Lecture 25

F, G as before. $Z =$ centre of G.

Def Say (π, V) is an irred smooth admissible rep of $G(F)$. We say it is <u>square integrable</u> (resp. <u>tempered</u>) if it has unitary central character and all matrix coeffs lie in $L^2(G(F)/Z(F))$ (resp $L^{2+\varepsilon}(G(F)/Z(F))$)

Rmk • A supercuspidal rep with unitary central char is square integrable (easy).

• Square integrable $\Rightarrow$ tempered (less easy, c.f. Lecture 15)

Exercise A square integrable representation is unitarizable.

Thm (π, V) is an irred sm adm rep of $G(F)$. Then it is tempered $\Leftrightarrow$ $\exists$ a parabolic subgroup $P \leq G$, with Levi decomp $P = MN$ and a sq. int. rep (σ, W) of $M(F)$ s.t. (π, V) is a subrep of $n\mathrm{Ind}_{P(F)}^{G(F)} \sigma$.

In part (π, V) is unitarizable.

Eg Until specified otherwise

$$G = GL_2, \quad B = \left\{ \begin{pmatrix} * & * \\ & * \end{pmatrix} \right\}$$

Any non-supercuspidal rep is a subquotient of

$$V(\chi_1, \chi_2) := n\mathrm{Ind}_{B(F)}^{GL_2(F)} (\chi_1 \otimes \chi_2), \quad \chi_i : F^\times \to \mathbb{C}^\times \text{ sm chars}$$

$V(\chi_1, \chi_2)$ is irred unless $\chi_1\chi_2^{-1} = |\ |^{\pm 1}$.

If $\chi_1\chi_2^{-1} \neq |\ |^{\pm 1}$ then $V(\chi_1, \chi_2)$ is irred and is tempered $\Leftrightarrow$ χ_1, χ_2 are unitary.

Now say $\chi_1 = |\ |^{-\frac{1}{2}}$, $\chi_2 = |\ |^{\frac{1}{2}}$.

Then $V(|\ |^{-\frac{1}{2}}, |\ |^{\frac{1}{2}}) = \{ f : B(F)\backslash GL_2(F) \to \mathbb{C} \text{ locally const} \}$

$$\cup\!|$$

$\mathbb{C}1 =$ space of constant functions $=$ triv rep.

$St := V(|\ |^{-\frac{1}{2}}, |\ |^{\frac{1}{2}}) / \mathbb{C}1$ is called the <u>Steinberg rep</u>.

It is irred and square integrable.

We have exact

$$0 \to \mathbb{C}1 \to V(|\ |^{-\frac{1}{2}}, |\ |^{\frac{1}{2}}) \to St \to 0 \qquad (1)$$

If $\chi_1 = |\ |^{\frac{1}{2}}$, $\chi_2 = |\ |^{-\frac{1}{2}}$, then St is a subrep of $V(|\ |^{\frac{1}{2}}, |\ |^{-\frac{1}{2}})$ and we have exact

$$0 \to St \to V(|\ |^{\frac{1}{2}}, |\ |^{-\frac{1}{2}}) \to \mathbb{C}1 \to 0 \qquad (2)$$

If $\chi_1\chi_2^{-1} = |\ |^{\pm 1}$, we can write

$$\chi_1 = \chi|\ |^{\pm\frac{1}{2}} \quad \chi_2 = \chi|\ |^{\mp\frac{1}{2}} \quad \text{for } \chi : F^\times \to \mathbb{C}^\times_{\text{sm}}$$

then $V(\chi_1, \chi_2) \cong V(|\ |^{\pm\frac{1}{2}}, |\ |^{\mp\frac{1}{2}}) \otimes \chi \circ \det$ and you tensor (1), (2) with $\chi \circ \det$.

Say $V(\chi_1, \chi_2)$ is irred and unitarizable.

Say $\langle\ ,\ \rangle$ is a Hermitian pairing on $V(\chi_1, \chi_2)$.

Complex conj of functions gives a $GL_2(F)$-equiv antilin iso

$$V(\chi_1, \chi_2) \to V(\bar{\chi}_1, \bar{\chi}_2)$$

OTOH, $\langle\ ,\ \rangle$ gives a lin iso

$$V(\bar{\chi}_1, \bar{\chi}_2) \cong V(\chi_1, \chi_2)^\vee \leftarrow \text{smooth dual}$$

$$\cong V(\chi_1^{-1}, \chi_2^{-1}) \qquad \text{Lecture 21}$$

So $\bar{\chi}_1 = \chi_1^{-1}$ and $\bar{\chi}_2 = \chi_2^{-1}$, i.e. χ_1 and χ_2 are unitary or $\bar{\chi}_1 = \chi_2^{-1}$ and $\bar{\chi}_2 = \chi_1^{-1}$. If this happens, write

$$\chi_1 = \chi|\ |^s \quad \text{with } \chi \text{ unitary}, \ s \in \mathbb{R}$$

Then $\chi_2 = \chi|\ |^{-s}$ and

$$V(\chi_1, \chi_2) = V(\chi|\ |^s, \chi|\ |^{-s}) \cong V(|\ |^s, |\ |^{-s}) \otimes \chi \circ \det$$

Can show, since χ is unitary, $V(\chi_1, \chi_2)$ is unitarizable $\Leftrightarrow$ $V(|\ |^s, |\ |^{-s})$ is unitarizable.

