Lecture 10 – Taylor–Wiles primes I

Fix again a global def problem

$$\mathcal{S} = (\bar\rho, S, \psi, \mathcal{O}, \{D_v\}_{v\in S})$$

where $\bar\rho : G_{F,S} \to GL_2(\mathbb{F})$ is rank 2

Def A Taylor–Wiles prime (for $\mathcal{S}$) is a prime v of F, $v \notin S$ such that
1. $q_v := Nm(v) \equiv 1 \mod p$
2. $\bar\rho(\mathrm{Frob}_v)$ has distinct $\mathbb{F}$-rational eigenvalues.

We say a Taylor–Wiles prime v has level N, $N \geq 1$, if further
1'. $q_v \equiv 1 \mod p^N$

Rmk • Can and do assume $\mathbb{F}$ is large enough so that all eigenvalues of all elements in $\bar\rho(G_{F,S})$ are defined $/\mathbb{F}$.

• In higher rank, the generalization of 2 varies depending on the context

Prop Let v be a Taylor–Wiles prime (for $\mathcal{S}$). For any $A \in \mathrm{CNL}_{\mathcal{O}}$ and any lift $\rho : G_{F_v} \to GL_2(A)$ of $\bar\rho|_{G_{F_v}}$, ρ is conjugate to a diagonal lift

$$\begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix}.$$

Proof Can reduce to the case where A is Artinian.
Fix $\Phi \in G_{F_v}$ a lift of Frob_v. Since $\bar\rho(\mathrm{Frob}_v)$ has distinct $\mathbb{F}$-rat eigvals, can find a basis for ρ s.t.

$$\rho(\Phi) = \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}$$

Since $\bar\rho(I_{F_v}) = 1$, $\rho(I_{F_v}) \subseteq 1 + M_2(m_A)$, so is pro-$p$
so $\rho|_{I_{F_v}}$ factors through tame inertia.
Fix a top gen t for tame inertia. It suffices to prove that in our fixed basis $\rho(t)$ is diagonal.

We induct on length(A). Can assume

$$\rho(t) = 1 + X \in 1 + M_n(m_A) \text{ with } X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad b, c \in m_A^n$$

and $m_A^{n+1} = 0$. Easy check shows that X^k is diagonal if $k \geq 2$.
We know that $\Phi^{-1} t \Phi = t^{q_v}$

$$\Rightarrow 0 = \rho(\Phi^{-1})\rho(t)\rho(\Phi) - \rho(t)^{q_v}$$
$$= 1 + \begin{pmatrix} a & \alpha^{-1}\beta b \\ \alpha\beta^{-1}c & d \end{pmatrix} - 1 + q_v\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \text{diagonal}$$
$$= \begin{pmatrix} 0 & (\alpha^{-1}\beta - 1)b \\ (\alpha\beta^{-1}-1)c & 0 \end{pmatrix} + \text{diagonal}', \text{ since } (q_v-1)b = (q_v-1)c = 0$$

But $\alpha^{-1}\beta - 1$ and $\alpha\beta^{-1} - 1$ are units in A, since
$\alpha \bmod m_A$, $\beta \bmod m_A$
are the distinct eigenvalues of $\bar\rho$.
$\Rightarrow b = c = 0$. ∎

Say v is a Taylor-Wiles prime for S.
Let $R_v^{\square,\chi}$ be the universal lifting ring for $\bar\rho|_{G_{F_v}}$ with fixed det χ, and let ρ^χ be the universal lift.

By the prop, ρ^χ is conj to $\begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix}$, $\chi_i : G_{F_v} \to (R_v^{\square,\chi})^\times$ and $\chi_1\chi_2 = \chi$.
In particular, since χ is unramified at v,
$$\chi_1|_{I_{F_v}} = \chi_2|_{I_{F_v}}^{-1}$$

Since $\bar\rho$ is unramified, $\chi_1|_{I_{F_v}}$ is a pro-p character of
$$I_{F_v^{ab}/F_v} \cong k_v^\times \times \mathbb{Z}_q^d \times (\text{fin } q\text{-group})$$
where q = res char of v, k_v = res fld of F at v.
Let Δ_v = max p-power quotient of $k_v^\times$,
$\mathcal{O}[\Delta_v]$ = group alg
$\mathfrak{a}_v$ = aug ideal.

$\chi_1|_{I_{F_v}}$ determines an $\mathcal{O}[\Delta_v]$-alg structure on $R_v^{\square,\chi}$

Moreover, note $\exists$ a natural surjection
$$R_v^{\square,\chi} \to R_v^{ur,\chi} = \text{universal lifting ring for } \bar\rho|_{G_{F_v}} \text{ of lifts } \rho \text{ s.t. } \rho(I_{F_v}) = 1 \text{ and } \det\rho = \chi$$
and its kernel is
$$\mathfrak{a}_v R_v^{\square,\chi}$$
since any unramified det $=\chi$ lift to A determines a map
$$\phi : R^{\square,\chi} \to A \quad \text{s.t. } \phi(\mathfrak{a}_v) = 0$$

$$\Rightarrow R_v^{\square,\psi}/\mathfrak{a}_v R_v^{\square,\psi} \twoheadrightarrow R_v^{w,\psi}$$

and conversely the universal unramified $R_v^{\square,\psi}/\mathfrak{a}_v R_v^{\square,\psi}$-valued lift is unramified.

$$\Rightarrow R_v^{\square,\psi} \to R_v^{\square,\psi}/\mathfrak{a}_v R_v^{\square,\psi}$$

factors through $R_v^{w,\psi}$.

