Lecture 13

$G = \mathbb{G}(k)$ or $\mathbb{G}(k)^c$ with $\mathbb{G}$ connected reductive, $k = \mathbb{R}$ or $\mathbb{C}$

π an irred unitary rep on a Hilbert space H

<u>Reduction to semisimple case</u> $Z = Z(G)$. Then since H is irred, Schur's Lemma $\Rightarrow$ Z acts by a character

$$\chi : Z \to \mathbb{C}^\times$$

But π unitary $\Rightarrow$ χ is unitary, i.e. $|\chi(z)| = 1$ $\forall z \in Z$.

Let $L^2(G, \chi) := \{ f : G \to \mathbb{C}$ measurable $|$

$$f(zg) = \chi(z) f(g) \quad \forall z \in Z, g \in G$$

$$\text{and } \int_{G/Z} |f(x)|^2 dx < \infty \}$$

And G/Z is a semisimple Lie group, so integrability conditions reduce to those on a semisimple group.

Recall that a matrix coefficient of H is a function

$$\varphi_{v,w} : g \longmapsto \langle \pi(g) v, w \rangle \qquad \text{for fixed } v, w \in H,$$

and if v, w are K-finite, we say $\varphi_{v,w}$ is <u>K-finite</u>.

<u>Thm</u> The following are equivalent.

1. Some nonzero K-finite matrix coefficient is in $L^2(G, \chi)$
2. All matrix coefficients are in $L^2(G, \chi)$
3. π is isomorphic to a direct summand of $L^2(G, \chi)$.

<u>Def</u> If π satisfies these equivalent conditions, we say π is <u>discrete series</u>.

Proof sketch Let V be the set of K-finite vectors in H.
Recall, H is admissible by Harish-Chandra.
Consider
2'. All K-finite matrix coeffs are in $L^2(G,\psi)$.
Clearly $2 \Rightarrow 2' \Rightarrow 1$.

$1 \Rightarrow 2'$ Let $\varphi_{v,w}(g) = \langle \pi(g)v, w\rangle \in L^2(G,\psi)$, $v,w \in V$.
Let $C_K := \{ f \in C^\infty_{comp}(G) \mid f \text{ is right and left } K\text{-finite}\}$,
an algebra under convolution

$$(g * f)(x) = \int_G g(xy^{-1}) f(y)\, dy$$

and acts on H by

$$\pi(f)v = \int_G f(x)\, \pi(x) v\, dx$$

Can check that $\{\pi(f)v \mid f \in C_K\}$ is a $(\mathfrak{g}, K)$-submodule of V. But by Harish-Chandra, H irred $\Rightarrow$ V irred, so

$$C_K V = V.$$

It then suffices to prove for any $f, h \in C_K$,

$$g \mapsto \langle \pi(g)\pi(f)v, \pi(h)w\rangle \in L^2(G,\psi)$$

Fix $f, h \in C_K$ with support in $U \subseteq G$ compact.
Set $f^*(x) = \overline{f(x^{-1})}$.
Then

$$\int_G |\langle \pi(f^*)\pi(x)\pi(h)v, w\rangle|^2 dx$$

$$= \int_G \left| \int_{U\times U} f^*(y) h(y') \langle \pi(yxy')v, w\rangle\, dy\, dy' \right|^2 dx$$

$$\leq \int_G \left[\int_{U\times U} |f^*(y)h(y')|^2 dy\,dy'\right]\left[\int_{U\times U} |\langle \pi(yxy')v, w\rangle|^2 dy\,dy'\right] dx$$

by Cauchy-Schwartz

$$\leq \|f\|\,\|h\| \int_{G\times U\times U} |\langle \pi(yxy')v, w\rangle|^2 dy\,dy'\,dx$$

substitute $yxy' \to x$

$$\leq \|f\|\,\|h\|\, \operatorname{meas}(U)^2 \int_G |\langle \pi(x)v, w\rangle|^2 dx < \infty$$

$\underline{2' \Rightarrow 2+3}$: Fix $0 \neq w \in V$, Define

$$B : V \to L^2(G, \chi)$$
$$v \mapsto \varphi_{v,w} : x \mapsto \langle \pi(x)v, w\rangle$$

Check that for $X \in \mathfrak{g}$

$$B(\pi(X)v)(g)$$
$$= \langle \pi(g)\pi(X)v, w\rangle$$
$$= \langle \pi(X)v, \pi^*(g)w\rangle$$
$$= \left\langle \lim_{t\to 0} \frac{\pi(e^{tX})v - v}{t}, \pi^*(g)w\right\rangle$$
$$= \lim_{t\to 0} \frac{\langle \pi(e^{tX})v, \pi^*(g)w\rangle - \langle v, \pi^*(g)w\rangle}{t}$$
$$= \lim_{t\to 0} \frac{\langle \pi(ge^{tX})v, w\rangle - \langle \pi(g)v, w\rangle}{t}$$
$$= (R(X)\varphi_{v,w})(g)$$

Similarly, B is equivariant for the K-action.

