Modularity of some $PGL_2(\mathbb{F}_5)$-representations

(joint with Khare and Thorne)

k = finite field, p = char(k), $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

Theorem (Khare – Wintenberger)
If $\rho : G_{\mathbb{Q}} \to GL_2(k)$ absolutely irreducible and odd, then ρ is modular.

ρ is odd if $\det \rho(c) = -1$
$\iff \rho(c)$ is nontrivial in $PGL_2(k)$ $(p > 2)$

Proof proceeds by induction on p and the conductor $N(\bar{\rho})$ using for the base case(s):

Theorem (Tate-Serre)
There are no abs irred odd ρ with $p = 2, 3$ and $N(\rho) = 1$.

Let F = totally real

Conj An abs irred totally odd (odd at every complex conj) $\rho : G_F \to GL_2(k)$ comes from a Hilbert modular form.

Problem: Serre-Tate result no longer true.

Known Cases

(a) (Hecke) Proj image of ρ = dihedral group

(b) (Langlands-Tunnell, Wiles) $k = \mathbb{F}_3$
(uses that $GL_2(\mathbb{F}_3)$ is solvable)

(c) (Wiles, Shepherd-Barron-Taylor) $k = \mathbb{F}_5$ and
$\det \rho = \overline{\varepsilon}_5$ = mod 5 cyclotomic character

(d, e, f) $\det \rho = \overline{\varepsilon}_p$ and
$|k| = 4$ (Shepherd-Barron-Taylor)
$|k| = 7$ (Manoharmayum)
$|k| = 9$ (Ellenberg)

Sketch of (c) ($k = \mathbb{F}_5$, $\det \rho = \epsilon_5$)

$$X = \{(E, \phi : \rho \xrightarrow{\sim} \rho_{E,5} \curvearrowright E[5])\}^{cpt}/_{\cong}, \text{ twist of } X(5)$$
$$\cong \mathbb{P}^1_{/F}$$

$\Rightarrow \exists\ X(F) \ni x \rightsquigarrow E_{/F}$ s.t. $E[3] \curvearrowleft \rho_{E,3}$ satisfies assumptions of a modularity lifting theorem (MLT)

(b) $\Rightarrow$ $\rho_{E,3}$ is modular

MLT $\Rightarrow$ E is modular

$\Rightarrow$ $\rho_{E,5} \cong \rho$ is modular $\square$

Consider

$$\begin{tikzcd} & & GL_2(\bar{k}) \arrow[d] \\ \bar{\rho} : G_F \arrow[r] \arrow[urr, dashed, "\rho"] & PGL_2(k) \subseteq & PGL_2(\bar{k}) \end{tikzcd}$$

Tate $\Rightarrow \exists\, \rho$

ρ, ρ' two lifts of $\bar{\rho}$, then

ρ modular $\Leftrightarrow \rho'$ is modular

Question: Can we prove modularity of totally odd $\bar{\rho} : G_F \to PGL_2(k)$ for small k?

Theorem (A.–Khare–Thorne)
Let $\rho : G_F \to PGL_2(\mathbb{F}_5)$ be totally odd with nonsolvable image. Assume that $\Delta : G_F \to \{\pm 1\}$, $\Delta = \rho \bmod PSL_2(\mathbb{F}_5)$, is totally even or totally odd and does not cut out $F(\zeta_5)$. Then ρ is modular.

Idea $\exists$ solvable K/F that is totally real or CM (Δ totally even or odd) s.t. $\bar{\rho}|_{G_K}$ lifts to

$$\rho_K : G_K \longrightarrow GL_2(\mathbb{F}_5) \text{ with } \det \rho_K = \bar{\varepsilon}_5$$

SB-T or A-K-T $\Rightarrow$ ρ_K is modular

$\bar{\rho}$ modular by mod p solvable descent
(Tate + Khare–Wintenberger method over F) $\square$

Thank You!