Lecture 14

G, K as before. (π, H) a unitary irred rep of G with central character $\psi: Z = Z(G) \to S^1 \subseteq \mathbb{C}^\times$

We said (π, H) is <u>discrete series</u>

$\iff$ some K-finite matrix coeff is in $L^2(G,\psi)$

$\iff$ every matrix coeff is in $L^2(G,\psi)$

$\iff$ (π, H) is $\cong$ to a subrep of $L^2(G,\psi)$ that is a direct summand.

If the centre of G is compact, then we don't need to fix ψ as above because

$$\int_G = \int_Z \int_{G/Z} \quad \text{and } Z \text{ compact}$$

$$\Rightarrow \int_G \text{ converges} \iff \int_{G/Z} \text{ converges}$$

Let's assume for now that Z is compact.

<u>Thm</u> (Harish-Chandra) G admits discrete series $\iff$ $\operatorname{rank} G = \operatorname{rank} K$, i.e. G has a compact Cartan subgroup

(a Cartan subgroup is a subgroup of G that is the normalizer of a max abelian subalg of the Lie alg)

<u>Eg</u> This holds for · $SL_2(\mathbb{R}) \supset SO(2)$ both have rank 1

· $Sp_{2g}(\mathbb{R}) \supseteq U(g)$ both have rank g

But does not hold for · $G = SL_3(\mathbb{R})$ has rank 2 and max compact $K = SO(3)$ has rank 1 with Cartan subgroup $\left\{\begin{pmatrix} A & \\ & \alpha \end{pmatrix} \middle| A \in O(2), \alpha = \pm 1, \det A = \alpha\right\}$

· $SL_2(\mathbb{C})$ has rank 2 since $\mathbb{C}^\times \cong S^1 \times \mathbb{R}_{>0}$ is 2-real dim and $K = SU(2)$ has ran 1 with Cartan S^1

<u>Exercise</u> Compare this with properties of $\mathbb{G}(\mathbb{R})$ for $\mathbb{G}$ a connected red group admitting a Sh var.

Assume we have $\operatorname{rank} G = \operatorname{rank} K$. Choose a Cartan subalg

$$\mathfrak{t} \subseteq \mathfrak{k} \subseteq \mathfrak{g}$$

Δ = set of roots for $\mathfrak{t}^{\mathbb{C}} = \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C}$ in $\mathfrak{g}^{\mathbb{C}}$

Δ_K = set of roots for $\mathfrak{t}^{\mathbb{C}}$ in $\mathfrak{k}^{\mathbb{C}}$

$\rightsquigarrow$ W_G the Weyl group for Δ in $\mathfrak{g}^{\mathbb{C}}$

W_K the Weyl group for Δ_K in $\mathfrak{k}^{\mathbb{C}}$

Say we are given an $\mathbb{R}$-lin map

$$\lambda: i\mathfrak{t} \to \mathbb{R} \quad \left(\iff \lambda: \mathfrak{t}^{\mathbb{C}} \to \mathbb{C} \text{ s.t. } \lambda(\mathfrak{t}) \subseteq i\mathbb{R}\right)$$

such that $\langle \lambda, \alpha \rangle \neq 0 \ \forall \alpha \in \Delta$.

Define $\Delta^+ = \{\alpha \in \Delta \mid \langle \lambda, \alpha \rangle > 0\} \rightsquigarrow \delta_G = \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$

$\Delta_K^+ = \{\alpha \in \Delta_K \mid \alpha \in \Delta^+\} \rightsquigarrow \delta_K = \frac{1}{2}\sum_{\alpha \in \Delta_K^+} \alpha$

Assume that $\lambda + \delta_G$ is <u>analytically integral</u>:

If $T \subseteq K$ is the closed connected subgroup with $\operatorname{Lie} T = \mathfrak{t}$, then $\exists$ a char $\chi: T \to \mathbb{C}^\times$ s.t. $\chi(e^H) = e^{(\lambda+\delta_G)(H)}$

<u>Rmk</u> This implies $\lambda + \delta_G$ is alg, i.e. $\dfrac{2\langle \lambda + \delta_G, \alpha\rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z} \ \forall \alpha \in \Delta$

<u>Thm</u> (Harish-Chandra) With the setup as above, $\exists$ a discrete series rep π_λ satisfying the following

