Lecture 20

Let G be a Hausdorff locally profinite group, i.e. every nbhd of $1 \in G$ contains a compact open subgroup. We will assume G has a countable basis of open subsets.

Eg
- Profinite groups.
- $G = \mathbb{G}(K)$ for $\mathbb{G}$ connected red over a nonarchimedean field K. Indeed, fix $\mathbb{G} \hookrightarrow GL_N$. Then a basis of compact open subgroups is
$$U_n := \mathbb{G}(K) \cap \left(1 + m_{\mathcal{O}_K}^n M_{N\times N}(\mathcal{O}_K)\right)$$
- $G = \mathbb{G}(\mathbb{A}_F^\infty)$, $F = \#$ fld.

Rmk Compact locally profinite $\Leftrightarrow$ profinite.

Def A representation (π, V) on a $\mathbb{C}$-vs V, of G is called smooth if $\forall\, 0 \neq v \in V$, $\mathrm{Stab}_G(v)$ is open.

We say (π, V) is admissible if for any open compact subgroup $K \leq G$, $\dim_{\mathbb{C}} V^K < \infty$.

We say (π, V) is unitarizable/unitary if it can be/is equipped with a G-inv inner product.

Rmk 1. Smooth is equivalent to
$$G \times V \to V$$
$$(g, v) \mapsto \pi(g)v$$
is continuous if we give V the discrete top.

If G ~~is compact and~~ V is fin dim, say
$$G \times V \to V$$
cts for the usual top on V. Then we have a cts hom
$$\phi : G \to GL_N(\mathbb{C}) \qquad N = \dim_{\mathbb{C}} V.$$
No small subgroups: $\exists$ open subset $U \subseteq GL_N(\mathbb{C})$ containing 1 st. $\{1\} \subseteq U$ is the unique closed subgroup in U.

Then $\phi^{-1}(U)$ is open in G, so contains an open compact subgroup K. Then $\phi(K) \subseteq U \Rightarrow \phi(K) = \{1\}$, and $\ker\phi$ is open. In part, if G is compact, ϕ factors through a finite quotient.

2. (π, V) smooth admissible $\Longleftrightarrow$ for any compact open subgroup $K \leq G$,
$$V \cong \bigoplus_{\sigma \in \hat{K}} V_\sigma$$
where $\sigma \in \hat{K}$ runs over irred fin dim reps of K (or iso classes of), V_σ is the σ-isotypic part. And each V_σ is fin dim.

[Rmk If K_1 and K_2 are two open compact subgroups, then $K_1 \cap K_2$ is open (and compact), nontrivial, hence finite index in each K_1 and K_2 by compactness.]

Sketch $\Leftarrow$ Look at $\sigma =$ triv.

$\Rightarrow$ Any σ factors through a finite quotient K/U.

So $\dim_{\mathbb{C}} V^U < \infty \Rightarrow \dim_{\mathbb{C}} V_\sigma < \infty$.

Now for $0 \neq v \in V$, $\mathrm{Stab}(v)$ is open, so $\exists$ open normal $U \leq K$ s.t. $v \in V^U$.

Then the K-translates of v span a fin dim space that can be decomposed wrt th action of the fin group G/U. Hence $v \in \bigoplus_{\sigma \in \hat{K}} V_\sigma$.

3. Assumptions as above, (π, V) smooth adm $\Longrightarrow \dim_{\mathbb{C}} V$ is countable.

Indeed let $\{K_n\}_{n \geq 1}$ be a base of nbhds of 1 of compact open subgroups.

V smooth $\Rightarrow V = \bigcup_{n \geq 1} V^{K_n}$

admissible $\Rightarrow \dim_{\mathbb{C}} V^{K_n} < \infty \ \forall n \geq 1$.

4. In fact, smooth + irred $\Rightarrow$ admissible.

Prop (Schur's Lemma) If (π, V) is smooth adm irred, then $\mathrm{End}_{\mathbb{C}[G]}(V) = \mathbb{C}1$.

Proof Fix $K \leq G$ open compact. Then $V \cong \bigoplus_{\sigma \in \hat{K}} V_\sigma$. Choose σ with $V_\sigma \neq 0$. Then $T : V \to V$ G-equiv preserves V_σ, so we can choose an eigenvalue α for T on the fin dim V_σ. $\ker(T - \alpha) \neq 0 \Rightarrow T = \alpha$ since V is irred. $\square$

Cor If (π, V) is irred smooth admissible, the centre Z of G acts by a smooth character

$$\omega : Z \to \mathbb{C}^\times$$

Now say K is a nonarch local fld with ring of integers $\mathcal{O}_K$, residue fld k, and abs value

$$|\ |: K \to \mathbb{R}_{\geq 0} \text{ normalized by } |\varpi_K|^{-1} = \#k.$$

Let $G = \mathbb{G}(K)$ with $\mathbb{G}$ connected reductive $/K$.

