Lecture 12

Thm (Harish-Chandra)

$$\gamma = \sigma_{\Delta^+} \circ \gamma_{\Delta^+} : Z(\mathfrak{g}_{\mathbb{C}}) \xrightarrow{\sim} \mathcal{T}^W, \quad \mathcal{T} = U(\mathfrak{t})$$

Proof that $\mathrm{im}(\gamma) \subseteq \mathcal{T}^W$: From last time it only remains to show that if $\lambda - \delta \in \mathfrak{t}^\vee$ is dominant integral and $\alpha \in \Delta^+$ is simple, then

$$V(s_\alpha \lambda) \subseteq V(\lambda)$$

where $V(\mu) := U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{b})} \mathbb{C}(\mu - \delta)$

$$\Rightarrow \gamma(z)(s_\alpha \lambda) = \gamma(z)(\lambda) \quad \forall \text{ simple } \alpha \text{ and } \lambda \in \mathfrak{t}^\vee$$
st. $\lambda - \delta$ is dominant.

$$\Rightarrow \mathrm{im}(\gamma) \subseteq \mathcal{T}^W$$

Set $\omega = \lambda - \delta$, $v_0 = 1 \otimes 1 \in V(\lambda)$ is a highest weight vector on which $\mathfrak{t}$ acts by ω.

Fact $\Delta^+ = \{\alpha_1, \ldots, \alpha_n\}$, $\exists$ a basis $\{E_{-\alpha_i}, H_{\alpha_i}, E_{\alpha_i}, 1 \le i \le n\}$ for $\mathfrak{g}_{\mathbb{C}}$ such that

- $0 \neq E_{-\alpha_i} \in \mathfrak{g}_{-\alpha_i}$, $0 \neq E_{\alpha_i} \in \mathfrak{g}_{\alpha_i}$
- $H_i = [E_{\alpha_i}, E_{-\alpha_i}]$
- $\forall \mu \in \mathfrak{t}^\vee$, $\mu(H_{\alpha_i}) = 2 \dfrac{\langle \mu, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}$

Then note $\mathbb{C}E_{-\alpha_i} + \mathbb{C}H_{\alpha_i} \oplus \mathbb{C}E_{\alpha_i} \cong \mathfrak{sl}_2$

Since ω is dominant integral,

$$\omega(H_\alpha) = m \in \mathbb{Z}_{\geq 0}$$

We consider the vector

$$w = E_{-\alpha}^{m+1} v_0$$

<u>Claim</u> w is a highest weigh vector, i.e. $E_\beta w = 0$ $\forall$ simple $\beta \in \Delta^+$.

If $\beta \neq \alpha$ is a simple root, $[E_\beta, E_{-\alpha}] = 0$ (since $[E_\beta, E_{-\alpha}] \in \mathfrak{g}_{\beta-\alpha}$ but $\beta - \alpha$ is not a root)

Then $\quad E_\beta w = E_\beta E_{-\alpha}^{m+1} v_0 = E_{-\alpha}^{m+1} E_\beta v_0 = 0.$

Show by induction that

$$\begin{aligned} E_\alpha w &= E_\alpha E_{-\alpha}^{m+1} v_0 \\ &= \left[(m+1) E_{-\alpha}^{m+1} (H_\alpha - m) + E_{-\alpha}^{m+1} E_\alpha\right] v_0 \\ &= (m+1) E_{-\alpha}^{m+1} (m-m) v_0 + E_{-\alpha}^{m+1} E_\alpha v_0 \\ &= 0 \end{aligned}$$

So w is a highest weight vector and $\mathfrak{h}$ acts on it by

$$\begin{aligned} \omega - (m+1)\alpha &= \omega - (\omega(H_\alpha) + 1)\alpha \\ &= \lambda - \delta - ((\lambda - \delta)(H_\alpha) + 1)\alpha \\ &= \lambda - \delta - \lambda(H_\alpha)\alpha \\ &= s_\alpha \lambda - \delta \end{aligned}$$

$\Rightarrow \exists$ a map $\mathcal{U}(s_\alpha\lambda) \to \mathcal{U}(\lambda)$ and can check it is an
$1\otimes 1 \longmapsto w$

injection.

<u>Independence of Δ^+</u> Say we have two choices Δ_1^+ and Δ_2^+ of positive roots. Want to check

$$\sigma_{\Delta_1^+} \circ \gamma_{\Delta_1^+}(Z) = \sigma_{\Delta_2^+} \circ \gamma_{\Delta_2^+}(Z)$$

WLOG, can assume $\Delta_2^+ = s_\alpha \Delta_1^+$ with α a simple root in Δ_1^+ and $s_\alpha \in W$. Then if $\{\alpha, \alpha_2, \dots, \alpha_m\}$ is a base for Δ_1^+, a base for Δ_2^+ is $\{-\alpha, \alpha_2, \dots, \alpha_m\}$.

