Lecture 21

G = locally profinite Hausdorff group.

Let (π, V) be a smooth rep of G.

The $\mathbb{C}$-linear dual
$$V^* := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$$
has G-action by
$$(g\lambda)(v) = \lambda(g^{-1}v).$$

V^* may not be smooth, so we define the <u>smooth dual</u> to be
$$V^\vee := \bigcup_{\substack{K \text{ open compact} \\ \text{in } G}} (V^*)^K$$

Then $V^\vee$ is smooth by construction. If $K \leq G$ is a fixed open compact
$$V \cong \bigoplus_{\tau \in \hat{K}} V_\tau \quad , \quad V_\tau = K\text{-isotypic piece.}$$
$$= V^K \oplus \bigoplus_{1 \neq \tau \in \hat{K}} V_\tau$$

So any $\lambda : V^K \to \mathbb{C}$ linear, we can extend it to V by letting it be 0 on $\bigoplus_{1 \neq \tau \in \hat{K}} V_\tau$. This is inverse to the restriction map $\lambda \mapsto \lambda|_{V^K}$ and gives an iso
$$(V^K)^* \cong (V^\vee)^K$$

Hence, if V is smooth admissible

- so is $V^\vee$
- the canonical map $V \to (V^\vee)^\vee$ is an iso
- V irred $\Rightarrow V^\vee$ is irred.

<u>Rmk</u> If V is not admissible it can happen that V is irred but $V^\vee$ is not.

<u>Exercise</u> G is connected reductive over a nonarch local field F, $P \leq G$ is a parabolic subgroup with Levi decomp $P = MN$, and (σ, W) is a smooth adm rep of $M(F)$. Show
$$\left(n\mathrm{Ind}_{P(F)}^{G(F)} W\right)^\vee \cong n\mathrm{Ind}_{P(F)}^{G(F)} W^\vee$$

<u>Hint</u> Let K be a max compact subgroup of $G(F)$ and define the pairing
$$n\mathrm{Ind}_{P(F)}^{G(F)} W \times n\mathrm{Ind}_{P(F)}^{G(F)} W^\vee \longrightarrow \mathbb{C}$$
$$(f, \phi) \longmapsto \int_K \phi(k)(f(k))\, dk$$

Back to generalities.

For $v \in V$, $\lambda \in V^\vee$, we form the <u>matrix coefficient</u>
$$m_{v,\lambda} : G \to \mathbb{C}$$
$$g \mapsto \lambda(gv)$$

<u>Def</u> A smooth admissible G-rep (π, V) is called <u>supercuspidal</u> (or just <u>cuspidal</u>) if all its matrix coefficients are compactly supported mod centre, i.e. $\exists$ a compact subset $\Omega \subseteq G$ s.t.
$$\mathrm{supp}(m_{v,\lambda}) \subset Z\Omega.$$

<u>Rmk</u> If (π, V) above is irred, it suffices to check a single nonzero matrix coeff $m_{v,\lambda}$ has compact supp mod centre. We saw above that $V^\vee$ is also irred, so for any $v' \in V$, resp. $\lambda' \in V^\vee$, is a lin comb of elements of the form gv, resp. $h\lambda$, $g, h \in G$. Then $m_{v',\lambda'}$ is a lin comb of the matrix coeffs
$$m_{gv, h\lambda} : x \longmapsto \lambda(h^{-1}xgv) \quad \text{has compact supp mod centre.}$$

Now say G connected red over a nonarch local fld F. We'll apply above to $G(F)$.

<u>Prop</u> Let H be an open subgroup of $G(F)$ containing the centre, and compact mod centre. Let (σ, W) be an irred fin dim rep of H.

If

$$c\mathrm{Ind}_H^{G(F)} W := \{ f : G \to W \mid f \text{ has compact supp mod centre and s.t. } f(hg) = \sigma(h) f(g) \text{ for all } h \in H, g \in G(F) \}$$

is irred and admissible, then it is supercuspidal. *Proof included at the end.*

<u>Folklore Conj</u> All supercuspidals arise this way.

To prove the prop, suffices to exhibit a single matrix coeff that is compactly supp mod centre.

