Lecture 7

If G is a locally compact group, it has left, resp right, Haar measure: $\exists$ a nontrivial Borel measure $d_\ell x$, resp. $d_r x$, s.t. $\forall g \in G$ and $f: G \to \mathbb{R}$ measurable,

$$\int_G f(gx)\, d_\ell x = \int_G f(x)\, d_\ell x \qquad \int_G f(xg)\, d_r x = \int_G f(x)\, d_r x$$

and $d_r x$, $d_\ell x$ are unique up to scalar

~~Eg~~ · $G = \mathbb{R}$, $d_\ell x = d_r x = dt$ = Lebesgue meas

· $G = \mathbb{R}_{>0}$, $d_\ell x = d_r x = \frac{dt}{t}$, $G = \mathbb{R}^\times_n$, $d_\ell x = d_r x = \frac{dt}{|t|}$

· $G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| a, b \in \mathbb{R}, a > 0 \right\}$ Guess: $d_\ell x = d_r x = \frac{1}{a}\, da\, db$

$d_r x = \frac{1}{a}\, da\, db$ but $d_\ell x = \frac{1}{a^2}\, da\, db$

Why? $U \leq G$ is meas, $g = \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \in G$

$$Ug = \left\{ \begin{pmatrix} \alpha a & b + \beta a \\ 0 & 1 \end{pmatrix} \middle| \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in U \right\} \qquad gU = \left\{ \begin{pmatrix} \alpha a & ab + \beta \\ 0 & 1 \end{pmatrix} \middle| \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in U \right\}$$

Note for any $g \in G$, $d_r(gx)$ is right inv, so $\exists\ \delta_G(g) \in \mathbb{R}_{>0}$ st $d_r(gx) = \delta_G(g)\, d_r x$, $\delta_G : G \longrightarrow \mathbb{R}_{>0}$ cts and a homomorphism, called the <u>modular character</u> of G. If $\delta_G = 1$, we say G is <u>unimodular</u>. Up to constants,

$$d_r x = \delta_G(x)\, d_\ell x$$

Eg · $G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| a, b \in \mathbb{R}, a > 0 \right\}$, $\delta_G \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = a$

- G abelian $\Rightarrow \delta_G = 1$
- G compact $\Rightarrow \delta_G = 1$, $\{1\} \subseteq \mathbb{R}_{>0}$ is the only compact subgroup

Fact If G is a connected Lie group, then

$$\delta_G(g) = \det Ad(g)$$

Consequence If G is connected reductive, i.e $G = \mathbb{G}(k)^\circ$, $\mathbb{G}$ a reductive group over $k = \mathbb{R}$ or $\mathbb{C}$, then $\delta_G = 1$

Sketch Cartan decomp $\Rightarrow$ $G \cong \mathbb{R}^n_{>0} \times H$ with H reductive with compact centre. Show only its hom $H \to \mathbb{R}_{>0}$ is trivial.

Fact 2 If G is a connected Lie group and H_1, H_2 are closed sub Lie groups of G with $H_1 \cap H_2$ is compact and $\{h_1 h_2 \mid h_1 \in H_1, h_2 \in H_2\} = G$ up to a set of meas 0, Then we can normalize Haar measures such that

$$\int_G f(x)\, dx = \int_{H_1 \times H_2} f(h_1 h_2) \frac{\det Ad_{H_1}(h_1)}{\det Ad_G(h_2)}\, dh_1\, dh_2$$

$d = d_\ell$ everywhere, or d_r everywhere

Consequence If G is unimodular,

$$dx = d_\ell h_1\, d_r h_2$$

In part, if G is connected reductive Lie group and $G = ANK$ is the Iwasawa decomp

$$dx = d_\ell(an)\, dk$$
$$= dk\, d_r(an)$$

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A (separable) <u>Hilbert space</u> is a $\mathbb{C}$-vect space V with an inner product $\langle , \rangle$ s.t. V is complete wrt $\|x\| = \sqrt{\langle x, x \rangle}$ and s.t. V contains a countable dense subset.

An operator $T : V \to V$ on a Hilbert space is <u>bounded</u> if $\exists\ c > 0$ s.t. $\|Tx\| \leq c\|x\|\ \forall x \in V$. T is <u>unitary</u> if $\langle Tx, Ty \rangle = \langle x, y \rangle\ \forall x, y \in V$

$$\iff \|Tx\| = \|x\|\ \forall x, y \in V.$$

A <u>representation</u> of G on a Hilbert space V is a hom

$$\pi : G \to GL(V) := \text{invertible ops } T \text{ s.t. } T \text{ and } T^{-1} \text{ are bounded}$$

s.t. $\underset{(g,v) \mapsto \pi(g)v}{G \times V \to V}$ is continuous. We say π is unitary if $\pi(g)$ is for every $g \in G$.

