Automorphic Representations

Lecture 1 - Intro

Automorphic representations tie together 2 threads.

Thread 1 : Various types of modular forms.

Elliptic modular forms :

$$\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\} = \mathcal{H} \circlearrowleft \underset{\substack{\cup\!\!\!\mid \\ SL_2(\mathbb{Z})}}{SL_2(\mathbb{R})} \qquad \gamma z = \frac{az+b}{cz+d}, \ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

A modular form of level 1 and weight $k \geq 1$ is a holomorphic fcn

$$f : \mathcal{H} \to \mathbb{C}$$

s.t. 1. $f(\gamma z) = (cz+d)^k f(z) \quad \forall \gamma \in SL_2(\mathbb{Z})$

2. $f(z)$ is bounded as $\mathrm{Im}(z) \to \infty$

Siegel modular forms : $g \geq 1$

$$\{Z \in M_{g \times g}(\mathbb{C}) \mid {}^tZ = Z, \ \mathrm{Im}(Z) \text{ is +ve definite}\}$$

$$\| $$

$$\mathcal{H}_g \circlearrowleft Sp_{2g}(\mathbb{R}) = \{\gamma \in GL_{2g}(\mathbb{R}) \mid {}^t\gamma \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} \gamma = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}\}$$

= aut that preserve the symplectic form

$$\langle x, y \rangle = \sum_{i=1}^{g} x_i y_{g+i} - \sum_{i=1}^{g} x_{i+g} y_i$$

$$\gamma Z = (AZ+B)(CZ+D)^{-1}, \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

A Siegel mod form of level 1 and weight $k \geq 1$ is a holomorphic fcn

$$f : \mathcal{H}_g \to \mathbb{C}$$

such that $f(\gamma Z) = \det(CZ+D)^k f(Z) \quad \forall \gamma \in Sp_{2g}(\mathbb{Z})$

+ growth condition if $g = 1$.

But, there are more general weights here.

Let $\rho : GL_g(\mathbb{C}) \to GL(V) = GL_N(\mathbb{C})$ be a rational or algebraic representation, i.e. hom ρ is given by polynomials in the entries of the matrices

<u>Eg</u> • $k \in \mathbb{Z}$, $\det^k : GL_g(\mathbb{C}) \to \mathbb{C}^\times$
$$\gamma \mapsto (\det \gamma)^k$$

• $k \in \mathbb{Z}_{\geq 0}$, $\mathrm{Sym}^k : GL_g(\mathbb{C}) \to GL(\mathrm{Sym}^k \mathbb{C}^g) = GL_{\binom{g+k-1}{k}}(\mathbb{C})$

If $\mathbb{C}^g$ has basis $e_1, \ldots, e_g$, then $\mathrm{Sym}^k \mathbb{C}^g$ has basis $\{ e_1^{i_1} \cdots e_g^{i_g} \mid i_1 + \cdots + i_g = k \}$

• The irred rational reps of $GL_1(\mathbb{C}) = \mathbb{C}^\times$ are

$$\mathbb{C}^\times \to \mathbb{C}^\times$$
$$z \mapsto z^k \qquad k \in \mathbb{Z}$$

A Siegel modular form of weight ρ is a holomorphic fcn $\quad f : \mathcal{H}_g \to V$

s.t. $f(\gamma Z) = \rho(CZ+D) f(Z) \quad$ + growth cond if $g = 1$

Hilbert modular forms: $F/\mathbb{Q}$ fin totally real, $d=[F:\mathbb{Q}]$

$$\mathcal{H}^d = \prod_{i=1}^{d} \mathcal{H} \circlearrowleft \prod_{i=1}^{d} GL_2(\mathbb{R})^+$$

$$\rotatebox{90}{$=$}$$

$$\{\gamma \in GL_2(\mathbb{R}) \mid \det \gamma > 0\}$$

Note $GL_2(\mathcal{O}_F)^+ \hookrightarrow \prod_{i=1}^{d} GL_2(\mathbb{R})^+$

$$\gamma \longmapsto (\sigma_1(\gamma), \ldots, \sigma_d(\gamma)) \qquad \mathrm{Hom}(F, \mathbb{R}) = \{\sigma_1, \ldots, \sigma_d\}$$

If $k = (k_1, \ldots, k_d)$ with $k_i \geq 1$, $k_i \equiv k_j \bmod 2$, then a Hilbert modular form of wt k and level 1 is a holomorphic fcn

$$f : \mathcal{H}^d \to \mathbb{C}$$

such that

$$\forall \gamma \in GL_2(\mathcal{O}_F)^+, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$f(\sigma_1(\gamma) z_1, \ldots, \sigma_d(\gamma) z_d) = \prod_{i=1}^{d} (\det \sigma_i(\gamma))^{-k_i/2} (\sigma_i(c) z_i + \sigma_i(d))^{k_i} f(z_1, \ldots, z_d)$$

+ growth if $d=1$.

But not good enough for Hecke theory if the strict class # of F is >1.

In this case, better to consider a tuple

$$F = (f_1, \ldots, f_h) \qquad h = \text{strict class \# of } F$$

with f_i as above but $\forall \gamma \in \left\{ \begin{pmatrix} \mathcal{O}_F & \mathfrak{c}_i^{-1} \\ \mathfrak{c}_i & \mathcal{O}_F \end{pmatrix} \,\middle|\, ad - bc \in \mathcal{O}_F^{\times,+} \right\}$

with $\{[\mathfrak{c}_1], \dots, [\mathfrak{c}_h]\}$ = strict class group.

