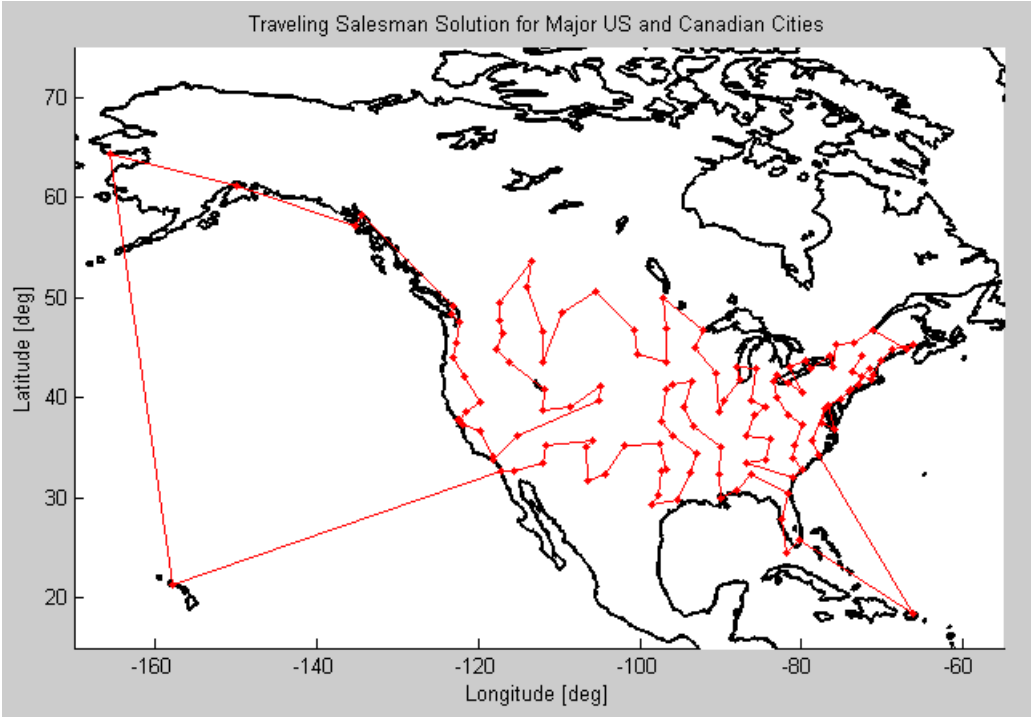
Module 11: Additional Topics
Graph Theory and Applications

Topics:

- Introduction to Graph Theory
- Representing (undirected) graphs
- Basic graph algorithms

Consider the following:

- Traveling Salesman Problem (TSP): Given N cities and the distances between them, find the shortest path to visit all cities and return to the start.

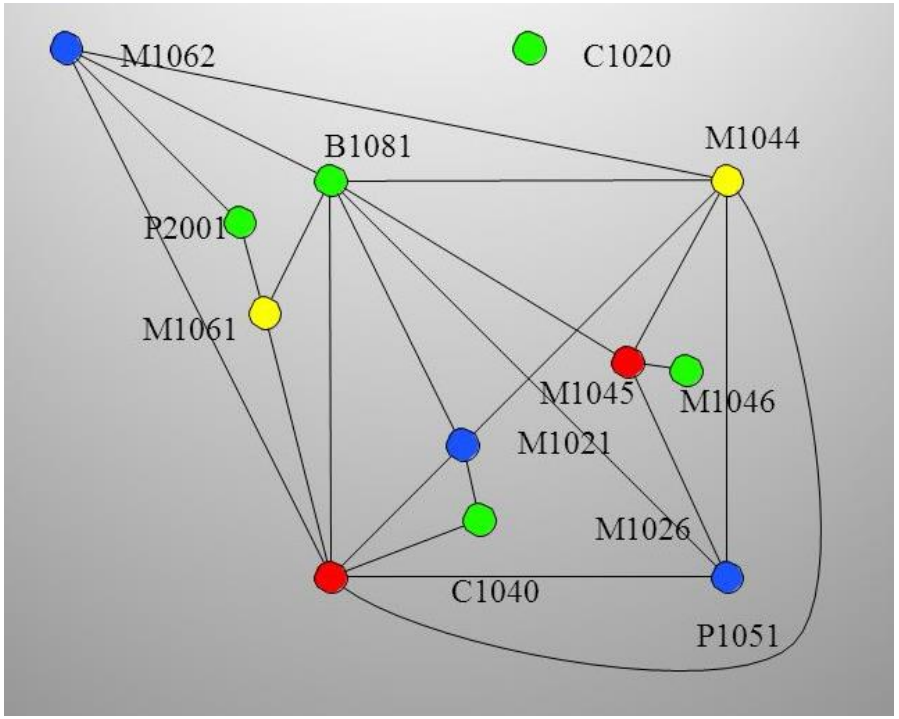


What does the TSP have in common with the following problems?

