Lecture 19

G = connected reductive / F = # fld.

$$\mathcal{A}(G) := \mathcal{A}(G(F)\backslash G(\mathbb{A}_F))$$

and similarly $\mathcal{A}_0(G) = \mathcal{A}^\circ(G)$ (change to subscript) the subspace of cusp forms.

Thus in the context of $\text{(discrete sub)}\backslash\text{(Lie group)}$ can be translated to the adelic framework as follows.

<u>Thm</u> (Borel) Let U be an open compact subgroup of $G(\mathbb{A}_F^\infty)$. Then

$$G(F)\backslash G(\mathbb{A}_F^\infty)/U \quad \text{is finite.}$$

<u>Rmk</u> For $G = \mathbb{G}_m$, this is the finiteness of (say) class groups.

Say we fix representatives $t_1, \dots, t_d \in G(\mathbb{A}_F^\infty)$ of

$$G(F)\backslash G(\mathbb{A}_F^\infty)/U.$$

Define $\Gamma_i = G(F) \cap t_i U t_i^{-1}$

Then we have a homeomorphism

$$G(F)\backslash G(\mathbb{A}_F)/U \xrightarrow{\sim} \bigsqcup_{i=1}^{d} \Gamma_i \backslash G(F_\infty)$$

$$G(F) g_i t_i U \longleftarrow\!\!\shortmid\ \Gamma_i g_i$$

Here $F_\infty = \mathbb{A}_{F,\infty} = F \otimes_{\mathbb{Q}} \mathbb{R}$.

<u>Thm</u> (Harish-Chandra)

Let $J \subseteq Z(\mathfrak{g})$ be an ideal of fin codim.

Let τ be an irred unitary rep of our fixed max compact sub $K_\infty \leq G(F_\infty)$.

Let $K^\infty \leq G(\mathbb{A}_F^\infty)$ be some open compact sub.

Then

$$\mathcal{A}(J, \tau, K^\infty) := \{\phi \in \mathcal{A}(G) \mid \cdot\ J \text{ annihilates } \phi \quad \cdot\ K_\infty \text{ translates of } \phi \text{ span a rep} \cong \tau \ (\text{if } \phi \neq 0) \quad \cdot\ K^\infty \phi = \phi\}$$

is finite dimensional.

Let's fix a unitary character

$$\omega : Z(F)\backslash Z(\mathbb{A}_F) \longrightarrow S^1 \subseteq \mathbb{C}^\times$$

where Z = centre of G.

<u>Rmk</u> In what follows, it suffices to fix ω on the smaller group

$$A_G = C(\mathbb{R})^\circ = \text{connected component of } C(\mathbb{R}), \text{ where } C$$

is the max $\mathbb{Q}$-split subtorus of $\mathrm{Res}_{F/\mathbb{Q}} Z$.

What's going on? A_G is the smallest part of $Z(\mathbb{A}_F)$ such that $G(F)A_G\backslash G(\mathbb{A}_F)$ has finite volume.

For eg if $G = \mathbb{G}_m$, then

$$A_G = \mathbb{R}_{>0} \xhookrightarrow[\text{diag}]{} \mathbb{G}_m(F_\infty) = (F \otimes \mathbb{R})^\times = \prod_{v \text{ real}} \mathbb{R}^\times \times \prod_{v \text{ complex}} \mathbb{C}^\times$$

and Dirichlet's Unit Thm (or its proof) shows

$$\mathcal{O}_F^\times \mathbb{R}_{>0} \backslash (F \otimes \mathbb{R})^\times \quad \text{is finite volume.}$$

(Feel free to ignore the above Rmk)

Let's let $\mathcal{A}(G,\omega)$ and $\mathcal{A}_0(G,\omega)$ be the subspaces of $\mathcal{A}(G)$ and $\mathcal{A}_0(G)$, resp., such that

$$(*) \qquad \phi(zg) = \omega(z)\phi(g) \quad \forall z \in Z(\mathbb{A}_F),\ g \in G(\mathbb{A}_F)$$

Let's also define

$L^2(G,\omega) :=$ L^2-space of measurable $\phi : G(F)\backslash G(\mathbb{A}_F) \to \mathbb{C}$ s.t.

$$\phi(zg) = \omega(z)\phi(g) \quad \forall z \in Z(\mathbb{A}_F),\ g \in G(\mathbb{A}_F)$$

$$\text{and } \|\phi\| = \int_{Z(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)} |\phi(x)|^2\, dx < \infty \}$$

$L^2(G,\omega) \supseteq L^2_0(G,\omega)$ = cuspidal subspace, i.e.

ϕ such that for all proper parabolic subgroups $P \subsetneq G$, N = unipotent radical, and all $g \in G(\mathbb{A}_F)$

$$\int_{N(F)\backslash N(\mathbb{A}_F)} \phi(ng)\, dn = 0$$

Thm (Gelfand–Piatetski-Shapiro)
$L^2_0(G,\omega)$ is the Hilbert space direct sum of closed irreducible subrepresentations each with finite multiplicity.

