Lecture 17

I want to correct/redo something from last time.

$k, N \geq 1$ integers.

$$\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$$

$f \in S_k(\Gamma_0(N), \chi)$ i.e. $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$,

$$f(\gamma(z)) = \chi(d)\underbrace{(cz+d)^k}_{j(\gamma,z)^k} f(z)$$

Define $\omega_\chi = \omega : \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$ as follows.

$$\mathbb{A}_\mathbb{Q}^\times = \mathbb{Q}^\times \times \mathbb{R}_{>0} \times \hat{\mathbb{Z}}^\times$$

So

$$\begin{tikzcd} \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times \arrow[r, two heads] \arrow[rr, bend right, "\mathrm{pr}"'] & \hat{\mathbb{Z}}^\times \arrow[r, two heads] & (\mathbb{Z}/N\mathbb{Z})^\times \end{tikzcd}$$

Set $\omega_\chi = \chi^{-1} \circ \mathrm{pr}$

By def $\omega = \omega_\chi$ is trivial on $\mathbb{R}_{>0}$ and we can write it

$$\omega = \prod_{p \neq \infty} \omega_p \text{ with } \omega_p : \mathbb{Z}_p^\times \to \mathbb{C}^\times, \ \omega_p = \omega|_{\mathbb{Z}_p^\times}$$

In part if $p \nmid N$, then $\omega_p = 1$. Say $p \nmid N$, and consider

$$p_p = (1, \dots, 1, \underset{\uparrow p^{th} \text{ place}}{p}, 1, \dots)$$

$$= p\, p_\infty^{-1} (p^{(p\infty)})^{-1}$$

with $p = (p, p, \dots) \in \mathbb{Q}^\times$, $p_\infty = (p, 1, \dots) \in \mathbb{R}_{>0}$

$$p^{(p\infty)} = (\underset{\uparrow \infty \text{ place}}{1}, p, \dots, \underset{\uparrow p^{th} \text{ place}}{p, 1}, p, \dots) \in \hat{\mathbb{Z}}^\times$$

$$\omega(p_p) = \omega(p\, p_\infty^{-1} (p^{(p\infty)})^{-1})$$

$$= \omega(p^{(p\infty)})^{-1}$$

$$= \prod_{q \mid N} \omega_q(p)^{-1} \qquad \omega_q : \mathbb{Z}_q^\times \to \mathbb{C}^\times$$

$$\begin{tikzcd} \mathbb{Z}_q^\times \arrow[r] \arrow[d] & \mathbb{C}^\times \\ (\mathbb{Z}/N\mathbb{Z})^\times \arrow[ur, "\chi^{-1}"'] & \end{tikzcd} \qquad N = \prod_{q \mid N} q^{e_q}$$

$$= \chi(p)$$

$$\omega_q = \mathbb{Z}_p^\times \to (\mathbb{Z}/q^{e_q}\mathbb{Z})^\times \hookrightarrow (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\chi^{-1}} \mathbb{C}^\times$$

Now given $g \in GL_2(\mathbb{A}_\mathbb{Q})$, writing

$$g = \gamma h u \quad \text{with } \gamma \in GL_2(\mathbb{Q})$$
$$h \in GL_2(\mathbb{R})^+$$
$$u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_0(N)$$

We define

$$\phi_{f,s} = (\det h)^s j(h, i)^{-k} \omega_\chi(d) f(h(i)).$$

Computation from last times shows its well defined and for $z \in \mathbb{A}_\mathbb{Q}^\times$, $\phi_{f,s}(zg) = |z|^{2s-k} \omega_\chi(z) \phi_{f,s}(g)$.

As last time, taking $s = \frac{k}{2}$ gives a unitary central character.

Let's now take $s = 1$ (not $s = k-1$, as I said last time). This gives central character

$$z \mapsto |z|^{2-k} \omega_\chi$$

Note we can act on $\phi_f = \phi_{f,1}$ $GL_2(\mathbb{A}_\mathbb{Q})$ by

$$(g\phi_f)(x) = \phi_f(xg).$$

Take $p \nmid N$, let $K_p = GL_2(\mathbb{Z}_p)$. Note ϕ_f is invariant under K_p, so we can define a double coset operator

$$T_p = \left[K_p \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}_p K_p\right]$$

$$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}_p = (1, \dots, \underset{\uparrow p^{th}}{\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}}, 1, \dots)$$

Apply to ϕ_f by writing

$$\left[K_p \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}_p K_p\right] = \bigcup_{b=0}^{p-1} \begin{pmatrix} p & -b \\ 0 & 1 \end{pmatrix}_p K_p \cup \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}_p K_p$$

Write $g = \gamma h u$ as above. Then

$$(T_p \phi_f)(g) = \sum_{b=0}^{p-1} \phi_f\left(g \begin{pmatrix} p & -b \\ 0 & 1 \end{pmatrix}_p\right) + \phi_f\left(g \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}_p\right)$$

$$hu \begin{pmatrix} p & -b \\ 0 & 1 \end{pmatrix}_p = \gamma' \begin{pmatrix} p^{-1} & p^{-1}b \\ 0 & 1 \end{pmatrix}_\infty h u'$$

where $\gamma' = \begin{pmatrix} p & -b \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{Q})$

$$u' = \left(1, \begin{pmatrix} p^{-1} & p^{-1}b \\ 0 & 1 \end{pmatrix}, \dots, \underset{\uparrow p^{th}}{1}, \begin{pmatrix} p^{-1} & p^{-1}b \\ 0 & 1 \end{pmatrix}, \dots\right)$$

