Lecture 3

Eg • $G = GL_n$

$\mathfrak{gl}_n := \mathrm{Lie}(G) \cong M_n(k)$. Why

$g \in L(G) = \ker(GL_n(k[\varepsilon]) \to GL_n(k))$

$g = I + \varepsilon X$, $X \in M_n(k)$, for any such X,

$I + \varepsilon X \in L(G)$ since $(I+\varepsilon X)(I - \varepsilon X) = I + \varepsilon(X - X) + \varepsilon^2 X$

$= I$ since $\varepsilon^2 = 0$

• $G = SL_n$, $\mathfrak{sl}_n := \mathrm{Lie}(SL_n) \subseteq M_n(k)$

$I + \varepsilon X$, $X \in M_n(k)$, $\det(I + \varepsilon X) = \det\begin{pmatrix} 1+\varepsilon x_{11} & \varepsilon x_{12} & \cdots & \varepsilon x_{1n} \\ \varepsilon x_{21} & 1+\varepsilon x_{22} & \cdots & \varepsilon x_{2n} \\ \vdots & & \ddots & \\ \varepsilon x_{n1} & \varepsilon x_{n2} & \cdots & 1+\varepsilon x_{nn} \end{pmatrix}$

$= \prod_{i=1}^{n}(1 + \varepsilon x_{ii}) + 0$

$= I + \varepsilon\left(\sum_{i=1}^{n} x_{ii}\right)$

$\mathfrak{sl}_n = \{X \in M_n(k) \mid \mathrm{tr}\, X = 0\}$

• $G = Sp_{2n} = \{g \in GL_{2n} \mid {}^t g J g = J\}$ $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$

$\mathfrak{sp}_{2n} := \mathrm{Lie}(G)$, $g = I + \varepsilon X$

$J = {}^t g J g = {}^t(I + \varepsilon X) J (I + \varepsilon X)$

$= (I + \varepsilon\, {}^t X) J (I + \varepsilon X)$

$= J + \varepsilon({}^t X J + J X)$

$\mathfrak{sp}_{2n} = \{X \in M_{2n}(k) \mid {}^t X J + J X = 0\}$

$$\boxed{\begin{array}{l}\text{Aside} \\ e^{\varepsilon X} := I + \varepsilon X\end{array}}$$

Can check $G \mapsto \mathrm{Lie}(G)$ is functorial in G.
Also, can replace $k[\varepsilon]$ above with $R[\varepsilon] := R[X]/(X^2)$ for any k-alg R, and we get a functor

$$\mathfrak{g} : k\text{-alg} \to k\text{-vector spaces}$$
$$R \mapsto \mathfrak{g}(R) := \mathrm{Lie}(G) \otimes_k R$$

Note $R \subseteq R[\varepsilon]$, so $G(R) \subseteq G(R[\varepsilon])$ acts by conj on

$$\ker\left(G(R[\varepsilon]) \to G(R)\right)$$

This action is R-linear and functorial in R

$$\Rightarrow \mathrm{Ad} : G \to GL_{\mathfrak{g}} \cong GL_{\dim \mathfrak{g}}$$

called the adjoint action.
Applying Lie, we get

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{Lie}(GL_{\mathfrak{g}}) = \mathrm{End}(\mathfrak{g})$$

Define for $x, y \in \mathfrak{g}$

$$[x, y] = \mathrm{ad}(x)(y)$$

<u>Exercise</u> Check that for GL_n, this is

$$[X, Y] = XY - YX$$

<u>Rmk</u> For arbitrary G, $G \hookrightarrow GL_n \Rightarrow \mathfrak{g} \hookrightarrow \mathfrak{gl}_n$ that takes $[\,,]$ on $\mathfrak{g}$ to $[\,,]$ on $\mathfrak{gl}_n$.

<u>Fact</u> If $k = \mathbb{R}$ or $\mathbb{C}$ and $K = \mathbb{C}$, can show $\mathfrak{g}$ agrees with the usual construction in Lie theory.

Rmk from last time: How do we know when $g \in G(K)$ is semisimple (resp unipotent) from $K[G]$?

Recall, we have an action ρ of G on $K[G]$ by

$$\rho_g : K[G] \to K[G] \text{ given by } (gf)(x) = f(xg)$$

This is a linear action of G on $K[G]$. This action is locally finite: any fin dim sub K-vector space W of $K[G]$ is contained in a fin dim G-stable subspace V of $K[G]$. Then g is semisimple (resp unipotent) if it is so on any fin dim G-stable subspace of $K[G]$.

