Notation as above/before. Set $\delta = \delta_B$

Recall, the Satake transform of $f \in \mathcal{H}(G(F), K)$ is

$$Sf(t) = \delta^{\frac{1}{2}}(t) \int_{N(F)} f(tn)\,dn$$

$$= \delta^{-\frac{1}{2}}(t) \int_{N(F)} f(nt)\,dn$$

We want to show (sketch) that it induces an isomorphism

$$\mathcal{H}(G(F), K) \cong \mathcal{H}(T(F), T(\mathcal{O}_F))^W$$

To show it is an algebra homomorphism, i.e. respects convolution, is a computation using the integration formulas

$$\int_G f(g)\,dg = \int_{B(F)} \int_K f(bk)\,dk\,d_\ell b$$

$$= \int_{T(F)} \int_{N(F)} \int_K f(tn\,k)\,dk\,dn\,dt$$

for any $f \in C_c(G(F))$.

Image $\subseteq$ W-invariants Can choose reps for W in $N(T) \cap K$

We want to show that for any $x \in N(T) \cap K$, we have

$$Sf(xtx^{-1}) = Sf(t). \quad (\ast)$$

Let $\mathfrak{n} = \mathrm{Lie}\,N$ and consider the fcn

$$A : T(F) \to F$$

$$t \mapsto \det(\mathrm{Ad}_{\mathfrak{n}}(t) - 1)$$

If $A(t) \neq 0$, we say t is regular. Since A is a nonzero polynomial function, regular $t \in T(F)$ are dense and it suffices to prove (★) for such t.

Define $\Delta(t) = |A(t)|$. Can construct a filtration

$$N(F) = N_0 \supseteq N_1 \supset \cdots \supseteq N_s = 1$$

stable under $T(F)$ such that each N_i / N_{i+1} is an F-vect sp with $T(F)$ acting linearly.

(Eg In GL_3: $\begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \supseteq \begin{pmatrix} 1 & 0 & * \\ & 1 & 0 \\ & & 1 \end{pmatrix} \supseteq \begin{pmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$)

Using this filtration, can show for any $\phi \in C_c(N(F))$,

$$\int_{N(F)} \phi(n)\,dn = \Delta(t) \int_{N(F)} \phi(ntn^{-1}t^{-1})\,dn \quad \text{if } t \text{ is regular}$$

(Eg GL_2: if $a \neq d$, then

$$\int_F \phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} a & \\ & d \end{pmatrix}\begin{pmatrix} 1 & -x \\ & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & \\ & d^{-1} \end{pmatrix}\right) dx = \int_F \phi\begin{pmatrix} 1 & (1-\frac{a}{d})x \\ & 1 \end{pmatrix} dx$$

$$= \left|\frac{a}{d} - 1\right|^{-1} \int_F \phi\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} dx \;)$$

Claim For regular t,

$$\int_{N(F)} f(nt)\,dn = \Delta(t) \int_{G(F)/T(F)} f(gtg^{-1})\,dg \qquad (\diamond)$$

Consider the function on $G(F)$: $g \mapsto f(gtg^{-1})$.

$$\int_{G(F)} f(gtg^{-1})\,dg = \int_{B(F)} \int_K f(kbtb^{-1}k^{-1})\,dk\,db$$

$$= \int_{B(F)} f(btb^{-1})\,db \qquad \text{since } f \in \mathcal{H}(G(F),K)$$

$$= \int_{N(F)} \int_{T(F)} f(nsts^{-1}n^{-1})\,ds\,dn$$

$$= \int_{N(F)} \int_{T(F)} f(ntn^{-1})\,ds\,dn$$

so the RHS of $(\Diamond)$ is

$$\Delta(t) \int_{N(F)} f(ntn^{-1})\,dn$$

$$= \Delta(t) \int_{N(F)} (tf)(ntn^{-1}t^{-1})\,dn$$

$$= \int_{N(F)} (tf)(n)\,dn = \int_{N(F)} f(nt)\,dt$$

Then for regular t,

$$(Sf)(t) = \delta^{-\frac{1}{2}}(t) \int_{N(F)} f(nt)\,dt$$

$$= D(t) \int_{G(F)/T(F)} f(gtg^{-1})\,dg \qquad (\dagger)$$

with $D(t) = \Delta(t)\delta^{-\frac{1}{2}}(t)$. Now

$$D(t)^2 = |\det(\mathrm{Ad}_n(t) - 1)|^2\, |\det \mathrm{Ad}_n(t)|^{-1}$$

$$= |\det(\mathrm{Ad}_{\mathfrak{n}}(t)-1)|\,|\det(\mathrm{Ad}_{\mathfrak{n}}(t^{-1})-1)|$$
$$= |\det(\mathrm{Ad}_{\mathfrak{n}}(t)-1)|\,|\det(\mathrm{Ad}_{\mathfrak{n}^-}(t)-1)|$$

where $\mathfrak{n}^- = \mathrm{Lie}\, N^-$ and N^- is the unipotent radical of the opposite Borel subgroup $B^- = TN^-$ (i.e. roots of T in N^- are $-\Phi^+$).

