

Modularity of some
 $\mathrm{PGL}_2(\mathbb{F}_6)$ -representations
(joint with Khare - Thorne)

$k = \text{finite field}$, $G_{\mathbb{Q}} := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

Theorem (Khare - Wintenberger)

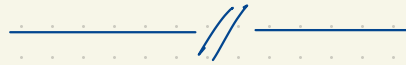
If $\rho: G_{\mathbb{Q}} \rightarrow \text{GL}_2(k)$ absolutely irreducible and odd, then ρ is modular.

We say ρ is odd if the image of complex conjugation in $\text{PGL}_2(k)$ is nontrivial.

Proof proceeds by induction on $p = \text{char}(k)$ and the conductor $N(\bar{\rho})$ of $\bar{\rho}$ using for the base case(s):

Theorem (Tate-Serre)

There are no abs irred ρ with $p=2,3$
and $N(\rho)=1$.



Let $F =$ totally real

Conj An abs irred totally odd (odd at every complex conj) $\rho: G_F \rightarrow GL_2(k)$ comes from a Hilbert modular form.

Problem: Serre-Tate result no longer true.

Known Cases

- (a) (Hsueks) Proj image of $\rho =$ dihedral group
- (b) (Langlands - Tunnell, Wiles) $k = \mathbb{F}_3$
(uses that $GL_2(\mathbb{F}_3)$ is solvable)
- (c) (Wiles, Shepherd-Barron - Taylor) $k = \mathbb{F}_5$ and
 $\det \rho = \bar{\epsilon}_5 = \text{mod } 5 \text{ cyclotomic character}$
- (d, e, f) $\det \rho = \bar{\epsilon}_p$ and
 - $|k| = 4$ (Shepherd-Barron - Taylor)
 - $|k| = 7$ (Manohar mayum)
 - $|k| = 9$ (Ellenberg)

Sketch of (c) ($k = \mathbb{F}_5$, $\det \rho = \bar{5}_5$)

$$X = \{ (E, \varphi: \rho \xrightarrow{\sim} \rho_{E,5} \xrightarrow{\sim} E[5]) \}_{/\cong}^{\text{cpt}}, \text{ twist of } X(5) \\ \cong \mathbb{P}_{\mathbb{F}}'$$

$\Rightarrow \exists X(F) \ni X \hookrightarrow E_F$ s.t. $E[3] \hookrightarrow \rho_{E,3}$ satisfies assumptions of a modularity lifting theorem (MLT)

(b) $\Rightarrow \rho_{E,3}$ is modular

MLT $\Rightarrow E$ is modular

$\Rightarrow \rho_{E,5} \cong \rho$ is modular

□

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Consider

$$\begin{array}{ccc} & \rho & \dashrightarrow GL_2(\bar{k}) \\ & \downarrow & \\ \bar{\rho}: G_F & \dashrightarrow & PGL_2(k) \subseteq PGL_2(\bar{k}) \end{array}$$

Tate $\Rightarrow \exists \rho$

ρ, ρ' two lifts of $\bar{\rho}$, then
 ρ modular $\Leftrightarrow \rho'$ is modular

Question: Can we prove modularity of totally odd $\bar{\rho}: G_F \rightarrow PGL_2(k)$ for small k ?

Theorem (A. - Kharas-Thorne)

Let $\rho: G_F \rightarrow \mathrm{PGL}_2(\mathbb{F}_5)$ be totally odd with nonsolvable image. Assume that $\Delta: G_F \rightarrow \{\pm 1\}$, $\Delta = \rho \bmod \mathrm{PSL}_2(\mathbb{F}_5)$, is totally even or totally odd and does not cut out $F(\zeta_5)$. Then ρ is modular.

Idea \exists solvable K/F that is totally real or CM (Δ totally even or odd) s.t. $\bar{\rho}|_{G_K}$ lifts to

$$\rho_K: G_K \rightarrow \mathrm{GL}_2(\mathbb{F}_5) \text{ with } \det \rho_K = \bar{\epsilon}_5$$

SB-T or A-K-T $\Rightarrow \rho_K$ is modular

$\bar{\rho}$ modular by mod p solvable descent

(Tate + Kharas-Wintenberger method over F) □

Thank You!