Aided Design (CAD) is a method for virtual representation of real objects using computers [12]. Mostly it is used for engineering purposes and it is not very common in mathematics visualization. It is not used in mathematics because, in the most of the cases, mathematics professors are not familiar with CAD software's. Basic knowledge of working principles of CAD software is necessary to represent complicated mathematical shapes. If a professor wants to use methodology presented in this paper for real physical visualization using CAD and 3D printing, authors advice is to seek help from his own students to whom he teaches mathematics. These students usually have basic knowledge of CAD software's from first year of faculty or before faculty education. It is a great way to additionally interest students for mathematics. Figure 3. shows developed CAD model for mathematical problem defined in chapter 2. SolidWorks software was used. Looking to the Figure 1, 2 and 3 it is already noticeably that visualization can be significantly improved using developed CAD model. Especially if this model is opened using any type of CAD software, in CAD software developed model can be rotated and zoomed.

B. Manufacturing of Real Physical Model

Real physical model is manufactured using one of the rapid prototyping technology which is also known as additive manufacturing technology (3D printing). In this research low cost 3D printing machines are used because they are available to wide number of schools and faculties around the world, which increases applicability of methodology presented in this research. Rapid prototyping technologies are in the focus of research for many authors these days. One of the main manufacturing technology for rapid prototyping is additive manufacturing, especially fused deposition modelling (FDM), which is used in this research for manufacturing of real physical prototypes.

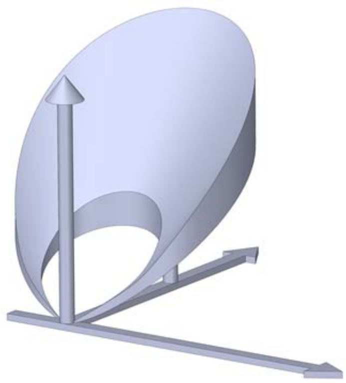

Figure 3. Developed CAD model for defined mathematical problem

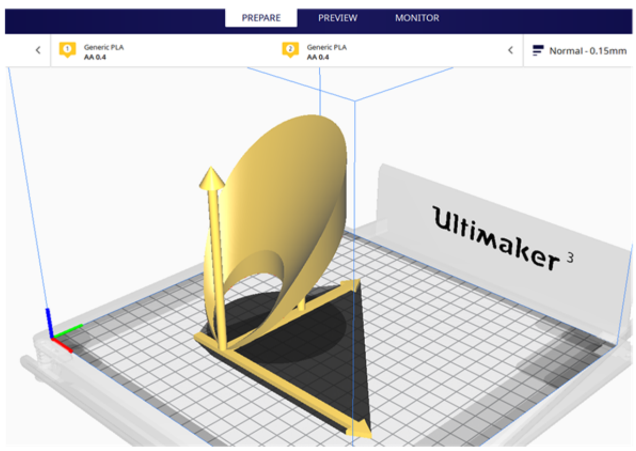

Figure 4. Preparing models for 3D printing

Machines for FDM have low cost and they become available for regular people out of the universities and big companies. Detail reviews about rapid prototyping and additive manufacturing can be found in papers [13], [14]. After development of CAD model first step in manufacturing process is to save developed model as STL file and to prepare it for 3D printing using software for preparing 3D model for 3D printing. In this case CURA software, from Ultimaker company was used (Figure 4).

For 3D printing process Ultimaker 3 3D printer is selected. This printer has two extrusion heads, so it can manufacture models using two materials.

In this case coordinate system is manufactured in one color and complex mathematical shape (3D model) in another. Manufactured model is showed at Figure 5.

Figure 5. 3D printed model of complex mathematical shape

4. Survey Analysis of Improvement of Teaching and Visualization

To test this type of visualization, five case studies (including problem solved in previous chapter) are selected, 3D printed and presented to the students.

Survey with specially selected questions is given to the students to find out is visualization improved.

Selected Case Studies

Problem 1: Calculate the volume of a shape which is limited by paraboloid $z = 6 - x^2 - y^2$, cone $z^2 = x^2 + y^2$ and $z \geq 0$

Problem 2: Calculate the volume of a shape which is limited by cylinder $x^2 + y^2 = 8$, and by planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 4$.