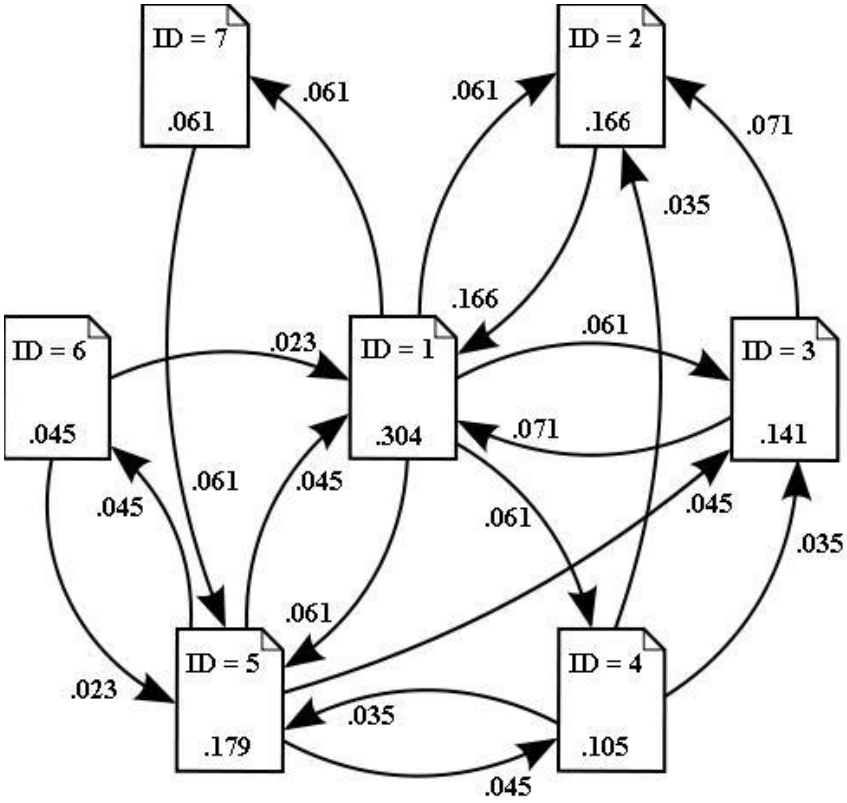
- Placement of new fire stations in a city to provide best coverage to all residents
- Ranking of "importance" of web pages by Google's PageRank algorithm
- Scheduling of final exams so they do not conflict
- Arranging components on a computer chip
- Analyzing strands of DNA
- Binary Search Trees

They all fall within the field of *GRAPH THEORY*

Non-conflicting exams

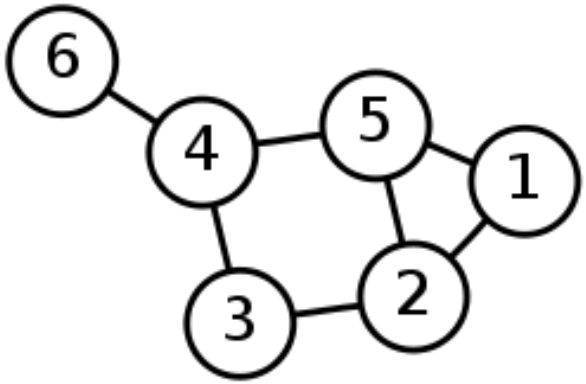


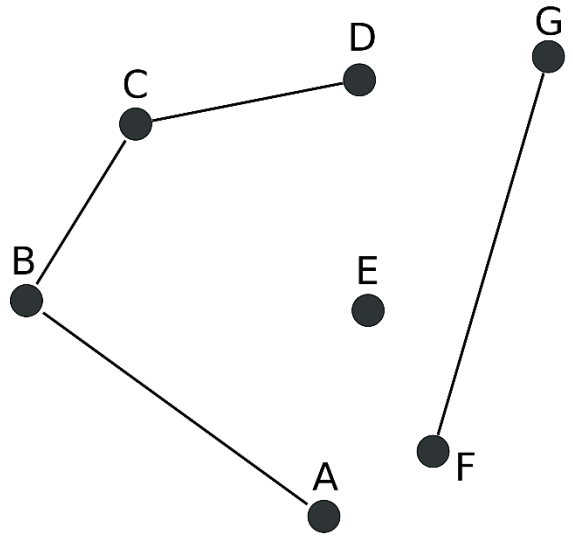
PageRank Algorithm



Undirected Simple Graphs

An undirected simple ***graph*** G is a set V, of ***vertices***, and a set E, of unordered distinct pairs from V, called ***edges***. We write G=(V,E).





Graph Terminology

- If (v_k, v_p) is an edge, we say that v_k and v_p are ***neighbours,*** and are ***adjacent.*** Note that k and p must be different.
- The number of neighbours of a vertex is also called its ***degree***
- A sequence of nodes $v_1, v_2, \ldots, v_k$ is a ***path*** of length k-1 if (v_1, v_2),(v_2, v_3),…,(v_{k-1}, v_k) are all edges
  - If $v_1 = v_k$, this is called a ***cycle***
- A graph G is ***connected*** if there exists a path through all vertices in G

Interesting Results on Graphs

Let n = number of vertices,

and m = number of edges:

1. $m \leq n(n-1)/2$
2. The number of graphs on n vertices is $2^{n(n-1)/2}$
3. The sum of the degrees over all vertices is $2m$.

