Lecture 4

$k \subseteq K$ are fields with K alg closed
G = a lin alg group / k
Let $G_K :=$ G viewed / K

For all of today: We assume G_K ~~(equiv G)~~ is connected

<u>Rmk</u> We have been using classical conventions (i.e. varieties not schemes) so our alg groups are reduced by convention. In general can show a group scheme is smooth $\Leftrightarrow$ it is reduced and that is automatic in char 0.

Last time: G_K is solvable, then

$$\exists \; G_K \hookrightarrow B_n := \text{upper triangular matrices in } GL_n$$
$$\text{IU}$$
$U_n :=$ upper triangular matrices in GL_n with 1's on the diagonal, the normal subgroup of unipotent elements.

<u>Def</u> The <u>radical</u> (resp. <u>unipotent radical</u>) of G_K is the maximal smooth connected solvable (resp. unipotent) normal subgroup of G_K.

The radical is denoted by $R(G_K)$ and the unipotent radical by $R_u(G_k)$.

Eg • $R(GL_n) =$ scalar matrices $\cong \mathbb{G}_m$

$R_u(GL_n) = 1$

• Can check by hand that the maximal normal solvable subgroup of SL_n is the scalar matrices in SL_n, i.e. $\mu_n = \{n^{th} \text{ roots of } 1\}$
This is not connected if $\mathrm{char}(k) \nmid n$, and not smooth if $\mathrm{char}(k) \mid n$.

$\Rightarrow R(SL_n) = 1$.

• $n = n_1 + n_2$, let

$$P = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in GL_n \;\middle|\; \begin{array}{c} A \in GL_{n_1}, D \in GL_{n_2} \\ B \in M_{n_1 \times n_2} \end{array} \right\}$$

Then $$R(P) = \left\{ \begin{pmatrix} aI_{n_1} & B \\ 0 & dI_{n_2} \end{pmatrix} \;\middle|\; \begin{array}{c} a, d \in \mathbb{G}_m, \\ B \in M_{n_1 \times n_2} \end{array} \right\}$$

$$R_u(P) = \left\{ \begin{pmatrix} I_{n_1} & B \\ 0 & I_{n_2} \end{pmatrix} \right\}$$

<u>Def</u> G_K is <u>reductive</u> (resp <u>semisimple</u>) if $R_u(G_K) = 1$ (resp. $R(G_K) = 1$). We say G is reductive (resp semisimple) if G_K is.

Eg • GL_n is reductive and SL_n is semisimple

• (More work) Sp_{2n} is semisimple, and

$$GSp_{2n} = \{ g \in GL_{2n} \mid {}^t g J g = \lambda J,\ \lambda \in \mathbb{G}_m \}$$

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

is reductive.

2^{nd} part follows from checking the natural representation of Sp_{2n} or GSp_{2n} on K^{2n} is irred +

Thm 1. G is reductive $\Leftrightarrow$ it admits a faithful semisimple (alg) representation over K.

2. If char $k = 0$, then G is reductive $\Leftrightarrow$ every (alg) representation of G over K is semisimple.

Proof of 1: Let $r : G_K \hookrightarrow GL(V)$ be a faithful semisimple representation of G_K. Let W be a simple subrepresentation of V.

Let $U = R_u(G_K)$. U normal in $G \Rightarrow W^U$ is G-stable. But U unipotent $\Rightarrow W^U \neq \{0\}$ by last time. Thus $W = W^U$ and the image of U in $GL(W)$ is trivial. But V is the direct sum of its simple subreps and V is faithful, $U = 1$. □

Thm Let G be a reductive group. Then $R(G_{\bar{k}})$ is the connected component of the centre of $G_{\bar{k}}$. In particular, a reductive group with finite centre is semisimple.

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Until specified otherwise, say $k = \bar{k}$, i.e. k is alg closed. Let G be reductive.

Let $T \subseteq G$ be a maximal torus.

We have the representation

$$\mathrm{Ad}: G \to GL_{\mathfrak{g}}, \quad \mathfrak{g} = \mathrm{Lie}(G).$$

$T \cong \mathbb{G}_m^r$. Let

$$X(T) := \mathrm{Hom}_{k\text{-groups}}(T, \mathbb{G}_m) \cong \mathbb{Z}^r$$

$$\left((t_1, \dots, t_r) \mapsto t_1^{n_1} t_2^{n_2} \cdots t_r^{n_r}\right) \leftarrow\!\shortmid (n_1, \dots, n_r)$$

called the <u>character group of T</u>

Consider the action of T on $\mathfrak{g}$, via Ad.

Since T is diagonalizable

$$\Rightarrow \mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\substack{\alpha \in X(T) \\ \alpha \neq 0}} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_0 = \mathfrak{g}^T$.

The <u>roots</u> of (G, T), denoted $\Phi(G, T)$ are the $0 \neq \alpha \in X(T)$ such that $\mathfrak{g}_\alpha \neq 0$.

<u>Eg</u> • GL_2, $T = \left\{ \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \right\}$, $X(T) = \mathbb{Z} e_1 \oplus \mathbb{Z} e_2$

where $e_1 \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} = t_1$ and $e_2 \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} = t_2$.

