Lecture 2 - Algebraic groups

Fix $k \subseteq K$ fields with K algebraically closed (e.g. $K = \bar{k}$ = alg closure of k; $k = \mathbb{Q}$, $K = \mathbb{C}$)

$\mathbb{A}^n = \mathbb{A}^n_K = K^n$ with the Zariski topology, i.e. closed sets are

$$Z(f_1, \ldots, f_m) := \{ P \in \mathbb{A}^n \mid f_1(P) = \cdots = f_m(P) = 0 \}$$
$$f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$$

or
$$Z(I) := \{ P \in \mathbb{A}^n \mid f(P) = 0 \quad \forall f \in I \}$$
$$I \text{ an ideal of } K[x_1, \ldots, x_n]$$

An <u>affine variety</u> is a closed subset of $\mathbb{A}^n$ (some n)

Morphisms of varieties $\phi : X \to Y$ are given locally by tuples of rational functions f/g, $f, g \in K[x_1, \ldots, x_n]$

<u>Hilbert's Nullstellensatz</u> $\Rightarrow$ $I \longmapsto Z(I)$ is a bijection from radical ideals of $K[x_1, \ldots, x_n]$ and closed subvarieties X of $\mathbb{A}^n$, and setting $K[X] := K[x_1, \ldots, x_n] / I(X)$, there is an antiequivalence of categories

$$\{\text{Affine varieties} / K\} \longrightarrow \{\text{Finitely} \overset{\text{commutative}}{\text{ generated}}\ k\text{-algs}\}$$
$$X \longmapsto K[X]$$

We say X is defined over k if we are given the data of a finitely generated k-alg $k[X]$ and an iso

$k[X] \otimes_k K \cong K[X]$, and a morphism $\varphi: X \to Y$ of affine varieties over k is defined over k if the K-alg morph $\varphi^*: K[Y] \to K[X]$ is induced by a k-alg morph $k[Y] \to k[X]$.

An <u>(affine) algebraic group</u> over k is an affine variety G over k and morphisms
$$m: G \times G \to G, \quad i: G \to G, \quad e \in G$$
defined over k, making G a group.

<u>Eg</u> • $\mathbb{G}_a(K) := K$ with addition is an alg group.
$$= \mathbb{A}^1$$

• $\mathbb{G}_m(K) = K^\times$ with mult is an alg group
$$= Z(xy-1) \subseteq \mathbb{A}^2$$

• $SL_n(K) = Z(\det(x_{ij}) - 1) \subseteq \mathbb{A}^{n^2}$ and
$GL_n(K) = Z(y\det(x_{ij}) - 1) \subseteq \mathbb{A}^{n^2+1}$
are alg groups. Indeed m and i are given by poly eqns (Cramer's rule for i)

• We can then define (Zariski) closed subgroups of GL_n:

– $U_n(K) = \left\{ \begin{pmatrix} 1 & * & \cdots & * \\ & 1 & \ddots & \vdots \\ & & \ddots & * \\ 0 & & & 1 \end{pmatrix} \in GL_n(K) \right\}$

- If $n=2m$ is even

$$Sp_n(K)=\{g\in GL_n(K) \mid {}^t g J g = J\} \quad J=\begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$$

$$GSp_n(K)=\{g\in GL_n(K) \mid {}^t g J g = \lambda J \text{ for some } \lambda\in K^\times\}$$

If G is an alg group over k, the equiv of cats above

$\exists$ k-alg morphisms

$$\mu: k[G]\to k[G\times G]:=k[G]\otimes_k k[G]$$
$$\iota: k[G]\to k[G]$$
$$\varepsilon: k[G]\to k$$

Satisfying axioms that correspond to group axioms under our antiequivalence of categories.

Then μ, ι, ε, equip $\mathrm{Hom}_{k\text{-alg}}(k[G],R)$ with the structure of a group for any k-alg R.

We get a functor

$$\begin{aligned} G: k\text{-Alg} &\mapsto \text{Grps} \\ R &\mapsto \mathrm{Hom}_{k\text{-alg}}(k[G],R) \end{aligned}$$

In particular, let $g,h\in G(R):=\mathrm{Hom}_{k\text{-alg}}(k[G],R)$, then $gh\in G(R)$ is the hom

$$k[G]\xrightarrow{\mu} k[G]\otimes_k k[G] \xrightarrow{g\otimes h} R$$
$$x\otimes y \longmapsto g(x)h(y)$$

Eg • $\mathbb{G}_a$ is defined over k, $k[\mathbb{G}_a] = k[x]$

$$\mu: k[x] \to k[x] \otimes k[x] \qquad \iota: k[x] \to k[x] \quad \varepsilon(x) = 0$$
$$x \mapsto x \otimes 1 + 1 \otimes x \qquad\qquad x \mapsto -x$$

$\mathbb{G}_a(R) = R$ with addition

• $\mathbb{G}_m$ has $k[\mathbb{G}_m] = k[x,y]/(xy-1) \cong k[t, t^{-1}]$

$\mu(t) = t \otimes t$, $\iota(t) = t^{-1}$ $\quad \varepsilon(t) = 1$

$\mathbb{G}_m(R) = R^\times$ under mult.

