Lecture 23

Let G be spit connected red group over a nonarch local field F, $G \supseteq T \cong \mathbb{G}_m^r$ a max torus.

$F \supseteq \mathcal{O}_F$ = ring of int $\ni \varpi$ = uniformizer, $k = \mathcal{O}_F/\varpi$.

Recall assoc to G we have a root datum

$$\Psi = (X, \Phi, X^\vee, \Phi^\vee)$$

$X = \mathrm{Hom}(T, \mathbb{G}_m)$ $X^\vee = \mathrm{Hom}(\mathbb{G}_m, T)$

Φ = set of roots, $\Phi^\vee$ = set of coroots.

Choose a Borel subgroup $B \supseteq T$.

$\Rightarrow$ Φ^+ = set of +ve roots in Φ.

In fact, $\exists$ a smooth group scheme $\mathcal{G}$ over $\mathcal{O}_F$, an isomorphism $\mathcal{G} \times_{\mathcal{O}_F} F \cong G$ and such that $\mathcal{G} \times_{\mathcal{O}_F} k$ is a split connected reductive group with same root datum.

(in fact $\mathcal{G}$ can be defined $/\mathbb{Z}$ such that all fibres are connected reductive, Chevalley group)

Eg GL_n is defined $/\mathbb{Z}$, same for SL_n, GSp_{2n}, Sp_{2n}...

Abusing notation, we will write $G(\mathcal{O}_F)$ for $\mathcal{G}(\mathcal{O}_F)$.

Set $K := \mathcal{G}(\mathcal{O}_F)$, called a hyperspecial maximal compact subgroup of $G(F)$.

We have the Cartan decomposition:

$$G(F) = \bigsqcup_{\lambda \in X^{\vee,+}} K\lambda(\varpi)K$$

where $X^{\vee,+} = \{\lambda \in X^\vee = \mathrm{Hom}(\mathbb{G}_m, T) \mid \langle \lambda, \alpha\rangle \geq 0 \ \forall \alpha \in \Phi^+\}$

Eg $G = GL_n$. Any compact subgroup U is conj to a subgroup of $GL_n(\mathcal{O}_F)$. Consider

$$U\mathcal{O}_F^n \subseteq F^n$$

Since U is compact, $\exists\, u_1, \dots, u_m \in U$ such that

$$U\mathcal{O}_F^n = u_1\mathcal{O}_F^n + \dots + u_m\mathcal{O}_F^n =: L \text{ is a lattice in } F^n.$$

Choosing $g \in GL_n(F)$ s.t. $gL = \mathcal{O}_F^n$,

$$gUg^{-1} \subseteq GL_n(\mathcal{O}_F).$$

Cartan decomposition is known as elementary divisors, i.e. any matrix in $GL_n(F)$ can be written as

$$k_1 t k_2 \quad \text{with } k_1, k_2 \in GL_n(\mathcal{O}_F)$$

$$t = \begin{pmatrix} \varpi^{e_1} & & \\ & \varpi^{e_2} & \\ & & \ddots \\ & & & \varpi^{e_n}\end{pmatrix} \quad e_1 \geq e_2 \geq \dots \geq e_n$$

Eg/Exercise In $SL_2(F)$ there are 2 conj classes of max compact subgroups, namely $SL_2(\mathcal{O}_F)$ and $\begin{pmatrix} \varpi & \\ & 1\end{pmatrix} SL_2(\mathcal{O}_F) \begin{pmatrix} \varpi^{-1} & \\ & 1\end{pmatrix}$

Today we consider

$$\mathcal{H}(G(F), K)$$

with K hyperspecial as above. Called the spherical Hecke alg.

Cartan decomp $\Rightarrow$ has a basis

$$c_\lambda := \mathbb{1}_{K\lambda(\varpi)K} \quad \lambda \in X^{\vee,+}$$

Prop $\mathcal{H}(G(F), K)$ is commutative.

