Lecture 2 - Hecke alg valued Gal reps for $GL_2(\mathbb{Q})$

Fix
- $E/\mathbb{Q}_p$ finite, $E \supset \mathcal{O} = $ ints $\ni \varpi = $ unif
  $\mathbb{F} = \mathcal{O}/(\varpi)$, $q = p^f = |\mathbb{F}|$
- $k \geq 2$, $\Gamma = \Gamma_1(N)$, $N \geq 4$
- $S = $ fin set of primes $\supseteq \{p, \infty\} \cup \{q \mid N\}$
  $\mathbb{Q}_S = $ max ext of $\mathbb{Q}$ (in some $\overline{\mathbb{Q}}$) unramified outside S
  $G_{\mathbb{Q},S} = \mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$
- An iso $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C} \rightsquigarrow S_k(\Gamma, \overline{\mathbb{Q}}_p) \underset{\iota}{\cong} S_k(\Gamma, \mathbb{C})$

Def/Notation $\quad \mathbb{T}^{S,\mathrm{univ}} := \mathbb{Z}[T_\ell, S_\ell]_{\underset{\text{prime}}{\ell \notin S}}$

If A is a commutative ring
$$\mathbb{T}_A^{S,\mathrm{univ}} := \mathbb{T}^{S,\mathrm{univ}} \otimes A$$
If M is a $\mathbb{T}_A^{S,\mathrm{univ}}$-module,
$$\mathbb{T}_A^S(M) = \mathbb{T}^S(M) := \mathrm{im}\left(\mathbb{T}_A^{\mathrm{univ}} \to \mathrm{End}_A(M)\right)$$

$\mathbb{T}_{\mathbb{C}}^{S,\mathrm{univ}}$ acts on $S_k(\Gamma, \mathbb{C})$ by
$$T_\ell = \left[\Gamma \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} \Gamma\right]$$
$$S_\ell = \left[\Gamma \begin{pmatrix} \ell & 0 \\ 0 & \ell \end{pmatrix} \Gamma\right] = \ell^{k-2} \langle \ell \rangle$$
and $S_k(\Gamma, \mathbb{C})$ is a semisimple $\mathbb{T}_{\mathbb{C}}^{S,\mathrm{univ}}$-module (Petersson inner product $\Rightarrow$ each T_ℓ is normal)
$$\Rightarrow \mathbb{T}^S(S_k(\Gamma, \mathbb{C})) \cong \prod_{\text{eigensystems}} \mathbb{C}$$

Then via $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$,
$$\mathbb{T}^S(S_k(\Gamma, \overline{\mathbb{Q}}_p)) \cong \prod_{\text{eigensystems}} \overline{\mathbb{Q}}_p$$
And for any eigensystem
$$\lambda : \mathbb{T}^S(S_k(\Gamma, \overline{\mathbb{Q}}_p)) \to \overline{\mathbb{Q}}_p$$
we have a Gal rep

$$\rho_\lambda : G_{\mathbb{Q},S} \to GL_2(\overline{\mathbb{Q}}_p)$$

s.t. $\forall\, \ell \notin S$ prime

$$\text{char poly } \rho_\lambda(\mathrm{Frob}_\ell) = X^2 - \lambda(T_\ell)X + \ell\lambda(S_\ell)$$

So we get

$$\rho = \prod_\lambda \rho_\lambda : G_{\mathbb{Q},S} \to GL_2(\mathbb{T}^S(S_k(\Gamma, \overline{\mathbb{Q}}_p)))$$

s.t. $\forall\, \ell \notin S$ prime

$$\text{char poly } \rho(\mathrm{Frob}_\ell) = X^2 - T_\ell X + \ell S_\ell$$

<u>Goal</u>: Integral version.

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<u>Thm</u> (Eichler - Shimura) There is an iso of $\mathbb{T}_{\mathbb{C}}^{S,\text{univ}}$-mods

$$M_k(\Gamma, \mathbb{C}) \oplus \overline{S_k(\Gamma, \mathbb{C})} \cong H^1(\Gamma, \mathrm{Sym}^{k-2}\mathbb{C}^2)$$

The action of a double coset operator $[\Gamma\alpha\Gamma]$, $\alpha \in GL_2(\mathbb{Q})$, on $H^i(\Gamma, \mathrm{Sym}^{k-2}\mathbb{C}^2)$ is

$$\begin{aligned} H^i(\Gamma, \mathrm{Sym}^{k-2}\mathbb{C}^2) &\xrightarrow{\text{res}} H^i(\Gamma \cap \alpha^{-1}\Gamma\alpha, \mathrm{Sym}^{k-2}\mathbb{C}^2) \\ &\xrightarrow{\alpha_*} H^i(\alpha\Gamma\alpha^{-1} \cap \Gamma, \mathrm{Sym}^{k-2}\mathbb{C}^2) \\ &\xrightarrow{\text{cores}} H^i(\Gamma, \mathrm{Sym}^{k-2}\mathbb{C}^2) \end{aligned}$$