Problem 3: Calculate the volume of a shape between cone $z^2 = x^2 + y^2$ and ball $x^2 + y^2 + z^2 = 4$.

Problem 4: Calculate the volume of a shape which is limited by cylinder $x^2 + y^2 = 2y$ and two paraboloids $z = 1 + x^2 + y^2$ and $z = -2 - x^2 - y^2$.

Problem 5: It is necessary to calculate the volume of a shape which is limited by cylinders $x^2 + y^2 = 2y$ and $x^2 + y^2 = 4y$, and by paraboloids $z = x^2 + y^2$ and $z = 5(x^2 + y^2)$. This problem is described in chapter 2 of the paper. It is used in chapter 2 to describe presented methodology

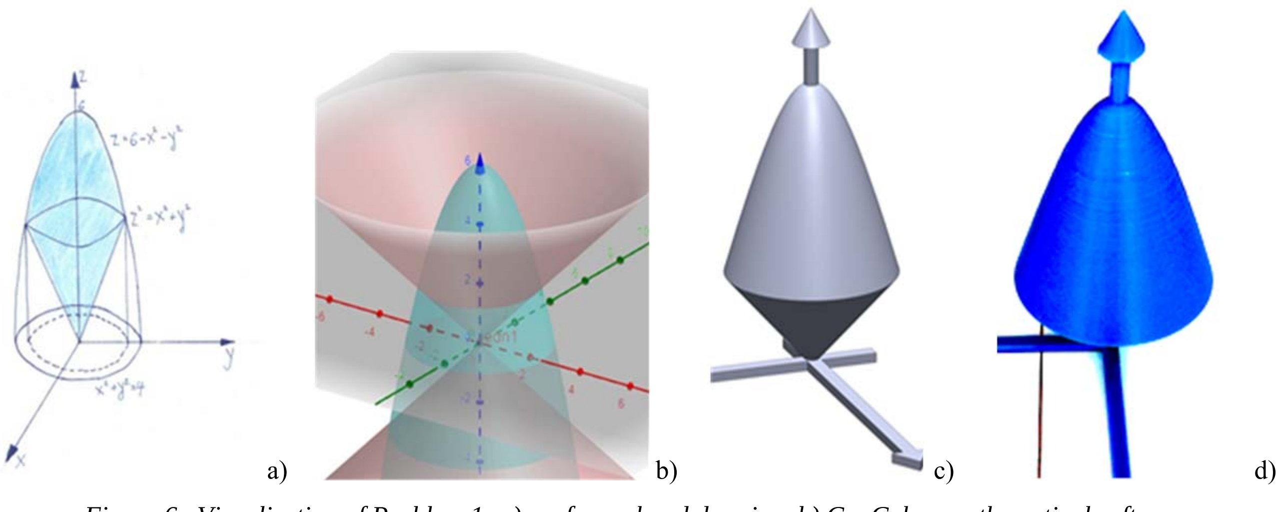

Figure 6. Visualization of Problem 1: a) professor hand drawing, b) GeoGebra mathematical software, c) CAD 3D model, d) 3D printed real model

TEM Journal. Volume 12, Issue 1, pages 13-21, ISSN 2217-8309, DOI: 10.18421/TEM121-02, February 2023.

