Lecture 19 – Totally real fields, base change, and JL

Previously
- Minimal modularity lifting as a consequence of an $R \cong \mathbb{T}$ theorem
- Non-minimal modularity lifting as a consequence of an $R^{red} \cong \mathbb{T}$ theorem provided we can show M_∞ has full support over

$$\operatorname{Spec} R_\infty = \operatorname{Spec}\left(\hat{\bigotimes_{v\in S}} R_v\right)[[x_1, \ldots, x_g]]$$

local lifting rings

This week we'll show how to do this in some cases and sketch the proof of

Thm Let F be a totally real fld and let $p \geq 5$ be a prime unramified in F. Let

$$\rho : G_F \to GL_2(\overline{\mathbb{Q}}_p)$$

be a cts irred rep satisfying the following.
1. ρ is unramified outside fin many primes
2. $\forall\, v|p$, $\rho|_{G_{F_v}}$ is crystalline with all labelled HT wts $= \{0,1\}$
3. $\bar{\rho}|_{G_{F(\zeta_p)}}$ is (abs) irred with adequate image.
4. $\bar{\rho} \cong \bar{\rho}_g$ for g a Hilbert modular cuspform of parallel wt 2 and level prime to p.

Then $\rho \cong \rho_f$ for f a Hilbert modular cuspform (of parallel wt 2).

Rmk Note, no assumptions on the ramification of ρ or level of g at $v \nmid p$.

We assume that we have a fixed iso $\mathbb{C} \cong \overline{\mathbb{Q}}_p$ in above and what follows.

Using cyclic base change (Saito, Shintani), we have

Thm Let L/F be a totally real solvable Galois ext. Let ρ and g be as above.

1. If $\rho_g|_{G_L}$ is irred, then $\exists$ a Hilbert modular cusp form h over L such that h is the base change of g. In particular
$$\rho_h \cong \rho_g|_{G_L}$$
2. If $\rho|_{G_L} \cong \rho_h$ for a Hilbert modular cuspform h over L, then $\rho \cong \rho_f$ for a Hilbert modular cuspform f over F.

Lemma Let K be a number field and let S be a finite set of places of K. For each $v \in S$, let K_v'/K_v be a finite ext. Then $\exists$ a finite solvable Galois ext L/K such that $\forall w$ of L above $v \in S$, $L_w \cong K_v'$ as K_v-algs.

Sketch It suffices to prove the Lemma with L given by a sequence of cyclic extensions, replacing it by its Galois closure if necessary. By induction, we are then reduced to the cyclic case, which is an application of the Grunwald-Wang Theorem. $\square$

Let $S_p = \{v|p \text{ in } F\}$, $S_\infty = \{v|\infty \text{ in } F\}$

Let Σ be a fin nonempty set of places of F containing all at which ρ or g is ramified and disjoint from S_p.

Let $M/F(\zeta_p)$ be the extension cut out by $\bar\rho|_{G_{F(\zeta_p)}}$.
Then M/F is finite Galois, so we can find a finite set V of finite places of F such that any nontrivial conj class in $\mathrm{Gal}(M/F)$ is Frob_v for some $v \in V$ and such that V is disjoint from $\Sigma \cup S_p$.

We apply the Lemma with $K = F$,
$$S = S_p \cup S_\infty \cup \Sigma \cup V$$
and (a) $v \in S_p$, $K_v' = F_v$
(b) $v \in S_\infty$, $K_v' = F_v \cong \mathbb{R}$
(c) $v \in \Sigma$, K_v'/F_v of even degree and such that $\rho|_{G_{K_v'}}$ is either unramified or unipotently ramified and similarly for ρ_g.
We assume moreover that the residue fld of K_v' has cardinality $\equiv 1 \pmod p$. (Will explain why next time)
(d) $v \in V$, $K_v' = F_v$.

Then we have L/F solvable Galois s.t.
(a) each $v | p$ in F splits completely in L, in part p is unramified in L.
(b) L/F is totally real
(c) If $\rho|_{G_L}$ is ramified at w, the ramification is unipotent.
And if g is ramified at w, g has Iwahori level
The residue field at any such w has cardinality $q_w \equiv 1 \pmod p$.
Moreover $[L:F]$ is even.
(d) $L \cap M = F$, so $\bar\rho|_{G_{L(\zeta_p)}}$ is abs irred with adequate image.

Applying the base change Thm and replacing F with L, we can assume

- $[F:\mathbb{Q}]$ is even
- letting Σ be the set of primes at which ρ or g is ramified, $\forall\, v \in \Sigma$
  - $\rho(I_v)$ is unipotent (may be trivial)
  - g has Iwahori or full level at v
  - $Nm(v) \equiv 1 \pmod p$

In particular, $\det\rho$ and $\det\rho_g$ are both finite unramified chars times ϵ^{-1}. Twisting, we can assume that
$$\det\rho = \det\rho_g = \eta\epsilon^{-1}$$
with η finite order and unramified.

