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Seminar Optimization
Introduction to Game Theory

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1 Introduction

Wieso ist das ganze interessant?

In this seminar report, we give an introduction to game theory, in particular static games. The content will be heavily inspired by Section 2 of [2]. However, in contrast to [2], we extend definitions and statements to an arbitrary number of players, if it is sensible.

In Section 2, we explain the setup, including a definition of a N-player static game along with some examples. In Section 3, we explore two different solution concepts for static games, namely Pareto optimum and Nash equilibrium. We state some basic properties and discuss the solutions to the example games in Section 2. We continue our analysis of the Nash equilibrium in Section 4, where we proof a fundamental existence result. In Section 5, we extend this result to a class of randomized strategies, which allows to drop some of the assumptions.

In the last two sections 2-player games are studied. Section 6 is about zero-sum games. We look at the notions of saddle points and value of the game and connect them to Nash equilibria. Finally, in Section 7, we look at cooperative-competitive solutions, which is a solution conecpt modeling the cooperation of two parties.

The appendix includes a brief summary of the other talks of the seminar.

2 Static Games

Before we give a definition of static game, we introduce the notation used throughout this report.

Notation 2.1. For a vector

$$x = (x_1, \ldots, x_n) \in \mathbb{R}^n$$

we write $x_i \in \mathbb{R}$ to represent the i-th component of that vector and

$$x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \mathbb{R}^{n-1}$$

to represent the rest. Sometimes, if for example subscripts become to crowded, we switch to superscripts x^i and x^{-i}.

To start with the theory of static games, we introduce the term formally.

Definition 2.2 (static game [5, Def. 1.1]). A static game of $N \in \mathbb{N}$ players is defined by

1. sets of strategies X_i
2. payoff functions $\Phi_i : \times_{i=1}^N X_i \to \mathbb{R}$

for each player index $1 \leqslant i \leqslant N$. We write $\Gamma = \{X_i, \Phi_i\}_{i=1}^N$ and call Γ a N-player game. The goal of each player is to maximize her payoff, i.e.

$$\max_{x_i \in X_i} \Phi_i(x_i, x_{-i}).$$

Let us also define

$$\mathcal{X} := \bigtimes_{i=1}^{N} X_i \quad \text{and} \quad \mathcal{X}^{-i} := \bigtimes_{j \neq i} X_j$$

for convenience.

For our analysis, we make the following assumptions.

Assumption 2.3. Given a N-player game $\Gamma = \{X_i, \Phi_i\}_{i=1}^N$, we assume each X_i to be a compact metric space and each Φ_i to be continuous.

To illustrate Definition 2.2 and future discusson, we look at some examples. For simplicity, we only choose examples that involve two players and finite strategy sets. However, with the exeptions of Sections 6 and 7, the theory is by no means limited by the number of players or strategies.

Example 2.4 (Prisoner's Dilemma [2, Ex. 3]). Two men are caught by the police after robbing a bank. They are kept in different cells with no means of communication with one another. The following outcomes are possible:

		Player 2	
		N	C
Player 1	N	-2,-2	-10,-1
	C	-1,-10	-5,-5

Table 1: Bi-matrix for the Prisoner's Dilemma, where the first number is the payoff of Player 1 and the second number is the payoff of Player 2.

		Player 2		
		rock	paper	scissors
	rock	0,0	-1,1	1,-1
Player 1	paper	1,-1	0,0	-1,1
	scissors	-1,1	1,-1	0,0

Table 2: Bi-matrix for Rock-Paper-Scissors, where the first number is the payoff of Player 1 and the second number is the payoff of Player 2.

- If only one suspect admits the robbery, he can blame the other for it. In this case the confessing suspect gets a prison sentence of 1 year, whereas the other suspect gets a prison sentence of 10 years.
- If both suspects confess and blame each other, they both get a prison sentence of 5 years.
- If none of the suspects confesses, the police can only hold them for illegal gun possession. This results in a 2 years prison sentence for each of the suspects.

This situation can be modelled as a game of 2 players in the following way: The strategy sets of the players are $X_1 = X_2 = \{C, N\}$, where C indicates confessing and N not confessing. The playoff functions are as follows:

$$\begin{aligned} &\Phi_1(N,N) = -2, \quad \Phi_1(N,C) = -10, \quad \Phi_1(C,N) = -1, \quad \Phi_1(C,C) = -5, \\ &\Phi_2(N,N) = -2, \quad \Phi_2(N,C) = -1, \quad \Phi_2(C,N) = -10, \quad \Phi_2(C,C) = -5. \end{aligned}$$

Note that the payoffs of the players are negative. This is due to the goal of each player being to maximize her own payoff and thus, minimize her prison sentence.

