# Lecture 22

Let $G$ be a locally profinite (Hausdorff) group.

Assume $G$ is unimodular, i.e. left Haar meas = right Haar measure.

The <u>Hecke algebra</u> for $G$ is

$$\mathcal{H}(G) := C_c^\infty(G)$$
$$= \{ f : G \to \mathbb{C} \text{ locally constant and compact support} \}$$

multiplication by convolution:

$$(f_1 * f_2)(x) = \int_G f_1(xy^{-1}) f_2(y)\, dy$$

Can check it is associative.

<u>Rmk</u> 1. $\mathcal{H}(G)$ does not have a mult unit unless $G$ is compact.

2. $G$-acts on $\mathcal{H}(G)$ by $(gf)(x) = f(xg)$ and this is smooth.

Let $K$ be an open compact subgroup of $G$.

We define

$$\mathcal{H}(G,K) = \{ f \in \mathcal{H}(G) \text{ such that } \forall k_1, k_2 \in K,\ x \in G,\ f(k_1 x k_2) = f(x) \}$$

$\mathcal{H}(G,K)$ is a subalg of $\mathcal{H}(G)$.

Indeed if $f_1, f_2 \in \mathcal{H}(G,K)$, $x \in G$, $k_1, k_2 \in K$,

$$(f_1 * f_2)(k_1 x k_2) = \int_G f_1(k_1 x k_2 y^{-1}) f_2(y)\, dy$$

$$= \int_G f_1(x k_2 y^{-1}) f_2(y)\, dy \qquad f_1 \text{ is left } K\text{-inv}$$

$$= \int_G f_1(x y^{-1}) f_2(y k_2^{-1})\, dy \qquad y \mapsto y k_2^{-1}$$

$$= \int_G f_1(x y^{-1}) f_2(y)\, dy \qquad f_2 \text{ is right } K\text{-inv}$$

$$= (f_1 * f_2)(x)$$

Set $e_K = \frac{1}{\text{meas}(K)} \mathbb{1}_K \in \mathcal{H}(G,K)$

Then
$$(e_K * e_K)(x) = \frac{1}{\text{meas}(K)^2} \int_G \mathbb{1}_K(xy^{-1}) \mathbb{1}_K(y)\, dy$$

$$= \frac{1}{\text{meas}(K)^2} \int_K \mathbb{1}_K(xy^{-1})\, dy$$

$$= \frac{1}{\text{meas}(K)} \begin{cases} 1 & \text{if } x \in K \\ 0 & \text{if } x \notin K \end{cases}$$

$$= e_K(x)$$

Moreover, if $f \in \mathcal{H}(G,K)$, can check

$$e_K * f = f * e_K = f$$

So $\mathcal{H}(G,K)$ is a ring with unit $e_K$.

For $f \in \mathcal{H}(G)$, $x \in G$, $k \in K$,

$$(e_K * f)(kx) = \int_G e_K(kxy^{-1}) f(y)\, dy$$
$$= \int_G e_K(xy^{-1}) f(y)\, dy = (e_K * f)(x)$$

$$(f * e_K)(xk) = \int_G f(xky^{-1}) e_K(y)\, dy$$
$$= \int_G f(xy^{-1}) e_K(yk^{-1})\, dy$$
$$= \int_G f(xy^{-1}) e_K(y)\, dy = (f * e_K)(x).$$

$$\Rightarrow \mathcal{H}(G,K) = e_K \mathcal{H}(G) e_K.$$

Any element of $\mathcal{H}(G,K)$ can be written as a fin linear combination of elements of the form

$$\mathbb{1}_{KgK}$$

And $g, h \in G$, then writing

$$KgK = \bigsqcup_i g_i K \quad \text{and} \quad KhK = \bigsqcup_j h_j K$$

<u>Exercise</u> $\mathbb{1}_{KgK} * \mathbb{1}_{KhK} = \sum_{i,j} \mathbb{1}_{K g_i h_j K}$