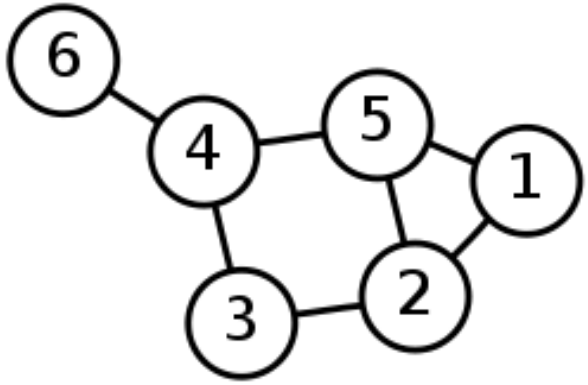
How can we store information about graphs in Python?

- We need to store labels for the vertices
  - These could be strings or integers
- We need to store both endpoints using the labels on the vertices.

- We will consider three different implementations for undirected, unweighted graphs

Implementation 1: Vertex and Edge Lists

- $V = [v_1, v_2, v_3, \ldots, v_m]$,
- $E = [e_1, e_2, e_3, \ldots, e_m]$, where edge $e_j = [a, b]$ when vertices a and b are connected by an edge



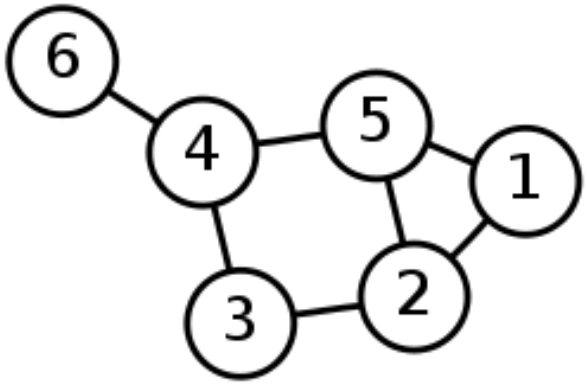

```
V=[6,4,5,3,2,1]
E=[[6,4], [4,5], [4,3], [3,2],
      [5,2], [1,2], [5,1]]
```

Implementation 2: Adjacency list

- For each vertex:
  - Store the labels on its neighbours in a list
- We will use a dictionary
  - Keys: labels of vertices
    - Recall: integers or strings can be keys
  - Associated values: List of neighbours (adjacent vertices)

Example:

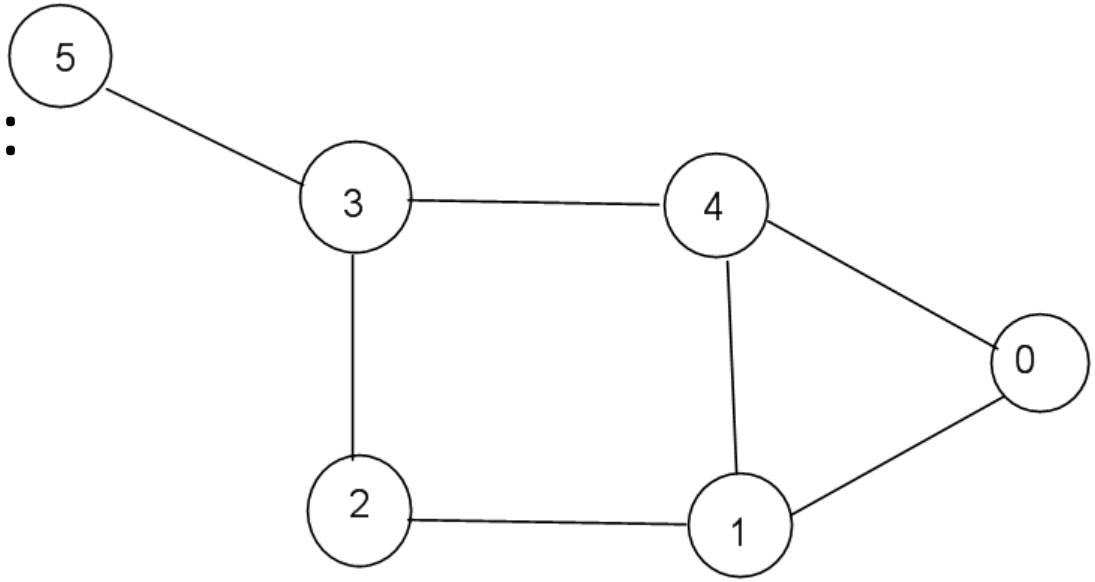
```
{1:[2,5],
 2:[1,3,5],
 3:[2,4],
 4:[3,5,6],
 5:[1,2,4],
 6:[4]}
```



Implementation 3: Adjacency Matrix

- For simplicity, assume vertices are labelled $0, \ldots, n-1$
- Create an n by n matrix for **`G`**
- If there is an edge connecting i and j:
  - Set **`G[i][j] = 1`**,
  - Set **`G[j][i] = 1`**
- Otherwise, set these values to 0

Example:



G:

vertex	*0*	*1*	*2*	*3*	*4*	*5*
0	[[0,	1,	0,	0,	1,	0],
1	[1,	0,	1,	0,	1,	0],
2	[0,	1,	0,	1,	0,	0],
3	[0,	0,	1,	0,	1,	1],
4	[1,	1,	0,	1,	0,	0],
5	[0,	0,	0,	1,	0,	0]]

Comparing the implementations on simple tasks

- Determine if two vertices are neighbours.
- Find all the neighbours of a vertex.

