Lecture 18

Eg. $G = GL_2/\mathbb{Q}$

$\mathcal{A}^\circ = \mathcal{A}^\circ(GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}_\mathbb{Q}))$ space of cuspidal aut forms

Let $k \geq 2$, $N \geq 1$ integers, $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ as before.

Recall we have

$$\begin{tikzcd} \omega = \omega_\chi : \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times \arrow[r] \arrow[d] & \mathbb{C}^\times \\ \hat{\mathbb{Z}}^\times \arrow[r] & (\mathbb{Z}/N\mathbb{Z})^\times \arrow[u, "\chi^{-1}"'] \end{tikzcd}$$

$$\omega(1, \dots, \underset{\uparrow p^{th}}{p}, 1, \dots) = \chi(p) \quad \forall p \nmid N$$

Let $\mathcal{A}^\circ(k, N, \omega)$ = subspace of $\mathcal{A}^\circ$ satisfying

1. $\phi(zg) = |z|^{2-k} \omega_\chi(z) \phi(g) \quad \forall g \in GL_2(\mathbb{A}_\mathbb{Q}), z \in \mathbb{A}_\mathbb{Q}^\times$
2. ϕ is invariant under $U_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\hat{\mathbb{Z}}) \;\middle|\; c, d-1 \equiv 0 \bmod N\hat{\mathbb{Z}} \right\}$
3. $Z(\mathfrak{gl}_2) \ni \Omega$ acts by $k(\frac{k}{2} - 1)$ on ϕ
4. $SO(2)$ acts by $\phi\left(g \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) = e^{ik\theta} \phi(g)$

Recall we have

$$S_k(\Gamma_0(N), \chi) \longrightarrow \mathcal{A}^\circ$$
$$f \longmapsto \phi_f$$

where $\phi_f(\gamma h u) = (\det h) j(h,i)^{-k} \omega_\chi(d) f(h(i))$

for $\gamma \in GL_2(\mathbb{Q})$, $h \in GL_2(\mathbb{R})^+$, $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_0(N)$.

Claim: This is an iso onto $\mathcal{A}^\circ(k, N, \omega_\chi)$.

Easy to see it's injective since $GL_2(\mathbb{R})^+$ acts transitively on $\mathbb{H}$.

The inverse map is

$$\phi \longmapsto f_\phi \text{ where } f_\phi(z) = (\det h)^{-1} j(h,i)^k \phi(h)$$

where $h \in GL_2(\mathbb{R})^+$ s.t. $h(i) = z$

Well-defined since $\kappa = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$$(\det(h\kappa))^{-1} j(h\kappa, i)^k \phi(h\kappa) = (\det h)^{-1} j(h,i)^k e^{-ik\theta} e^{ik\theta} \phi(h)$$
$$= (\det h)^{-1} j(h,i) \phi(h)$$

Take $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. Then in the adeles

$$\gamma = \gamma_\infty \gamma^\infty \text{ and } \gamma(z) = (\gamma_\infty h)(i)$$

$g \in GL_2(\mathbb{A}_\mathbb{Q})$

$g = g_\infty g^\infty$ with

$g_\infty \in GL_2(\mathbb{R})$

$g^\infty \in GL_2(\mathbb{A}_\mathbb{Q}^\infty)$

$$f_\phi(\gamma(z)) = (\det \gamma_\infty h)^{-1} j(\gamma_\infty h, i)^k \phi(\gamma_\infty h)$$
$$= (\det h)^{-1} j(\gamma_\infty, z)^k j(h,i)^k \phi(\gamma(\gamma^\infty)^{-1} h)$$

$z = h(i)$

$$= (\det h)^{-1} j(\gamma_\infty, z)^k j(h,i)^k \omega_\chi(d^\infty)^{-1} \phi(h)$$
$$= j(\gamma_\infty, z)^k \chi(d) f_\phi(z)$$

Let's check cuspidal at ∞. We're assuming

$$\int_{\mathbb{Q}\backslash\mathbb{A}} \phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) dx = 0 \quad \forall g \in GL_2(\mathbb{A}_\mathbb{Q})$$

Take $g = 1$. Recall/learn

$$\mathbb{A}_\mathbb{Q} \cong \mathbb{Q} + [0,1) + \hat{\mathbb{Z}}$$

$$\int_{\mathbb{Q}\backslash\mathbb{A}} \phi\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} dx = \int_0^1 \int_{\hat{\mathbb{Z}}} \phi\left(\begin{pmatrix} 1 & x_\infty \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & x^\infty \\ & 1 \end{pmatrix}\right) dx^\infty dx_\infty$$
$$= \int_0^1 \phi\begin{pmatrix} 1 & x_\infty \\ & 1 \end{pmatrix} dx_\infty$$
$$= \int_0^1 f_\phi(x+i)\, dx = a_0(f)$$

Finally: Why is f_ϕ holomorphic?

