Lecture 15

Correction from last time

For the Thm classifying disc series need to assume G is connected. If G is not connected, the discrete series are parametrized by pairs (λ, χ) where

$\lambda : i\mathfrak{t} \to \mathbb{R}$ is as before

$\chi : Z = Z(G) \to \mathbb{C}^\times$ a unitary character s.t.

$$\chi = e^{\lambda} \text{ on } Z \cap G^\circ$$

The discrete series reps for (λ, χ) and (λ', χ') are equivalent $\iff \chi' = \chi$ and $\lambda' = w\lambda$ for $w \in W(G, \mathfrak{t}) = N_G(\mathfrak{t})/Z_G(\mathfrak{t})$

Idea Have π_λ as last time for G°. Extend to ZG° using χ. Then form $\mathrm{Ind}_{ZG^\circ}^{G} \pi_\lambda$

Eg $SL_2^{\pm}(\mathbb{R})$ from last time, $Z = \{\pm 1\} \subseteq SL_2(\mathbb{R})$, so χ is redundant and $W(G, \mathfrak{t}) = \{1, w\}$.

Rmk about Weyl groups $\exists$ 3 different types of Weyl groups.

1. $W(\Delta)$ = Weyl group of a root system Δ, generated by reflections s_α
2. $W(\mathbb{G}, \mathbb{T}) = N_{\mathbb{G}}(\mathbb{T})/Z_{\mathbb{G}}(\mathbb{T})$ for $\mathbb{T}$ a max torus in a reductive group $\mathbb{G}$
3. $W(G, \mathfrak{t}) = N_G(\mathfrak{t})/Z_G(\mathfrak{t})$ for G a Lie group and $\mathfrak{t} \subseteq \mathfrak{g}$ a Cartan.

If $\mathbb{G}$ is (Zariski) connected and $\mathbb{T}$ is split, then

$$W(\mathbb{G}, \mathbb{T}) \cong W_\Delta \text{ for } \Delta = \text{rootsys of } \mathbb{T} \text{ on } \mathfrak{g}$$

If G is (Euclidean) connected, reductive, Δ = root system for $\mathfrak{t}$ in $\mathfrak{g}_0$ then $W(G, \mathfrak{t}) \subseteq W_\Delta$ and if $G = \mathbb{G}(\mathbb{R})$, $\mathfrak{t} = \mathrm{Lie}\,T$, T split, then

$$W(G, \mathfrak{t}) = W(\mathbb{G}, \mathbb{T}) = W(\Delta).$$

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Many nice groups do not have discrete series, e.g. $SL_2(\mathbb{C})$, $SL_n(\mathbb{R})$ $n \geq 3$.

Def An irred admissible rep (π, H) of G is tempered if every K-finite matrix coeff of (π, H) lies in

$$L^{2+\varepsilon}(G) \quad \forall \varepsilon > 0.$$

Fact If (π, H) irred admissible and $1 \leq p \leq q < \infty$, all K-finite matrix coeffs are in $L^p(G)$

$$\Rightarrow \text{all in } L^q(G).$$

Consequence A discrete series rep is tempered.

Assume $G \subseteq GL_n(\mathbb{C})$ such that it is stable under $\Theta = g \mapsto {}^t\bar{g}$.

Recall we have an Iwasawa decomp,

$$G = P_0 K = N_0 A_0 K, \text{ where}$$

K = max compact

N_0 = unipotent

$A_0 \cong \mathbb{R}^r_{>0}$

Eg $GL_n(\mathbb{R})$, $A_0 = \begin{pmatrix} r_1 & & \\ & \ddots & \\ & & r_n \end{pmatrix}$ $r_i > 0$.

$$N_0 = \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}$$

$K = O(n)$

Eg $GL_n(\mathbb{C})$, $A_0 = \begin{pmatrix} r_1 & & \\ & \ddots & \\ & & r_n \end{pmatrix}$ $r_i > 0$, $N_0 = \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}$

$K = U(n)$.

We consider (closed) parabolic subgroups, $P \supseteq N_0 A_0 M_0 =: P_0$, $M_0 = Z_K(A_0)$.

Given such a P, let $M' = P \cap \Theta P$, then

$$P = NM' \text{ with } N \text{ unipotent}$$
$$= NAM \text{ with } M = M' \cap K,\ A \cong \mathbb{R}^s_{>0}$$

and $N \times A \times M \to P$ is a diffeomorphism.

Say we have an admissible rep (σ, H) of M.

$\cdot\ \nu : \mathfrak{a} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C}$, $\mathbb{C}$-lin, $\mathfrak{a} = \mathrm{Lie}\,A$.

We get a rep $1 \otimes e^{\nu} \otimes \sigma$ of P on H by

$$1 \otimes e^{\nu} \otimes \sigma\,(n e^{H} m) = e^{\nu(H)} \sigma(m)$$

We have an adm rep

$H(P,\sigma,\nu) = $ meas $f: G \to H$ such that

$$f(n e^H m g) = e^{(\nu+\delta_P)(H)} \sigma(m) f(g)$$

$\forall\, n e^H m \in P,\ g \in G,$

$$\|f\| = \int_K \|f(k)\|\, dk < \infty$$

Here δ_P = modular char of P.

Thm Let (π, H) be an irred adm rep of G. Then π is tempered $\iff$ $\exists\, P, \sigma, \nu$ as above s.t. σ is discrete series and $\operatorname{Re}\nu = 0$ s.t. π is equiv to a subrep of $H(P,\sigma,\nu)$.

Eg For $SL_2(\mathbb{R})$ we have

1. Trivial rep, not tempered.
2. Unitary princ series: $n\operatorname{Ind}_B^{SL_2(\mathbb{R})} \chi$, χ = unitary char of $\mathbb{R}^\times = \mathbb{R}_{>0} \times \{\pm 1\}$ $= |a|^{is}$ or $\operatorname{sign}(a)|a|^{is}$, $s \in \mathbb{R}$

   Tempered.
3. Complimentary series: $n\operatorname{Ind}_B^{SL_2(\mathbb{R})} \chi$, $\chi\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix} = |a|^s$, $s \in \mathbb{R}$, $-1 < s < 1$, $s \neq 0$. Not tempered.
4. Discrete series. Tempered.
5. Limits of discrete series D_1^+, D_1^-, $D_1^+ \oplus D_1^- = n\operatorname{Ind}_B^{SL_2(\mathbb{R})} \chi$, $\chi\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix} = \operatorname{sign}(a)$. Tempered.

Rmk Using the computation of the Casimir op on these, its relation to the Laplacian Δ on $\mathbb{H}$, and the representations gen by cuspidal modular forms + cuspidal Maass forms, Selberg's $\lambda \geq \frac{1}{4}$ conj is equivalent to saying any nontrivial irred rep appearing discretely in $L^2(\Gamma \backslash SL_2(\mathbb{R}))$ (Γ congruence) is tempered.

Thm (Langlands) $P = NAM$ as above.

σ = irred adm tempered on M

$\nu: \mathfrak{a} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C}$ s.t. $\langle \operatorname{Re}\nu, \alpha \rangle > 0$ $\forall \alpha \in \Delta^+$

Then $\exists$ a unique irred quotient (called the Langland quotient) $J(P,\sigma,\nu)$ of $H(P,\sigma,\nu)$. Moreover

$$\{(P,[\sigma],\nu)\} \longrightarrow J(P,\sigma,\nu)$$

is a bijection between

- triples $(P,[\sigma],\nu)$ where P,σ,ν as above and $[\sigma]$ the equiv class of σ

and

- equiv classes of irred adm reps of G.