Note if $g \in G$ and $U$ is an open compact subgroup, the setting $K = U \cap gUg^{-1}$, we have

$$KgK \subseteq gUK \subseteq gU$$

So $\{KgK,\ K \leq G$ open compact subgroup$\}$ are a nbhd base around $g$. Then

$$\mathcal{H}(G) = \bigcup_{\substack{K \leq G \\ \text{open cpt}}} \mathcal{H}(G,K)$$

Now say $(\pi, V)$ is a smooth $G$-rep. We can give $V$ an $\mathcal{H}(G)$-module structure as follows: $f \in \mathcal{H}(G)$, $v \in V$, define

$$\pi(f)v := \int_G f(g)\pi(g)v\,dg = \sum_{g \in G/K} f(g)\pi(g)v$$

if $K \leq G$ open compact stabilizing $v$, and $f$ is right $K$-inv.

This gives $V$ the structure of an $\mathcal{H}(G)$-mod, and if $T: (\pi, V) \to (\sigma, W)$ is a $G$-equiv map of smooth $G$-reps, then $T: V \to W$ is an $\mathcal{H}(G)$-mod map.

Can check we get a functor from the cat of smooth $G$-reps to the cat of $\mathcal{H}(G)$-mods.

We say an $\mathcal{H}(G)$-mod $M$ is <u>nondegenerate</u> if $\forall m \in M$, $\exists\ f_1, \ldots, f_n \in \mathcal{H}(G)$, $m_1, \ldots, m_n \in M$ such that

$$m = f_1 m_1 + \cdots + f_n m_n.$$

An $\mathcal{H}(G)$-mod is nondeg $\iff$ for every $m \in M$, $\exists$ an open compact subgroup $K$ s.t. $e_K m = m$.

$\Leftarrow$ Immediate.

$\Rightarrow$ Write $m = f_1 m_1 + \cdots + f_n m_n$. We can find open compact $K \leq G$ such that $e_K * f_i = f_i\ \forall 1 \leq i \leq n$.

Then $e_K m = (e_K * f_1)m_1 + \cdots + (e_K * f_n)m_n = f_1 m_1 + \cdots + f_n m_n = m$.

In particular, the $\mathcal{H}(G)$ module $V$ for $(\pi, V)$ a smooth $G$-rep is nondegenerate, since if $K$ stabilizes $v$,

$$\pi(e_K)v = \int_G e_K(g)\pi(g)v\,dg = \frac{1}{\text{meas}(K)}\int_K \pi(g)v\,dg = v.$$

Conversely, say $M$ is a nondeg $\mathcal{H}(G)$-mod.

Nondegenerate $\Rightarrow$ $\mathcal{H}(G) \otimes_{\mathcal{H}(G)} M \to M$ is surj.

Say $\sum f_i \otimes m_i \in$ the kernel. Can choose $K \leq G$ open compact such that $f_i \in \mathcal{H}(G,K)$ and $e_K m_i = m_i$ for $1 \leq i \leq n$.

Then
$$\sum f_i \otimes m_i = \sum f_i \otimes e_K m_i = e_K \otimes \left(\sum f_i m_i\right) = 0.$$

So $\mathcal{H}(G) \otimes_{\mathcal{H}(G)} M \to M$ is an iso.

Then the smooth $G$-action on $\mathcal{H}(G)$ gives one on $M$.

<u>More explicitly</u> If $e_K m = m$, then we define

$$\pi(g)m := \frac{1}{\text{meas}(K)} \mathbb{1}_{gK} m$$

<u>Exercise</u> Check the above gives an equiv of cats between smooth $G$-reps and nondeg $\mathcal{H}(G)$-mods.