Can also see this geometrically. Say $k=2$. Then

$$H^1(\Gamma, \mathbb{C}) \cong H^1(Y(\Gamma), \mathbb{C}) \quad \text{where} \quad Y(\Gamma) = \Gamma \backslash \mathbb{H}$$

(uses $N \geq 4$)

holds for other coeffs

$$\begin{tikzcd} & Y(\Gamma\cap\alpha^{-1}\Gamma\alpha) \arrow[dl, "\pi_1"'] & Y(\alpha\Gamma\alpha^{-1}\cap\Gamma) \arrow[l, "\alpha"'] \arrow[d, "\pi_2"] \\ Y(\Gamma) & & Y(\Gamma) \end{tikzcd}$$

and $[\Gamma\alpha\Gamma]$ acts by $\pi_{2*}\circ\alpha^*\circ\pi_1^*$

Then $H^1(\Gamma, \mathrm{Sym}^{k-2}\mathbb{C}^2) \cong H^1(\Gamma, \mathrm{Sym}^{k-2}\mathbb{Z}^2)\otimes\mathbb{C}$
and $H^1(\Gamma, \mathrm{Sym}^{k-2}\mathbb{Z}^2)$ is a finitely generated abelian group.
$\rightsquigarrow H^1(\Gamma, \mathrm{Sym}^{k-2}\mathcal{O}^2) \cong H^1(\Gamma, \mathrm{Sym}^{k-2}\mathbb{Z}^2)\otimes\mathcal{O}$
a fin gen $\mathcal{O}$-mod and

$$H^1(\Gamma, \mathrm{Sym}^{k-2}\mathcal{O}^2)\otimes_{\mathcal{O}}\overline{\mathbb{Q}}_p \cong H^1(\Gamma, \mathrm{Sym}^{k-2}\overline{\mathbb{Q}}_p^2) \xrightarrow[\sim]{?} H^1(\Gamma, \mathrm{Sym}^{k-2}\mathbb{C}^2) \xleftarrow{ES} S_k(\Gamma,\mathbb{C})$$

all $\mathbb{T}^{S,\mathrm{univ}}$-equiv.

Choose a Hecke eigenform $g\in S_k(\Gamma,\mathbb{C})\cong S_k(\Gamma,\overline{\mathbb{Q}}_p)$ and let

$$\begin{tikzcd} \lambda_g : \mathbb{T}^S(H^1(\Gamma, \mathrm{Sym}^{k-2}\overline{\mathbb{Q}}_p^2)) \arrow[r] & \mathbb{T}^S(S_k(\Gamma,\overline{\mathbb{Q}}_p)) \arrow[r] & \overline{\mathbb{Q}}_p \\ \mathbb{T}^S(H^1(\Gamma, \mathrm{Sym}^{k-2}\mathcal{O}^2)) \arrow[u] \arrow[rr, "\lambda_g"] \arrow[rrd, "\overline{\lambda}_g"'] & & \mathcal{O} \arrow[u] \arrow[d] \\ & & \mathbb{F} \end{tikzcd}$$

(enlarging $\mathcal{O}$ if nec)

Then $m = \ker\overline{\lambda}_g$ is a maximal ideal of
$$\mathbb{T}^S(\Gamma, k) := \mathbb{T}^S(H^1(\Gamma, \mathrm{Sym}^{k-2}\mathcal{O}^2))$$
and assoc to m and λ_g is a cts rep
$$\overline{\rho}_m : G_{\mathbb{Q},S} \longrightarrow GL_2(\mathbb{F})$$
s.t. $\forall \ell\notin S$,
$$\begin{aligned} \mathrm{charpoly}\,\overline{\rho}_m(\mathrm{Frob}_\ell) &= X^2 - \overline{\lambda}_g(T_\ell)X + \ell\overline{\lambda}_g(S_\ell) \\ &= X^2 - T_\ell X + \ell S_\ell \mod m \end{aligned}$$

Def We say $\mathfrak{m}$ is <u>non-Eisenstein</u> if $\bar{\rho}_{\mathfrak{m}}$ is abs irred.

Prop If $\mathfrak{m}$ is non-Eisenstein, then $H^1(\Gamma, \mathrm{Sym}^{k-2}\mathcal{O}^2)_{\mathfrak{m}}$ is a finite free $\mathcal{O}$-module.