Since $\mathfrak{g} = \mathrm{Lie}\, G = \mathfrak{n} \oplus \mathfrak{t} \oplus \mathfrak{n}^-$, $\mathfrak{t} = \mathrm{Lie}\, T$, we have

$$D(t) = |\det(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{t}}(t) - 1)|^{\frac{1}{2}}$$

From this expression,

$$D(xtx^{-1}) = D(t) \quad \forall\, t \in T \text{ and } x \in N(T) \quad (\dagger\dagger)$$

On the other hand conj by any $x \in N(T) \cap K$ leaves Haar measure on both $G(F)$ and $T(F)$ invariant, so it leaves invariant the induced measure on $G(F)/T(F)$ and $f \in \mathcal{H}(G(F), K)$ implies $f(x^{-1}gx) = f(g)$ for any $g \in G(F)$.

Thus

$$\int_{G(F)/T(F)} f(g(xtx^{-1})g^{-1})\,dg = \int_{G(F)/T(F)} f((x^{-1}gx)t(x^{-1}gx)^{-1})\,dg$$
$$= \int_{G(F)/N(F)} f(gtg^{-1})\,dg \qquad (\dagger\dagger\dagger)$$

$(\dagger) + (\dagger\dagger) + (\dagger\dagger\dagger) \Rightarrow (*)$ for regular t, which is what we wanted to prove. (Note: The above also shows that the Satake transform is independent of the choice of Borel containing T.)

$\underline{S \text{ is bijective}}$ Recall we have the basis

$$c_\lambda = K\lambda(\varpi)K \qquad \lambda \in X^{\vee,+}$$

of $\mathcal{H}(G(F), K)$. Now any element of $X^\vee$ is W-conjugate to a unique element of $X^{\vee,+}$. So setting

$$d_\lambda = \frac{1}{\# \mathrm{Stab}_W(\lambda)} \sum_{w \in W} \mathbb{1}_{(w\lambda)(\varpi)T(\mathcal{O}_F)}$$

we get a basis $\{d_\lambda\}_{\lambda \in X^{\vee,+}}$ of $\mathcal{H}(T(F), T(\mathcal{O}_F))$

We then define, for $\lambda \in X^{\vee,+}$

$$Sc_\lambda = \sum_{\mu \in X^{\vee,+}} a(\mu,\lambda) d_\mu$$

Fix $\mu \in X^{\vee,+}$, set $t = \mu(\varpi)$ and $s = \lambda(\varpi)$

Then we have

$$\begin{aligned} a(\mu,\lambda) &= Sc_\mu(t) \\ &= \delta(t)^{\frac{1}{2}} \int_{N(F)} c_\lambda(tn)\,dn \\ &= \delta(t)^{\frac{1}{2}} \operatorname{meas}(N(F) \cap t^{-1}KsK) \end{aligned}$$

Note that if $t = s$, then

$$N(F) \cap t^{-1}KtK \supseteq t^{-1}N(\mathcal{O}_F)t$$

which is open in $N(F)$, so $a(\lambda,\lambda) \neq 0$

Also, Bruhat-Tits theory $\Rightarrow$

$$N(F) \cap t^{-1}KsK = \emptyset \text{ unless } \lambda - \mu \text{ is a lin comb}$$

of elements of $\Phi^{\vee,+}$ with nonnegative coefficients.

(Eg $G = GL_2$, $B = \begin{pmatrix} * & * \\ & * \end{pmatrix}$, $K = GL_2(\mathcal{O}_F)$

$$t = \mu(\varpi) = \begin{pmatrix} \varpi^{e_1} & \\ & \varpi^{e_2} \end{pmatrix}, \quad e_1 \geq e_2$$
$$s = \lambda(\varpi) = \begin{pmatrix} \varpi^{f_1} & \\ & \varpi^{f_2} \end{pmatrix}, \quad f_1 \geq f_2$$

Say $\exists\ n = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \in N(F)$ and $k_1, k_2 \in GL_2(\mathcal{O}_F)$ such that
$$tn = k_1 s k_2.$$
The positive coroot is $y \mapsto \begin{pmatrix} y & \\ & y^{-1} \end{pmatrix}$, so we want to show
$$(f_1, f_2) = (e_1, e_2) + a(1,-1) \quad \text{for some } a \geq 0$$
The entries of $k_1 s k_2$ have valuations f_1 or f_2, and the entries of tn have valuations $e_1, e_2, e_1 + \mathrm{val}(x)$

So $\min\{e_1, e_2, e_1 + \mathrm{val}(x)\} = f_2 \Rightarrow f_2 \leq e_2$.

Comparing valuations of determinants, we find
$$e_1 + e_2 = f_1 + f_2$$
So setting $a = e_2 - f_2$, we have
$$(f_1, f_2) = (e_1 + a, e_2 - a) \quad)$$

The relation
$$\lambda \geq \mu \iff \lambda - \mu \text{ is a nonnegative lin comb of positive coroots}$$
is a partial order that we can extend to a total order with a suitable lexicographic ordering.

Then $a(\mu, \lambda) = 0$ unless $\lambda \geq \mu$ and $a(\lambda, \lambda) \neq 0$

$\Rightarrow \{Sc_\lambda\}_{\lambda \in X^{\vee,+}}$ is a basis. $\square$