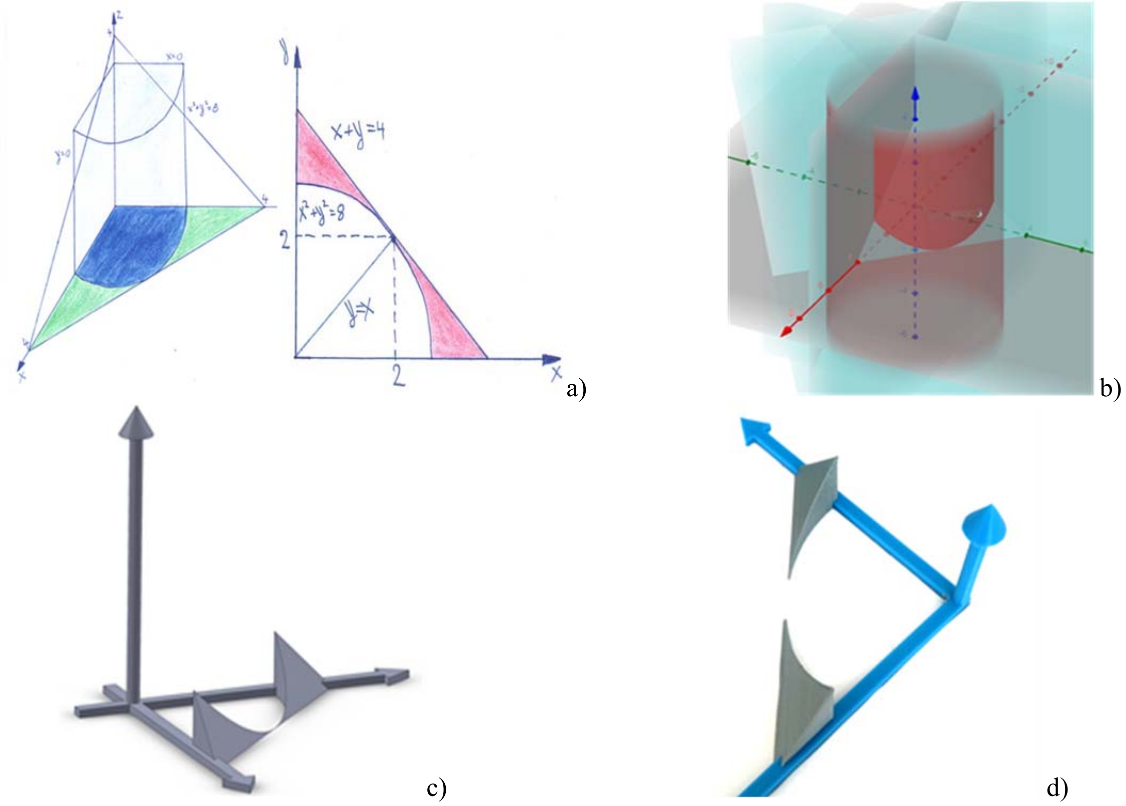

Figure 7. Visualization of Problem 2: a) professor hand drawing, b) GeoGebra mathematical software, c) CAD 3D model, d) 3D printed real model