We can also represent this 2-player game as a bi-matrix, see Table 1.

Example 2.5 (Rock-Paper-Scissors [5, Ex. 1.4])**.** Two players choose either rock, paper or scissors at the same time. The rules of the game are: paper covers rock, scissors cut paper, rock smashes scissors. Thus the strategy sets of both players are given by $X_1 = X_2 = \{\text{rock}, \text{paper}, \text{scissors}\}$. If we denote winning with a payoff of 1, losing with a payoff of -1 and a draw with a payoff of 0, we can represent the game as the bi-matrix in Table 2.

3 Solution Concepts

In this section we will look at two different solution concepts, namely Pareto optimum and Nash equilibrium. For the latter, we will analyze its properties and later prove an existence theorem in Section 4. We also revisit the examples of Section 2 to find their Pareto optima and Nash equilibria.

Let us first look at the Pareto optimum.

Definition 3.1 (Pareto optimum [2, p. 6])**.** Consider a N-player game $\Gamma = \{X_i, \Phi_i\}_{i=1}^N$. We call a strategy vector $x^* \in \mathcal{X}$ Pareto optimal iff there exists no other strategy vector $x \in \mathcal{X}$ and player index $1 \leqslant i \leqslant N$ such that

$$\Phi_i(x) > \Phi_i(x^*) \quad \text{and} \quad \Phi_j(x) \geqslant \Phi_j(x^*)$$

for all $1 \leqslant j \leqslant N$, $j \neq i$.

Remark 3.2. A strategy vector $x^* \in \mathcal{X}$ is called Pareto optimal iff it is not possible to strictly increase the payoff of one player, without decreasing the payoff of another player. Thus, we can interpret the Pareto optimum x^* to be favourable from a social perspective, where members of a group do not want to harm each other.

Let us find the Pareto optima in the examples of Section 2.

Example 3.3 (Prisoner's Dilemma)**.** The Pareto optimal solutions of the Prisoner's Dilemma (see Table 1) are the following: (C, N), (N, C) and (N, N).

Let us for example check (C, N). There is no pair of strategies that could (strictly) increase the payoff of Player 1. For Player 2 all other pairs of strategies strictly increase her payoff, but the payoff of player 1 gets strictly decreased for each of them. Thus (C, N) is Pareto optimal. By symmetry of the payoff functions, this also proves that (N, C) is Pareto optimal.

Example 3.4 (Rock-Paper-Scissors)**.** All pairs of strageties in Rock-Paper-Scissors (see Table 2) are Pareto optimal. The reason is, that the Rock-Paper-Scissors game is a so called zero-sum game. Indeed, we can show that in every zero-sum game all strategy vectors are Pareto optimal. A proof of this statement can be found in the following Remark 3.5.

Remark 3.5. We call a game $\Gamma = \{X_i, \Phi_i\}_{i=1}^N$ zero-sum iff the sum of the payoff functions is zero, i.e.

$$\sum_{i=1}^N \Phi_i(x) = 0 \quad \forall\, x \in \mathcal{X}.$$

In a N-player zero-sum game all strategy vectors are Pareto optimal. In Section 6 we explore zero-sum games of two players in more detail.

We continue with a short proof of the statement.

Proof. Assume there exists a strategy $\widetilde{x} \in \mathcal{X}$ that is not Pareto optimal, i.e. there exists $x \in \mathcal{X}$ and $1 \leqslant i \leqslant N$ such that

$$\Phi_i(x) > \Phi_i(\widetilde{x}) \quad \text{and} \quad \Phi_j(x) \geqslant \Phi_j(\widetilde{x})$$

for all $1 \leqslant j \leqslant N$, $j \neq i$. Using the zero-sum property of the game we get

$$1 = \sum_{j=1}^N \Phi_j(\widetilde{x}) = \Phi_i(\widetilde{x}) + \sum_{j \neq i} \Phi_j(\widetilde{x}) < \Phi_i(x) + \sum_{j \neq i} \Phi_j(x) = \sum_{j=1}^N \Phi_j(x) = 1,$$

which is a contradiction. □ □

We now define Nash equilibrium, a key concept for the theory of competitive games, which will accompany us through the following sections.

Definition 3.6 (Nash equilibrium [2, p. 8])**.** Consider a N-player game $\Gamma = \{X_i, \Phi_i\}_{i=1}^N$. A strategy vector $x^* \in \mathcal{X}$ is called a Nash equilibrium iff

$$\Phi_i(x_i, x_{-i}^*) \leqslant \Phi_i(x_i^*, x_{-i}^*) \quad \forall\, x_i \in X_i$$

for all $1 \leqslant i \leqslant N$.