Hence

$$R_v^{\square,\psi}/\mathfrak{a}_v R_v^{\square,\psi} \cong R_v^{w,\psi}.$$

Then, say Q is a finite set of Taylor-Wiles primes. Let $\Delta_Q = \prod_{v\in Q} \Delta_v$, $\mathcal{O}[\Delta_Q]$ and aug ideal $\mathfrak{a}_Q$.

We define the global def problem

$$\mathcal{S}_Q = (\bar{\rho}, S\cup Q, \psi, \mathcal{O}, \{D_v\}_{v\in S} \cup \{D_v^{\psi}\}_{v\in Q})$$

where for $v\in Q$, D_v^{ψ} is the def condition of all lifts of $\bar\rho|_{G_{F_v}}$ with det $= \psi|_{G_{F_v}}$.

Then, assuming $\mathrm{End}_{\mathbb{F}[G_{F,S}]}(\bar\rho) = \mathbb{F}$, we have

$$R_{\mathcal{S}_Q} \text{ and } R_{\mathcal{S}}$$

and also $R_{\mathcal{S}_Q}^{T}$ and $R_{\mathcal{S}}^{T}$ for any $T\subseteq S$.

$R_{\mathcal{S}_Q}^{T}$ has the structure of an $\mathcal{O}[\Delta_Q]$-alg, and the natural surjection

$R_{\mathcal{S}_Q}^{T} \to R_{\mathcal{S}}^{T}$ has kernel $\mathfrak{a}_Q R_{\mathcal{S}_Q}^{T}$.

(relations to $R_S^{T,\mathrm{loc}}$)

Recall for any (possibly empty) $T \subseteq S$, the tangent space of R_S^T is given by a cohom group

$$H^1_{S,T}(\mathrm{ad}^0 \bar{\rho})$$

and its dimension is

$$h^1_{S,T}(\mathrm{ad}^0\bar{\rho}) = h^1_{S^\perp,T}(\mathrm{ad}^0\bar{\rho}(1)) + \sum_{v\in S\setminus T}\left(\dim_{\mathbb{F}} L_v - h^0(F_v,\mathrm{ad}^0\bar{\rho})\right)$$
$$- \sum_{v\mid\infty} h^0(F_v,\mathrm{ad}^0\bar{\rho}) - h^0(F_S/F,\mathrm{ad}^0\bar{\rho}(1)) + \begin{cases} |T|-1 & \text{if } T\neq\emptyset \\ 0 & \text{if } T=\emptyset\end{cases}$$

where
- $h^1_{S^\perp,T}(\mathrm{ad}^0\bar{\rho}(1)) := \dim\ker\left(H^1(F_S/F,\mathrm{ad}^0\bar{\rho}(1)) \to \prod_{v\in S\setminus T} H^1(F_v,\mathrm{ad}^0\bar{\rho}(1))/L_v^\perp\right)$

- $L_v \subseteq H^1(F_v,\mathrm{ad}^0\bar{\rho})$ that is image of $D_v(\mathbb{F}[\varepsilon]) \cong \mathcal{L}_v \subseteq Z^1(F_v,\mathrm{ad}^0\bar{\rho})$
  $L_v^\perp \subseteq H^1(F_v,\mathrm{ad}^0\bar{\rho}(1))$ is the orthog complement of L_v under Tate duality.

Now assume that the following hold

1. $\bar{\rho}|_{G_{F(\zeta_p)}}$ is abs irred $\Rightarrow$ no nonscalar $G_{F,S}$-equiv homs $\bar{\rho}\to\bar{\rho}(1)$ $\Rightarrow H^0(F_S/F,\mathrm{ad}^0\bar{\rho}(1)) = 0$

2. F is totally real and $\det\bar{\rho}(c_v) = -1$ for all $v\mid\infty$ in F and c_v = complex conj at v.
   $\Rightarrow h^0(F_v,\mathrm{ad}^0\bar{\rho}) = 1$

3. $\forall\, v \mid p,\ v \in T,\ \dim_{\mathbb{F}} L_v - h^0(F_v, \mathrm{ad}^0 \bar{\rho}) = [F_v : \mathbb{Q}_p]$

<u>Eg</u> This is true if $\bar{\rho}|_{G_{F_v}} \cong \begin{pmatrix} \bar{\chi}_1 & * \\ 0 & \bar{\chi}_2 \end{pmatrix}$ with $\bar{\chi}_1|_{I_{F_v}} = 1$

and $\bar{\chi}_2|_{I_{F_v}} \neq 1$ and $D_v = D_v^{\mathrm{ord}, \psi}$ is the D_v^{ord} from

Lectures 6 and 7 + fixed det ψ.

4. $\forall\, v \in S \setminus \{v \mid p\},\ v \in T,\ \dim_{\mathbb{F}} L_v - h^0(F_v, \mathrm{ad}^0 \bar{\rho}) = 0$

<u>Eg</u> This is true if

$$\bar{\rho}|_{I_v} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \neq 1 \quad \text{or} \quad \bar{\rho}|_{G_{F_v}} = \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix} \quad \begin{matrix} \text{with } \chi_1(I_{F_v}) = 1 \\ \chi_2(I_{F_v}) \neq 1 \end{matrix}$$

and D_v = min def problem + fixed det from Lectures 6+7.

Under these assumptions

$$h^1_{S,T}(\mathrm{ad}^0 \bar{\rho}) = h^1_{S^\perp, T}(\mathrm{ad}^0 \bar{\rho}(1)) + \begin{cases} |T| - 1 & \text{if } T \neq \emptyset \\ 0 & \text{if } T = \emptyset. \end{cases}$$