G as before, K a max compact sub

If H is any top^l vs we can define what it means for $f: G \to H$ to be C^∞ as follows

- Say $f: \mathbb{R}^n \to H$ and $x_0 \in \mathbb{R}^n$, then f is diff at x_0 if $\exists$ linear $f'(x_0): \mathbb{R}^n \to H$ (nec unique) such that
$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{\|x - x_0\|} = 0$$
- Say f is C^1 on open $U \subseteq \mathbb{R}^n$ if the map $U \ni x \mapsto f'(x) \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, H) = H^n$ is cts. Iterating, we can define C^r $\forall r \geq 1$, and C^∞.
- $f: G \to H$, use a chart on G.

Now say G acts ctsly via π on H. We say $v \in H$ is smooth or C^∞ if
$$g \mapsto \pi(g)v \text{ is } C^\infty$$

A matrix coefficient for π is a function $G \to \mathbb{C}$ of the form
$$g \mapsto \lambda(\pi(g)v) \quad \text{for fixed } v \in H, \lambda \in H^\vee, \text{ i.e. } \lambda \text{ a cts linear functional}$$

Now say H is a Hilbert space.

Riesz rep $\Rightarrow$ matrix coeffs are of the form
$$g \mapsto \langle \pi(g)v, w \rangle \quad \text{fixed } v, w \in H$$

Say $(e_i)_{i \geq 1}^\infty$ an orthonormal basis for H.

Then
$$g \mapsto \pi(g)v = \sum_{n=1}^\infty \langle \pi(g)v, e_i \rangle e_i$$

Using this, v is smooth $\iff$ $\forall w \in V$, the matrix coeff
$$g \mapsto \langle \pi(g)v, w \rangle \text{ is smooth}$$

Say $X \in \mathfrak{g} = \mathrm{Lie}(G)$. Recall we have
$$e^X \in G \quad \left(e^X = \sum_{n=1}^\infty \frac{1}{n!} X^n\right)$$

If $v \in H$ is smooth,
$$\pi(X)v := \lim_{t \to 0} \frac{\pi(e^{tX})v - v}{t} \quad \text{exists}$$

Notation H^∞ is the subspace of smooth vectors.

Prop 1. $X \in \mathfrak{g}$, $v \in H^\infty$, then $\pi(X)v \in H^\infty$

2. If $V \subseteq H$ stable under $\pi(X)$ $\forall X \in \mathfrak{g}$, then $V \subseteq H^\infty$.

3. $(X, v) \mapsto \pi(X)v$ defines a Lie alg representation of $\mathfrak{g}$ on H^∞, i.e. each $\pi(X)$ is linear and
$$\pi([X,Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X)$$

Proof of 1 Take $v \in H^\infty$ and let $f(g) = \pi(g)v$, so f is C^∞.

Then
$$(Xf)(g) := \lim_{t \to 0} \frac{f(ge^{tX}) - f(g)}{t} \quad \text{exists}$$
and $g \mapsto (Xf)(g)$ is C^∞

But
$$\lim_{t \to 0} \frac{f(ge^{tX}) - f(g)}{t} = \lim_{t \to 0} \frac{\pi(ge^{tX})v - \pi(g)v}{t}$$
$$= \pi(g) \lim_{t \to 0} \frac{\pi(e^{tX})v - v}{t}$$
$$= \pi(g)\pi(X)v$$

So $g \mapsto \pi(g)\pi(X)v$ is C^∞ and $\pi(X)v \in H^\infty$ $\square$

Idea for 2 Can encode partial derivatives in $\mathfrak{g}$-action. Then V stable under $\mathfrak{g} \Rightarrow$ all partial derivatives of all orders exist, hence exist and are continuous So $V \subseteq H^\infty$.

3 is a tedious computation.

Let $f \in C^\infty_{comp}(G)$, i.e. $f: G \to \mathbb{R}$ that is smooth with compact support. By Riesz rep, we let

$$\pi(f)v = \int_G f(x)\pi(x)v\,dx \in H$$

is the vector with

$$\langle \pi(f)v, w\rangle = \int_G f(x)\langle \pi(x)v, w\rangle dx \quad \forall w \in H$$

Prop For $v \in H$ and $f \in C^\infty_{comp}(G)$,

$$\pi(f)v \in H^\infty$$

Proof Want to check that $g \mapsto \pi(g)\pi(f)v$ is C^∞.

But
$$\pi(g)\pi(f)v = \int_G f(x)\pi(gx)v\,dx$$
$$= \int_G f(g^{-1}x)\pi(x)v\,dx$$

Since f is C^∞ with compact supp, can differentiate under the integral sign, and it is C^∞ in g. $\square$

Prop (Gårding) H^∞ is dense in H.

Proof Fix $v \in H$ and $\varepsilon > 0$.

$U = \{g \in G \mid \|\pi(g)v - v\| < \varepsilon\}$ is open in G.