Can check $\delta_2 = \delta_1 - \alpha = s_\alpha \delta_1$

Say V is a fin dim irred rep of $\mathfrak{g}_\mathbb{C}$ and

$\lambda =$ highest weight wrt Δ_1^+

Then $s_\alpha\lambda =$ highest weight wrt Δ_2^+.

And $Z \in Z(\mathfrak{g}_\mathbb{C})$ acts on this space by

$$\gamma_{\Delta_1^+}(Z)(\lambda) = \gamma_{\Delta_2^+}(Z)(s_\alpha\lambda) = \gamma_{\Delta_2}(s_\alpha\lambda + \delta_2)$$

$$\shortparallel$$

$$\gamma_{\Delta_1}(Z)(\lambda + \delta_1) = \gamma_{\Delta_1}(Z)(s_\alpha(\lambda + \delta_1))$$

$$= \gamma_{\Delta_1}(Z)(s_\alpha\lambda + s_\alpha\delta_1)$$

$$= \gamma_{\Delta_1}(Z)(s_\alpha\lambda + \delta_1 - \alpha)$$

$$= \gamma_{\Delta_1}(Z)(s_\alpha\lambda + \delta_2)$$

So $\forall\ Z \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$ and λ dominant integral for Δ_1^+, we have

$$\gamma_{\Delta_1}(z)(s_\alpha\lambda+\delta_2) = \gamma_{\Delta_2}(z)(s_\alpha\lambda+\delta_1)$$
$$\Rightarrow \gamma_{\Delta_1}(z) = \gamma_{\Delta_2}(z).$$

Rmk In the above, we used the following 2 key properties of δ: if $\alpha\in\Delta^+$ is a simple root, then
$$s_\alpha\delta = \delta-\alpha \quad \text{and} \quad 2\frac{\langle\delta,\alpha\rangle}{\langle\alpha,\alpha\rangle} = 1$$

Proof that γ is mult σ_{Δ^+} is an alg iso $\mathcal{T}\xrightarrow{\sim}\mathcal{T}$, so suffices to check $\gamma_{\Delta^+}(z_1z_2) = \gamma_{\Delta^+}(z_1)\gamma_{\Delta^+}(z_2)$

Check
$$\begin{aligned} z_1z_2-\gamma_{\Delta^+}(z_1)\gamma_{\Delta^+}(z_2) &= z_1(z_2-\gamma_{\Delta^+}(z_2)) + \gamma_{\Delta^+}(z_2)(z_1-\gamma_{\Delta^+}(z_1)) \\ &\in U(\mathfrak{g}_{\mathbb{C}})\mathcal{P} \end{aligned}$$
$$\Rightarrow \gamma_{\Delta^+}(z_1z_2) = \gamma_{\Delta^+}(z_1)\gamma_{\Delta^+}(z_2)$$

Isomorphism: Injectivity not too difficult. Surjectivity is harder. See Knapp, Representations of Semisimple Groups for proof for $\mathfrak{gl}_n$. See also Getz-Hahn.

Say we have $\lambda:\mathfrak{t}\to\mathbb{C}$ a character, it extends to a alg hom $\lambda:\mathcal{T}\to\mathbb{C}$ and we get a character
$$\chi_\lambda := \lambda\circ\gamma : Z(\mathfrak{g}_{\mathbb{C}})\to\mathbb{C}.$$
Note that if $w\in W$, then $(w\lambda)(H) = \lambda(Ad(w^{-1})H)$, so $\chi_{w\lambda} = \chi_\lambda$, since $\mathrm{im}(\gamma)\in\mathcal{T}^W$.

Prop Any character $\chi: Z(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C}^\times$ is of the form
$\chi = \chi_\lambda = \lambda \circ \gamma$ for $\lambda: \mathfrak{t} \to \mathbb{C}$
and λ is uniquely determined up to W.

Proof It is a fact that T is a finite free T^W-alg of rank $|W|$. So if $\chi: T^W \to \mathbb{C}$ is a character, then the maximal ideal $m = \ker \chi \in T^W$ lies below a maximal ideal $\mathfrak{M}$ of T, which also has residue fld $\mathbb{C}$, so we can extend χ to $\lambda: T \to T/\mathfrak{M} \cong \mathbb{C}$.