Thm (Harish-Chandra)

$$A_0(G,\omega) \text{ is a dense subspace of } L^2_0(G,\omega).$$

Moreover if H is an irreducible summand of $L^2_0(G,\omega)$, then its $K = K_\infty \times K^\infty$-finite vectors ($K^\infty \leq G(\mathbb{A}_F^\infty)$ open compact) H_K are in $A_0(G,\omega)$.

Cor $A_0(G,\omega)$ decomposed as an (algebraic) direct sum of aut representations with fin mults. The association $H \longmapsto H_K$ is a bijection between irred subreps of $L^2_0(G,\omega)$ and cusp aut reps in $A_0(G,\omega)$.

Rmk What about the rest of $L^2(G,\omega)$?

$\exists$ a decomposition

$$L^2(G,\omega) = L^2_{disc}(G,\omega) \oplus L^2_{cont}(G,\omega)$$

where $L^2_{disc}(G,\omega)$ = the largest closed subspace that decomposes into a (Hilbert) direct sum of irreducible subreps.

$$L^2_{disc}(G,\omega) = L^2_0(G,\omega) \oplus L^2_{res}(G,\omega)$$

Both $L^2_{cont}(G,\omega)$ and $L^2_{res}(G,\omega)$ can be understood, as representations, using parabolic inductions and representations in $L^2_0(M,\omega')$ M is the Levi of the parabolic subgroup.

Recall that

$$G(\mathbb{A}_F) = G(F_\infty) \times G(\mathbb{A}_F^\infty)$$
$$= \prod_{v|\infty} G(F_v) \times \prod_{v\nmid\infty}' G(F_v)$$

where $g^\infty = (g_v) \in \prod_{v\nmid\infty} G(F_v)$ means $g_v \in G(\mathcal{O}_{F_v})$ for all but fin many v, $G(\mathcal{O}_{F_v}) = G(F_v) \cap GL_N(\mathcal{O}_{F_v})$ for some fixed $G \hookrightarrow GL_N$.

Assume K_∞ and K^∞ are chosen such that

$K_\infty = \prod_{v|\infty} K_v$ with K_v max compact in $G(F_v)$

$K^\infty = \prod_{v\nmid\infty} K_v$ with K_v open compact in $G(F_v)$.

Note $K_v = G(\mathcal{O}_{F_v})$ for all but fin many $v\nmid\infty$.
Indeed, $U := \prod_{v\nmid\infty} G(\mathcal{O}_{F_v})$ is open compact in $G(\mathbb{A}_F^\infty)$

So $V := U \cap K^\infty$ is open compact.

$\Rightarrow V$ is finite index in U and in K^∞.

But $V = \prod_{v\nmid\infty} V_v$ with $V_v = G(\mathcal{O}_{F_v}) \cap K_v$

$$\Rightarrow G(\mathcal{O}_{F_v}) \cap K_v = G(\mathcal{O}_{F_v})$$
$$= K_v$$

for all but fin many v.

Thm (Flath) If π is an (irred) automorphic rep of $G(\mathbb{A}_F)$, then

$$\pi \cong \pi_\infty \otimes \pi^\infty$$
$$\cong \bigotimes_{v|\infty} \pi_v \otimes \bigotimes_{v\nmid\infty}' \pi_v$$

Where π_v is

- an irred admissible $(\mathfrak{g}_v, K_v)$-mod for $v|\infty$ ($\mathfrak{g}_v = \mathrm{Lie}\, G(F_v)$) on $\mathbb{C}$-vect sp.
- an irred smooth admissible $G(F_v)$-rep (to be defined next time) on $\mathbb{C}$-vect sps.

Here $\bigotimes_{v\nmid\infty}'$ is defined as follows.

$\exists$ a finite set S_0 of $v\nmid\infty$ such that if $v \notin S_0$,

$$K_v = G(\mathcal{O}_{F_v}) \text{ and } \dim_{\mathbb{C}} \pi_v^{K_v} = 1$$

Choose $0 \neq x_v \in \pi_v^{K_v}$ for all $v\nmid\infty$, $v \notin S_0$.
Then for any $S' \supseteq S \supseteq S_0$ finite sets of fin places, define

$$\pi_S = \bigotimes_{v\in S} \pi_v \qquad \pi_{S'} = \bigotimes_{v\in S'} \pi_v$$

and a map

$$\pi_S = \bigotimes_{v \in S} \pi_v \longrightarrow \pi_{S'} = \bigotimes_{v \in S'} \pi_v$$

$$\bigotimes_{v \in S} y_v \longmapsto \left(\bigotimes_{v \in S} y_v\right) \otimes \left(\bigotimes_{v \in S' \setminus S} x_v\right)$$

Define

$$\bigotimes_{v \nmid \infty}' \pi_v := \varinjlim_{S} \pi_S$$