We now let D be the (unique up to iso) quaternion algebra $/F$ such that

- $\forall\, v|\infty$, $D\otimes_F F_v \cong \mathbb{H}$
- $\forall\, v\nmid\infty$, $D\otimes_F F_v \cong M_2(F_v)$

We fix a max order $\mathcal{O}_D$ of D and an iso
$$\mathcal{O}_D\otimes_{\mathbb{Z}}\hat{\mathbb{Z}} \cong M_2(\mathcal{O}_F\otimes_{\mathbb{Z}}\hat{\mathbb{Z}}) \cong \prod_{v\nmid\infty} M_2(\mathcal{O}_{F_v})$$
hence an iso
$$(D\otimes_F \mathbb{A}_F^\infty)^\times \cong GL_2(\mathbb{A}_F^\infty)$$
taking $(\mathcal{O}_D\otimes_{\mathbb{Z}}\hat{\mathbb{Z}})^\times$ to $GL_2(\mathcal{O}_F\otimes_{\mathbb{Z}}\hat{\mathbb{Z}}) \cong \prod_{v\nmid\infty} GL_2(\mathcal{O}_{F_v})$.

Fix an open compact subgroup U of $(\mathcal{O}_D\otimes_{\mathbb{Z}}\hat{\mathbb{Z}})^\times$, which we identify with one of $\prod_{v\nmid\infty} GL_2(\mathcal{O}_{F_v})$.
We will make a precise choice of U later.

Now choose $E/\mathbb{Q}_p$ finite with ring of integers $\mathcal{O}$ such that ρ takes values in $GL_2(\mathcal{O})$, conjugating if necessary.

For any $\mathcal{O}$-algebra A, define

$$S_{2,\eta}(U, A) := \{ f : D^\times \backslash (D \otimes \mathbb{A}_F^\infty)^\times \to A \text{ cts such that } f(guz) = \eta^{-1}(z) f(g) \text{ for all } g \in (D \otimes_F \mathbb{A}_F^\infty)^\times, u \in U, z \in (\mathbb{A}_F^\infty)^\times. \}$$

Abusing notation, we again write η for the (finite order) character

$$\eta \circ \mathrm{Art}_F : F^\times \backslash \mathbb{A}_F^\times \to \mathcal{O}^\times$$

For any finite place v of F such that $U_v = GL_2(\mathcal{O}_{F_v})$, the double coset operators

$$T_v = \left[GL_2(\mathcal{O}_{F_v}) \begin{pmatrix} \varpi_v & \\ & 1 \end{pmatrix} GL_2(\mathcal{O}_{F_v}) \right]$$

$$S_v = \left[GL_2(\mathcal{O}_{F_v}) \begin{pmatrix} \varpi_v & \\ & \varpi_v \end{pmatrix} GL_2(\mathcal{O}_{F_v}) \right]$$

act on $S_{2,\eta}(U, A)$.

Letting $S = \{v | p\} \cup \{v : U_v \neq GL_2(\mathcal{O}_{F_v})\}$, we thus have an action of

$$\mathbb{T}^{S,\mathrm{univ}} := \mathcal{O}[\{T_v, S_v\}_{v \notin S}]$$

on $S_{2,\eta}(U, A)$.

(Note that S_v simply acts by $\eta^{-1}(\varpi_v)$, so we could have omitted these operators.)

Thm (Jacquet-Langlands) Recall we have a fixed iso $\mathbb{C} \cong \overline{\mathbb{Q}}_p$.
We have an equality

$$\left\{ \begin{array}{l} \mathcal{O}\text{-alg homs } \lambda : \mathbb{T}^{S,\mathrm{univ}} \to \overline{\mathbb{Q}}_p \text{ s.t.} \\ \lambda \text{ is the eigensystem for a Hilbert} \\ \text{mod cuspform of parallel wt } 2, \\ \text{level } U \text{ and nebentypus } \eta \end{array} \right\} = \left\{ \begin{array}{l} \mathcal{O}\text{-alg homs } \lambda : \mathbb{T}^{S,\mathrm{univ}} \to \overline{\mathbb{Q}}_p \\ \text{s.t. } \lambda \text{ is the eigensystem for an} \\ \text{eigenform } f \in S_{2,\eta}(U, \overline{\mathbb{Q}}_p) \text{ not} \\ \text{factoring through the reduced norm of } D \end{array} \right\}$$

The Hecke eigensystems that factor through the reduced norm of D are Eisenstein, i.e. have associated Galois representations that are reducible.
It thus suffices to prove that

$$\rho \cong \rho_f \quad \text{for some } f \in S_{2,\eta}(U, \mathcal{O})$$

and we can assume that $\overline{\rho} \cong \overline{\rho}_g$ for some $g \in S_{2,\eta}(U, \mathcal{O})$ (enlarging $\mathcal{O}$ if necessary).