Lecture 24

Notation as before (G,T) split red $/F$

$G \supseteq B \supseteq T$, $K = G(\mathcal{O}_F)$, W = Weyl group
(B Borel)

$$\mathcal{S}: \mathcal{H}(G(F),K) \xrightarrow{\sim} \mathcal{H}(T(F),T(\mathcal{O}_F))^W$$

$$f \longmapsto \left(t \mapsto \delta^{\frac{1}{2}}(t)\int_{N(F)} f(tn)\,dn = \delta^{-\frac{1}{2}}(t)\int_{N(F)} f(nt)\,dn\right)$$

Sketch 1. Alg map
2. $\mathrm{Im} \subseteq$ Weyl inv
3. Bijection

} 2 approaches, one in notes using integral formula, another using an adjunction later today.

$\mathcal{S}$ is bijective: Recall we have a basis $\{c_\lambda\}_{\lambda \in X^{\vee,+}}$

$$c_\lambda = K\lambda(\varpi)K$$

for $\mathcal{H}(G(F),K)$. Note any element of $X^\vee$ is W-conj to an element of $X^{\vee,+}$, so setting

$$d_\lambda = \frac{1}{\#\mathrm{Stab}_W(\lambda)} \sum_{w \in W} \mathbb{1}_{T(\mathcal{O}_F)w\lambda(\varpi)T(\mathcal{O}_F)}, \quad \lambda \in X^{\vee,+}$$

is a basis for $\mathcal{H}(T(F),T(\mathcal{O}_F))^W$

For $\lambda \in X^{\vee,+}$, write

$$\mathcal{S}c_\lambda = \sum_{\mu \in X^{\vee,+}} a(\mu,\lambda)d_\mu$$

Fix $\mu \in X^{\vee,+}$ and set $t = \mu(\varpi)$, $s = \lambda(\varpi)$, $c_\lambda = \mathbb{1}_{KsK}$

Then

$$\begin{aligned} a(\mu,\lambda) &= \mathcal{S}c_\lambda(t) \\ &= \delta^{\frac{1}{2}}(t)\int_{N(F)} c_\lambda(tn)\,dn \\ &= \delta^{\frac{1}{2}}(t)\,\mathrm{meas}(N(F) \cap t^{-1}KsK) \end{aligned}$$

If $t = s$, then

$$N(F) \cap t^{-1}KtK \supseteq t^{-1}N(\mathcal{O}_F)t$$

is open in $N(F)$, so $a(\lambda,\lambda) \neq 0$.

Bruhat-Tits $\Rightarrow N(F) \cap t^{-1}KsK = \emptyset$ unless $\lambda - \mu$ is a lin comb of elements of $\Phi^{\vee,+}$ with nonneg coeffs.

Eg $G = GL_2$, $B = \left\{\begin{pmatrix} * & * \\ & * \end{pmatrix}\right\}$.

$$t = \begin{pmatrix} \varpi^{e_1} & \\ & \varpi^{e_2} \end{pmatrix} \qquad e_1 \geq e_2, \ \mu(x) = \begin{pmatrix} x^{e_1} & \\ & x^{e_2} \end{pmatrix}$$

$$s = \begin{pmatrix} \varpi^{f_1} & \\ & \varpi^{f_2} \end{pmatrix} \qquad f_1 \geq f_2 \quad \lambda(x) = \begin{pmatrix} x^{f_1} & \\ & x^{f_2} \end{pmatrix}$$

Assume $\exists\, n = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \in N(F)$ and $k_1,k_2 \in GL_2(\mathcal{O}_F)$ s.t. $tn = k_1 s k_2$. We want to show $\exists\, a \in \mathbb{Z}_{\geq 0}$ s.t. $\lambda - \mu = a\alpha$, $\alpha(x) = \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix}$, i.e. $(f_1,f_2) = (e_1+a, e_2-a)$.

The entries of

$$tn = \begin{pmatrix} \varpi^{e_1} & \\ & \varpi^{e_2} \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$$

have valuations $e_1, e_2, e_1 + \mathrm{val}(x)$.

The entries of $k_1\begin{pmatrix} \varpi^{f_1} & \\ & \varpi^{f_2} \end{pmatrix}k_2$ have valuations $\geq f_1$ or f_2, so

$$\min\{e_1, e_2, e_1+\mathrm{val}(x)\} = f_2 \Rightarrow e_2 \geq f_2$$

Comparing determinants

$$e_1 + e_2 = f_1 + f_2$$

So if $a = e_2 - f_2$, then

$$(f_1,f_2) = (e_1+a, e_2-a).$$

The relation $\lambda \geq \mu \Leftrightarrow \lambda - \mu$ is a nonnegative lin comb of elements of $\Phi^{\vee,+}$ is a partial order that can be extended to a total order with a suitable lexicographic ordering.