Which implementation to use?

- We'll use the adjacency list (a good case could also be made for the adjacency matrix).

Graph Traversals

- Determine all vertices of G that can be reached from a starting vertex
- There can be different types of traversals
- If you find all vertices starting from v, the graph is ***connected***
- If not all vertices can be reached, a ***connected component*** containing v has been found
- Must determine a way to ensure we do not cycle indefinitely

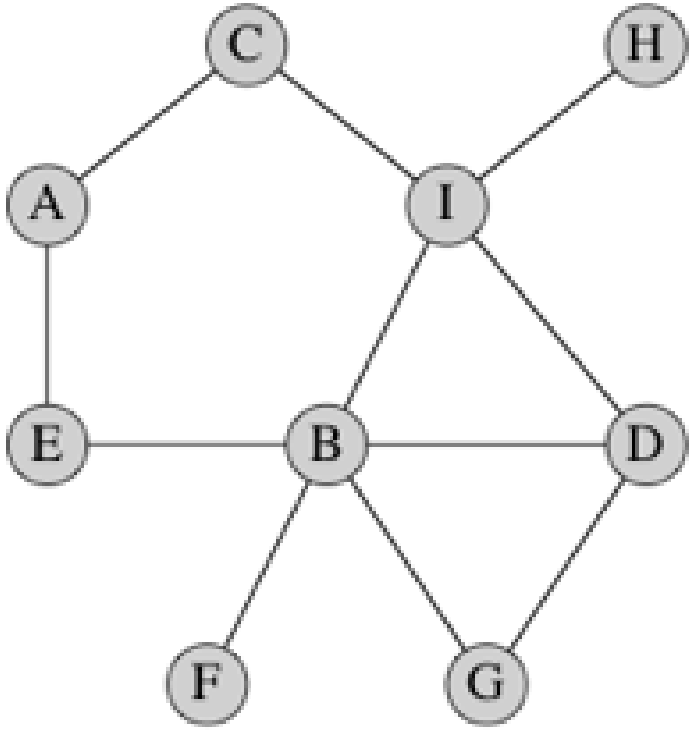
Applications of traversals

- Finding path between two vertices
- Finding connected components
- Tracing garbage collection in programs (managing memory)
- Shortest path between two points
- Planarity testing
- Solving puzzles like mazes
- Graph colouring

One approach: Breadth-first search Traversal (bfs)

- Choose a starting point v
- Visit all the neighbours of v
- Then, visit all of the neighbours of the neighbours of v, etc.
- Repeat until all reachable vertices are visited
- Need some way to avoid visiting edges more than once
- Note: there may be more than one bfs ordering of a graph, starting from v.

Sample bfs orders



- A, C, E, B, I, F, G, D, H
- A, E, C, I, B, H, D, G, F
- B, E, F, G, D, I, A, H, C
- B, D, E, F, G, I, A, C, H
- H, I, C, B, D, A, E, F, G

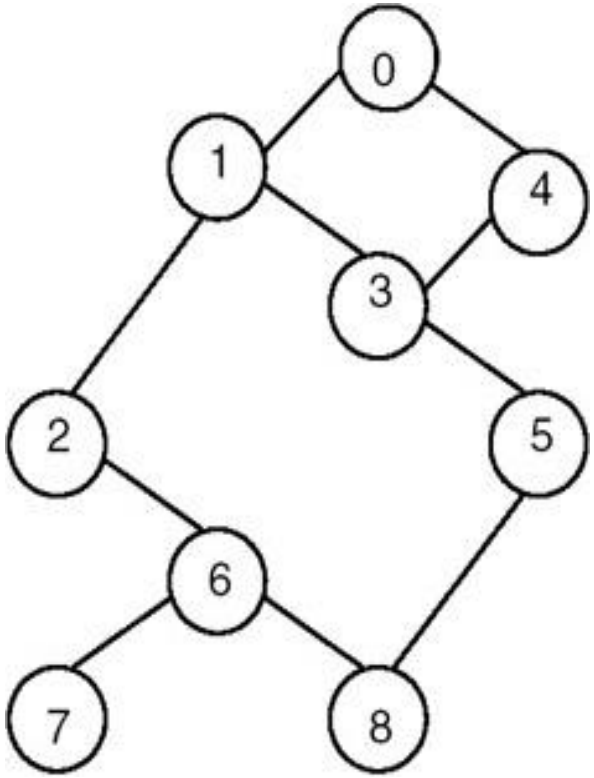
plus more …

Implementing bfs

- We will look at one implementation.
- Assumes an adjacency list representation.
- We use several lists:
  - **`all`** includes all "visited" vertices
    - Vertices are appended to the end
  - **Q** includes vertices waiting to be "visited" (it will grow and shrink as the algorithm progresses)
    - Vertices are appended to the end and removed from the front of **Q**