We say π is irreducible if the only <u>closed</u> G-inv subspaces are $\{0\}$ and V

Assume G is locally compact

Then we have unitary representations π_ℓ, resp. π_r, called the left-, resp right-, regular reps on

$$L^2(G, d_\ell x) \quad , \text{resp} \quad L^2(G, d_r x)$$

by $\quad (\pi_\ell(g)f)(h) = f(g^{-1}h) \quad (\pi_r(g)f)(h) = f(hg)$

<u>Unitary</u> for π_r

$$\|\pi_r(g)f\|^2 = \int_G |f(xg)|^2 d_r x = \int_G |f(x)|^2 d_r x = \|f\|^2$$

<u>Eg</u> $F = \mathbb{R}$ or $\mathbb{C}$, $G = GL_2(F)$ (or $GL_2(\mathbb{R})^+$ if $F = \mathbb{R}$)

Let $\chi_1, \chi_2 : F^\times \to \mathbb{C}$ cts characters. Consider the set of functions $\{ f : G \to \mathbb{C} \mid f \text{ is cts and } f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} x\right) = \chi_1(a)\chi_2(d)\left|\frac{a}{d}\right|^{\frac{1}{2}} f(x),$

$$\forall x \in G, \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in B \subseteq G \}$$

We can define an inner product on this space by

$$\langle f, h \rangle = \int_K f(k)\overline{h}(k)\,dk$$

Where $K = O(2)$ or $U(2)$ ($F = \mathbb{R}$ or $\mathbb{C}$) is the max compact

This is an inner product by the Iwasawa decomp.

$\langle f, f \rangle = 0 \Rightarrow f = 0$ on K

$\Rightarrow f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} k\right) = \chi_1(a)\chi_2(d)\left|\frac{a}{d}\right|^{\frac{1}{2}} f(k) = 0$

$\Rightarrow f = 0$ on G

Let $H(\chi_1, \chi_2)$ be the completion of the above space wrt $\| \ \|$

This is a Hilbert space rep of G by $(gf)(x) = f(xg)$

$\pi(\chi_1, \chi_2)$ the rep on $H(\chi_1, \chi_2)$

<u>Claim</u> If χ_1 and χ_2 are unitary, then $\pi(\chi_1, \chi_2)$ is unitary

Can check $\left|\frac{a}{d}\right| = \delta_B$, write $B = TN$, $T = \left\{\begin{pmatrix} a & \\ & d \end{pmatrix}\right\}$, $N = \left\{\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}\right\}$

Can further write $T = M \times A$, M compact, $A \cong \mathbb{R}^2_{>0}$

Choose $\varphi : B \to \mathbb{R}_{>0}$ cts such that $\int_B \varphi(b)\,d_r b = 1$

Averaging over $K \cap M$, can assume φ is right invariant under $K \cap M$. Then can extend φ to G by

$$\varphi(bk) = \varphi(b) \quad \forall b \in B, k \in K$$

Then $\int_B \varphi(bg)\,d_r b = 1 \quad \forall g \in G$

Now take $f \in H(\chi_1, \chi_2)$

$$\|\pi(g)f\| = \int_K |f(kg)|^2\,dk$$

$$= \int_K |f(kg)|^2 \, dk$$

$$= \int_K |f(kg)|^2 \left(\int_B \varphi(bk) \, d_r b \right) dk$$

$$= \int_{B \times K} |f(kg)|^2 \varphi(bk) \, d_r b \, dk$$

$$= \int_{B \times K} |f(kg)|^2 \varphi(bk) \, \delta_B(b) \, d_\ell b \, dk$$

$$= \int_{B \times K} |f(bkg)|^2 \varphi(bk) \, d_\ell b \, dk$$

since $|f(bkg)|^2 = \delta_B(b) |f(kg)|^2$ as χ_1, χ_2 are unitary

$$= \int_G |f(xg)|^2 \varphi(x) \, dx$$

$$= \int_G |f(x)|^2 \varphi(xg^{-1}) \, dx$$

$$= \int_{B \times K} |f(bk)|^2 \varphi(bkg^{-1}) \, d_\ell b \, dk$$

$$= \int_{B \times K} \delta_B(b) |f(k)|^2 \varphi(bkg^{-1}) \, d_\ell b \, dk$$

$$= \int_{B \times K} |f(k)|^2 \varphi(bkg^{-1}) \, d_r b \, dk$$

$$= \int_K |f(k)|^2 \left(\int_B \varphi(bg^{-1}) \, d_r b \right) dk$$

$$= \int_K |f(k)|^2 \, dk = \|f\|^2$$