Lecture 5

<u>Eg</u> Sp_4, $J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$, a maximal torus is

$$T = \left\{ \begin{pmatrix} a & & & \\ & b & & \\ & & a^{-1} & \\ & & & b^{-1} \end{pmatrix} \right\} \cong \mathbb{G}_m^2 \quad , \quad e_i(\mathrm{diag}(t_1,t_2,t_3,t_4)) = t_i$$

$$X(T) \cong \mathbb{Z}^2$$
$$= \mathbb{Z}^4 / \mathbb{Z}(e_1+e_3) \oplus \mathbb{Z}(e_2+e_4)$$

$$\mathfrak{sp}_4 = \{ X \in \mathfrak{gl}_4 \mid {}^tXJ + JX = 0 \}$$

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad {}^tXJ + JX = 0 \iff \begin{pmatrix} -{}^tC & {}^tA \\ -{}^tD & {}^tB \end{pmatrix} + \begin{pmatrix} C & D \\ -A & -B \end{pmatrix} = 0$$

$$\iff {}^tA = -D, \ {}^tB = B, \ {}^tC = C.$$

$$\mathfrak{sp}_4 = \mathfrak{g}_0 = \left\{ \begin{pmatrix} x & & & \\ & y & & \\ & & -x & \\ & & & -y \end{pmatrix} \right\} \circlearrowleft T \text{ acts trivially}$$

$$\oplus\ \mathfrak{g}_\alpha = \left\{ \begin{pmatrix} 0 & 0 & & \\ -x & 0 & & \\ & & 0 & x \\ & & 0 & 0 \end{pmatrix} \right\} \circlearrowleft T \text{ acts by}$$
$$\alpha = e_2 - e_1 = e_3 - e_4$$

$$\oplus\ \mathfrak{g}_\beta = \left\{ \begin{pmatrix} & & x & 0 \\ & & 0 & 0 \\ & & & \\ & & & \end{pmatrix} \right\} \circlearrowleft T \text{ acts by}$$
$$\beta = e_1 - e_3$$

$$\oplus\ \mathfrak{g}_{\alpha+\beta} = \left\{\begin{pmatrix} & & 0 & x \\ & & x & 0 \\ & & & \\ & & & \end{pmatrix}\right\} \circlearrowleft T \text{ by } e_1 - e_4 = e_2 - e_3$$
$$= \alpha + \beta$$

$$\oplus\ \mathfrak{g}_{2\alpha+\beta} = \left\{\begin{pmatrix} & & 0 & 0 \\ & & 0 & x \\ & & & \\ & & & \end{pmatrix}\right\} \circlearrowleft T \text{ by } e_2 - e_4$$
$$= 2\alpha + \beta$$

$$\oplus\ \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-\beta} \oplus \mathfrak{g}_{-\alpha-\beta} \oplus \mathfrak{g}_{-2\alpha-\beta}$$

$$\Phi(Sp_4, T) = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, -\alpha, -\beta, -\alpha-\beta, -2\alpha-\beta\}$$

Back to general picture.

$k \subseteq K$ flds, K = alg closed, G a connected alg group over k. Assume G is reductive.

The <u>rank</u> of G is r if a max torus T of G_K is $\cong \mathbb{G}_m^r$.

<u>Thm</u> If G_K is semisimple of rank 1, then

$G_K \cong SL_2$ or $G_K \cong PGL_2$.

Let $T \subseteq G$ be a max torus.

Let's assume $k = \bar{K}$, i.e. k is alg closed.

G has $X(T) = \mathrm{Hom}(T, \mathbb{G}_m)$

$$X^\vee(T) = \mathrm{Hom}(\mathbb{G}_m, T)$$

$$\langle\, , \rangle : X(T) \times X^\vee(T) \to \mathbb{Z}$$

$$\Phi(G,T) = \{ 0 \neq \alpha \in X(T) \mid \mathfrak{g}_\alpha \neq 0 \}$$

$$\Phi^\vee(G,T) = \{ 0 \neq \alpha^\vee \in X^\vee(T) \mid$$

$$\langle \alpha, \alpha^\vee \rangle = 2,\ T = \mathrm{im}(\alpha^\vee) T_\alpha \text{ where}$$

$$T_\alpha = \ker(\alpha)^\circ \}$$

And $G_\alpha = Z_G(\ker(\alpha))$ and $G_\alpha' =$ derived subgroup $\cong SL_2$ or PGL_2.