<u>Eg</u> $GL_2(F)$. Look at irreps of $GL_2(\mathbb{F}_q)$, $\mathbb{F}_q$ = res fld of F. Order is $(q^2-1)(q^2-q)$, and has q^2-1 conj classes. Let $\mathbb{F}_{q^2}/\mathbb{F}_q$ be the unique quad ext. Let $\Theta : \mathbb{F}_{q^2}^\times \to \mathbb{C}^\times$ be a character.

Fix a basis of $\mathbb{F}_{q^2}$ over $\mathbb{F}_q \Rightarrow \mathbb{F}_{q^2}^\times \hookrightarrow GL_2(\mathbb{F}_q)$

Assume $\Theta^q \neq \Theta$. Fix nontrivial

$$\psi : N(\mathbb{F}_q) \to \mathbb{C}^\times$$
$$N(\mathbb{F}_q) = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| x \in \mathbb{F}_q \right\}$$

Set $\Theta_\psi : \mathbb{F}_q^\times N(\mathbb{F}_q) \to \mathbb{C}^\times$

$$\begin{pmatrix} a & \\ & a \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \longmapsto \Theta|_{\mathbb{F}_q^\times}(a)\, \psi\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$$

Computing characters, can show

$$\mathrm{Ind}_{\mathbb{F}_{q^2}^\times}^{GL_2(\mathbb{F}_q)} \Theta$$

is a subrep of $\mathrm{Ind}_{\mathbb{F}_q^\times N(\mathbb{F}_q)}^{GL_2(\mathbb{F}_q)} \Theta_\psi$

The quotient is an irrep σ of $GL_2(\mathbb{F}_q)$ of dimension $q-1$, called a cuspidal rep of $GL_2(\mathbb{F}_q)$.

Inflate it to $GL_2(\mathcal{O}_F)$. Extend to $F^\times GL_2(\mathcal{O}_F)$ by lifting central char of σ to $\mathcal{O}_F^\times$ and extending to $F^\times$. Then

$$c\mathrm{Ind}_{F^\times GL_2(\mathcal{O}_F)}^{GL_2(F)} \sigma \text{ is a supercusp sm adm rep}$$

of $GL_2(F)$.

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Back to G connected red $/F$.

Let $P = MN$ be an F-parabolic.

Let (π, V) a sm adm rep of $G(F)$.

Set $V(N) := \mathrm{span}\{ \pi(n)v - v \mid n \in N(F) \}$

$$V_N := V / V(N)$$

$M(F)$ acts on V_N by $\pi|_{M(F)}$.

Define $J_P(\pi, V) := V_N$ with $M(F)$-action by

$$\pi_N := \pi|_{M(F)} \otimes \delta_P^{-\frac{1}{2}}$$

The <u>Jacquet module</u> of (π, V) wrt P.

This is a functor

$$\text{Smooth } G(F)\text{-reps} \to \text{Smooth } M(F)\text{-reps}.$$

<u>Prop</u> J_P is left adjoint to $n\mathrm{Ind}_{P(F)}^{G(F)}$.

<u>Sketch</u> (π, V) is a smooth rep of $G(F)$

(σ, W) " " $M(F)$

Check the following maps are inverses and functorial in (π, V) and (σ, W):

$$\mathrm{Hom}_{G(F)}(V, n\mathrm{Ind}_{P(F)}^{G(F)} W) \longrightarrow \mathrm{Hom}_{M(F)}(J_P(V), W)$$
$$\phi \longmapsto (v \mapsto \phi(v)(1))$$

$$\mathrm{Hom}_{M(F)}(J_P(V), W) \longrightarrow \mathrm{Hom}_{G(F)}(V, n\mathrm{Ind}_{P(F)}^{G(F)} W)$$
$$\psi \longmapsto (v \mapsto (g \mapsto \psi(gv)))$$

Thm (Jacquet)

1. J_P takes admissible reps to admissible reps.
2. An irred sm adm rep (π, V) is supercuspidal $\Leftrightarrow$ $J_P(V) = 0$ $\forall$ proper F-parabolics $P \subsetneq G$.