So we can find $f \geq 0$, $f \in C^\infty_{comp}(G)$ with $\mathrm{supp}(f) \subseteq U$, and $\int_G f(x)\,dx = 1$.

Then $\pi(f)v \in H^\infty$ by above and

$$\|\pi(f)v - v\| = \left\| \int_G f(x)\pi(x)v\,dx - v \right\|$$
$$= \left\| \int_G f(x)(\pi(x)v - v)\,dx \right\|$$
$$= \left\| \int_U f(x)(\pi(x)v - v)\,dx \right\|$$
$$\leq \int_U f(x)\|\pi(x)v - v\|\,dx$$
$$\leq \varepsilon \int_G f(x)\,dx = \varepsilon \qquad \square$$

Now say H is admissible, so H as a K-rep, is

$$H \cong \hat{\bigoplus_{\tau \in \hat{K}}} V_\tau^{n_\tau} \quad \text{with } n_\tau < \infty$$

$\hat{K}$ = set of iso classes of irred unitary K-rep.

If $\tau \in \hat{K}$ is s.t. the τ-isotypic part of H is $\neq 0$, i.e. $n_\tau > 0$, τ is called a <u>K-type</u> for H.

Prop If V is the subspace of K-finite vectors in H, $V \subseteq H^\infty$.

Proof If f is any K-finite function on K (e.g. the trivial function, or a matrix coeff of an irred rep) and $h \in C^\infty_{comp}(e^{\mathfrak{p}})$, $G = K \times e^{\mathfrak{p}}$ is the Cartan decomp, then $F \in C^\infty_{comp}(G)$ given by $F(ke^X) = f(k)h(e^X)$ is K-finite.

Arguing as in Gårding's prop above ~~g~~ shows

$$H^\infty \cap V \text{ is dense in } V = \bigoplus_{\tau \in \hat{K}} V_\tau^{n_\tau} \text{ an orthog direct sum}$$

$\Rightarrow H^\infty \cap V_\tau^{n_\tau}$ is dense in $V_\tau^{n_\tau}$

But $\dim(V_\tau^{n_\tau}) < \infty \Rightarrow H^\infty \cap V_\tau^{n_\tau} = V_\tau^{n_\tau}$,

and $H^\infty \cap V = V$. $\square$

Prop Let $V \subseteq H$ be as above. V is stable under $\mathfrak{g}$.

Proof Take $v \in V$. Let $W = \mathrm{span}(Kv)$ a fin dim sp, and has an action of $\mathfrak{k} = \mathrm{Lie}(K)$.

If $X \in \mathfrak{g}$, $Y \in \mathfrak{k}$, $w \in W$,

$$\pi(Y)\pi(X)w = \pi(X)\pi(Y)w + \pi([Y,X])w$$
$$\subseteq \mathrm{span}(\pi(\mathfrak{g})W) =: W'$$

So W' is stable under $\mathfrak{k}$ and finite dimensional, so we can exponentiate and W' is stable under K.

Then for any $X \in \mathfrak{g}$, $\pi(X)v \in W'$ fin dim and K-stable, so $\pi(X)v \in V$. $\square$

Rmk A theorem of Harish-Chandra shows that if $v \in H$ is K-finite and $w \in H$, $g \mapsto \langle \pi(g)v, w\rangle$ is real analytic.

Def A $(\mathfrak{g}, K)$-module or (Harrish-Chandra module) is a $\mathbb{C}$-vector space with representations of K and $\mathfrak{g}$ such that

1. Any $v \in V$ is K-finite
2. For $v \in V$ and $Y \in \mathfrak{k} \subseteq \mathfrak{g}$,

$$\lim_{t \to 0} \frac{e^{tY} \cdot v - v}{t} = Y \cdot v$$

$\uparrow$ K-action $\quad \uparrow$ $\mathfrak{g}$-action

3. $k \in K$, $X \in \mathfrak{g}$, $v \in V$

$$k \cdot (X \cdot v) = (Ad(k) X) \cdot (k \cdot v)$$

We say V is admissible if any irred rep of K appears with fin mult in V. We say V is unitary if $\exists$ a +ve def inner product $\langle , \rangle$ on V s.t.

$$\langle kv, kw \rangle = \langle v, w \rangle \quad \forall k \in K$$

$$\langle Xv, w \rangle = -\langle v, Xw \rangle$$

Thms (Harrish-Chandra)

1. (π, H) is an admissible Hilbert sp rep of G and (π, V) is the subsace of K-fin vectors, then (π, H) is irred $\Leftrightarrow$ (π, V) is irred.
2. Two unitary admissible Hilb sp reps (π_1, H_1), (π_2, H_2) are isomorphic $\Leftrightarrow$ (π_1, V_1) and (π_2, V_2) are iso.
3. An admissible $(\mathfrak{g}, K)$-module V is the space of K-finite vectors in a unitary admissible Hilbert space rep $\Leftrightarrow$ V is unitary.