Now $W(G_\alpha, T) = N_{G_\alpha}(T)/T \leq N_G(T)/T = W(G,T)$

$$\underset{\{1, s_\alpha\}}{\parallel}$$

Where s_α is the class in $N_{G_\alpha}(T)/T$ corr to

$$w_\alpha = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL_2 \text{ or } PGL_2$$

Recall that for any $\chi \in X(T)$, $w \in W(G,T)$,

$w\chi(t) := \chi(w^{-1} t w)$ and $W(G,T)$ preserves $\Phi(G,T)$

Consider $w = s_\alpha$. Then let $t \in T$ and write

$t = \alpha^\vee(a) b$ with $a \in \mathbb{G}_m$, $b \in T_\alpha$.

$$
\begin{aligned}
s_\alpha x(t) &= x(w_\alpha^{-1} t w_\alpha) \\
&= x(w_\alpha^{-1} \alpha^\vee(a) b w_\alpha) \\
&= x(b)\, x(w_\alpha^{-1} \alpha^\vee(a) w_\alpha) \qquad \text{since } w_\alpha \in Z(T_\alpha) \\
&= x(b)\, x(\alpha^\vee(a))^{-1} \\
&= x(t)\, x(\alpha^\vee(a))^{-2} \\
&= x(t)\, a^{-2\langle x, \alpha^\vee \rangle} \\
&= x(t)\, a^{-\langle \alpha, \alpha^\vee \rangle \langle x, \alpha^\vee \rangle} \\
&= x(t)\, \alpha(\alpha^\vee(a))^{-\langle x, \alpha^\vee \rangle} \\
&= x(t)\, \alpha(t)^{-\langle x, \alpha^\vee \rangle} \qquad \text{since } b \in \ker(\alpha)
\end{aligned}
$$

$$\Rightarrow s_\alpha x = x - \langle x, \alpha^\vee \rangle \alpha$$

On another note, say $\alpha, \beta \in \Phi(G, T)$ such that

$$n\alpha = m\beta \quad \text{for} \quad n, m \in \mathbb{Z} \setminus \{0\}$$

Then $T_\alpha = T_\beta$. Can show $\Phi(G_\alpha, T) = \{\pm \alpha\}$

Similarly $\Phi(G_\beta, T) = \{\pm\beta\}$ but $G_\alpha = Z_G(T_\alpha) = Z_G(T_\beta) = G_\beta$

$$\Rightarrow \alpha = \pm \beta.$$

<u>Def</u> A <u>root datum</u> is a quadruple

$$\Psi = (X, \Phi, X^\vee, \Phi^\vee)$$

such that $X, X^\vee$ are finite free abelian groups in duality under a perfect pairing $\langle , \rangle : X \times X^\vee \to \mathbb{Z}$,

$\Phi \subseteq X$ and $\Phi^\vee \subseteq X$ are finite subsets with a bijection $\Phi \ni \alpha \mapsto \alpha^\vee \in \Phi^\vee$, and such that if we define, for $\alpha \in \Phi$, endomorphisms

$$S_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha \qquad x \in X$$

$$S_{\alpha^\vee}(y) = y - \langle \alpha, y \rangle \alpha^\vee \qquad y \in X^\vee$$

the following hold:

1. $\langle \alpha, \alpha^\vee \rangle = 2 \quad \forall \alpha \in \Phi$
2. $\forall \alpha \in \Phi$, $S_\alpha(\Phi) \subseteq \Phi$ and $S_{\alpha^\vee}(\Phi^\vee) \subseteq \Phi^\vee$

We say Ψ is reduced if $\forall \alpha \in \Phi$,

$$\mathbb{Q}\alpha \cap \Phi = \{\pm \alpha\}$$

We proved above, most of

<u>Prop</u> $\Psi(G,T) := (X(T), \Phi(G,T), X^\vee(T), \Phi^\vee(G,T))$ is a reduced root datum.

Let's now drop the assumption that $k = \bar{k}$.

We say G is <u>split</u> (over k) if it has a max torus T over k with $T \cong \mathbb{G}_m^r$.