Unlike the Pareto optimum, a Nash equilibrium models a competitive situation, as described in the following remark.

Remark 3.7. A strategy vector $x^* \in \mathcal{X}$ is called Nash equilibrium iff no player can increase her payoff by changing her strategy, as long as the other players stick to the equilibrium strategy. Thus, the Nash equilibrium can be considered optimal in a competitive setting: Each player does not deviate from the equilibrium strategy in fear of the possibility of the other players choosing the equilibrium strategy and the loss that a deviation could cause in this situation.

Let us now find an equivalent formulation to the definition of Nash equilibrium. This will prove useful in the proof of the upcoming Theorem 4.3. We need the following.

Definition 3.8 (Best Response Map [5, Def. 1.10])**.** Consider a N-player game $\Gamma = \{X_i, \Phi_i\}_{i=1}^N$ and let $x \in \mathcal{X}$. For $1 \leqslant i \leqslant N$, we call the set-valued function

$$x_{-i} \mapsto S_i(x_{-i}) := \underset{x_i \in X_i}{\arg\max}\, \Phi_i(x_i, x_{-i})$$

the best response map of Player i. The best response map of the game Γ is given by

$$x \mapsto S(x) := S_1(x_1) \times \cdots \times S_N(x_N).$$

Lemma 3.9. [5, Thm. 1.11] *Consider a N-player game $\Gamma = \{X_i, \Phi_i\}_{i=1}^N$. A vector $x^* \in \mathcal{X}$ is a Nash equilibrium iff it is a fixed point of the best response map, i.e.*

$$x^* \in S(x^*).$$

Proof. Let $x^* \in \mathcal{X}$. For all $1 \leqslant i \leqslant N$ there holds

$$x_i^* \in S_i(x_{-i}^*) \Leftrightarrow x_i^* \in \underset{x_i \in X_i}{\arg\max}\, \Phi_i(x_i, x_{-i}^*) \Leftrightarrow \Phi_i(x_i, x_{-i}^*) \leqslant \Phi_i(x_i^*, x_{-i}^*)\ \forall x_i \in \mathcal{X}_i.$$

Thus, $x^* \in S(x^*)$ is equivalent to x^* being a Nash equilibrium. □

Let us find the Nash equilibria in the examples from Section 2.

Example 3.10 (Prisoner's Dilemma)**.** By checking, we find (C, C) as the unique Nash equilibrium. Combining this result with the result in example 3.3, we see that all strategies but the Nash equilibrium are Pareto optimal, i.e. optimal from a social point of view. This explains why the prisoner's dilemma is such a dilemma: The Nash equilibrium, which is optimal from a competitive point of view and thus will propably be choosen by the two prisoners given their situation, is the only strategy that is not favourable for the group.

		player 2		
		b_1	b_2	b_3
player 1	a_1	$0, 0$	$0, 0$	$5, 4$
	a_2	$3, 3$	$0, 0$	$0, 0$

Table 3: A bi-matrix game, where the first number is the payoff of player 1 and the second number is the payoff of player 2.

Example 3.11 (Rock-Paper-Scissors)**.** This game has no Nash equilibria.

In the following lemma, we state a few properties of Nash equilibria, some of which we have already seen in the preceding examples.

Lemma 3.12. [2, p. 8] *Consider a N-player game $\Gamma = \{X_i, \Phi_i\}_{i=1}^N$.*

1. *A Nash equilibrium may not exist.*
2. *A Nash equilibrium does not need to be unique.*
3. *Different Nash equilibria can yield different payoffs to each player and different total payoffs.*
4. *A Nash equilibrium may not be Pareto optimal.*

Proof.

1. By Example 3.11.
2. Consider the bi-matrix game in Table 3. We can check that both (a_1, b_3) and (a_2, b_1) are Nash equilibria. These yield not only different payoffs to each player but also different total payoffs.
3. The Nash equilibria from 2. have different payoffs.
4. By example 3.10.

□

4 Existence of Nash Equilibria

We continue with our analysis of Nash equilibria and prove an existence theorem. To do this, we need suitable assumptions on the payoff functions Φ_i, i.e. suitable convexity (or in this case concavity) assumptions. Let us first define a concave function.