• GL_n, $k[GL_n] = k[x_{ij}, \det(x_{ij})^{-1}]$

$\mu(x_{ij}) = \sum_{m=1}^{n} x_{im} \otimes x_{mj}$ $\quad \iota$ = complicated

$$\varepsilon(x_{ij}) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$GL_n(R) = GL_n(R)$

A morphism of alg groups $/k$
$$\varphi: G \to H$$
is a morphism of functors, i.e. group hom $\varphi_R: G(R) \to H(R)$ for every k-alg R that is natural in R.

Eg $\det: GL_n \to \mathbb{G}_m$ is a morphism

Thm An (affine) alg group over k is isomorphic to a closed subgroup, defined over k, of some GL_n.

Idea Can find $f_1, \dots, f_n \in k[G]$ such that $k[G] = k[f_1, \dots, f_n]$ and $\mu(f_j) = \sum_{i=1}^{n} f_i \otimes m_{ij}$ with $m_{ij} \in k[G]$ (takes some work).

For any $g \in G(K)$, $G(K) \to G(K)$ a morphism of varieties
$$h \mapsto hg$$
$\Rightarrow$ $\rho_g : K[G] \to K[G]$ a k-alg morphism, and
$$(\rho_g f_j)(h) = f_j(hg) = \sum_i f_i(h) \otimes m_{ij}(g) \in K$$
$\Rightarrow$ $\rho_g f_j = \sum_i m_{ij}(g) f_i$

We get a morphism of alg groups
$$\varphi : G \to GL_n \quad g \mapsto (m_{ij}(g))$$
defined $/k$ since the $m_{ij} \in k[G]$.

Further $\varphi^* : k[GL_n] = k[x_{ij}, (\det)^{-1}] \to k[G]$
$$x_{ij} \mapsto m_{ij}$$
and since $f_j(g) = f_j(eg) = \sum_i f_i(e) m_{ij}(g)$ we see that $f_j = \sum_{i=1}^n f_i(e) m_{ij}$. Hence
$$k[G] = k[f_1, \ldots, f_n] = k[m_{ij}] \text{ and } \varphi^* \text{ is surj.} \quad \square$$

<u>Aside</u> We can think of elements $f \in k[G]$ or $K[G]$ as functions on $G(K)$ as follows: $g \in G(K) = \mathrm{Hom}_{k\text{-alg}}(k[G], K)$,
$$f(g) := g(f)$$

Recall that for $g \in GL_n(K)$, $\exists$ unique factorization $g = g_s g_u$, called the Jordan decomposition, with g_s semisimple and g_u unipotent.

Thm (Jordan decomposition) Let G be an alg grp over k. Any $g \in G(K)$ has a unique factorization $g = g_s g_u$ in $G(K)$ that is the Jordan decomposition under any embedding $G \hookrightarrow GL_n$. If $g \in G(k)$ and $\operatorname{char}(k) = 0$, then $g_s, g_u \in G(k)$.

Also the decomposition is preserved under any hom $G \to H$ of alg groups over k.

Recall/learn, a Lie algebra over k is a k-vect space $\mathfrak{g}$ (for us: fin dim) equipped with a k-bilinear map

$$[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

that is
- $[x,x] = 0 \quad \forall\, x \in \mathfrak{g}$
- $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ $\forall\, x,y,z \in \mathfrak{g}$

Eg If A is any associative alg over k (not nec commutative), then $[x,y] = xy - yx$ defines a Lie alg structure on A.

Eg $A = M_n(k) = n \times n$-matrices over k.

Let G be an alg group over k and let

$k[\varepsilon] = k[X]/(X^2)$, so $\varepsilon^2 = 0$ and we have a k-alg hom $k[\varepsilon] \to k$, $\varepsilon \mapsto 0$. Let

$$L(G) := \ker\left(G(k[\varepsilon]) \to G(k)\right)$$

$$= \{\varphi : k[G] \to k[\varepsilon] \mid \varphi \text{ composed with } \underset{\varepsilon \mapsto 0}{k[\varepsilon] \to k} \text{ is the counit } k[G] \xrightarrow{\varepsilon} k\}$$

(apologies for the two ε's!)

If $m_e = \ker(\text{counit})$, then φ above maps m_e to $\varepsilon k[\varepsilon]$ and factors through

$$k[G]/m_e^2 \cong k \oplus m_e/m_e^2 \ni (a,b) \xmapsto{\varphi} a + D(b)\varepsilon$$

with $D(b) \in k$. The map $\varphi \mapsto D$ is a bijection, so

$$\mathrm{Lie}(G) := \mathrm{Hom}_{k\text{-lin}}(m_e/m_e^2, k) \cong L(G)$$

a fin dim k-vect sp.