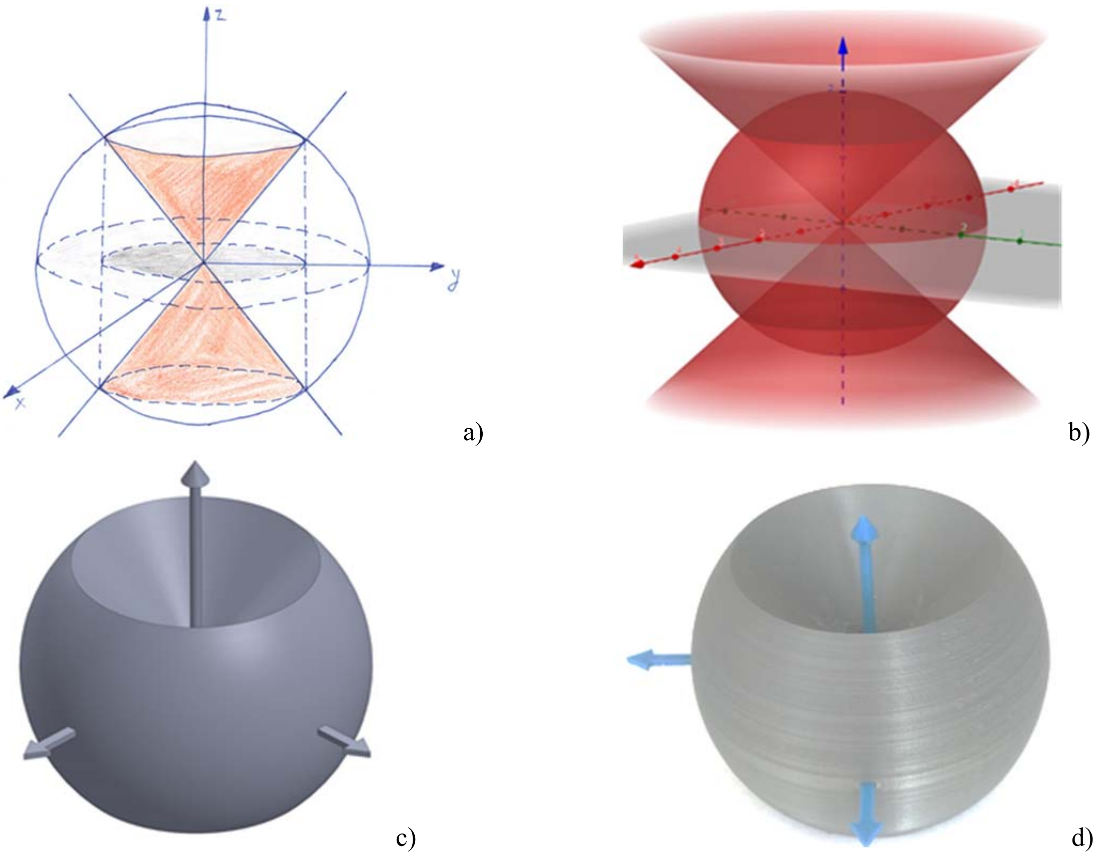

Figure 8. Visualization of Problem 3: a) professor hand drawing, b) GeoGebra mathematical software, c) CAD 3D model, d) 3D printed real model

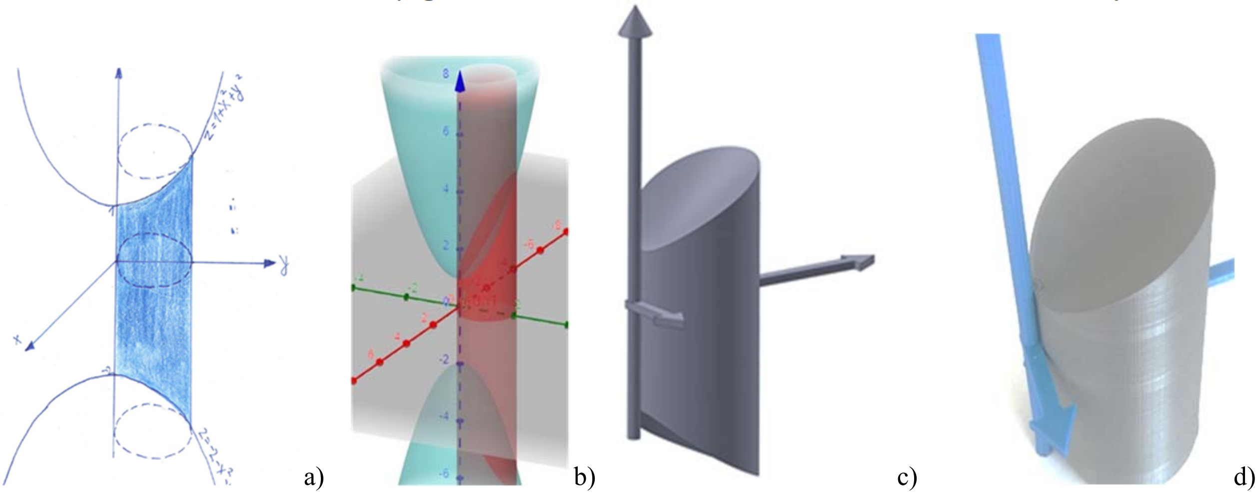

Figure 9. Visualization of Problem 4: a) professor hand drawing, b) GeoGebra mathematical software, c) CAD 3D model, d) 3D printed real model

A. Survey Methodology

Survey is given to 57 students who already pass Mathematics 1 and Mathematics 2 course at Faculty of Mechanical Engineering. Same question is asked from all students for all above presented mathematical problems:

Question: With the grade from 1 to 10 mark how well you can visualize and understood the shape which needs to be calculated by:

a) Reading the text of the problem.
b) Looking to the picture of professor hand drawing.
c) Looking to the picture of developed real physical model.

It is important to underline following notes:

- Questions are given in the above mentioned order a, b and c. Text of the problem is given first without any picture and student give the grade of visualization, after that, professor hand drawing is given, students give second grade and as final question picture of real physical model is given and final grade is given by the students.
- Initial plan was to present live, in classroom, GeoGebra generated model and CAD developed model to the students, including rotation, zoom and translation of the model, but because of Corona virus students was not attending at the faculty and live presentation was not possible. In addition, initial plan was to give developed real physical models to the students, so they can take it in their hand and after that give a grade for visualization, but that also was not possible, only pictures of real physical models are shown to the students.