1. Infinitesimal character of π_λ is $\chi_\lambda: Z(\mathfrak{g}) \to \mathbb{C}^\times$

$$\begin{tikzcd} Z(\mathfrak{g}) \arrow[r] \arrow[d, "\wr\ \mathrm{HC\ iso}"'] & \mathbb{C}^\times \\ U(\mathfrak{t}^{\mathbb{C}})^W \arrow[ur, "\lambda"'] & \end{tikzcd}$$

(same inf char as the irred rep of $\mathfrak{g}^{\mathbb{C}}$ with highest wt $\lambda + \delta_G$)

2. π_λ contains with mult 1 the irred rep of K with highest wt
$$\Lambda = \lambda + \delta_G - 2\delta_K \quad \text{(minimal } K\text{-type)}$$

3. If Λ' is a highest wt of an irred K-rep in π_λ, then
$$\Lambda' = \Lambda - \sum_{\alpha \in \Delta^+} n_\alpha \alpha, \quad n_\alpha \geq 0.$$

Moreover, all discrete series reps are of this form and $\pi_\lambda \cong \pi_{\lambda'}$ $\iff$ λ and λ' are conj by W_K

<u>Eg</u> $G = SL_2(\mathbb{R})$, $K = SO(2)$

$\mathfrak{t} = \operatorname{Lie} SO(2)$ and integral weights are $\cong \mathbb{Z}$. Take

$$\lambda: i\mathfrak{t} \to \mathbb{R}$$
$$\begin{pmatrix} & i \\ -i & \end{pmatrix} \mapsto r \neq 0 \iff \langle \lambda, \alpha \rangle = \lambda(iH_\alpha) = \lambda\begin{pmatrix} & i \\ -i & \end{pmatrix} \neq 0$$

So $r > 0$ or $r < 0$

If $r > 0$, then under the ident of wts $\cong \mathbb{Z}$, $\delta_G = 1$

$r < 0$, " " $\delta_G = -1$

Note $W_K = 1$ here. Can check that analytically integral assumption is satisfied if $\lambda + \delta_G \in \mathbb{Z}$.

Thm $\Rightarrow$ $\exists$ a discrete series rep for each $n \in \mathbb{Z}_{\geq 2}$ and $n \in \mathbb{Z}_{\leq -2}$

These are D_n^+ and D_n^-, $n \geq 2$, the holomorphic and antihol disc series from last time

What do you do if G does not have compact centre?

Let ${}^\circ G = \{ g \in G \mid |\chi(g)| = 1 \ \forall$ cts homs $\chi : G \to \mathbb{R}^\times \}$

Let Z = centre of G, this is the $\mathbb{R}$-pts of a torus, or a connected comp, so $Z = C \times A$ with C compact and $A = \mathbb{R}_{>0}^r$, some r. Can show that

$${}^\circ G \times A \longrightarrow G \text{ is an iso of Lie groups}$$

and ${}^\circ G$ has compact centre

If you have an irred Hilbert space rep (π, H) of G, its restriction to ${}^\circ G$ is also irred. We say (π, H) is <u>essentially discrete series</u> if it is discrete series when restricted to ${}^\circ G$. We drop "essentially" if (π, H) is unitary

<u>Eg</u> $G = GL_2(\mathbb{R})$, $K = O(2)$

$G = {}^\circ G \times A$ with $A = \mathbb{R}_{>0}$, ${}^\circ G = SL_2^{\pm}(\mathbb{R}) = \{ g \in GL_2(\mathbb{R}) \mid \det g = \pm 1 \}$

$T = SO(2) \subseteq O(2) \subseteq GL_2(\mathbb{R})$

$W_K = \{1, w\} = W_G$

So the λ are as above $\rightsquigarrow (\pi_n, D_n)$ and (π_{-n}, D_{-n}) for $n \geq 2$ ~~...~~ But now $\pi_n \cong \pi_{-n}$. So for $GL_2(\mathbb{R})$ or $SL_2^{\pm}(\mathbb{R})$, we get one iso class of disc series reps (π_n, D_n) for $n \geq 2$

What happens is $(\pi_n, D_n)\big|_{SL_2(\mathbb{R})} \cong (\pi_n^+, D_n^+) \oplus (\pi_n^-, D_n^-)$

<u>Rmk</u> Compare with the fact that

$$GL_2(\mathbb{R}) / \mathbb{R}_{>0} SO(2) \cong \mathbb{C} \setminus \mathbb{R} = \mathcal{H}^+ \sqcup \mathcal{H}^-$$

$$GL_2(\mathbb{R}) / \mathbb{R}_{>0} O(2) \cong \mathcal{H}^+ \text{ by identifying } z \text{ with } -z$$