So setting $z = h(i)$,

$$\phi_f\left(g \begin{pmatrix} p & -b \\ 0 & 1 \end{pmatrix}_p\right) = p^{-1} (\det h)\, j(h, i)^{-k} \omega_\chi(u)\, f\left(\frac{z+b}{p}\right)$$

$hu\begin{pmatrix}1&0\\0&p\end{pmatrix}_p = \gamma''\begin{pmatrix}1&0\\0&p^{-1}\end{pmatrix}_\infty huu''$

$\gamma' = \begin{pmatrix}1&0\\0&p\end{pmatrix} \in GL_2(\mathbb{Q})$

$u'' = \left(1, \begin{pmatrix}1&0\\0&p^{-1}\end{pmatrix}, \dots, \underset{\uparrow p^{th}}{1}, \begin{pmatrix}1&0\\0&p^{-1}\end{pmatrix}, \dots\right)$

So $\phi_p\left(g\begin{pmatrix}1&0\\0&p\end{pmatrix}_p\right) = p^{-1}(\det h)\, p^k\, j(h,i)^{-k}\, \omega_\chi(u)\, \chi(p)\, f(pz)$

$\Rightarrow (T_p\phi_f)(g) = (\det h)\, j(h,i)^{-k}\, \omega_\chi(u) \left[p^{-1}\sum_{b=0}^{p-1} f\left(\frac{z+b}{p}\right) + p^{k-1}\chi(p) f(pz)\right]$

$= (\det h)\, j(h,i)^{-k}\, \omega_\chi(u)\, (T_p f)(h(i))$

$= \phi_{T_p f}(g)$

Rmk If we used the unitary normalization, we would have $T_p\phi_p = p^{1-\frac{k}{2}}\phi_{T_p f}$

Note the Hecke action, for $p \nmid N$, is encoded in the $GL_2(\mathbb{Q}_p)$-action on ϕ_p. Now for $p \mid N$, the $GL_2(\mathbb{Q}_p)$-action encodes much more that U_p.

Back to generalities.

G = connected reductive group over a # fld F.

$G(\mathbb{A}_F) = G(\mathbb{A}_{F,\infty}) \times G(\mathbb{A}_F^\infty)$

Set $G_\infty := G(\mathbb{A}_{F,\infty}) = G(F \otimes_{\mathbb{Q}} \mathbb{R})$

Choose a max compact subgroup K_∞ of G_∞.

$\mathfrak{g} = \mathrm{Lie}\, G_\infty$

$Z(\mathfrak{g})$ = centre of $U(\mathfrak{g}_{\mathbb{C}})$

Choose an open compact subgroup K^∞ of $G(\mathbb{A}_F^\infty)$, and set $K = K_\infty \times K^\infty$.

Choose a closed embedding $G \overset{\sigma}{\hookrightarrow} GL_n$ and define a norm on $G(\mathbb{A}_F)$ by

$$\|g\| := \sup_{v \text{ place of } F} \max_{1\le i,j\le n} \{ |\sigma(g)_{i,j}|_v, |\sigma(g^{-1})_{i,j}|_v \}$$

Def An automorphic form on $G(\mathbb{A}_F)$ is a function $f: G(\mathbb{A}_F) \to \mathbb{C}$ satisfying the following.

1. f is smooth, i.e. as a function of $x \in G_\infty$ and $y \in G(\mathbb{A}_F^\infty)$, f is C^∞ wrt x and locally constant in y.
2. $f(\gamma g) = f(g)$ $\forall \gamma \in G(F)$, $g \in G(\mathbb{A}_F)$.
3. f is K-finite, i.e. the K-translates $g \mapsto f(gk)$, $k \in K$, span a fin dim space. $(K = K_\infty \times K^\infty)$
4. f is $Z(\mathfrak{g})$ finite, i.e. the $Z(\mathfrak{g})$ translates span a fin dim space.
5. f is slowly increasing, i.e. $\exists\, C > 0$, $n \in \mathbb{Z}_{\ge 0}$, such that $|f(g)| < C\|g\|^n$ $\forall g \in G(\mathbb{A}_F)$.

We further say f is cuspidal if

6. For every proper parabolic $P \subsetneq G$, N = unipotent radical of P, and every $g \in G(\mathbb{A}_F)$,
$$\int_{N(F)\backslash N(\mathbb{A}_F)} f(ng)\,dn = 0$$

We denote by $\mathcal{A}(G(F)\backslash G(\mathbb{A}_F))$, resp. $\mathcal{A}^\circ(G(F)\backslash G(\mathbb{A}_F))$ the space of automorphic forms, resp. cuspidal automorphic forms, on $G(\mathbb{A}_F)$

These spaces are $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty)$-modules.

We abuse terminology and call this a $G(\mathbb{A}_F)$-rep, but note G_∞ doesn't act because preserve K_∞-finiteness for our fixed choice of K_∞.

Eg If $f \in S_k(\Gamma_0(N), \chi)$ as before, $\phi_{f,s}$ is a cuspidal automorphic form.

<u>Exercise</u> $G = GL_2$, up to conj the only proper parabolic is $B = \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\} \supsetneq N = \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\}$. So ϕ_f being cuspidal means

$$\int_{\mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}}} \phi_f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx = 0 \quad \forall g \in GL_2(\mathbb{A}_{\mathbb{Q}}).$$

Show this using the fact that f is a cusp form.

A (<u>cuspidal</u>) <u>automorphic representation</u> of $G(\mathbb{A}_F)$ is an irreducible subrepresentation of $\mathcal{A}(G(F)\backslash G(\mathbb{A}_F))$ (resp. $\mathcal{A}^{\circ}(G(F)\backslash G(\mathbb{A}_F))$).