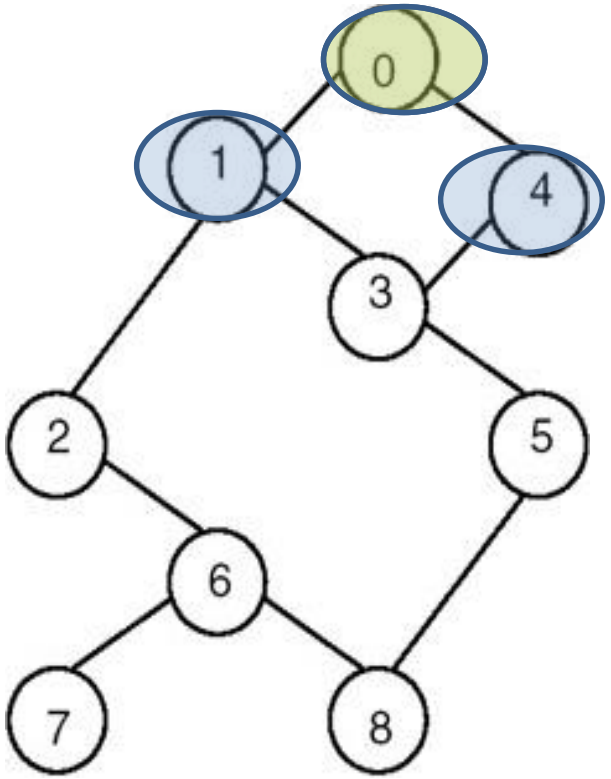
Implementation of bfs traversal

```
def bfs(graph, v):
    all = []
    Q = []
    Q.append(v)
    while Q != []:
        v = Q.pop(0)
        all.append(v)
        for n in graph[v]:
            if n not in Q and\
                n not in all:
                Q.append(n)
    return all
```



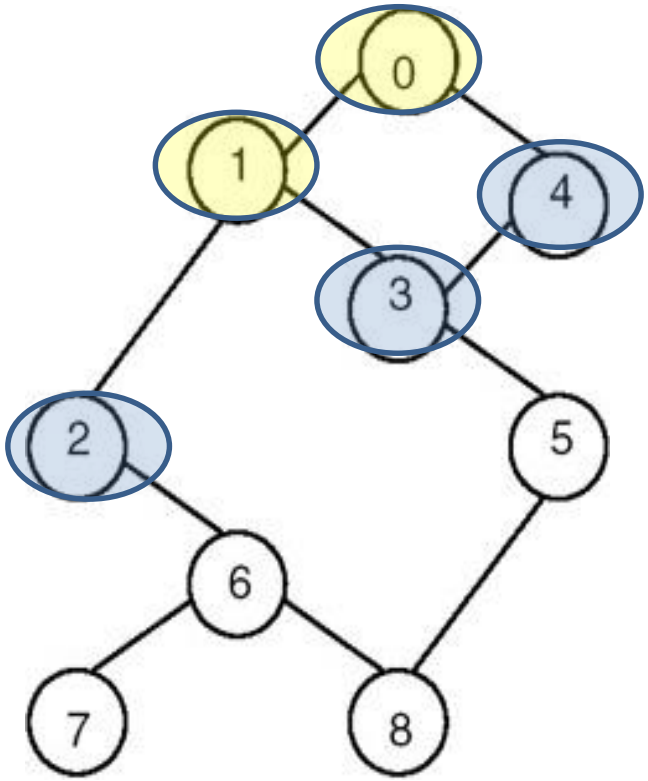
Starting bfs from 0 (1)

- Start from v=0
- all = []
- Q = [0]
  - v = 0
  - all = [0]
  - Neighbours of 0: 1,4
    - Q = [1,4]



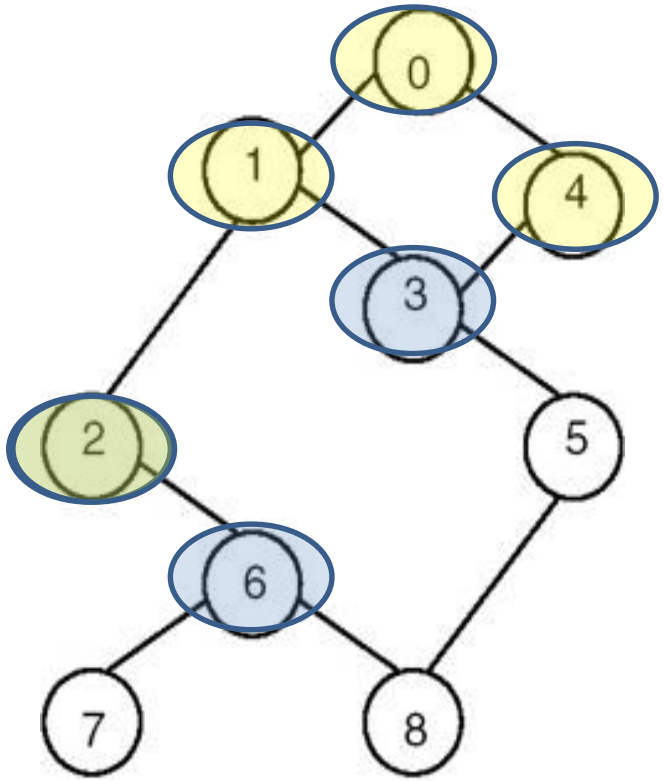
Continuing bfs (2)