__Parabolic Induction__ $\mathbb{P} \subseteq \mathbb{G}$ a parabolic subgroup $/K$, $\mathbb{N}$ = unipotent radical of $\mathbb{P}$, $\mathbb{M}$ = Levi.

$P = \mathbb{P}(K)$, $M = \mathbb{M}(K)$, $N = \mathbb{N}(K)$.

We again have the modulus character

$$\delta_P : P \to \mathbb{R}_{>0} \quad \text{by}$$

$$d_P(px) = \delta_P(p)\, d_P x \quad \text{with } d_P x = \text{right Haar meas on } P$$

and $\delta_P(p) = |\det(\mathrm{Ad}(p) : \mathrm{Lie}\, P \to \mathrm{Lie}\, P)|$

__Eg__ $G = GL_2(K)$, $P = B = \left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right\}$, then

$$\delta_B\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \left|\frac{a}{d}\right|$$

$G = GL_n(K)$, $P = B = \{(\nabla)\}$, then

$$\delta_B\begin{pmatrix} t_1 & & * \\ & t_2 & \\ 0 & & \ddots \\ & & & t_n \end{pmatrix} = \left| t_1^{n-1} t_2^{n-3} \cdots t_{n-1}^{3-n} t_n^{1-n} \right|$$

If (σ, W) is a smooth admissible rep of M, we form the parabolic induction

$$n\mathrm{Ind}_P^G \sigma = n\mathrm{Ind}_P^G W$$

$$= \{ f : G \to W \mid f \text{ is locally constant and } f(mng) = \delta_P(m)^{\frac{1}{2}} \sigma(m) f(g) \text{ for all } m \in M, n \in N, g \in G \}$$

with G-action

$$(gf)(x) = f(xg).$$

__Note__ 1. $n\mathrm{Ind}_P^G \sigma$ is smooth. Indeed, any $f \in n\mathrm{Ind}_P^G \sigma$ is locally constant, so if $g \in G$, $\exists$ open subgroup U such that f is constant on $gU \Rightarrow U$ fixes f. * (needs more here. Use Iwasawa below & intersect fin many such U.)

2. $n\mathrm{Ind}_P^G \sigma$ is admissible. Choose a max open compact subgroup U_{max}.

__Iwasawa__ $G = P U_{max}$

Then for any open compact U,

$$X_U = P \backslash G / U \quad \text{is a finite set.}$$

Then $(n\mathrm{Ind}_P^G \sigma)^U \ni f$ is determined by its values on the fin set X_U, and on $x \in X_U$ takes values in the fin dim space $W^{M \cap x U x^{-1}}$

$$\Rightarrow \dim_{\mathbb{C}} (n\mathrm{Ind}_P^G \sigma)^U < \infty.$$

3. If (σ, W) is unitarizable, so is $n\mathrm{Ind}_P^G \sigma$ (some computation as in arch case).

In part, if $\mathbb{P} = \mathbb{B}$ is a Borel subgroup and $\mathbb{M} = \mathbb{T}$ is a maximal torus. Then can take

$$\sigma = \chi : T \to \mathbb{C}^\times$$

a smooth character, i.e. one with open kernel. Further, if $\mathbb{T} \cong \mathbb{G}_m^r$, then

$$\chi = \chi_1 \times \cdots \times \chi_r \text{ with } \chi_i : K^\times \to \mathbb{C}^\times$$

smooth hom. In this case,

$n\mathrm{Ind}_B^G(\chi_1 \times \cdots \times \chi_r)$ are called __principal series__ when they are irred.

$\leftarrow$ Let's justify this correctly.

Let U be a maximal open compact subgroup and take $f \in n\mathrm{Ind}_P^G \sigma$. Since f is locally constant, for every $x \in U$, $\exists$ open compact $U_x \leq U$ such that $f(xU_x) = f(x)$. Since $U = \bigcup_{x \in U} U_x$ and U is compact, $\exists$ a finite subcover $U_1, \dots, U_n$. Setting $H = \bigcap_{i=1}^n U_i$, we see that $f|_U$ is invariant under H.

Now take any $g \in G$ and $h \in H$. By the Iwasawa decomposition, we can write $g = mnu$ with $m \in M$, $n \in N$, $u \in U$. Then

$$f(gh) = f(puh) = \delta_P^{\frac{1}{2}}(m) \sigma(m) f(uh)$$

$$= \delta_P^{\frac{1}{2}}(m) \sigma(m) f(u)$$

$$= f(g).$$