<u>Rmk</u> If k is alg closed, any reductive group is split.

<u>Thm</u> For any reduced root datum Ψ, $\exists$ a

unique up to iso split reductive group (G, T) over k such that

$$\Psi(G_K, T_K) \cong \Psi.$$

Rmk Let $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$ be a reduced root datum with $\Phi \neq \emptyset$. Let $Q = \mathbb{Z}\Phi \subseteq X$, $V = Q \otimes_{\mathbb{Z}} \mathbb{R}$ and identify Φ with $\Phi \otimes 1$. Then $\{V, \Phi\}$ is a reduced root system in the sense of Lie algebras.
An reduced root system is irreducible if it cannot be written as a direct sum of proper root systems and any root system is a product of irreducible ones.
The irreducible ones are completely classified.
There are 4 infinite families

$$\underset{n \geq 1}{A_n}, \underset{n \geq 2}{B_n}, \underset{n \geq 3}{C_n}, \underset{n \geq 4}{D_n}$$

$(A_1 \cong B_1 \cong C_1$, $B_2 \cong C_2$, D_1 is degenerate, $D_2 \cong A_1 \oplus A_1$, $D_3 \cong A_3)$

and 5 sporadic ones

$$E_6, E_7, E_8, F_4, G_2$$

Eg
- GL_{n+1}, SL_{n+1}, PGL_{n+1} are type A_n
- $O_{2n+1} = \{g \in GL_{2n+1} \mid {}^t g g = I\}$, $SO_{2n+1} = \{g \in O_{2n+1} \mid \det g = 1\}$ are type B_n
- GSp_{2n}, Sp_{2n} are type C_n

• O_{2n}, SO_{2n} are type D_n

<u>Rmk</u> n in the above examples is $\dim V$ = rank of the group if G is semisimple

<u>Rmk</u> If $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$ is a root datum, it is easy to see that

$$\Psi^\vee = (X^\vee, \Phi^\vee, X, \Phi)$$

is also a root datum.

<u>Consequence</u> If G is a reductive group, $\exists$ a split reductive group $G^\vee$ such that

$$\Psi(G^\vee_{\bar{k}}, T^\vee_{\bar{k}}) = \Psi(G_{\bar{k}}, T_{\bar{k}})^\vee$$

<u>Eg</u> 1. $GL_n^\vee = GL_n$, $SL_n^\vee = PGL_n$, $PGL_n^\vee = SL_n$.

<u>Exercise</u> Show these for $n=2$.

2. If G is type B_n, $G^\vee$ is type C_n.

$$GSp_{2n}^\vee = GSpin_{2n+1}$$

If G and H are reductive groups over k with G split and $G_{\bar{k}} \cong H_{\bar{k}}$, we say H is a <u>form</u> of G

<u>Eg</u> 1. If B is a quaternion alg, then $H = B^\times$ define an alg group and is a form of GL_2

since over K, $B \otimes_k K \cong M_2(K)$, so

$$H_K = (M_2(K))^\times = GL_2(K).$$

More generally, if B is a central simple algebra of dim n^2 over k, $G = B^\times$ is a form of GL_n

Here $G(R) = (B \otimes_k R)^\times$.

2. $U(n) = \{ g \in GL_n(\mathbb{C}) \mid {}^t g^c g = 1 \}$ $\quad g^c$ = complex conj on each element.

This is an alg group $/\mathbb{R}$, functor of pts is

$$R \mapsto U(n)(R) = \{ g \in GL_n(\mathbb{C} \otimes_{\mathbb{R}} R) \mid {}^t g^{c \otimes 1} g = 1 \}$$

$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^2$ $\quad$ and $c \otimes 1$ on $\mathbb{C}^2$ is $(x,y) \mapsto (y,x)$

$a \otimes b \mapsto (ab, \bar{a} b)$

$$U(n)(\mathbb{C}) = \{ g = (g_1, g_2) \in GL_n(\mathbb{C})^2 \mid {}^t g_2 g_1 = {}^t g_1 g_2 = 1 \}$$

$$\begin{array}{c} \cong \\ GL_n(\mathbb{C}) \end{array} \quad \begin{array}{c} (g, {}^t g^{-1}) \\ \uparrow \\ \ni g \end{array}$$