- Q = [1, 4]
- v = 1
- all = [0,1]
- Neighbours of 1: 0,2,3
  - Q = [4,2,3]



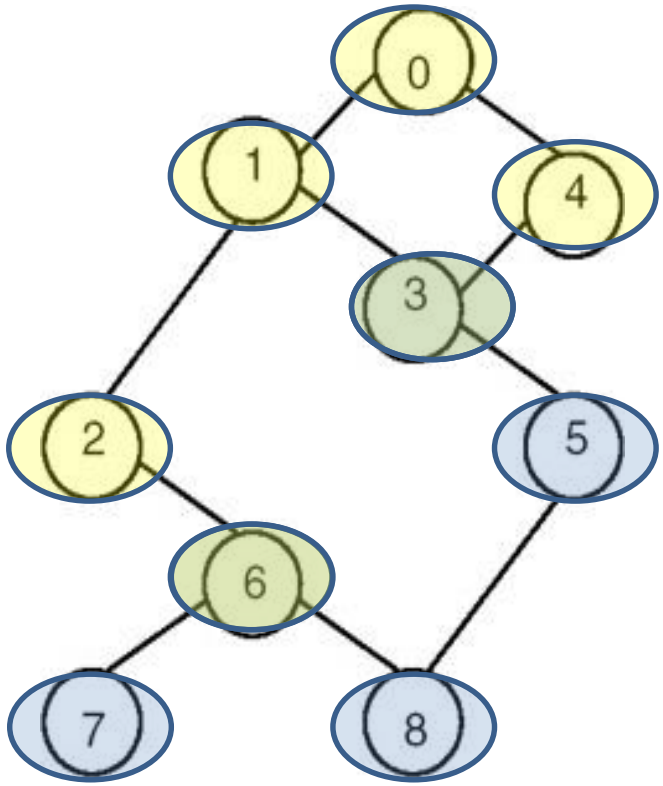
Continuing bfs (3)

- Q: [4, 2, 3]
- v = 4
- all = [0,1,4]
- Neighbours of 4: 0,3
  - No vertices added to Q
- Q= [2,3]
- v = 2
- all = [0,1,4,2]
- Neighours of 2: 1,6 → Q = [3,6]



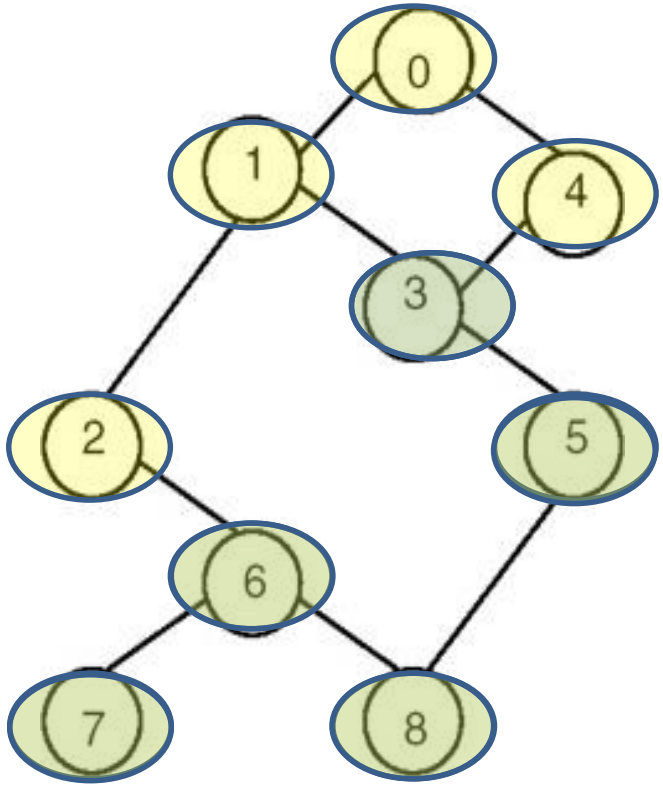
Continuing bfs (4)

- Q: [3, 6]
- v = 3
- all = [0,1,4,2,3]
- Neighbours of 3: 1,4,5
  - Q = [6,5]
- Q = [6, 5]
- v = 6
- all = [0,1,4,2,3,6]
- Neighbours of 6: 2,7,8
  - Q = [5,7,8]



Continuing bfs (5)