Fact: We can write $\phi = \sum_{i=1}^r \phi_i$ where each ϕ_i belongs to an irreducible sub $GL_2(\mathbb{A}_\mathbb{Q})$-rep in $\mathcal{A}^\circ$

Then Ω acts on each ϕ_i by $k(\frac{k}{2} - 1)$

Recall $k \geq 2$, so by the classification of irred $(O(2), \mathfrak{gl}_2)$-mods, ϕ_i as an $(O(2), \mathfrak{gl}_2)$-rep is in the weight k discrete series

$$D_k.$$

Action of $\mathfrak{gl}_2$, it decomposes as

$$D_k = D_k^- \oplus D_k^+$$

Let $W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $E^+ = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$, $E^- = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$

We can describe D_k as follows

$\exists$ unique, up to scalar, generator v_0 such that

- $e^{\theta W} v_0 = e^{ik\theta} v_0$
- $E^- v_0 = 0$
- $D_k^+ = \bigoplus_{n \geq 0} \mathbb{C} v_n$ with $v_n = (E^+)^n v_0$, and $e^{\theta W}$ acts on v_n by $e^{i(k+2n)\theta}$
- $D_k^- = \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{O(2)} D_k^+ = \bigoplus_{n \geq 0} \mathbb{C} v_n^-$ with $E^+ v_0^- = 0$,

$v_n = (E^-)^n v_0$ and $e^{\theta W}$ acts on v_n by $e^{-i(k+2n)\theta}$

By this description, since $SO(2)$ acts on ϕ, hence each ϕ_i, by $\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\cdot\phi_i = e^{ik\theta}\phi_i$, we must have that each ϕ_i corr to v_0 in D_k, up to scalar.

$$\Rightarrow E^-\phi_1 = 0 \Rightarrow E^-\phi = 0$$

In the coordinates

$$GL_2(\mathbb{R})^+ \ni h = \begin{pmatrix} z & \\ & z\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}y^{\frac12} & \\ & y^{-\frac12}\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$$

one computes

$$E^- = e^{-2i\theta}\left[-2iy\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + i\frac{\partial}{\partial\theta}\right]$$

So if $h \in SL_2(\mathbb{R})$,

$$\phi(h) = y^{\frac{k}{2}} e^{ik\theta} f_\phi(z) \qquad z = h(i) = x+iy$$

We have

$$0 = (E^-\phi)(h)$$
$$= e^{-2i\theta}\left[-2iy\, y^{\frac{k}{2}} e^{ik\theta}\frac{\partial f_\phi}{\partial x}(z) + \cancel{ky^{\frac{k}{2}}e^{ik\theta}f_\phi(z)} + 2y^{\frac{k}{2}+1}e^{ik\theta}\frac{\partial f_\phi}{\partial y}(z) - \cancel{ky^{\frac{k}{2}}e^{ik\theta}f_\phi(z)}\right]$$
$$= -2y^{\frac{k}{2}+1}e^{i(k-2)\theta}\left[\frac{\partial f_\phi}{\partial x} + i\frac{\partial f_\phi}{\partial y}\right] = (\ldots)\,\frac{\partial f_\phi}{\partial \bar z}$$
$$\Rightarrow \frac{\partial f_\phi}{\partial \bar z} = 0, \text{ so } f_\phi \text{ is holomorphic}$$

<u>Rmk</u> Say we have a ϕ as above, will correspond to some modular form $f \in S_k(\Gamma_0(N), \chi)$. What do the translates of ϕ by $(O(2), \mathfrak{gl}_2) \times GL_2(\mathbb{A}_\mathbb{Q}^\infty)$ correspond to? The $GL_2(\mathbb{A}_\mathbb{Q}^\infty)$-action corresponds to viewing f as an oldform in higher levels in various ways. The $\mathfrak{gl}_2$-action correspond various diff operators that arise in the classical, e.g. Maass-Shimura diff op.