Then $a(\mu,\lambda) = 0$ unless $\lambda \geq \mu$ and $a(\lambda,\lambda) \neq 0$

$\Rightarrow \mathcal{S}$ is an iso.

Say we have an unramified character
$$\chi: T(F) \to \mathbb{C}^\times$$
From the Iwasawa decomp $G(F) = B(F)K$, we have a unique $\underbrace{\text{unramified}}_{K\text{-fixed}}$ line in $n\mathrm{Ind}_{B(F)}^{G(F)} \chi = n\mathrm{Ind}\,\chi$ given by
$$\phi_\chi(tnk) = \delta^{\frac{1}{2}}(t)\chi(t) \qquad t \in T(F), n \in N(F), k \in K$$
Take $f \in \mathcal{H}(G(F), K)$. Then the action of f on ϕ_χ is given by
$$\langle f, n\mathrm{Ind}\,\chi \rangle := \int_{G(F)} f(g)\pi_\chi(g)\phi_\chi(1)\,dg$$
$$= \int_{T(F)}\int_{N(F)}\int_K f(tnk)\phi_\chi(tnk)\,dk\,dn\,dt$$
$$= \int_{T(F)}\int_{N(F)} f(tn)\delta^{\frac{1}{2}}(t)\chi(t)\,dn\,dt$$
$$= \int_{T(F)} (Sf)(t)\chi(t)\,dt$$
$$=: \langle Sf, \chi \rangle$$
__Thm__ (Casselman) If (π, V) is an irred smooth adm rep of $G(F)$ and is unramified, then $\exists$ an unramified character $\chi: T(F) \to \mathbb{C}^\times$ s.t. π is iso to a subrep of $n\mathrm{Ind}\,\chi$.

__Eg__ $G = GL_2(F)$, consider
$$\mathbb{1}_{K\left(\begin{smallmatrix}\varpi & \\ & 1\end{smallmatrix}\right)K} \qquad \mathbb{1}_{K\left(\begin{smallmatrix}\varpi & \\ & \varpi\end{smallmatrix}\right)K}$$
Writing $K\left(\begin{smallmatrix}\varpi & \\ & 1\end{smallmatrix}\right)K = \left(\begin{smallmatrix}1 & \\ & \varpi\end{smallmatrix}\right)K \sqcup \bigsqcup_{x \bmod \varpi} \left(\begin{smallmatrix}\varpi & x \\ & 1\end{smallmatrix}\right)K$
$$\langle \mathbb{1}_{K\left(\begin{smallmatrix}\varpi & \\ & 1\end{smallmatrix}\right)K}, n\mathrm{Ind}\,\chi \rangle$$
$$= \int_{K\left(\begin{smallmatrix}\varpi & \\ & 1\end{smallmatrix}\right)K} \phi_\chi(g)\,dg = \sum_{\gamma \in K\left(\begin{smallmatrix}\varpi & \\ & 1\end{smallmatrix}\right)K/K} \int_K \phi_\chi(\gamma k)\,dk$$
$$= \sum_{\gamma \in K\left(\begin{smallmatrix}\varpi & \\ & 1\end{smallmatrix}\right)K/K} \phi_\chi(\gamma)$$
$$= (\delta^{\frac{1}{2}}\chi)\left(\begin{smallmatrix}1 & \\ & \varpi\end{smallmatrix}\right) + q(\delta^{\frac{1}{2}}\chi)\left(\begin{smallmatrix}\varpi & \\ & 1\end{smallmatrix}\right)$$
$q = \#\mathcal{O}_F/\varpi$. If $\chi = \chi_1 \times \chi_2$ and $\chi_i(\varpi) = \alpha_i \in \mathbb{C}^\times$, This equals
$$q^{\frac{1}{2}}(\alpha_1 + \alpha_2)$$
$$\langle \mathbb{1}_{K\left(\begin{smallmatrix}\varpi & \\ & \varpi\end{smallmatrix}\right)K}, n\mathrm{Ind}\,\chi \rangle = \alpha_1\alpha_2$$
$$\mathcal{H}(GL_2(F), GL_2(\mathcal{O}_F)) \to \mathcal{H}(\mathbb{G}_m(F)^2, \mathbb{G}_m(\mathcal{O}_F)^2)^W$$
$$\mathbb{1}_{K\left(\begin{smallmatrix}\varpi & \\ & 1\end{smallmatrix}\right)K} \mapsto q^{\frac{1}{2}}(t_1 + t_2) \in \mathbb{C}[t_1, t_2]^W \cong \mathbb{C}[t_1 + t_2, t_1t_2]$$
$$\mathbb{1}_{K\left(\begin{smallmatrix}\varpi & \\ & \varpi\end{smallmatrix}\right)K} \mapsto t_1t_2$$