Definition 4.1. A set $X \subset \mathbb{R}^n$ is called convex, iff for all $x, y \in X$ and $\lambda \in (0, 1)$, we also have

$$\lambda x + (1 - \lambda)y \in X.$$

Definition 4.2. Let $X \subset \mathbb{R}^n$ be a convex set. A function $f : X \to \mathbb{R}$ is called

- convex (on X), iff for all $x, y \in X$ and $\lambda \in (0, 1)$, there holds

$$f(\lambda x + (1 - \lambda)y) \leqslant \lambda f(x) + (1 - \lambda)f(y).$$

- concave (on X), iff $-f$ is convex, i.e. for all $x, y \in X$ and $\lambda \in (0,1)$, there holds
$$\lambda f(x) + (1-\lambda) f(y) \leqslant f(\lambda x + (1-\lambda) y).$$

Let us now formulate the existence theorem, which is the N-player equivalent of the one found in [2].

Theorem 4.3 (Existence of Nash equilibria)**.** *Consider the N-player game $\Gamma = \{X_i, \Phi_i\}_{i=1}^N$. Assume $X_i \subset \mathbb{R}^n$ are compact, convex and Φ_i are continuous. Further assume*

$$x_i \mapsto \Phi_i(x_i, x_{-i}) \text{ is concave for all } x_{-i} \in \mathcal{X}^{-i}$$

for all $1 \leqslant i \leqslant N$. Then, the game Γ admits a Nash equilibrum.

To prove this result, we use the fixpoint theorem of Kakutani. Recall that by Lemma 3.9, $x^* \in \mathcal{X}$ is a Nash equilibrium iff it is a fixed point of the best reply map S. First, a remark on notation.

Remark 4.4. For a set-valued function $f : X \to \mathfrak{P}(Y)$ we write $f : X \rightrightarrows Y$, where $\mathfrak{P}(Y)$ is the power set of Y. If f is set-valued we also call it a multifunction.

Theorem 4.5 (Fixpoint Theorem of Kakutani [2, Cor. A.6])**.** *Let $K \subset \mathbb{R}^n$ be compact and convex. Let $F : K \rightrightarrows \mathbb{R}^n$ be an upper semicontinuous multifunction with compact, convex values. Assume that $F(x) \subset K$ for all $x \in K$. Then there exists $x^* \in K$ such that*
$$x^* \in F(x^*).$$

To cope with the notion of a upper semicontinuous multifunction in the preceding theorem, we are going to use the following characterization [2].

Lemma 4.6. *Let $F : X \rightrightarrows \mathbb{R}^n$ be a bounded multifunction with compact values. Then the following are equivalent:*

1. *F is upper semicontinuous, i.e. $\forall x \in X\, \forall \varepsilon > 0\, \exists \delta > 0$ such that*
$$F(x') \subset B_\varepsilon(F(x))$$
for all $d(x', x) < \delta$, where $B_\varepsilon(F(x)) = \{z : \exists y \in F(x), d(y,z) < \varepsilon\}$.
2. $\operatorname{graph}(F) := \{(x,y) \in X \times Y : y \in F(x)\}$ *is closed.*

Proof (of Theorem 4.3). We want to make statements about properties of the best reply map S. Let us first look at its parts

$$\begin{aligned} S_i(x_i) &= \arg\max_{x_i \in X_i} \Phi_i(x_i, x_{-i}) \\ &= \{x_i \in X_i : \Phi_i(x_i, x_{-i}) = \max_{\omega \in X_i} \Phi_i(\omega, x_{-i})\} \\ &= \{x_i \in X_i : \Phi_i(x_i, x_{-i}) = m_i(x_{-i})\} \end{aligned}$$

where $m_i(x_{-i}) := \max_{x_i \in X_i} \Phi(x_i, x_{-i})$. Note that m_i is continuous, because of the compactness of $\mathcal{X}^{-i}$ and continuity if Φ_i. We claim:

1. $S_i(x_{-i})$ is compact.
2. $S_i(x_{-i})$ is convex.

Let us first assume both these claims are true. We now consider

$$S : \mathcal{X} \rightrightarrows \mathcal{X},\, x \mapsto S(x) := S_1(x_{-1}) \times \cdots \times S_N(x_{-N}).$$

The properties of $S_i(x_{-i})$ (i.e. compact and convex) also hold for the cartesian product. Thus, S has compact and convex values. With that, we satisfy the conditions of lemma 4.6. Note that $\mathcal{X}$ is bounded, thus S is also bounded. Let us now look at

$$\operatorname{graph}(S) = \{(x, x') \in \mathcal{X} \times \mathcal{X} : x' \in S(x)\}.$$

Let us take a convergent sequence $(\omega_n, \omega_n')_{n\in\mathbb{N}} \subset \operatorname{graph}(S)$, i.e.

$$(\omega_n, \omega_n') \to (\omega, \omega') \in \mathcal{X} \times \mathcal{X} \quad \text{for} \quad n \to \infty.$$