B. Results

Results for Problem 5 are presented in diagrams showed on Figures 10, 11 and 12. Diagrams are presented only for problem 5 because of the length of the paper, results for other problems are given in Table 1. In addition, problem 5 have the most complex shape of geometrical for students to understand and because of that it is a good example to show efficiently of presented methodology for improvement of mathematical education.

From Figure 10 it can be noticed that in the case when students have only text of the mathematical problem given grades differ a lot, all grades have similar percentage. It is clear that most of the students do not understand the shape which needs to be calculated. Only 4 students (7,14%) answer with a grade of 10, only 2 students (3,57%) answer with a grade of 9 and only 4 students (7,14%) answer with a grade of 8. If grades 8, 9 and 10 are taken as grades which represents a good understanding of the problem then in total only 10 students (17,85%) from

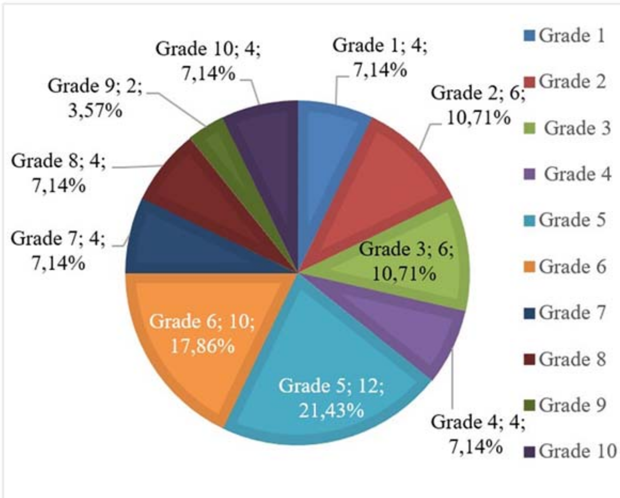

Figure 10. Results of answers for Problem 5 for a) Reading the text of the problem

57 in total can understood the geometrical shape which needs to be calculated by just reading the text of the problem.

In the case when picture of professor hand drawing is presented to the students (Figure 11), significant improvement of students understanding of the problem can be noticed. Grades 8, 9 and 10 have biggest percentages. 9 students (16.36%) answer with the grade of 10 which means that they completely understood the problem. 8 students (14,55%) answer with a grade of 9 and 6 students (10,91%) answer with a grade of 8. This result shows significant improvement, in total 23 students (41,28%) answer with a grades 8, 9 and 10 which is improvement in amount of 23,97% in comparison to the summary results for grades 8, 9 and 10 from Figure 10 (reading just the text of the problem).

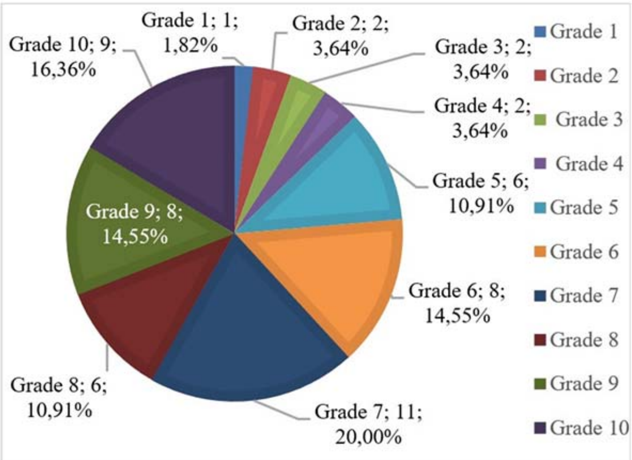

Figure 11. Results of answers for Problem 5 for b) Looking to the picture of professor hand drawing

This results are improvement but it is still small percentages taking in consideration that this is usually the way how mathematical problems are presented to the students in the most of the schools and faculties worldwide. This means that less than half of the students understand this problem completely.

