Lecture 16

We saw $f \in S_k(\Gamma_1(N)) \rightsquigarrow \phi_f \in L^2(\Gamma_1(N)\backslash SL_2(\mathbb{R}))$ and the weight of f is encoded in action of $Z(\mathfrak{sl}_2)$.

How do detect Hecke action? We want $\mathbb{Q}_p$ $\forall p$ as well.

Recall, F is a # fld, the adeles of F are

$$\mathbb{A}_F = \mathbb{A}_{F,\infty} \times \mathbb{A}_F^\infty$$

where
- $\mathbb{A}_{F,\infty} = F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^r \times \mathbb{C}^s$, $r = \#$ real $F \hookrightarrow \mathbb{R}$, $2s = \#$ complex $F \hookrightarrow \mathbb{C}$ $\not\subseteq \mathbb{R}$
- $\mathbb{A}_F^\infty = \mathbb{A}_{F,f} = \prod'_{v\nmid\infty} F_v$ where the product is over all finite places v of F, F_v is the completion at v, and $\prod'_{v\nmid\infty} F_v := \{(x_v) \in \prod_{v\nmid\infty} F_v \mid x_v \in \mathcal{O}_{F_v} \text{ for all but fin many } v\}$

Rmk: $\mathbb{A}_F^\infty = \left(\prod_{v\nmid\infty} \mathcal{O}_{F_v}\right) \otimes_{\mathcal{O}_F} F$

We topologize $\mathbb{A}_F^\infty$ so that $\prod_{v\nmid\infty} \mathcal{O}_{F_v}$ is open compact.

Then $\mathbb{A}_F$ is a topological ring.

Let G be a connected reductive (alg) group over F.

$F \hookrightarrow \mathbb{A}_F$ diag, so $\mathbb{A}_F$ is an F-alg and we can consider

$$G(\mathbb{A}_F)$$

We topologize $G(\mathbb{A}_F)$ as follows. Choose

$$G \underset{\substack{\text{closed emb}\\ \text{of alg groups}}}{\hookrightarrow} GL_N \underset{\substack{\text{closed}\\ \text{emb of } F\text{-vars}}}{\hookrightarrow} \mathrm{Aff}^{N^2+1}/F \quad = \text{affine } N^2+1\text{-space}/F$$

We get $G(\mathbb{A}_F) \underset{\text{closed}}{\subseteq} GL_N(\mathbb{A}_F) \underset{\text{closed}}{\subseteq} \mathbb{A}_F^{N^2+1}$ and we give $G(\mathbb{A}_F)$ the subspace top.

This makes $G(\mathbb{A}_F)$ a locally compact topological group.

If we set $G(\mathcal{O}_{F_v}) := G(F_v) \cap GL_N(\mathcal{O}_{F_v})$ for $v\nmid\infty$. Then

$$G(\mathbb{A}_F) = G(\mathbb{A}_{F,\infty}) \times G(\mathbb{A}_F^\infty) \text{ and}$$

$$G(\mathbb{A}_F^\infty) = \prod'_{v\nmid\infty} G(F_v)$$
$$= \{(g_v) \in \prod_{v\nmid\infty} G(F_v) \mid g_v \in G(\mathcal{O}_{F_v}) \text{ for all but fin many } v\}$$

Exercise Check this.

$F \hookrightarrow \mathbb{A}_F \Rightarrow G(F) \hookrightarrow G(\mathbb{A}_F)$ and $G(F)$ is a discrete subgroup of $G(\mathbb{A}_F)$.

($F^{N^2+1} \subseteq \mathbb{A}_F^{N^2+1}$ is discrete)

Let's focus now on SL_2 and GL_2 over $\mathbb{Q}$.

Thm (Strong approx for SL_2)

$SL_2(\mathbb{Q})$ is dense in $SL_2(\mathbb{A}_{\mathbb{Q}}^\infty)$

Proof For any $N \in \mathbb{Z}_{\geq 1}$, $SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z})$ is surjective. So $SL_2(\mathbb{Z}) \subseteq SL_2(\hat{\mathbb{Z}})$ is dense.