There holds

$$\begin{aligned} & \omega_n' \in S(\omega_n) \\ \Leftrightarrow\; & \omega_{n,i}' \in S_i(\omega_{n,i}) && \forall 1 \leqslant i \leqslant N \\ \Leftrightarrow\; & \Phi_i(\omega_{n,i}', \omega_{n,-i}) = m_i(\omega_{n,-i}) && \forall 1 \leqslant i \leqslant N. \end{aligned}$$

Because Φ_i and m_i are both continuous, the above is equivalent to

$$\begin{aligned} & \Phi_i(\omega_i', \omega_{-i}) = m_i(\omega_{-i}) && \forall 1 \leqslant i \leqslant N \\ \Leftrightarrow\; & \omega' \in S(\omega). \end{aligned}$$

Thus, $(\omega, \omega') \in \operatorname{graph}(S)$. This means $\operatorname{graph}(S)$ is closed and, by lemma 4.6, S is uppersemicontinuous. The Fixpoint Theorem of Kakutani yields a fixed point $x^* \in S(x^*)$ which is, by lemma 3.9, a Nash equilibrium.

It remains to proof the claims from above:

1. Note that $S_i(x_{-i})$ is bounded as a subset of X_i, which itself is bounded.

   Let us take a convergent sequence $(\omega_n)_{n\in\mathbb{N}} \subset S_i(x_{-i})$, i.e.

   $$\omega_n \to \omega \quad \text{for} \quad n \to \infty$$

   for some $\omega \in X_i$. We show $\omega \in S_i(x_{-i})$. As for $n \in \mathbb{N}$, $\omega_n \in S_i(x_{-i})$ it holds that $\Phi_i(\omega_n, x_{-i}) = m_i(x_{-i})$ and, as Φ_i is continuous, $\Phi_i(\omega, x_{-i}) = m_i(x_{-i})$. Thus, $\omega \in S_i(x_{-i})$ which means $S_i(x_{-i})$ is closed.

   In total, $S_i(x_{-i})$ is compact.

2. Let $\omega_i, \omega_2 \in S_i(x_{-i})$, i.e.

   $$\Phi_i(\omega_1, x_{-i}) = \Phi_i(\omega_2, x_{-i}) = m_i(x_{-i}).$$

   Let $\lambda \in (0, 1)$. Because $x_i \mapsto \Phi_i(x_i, x_{-i})$ is concave, there holds

   $$\begin{aligned} m_i(x_{-i}) &\geqslant \Phi_i(\lambda\omega_1 + (1-\lambda)\omega_2, x_{-i}) \\ &\geqslant \lambda\Phi_i(\omega_1, x_{-i}) + (1-\lambda)\Phi_i(\omega_2, x_{-i}) \\ &= m_i(x_{-i}). \end{aligned}$$

   Thus, all inequalities are equal and $\lambda\omega_1 + (1-\lambda)\omega_2 \in S_i(x_{-i})$.

   This means $S_i(x_{-i})$ is convex.

□

5 Randomized Strategies

As we have seen in theorem 4.3, one can guarantee the existence of a Nash equilibrium under suitable convexity assumptions. To achive a general existence result, we relax our notion of strategy. Let us look at the following:

Definition 5.1. Consider a N-player game $\Gamma = \{X_i, \Phi_i\}_{i=1}^N$. A randomized strategie is a probability measure $\mu_i \in \mathcal{P}(X_i)$, where $\mathcal{P}(X_i)$ is the set of all probability measures on the set of strategies X_i. The corresponding payoff functions are defined as

$$\widetilde{\Phi}_i(\mu_i) = \int_{\mathcal{X}} \Phi_i(x_i, x_{-i}) \, \mathrm{d}(\mu_i, \mu_{-i}).$$

We define $\widetilde{\Gamma} = \{\mathcal{P}(X_i), \widetilde{\Phi}_i\}_{i=1}^N$ as the game with respect to randomized strategies.

Remark 5.2.
1. Randomized strategies allow players to not only deterministically choose a strategy, but to choose each strategy with a certain probability.
2. The payoffs w.r.t. randomized strategies $\widetilde{\Phi}_i$ are the expected values of the payoffs Φ_i if the players choose random strategies independently, according to the probability measures μ_i.
3. We call $x_i \in X_i$ a pure strategy. Note that for each pure strategy there is a $\mu_i \in \mathcal{P}(X_i)$ that concentrates at x_i. Thus, the set of pure strategies is a subset of the set of randomized strategies.

With this relaxed notion of strategy, we will be able to prove the following. Again, we refer to [2] for the 2-player equivalent.