Looking to the picture of developed real physical model (Figure 12), understanding of the mathematical body which needs to be calculated improves significantly. 29 students (52,73%) answer with the grade of 10, 6 students (10,91%) answer with a grade of 9 and 7 students (12,73%) answer with the grade of 8. In total, 42 students answer with grades 8, 9 and 10, which is (76,37%). This means that two thirds of the students completely understood the shape which needs to be calculated for Problem 5. This is significant improvement in comparison to the results from Figure 12 for professor hand drawing. Summary results for all five problems presented in chapter 4.1 are given in Table 1. From the results from Table 1 it can be seen that understanding of the problems is increased in the case of all problems. This conclusion can be especially visible by calculating of average grade for all scenarios and problems (Figure 13). Increasing of average grade can be seen for all the cases. Average grade is increased from around 5 to around 8 for all five problems.

Table 1. Results of survey with average grade

Grades	Problem 1			Problem 2			Problem 3			Problem 4			Problem 5		
	Text	Hand drawing	Real physical model	Text	Hand drawing	Real physical model	Text	Hand drawing	Real physical model	Text	Hand drawing	Real physical model	Text	Hand drawing	Real physical model
Grade 1	6	0	1	7	0	2	5	0	2	5	0	1	4	1	2
Grade 2	4	2	0	6	2	1	3	1	0	6	2	2	6	2	1
Grade 3	7	2	0	5	3	1	8	1	0	8	2	0	6	2	0
Grade 4	3	1	2	5	4	0	3	4	1	4	5	1	4	2	0
Grade 5	6	3	1	10	4	0	8	2	3	5	2	0	12	6	4
Grade 6	5	2	5	4	4	5	4	6	4	12	7	4	10	8	2
Grade 7	7	3	2	4	11	4	9	8	1	2	11	3	4	11	4
Grade 8	7	11	4	10	9	5	6	9	5	3	4	6	4	6	7
Grade 9	4	10	3	2	8	8	4	10	6	4	6	7	2	8	6
Grade 10	8	22	37	3	11	29	6	14	34	7	17	31	4	9	29
Average grade	5.8	8.2	8.9	5.1	7.2	8.5	5.7	7.7	8.7	5.3	7.4	8.6	5.1	7.0	8.4

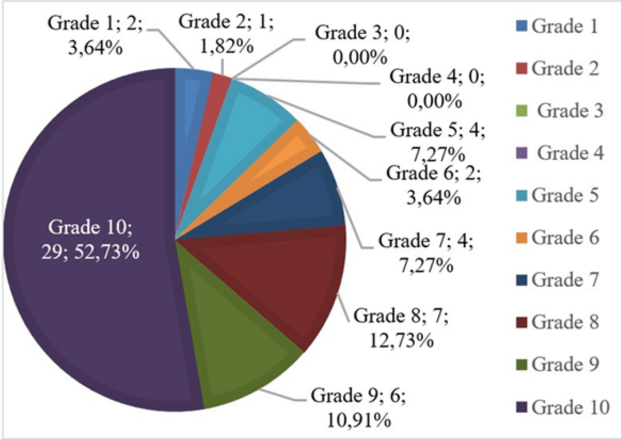

Figure 12. Results of answers for Problem 5 for c) Looking to the picture of developed real physical model

