$k = \mathbb{R}, \mathbb{C}$, $G = \mathbb{G}(k)$ or $\mathbb{G}(k)^\circ$, $\mathfrak{g} = \mathrm{Lie}\, G = \mathrm{Lie}\, \mathbb{G}$

$$\mathfrak{g}_\mathbb{C} := \mathfrak{g} \otimes_k \mathbb{C} = \mathrm{Lie}\, \mathbb{G}(\mathbb{C})$$

$K \subseteq G$ max compact. Assume for now $\mathbb{G}$ is semisimple.

$T \subseteq G(\mathbb{C})$ max torus.

$\mathfrak{t} = \mathrm{Lie}(T)$.

Δ^+ = system of +ve roots for T acting on $\mathfrak{g}_\mathbb{C}$.

$$\mathcal{T} = U(\mathfrak{t}), \quad \mathcal{P} = \sum_{\alpha \in \Delta^+} U(\mathfrak{g}_\mathbb{C}) E_\alpha$$

where E_α is an eigenvector with eigval α.

<u>Facts</u> The fin dimensional irred representations of $\mathfrak{g}_\mathbb{C}$ are in bijection with dominant integral $\lambda \in \mathfrak{t}^\vee$

where - integral means that

$$2\frac{\lambda(\alpha)}{\|\alpha\|^2} = 2\frac{\langle \lambda, \alpha\rangle}{\langle \alpha, \alpha\rangle} \in \mathbb{Z} \quad \forall \text{ roots } \alpha$$

- dominant if

$$2\frac{\langle \lambda, \alpha\rangle}{\langle \alpha, \alpha\rangle} \geq 0 \quad \forall \alpha \in \Delta^+$$

In part a polynomial function on $\mathfrak{t}^\vee$ that vanishes at all dominant integral λ, it is 0.

This bijection sends rep V to its highest weight, defined as follows: A <u>higest weight vector</u> $v \in V$ is a nonzero vector v_0 s.t. $E_\alpha v = 0 \quad \forall \alpha \in \Delta^+$.

Since V is irred, v_0 will be unique up to scalar, and $\mathfrak{t}$ will act on v_0 by a dominant integral $\lambda \in \mathfrak{t}^\vee$, this is called the <u>highest weight</u> of V.

Can construct V as a quotient of the <u>Verma module</u>

$$V(\lambda+\delta) : U(\mathfrak{g}_\mathbb{C}) \otimes_{U(\mathfrak{b})} \mathbb{C}(\lambda) \cong U(\mathfrak{n}^-) \otimes_\mathbb{C} \mathbb{C}(\lambda)$$

where - $\mathfrak{b}$ is the Borel $\subseteq \mathfrak{g}_\mathbb{C}$ corr to Δ^+, i.e. $\mathfrak{b} = \mathfrak{t} \oplus \left(\bigoplus_{\alpha \in \Delta^+} \mathbb{C} E_\alpha\right)$

- $\mathfrak{b}$ acts on $\mathbb{C}(\lambda)$ by $\mathfrak{b} \twoheadrightarrow \mathfrak{t} \xrightarrow{\lambda} \mathbb{C}$.

This has highest wt vector $v_0 = 1 \otimes 1$, with weight λ, and $V(\lambda+\delta)$ is universal wrt this property.

$$\delta = \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$$

and $\mathfrak{n}^- = \sum_{\alpha \in \Delta^+} \mathbb{C} E_{-\alpha}$

Back to stuff from Monday.

<u>Lemma</u> $\mathcal{T} \cap \mathcal{P} = 0$ and $Z(\mathfrak{g}_\mathbb{C}) \subseteq \mathcal{T} \oplus \mathcal{P}$

<u>Proof</u> Take a dominant integral λ corr to the irred rep V_λ of $\mathfrak{g}_\mathbb{C}$. Let $v_\lambda \in V_\lambda$ be a highest vector.

$$E_\alpha v_\lambda = 0 \quad \forall \alpha \in \Delta^+$$
$$\Rightarrow X v_\lambda = 0 \quad \forall X \in \mathcal{P}$$

So if $X \in \mathcal{T} \cap \mathcal{P}$, then $\lambda(X) = 0 \quad \forall$ dominant integral λ.

But if we view X as a poly on the space $\mathfrak{t}^\vee$, it vanishes on all lattice points in some cone $\Rightarrow X = 0$.

Now let $Z \in Z(\mathfrak{g}_\mathbb{C})$. We can write it as a lin comb of elements of the form

$$(*) \quad E_{-\alpha_1}^{p_1} E_{-\alpha_2}^{p_2} \cdots E_{-\alpha_m}^{p_m} T_1^{e_1} \cdots T_r^{e_r} E_{\alpha_1}^{q_1} \cdots E_{\alpha_m}^{q_m}$$

where $\{\alpha_1, \dots, \alpha_m\}$ is a basis for Δ^+, $T_1, \dots, T_r$ a basis for $\mathfrak{t}$, by the Poincare-Birkhoff-Witt Thm.