Bianchi modular forms: Here $F/\mathbb{Q}$ is imag quad and we use $SL_2(\mathbb{C}) \supset SL_2(\mathcal{O}_F)$ or $GL_2(\mathbb{C}) \supseteq GL_2(\mathcal{O}_F)$

acting on

$$\mathbb{H}^3 = \{(x_1, x_2, y) \in \mathbb{R}^3 \mid y > 0\}$$

via complicated formula.

Automorphic representations provide a unifying framework.

<u>Thread</u> Representations of Lie groups.

If G is a finite group, then any irred representation V of G is finite dimensional and unitarizable (i.e. admits a G-inv inner product)

Also, if

$$R[G] = \{f : G \to \mathbb{C}\} \quad \text{with } (gf)(h) = f(hg)$$

Then $R[G] \cong \bigoplus_{V \in \hat{G}} V^{\dim V}$

$\hat{G}$ = the set of iso classes of irred reps of G

Useful for above: Can average over G, $\sum_{g \in G} \sim$

Now say G is a compact group.
Then G admitts a unique up to scaler left invariant measure μ, called Haar measure, i.e.

$$\forall g \in G,\ U \subseteq G \text{ measurable},\ \mu(gU) = \mu(U)$$

(True for any locally compact group.)
Then the Peter-Weyl Theorem tells us that any irred rep of G is fin dim and unitarizable and

$$L^2(G) \cong \widehat{\bigoplus_{V \in \widehat{G}}} V^{\dim V}$$

Useful for above: Can average over G, $\int_G \dots dg$

If G is no longer compact, these things are not true. In particular, there are interesting ∞-dim irred Hilbert space reps not appearing in $L^2(G)$.
For example if $G = SL_2(\mathbb{R})$, then $\exists$ interesting representations appearing in

$$L^2(SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R})) \qquad (gf)(h) = f(hg)$$

that do not appear in $L^2(SL_2(\mathbb{R}))$.
One reason: $SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R})$ has finite volume.
Turns out that $L^2(SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R})$ can be

described in terms of classical modular forms (and anti-holomorphic versions) and Maass forms of level 1.

Say $\Gamma \leq SL_2(\mathbb{Z})$ is a congruence subgroup.

Then $L^2(\Gamma \backslash SL_2(\mathbb{R}))$ can be described similarly.

Note if $\Gamma' \leq \Gamma \leq SL_2(\mathbb{Z})$ are congruence, then $\Gamma' \backslash SL_2(\mathbb{R}) \twoheadrightarrow \Gamma \backslash SL_2(\mathbb{R})$

and $L^2(\Gamma \backslash SL_2(\mathbb{R})) \hookrightarrow L^2(\Gamma' \backslash SL_2(\mathbb{R}))$

Can show $\varprojlim_{\Gamma \text{ congruence}} \Gamma \backslash SL_2(\mathbb{R}) \cong SL_2(\mathbb{Q}) \backslash SL_2(\mathbb{A})$

$$\mathbb{A} = \text{adeles of } \mathbb{Q}$$

$$= \mathbb{R} \times \prod_p{}' \mathbb{Q}_p = \prod_{p \leq \infty} \mathbb{Q}_p$$

Indicates that

$$L^2(SL_2(\mathbb{Q}) \backslash SL_2(\mathbb{A}))$$

is interesting.

Can do this for any reductive group G

<u>Eg</u>
- SL_2 or SL_n
- GL_2 or GL_n
- Sp_{2g} or GSp_{2g}

and over any # field F.

Then elliptic mod forms $\leftrightsquigarrow SL_{2/\mathbb{Q}}, GL_{2/\mathbb{Q}}$

Siegel mod forms $\leftrightsquigarrow Sp_{2g/\mathbb{Q}}$ or $GSp_{2g/\mathbb{Q}}$

Hilbert mod forms $\leftrightsquigarrow GL_{2/F}$, F = tot real

Bianchi mod forms $\leftrightsquigarrow SL_{2/F}$ or $GL_{2/F}$, F = imag quad

For general $G_{/F}$, roughly an aut rep π of $G(\mathbb{A}_F)$, $\mathbb{A}_F = \mathbb{A} \otimes_{\mathbb{Q}} F$, is an irred rep of $G(\mathbb{A}_F)$ appearing in

$$L^2(G(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)) \qquad Z = \text{centre of } G$$

For such π, since

$$G(\mathbb{A}_F) \cong \prod_{\substack{\text{places } v \\ \text{of } F}}{}' G(F_v) \qquad F_v = \text{completion of } F \text{ at } v$$

you can prove (and we hope to)

$$\pi = \bigotimes_v{}' \pi_v \quad \text{with } \pi_v \text{ an irred rep of } G(F_v)$$

and

$\pi_\infty = \bigotimes_{v \mid \infty} \pi_v$ encodes + refines the weight and holomorphicity or not

$\pi^\infty = \bigotimes_{v \nmid \infty}{}' \pi_v$ encodes + refines the level and Hecke action.

For $G = GL_n$, such π have associated L-functions that admit meromorphic continuation to $\mathbb{C}$ with functional equation and the philosophy of Langlands predicts that many L-functions of number theoretic interest (e.g. Hasse-Weil L-fcns of smooth proper varieties (# flds, Artin L-fcns) equal automorphic L-functions.