Last time: $k = \mathbb{R}$ or $\mathbb{C}$, $G = GL_2(k)$, $\chi_1, \chi_2 : k^\times \to \mathbb{C}^\times$ cts

$H(\chi_1,\chi_2)$ = completion of
$$\left\{ f: G \xrightarrow{\text{cts}} \mathbb{C} \;\middle|\; f\left(\begin{pmatrix} a & b \\ 0 & d\end{pmatrix} g\right) = \chi_1(a)\chi_2(d)\left|\frac{a}{d}\right|^{\frac12} f(g)\right\}$$
wrt $\|f\|^2 = \int_{K = O(2) \text{ or } U(2)} |f(k)|^2 dk$

This is a Hilb space rep of G and is unitary if χ_1, χ_2 are unitary.

This construction is more general. Say $\mathbb{G}$ is a connected reductive group over $k = \mathbb{R}$ or $\mathbb{C}$, $\mathbb{P} \leq \mathbb{G}$ a parabolic subgroup (defined over k), $G = \mathbb{G}(k)$, $P = \mathbb{P}(k)$.

If (σ, V) a Hilbert space rep of P on V, we let $H(P,\sigma)$ be the completion of the space
$$\{ f: G \xrightarrow{\text{cts}} V \mid f(px) = \sigma(p)\,\delta_p(p) f(x) \ \forall p \in P, x \in G\}$$
wrt $\|f\|^2 = \int_K \|f(k)\|^2 dk$

with G-action
$$(\pi(g)f)(x) = f(xg).$$

__Exercise__ Check that (σ, V) unitary $\Rightarrow$ $(\pi, H(P,\sigma))$ is unitary

__Eg__ Let $f \in S_k(\Gamma)$ be a cuspidal modular form of wt $k \geq 1$ and (congruence) level Γ.

Note $G := SL_2(\mathbb{R})$ acts transitively on $\mathbb{H} = \{x+iy \mid y > 0\}$ by $\begin{pmatrix} a & b \\ c & d\end{pmatrix} z = \frac{az+b}{cz+d}$

And the stabilizer of i is $SO(2) = \left\{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\right\}$.

So $SL_2(\mathbb{R})/SO(2) \xrightarrow{\sim} \mathbb{H}$
$$g \longmapsto g(i)$$

So we can define $\phi_f : SL_2(\mathbb{R}) \to \mathbb{C}$ by
$$\phi_f(g) = f(g(i))\, j(g,i)^{-k}$$
$$j\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}, z\right) = cz+d.$$

For $\gamma \in \Gamma$, $g \in G$,
$$\begin{aligned}\phi_f(\gamma g) &= f(\gamma g(i))\, j(\gamma g, i)^{-k} \\ &= j(\gamma, g(i))^k f(g(i))\, j(\gamma, g(i))^{-k} j(g,i)^{-k} \\ &= \phi_f(g)\end{aligned}$$

So $\phi_f : \Gamma\backslash G \to \mathbb{C}$. Haar measure on G induces a unique measure on $\Gamma\backslash G$ s.t.
$$\int_G \phi\, dg = \sum_{\gamma \in \Gamma} \int_{\Gamma\backslash G} \phi(\gamma g)\, dg$$

$SL_2(\mathbb{R}) = BK = NAK$
$$g = \begin{pmatrix} y^{\frac12} & xy^{-\frac12} \\ & y^{-\frac12}\end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix} \qquad y > 0$$
$$\begin{aligned} dg &= d_e b\, dk \\ &= \left(\frac{1}{y^2} dx\,dy\right)\left(\frac{1}{2\pi} d\theta\right)\end{aligned}$$

Also for $k = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix} \in SO(2)$, $g \in G$
$$\begin{aligned}\phi_f(gk) &= f(gk(i))\, j(gk, i)^{-k} \\ &= f(g(i))\, j(g,i)^{-k} j(k,i)^{-k} \\ &= \phi_f(g)\, e^{-ki\theta}\end{aligned}$$