Note for $w \in W$, $(w\chi)(t) = \chi(w^{-1}tw)$ is an unramified char.

__Prop__ $\exists$ a nonzero $G(F)$-equiv lin map
$$T_w: n\mathrm{Ind}\,\chi \to n\mathrm{Ind}(w\chi)$$
In part, $n\mathrm{Ind}\,\chi \cong n\mathrm{Ind}(w\chi)$ if they are irred.

__Sketch for GL_2__ $\chi = \chi_1 \times \chi_2$, $w = \begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}$, $w\chi = \chi_2 \times \chi_1$. Take $\phi \in n\mathrm{Ind}\,\chi_1 \times \chi_2$

Define
$$T_w(\phi)(g) = \int_F \phi\left(\begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & x \\ & 1\end{pmatrix}g\right)dx$$
For $n = \begin{pmatrix}1 & y \\ & 1\end{pmatrix}$,
$$T_w(\phi)(ng) = \int_F \phi\left(\begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & x \\ & 1\end{pmatrix}\begin{pmatrix}1 & y \\ & 1\end{pmatrix}g\right)dx$$
$$= T_w(\phi)(g) \qquad \text{by } x = y + x$$
For $t = \begin{pmatrix}a & \\ & d\end{pmatrix} \in T(F)$
$$T_w(\phi)(tg) = \int_F \phi\left(\begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & x \\ & 1\end{pmatrix}\begin{pmatrix}a & \\ & d\end{pmatrix}g\right)dx$$
$$= \int_F \phi\left(\begin{pmatrix}d & \\ & a\end{pmatrix}\begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & da^{-1}x \\ & 1\end{pmatrix}g\right)dx$$
$$= \int_F \left(\left|\frac{d}{a}\right|^{\frac{1}{2}}\chi_1(d)\chi_2(a)\right)\phi\left(\begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & x \\ & 1\end{pmatrix}g\right)\left|\frac{a}{d}\right|dx$$

$$= \left|\frac{a}{d}\right|^{\frac{1}{2}} \chi_2(a)\chi_1(d) \int_F \phi\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) dx$$

$$= \left|\frac{a}{d}\right|^{\frac{1}{2}} \chi_2(a)\chi_1(d)\, T_w(\phi)(g).$$

So $T_w(\phi) \in n\mathrm{Ind}(w\chi)$

To show $T_w \neq 0$, define

$$\phi : G(F) \to \mathbb{C}$$

by $\phi\left(\begin{pmatrix} a & y \\ & d \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) = \left|\frac{a}{d}\right|^{\frac{1}{2}} \chi_1(a)\chi_2(d)$

if $a, d \in F^\times$, $y \in F$, $x \in \mathcal{O}_F$, and

$\phi(g) = 0$ if g is not of this form.

Then $\phi \in n\mathrm{Ind}(\chi_1 \times \chi_2)$ and if $g = 1$,

$$T_w(\phi)(1) = \int_F \phi\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) dx = \int_{\mathcal{O}_F} 1\, dx = 1. \quad \square$$

<u>Rmk</u> Did not justify convergence of integral defining T_w. Can show it converges on some half plane (writing $\chi_i = (\text{unitary char})|\,|^s$) and then analytically continue.

<u>Rmk</u> It is completely understood when $n\mathrm{Ind}\,\chi$ are reducible / irred, relies on understanding T_w explicitly.

For ex 1. If χ is unitary and $w\chi \neq \chi$ $\forall w \neq 1$, then $n\mathrm{Ind}\,\chi$ is irred.

2. $G = GL_2(F)$, $\chi = \chi_1 \times \chi_2$, $n\mathrm{Ind}\,\chi$ is reducible $\iff \chi_1\chi_2 = |\,|^{\pm 1}$