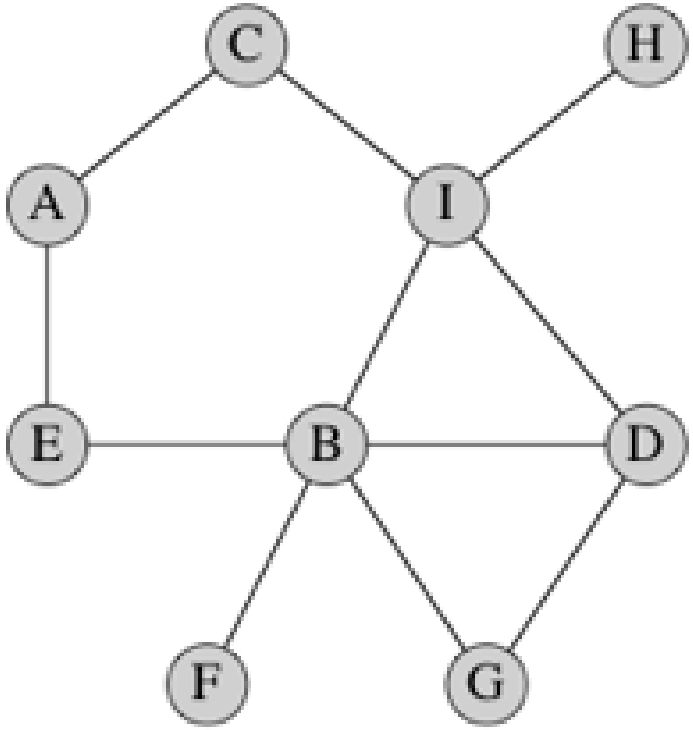
- Q: [5, 7, 8]
- v = 5
- all = [0,1,4,2,3,6,5]
- Neighbours of 5: 3,8 (Q unchanged)
- Q = [7,8]
- v = 7
- all = [0,1,4,2,3,6,5,7]
- Neighbours of 7: 6 (Q unchanged)
- Q = [8]
- v = 8
- all = [0,1,4,2,3,6,5,7,8]
- Neighbours of 8: 5,6 (Q unchanged)
- Q is empty



Another approach: depth-first traversal (dfs)

- Choose a starting point v
- Proceed along a path from v as far as possible
- Then, backup to previous (most recently visited) vertex, and visit its unvisited neighbour (this is called *backtracking*)
  - Repeat while unvisited, reachable vertices remain
- Note: there may be more than one dfs ordering of a graph, starting from v.

Sample dfs orders



- A, C, I, H, B, F, D, G, E
- A, E, B, G, D, I, H, C, F
- B, F, G, D, I, C, A, E, H
- F, B, G, D, I, H, C, A, E
- H, I, B, F, G, D, E, A, C

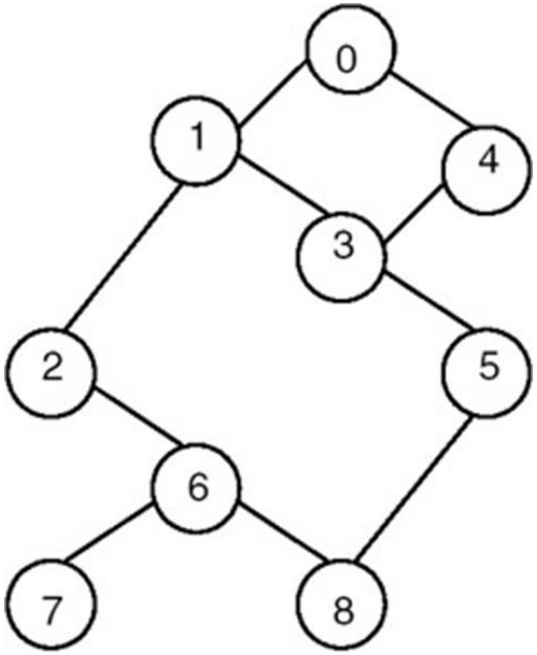
plus more …

Implementing dfs

- We will look at one implementation.
- Assumes an adjacency list representation.
- We use several lists:
  - **`all`** includes all "visited" vertices
    - Vertices are appended to the end
  - **`S`** includes vertices waiting to be "visited" (it will grow and shrink as the algorithm progresses)
    - Vertices are appended to the end and removed from the end of **`S`** as well

A depth first search traversal solution

```
def dfs(graph, v):
    all = []
    S = [v]
    while S != []:
        v = S.pop()
        if v not in all:
            all.append(v)
            for w in graph[v]:
                if w not in all:
                    S.append(w)
    return all
```

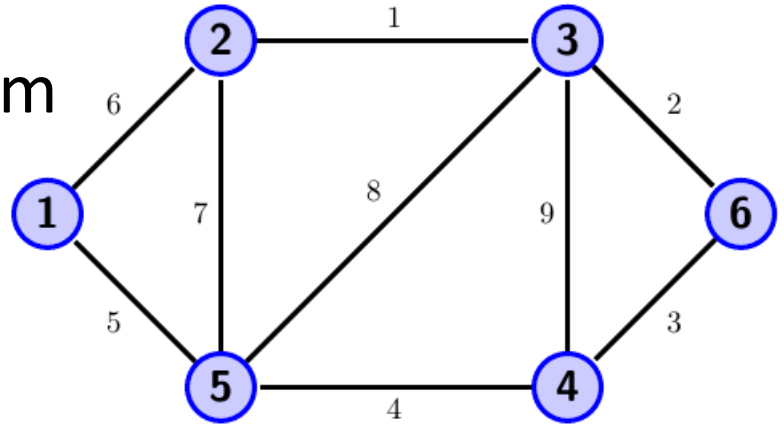


Breadth first vs depth first Searches

- Both need an additional list to store needed information:
  - BFS uses Q:
    - Add to the end and remove from the front
    - Called a Queue
  - DFS uses S:
    - Add to the end and remove from the end
    - Called a Stack
  - Stacks and Queues are both very useful in CS