So
$$\begin{aligned}\|\phi_f\|_{\Gamma\backslash G} &= \int_{\Gamma\backslash G} |\phi_f(g)|^2 dg \\ &= \int_{\Gamma\backslash BK} |\phi_f(bk)|^2 d_e b\, dk \\ &= \int_{\Gamma\backslash BK/K} |\phi_f(b)|^2 d_e b \\ &= \int_{\Gamma\backslash \mathbb{H}} |f(x+iy)|^2 y^{k-2} dx\,dy \\ &= \|f\| = \text{The Petersson norm of } f \\ &< \infty \text{ since } f \text{ is cuspidal.}\end{aligned}$$

$\Rightarrow \phi_f \in L^2(\Gamma\backslash G)$ and G acts by $(g\phi_f)(x) = \phi_f(xg)$ and we can consider the Hilbert space rep generated by it in $L^2(\Gamma\backslash G)$.

Exercise Do this for other examples of "modular forms" such as Hilbert modular forms, Siegel modular forms, etc.

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Say $\mathbb{G}$ is connected reductive / $k = \mathbb{R}$ or $\mathbb{C}$,

$G = \mathbb{G}(k)$ (or $\mathbb{G}(k)^\circ$). $K \leq G$ max compact sub.

(π, H) is a Hilbert space rep of G on H.

Say K acts by unitary operators.

Peter-Weyl $\Rightarrow$ $H|_K \cong \widehat{\bigoplus_{\tau \in \hat{K}}} V_\tau^{n_\tau}$ where

- $\hat{K}$ = set of iso classes of unitary irred reps of K
- $n_\tau \geq 0$ or $n_\tau = \infty$.

Def H is <u>admissible</u> if each τ occurs with finite multiplicity in $(\pi|_K, H|_K)$. We say $v \in H$ is <u>K-finite</u> if the K-translates of v span a fin dim vect space.

Eg 1. $\phi_f \in L^2(\Gamma \backslash SL_2(\mathbb{R}))$ above is K-finite, $f \in S_k(\Gamma)$.

2. K-finite vectors in $H(P, \sigma)$ above are
$\{ f: G \xrightarrow{cts} V \mid f(pg) = \sigma(p)\delta_p(p) f(g) \text{ and } f \text{ is } K\text{-finite} \}$

Thm (Harish-Chandra) 1. If (π, H) is irreducible and unitary, then it is admissible.

2. If (π, H) is admissible, then the K-finite vectors are C^∞, i.e. if $S \subseteq \hat{K}$ is a finite subset and $v \in \bigoplus_{\tau \in S} H_\tau$, then

$$G \longrightarrow \bigoplus_{\tau \in S} H_\tau \quad \text{is real analytic.}$$
$$g \longmapsto \mathrm{proj}_{\bigoplus_{\tau \in S} H_\tau}(gv)$$

~~If (π, H) is~~

Equivalently for any K-finite $v, w \in H$,

$$G \to \mathbb{C}$$
$$g \mapsto \langle \pi(g)v, w \rangle$$

is C^∞.

<u>Sketch of 2</u> Let f be a K-finite function of K, smooth.

Recall we have the Cartan decomposition, a diffeomorphism

$$K \times \mathfrak{p} \longrightarrow G$$
$$(k, X) \longmapsto k e^X$$

Take $h \in C^\infty_{comp}(e^{\mathfrak{p}}) \subseteq G$. Put

$$F(k e^X) = f(k) h(e^X)$$

Then $F \in C^\infty_{comp}(G)$ and we have an operator on H

$$\pi(F) v = \int_G F(g) \pi(g) v \, dg \in H$$

Can check that

$$\pi(F) v \text{ is } C^\infty.$$

If v is K-finite, then $\pi(F)v$ is also K-finite since

$$\pi(k)\pi(F) v = \int_G F(g) \pi(kg) v \, dg$$
$$= \int_G F(k^{-1} g) \pi(g) v \, dg$$

and F is K-finite on the left.

An approx argument $\Rightarrow$ C^∞ K-finite vectors are dense in the space of all K-finite vectors.

Since spaces for various K-types (i.e. $\tau \in \hat{K}$) are orthogonal, C^∞ vectors for each K-type τ are dense in the space H_τ.

Admissibility $\Rightarrow$ H_τ is fin dim, hence every K-fin vector is C^∞.