$\mathfrak{g} = \mathfrak{gl}_2 = M_2(K)$ and $\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}$ acts on $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by

$$\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t_1^{-1} & \\ & t_2^{-1} \end{pmatrix} = \begin{pmatrix} a & \frac{t_1}{t_2} b \\ \frac{t_2}{t_1} c & d \end{pmatrix}$$

$\mathfrak{gl}_2 = \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ where

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right\}, \quad \mathfrak{g}_\alpha = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \right\}, \quad \mathfrak{g}_{-\alpha} = \left\{ \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \right\}$$

$\alpha = e_1 - e_2 \in X(T)$, $-\alpha = e_2 - e_1$

$$\Phi(G,T) = \{\pm \alpha\}$$

• $G = SL_2$, $T = \left\{ \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \right\}$. $X(T) = \mathbb{Z} e$ with $e \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} = t$. Then

$\mathfrak{g} = \mathfrak{sl}_2 = \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$

$\alpha = 2e$ since $\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} t^{-1} & \\ & t \end{pmatrix} = \begin{pmatrix} a & t^2 b \\ t^{-2} c & -a \end{pmatrix}$

$$\Phi(G,T) = \{\pm \alpha\}.$$

• $G = PGL_2$, $X(T) = \mathbb{Z} x$ $\quad x \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} = t_1 / t_2$

$$\Phi(G,T) = \{\pm x\}.$$

- GL_n, $T = \left\{\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}\right\}$, $X(T) = \bigoplus_{i=1}^{n} \mathbb{Z} e_i$

with $e_i\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} = t_i$. Then

$$\Phi(G,T) = \{ e_i - e_j \mid 1 \le i \ne j \le n \}$$

The <u>Weyl group</u> of (G,T) is

$$W(G,T) := \underbrace{N_G}_{\text{normalizer of } T \text{ in } G}(T)/T$$

<u>Eg</u> $G = GL_n$, $N_G(T)$ = group gen by T and all permutation matrices.
T = diagonal

So $W(G,T) \cong S_n$ = symmetric group on n elements

Note $W(G,T)$ acts on $X(T)$ by
$(w\chi)(t) = \chi(g^{-1}tg)$ if $g \in N_G(T)$ lifting $w \in W(G,T)$.

<u>Prop</u> $W(G,T)$ stabilizes $\Phi(G,T)$.

<u>Proof</u> Choose $g \in N_G(T)$ lifting $w \in W(G,T)$.
If $\alpha \in \Phi(G,T)$ and $X \in \mathfrak{g}_\alpha$,

$$t \cdot (g \cdot X) = t \cdot (gXg^{-1}) = g(g^{-1}tg)X = g\,\alpha(g^{-1}tg)X$$
$$= \alpha(g^{-1}tg)\,gX = (w\alpha)(t)(gX)$$

So T acts on $g\mathfrak{g}_\alpha$ by $w\alpha$. $\square$

Let $X^\vee(T) = \mathrm{Hom}_{k\text{-groups}}(\mathbb{G}_m, T)$, the cocharacter group of T. There is a perfect pairing

$$\langle\, ,\, \rangle : X(T) \times X^\vee(T) \to \mathbb{Z} \cong \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m)$$
$$n \mapsto (t \mapsto t^n)$$

$$\langle x, \mu \rangle = n \text{ if } x \circ \mu(t) = t^n$$

Let $\alpha \in \Phi(G,T)$ and let $T_\alpha = (\ker \alpha)^\circ \subseteq T$, a subtorus of codim 1.

Let G_α = centralizer of T_α in G.

<u>Fact</u>: G_α is a connected reductive group with maximal torus T whose derived (aka commutator) subgroup is $\cong SL_2$ or PGL_2 and $\exists$ a unique hom

$$\alpha^\vee : \mathbb{G}_m \to G_\alpha \text{ such that}$$

$$T = \alpha^\vee(\mathbb{G}_m)\, T_\alpha \text{ and } \langle \alpha, \alpha^\vee \rangle = 2.$$

<u>Eg</u>. $G = GL_n$, T = diag torus. $\alpha = \alpha_{12} = e_1 - e_2$, i.e.

$$\alpha_{12}\begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_n \end{pmatrix} = t_1 t_2^{-1}$$

$$T_{\alpha_{12}} = (\ker \alpha_{12}) = \left\{ \begin{pmatrix} t & & & & \\ & t & & & \\ & & t_3 & & \\ & & & \ddots & \\ & & & & t_n \end{pmatrix} \right\}$$

$G_{\alpha_{12}}$ = centralizer of $T_{\alpha_{12}}$

$$= \left\{ \begin{pmatrix} * & * & 0 & \cdots & 0 \\ * & * & 0 & \cdots & 0 \\ 0 & 0 & * & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & * \end{pmatrix} \right\}$$

The commutator subgroup of $G_{\alpha_{12}} \cong$ commutator subgroup of $GL_2 \cong SL_2$

$$\alpha^\vee : \mathbb{G}_m \to T$$
$$t \mapsto \begin{pmatrix} t & & & & \\ & t^{-1} & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

Then in general, we set

$$\Phi^\vee(G,T) = \{\alpha^\vee \mid \alpha \in \Phi(G,T)\}$$

called the <u>coroots</u> of (G,T)