Thm If (π, V) is an irred smooth adm rep of $G(F)$, then $\exists$ an F-parabolic subgroup $P \leq G$ with Levi decomp $P = MN$ and a supercuspidal rep (σ, W) of $M(F)$ such that (π, V) is isomorphic to a subrep of

$$n\operatorname{Ind}_{P(F)}^{G(F)} \sigma$$

Proof Since V is irreducible, it suffices to show $\exists$ nonzero G-equiv map

$$V \to n\operatorname{Ind}_{P(F)}^{G(F)} W$$

with (σ, W) as in the statement of the Thm.

We induct on the dimension of G.

If $\dim G = 1$, it is a torus and equals its centre so any function on $G(F)$ is compact mod centre.

Now assume $\dim G > 1$.

First assume there are no G-equivariant maps

$$V \to n\operatorname{Ind}_{P(F)}^{G(F)} W$$

for any proper parabolic $P = MN$ and smooth admissible rep (σ, W) of $M(F)$. Then

By adjointness of J_P to $n\operatorname{Ind}_{P(F)}^{G(F)}$ and the first part of Jacquet's Thm (that $J_P(V)$ is admissible) we have $J_P(V) = 0$ for all proper parabolic subgroups P.

By the 2^{nd} part of Jacquet's Thm above, (π, V) is supercuspidal.

Now assume there is a proper F-parabolic $P = MN$, a smooth admissible rep (σ, W) of $M(F)$, and a nonzero $G(F)$-equiv map

$$V \to n\operatorname{Ind}_{P(F)}^{G(F)} W$$

hence (by adjointness) a nonzero $M(F)$-equiv map

$$J_P(V) \to W \qquad (*)$$

Since P is proper, $\dim M < \dim G$, and our induction hypothesis implies that $\exists$ a parabolic subgroup Q of M with Levi subgroup L, a supercuspidal representation (ρ, U) of $L(F)$, and a nonzero $M(F)$-equiv map

$$W \to n\operatorname{Ind}_{Q(F)}^{M(F)} U$$

Composing with $(*)$ and applying the adjunction, we have a nonzero $G(F)$-equiv map

$$V \to n\operatorname{Ind}_{P(F)}^{G(F)}\left(n\operatorname{Ind}_{L(F)}^{M(F)} U\right)$$

Now QN is a parabolic subgroup of G with Levi subgroup L and transitivity of induction (we didn't prove this, but it's true) gives

$$n\operatorname{Ind}_{P(F)}^{G(F)}\left(n\operatorname{Ind}_{Q(F)}^{M(F)} U\right) \cong n\operatorname{Ind}_{(QN)(F)}^{G(F)} U \qquad \square$$

Proof that $c\operatorname{Ind}_H^{G(F)} W$ is supercuspidal if it is irred and admissible:

It suffices to check a single matrix coeff is nonzero.

Fix $0 \neq w \in W$ and $0 \neq \lambda \in W^*$ such that $\lambda(w) \neq 0$.

Define $f_w \in c\operatorname{Ind}_H^{G(F)} W$ by

$$f_w(g) = \begin{cases} \sigma(g)w & \text{if } g \in H \\ 0 & \text{otherwise} \end{cases}$$

Define $f_\lambda \in c\operatorname{Ind}_H^{G(F)} W^*$ similarly. Given any $f \in c\operatorname{Ind}_H^{G(F)} W$, define $\langle f_\lambda, f\rangle = \langle f_\lambda(1), f_w(1)\rangle$. This realizes f_λ as an element of $(c\operatorname{Ind}_H^{G(F)} W)^\vee$ and we can form the matrix coefficient

$$m_{f_w, f_\lambda} : g \longmapsto \langle f_\lambda, g f_w\rangle$$

Note

$$\begin{aligned} m_{f_w, f_\lambda}(g) &= \langle f_\lambda, g f_w\rangle \\ &= \langle f_\lambda(1), (g f_w)(1)\rangle \\ &= \langle \lambda, f_w(g)\rangle \end{aligned}$$

Thus $m_{f_w, f_\lambda}(1) = \langle \lambda, w\rangle \neq 0$ and $\operatorname{supp}(m_{f_w, f_\lambda}) \subseteq H$. $\square$