Now say we have $\lambda, \mu: T \to \mathbb{C}$ such that $\mu \neq w\lambda$ for any $w \in W$. Want to check $\chi_\lambda \neq \chi_\mu$. Can find a polynomial on $\mathfrak{t}^\vee$ that is 1 on $W\lambda$ and 0 on $W\mu$. Then

$$q = \frac{1}{|W|} \sum_{w \in W} wp$$

has the same property and is W-invariant.
Harish-Chandra iso $\Rightarrow \exists\ Z \in Z(\mathfrak{g}_{\mathbb{C}})$ with $\gamma(Z) = q$.
Then $\chi_\lambda(Z) = q(\lambda) = 1$ and $\chi_\mu(Z) = q(\mu) = 0$. $\square$

Def If V is a $(\mathfrak{g}, K)$-module on which $Z(\mathfrak{g}_{\mathbb{C}})$ acts by a character χ (e.g. V is irred) we call χ or $\lambda \in \mathfrak{t}^\vee$ if $\chi = \chi_\lambda$ the <u>infinitessimal character</u> of V

Eg V is an irred finite dimensional representation with highest weight λ (for some choice Δ^+ of $\mathfrak{g}_{\mathbb{C}}$), then the infinitessimal character is $\lambda+\delta$, $\delta = \frac{1}{2}$ sum of +ve roots.

Eg For SL_2, representation generated by ϕ_f with f a modular form of wt $k \geq 2$.

We saw $Z(\mathfrak{sl}_2) \cong \mathbb{C}[\Omega] \xrightarrow[\gamma]{\sim} \mathbb{C}[H^2]$ $\quad H = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

$$\Omega \longmapsto \tfrac{1}{2}H^2 - \tfrac{1}{2}$$

and $\Omega = -2\Delta$, with $\Delta\phi_f = -\frac{k}{2}\left(\frac{k}{2}-1\right)$

So $\Omega\phi_f = k\left(\frac{k}{2}-1\right)$ and via γ, $H^2 = 2\Omega + 1$ acts by

$$k(k-2)+1 = (k-1)^2$$

The irred fin reps of $\mathfrak{sl}_2$ are $\mathrm{Sym}^n \mathbb{C}^2$, $n \geq 0$, which have highest weight $H \mapsto n$. Here $\delta(H) = 1$, so the inf char on $\mathrm{Sym}^n\mathbb{C}^2$ is given by

$$H^2 \longmapsto (n+1)^2 \quad \text{for } n \geq 0$$

So if $k \geq 2$, then ϕ_f has the same inf char as the fin dim rep $\quad \mathrm{Sym}^{k-2}\mathbb{C}^2$.

We'll see soon that if F is a number field and π is an automorphic representation of $GL_n(\mathbb{A}_F)$, then part of the data of π is a $(\mathfrak{g}, K)$-module for the $\mathbb{R}$-Lie group

$$G := GL_n(F \otimes_{\mathbb{Q}} \mathbb{R}) \cong \prod_{v \text{ real}} GL_n(\mathbb{R}) \times \prod_{v \text{ complex}} GL_n(\mathbb{C}).$$

We say π is <u>regular algebraic</u> if it has the same infinitesimal character as an irred fin dimensional rep of $\mathfrak{g}_{\mathbb{C}}$.

$\mathfrak{g} = \mathrm{Lie}\, G = \prod_{v \text{ real}} \mathfrak{gl}_{n/\mathbb{R}} \times \prod_{v \text{ complex}} \mathfrak{gl}_{n/\mathbb{C}}$

$$\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \prod_{v \text{ real}} \mathfrak{gl}_{n/\mathbb{C}} \times \prod_{v \text{ complex}} (\mathfrak{gl}_{n/\mathbb{C}} \times \mathfrak{gl}_{n/\mathbb{C}})$$

$$= \prod_{\tau: F \hookrightarrow \mathbb{C}} \mathfrak{gl}_{n/\mathbb{C}}$$

Irred fin reps of $\mathfrak{gl}_{n/\mathbb{C}}$ are parametrized by

$$\mathbb{Z}^n_+ = \{ (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \}$$

$\lambda = (\lambda_1, \dots, \lambda_n)$ corr to the weight $\mathrm{diag}(t_1, \dots, t_n) \mapsto t_1^{\lambda_1} t_2^{\lambda_2} \cdots t_n^{\lambda_n}$.

The <u>weight of π</u> is then the tuple

$$(\lambda_\tau)_{\tau: F \hookrightarrow \mathbb{C}} \in (\mathbb{Z}^n_+)^{\mathrm{Hom}(F, \mathbb{C})}$$

<u>Eg</u> $n=2$, $\lambda = (a, b)$ with $a \geq b$, the fin. dim rep is

$$\mathrm{Sym}^{a-b} \mathbb{C}^2 \otimes \det{}^b$$