$\hat{\mathbb{Z}} := \varprojlim_N \mathbb{Z}/N\mathbb{Z} \cong \prod_p \mathbb{Z}_p$.

Let $L = \hat{\mathbb{Z}}^2 \subseteq (\mathbb{A}_{\mathbb{Q}}^\infty)^2 = V$. For $g \in SL_2(\mathbb{A}_{\mathbb{Q}}^\infty)$, gL is a free rank 2 $\hat{\mathbb{Z}}$-mod, choose a basis $\{e_1, e_2\}$ for gL. Since $\mathbb{Q}$ is dense in $\mathbb{A}_F^\infty$, we can assume $e_1, e_2 \in \mathbb{Q}^2$. Set $h = (e_1\ e_2) \in GL_2(\mathbb{Q})$.

Since $hL = gL \Rightarrow g^{-1}h \in \mathrm{Aut}_{\hat{\mathbb{Z}}}(L) = GL_2(\hat{\mathbb{Z}})$.

Then $\det(h) = \det(g^{-1}h) \in \hat{\mathbb{Z}}^\times \cap \mathbb{Q}^\times = \{\pm 1\}$

Replacing e_1 be $-e_1$, if nec, can assume $\det h = 1$.

Then $g^{-1}h \in SL_2(\hat{\mathbb{Z}})$.

Now for any open subgroup U in $SL_2(\hat{\mathbb{Z}})$, we can find $\gamma \in SL_2(\mathbb{Z})$ such that

$$g^{-1}h \in U\gamma \Rightarrow \underbrace{h\gamma^{-1}}_{SL_2(\mathbb{Q})} \in \underbrace{gU}_{\text{a nbhd of } g}$$

$\square$

Rmk For a general reductive group G over a # fld F and S a fin set of places of F, $G(F) \hookrightarrow G(\mathbb{A}_F^S)$ ($\mathbb{A}_F$ with places in S left out) is dense $\Leftrightarrow$ the following 2 conds hold ($S \neq \emptyset$)

1. G is semisimple and simply connected
2. $G(F_v)$ is not compact for at least one $v \in S$.

See Platonov-Rapinchuk.

Cor Let U be an open compact subgroup of $GL_2(\hat{\mathbb{Z}})$ such that $\det(U) = \hat{\mathbb{Z}}^\times$. Then

$$GL_2(\mathbb{A}_{\mathbb{Q}}) = GL_2(\mathbb{Q})GL_2(\mathbb{R})^+ U$$

Proof Fix $g \in GL_2(\mathbb{A}_\mathbb{Q})$ and write

$$g = g_\infty g^\infty \text{ with } g_\infty \in GL_2(\mathbb{R}), \quad g^\infty \in GL_2(\mathbb{A}_\mathbb{Q}^\infty)$$

Mult g on the left by an element of $GL_2(\mathbb{Q})$, if nec, we can assume $\det(g_\infty) > 0$ and that $\det(g^\infty) \in \hat{\mathbb{Z}}^\times$.
By assumption $\exists\, u \in U$ such that $\det(u) = \det(g^\infty)$.
So $g^\infty u^{-1} \in SL_2(\mathbb{A}_\mathbb{Q}^\infty)$ and setting $V = U \cap SL_2(\mathbb{A}_\mathbb{Q}^\infty)$, $\exists\, \gamma \in SL_2(\mathbb{Q})$ such that

$$g^\infty u^{-1} \in \gamma^\infty V \quad \text{by above Thm.}$$

Then $g = g_\infty g^\infty \in g_\infty \gamma^\infty U = \gamma \underbrace{(\gamma_\infty^{-1} g_\infty)}_{GL_2(\mathbb{R})^+} U$ $\square$