Theorem 5.3 (Existence of Nash equilibria for randomized strategies)**.** *Consider a N-player game in randomized strategies $\widetilde{\Gamma} = \{\mathcal{P}(X_i), \widetilde{\Phi}_i\}_{i=1}^N$. Let X_i be compact metric spaces and let the underlying payoff functions Φ_i be continuous. Then there exists a Nash equilibrium in randomized strategies, i.e. there exists $\mu_i^* \in \mathcal{P}(X_i)$ such that*

$$\widetilde{\Phi}_i(\mu_i, \mu_{-i}^*) \leqslant \widetilde{\Phi}_i(\mu_i^*, \mu_{-i}^*) \quad \forall \mu_i \in \mathcal{P}(X_i)$$

for all $1 \leqslant i \leqslant N$.

Proof. Note that we will sometimes switch from subscript to superscript as purposed in notation 2.1.

1. Let us first consider the following (simple) case where each X_i is finite, i.e.
$$X_i = \{x_1^i, \ldots, x_{n_i}^i\}$$
with $n_i \in \mathbb{N}$ for $1 \leqslant i \leqslant N$. For the payoff functions we define the abbrevation
$$\Phi^i_{\boldsymbol{k}_i, \boldsymbol{k}_{-i}} := \Phi^i(x^i_{\boldsymbol{k}_i}, x^{-i}_{\boldsymbol{k}_{-i}})$$
where $\boldsymbol{k} \in \mathcal{I} := \{k = (k_1, \ldots, k_N) \in \mathbb{N}^N : 1 \leqslant k_i \leqslant n_i \text{ for all } 1 \leqslant i \leqslant N\}$ is a (feasible) multiindex.

In this setting, a randomized strategy for player i, i.e. probability measure on X_i, is uniquely determined by a vector $\hat{x}^i = (\hat{x}^i_1, \dots, \hat{x}^i_{n_i}) \in \Delta_{n_i}$ where

$$\Delta_{n_i} := \{\hat{x} = (\hat{x}_1, \dots, \hat{x}_{n_i}) : \hat{x}_j \in [0,1], \sum_{j=1}^{n_i} \hat{x}_j = 1\}.$$

Here, $\hat{x}^i_j$ is the probability that player i chooses the strategy x^i_j. We can identify the sets of randomized strategies $\mathcal{P}(X_i)$ and Δ_{n_i} from above.

For the payoff functions $\widetilde{\Phi}^i : \bigtimes_{i=1}^N \Delta_{n_i} \to \mathbb{R}$ there holds

$$\widetilde{\Phi}^i(\hat{x}^i, \hat{x}^{-i}) = \sum_{\boldsymbol{k} \in \mathcal{I}} \left(\Phi^i_{\boldsymbol{k}_i, \boldsymbol{k}_{-i}} \prod_{j=1}^N \hat{x}^j_{\boldsymbol{k}_j} \right).$$

We see that $\widetilde{\Phi}^i$ are linear in each component, hence continuous. In particular $\widetilde{\Phi}^i$ is linear in its i-th component, thus also concave in its i-th component. We can apply theorem 4.3 to optain the existence of a Nash equilibrium $\hat{x}^* \in \Delta_{n_1} \times \cdots \times \Delta_{n_N}$.

2. Using an approximation argument we extend the result from above to the general case where X_i are compact metrix spaces.

   Let $\{x^i_j\}_{j\in\mathbb{N}}$ be a sequence of points dense in X_i. For each $n \geqslant 1$, we consider the game where players may only choose among the strategies $X'_i = \{x^i_1, \dots, x^i_n\}$. By step 1, this game admits a Nash equilibrium solution in randomized strategies $\mu^i_n \in \mathcal{P}(X'_i)$.

   Since X_i are compact (and by possibly extracting a subsequence) we can achive weak convergence

$$\mu^i_n \rightharpoonup \mu^i_* \text{ as } n \to \infty \tag{1}$$

   for some probability measure $\mu^i_* \in \mathcal{P}(X_i)$.

3. We claim that μ_* is a Nash equilibrium solution to the original game $\widetilde{\Gamma}$. We must show that

$$\int_{\mathcal{X}} \Phi^i(x_i, x_{-i}) \mathrm{d}(\mu^i_*, \mu^{-i}_*) = \max_{\mu \in \mathcal{P}(X_i)} \int_{\mathcal{X}} \Phi^i(x_i, x_{-i}) \mathrm{d}(\mu, \mu^{-i}_*)$$

   for all $1 \leqslant i \leqslant N$.