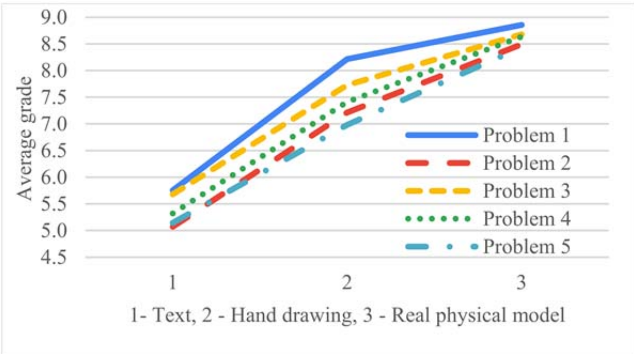

Figure 13. Diagram of average grades

5. Conclusion

Teaching mathematics in engineering schools and faculties can be difficult and challenging job. Engineering students prefers to see everything in three dimensional space or in real “hands on” examples for better understanding. This research showed that computer aided design (CAD) and low cost rapid prototyping technology (3D printing) can be used to develop and manufacture real physical models of mathematical problems for better students understanding. This methodology can be used in all levels of education, from primary schools to universities. In addition, this methodology is a great example how to increase students’ interest in mathematics, especially in engineering schools and faculties. If the professor is not familiar with CAD software’s he can work together with its engineering students to develop and manufacture those physical models. Today, almost every school or faculty have one or more devices for rapid prototyping (3D printers), these devices can be used to increase quality of education on several subjects and courses, usually they are primarily used for engineering classes and courses, this research showed that they can be used for mathematics education also. Future research can include more different problems for various mathematics or physics courses. Also, future research can be oriented to the development of web based database of developed CAD models for more mathematical problems. Database will have open access for upload and download of all modes so they can be downloaded and used from all around the world.

References

[1]. Wood, J., Campbell, M., Wood, K., & Jensen, D. (2005). Enhancing the teaching of machine design by creating a basic hands-on environment with mechanical ‘breadboards’. *International Journal of Mechanical Engineering Education*, *33*(1), 1-25. https://doi.org/10.7227/IJMEE.33.1.1

[2]. Beckmann, G., & Krause, D. (2010). Improving the mechanical design education by hands-on experience with machine parts. In *DS 62: Proceedings of E&PDE 2010, the 12th International Conference on Engineering and Product Design Education-When Design Education and Design Research meet..., Trondheim, Norway, 02.-03.09. 2010*.

[3]. Beckmann, G., & Krause, D. (2011). Machine part exhibition and functional mock-ups to enrich design education. In *DS 68-8: Proceedings of the 18th International Conference on Engineering Design (ICED 11), Impacting Society through Engineering Design, Vol. 8: Design Education, Lyngby/Copenhagen, Denmark, 15.-19.08. 2011* (pp. 130-139).

[4]. Wood, J., Jensen, D., & Wood, K. (2005, June). Enhancing Machine Design Courses Through Use Of A Multimedia Based Review Of Mechanics Of Materials. In *2005 Annual Conference* (pp. 10-572).

[5]. Witzke, I., & Hoffart, E. (2018). 3D-Drucker: Eine Idee für den Mathematikunterricht? Mathematikdidaktische Perspektiven auf ein neues Medium für den Unterricht. *Beiträge zum Mathematikunterricht 2018*.

[6]. Horvath, J. (2014). Printers in the Classroom. In *Mastering 3D Printing* (pp. 151-164). Apress, Berkeley, CA. https://doi.org/10.1007/978-1-4842-5842-2_13

[7]. Dilling, F., & Witzke, I. (2020). The use of 3D-printing technology in calculus education: Concept formation processes of the concept of derivative with printed graphs of functions. *Digital Experiences in Mathematics Education*, *6*(3), 320-339. https://doi.org/10.1007/s40751-020-00062-8

[8]. Ng, O. L. (2017). Exploring the use of 3D computer-aided design and 3D printing for STEAM learning in mathematics. *Digital Experiences in Mathematics Education*, *3*(3), 257-263.

[9]. Dilling, F. (2019). *Der Einsatz der 3D-Druck-Technologie im Mathematikunterricht*. Springer Fachmedien Wiesbaden.

[10]. Fatma, N., Haleem, A., Javaid, M., & Khan, S. (2021). Comparison of fused deposition modeling and color jet 3D printing technologies for the printing of mathematical geometries. *Journal of Industrial Integration and Management*, *6*(01), 93-105. https://doi.org/10.1142/S2424862220500104

[11]. Presmeg, N. (2020). Visualization and learning in mathematics education. *Encyclopedia of mathematics education*, 900-904. https://doi.org/10.1007/978-3-030-15789-0_161

[12]. Sarcar, M. M. M., Rao, K. M., & Narayan, K. L. (2008). *Computer aided design and manufacturing*. PHI Learning Pvt. Ltd..

[13]. Mahindru, D. V., Priyanka Mahendru, S. R. M. G. P. C., & Tewari Ganj, L. (2013). Review of rapid prototyping-technology for the future. *Global journal of computer science and technology*.

[14]. Wong, K. V., & Hernandez, A. (2012). A review of additive manufacturing. *International scholarly research notices*, *2012*.