Now let $k, N \geq 1$ be integers. Define open compact subgroups of $GL_2(\hat{\mathbb{Z}})$ by

$$U_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\hat{\mathbb{Z}}) \;\middle|\; c \equiv 0 \bmod N\hat{\mathbb{Z}} \right\}$$

$$U_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_0(N) \;\middle|\; d \equiv 1 \bmod N\hat{\mathbb{Z}} \right\}$$

Note $U_0(N)/U_1(N) \cong (\hat{\mathbb{Z}}/N\hat{\mathbb{Z}})^\times \cong (\mathbb{Z}/N\mathbb{Z})^\times$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto d \bmod N.$$

So if $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ is a Dirichlet char, we can define $\chi : U_0(N) \to \mathbb{C}^\times$ by $\chi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi(d \bmod N)$.

Let $f \in S_k(\Gamma_1(N))$ such that

$$\langle d \rangle f = \chi(d)^{-1} f \quad \forall\, d \in (\mathbb{Z}/N\mathbb{Z})^\times$$

Say $s \in \mathbb{C}$. Define

$$\phi_f = \phi_{f,s} : GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}_\mathbb{Q}) \longrightarrow \mathbb{C}$$

by $\phi_{f,s}(\gamma h u) = (\det h)^s \chi(u)\, j(h,i)^{-k} f(h(i))$

$$\gamma \in GL_2(\mathbb{Q}),\ h \in GL_2(\mathbb{R})^+,\ u \in U_0(N)$$

Well-defined $GL_2(\mathbb{Q}) \cap (GL_2(\mathbb{R})^+ U_0(N)) = \Gamma_0(N)$.
$\gamma h u = (\gamma \delta^{-1})(\delta_\infty h)(\delta^\infty u)$ for $\delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

$$\phi_{f,s}((\gamma\delta^{-1})(\delta_\infty h)(\delta^\infty u))$$
$$= \det(\delta_\infty h)^s \chi(\delta^\infty u)\, j(\delta_\infty h, i)^{-k} f(\delta_\infty h(i))$$

- $\det(\delta_\infty h)^s = \det(h)^s$
- $\chi(\delta^\infty u) = \chi(d)\chi(u)$
- $j(\delta_\infty h, i)^{-k} = j(\delta_\infty, h(i))^{-k} j(h,i)^{-k}$
- $f(\delta_\infty h(i)) = \chi^{-1}(d)\, j(\delta_\infty, h(i))^k f(h(i))$

Taking product $= (\det h)^s \chi(u)\, j(h,i)^{-k} f(h(i))$
$= \phi_{f,s}(\gamma h u)$.

What's with s?

Note for any $\phi : GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}$ and $s \in \mathbb{C}$, we have a function

$$\phi_s : GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}$$

by $\phi_s(g) = \phi(g) |\det g|^s$

where $|(x_p)_{p \leq \infty}| = \prod_{p \leq \infty} |x|_p$.

And if $g = \gamma h u$ as above, $|\det g|^s = (\det h)^s$

In the construction $f \rightsquigarrow \phi_f$, there are 2 standard choices for s

1. $s = \frac{k}{2}$
2. $s = k - 1$

Rmk
- The two choices are just twists of each other by $g \mapsto |\det g|^s$.
- They are the same when $k = 2$.
- Related to the different normalizations for the wt k slash operator on modular forms.

Exercise For any $g \in GL_2(\mathbb{A}_\mathbb{Q})$ and $z \in \mathbb{A}_\mathbb{Q}^\times$,

$$\phi_{f,s}(zg) = |z|^{2s-k} \chi(z)\, \check{f}(g)$$

So $s = \frac{k}{2} \Rightarrow \phi_{f,s}$ has unitary central char.

$\Rightarrow |\phi_f|$ descends to $\mathbb{A}_{\mathbb{Q}}^\times GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}_{\mathbb{Q}})$
which has fin volume, so can define an inner prod and talk about unitary, etc.

But $s = k - \underline{1}$ is more natural from Hecke side.
(Next time).