Lecture 15 - Taylor-Wiles primes on modular forms 3

Recall we have

$\bar{\rho} : G_{\mathbb{Q},S} \to GL_2(\mathbb{F})$ abs irred w/ $\mathfrak{m} \subset \mathbb{T}^{S,\text{univ}}$ non-Eis

$(Q, \{\alpha_v\}_{v \in Q})$ a Taylor-Wiles datum w/ Δ_Q

$\mathfrak{m}_Q = \mathfrak{m}_\alpha = (\mathfrak{m}, \{U_v - \tilde{\alpha}_v\}_{v \in Q}) \subset \mathbb{T}_Q^{S \cup Q, \text{univ}}$, $\tilde{\alpha}_v \in \mathcal{O}$ lifting α_v

$\Gamma_Q < \Gamma_0(Q) < \Gamma$, $Y = Y(\Gamma)$, $Y_0(Q) = Y(\Gamma_0(Q))$, $Y_Q = Y(\Gamma_Q)$

$H^1(\Gamma, \mathbb{F})_\mathfrak{m} \neq 0$, $\Gamma_0(Q)/\Gamma_Q \cong \Delta_Q$, and

__Prop 1__ The natural map
$$H_1(Y_0(Q), \mathcal{O}) \longrightarrow H_1(Y, \mathcal{O})$$
induces an iso
$$H_1(Y_0(Q), \mathcal{O})_{\mathfrak{m}_Q} \cong H_1(Y, \mathcal{O})_\mathfrak{m}$$

Now we prove

__Prop 2__ $H_1(Y_Q, \mathcal{O})_{\mathfrak{m}_Q}$ is a free $\mathcal{O}[\Delta_Q]$-module and the natural map
$$H_1(Y_Q, \mathcal{O})_{\mathfrak{m}_Q} \longrightarrow H_1(Y_0(Q), \mathcal{O})_{\mathfrak{m}_Q}$$
induces an iso from the Δ_Q-coinvs of $H_1(Y_Q, \mathcal{O})_{\mathfrak{m}_Q}$ to $H_1(Y_0(Q), \mathcal{O})_\mathfrak{m}$.

Combining Prop 1 + Prop 2, we get

<u>Prop 3</u> $H_1(Y_Q, \mathcal{O})_{m_Q}$ is a free $\mathcal{O}[\Delta_Q]$-module and the natural map
$$H_1(Y_Q, \mathcal{O})_{m_Q} \longrightarrow H_1(Y_0(Q), \mathcal{O})_{m_Q}$$
induces an iso from the Δ_Q-coinvs of $H_1(Y_Q, \mathcal{O})_{m_Q}$ to $H_1(Y_0(Q), \mathcal{O})_m$.

To prove Prop 1, first recall that if $i \neq 1$,
$$H_i(Y_Q, \mathbb{F})_{m_Q} = \mathrm{Hom}(H^i(Y_Q, \mathcal{O})_{m_Q}, \mathbb{F}) = 0$$
and as a consequence
$$H_i(Y_Q, \mathcal{O})_{m_Q} = \begin{cases} 0 & \text{if } i \neq 1 \\ \mathcal{O}\text{-free} & \text{if } i = 1 \end{cases}$$

<u>Proof of Prop 3</u> (We switch to group homology for the proof.)

The Hochschild-Serre spectral sequence gives
$$H_i(\Delta_Q, H_j(\Gamma_Q, \mathcal{O})) \Rightarrow H_{i+j}(\Gamma_0(Q), \mathcal{O})$$
Localizing at m and using above we get
$$H_0(\Delta_Q, H_1(\Gamma_Q, \mathcal{O})_m) \cong H_1(\Gamma_0(Q), \mathcal{O})_m$$

It remains to prove that $H_1(\Gamma_Q, \mathcal{O})_{m_Q}$ is free $/ \mathcal{O}[\Delta_Q]$

Fact from Comm Alg: Since $\mathcal{O}[\Delta_Q]$, an $\mathcal{O}[\Delta_Q]$-module M is free $\Leftrightarrow$ it is flat $\Leftrightarrow$ $\mathrm{Tor}_1^{\mathcal{O}[\Delta_Q]}(M, \mathbb{F}) = 0$.