Prop $V(|\ |^s, |\ |^{-s})$ is unitarizable $\Leftrightarrow$ $-\frac{1}{2} < s < \frac{1}{2}$.

Idea The pairing gives an iso $V(|\ |^s, |\ |^{-s}) \to V(|\ |^s, |\ |^{-s})^\vee$

by $\phi \mapsto (\psi \mapsto \langle \phi, \bar{\psi} \rangle)$

Since its irred, such a map is unique up to scalar.

We saw one last time

$$T_w(\phi)(g) = \int_F \phi\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) dx$$

So if $\langle , \rangle$ exists it must equal

$$\langle \phi, \psi \rangle = \int_K (\alpha T_w(\phi)(k)) \bar{\psi}(k)\, dk$$

for some α. Hard part: Show this is true def for some α $\iff -\frac{1}{2} < s < \frac{1}{2}$.

Upshot $V(\chi_1, \chi_2)$ is irred and unitorizable
$\iff$ either 1. χ_1 and χ_2 are unitary
2. $\chi_1 = \chi|\,|^s$, $\chi_2 = \chi|\,|^{-s}$ with χ unitary and $-\frac{1}{2} < s < \frac{1}{2}$ real. (called complimentary series is $s \neq 0$)

Now let's say $V(\chi_1, \chi_2)$ is unitorizable and unramified.
Let $\alpha_i = \chi_i(\varpi)$. Unitary $\Rightarrow$ $\chi_1\chi_2$, central char, is unitary
So $|\alpha_1\alpha_2| = 1$. By last time, the Hecke operator

$\mathbb{1}_{K(\varpi\ 1)K}$, $K = GL_2(\mathcal{O}_F)$, acts by

$$q^{\frac{1}{2}}(\alpha_1 + \alpha_2) \qquad q = \# \mathcal{O}_F/\varpi$$

Now $V(\chi_1, \chi_2)$ is tempered $\iff$ χ_1, χ_2 are unitary
$\iff |\alpha_1| = |\alpha_2| = 1$
$\Rightarrow |q^{\frac{1}{2}}(\alpha_1 + \alpha_2)| \leq 2q^{\frac{1}{2}}$

If $V(\chi_1, \chi_2)$ is not tempered, then
$\chi_1 = \chi|\,|^s$, $\chi_2 = \chi|\,|^{-s}$ with χ unitary and $-\frac{1}{2} < s < \frac{1}{2}$
$s \neq 0$.
So $\alpha_1 = \beta q^s$ $\alpha_2 = \beta q^{-s}$ with $|\beta| = 1$ and
$$|q^{\frac{1}{2}}(\alpha_1 + \alpha_2)| = q^{\frac{1}{2}}(q^s + q^{-s}) > 2q^{\frac{1}{2}}$$

Thus The irred unitorizable unramified principal series $V(\chi_1, \chi_2)$ is tempered $\iff$ the eigenvalue of $\mathbb{1}_{K\left(\begin{smallmatrix} \varpi & 0 \\ 0 & 1 \end{smallmatrix}\right)K}$ is $\leq 2q^{\frac{1}{2}}$ in abs value.

Now let's say $f \in S_k(\Gamma_1(N))$ is a normalized newform of wt $k \geq 1$, level $\Gamma_1(N)$.
It can be shown that since f is a newform, it generates an irreducible cuspidal aut rep π of $GL_2(\mathbb{A}_\mathbb{Q})$
Lecture 19 $\Rightarrow \pi \cong \pi_\infty \otimes \bigotimes_p' \pi_p$

If $p \nmid N$, then π_p is unramified and can be shown to be ∞ dim, so is an unramified principal series $V(\chi_{p,1}, \chi_{p,2})$. Let's normalize π so it has unitary central character. Then we discussed that $\pi \hookrightarrow L^2(GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}_\mathbb{Q}))$.
So π, hence also π_p is unitorizable.

If $a_p = p^{th}$ Fourier coeff of f
$= T_p$-eigval of f
$= p^{\frac{k}{2}-1}$ (eigvalue of $\mathbb{1}_{K\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)K}$ on π_p)
$= p^{\frac{k-1}{2}}(\alpha_{p,1} + \alpha_{p,2})$ (Lecture 17)

Conclusion Ramanujan-Peterson Conj is true for f
$\iff \pi_p$ is tempered at all unramified p.

Conjecture (Generalized Ramanujan Conjecture)
Let F be a number field and let π be an irreducible cuspidal automorphic representation with unitary central char of $GL_n(\mathbb{A}_F)$.
Then $\pi \cong \bigotimes_v' \pi$ is tempered $\forall$ places v.

Rmk 1. If $F = \mathbb{Q}$, the up to twist, every irred cusp aut rep of $GL_2(\mathbb{A}_\mathbb{Q})$ is generated by either (a) a cuspidal modular form, or (b) a cuspidal Maass form.

(a) As above (up to ramified places) GRC at finite places $\Leftrightarrow$ RPC for the mod form.
At ∞ places, follows from fact that π_∞ is discrete series or limit of discrete series.

(b) GRC at ∞ places $\Leftrightarrow$ Selberg's eigenval $\frac{1}{4}$ Conjecture for the Maass form.

At fin places $\Leftrightarrow$??

2. GRC is false for other groups, e.g. GSp_4.