Say $\mathfrak{g}$ is a Lie alg over $k=\mathbb{R}$ or $\mathbb{C}$.

The <u>universal enveloping algebra</u> is the assoc k-alg

$$U(\mathfrak{g}) := T(\mathfrak{g})/I(\mathfrak{g})$$

where
- $T(\mathfrak{g}) = \bigoplus_{n\geq 0} \mathfrak{g}^{\otimes n}$, the tensor alg
- $I(\mathfrak{g})$ is the 2-sided ideal gen by
$$\{X\otimes Y - Y\otimes X - [X,Y] \mid X, Y \in \mathfrak{g}\}$$

It is easy to see that $U(\mathfrak{g})$ represents the functor

$$\mathrm{AssocAlg}_{/k} \longrightarrow \mathrm{Sets}$$
$$A \longmapsto \mathrm{Hom}_{k\text{-LieAlg}}(\mathfrak{g}, A)$$

So we see that

{Lie alg reps of $\mathfrak{g}$ on complex vector spaces}

$\wr\wr$

{Lie alg reps of $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g}\otimes_k \mathbb{C}$ on complex vector spaces}

$\wr\wr$

{Assoc alg reps of $U(\mathfrak{g}_{\mathbb{C}})$ on complex vector spaces}

<u>Why do we care</u>? $U(\mathfrak{g}_{\mathbb{C}})$ has structure not seen in $\mathfrak{g}_{\mathbb{C}}$.

<u>Eg</u> $\mathrm{Lie}(SL_2) = \mathfrak{sl}_2 =$ trace 0 matrices in $M_2(k)$.

This is a simple nonab Lie alg.

$$E=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad F=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad H=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$[E,F]=H,\quad [H,F]=-2F,\quad [H,E]=2E$$

Consider $\Omega = \frac{1}{2}H^2 + EF + FE \in U(\mathfrak{sl}_2)$
$= \frac{1}{2}H^2 + H + 2FE$, called the <u>Casimir element</u>.

<u>Claim</u> $\Omega \in Z(U(\mathfrak{sl}_2))$

Since $U(\mathfrak{sl}_2)$ is generated by H, E, F, it suffices to check Ω commutes with each of these.

$$\begin{aligned}
H\Omega &= \tfrac{1}{2}H^3 + H^2 + 2HFE\\
&= \tfrac{1}{2}H^3 + H^2 + 2(-2F + FH)E\\
&= \tfrac{1}{2}H^3 + H^2 - 4FE + 2FHE\\
&= \tfrac{1}{2}H^3 + H^2 - 4FE + 2F(2E + EH)\\
&= \tfrac{1}{2}H^3 + H^2 + 2FEH\\
&= \Omega H
\end{aligned}$$

$$\begin{aligned}
E\Omega &= \tfrac{1}{2}EH^2 + EFE + E^2F\\
&= \tfrac{1}{2}(-2E + HE)H + EFE + E(H + FE)\\
&= \tfrac{1}{2}HEH + 2EFE\\
&= \tfrac{1}{2}H(-2E + HE) + 2EFE\\
&= \tfrac{1}{2}H^2E - HE + 2EFE\\
&= \tfrac{1}{2}H^2E - (EF - FE)E + 2EFE\\
&= \tfrac{1}{2}H^2E + FE^2 + EFE = \Omega E
\end{aligned}$$

Sim, $F\Omega = \Omega F$.

Now say $\Gamma \leq SL_2(\mathbb{Z})$ is a congruence subgroup and f is a cusp modular form of weight $k\geq 1$ and level Γ. Recall we defined

$$\phi = \phi_f \quad \text{by } \phi_f(g) = f(g(i))\, j(g,i)^{-k}$$

$g \in SL_2(\mathbb{R})$, $j\left(\begin{pmatrix}a&b\\c&d\end{pmatrix}, z\right) = cz+d$

And we saw $\phi \in L^2(\Gamma\backslash SL_2(\mathbb{R}))$

Now for any $\psi \in L^2(\Gamma\backslash SL_2(\mathbb{R}))$, consider the matrix coeff

$$g \longmapsto \langle g\phi, \psi\rangle = \int_{\Gamma\backslash SL_2(\mathbb{R})} \phi(xg)\overline{\psi(x)}\,dx$$

f holomorphic $\Rightarrow$ ϕ is smooth on G
$\Rightarrow$ $g \mapsto \langle g\phi,\psi\rangle$ is smooth on G
$\Rightarrow$ ϕ is a smooth vector in $L^2(\Gamma\backslash SL_2(\mathbb{R}))$

We have coordinates

$$g = \begin{pmatrix} y^{\frac12} & xy^{-\frac12}\\ & y^{-\frac12}\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}, \quad g\in SL_2(\mathbb{R})$$

A computation shows that

$$\begin{aligned}
\Omega &= -2\Delta\\
&= -2\left[-y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - y\frac{\partial^2}{\partial x\partial\theta}\right]
\end{aligned}$$