   Let $\varepsilon > 0$. By the continuity assumption, there exists $\delta > 0$ such that

$$\mathrm{d}((x_i, x_{-i}), (x'_i, x'_{-i})) \leqslant \delta \quad \text{imply} \quad |\Phi^i(x_i, x_{-i}) - \Phi^i(x'_i, x'_{-i})| < \varepsilon.$$

   Since the sequence $\{x^i_j\}_{j\in\mathbb{N}}$ is dense in X_i, we can find $M_i = M_i(\delta) \in \mathbb{N}$ such that X_i is covered by the union of open balls $B(x^i_j, \delta)$, $j = 1, \dots, M_i$, centered at x^i_j with radius δ. Let $M := \max_{1 \leqslant i \leqslant N} M_i$. Then

$$\mathcal{B}^i := \{B(x^i_j, \delta), j = 1, \dots, M\}$$

   is still a covering of X_i.

Let $\{\varphi_1^i, \ldots, \varphi_M^i\}$ be a continuous partition of unity on X_i, subordinate to the covering $\mathcal{B}^i$. For further reading on paritions of unity and their properties we refer to [1].

Any probability measure $\mu^i \in \mathcal{P}(X_i)$ can be approximated by a probability measure $\hat{\mu}^i$ supported on $X_i' = \{x_1^i, \ldots, x_M^i\}$. This approximation is unique by setting

$$\hat{\mu}^i(\{x_j^i\}) \coloneqq \int \varphi_j^i \mathrm{d}\mu^i \quad \forall j = 1, \ldots, M.$$

For every vector of randomized strategies $\mu \in \times_{i=1}^N \mathcal{P}(X_i)$, (3) yields

$$\begin{aligned}
&\left| \int_{\mathcal{X}} \Phi^i(x_i, x_{-i}) \ \mathrm{d}(\mu^i, \mu^{-i}) - \int_{\mathcal{X}} \Phi^i(x_i, x_{-i}) \ \mathrm{d}(\hat{\mu}^i, \hat{\mu}^{-i}) \right| \\
&\leqslant \int_{\mathcal{X}} \sum_{\boldsymbol{k} \in \mathcal{I}} \prod_{j=1}^N \varphi_{k_j}^j(x_{k_j}^j) |\Phi^i(x_i, x_{-i}) - \Phi^i(x_{k_i}^i, x_{k_j}^j)| \ \mathrm{d}(\mu^i, \mu^{-i}) \qquad (2) \\
&\leqslant \int_{\mathcal{X}} \varepsilon \ \mathrm{d}(\mu^i, \mu^{-i}) = \varepsilon
\end{aligned}$$

where we used that

$$\sum_{\boldsymbol{k} \in \mathcal{I}} \left(\prod_{j=1}^N \varphi_{k_j}^j(x_{k_j}^j) \right) = \prod_{j=1}^N \left(\sum_{1 \leqslant k_j \leqslant n_j} \varphi_{k_j}^j(x_{k_j}^j) \right) = 1$$

by the partition of unity property.

4. For all $j = 1, \ldots, M$, as $n \to \infty$ the weak convergence (1) yields

$$\hat{\mu}_n^i(\{x_j^i\}) = \int \varphi_j^i \ \mathrm{d}\mu_n^i \to \int \varphi_j^i \ \mathrm{d}\mu_*^i = \hat{\mu}_*^i(\{x_j^i\}). \qquad (3)$$

Recall that, for $\mu^i \in \mathcal{P}(X_i)$ and $n \geqslant M$, $\hat{\mu}$ is supported on the finite set $\{x_1^i, \ldots, x_M^i\}$ and μ_n^i provides a Nash equilibrium to the game restricted to the sets $X_i' = \{x_1^i, \ldots, x_n^i\}$. Thus,

$$\widetilde{\Phi}^i(\hat{\mu}^i, \mu_n^{-i}) \leqslant \widetilde{\Phi}^i(\mu^i, \mu_n^{-i}). \qquad (4)$$

Using (2), (3) and (4) we find for every $\mu^i \in \mathcal{P}(X_i)$ there holds

$$\begin{aligned}
\widetilde{\Phi}^i(\mu, \mu_*^{-i}) - \varepsilon &\leqslant \widetilde{\Phi}^i(\hat{\mu}^i, \hat{\mu}_*^{-i}) \\
&= \lim_{n \to \infty} \widetilde{\Phi}^i(\hat{\mu}^i, \hat{\mu}_n^{-i}) \\
&\leqslant \limsup_{n \to \infty} \widetilde{\Phi}^i(\hat{\mu}^i, \mu_n^{-i}) + \varepsilon \\
&\leqslant \lim_{n \to \infty} \widetilde{\Phi}^i(\mu_n^i, \mu_n^{-i}) + \varepsilon = \widetilde{\Phi}^i(\mu_*^i, \mu_*^{-i}) + \varepsilon.
\end{aligned}$$

As $1 \leqslant i \leqslant N$, $\mu^i \in \mathcal{P}(X_i)$ and $\varepsilon > 0$ were arbitrary, this proves the claim.