Proof for GL_n (Gelfand's trick)

Consider $g \mapsto {}^t g$ on $GL_n(F)$. This induces an aut of $\mathbb{C}$-vect spaces

$$\mathcal{H}(GL_n(F), GL_n(\mathcal{O}_F)) \circlearrowleft \sigma$$

by $f^\sigma(x) = f({}^t x)$.

$$\begin{aligned}
(f_1 * f_2)^\sigma(x) &= \int_G f_1({}^t x y^{-1}) f_2(y)\, dy \\
&= \int_G f_1^\sigma({}^t y^{-1} x) f_2^\sigma({}^t y)\, dy \\
&= \int_G f_1^\sigma(y^{-1} x) f_2^\sigma(y)\, dy \qquad y \mapsto {}^t y \\
&= \int_G f_1^\sigma(y) f_2^\sigma(xy^{-1})\, dy \qquad y \mapsto xy^{-1} \\
&= (f_2^\sigma * f_1^\sigma)(x)
\end{aligned}$$

So σ is an involution on $\mathcal{H}(GL_n(F), GL_n(\mathcal{O}_F))$.
On the other hand σ acts as id on the basis

$$GL_n(\mathcal{O}_F)\begin{pmatrix} \varpi^{e_1} & & \\ & \varpi^{e_2} & \\ & & \ddots \end{pmatrix}GL_n(\mathcal{O}_F)$$

$\Rightarrow \sigma = \mathrm{id}$ and $\mathcal{H}$ is commutative. $\square$

Cor Let π be an irred smooth admissible $G(F)$-rep. Then $\dim \pi^K \leq 1$.

Proof If $\pi^K \neq 0$, then it is a simple $\mathcal{H}(G,K)$-module by a result from last time. But $\mathcal{H}(G,K)$ commutative $\Rightarrow$ its simple modules are 1-dim. $\square$

Def An irred smooth admissible $G(F)$-rep π with $\pi^K \neq 0$ (hence of dim 1) is called <u>unramified</u>.

Eg Assume $\chi : T(F) \to \mathbb{C}^\times$ is a char with $T(\mathcal{O}_F)$ in its kernel (an <u>unramified character</u> of $T(F)$).
If $n\text{-}\mathrm{Ind}_{B(F)}^{G(F)} \chi$ is irred, then it is unramified.
Consider $\phi \in n\text{-}\mathrm{Ind}_{B(F)}^{G(F)} \chi$ given by

$$\phi(bk) = \delta_B^{\frac{1}{2}}(b)\chi(b) \quad b \in B(F),\ k \in K.$$

This is well-defined since

$$B(F) \cap K \subseteq T(\mathcal{O}_F)N(F) \subseteq \ker(\delta_B^{\frac{1}{2}}\chi).$$

We can say more: Consider

$$\mathcal{H}(T(F), T(\mathcal{O}_F)) = \mathbb{C}[T(F)/T(\mathcal{O}_F)]$$

by $\mathbb{1}_{\lambda(\varpi)T(\mathcal{O}_F)} \in \mathcal{H}(T(F), T(\mathcal{O}_F))$

and $\mathbb{C}[T(F)/T(\mathcal{O}_F)] \cong \mathbb{C}[x_1^{\pm}, \dots, x_r^{\pm}] \quad T \cong \mathbb{G}_m^r$

We define the <u>Satake transform</u>

$$S : \mathcal{H}(G(F), K) \to \mathcal{H}(T(F), T(\mathcal{O}_F))$$
$$f \mapsto \left(t \mapsto \delta_B(t)^{\frac{1}{2}} \int_{N(F)} f(tn)\,dn\right)$$

Thm (Satake) S induces an isomorphism

$$\mathcal{H}(G(F), K) \cong \mathcal{H}(T(F), T(\mathcal{O}_F))^W$$

W = Weyl group of $(G,T) = N_G(T)/T$