Each such monomial is an eigenvector for $\mathfrak{t}$ with eigenvalue

$$\sum q_i \alpha_i - \sum p_i \alpha_i$$

But $\mathfrak{t}$ acts trivially on $Z(\mathfrak{g}_\mathbb{C})$. So each $\sum (q_i - p_i)\alpha_i = 0$

Thus if an element of $(*)$ has a $E_{-\alpha}$, it also has E_α.

$\Rightarrow Z \in \mathcal{T} \oplus \mathcal{P}$ $\square$

Now let

$$\begin{tikzcd} Z(\mathfrak{g}_\mathbb{C}) \arrow[r, "\gamma_{\Delta^+}"] \arrow[d, phantom, "\cap" description] & \mathcal{T} \\ \mathcal{T} \oplus \mathcal{P} \arrow[ur, "\mathrm{proj}"'] & \end{tikzcd}$$

Let $\gamma : Z(\mathfrak{g}_\mathbb{C}) \to \mathcal{T}$ be $\gamma = \sigma_{\Delta^+} \circ \gamma_{\Delta^+}$

where $\sigma_{\Delta^+}(T) = T - \delta \quad \forall T \in \mathfrak{t}$.

<u>Thm</u> (Harrish-Chandra) γ induces an iso

$$\gamma : Z(\mathfrak{g}_\mathbb{C}) \to \mathcal{T}^W, \quad W = \text{Weyl group}.$$

<u>Eg</u> $\mathfrak{g}_\mathbb{C} = \mathfrak{sl}_2$.

$$\Omega = \frac{1}{2}H^2 + EF + FE, \qquad H = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$= \frac{1}{2}H^2 + H + 2FE$$

$$= \frac{1}{2}H^2 - H + 2EF$$

Let $\Delta^+ =$ eigval for H on E, i.e. $\alpha(H)=2$.

$$\gamma_{\Delta^+}(\Omega) = \tfrac{1}{2}H^2 + H$$

$\delta = \tfrac{1}{2}\alpha$, $\sigma_{\Delta^+}(H) = H - \tfrac{1}{2}\alpha(H) = H-1$

Then $\gamma(\Omega) = \tfrac{1}{2}(H-1)^2 + (H-1)$

$$= \tfrac{1}{2}H^2 - \tfrac{1}{2}$$

Now $W = \{1, w\}$ with $w(H) = -H$, so

$\mathcal{T} = \mathbb{C}[H]$ and $\mathcal{T}^W = \mathbb{C}[H^2] \cong \mathbb{C}[\Omega]$

Note if we chose $\Delta^+ = \{-\alpha\}$, then

$\gamma_{\Delta^+}(\Omega) = \tfrac{1}{2}H^2 - H$, but now $\sigma_{\Delta^+}(H) = H+1$

and again $\gamma(\Omega) = \tfrac{1}{2}H^2 - \tfrac{1}{2}$

<u>Eg</u> $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}_2 = \mathbb{C}Z \oplus \mathfrak{sl}_2$ with $Z = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$

$Z(\mathfrak{gl}_2) \cong \mathbb{C}[Z, \Omega]$

$\cong \mathbb{C}[T_1, T_2]^W$ $\quad T_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $T_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\cong \mathbb{C}[T_1 + T_2, T_1 T_2]$

<u>Some ideas of Thm</u>

Let's sketch that $\operatorname{im}(\gamma) \subseteq \mathcal{T}^W$. For any $Z \in Z(\mathfrak{g}_{\mathbb{C}})$ and $\lambda \in \mathfrak{t}^\vee$,

$$\gamma(Z)(\lambda) = \gamma_{\Delta^+}(Z)(\lambda - \delta)$$

It suffices to show that

$$\gamma_{\Delta^+}(Z)(\lambda - \delta) = \gamma_{\Delta^+}(Z)(w\lambda - \delta)$$

$\forall w \in W$. It further suffices to show this for $w = s_\alpha$, α a simple root in Δ^+. And because this is an equality of polys, can restrict to dominant integral λ.

Consider $V(\lambda) = U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{b})} \mathbb{C}(\lambda - \delta)$, $v_0 = 1 \otimes 1$.

Since $P v_0 = 0$, Z acts on v_0 by

$$\gamma_{\Delta^+}(Z)(\lambda - \delta)$$

But $Z \in Z(\mathfrak{g}_{\mathbb{C}})$, so it acts all of $V(\lambda)$ by

$$\gamma_{\Delta^+}(Z)(\lambda - \delta).$$

[Using here that $V(\lambda) = U(\mathfrak{g}) v_0$]

Idea now is to show that

$$V(s_\alpha \lambda) \subseteq V(\lambda).$$

Then Z acts on $V(s_\alpha\lambda)$ by $\gamma_{\Delta^+}(Z)(s_\alpha\lambda - \delta)$ and by $\gamma_{\Delta^+}(Z)(\lambda - \delta)$.