For g as above and $z = x+iy \in \mathbb{H}$,

$$\phi(g) = y^{\frac{k}{2}} f(z)\, e^{-ik\theta}$$

We compute

$$\Delta\phi = \left(-y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - y\frac{\partial^2}{\partial x\partial\theta}\right)\left(y^{\frac{k}{2}} f(z)\, e^{-ik\theta}\right)$$

$$\Delta\phi = \left(-y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) - y\frac{\partial^2}{\partial x\partial\theta}\right)\left(y^{\frac{k}{2}} f(z) e^{-ik\theta}\right)$$

$$= \left(iky^{\frac{k}{2}+1} e^{-ik\theta}\frac{\partial f}{\partial x}\right)$$

$$+ e^{-ik\theta}\left(-\frac{k}{2}\left(\frac{k}{2}-1\right)y^{\frac{k}{2}} f(z) - y^{\frac{k}{2}+2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)f(z) - ky^{\frac{k}{2}+1}\frac{\partial f}{\partial y}\right)$$

$$= -\frac{k}{2}\left(\frac{k}{2}-1\right)\phi + iky^{\frac{k}{2}+1}e^{-ik\theta}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)f(z)$$

$$- y^{\frac{k}{2}+2}e^{-ik\theta}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)f(z)$$

But f holomorphic $\Rightarrow \left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)f(z) = 0$

So $\Delta\phi = -\frac{k}{2}\left(\frac{k}{2}-1\right)\phi$

<u>Upshot</u> Action of $Z(U(\mathfrak{sl}_2))$ encodes the weight of the modular form.

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Back to $G = \mathbb{G}(k)$, $\mathbb{G}$ connected red / $k = \mathbb{R}$ or $\mathbb{C}$.

$\mathfrak{g} = \operatorname{Lie} G$, $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes_k \mathbb{C} = \operatorname{Lie} \mathbb{G}(\mathbb{C})$.

K = max compact subgroup.

<u>Schur's Lemma</u> Let V be an irred admissible $(\mathfrak{g}, K)$-module and let $L: V \to V$ be a lin trans, equivariant for $\mathfrak{g}$ and K. Then L is scalar.

<u>Proof</u> By def, $\exists$ not trivial fin dim K-stable subspace s.t. K acts by a fixed irred rep τ, let V_τ be the τ-isotypic piece in V. Then L commuting with K-action $\Rightarrow L(V_\tau) \subseteq V_\tau$. Then V_τ fin dim $\Rightarrow \exists$ an eigenvalue λ for L on V_τ.

Consider $\ker(L-\lambda) \subseteq V$. It is stable under $\mathfrak{g}$ and K, so is a nontrivial $(\mathfrak{g}, K)$-submod.

V irred $\Rightarrow \ker(L-\lambda) = V$ and $L = \lambda$. □

<u>Cor</u> If V is an irred admissible $(\mathfrak{g}, K)$-module, then $Z(\mathfrak{g}_\mathbb{C}) := Z(U(\mathfrak{g}_\mathbb{C}))$ acts by a character on V.

<u>Proof</u> $Z(\mathfrak{g}_\mathbb{C})$ commutes with $\mathfrak{g}$ and with K because $\exp: \mathfrak{k} \to K$ is surj as $\mathfrak{k}$ is compact. By Schur's Lemma, each $z \in Z(\mathfrak{g}_\mathbb{C})$ acts on V as a scalar and easy to check it is a hom. □

Say we fix a system of +ve roots Δ^+ for $\mathfrak{g}_\mathbb{C}$.

Let T be a maximal torus for $\mathbb{G}(\mathbb{C})$

Let $\mathfrak{t} = \operatorname{Lie} T \subseteq \mathfrak{g}_\mathbb{C}$

Let $\mathcal{T} = U(\mathfrak{t}) = \operatorname{Symm}(\mathfrak{t})$

The Weyl group $W = W(G, T)$ acts on $\mathfrak{t}$ and hence on $\mathcal{T}$.

Set $\mathcal{P} = \sum_{\alpha\in\Delta^+} U(\mathfrak{g}_\mathbb{C}) E_\alpha$

where $E_\alpha \in \mathfrak{g}$ is eigenvector for α.

<u>Lemma</u> $\mathcal{T} \cap \mathcal{P} = 0$ and $Z(\mathfrak{g}_\mathbb{C}) \subseteq \mathcal{T} \oplus \mathcal{P}$

<u>Proof</u>: Omitted.

Let P be the projection of $Z(\mathfrak{g}_\mathbb{C}) \subseteq \mathcal{T} \oplus \mathcal{P}$ onto $\mathcal{T}$.

Let $\delta = \frac{1}{2}\sum_{\alpha\in\Delta^+} \alpha$

Let $S: \mathcal{T} \to \mathcal{T}$ by $S(H) = H - \delta(H)1$ for all $H \in \mathfrak{t}$.

Define the Harish-Chandra homomorphism

$$F: Z(\mathfrak{g}_\mathbb{C}) \to \mathcal{T} \text{ by } F = S \circ P$$

<u>Thm</u> (Harish-Chandra) F is an isomorphism onto $\mathcal{T}^W$ and does not depend on the choice of Δ^+.

<u>Eg</u>
- SL_2, $Z(\mathfrak{sl}_2) \cong \mathbb{C}[\Omega]$ since the max torus is rank 1.
- GL_n, $Z(\mathfrak{gl}_n) \cong \mathbb{C}[x_1, \ldots, x_n]^{S_n}$ $\cong \mathbb{C}[e_1, \ldots, e_n]$ with $e_1, \ldots, e_n$ the std symm functions.