Since $\mathbb{T}(\Gamma, k)_{\mathfrak{m}} \subset \mathrm{End}_{\mathcal{O}}(H^1(\Gamma, \mathrm{Sym}^{k-2}\mathcal{O}^2)_{\mathfrak{m}})$, we get

Cor $\mathfrak{m}$ non-Eisenstein $\Rightarrow$ $\mathbb{T}(\Gamma, k)_{\mathfrak{m}}$ is $\mathcal{O}$-flat

Proof of Prop (when $k=2$) WTS that $H^1(\Gamma, \mathcal{O})_{\mathfrak{m}}$ is p-torsion free.
Taking cohom of the exact sequence

$$0 \to \mathcal{O} \xrightarrow{\varpi} \mathcal{O} \to \mathbb{F} \to 0$$

and localizing at $\mathfrak{m}$, we get

$$H^0(\Gamma, \mathbb{F})_{\mathfrak{m}} \to H^1(\Gamma, \mathcal{O})_{\mathfrak{m}} \xrightarrow{\varpi} H^1(\Gamma, \mathcal{O})_{\mathfrak{m}}$$

Suffices to show

$$H^0(\Gamma, \mathbb{F})_{\mathfrak{m}} = 0$$

A double coset op $[\Gamma\alpha\Gamma]$ acts on $H^0(\Gamma, \mathbb{F})$ by

$$\begin{array}{ccccccc} H^0(\Gamma, \mathbb{F}) & \xrightarrow{\mathrm{res}} & H^0(\Gamma \cap \alpha^{-1}\Gamma\alpha, \mathbb{F}) & \xrightarrow{\alpha} & H^0(\alpha\Gamma\alpha^{-1} \cap \Gamma, \mathbb{F}) & \xrightarrow{\mathrm{cores}} & H^0(\Gamma, \mathbb{F}) \\ \| & & \| & & \| & & \| \\ \mathbb{F} & \xrightarrow{\mathrm{id}} & \mathbb{F} & \xrightarrow{\mathrm{id}} & \mathbb{F} & \xrightarrow[{[\Gamma : \alpha\Gamma\alpha^{-1}\cap\Gamma]}]{\text{mult by}} & \mathbb{F} \end{array}$$

For any $\ell \notin S$, T_ℓ acts on $H^0(\Gamma, \mathbb{F})$ by $1+\ell$
S_ℓ " " 1

So if $H^0(\Gamma, \mathbb{F})_{\mathfrak{m}} \neq 0 \Rightarrow T_\ell \equiv 1+\ell \mod \mathfrak{m}$
$S_\ell \equiv 1 \mod \mathfrak{m}$

By Chebotarev $\bar{\rho}_{\mathfrak{m}} = 1 \oplus \bar{\varepsilon}$, $\bar{\varepsilon}$ = mod p cyclotomic char, contradicting the fact that $\mathfrak{m}$ is non-Eisenstein. $\square$

Then $\mathbb{T}^S(\Gamma,k)_m \hookrightarrow \mathbb{T}^S(\Gamma,k)_m \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}_p = \prod_{\substack{\text{eigensystems} \\ \text{above } m}} \overline{\mathbb{Q}}_p$

so we have

$$\rho = \prod \rho_i : G_{\mathbb{Q},S} \to GL_2(\mathbb{T}^S(\Gamma,k)_m \otimes \cancel{\overline{\mathbb{Q}}_p}\ \overline{\mathbb{Z}}_p)$$

s.t. charpoly $\rho(\mathrm{Frob}_\ell) = X^2 - T_\ell X + \ell S_\ell \in \mathbb{T}(\Gamma,k)_m[X]$

This rep descends to

$$\rho_m : G_{\mathbb{Q},S} \to GL_2(\mathbb{T}^S(\Gamma,k)_m)$$

by a

Thm (Carayol) Let A be a local ring with residue fld F such that the Brauer group of F is trivial, and let R be an A-alg. (e.g. for us, $A = \mathbb{T}^S(\Gamma,k)_m$, F = finite, R = the group algebra $\mathbb{T}^S(\Gamma,k)_m[G_{\mathbb{Q},S}]$)
Let $A \subset A' = \prod_i A'_i$ be a semilocal ext with A'_i local with max ideals m'_i and res flds F'_i ($A' = \mathbb{T}^S(\Gamma,k)_m \otimes_{\mathbb{Z}} \overline{\mathbb{Z}}_p$)
Assume we have an A-alg rep

$$\rho' = \prod \rho'_i : R \otimes_A A' \to M_n(A') = \prod_i M_n(A'_i)$$

s.t.

1. $\mathrm{tr}\, \rho(r \otimes 1) \in A \quad \forall r \in R$
2. $\bar{\rho}_i : R \otimes_A F'_i \to M_n(F'_i)$ are all abs irred and s.t. $\mathrm{tr}\, \bar{\rho}'_i(r \otimes 1) \in F$ and indep of i

Then ρ' is conj to the scalar ext $\otimes_A A'$ of a rep

$$\rho : R \to M_n(A).$$