Lecture 22 - General number fields, II

Proof of Thm 1. The S_∞ action on $H_*(C_\infty)$ factors through R_∞ and $H_*(C_\infty)$ is a finite R_∞-mod, so

$$\operatorname{depth}_{S_\infty} H_i(C_\infty) \leq \operatorname{depth}_{R_\infty} H_i(C_\infty) \leq \dim_{R_\infty} H_i(C_\infty) \leq \dim R_\infty = \dim S_\infty - \delta \quad (*)$$

Claim $\operatorname{depth}_{S_\infty} H_{q_0}(C_\infty) = \dim S_\infty - \delta$

Assuming this, all $\leq$ in $(*)$ are $=$ and 1 follows.

2. $\forall\, p \in \operatorname{Spec} R_S$, letting p_∞ be its pullback to R_∞, $H_{q_0}(C_\infty)_{p_\infty} \neq 0$ by assumption

$$\Rightarrow H_{q_0}(Y(U), \mathcal{O})_p = H_{q_0}(C_\infty \otimes_{S_\infty} \mathcal{O})_p$$
$$= \left(C_{\infty, q_0} / (\operatorname{im} d_{q_0+1}, \mathfrak{a})\right)_p$$
$$= \left(H_{q_0}(C_\infty)/\mathfrak{a}\right)_p$$
$$= H_{q_0}(C_\infty)_{p_\infty}/\mathfrak{a} \neq 0 \text{ by Nak.}$$

3. Since R_∞ is regular and $\dim_{R_\infty} H_{q_0}(C_\infty) = \operatorname{depth}_{R_\infty} H_{q_0}(C_\infty)$, the Auslander-Buchsbaum formula

$\Rightarrow H_{q_0}(C_\infty)$ is a projective, hence free, R_∞-mod

$\Rightarrow H_{q_0}(Y(U), \mathcal{O})_m$ is a free $R_\infty/\mathfrak{a}$-mod.

This action of $R_\infty/\mathfrak{a}$ factors through R_S and $R_\infty/\mathfrak{a} \cong R_S$ $\square$

After shifting the claim follows from the following Lemma.

Lemma Let S be a local regular Noeth ring, $n = \dim S$.
Let $P = P_\bullet$ be a (homological) complex of finite free S-mods concentrated in degrees $[0, \delta]$. Then $\dim H_0(P) \geq n - \delta$ and if $=$ holds
1. P is a proj resolution of $H_0(P)$
2. $H_0(P)$ has depth $n - \delta$.

Proof Let $d_n : P_n \to P_{n-1}$ be the differential, and let $m \geq 0$ be the largest integer s.t. $H_m(P) \neq 0$. Then

$$0 \to P_\delta \to P_{\delta-1} \to \cdots \to P_m$$

is exact until the final term, so is a projective resolution of

$$M := P_m / \operatorname{im} d_{m+1}$$

Thus $\operatorname{projdim} M = \delta - m$. On the other hand

$$H_m(P) = \ker d_m / \operatorname{im} d_{m+1} \subseteq M$$

so

$$\dim H_m(P) \geq \operatorname{depth} M \qquad \text{(see Sublemma below)}$$

Then we have

$$\begin{aligned} \dim H_m(P) &\geq \operatorname{depth} M \\ &= n - \operatorname{projdim} M \qquad \text{by Auslander-Buchsbaum form} \\ &\geq n - \delta + m \end{aligned}$$

Now if $\dim H_0(P) \leq n - \delta$, the above $\Rightarrow m = 0$, P is a proj resolution of $H_0(P)$ and $\geq$ are $=$ in above. $\square$

Sublemma Let S be a local Noeth ring and let $0 \neq N \subseteq M$ be finite S-mods. Then $\dim N \geq \operatorname{depth} M$.

Proof Let $\mathfrak{p} \in \operatorname{Ass} N$, so $\mathfrak{p} = \operatorname{Ann}(x)$ for some $0 \neq x \in N$.

Now $\mathfrak{p}$ is also an assoc prime, and since S is local Noeth ring and M is fin gen

$$\operatorname{depth} M \leq \min_{\mathfrak{q} \in \operatorname{Ass} M} \dim S/\mathfrak{q} \leq \dim S/\mathfrak{p} = \dim(S x) \leq \dim N.$$

$\square$

So we want to construct

$$\begin{array}{ccc} S_\infty \to R_\infty & \curvearrowright & H_*(C_\infty) \\ \downarrow & & \\ R_S =: R & \curvearrowright & H_*(C) := H_*(Y(U), \mathcal{O})_{\mathfrak{m}} \end{array}$$

where

- S_∞ = power series ring / $\mathcal{O}$ with $\mathfrak{a}$ = aug ideal
- $\dim R_\infty - \dim S_\infty = -\delta$
- C_∞ is a complex of finite free S_∞-mods concentrated in degrees $[q_0, q_0+\delta]$ and $C \cong C_\infty \bmod \mathfrak{a}$ is a complex of fin free $\mathcal{O}$-mods with $H_*(C) = H_*(Y(U), \mathcal{O})_{\mathfrak{m}}$
- $H_*(C_\infty)$ is a finite R_∞-mod.

