Lecture 26

$F = \#$ fld.

<u>Generalized Ramanujan Conj</u> π a unitarizable cusp aut rep of $GL_n(\mathbb{A}_F)$, then π is tempered at all places.

<u>Known cases</u>

1. $n=2$, $F=\mathbb{Q}$, $\pi=\pi_f$ with $f\in S_k(\Gamma_1(N))$ a cuspidal eigenform.
2. $n\geq 2$, $F=$ totally real or CM, $F^+\subseteq F$ max tot real subfield, $\mathrm{Gal}(F/F^+)=\langle c\rangle$. If π_∞ is regular algebraic (Lecture 12) and
$$\pi^\vee\circ c\cong \pi\otimes\chi\circ\det \text{ for some smooth } \chi:(F^+)^\times\backslash\mathbb{A}_{F^+}^\times\to\mathbb{C}^\times.$$
3. $n=2$, $F=$ CM, π_∞ reg alg of wt 0.

<u>Not Known</u>

1. $n=2$, $F=\mathbb{Q}$, π comes from a Maass form. Unknown for $v|\infty$ or $v\nmid\infty$ there are nontrivial bounds.
2. $F\neq\mathbb{Q}$ is totally real and π arises from a Hilbert modular form of partial weight one (e.g. $(1,3)$).

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Let v be a finite place of F. Again π is a unitary ~~cusp~~ aut rep of $GL_n(\mathbb{A}_F)$. Say π_v is unramified.

iso class of $\pi_v \overset{\sim}{\longleftrightarrow}$ $\mathbb{C}$-alg hom $\mathcal{H}(GL_n(F_v),GL_n(\mathcal{O}_{F_v}))\to\mathbb{C}$

$\overset{\sim}{\longleftrightarrow}$ $\mathbb{C}$-alg hom $\mathbb{C}[T(F_v)/T(\mathcal{O}_{F_v})]^W\to\mathbb{C}$,

$\mathbb{C}[T(F_v)/T(\mathcal{O}_{F_v})]^W \cong \mathbb{C}[x_1^{\pm},\dots,x_n^{\pm}]^{S_n}$

$T=$ diag max torus, $W=$ Weyl group

$\overset{\sim}{\longleftrightarrow}$ $\alpha_{v,1},\dots,\alpha_{v,n}\in\mathbb{C}^\times$ up to order

We call $\alpha_{v,1},\dots,\alpha_{v,n}$ the <u>Satake parameters</u> of π_v

<u>Langlands</u> Set
$$L_v(\pi,s)=\frac{1}{(1-\alpha_{v,1}q_v^{-s})\cdots(1-\alpha_{v,n}q_v^{-s})}\qquad q_v=Nm(v)$$

If $S=\{v|\infty\}\cup\{v\nmid\infty \text{ s.t. } \pi_v \text{ is ramified}\}$
$$L^S(\pi,s):=\prod_{v\notin S}L_v(\pi,s)\quad \text{converges in some half plane.}$$

There should be a way to define $L_v(\pi,s)$ for $v\in S$ s.t.
$$L(\pi,s)=\prod_v L_v(\pi,s)$$
has meromorphic continuation, holomorphic for $\mathrm{Re}(s)>1$, nonvanishing on $\mathrm{Re}(s)=1$, entire if π is cuspidal, satisfying a functional equation relating
$$L(\pi,s) \text{ and } L(\pi^\vee,1-s)$$

<u>Thm</u> This is true. ($n=1$: Hecke, Tate, $n=2$: Jacquet-Langlands, $n>2$: Godement-Jacquet)

What about for more general G (connected reductive$/F$)?

<u>Note</u> $\alpha_1,\dots,\alpha_n\in\mathbb{C}^\times$ up to order

$\Longleftrightarrow$ a semisimple conj class in $GL_n(\mathbb{C})$.

Say (G,T) is a split connected red grp$/F$.

$(G,T)\overset{\sim}{\longleftrightarrow}$ root datum $\Psi=(X,\Phi,X^\vee,\Phi^\vee)$ where
$$X=\mathrm{Hom}(T,\mathbb{G}_m)\quad X^\vee=\mathrm{Hom}(\mathbb{G}_m,T)$$

Then $\Psi^\vee = (X^\vee,\Phi^\vee,X,\Phi)$ is another root datum, so defines a unique (up to iso) conn red grp $G^\vee$ over $\mathbb{C}$.

<u>Eg</u>

G	$G^\vee$
GL_n	GL_n
SL_n	PGL_n
Sp_{2n}	SO_{2n+1}
SO_{2n}	SO_{2n}
GSp_{2n}	$GSpin_{2n+1}$

Can show $W(G^\vee,T^\vee)\cong W(G,T)$

Then

π_v is an unramified sm adm irred rep of $G(F_v)$

$\longleftrightarrow$ $\mathbb{C}$-alg hom $\mathcal{H}(G(F_v),G(\mathcal{O}_{F_v}))\to\mathbb{C}$

$\longleftrightarrow$ $\mathbb{C}$-alg hom $\mathbb{C}[T(F_v)/T(\mathcal{O}_{F_v})]^W\to\mathbb{C}$

$\mathbb{C}[T(F_v)/T(\mathcal{O}_{F_v})]^W \cong \mathbb{C}[X^\vee(T)]^W\cong\mathbb{C}[X(T^\vee)]^W$

$\longleftrightarrow$ W-conj class in $T^\vee(\mathbb{C})$ by

$\mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{C}[X(T^\vee)], \mathbb{C})$

$\cong \mathrm{Hom}(X(T^\vee), \mathbb{C}^\times)$

$\cong \mathrm{Hom}(X(T^\vee), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}^\times$

$\cong X^\vee(T^\vee) \otimes_{\mathbb{Z}} \mathbb{C}^\times$

$\cong T^\vee(\mathbb{C})$

$\longleftrightarrow$ a semisimple conj class $\alpha_v \in G^\vee(\mathbb{C})$.