So B is an iso onto a $(\mathfrak{g}, K)$-submodule W of $L^2(G,\Psi)$. Some analytical arguments show that the closure $\overline{W}$ of W in $L^2(G,\Psi)$ is G-stable, irred (Harish-Chandra) and B extends to a unitary equiv, up to scalar. (Also should check it is a direct summand.)

<u>$3 \Rightarrow 1$</u>: WLOG H is a direct summand of of $L^2(G,\Psi)$, and let $P: L^2(G,\Psi) \to H$ the orthog projection. Choose $h_0 \in C_{comp}(G)$ K-finite such that $h := P(h_0) \neq 0$.

Can show that $g \mapsto \langle R(g)h, h\rangle$ is in $L^2(G,\Psi)$.

<u>Thm</u> (Bargmann) The irreducible unitary reps of $SL_2(\mathbb{R})$ are classified as follows.

1. The trivial one dim rep.
2. The class 1 principal series π_s^+, $s \in i\mathbb{R}$, the normalized induction of

$$\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix} \mapsto |a|^s$$

$$\pi_s^+ \cong \pi_{-s}^+$$

3. The non class 1 principal series π_s^-, $s \in i\mathbb{R} \setminus \{0\}$ the normalized induction of

$$\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix} \mapsto \operatorname{sign}(a)\,|a|^s$$

Here $\pi_s^- \cong \pi_{-s}^-$

4. The complementary series, $\hat{\pi}_s^c$, $s \in \mathbb{R}$, $-1 < s < 1$, the normalized induction of

$$\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix} \mapsto |a|^s$$

but with a different innerproduct (!).

5. Limits of discrete series, $\hat{\pi}_0^{\pm}$, normalized induction of $\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix} \mapsto \text{sign}(a)$ decomposes as $\pi_0^- \oplus \pi_0^+$

6. The discrete series reps π_k^+, π_k^-, $k \in \mathbb{Z}_{\geq 2}$.

Space for π_k^+ is

$$\mathcal{D}_k^+ = \{ f : \mathbb{H} \to \mathbb{C} \text{ holomorphic} \mid \int_{\mathbb{H}} |f(z)|^2 y^{k-2}\, dx\, dy < \infty \}$$

where $\mathbb{H}$ = upper half plane and

$$\hat{\pi}_k^+ \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(z) = (bz+d)^{-k} f\left(\frac{az+c}{bz+d}\right)$$

and π_k^- is defined similarly using complex conj, i.e. lower half plane, antiholomorphic, etc...

Rmk Can show the normalized induction of

$$\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix} \mapsto \text{sign}(a) |a|^{k-1} \text{ decomposes as}$$

$$\pi_k^- \oplus \text{Sym}^{k-2} \mathbb{C}^2 \oplus \pi_k^+$$

The Casimir operator $\Omega = -2\Delta$ acts on these by

- $\frac{1}{2}(s^2-1)$ on $\pi_s^{\pm}$, $s \in i\mathbb{R}$, or π_s^c, $s \in (-1,1)$
- $\frac{1}{2}k(k-2)$ on $\pi_k^{\pm}$, $k \geq 2$

<u>Exercise</u> Compute the action of Ω on the normalized of a character $\chi\begin{pmatrix} a & b \\ & a^{-1}\end{pmatrix} = \chi(a) \in \mathbb{C}^{\times}$, assuming it acts by a scalar.

Recall that if Γ is a congruence subgroup of $SL_2(\mathbb{Z})$, and $f \in S_k(\Gamma)$, $k \geq 2$, we defined

$$\phi_f \in L^2(\Gamma \backslash SL_2(\mathbb{R}))$$

and it generates a closed subspace $H(f)$ that one can show decomposes as a finite direct sum

$$H(f) = \bigoplus_i H_i$$

But we computed earlier that

$$\Omega \phi_f = \frac{1}{2}k(k-2)$$

By the classification, each H_i is discrete series, in fact π_k^{+}.

<u>Rmk</u> Maass form generate direct summands of $L^2(\Gamma \backslash G)$ that are not discrete series reps.

Conj (Selberg) If f is a cuspidal Maas form with Δ-eigenvalue λ, then $\lambda \geq \frac{1}{4}$.

$\Leftrightarrow$ the reps in $L^2(\Gamma \backslash SL_2(\mathbb{R}))$ they generate are not complementary series.