<u>Galois side</u>: Just as before, if $\bar{\rho}|_{G_{F(\zeta_p)}}$ is abs irred with adequate image, then for any $N \geq 1$, we can find a TW datum Q_N of level N such that

$$h^1_{\mathcal{S}_{Q_N}^\perp}(\operatorname{ad}^\circ \bar{\rho}(1)) = 0 \quad \text{and} \quad |Q_N| = q \text{ indep of } N$$

$\Rightarrow R^S_{\mathcal{S}_{Q_N}}$ is a quotient of $R_\infty := R^{loc}_S [\![x_1, \ldots, x_g]\!]$ with

$$g = h^1_{\mathcal{S}_{Q_N}, S}(\operatorname{ad}^\circ \bar{\rho}) = q + |S| - 1 - \delta$$

$$\Rightarrow \dim R_\infty = \dim S_\infty - \delta$$

Automorphic side.

$G := PGL_2$, $X = G(\mathbb{R})/(\text{max cpt})$, $U \leq G(\mathbb{A}_F^\infty)$ suff small
$\rightsquigarrow Y(U) = G(F) \backslash X \times G(\mathbb{A}_F^\infty)/U$

Given a TW datum Q, can still define
$$U_Q \leq U_0(Q) \leq U$$
$$\uparrow \text{Iwahori at } v \in Q$$
$$U_0(Q)/U_Q \cong \Delta_Q$$

Can still define $m_Q \in \mathbb{T}_Q^{S,\text{univ}}$, but

Problem:
$$H_*(Y(U_Q), \mathcal{O})_{m_Q} \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O} \cong H_*(Y_0(Q), \mathcal{O}[\Delta_Q])_{m_Q} \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O}$$
$$\not\cong H_*(Y_0(Q), \mathcal{O})_m$$
because H_* and $\otimes$ don't commute if H_* is not concentrated in a single degree.

Solution: Instead use a complex C_Q of $\mathcal{O}[\Delta_Q]$-mods that computes $H_*(Y(U_Q), \mathcal{O})_{m_Q}$. Then $C_Q \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O}$ computes $H_*(Y(U_0(Q)), \mathcal{O})_{m_Q}$.

Problem Want a Hecke action, and then an action of R_{S_Q} via a map $R_{S_Q} \to \mathbb{T}^{S_Q}(?)$. Could use singular chains for C_Q to get a Hecke action.

<u>But</u> For patching, need C_Q to be a bounded complex of fin free $\mathcal{O}[\Delta_Q]$-mods. (want fin many iso classes of "level N patching data").

<u>Want</u> C_Q to be both singular chains and simplicial chains coming from a finite triangulation at the same time.

<u>Solution:</u> Work in $D(\mathcal{O})$, resp. $D(\mathcal{O}[\Delta_Q])$, the derived cat of $\mathcal{O}$-mods, resp. $\mathcal{O}[\Delta_Q]$-mods.

<u>Some quick facts</u> Let R be a ring and let $Ch(R)$ be the

(it is often convenient to identify a chain complex $C_\bullet$ with the cochain complex $C^\bullet$ by $C^i = C_{-i}$)

$\rightsquigarrow$ $K(R)$ = cat with $Ob_{K(R)} = Ob_{Ch(R)}$ and
$\mathrm{Hom}_{K(R)}(X, Y) = \mathrm{Hom}_{Ch(R)}(X, Y)/\sim$, $\sim$ = chain homotopy

Then roughly $D(R)$ is cat with same objects and inverting quasi-iso

So $f \in \mathrm{Hom}_{D(R)}(X, Y)$ is repd by

$$\begin{tikzcd} & Z \arrow[dl, "\text{quasi iso}"'] \arrow[dr, "\text{chain map}"] & \\ X & & Y \end{tikzcd}$$

Also have $K(R)^-$ and $D(R)^-$ and $K(R)^{-,proj} \subset K(R)^-$ and $K(R)^{-,proj} \to D(R)^-$ is an equiv of cats

Can also define $K(R)^+$, $D(R)^+$, $D(R)^b$