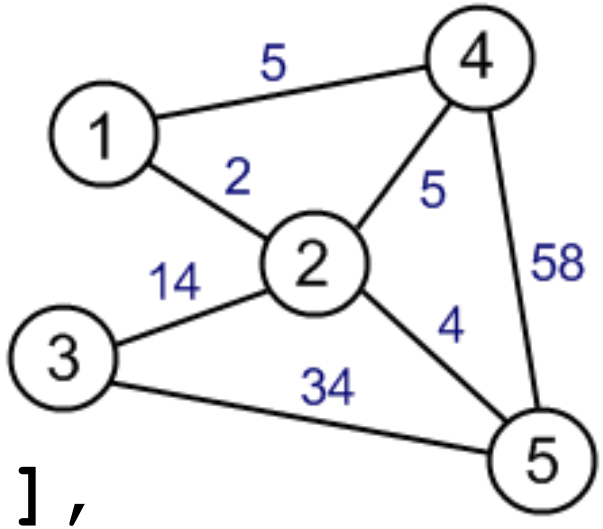
Extension: Weighted edges

- Each edge has an associated weight. It might represent:
  - Distance between cities
  - Cost to move between locations
  - Capacity of a route
  - Probability of moving from one web page to another



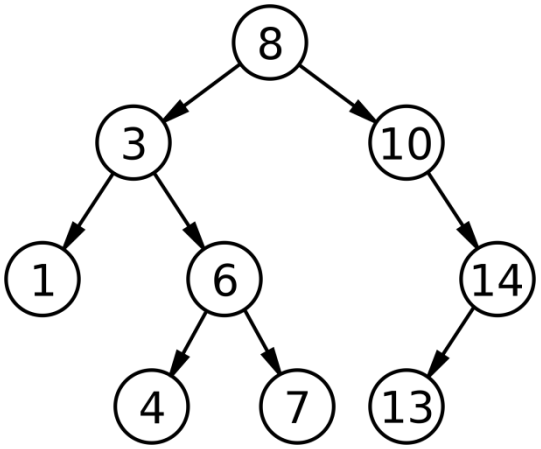
Adjust adjacency list to include weights

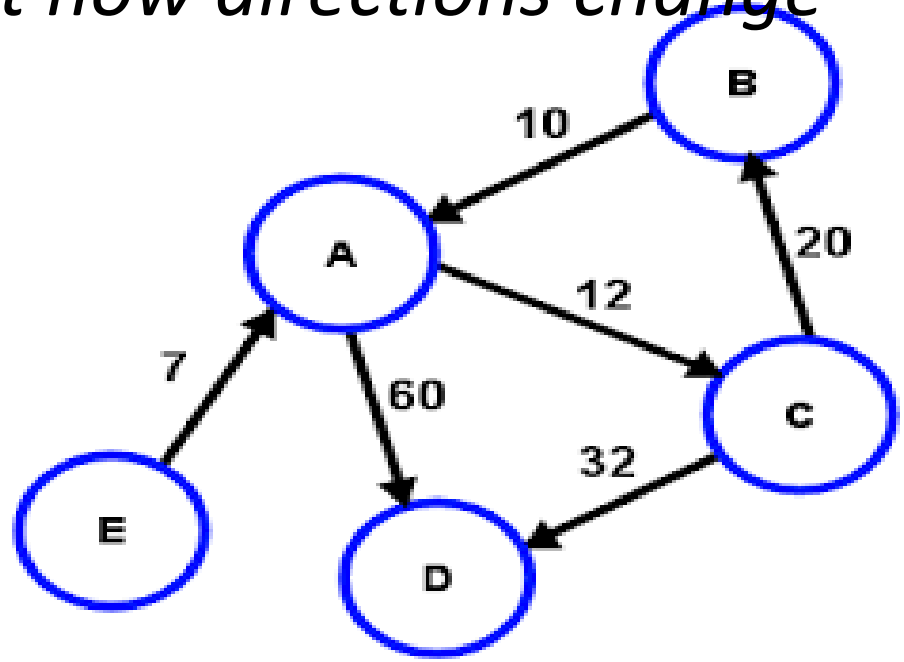
- Adjust our adjacency list to store weights with each edge
-
```
{1:[[2,2],[4,5]],
  2:[[1,2],[3,14],
     [4,5],[5,4]],
  3:[[2,14],[5,34]],
  4:[[1,5],[2,5],[5,58]],
  5:[[2,4],[3,34],[4,58]]}
```



Other types of graphs

- Edges can be directed – from one vertex to another
- Directed edges can have weights as well
- *Exercise: Think about how directions change our representations*





Goals of Module 11

- Understand basic graph terminology
- Understand representation of graphs in Python
- Understand breadth-first and depth-first search traversals