Archimedean theory

$k = \mathbb{R}$ or $\mathbb{C}$. Let $\mathbb{G}$ be a connected (affine) alg group over k and let $G = \mathbb{G}(k)$.

Then G is a Lie group. Indeed, if we choose $\mathbb{G} \hookrightarrow GL_n$, then G is a closed subgroup of $GL_n(k)$.

Let $\Theta : GL_n(k) \to GL_n(k)$

$$g \longmapsto {}^t\bar{g}^{-1}$$

Say $\mathbb{G}$ is Θ-stable and let $\mathcal{U} = R_u(\mathbb{G})$, the unipotent radical of $\mathbb{G}$.

Take $g \in \mathcal{U}(\mathbb{C})$. An automorphism of $\mathbb{G}$ fixes $\mathcal{U}$, so $\Theta(g) \in \mathcal{U}$ and $h := \Theta(g) g^{-1} \in \mathcal{U}$. But

$$h = {}^t\bar{g}^{-1} g^{-1} \quad \text{so} \quad {}^t\bar{h} = h \Rightarrow h \text{ is diagonalizable}$$

h semisimple + unipotent $\Rightarrow h = I$.

Conversely:

<u>Thm</u> (Mostow) If $\mathbb{G}$ is reductive, $\exists\ \mathbb{G} \hookrightarrow \mathbb{G}L_n$ such that $\mathbb{G}$ is stable under Θ.

Assume from now on that we have such an embedding.

So Θ restricts to G and is called a <u>Cartan involution</u>.

Set $K = \{ g \in G \mid \Theta g = g \}$

$= G \cap U(n)$, a compact subgroup of G.

On the Lie algebra

$$\mathfrak{g} = \text{Lie}(G) = \text{Lie}(\mathbb{G})$$

Θ induces $X \mapsto -{}^t\overline{X}$, an involution

$$\Rightarrow \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \begin{array}{l} \mathfrak{k} = +1 \text{ eigenspace} \\ \mathfrak{p} = -1 \text{ eigenspace} \end{array}$$

called a <u>Cartan decomposition</u> of $\mathfrak{g}$, and

$$\mathfrak{k} = \text{Lie}(K)$$

Note for $X \in \mathfrak{p}$, can form $e^X := \sum_{n=0}^{\infty} \frac{1}{n!} X^n \in G$

<u>Prop</u> K is a maximal compact subgroup of G meeting all connected components and the map

$$\begin{aligned} K \times \mathfrak{p} &\to G \\ (k, X) &\mapsto k e^X \end{aligned}$$

is a diffeomorphism.

<u>Proof when G is connected</u>

Let $g \in G$. By the polar decomposition, we can write $g = kh$ with $k \in U(n)$, h = +ve def Hermitian.

and $h = e^X$ for a Hermitian matrix X (diagonalize h).

Then $\Theta(g) = k e^{-X}$, so $(\Theta g)^{-1} g = e^{2X} \in G$

<u>Claim</u> $X \in \mathfrak{g}$.

It suffices to show $e^{tX} \in G \ \forall t \in \mathbb{R}$.

Conjugating, we can assume $e^{tX} = \mathrm{diag}(e^{ta_1}, \ldots, e^{ta_n})$

Since $G = G(k)$ is defined by polynomials, it suffices to show $e^{mX} = \mathrm{diag}(e^{ma_1}, \ldots, e^{ma_n}) \in G$ for ∞ly many integers m.

But $e^{2X} \in G \Rightarrow e^{2rX} \in G \ \forall r \in \mathbb{Z}$.

So $X \in \mathfrak{g}$.

Then $k = g(e^X)^{-1} \in G$, so

$$K \times \mathfrak{p} \to G$$

is surjective. It is injective by uniqueness of the polar decomposition, and smooth because mult and exp are smooth. Can check the inverse is also smooth. □

Recall,

$$\mathfrak{g} = \mathfrak{g}_0 \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

$$= \mathfrak{k} \oplus \mathfrak{p}$$

Can check that $\Theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ and $\Theta(\mathfrak{g}_0) = \mathfrak{g}_0$.

So $\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}$ with $\mathfrak{m} = \mathfrak{g}_0 \cap \mathfrak{k}$ and $\mathfrak{a} = \mathfrak{g}_0 \cap \mathfrak{p}$.

Fix +ve roots Φ^+ of Φ and let $\mathfrak{n} = \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$.
Then for any $X \in \mathfrak{g}$, we can write

$$X = H + X_0 + \sum_{\alpha\in\Phi} X_\alpha, \quad H \in \mathfrak{a},\ X_0 \in \mathfrak{m},\ X_\alpha \in \mathfrak{g}_\alpha$$

$$= \left\{ X_0 + \sum_{\alpha\in\Phi^+} (X_{-\alpha} + \theta X_{-\alpha}) \right) + H + \left(\sum_{\alpha\in\Phi^+} X_\alpha - \theta X_{-\alpha} \right)$$

$$\in \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$$

and can check

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$$

(use that proj of $X \in \mathfrak{k}$ to $\mathfrak{g}_\alpha$ is nonzero $\Leftrightarrow$ proj to $\mathfrak{g}_{-\alpha}$ is non zero)

Now $\exists$ a Lie subgroup A of G with $A \cong \mathbb{R}_{>0}^{\dim\mathfrak{a}}$

$$A = e^{\mathfrak{a}} \text{ a closed subgroup of } G$$
$$\cong \mathbb{R}_{>0}^{\dim\mathfrak{a}}$$

and N unipotent with Lie alg $\mathfrak{n}$

<u>Eg</u> • $G = SL_2(\mathbb{R})$

$K = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right\}$, $A = \left\{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \middle| a > 0 \right\}$,

$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\}$

• $SL_2(\mathbb{C})$, $K = SU(2)$, $A = \left\{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \middle| a \in \mathbb{R}_{>0} \right\}$

$N = $ ~

- $Sp_4(\mathbb{R})$

$$K = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \,\middle|\, A + iB \in U(2) \right\} \cong U(2)$$

$$A = \left\{ \begin{pmatrix} a & & & \\ & b & & \\ & & a^{-1} & \\ & & & b^{-1} \end{pmatrix} \,\middle|\, a, b \in \mathbb{R}_{>0} \right\} \qquad N = \left\{ \begin{pmatrix} 1 & 0 & * & * \\ * & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \right\} \cap Sp_4(\mathbb{R})$$

<u>Thm</u> (Iwasawa decomposition)

$$\begin{aligned} \phi : K \times A \times N &\to G \\ (k, a, n) &\mapsto kan \end{aligned}$$

is a diffeomorphism.