Call this the Satake parameter of π_v.

Let $r: G^\vee \to GL_n$ a (alg) rep of $G^\vee$.

Langlands There should be an L-function

$$L(\pi, r, s) = \prod_v L_v(\pi, r, s)$$

s.t. for $v \nmid \infty$ and π_v unramified,

$$L_v(\pi, r, s) = \frac{1}{\det(1 - r(\alpha_v) q_v^{-s})}$$

with meromorphic cont, hol for $Re(s) > 1$ and nonvanishing for $Re(s) = 1$, with functional eqn relating

$L(\pi, r, s)$ and $L(\pi, r^\vee, 1-s)$.

Note • If $G = GL_n$, $r = st$ = standard rep $GL_n \xrightarrow{id} GL_n$,

$L(\pi, st, s) = L(\pi, s)$ from before, and

$L(\pi, st^\vee, s) = L(\pi^\vee, st, s)$.

• Maybe $L(\pi, r, s) = L(\Pi, s)$ for some aut rep Π of $GL_n(\mathbb{A}_F)$.

Conj (Langlands Functoriality, split case)
Let G and H be split red groups $/F$ and $r: G^\vee \to H^\vee$ a hom of red groups $/\mathbb{C}$ and π an aut rep of $G(\mathbb{A}_F)$. Then $\exists$ an aut rep Π of $H(\mathbb{A}_F)$ such that if v is a fin place of F at which π is unramified and $\alpha_v \in G^\vee(\mathbb{C})$ is the Satake parameter of π_v, then Π_v is unramified with Satake par $r(\alpha_v)$.

Eg • $G = PGL_n$, $H = GL_n$, $r: SL_n \hookrightarrow GL_n$.

$\Rightarrow$ transfer is viewing a $PGL_n(\mathbb{A}_F)$-rep as a $GL_n(\mathbb{A}_F)$ rep with trivial central char.

• $G = GL_2$, $H = SL_2$, $r: GL_2 \to PGL_2$ canonical.

$\Rightarrow$ transfer is understand irred constituents of $\pi|_{SL_2(\mathbb{A}_F)}$ (Langlands–Labesse)

• $G = GL_2$, $H = GL_{m+1}$, $r = Sym^m: GL_2 \to GL_{m+1}$

Then $\pi = \otimes_v' \pi_v$ on $GL_2(\mathbb{A}_F)$ should have a functorial transfer to $\Pi = \otimes_v \Pi$ on $GL_{m+1}(\mathbb{A}_F)$ s.t. if α_v, β_v are Satake par of π_v, then

$\alpha_v^m, \alpha_v^{m-1}\beta_v, \dots, \beta_v^m$ are Satake par of Π_v.

Known if $m = 2, 3, 4$ (Gelbart–Jacquet, Kim–Shahidi, Kim)

• $F = \mathbb{Q}$, $\pi = \pi_f$, f a mod form, any $m \geq 1$ (Newton–Thorne)

• $Sp_{2n} \hookrightarrow GL_{2n}$ or $SO_{2n+1} \hookrightarrow GL_{2n+1}$,

$SO_{2n} \hookrightarrow GL_{2n}$ gives functorial lifts from

SO_{2n+1} to GL_{2n}, Sp_{2n} to GL_{2n+1},

SO_{2n} to GL_{2n} (Arthur)

Thm Symmetric power functoriality for GL_n

$\Rightarrow$ Generalized Ramanujan Conj for GL_n.

Proof for unramified fin places Let π be a unitarizable cusp aut rep of $GL_n(\mathbb{A}_F)$. Let v be a place at which π is unramified and $\alpha_1, \dots, \alpha_n \in \mathbb{C}^\times$ the Satake parameters of π_v.

Want $|\alpha_i| = 1 \quad \forall 1 \leq i \leq n$.

Jacquet–Shalika: $|\alpha_i| < q_v^{\frac{1}{2}} \quad 1 \leq i \leq n$.

Now take $m \geq 1$ and assume the m^{th} symm power Π_m of π exists. For simplicity assume Π_m is cuspidal.

The Satake parameters at v are

$$\{\alpha_1^{e_1} \cdots \alpha_n^{e_n} \mid e_i \geq 0, e_1 + \cdots + e_n = m\}$$

Apply Jacquet–Shalika to Π_m

$\Rightarrow |\alpha_i^m| < q_v^{\frac{1}{2}}$ for each i

$\Rightarrow |\alpha_i| < q_v^{\frac{1}{2m}} \Rightarrow |\alpha_i| \leq 1$ by $m \to \infty$

But π unitary $\Rightarrow \alpha_1 \cdots \alpha_n = 1 \Rightarrow |\alpha_i| = 1 \ \forall i$. $\square$