First, again using Hochschild–Serre,

$$0 = H_2(\Gamma_0(Q), \mathcal{O})_{m_Q} = H_1(\Delta_Q, H_1(\Gamma_Q, \mathcal{O})_{m_Q})$$
$$= \mathrm{Tor}_1^{\mathcal{O}[\Delta_Q]}(H_1(\Gamma_Q, \mathcal{O})_{m_Q}, \mathcal{O})$$

Then, tensoring

$$0 \to \mathcal{O} \xrightarrow{\varpi} \mathcal{O} \to \mathbb{F} \to 0$$

over $\mathcal{O}[\Delta_Q]$ with $H_1(\Gamma_Q, \mathcal{O})_{m_Q}$ and using above, we have an exact sequence

$$\begin{tikzcd} 0 = \mathrm{Tor}_1^{\mathcal{O}[\Delta_Q]}(H_1(\Gamma_Q, \mathcal{O})_{m_Q}, \mathcal{O}) \arrow[r] & \mathrm{Tor}_1^{\mathcal{O}[\Delta_Q]}(H_1(\Gamma_Q, \mathcal{O})_{m_Q}, \mathbb{F}) \arrow[r] & {} \\ H_1(\Gamma_Q, \mathcal{O})_{m_Q} \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O} \arrow[r, "\varpi"] \arrow[d, "\cong"] & H_1(\Gamma_Q, \mathcal{O})_{m_Q} \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O} \arrow[r] \arrow[d, "\cong"] & H_1(\Gamma, \mathbb{F})_{m_Q} \arrow[r] & 0 \\ H_1(\Gamma_0(Q), \mathcal{O})_{m_Q} \arrow[r, "\varpi"] & H_0(\Gamma_0(Q), \mathcal{O})_{m_Q} \arrow[ur] & & \end{tikzcd}$$

But $H_1(\Gamma_0(Q), \mathcal{O})_{m_Q} \xrightarrow{\varpi} H_1(\Gamma_0(Q), \mathcal{O})_{m_Q}$ is injective, so

$$\mathrm{Tor}_1^{\mathcal{O}[\Delta_Q]}(H_1(\Gamma_Q, \mathcal{O})_m, \mathbb{F}) = 0$$

$\square$

Recall that if we have a global def datum

$$\mathcal{S} = (\bar{\rho}, S, \psi, \mathcal{O}, \{D_v\}_{v \in S})$$

Then we have an augmented def datum

$$\mathcal{S}_Q = (\bar{\rho}, S, \psi, \mathcal{O}, \{D_v\}_{v \in S} \cup \{D_v^{\square, \psi}\}_{v \in Q})$$

and $R_{\mathcal{S}_Q}$ is an $\mathcal{O}[\Delta_Q]$-alg s.t.

$$R_{\mathcal{S}_Q} / \mathfrak{a}_Q \cong R_{\mathcal{S}}$$

with $\mathfrak{a}_Q =$ aug ideal.

We also have Galois reps

$$\rho_{m} : G_{\mathbb{Q},S} \to GL_2(\mathbb{T}^S(\Gamma)_m)$$

and $$\rho_{m_Q} : G_{\mathbb{Q},S} \to GL_2(\mathbb{T}_Q^{S\cup Q}(\Gamma)_{m_Q})$$

If they are of type S and S_Q, resp, then we have

$$\begin{array}{ccc} R_{S_Q} & \curvearrowright & H_1(Y_Q, \mathcal{O})_{m_Q} \\ {\scriptstyle \bmod \mathfrak{a}_Q} \downarrow & & \downarrow {\scriptstyle \bmod \mathfrak{a}_Q} \\ R_S & \curvearrowright & H_1(Y, \mathcal{O})_{m_Q} \end{array}$$