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Thm Let $H \leq G$ be a (affine) alg subgroup over k, closed in G. Then $\exists$ a smooth quasi-projective (i.e. open in a projective variety) variety G/H over k and a morphism $\pi: G \to G/H$ of k-varieties satisfying the following

1. π is surjective on K-points and universal for morphisms from G that are constant on cosets of H
2. If H is normal in G, then G/H is an (affine) alg k-group.

Warning In 2, $G(K)/H(K) = (G/H)(K)$ but may have

$$G(k)/H(k) \subsetneq (G/H)(k)$$

Eg Say $\operatorname{char}(k) \nmid n$, μ_n = group of n^{th} roots of 1 in K

$$= \{x^n - 1 = 0\}$$

Note μ_n is the center of SL_n. Can show that
$$SL_n/\mu_n \cong PGL_n$$
Note $SL_n(K) \twoheadrightarrow PGL_n(K)$ because any $g \in GL_n(K)$ by taking an n^{th} root α of $\det g$, we have
$$g = \alpha h, \quad h \in SL_n(K)$$
And $\ker(SL_n(K) \to PGL_n(K))$ is μ_n.
But say $k = \mathbb{Q}$ and $n = 2$, then $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in PGL_2(\mathbb{Q})$ is not in the image of $SL_2(\mathbb{Q})$.

<u>Eg</u> Say $B_2 = \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\} \subseteq GL_2$. Then $GL_2/B_2 \cong \mathbb{P}^1$
$GL_2(K)$ acts on the set of lines in K^2 and $B_2(K)$ is the stabilizer of $K\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
More generally, if
$$H = \left\{\begin{pmatrix} * & * & \cdots & * \\ 0 & & & \\ \vdots & & \boxed{GL_{n-1}} & \\ 0 & & & \end{pmatrix}\right\} \subseteq GL_n, \quad GL_n/H \cong \mathbb{P}^{n-1}$$

<u>Def</u> An alg group G is
1. <u>Solvable</u> if $\exists$ closed subgroups
$$0 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_r = G$$
with G_{i-1} normal in G_i and G_i/G_{i-1} commutative.
2. <u>Unipotent</u> if every $g \in G(K)$ is unipotent.
3. A <u>torus</u>, if $G \cong \mathbb{G}_m^r$ over K. We say G is a <u>split torus</u>, if $G \cong \mathbb{G}_m^r$ over k.

Eg • $U_n = \left\{ \begin{pmatrix} 1 & & * \\ & 1 & \\ 0 & & \ddots \\ & & & 1 \end{pmatrix} \right\} \subseteq GL_n$ is unipotent

• $B_n := \left\{ \begin{pmatrix} * & * & \cdots & * \\ & * & & * \\ 0 & & \ddots & \vdots \\ & & & * \end{pmatrix} \right\} \subseteq GL_n$ is solvable

$$0 \subseteq G_1 = \left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 & * \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \right\} \subseteq G_2 \left\{ \begin{pmatrix} 1 & 0 & \cdots & * & * \\ 0 & 1 & \cdots & 0 & * \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \right\}$$

$$\subseteq \cdots \subseteq G_{n-1} = U_n \subseteq B_n$$

Note, here $G_i / G_{i-1} \cong \mathbb{G}_a^i$ if $i < n$

$$G_n / G_{n-1} = B_n / U_n \cong \mathbb{G}_m^n$$

• $k = \mathbb{R}$, $K = \mathbb{C}$, define

$$G = SO_2 = \{ x^2 + y^2 = 1 \} \subseteq \mathbb{C}^2$$

is an alg $\mathbb{R}$-group with $\mathbb{R}[G] = \mathbb{R}[x,y]/(x^2+y^2-1)$

and group structure $\hookrightarrow SL_2$

$$(x,y) \longmapsto \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

<u>Exercise</u> SO_2 is a torus that is not split over $\mathbb{R}$.

(In part, $SO_2(\mathbb{C}) \cong \mathbb{C}^\times$ but $SO_2(\mathbb{R}) \not\cong \mathbb{R}^\times$)

as alg groups not iso as abstract groups

<u>Thm</u> (Lie-Kolchin) Let G be a connected alg subgroup of GL_n.