Fix $K \leq G$ open compact and let $(\pi, V)$ be a smooth $G$-rep. Note that, for $k \in K$

$$\begin{aligned}
\pi(k)\pi(e_K)v &= \pi(k)\left(\frac{1}{\text{meas}(K)}\int_G \mathbb{1}_K(g)\pi(g)v\,dg\right) \\
&= \frac{1}{\text{meas}(K)}\int_G \mathbb{1}_K(g)\pi(kg)v\,dg \\
&= \frac{1}{\text{meas}(K)}\int_K \pi(kg)v\,dg \\
&= \frac{1}{\text{meas}(K)}\int_K \pi(g)v\,dg \quad \text{since } K \text{ is unimodular} \\
&= \frac{1}{\text{meas}(K)}\int_G \mathbb{1}_K(g)\pi(g)v\,dg \\
&= \pi(e_K)v
\end{aligned}$$

So $\pi(e_K): V \to V^K$ is the $K$-equiv projection.

Then $V^K = \pi(e_K)V$ is an $\mathcal{H}(G,K) = e_K\mathcal{H}(G)e_K$-mod.

**Thm** Let $(\pi, V)$ be an irred smooth $G$-rep.

1. $V^K$ is either $0$ or a simple $\mathcal{H}(G,K)$-mod.
2. This give a bijection between iso classes of smooth irred $G$-reps with $V^K \neq 0$ and iso classes of simple $\mathcal{H}(G,K)$-mods.

**Rmk** If $V$ is further admissible, then this reduces understanding $V$ with $V^K \neq 0$, to understanding the fin dim $\mathcal{H}(G,K)$-mod $V^K$. Problem: $\mathcal{H}(G,K)$ can be hard to understand.

Proof of Thm:

1. Assume $V^K \neq 0$ and let $M$ be a simple $\mathcal{H}(G,K)$-submodule of $V^K$. Then $\pi(\mathcal{H}(G))M$ is a $G$-stable subspace of $V$, so equals $V$ by irreducibility. Then

$$V^K = \pi(e_K)V = \pi(e_K)\pi(\mathcal{H}(G))M = \pi(\mathcal{H}(G,K))M = M$$

So $V^K$ is simple.

2. Let $M$ be a simple $\mathcal{H}(G,K)$-module and consider the smooth $G$-representation

$$W = \mathcal{H}(G) \otimes_{\mathcal{H}(G,K)} M$$

Note that
$$\begin{aligned} W^K &= e_K \mathcal{H}(G) \otimes_{\mathcal{H}(G,K)} M \\ &= e_K \mathcal{H}(G) \otimes_{\mathcal{H}(G,K)} e_K M \\ &= \mathcal{H}(G,K) \otimes_{\mathcal{H}(G,K)} M \\ &= M \end{aligned}$$

Zorn's Lemma $\Rightarrow \exists$ a maximal $G$-stable subspace $X \subseteq W$ such that $X^K = 0$. Say $Y$ is another $G$-stable subspace with $Y^K = 0$. Decomposing $X$ and $Y$ in terms of $K$-types, we have

$$(X+Y)^K = X^K + Y^K = 0$$

So $X+Y = X$ and $Y \subseteq X$ by maximality of $X$. It then follows that $X$ is the unique maximal $G$-stable subspace of $W$ such that $X^K = 0$.

Then any $G$-stable subspace $Z$ of $W$ strictly containing $X$ must meet, hence contain, the simple $\mathcal{H}(G,K)$-module $e_K \otimes M$, which implies $Z = W$.

Then $X$ is a maximal $G$-stable subspace and

$$V = W/Z$$

is irreducible.

Now if $\phi: M \to M'$ is an iso of $\mathcal{H}(G,K)$-modules, then $1 \otimes \phi: W \xrightarrow{\sim} W' := \mathcal{H}(G) \otimes_{\mathcal{H}(G,K)} M'$ must take $X$ to the the unique maximal $G$-stable subspace $X'$ of $W'$ satisfying $(X')^K = 0$. Thus $\phi$ induces an isomorphism

$$V \xrightarrow{\sim} V' := W'/X'.$$

$\square$