1. If G is unipotent, then it is conjugate in $GL_n(K)$ to a subgroup of U_n.

2. If G is solvable, then it is conjugate in $GL_n(K)$ to a subgroup of B_n.

Proof 1. Suffices to show that if V is a fin dim K-vector space with linear action of $G = G(K)$, the $\exists$ a basis for V, for which image of G in $GL(V) \cong GL_{\dim V}(K)$ is in $U_{\dim V}(K)$.

Then by induction on the dimension of V, it suffices to show that the only irred rep of G is the trivial one.

Take an irred rep V of G. Take $g \in G$ and write $g = 1 + n$ with n nilpotent.

Then $tr_V(g) = tr_V(1) + tr_V(n) = \dim V$

Then this holds $\forall\, g \in G$. In part, if $g' \in G$ is another element,

$(*)$ $tr_V(ng') = tr_V((g-1)g') = tr_V(gg') - tr_V(g') = 0$

Burnside's Thm says that since V is irred,

$$\mathrm{End}_K(V) = K\text{-span of } g \in G \text{ (acting on } V)$$

Then $(*) \Rightarrow tr_V(nh) = 0 \quad \forall\, h \in \mathrm{End}_K(V)$.

$\Rightarrow n = 0$ on V

$\Rightarrow g = 1$ on V $\forall g \in G$. $\square$

2. Since $G \subseteq GL_n$ acts on $\mathbb{P}^{n-1}$, 2 is a consequence of Borel's Fixed Point Theorem: a solvable connected alg group acting on a complete (e.g. projective) variety has a fixed point. + Induction. $\square$

<u>Defs</u> • A <u>maximal torus</u> (resp. <u>k-torus</u>) in G is a closed K-subgroup (resp. k-subgroup) that is a torus and maximal wrt these properties.

• A <u>Borel subgroup</u> is a closed connected solvable K-subgroup of G that is maximal wrt these properties.

• A <u>parabolic subgroup</u> is a closed K-subgroup s.t. G/P is projective.

<u>Thms</u> 1. A maximal k-torus is a maximal torus.

2. A maximal torus is contained in a Borel subgroup and is the maximal torus of the Borel subgroup.

3. Two pairs (T, B) and (T', B') of maximal tori in Borel subgroups are conjugate by an element of $G(K)$.

4. A closed subgroup P of G over k is parabolic $\Leftrightarrow$ it contains a Borel subgroup. In particular, a Borel subgroup is a minimal parabolic (G/B is called the <u>flag variety</u> of G)

5. If P is a parabolic subgroup, $N_G(P) = P$.

<u>Eg</u> • In GL_n (or SL_n), the diagonal matrices are a maximal torus. Upper triangular matrices B_n are a Borel. The parabolics are stabilizers of flags $0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_r = K^n$

In $GL_n(K)$, this is conj to

$$\left\{ \begin{pmatrix} \boxed{GL_{n_1}} & & * \\ & \boxed{GL_{n_2}} & \\ 0 & & \ddots \; \boxed{GL_{n_r}} \end{pmatrix} \right\} \subseteq GL_n, \quad n_1 + \cdots + n_r = n$$

• In Sp_{2n} or GSp_{2n}, the parabolics are stabilizers of isotropic flags

$$0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_r = K^{2n}$$

isotropic means the bilinear pairing $\langle x, y\rangle = {}^t x J y$ is identically 0 on all F_i with $i < r$.

<u>EgEg</u> Sp_4. There are 2 conj of proper maximal Parabolics. $K^4 = Ke_1 \oplus Ke_1 \oplus Ke_3 \oplus Ke_4$.

Siegel parabolic is the stabilizer of

$$0 \subsetneq Ke_1 \oplus Ke_2 \subsetneq K^4, \text{ call it } P_{Si}$$

Klingen parabolic is the stabilizer of

$$0 \subsetneq Ke_2 \subsetneq K^4, \text{ call it } P_{Kl}$$

Borel is stabizer of

$$0 \subsetneq Ke_2 \subsetneq Ke_1 \oplus Ke_2 \subsetneq K^4, \text{ call it } B$$

$$P_{Si} = Sp_4 \cap \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \right\}$$

$$P_{Kl} = Sp_4 \cap \left\{ \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \right\}$$

$$B = Sp_4 \cap \left\{ \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \right\}$$