$\square$

6 Zero-Sum Games

Because the following theory seems not to be available in the literature for an arbitraty number of players, we are going to restrict outselves to 2 player games. For example, we could not find an equivalent result to the Von Neumann Minimax Theorem (see corollary 6.10).

Let us first define what a 2-player zero-sum game is.

Definition 6.1. Consider a 2 player game $\Gamma = \{X_i, \Phi_i\}_{i=1,2}$. In the special case where the sum of the payoff functions is zero, i.e.

$$\Phi_1 + \Phi_2 = 0$$

we call this game zero-sum. We define $\Phi := \Phi_1 = -\Phi_2$ as the single payoff function and write $\Gamma_0 = (X_1, X_2, \Phi)$ to put emphasis on the zero-sum property of the game.

Remark 6.2. The goal of player 1 is

$$\max_{x_1 \in X_1} \Phi(x_1, x_2).$$

and the goal of player 2 is

$$\min_{x_2 \in X_2} \Phi(x_1, x_2).$$

Note that the latter is equivalent to the original goal of player 2, i.e.

$$\max_{x_2 \in X_2} -\Phi(x_1, x_2).$$

We see that in a 2 player zero-sum game the gain of one player is anothers loss, i.e. $\Phi_1 = -\Phi_2$. In this sense, a zero-sum game can be considered purely competitive.

The general assumption is, that each player chooses her strategy at exactly the same time, without knowledge of the other participants choices. Let us assume for a moment that this is not the case. There are two possible cases

1. Player 1 chooses a strategy $x_1 \in X_1$. Then player 2 makes her choice depending on x_1. The best reply of player 2 will be some $\beta(x_1) \in S_2(x_1)$, such that
$$\Phi(x_1, \beta(x_1)) = \min_{x_2 \in X_2} \Phi(x_1, x_2).$$
Knowing that player 2 will respond with $\beta(x_1)$, the maximum payoff player 1 can achive is
$$V^- := \max_{x_1 \in X_1} \Phi(x_1, \beta(x_1)) = \max_{x_1 \in X_1} \min_{x_2 \in X_2} \Phi(x_1, x_2).$$

2. Player 2 chooses a strategy $x_2 \in X_2$. Then player 1 makes her choice depending on x_2. Again, the best reply for player 1 will be some $\alpha(x_2) \in S_1(x_2)$, such that
$$\Phi(\alpha(x_2), x_2) = \max_{x_1 \in X_1} \Phi(x_1, x_2).$$
Knowing that player 1 will respond with $\alpha(x_2)$, the minimum payoff player 2 can achive is
$$V^+ := \min_{x_2 \in X_2} \Phi(\alpha(x_2), x_2) = \min_{x_2 \in X_2} \max_{x_1 \in X_1} \Phi(x_1, x_2).$$

We can show the following

Lemma 6.3. *In the above setting, there holds*

$$V^- := \max_{x_1 \in X_1} \min_{x_2 \in X_2} \Phi(x_1, x_2) \leqslant \min_{x_2 \in X_2} \max_{x_1 \in X_1} \Phi(x_1, x_2) =: V^+.$$

Proof. Consider

$$V^- = \sup_{x_1 \in X_1} \Phi(x_1, \beta(x_1)).$$

Note that we write sup instead of max, because the map $x \mapsto \beta(x)$ may be discontinuous. By definition of the supremum, for any $\varepsilon > 0$ there exists some $x_\varepsilon \in X_1$ such that

$$\Phi(x_\varepsilon, \beta(x_\varepsilon)) > V^- - \varepsilon.$$

This implies

$$V^+ = \min_{x_2 \in X_2} \max_{x_1 \in X_1} \Phi(x_1, x_2) \geqslant \min_{x_2 \in X_2} \Phi(x_\varepsilon, x_2) = \Phi(x_\varepsilon, \beta(x_\varepsilon)) > V^- - \varepsilon$$

where we used the definition of β from above. As $\varepsilon > 0$ was arbitrary, this completes our proof. $\square$

Based on this, we make the following definition.

Definition 6.4. In the case that $V^- = V^+$ we define

$$V := V^- = V^+$$

and call it the value of the game.

Let us make another definition before we start to discuss how the above, the following definition and the Nash equilibrium are linked to one another.

Definition 6.5. Consider a 2-player zero-sum game $\Gamma_0 = (X_1, X_2, \Phi)$. If there exists $(x_1^*, x_2^*) \in X_1 \times X_2$ such that

$$\min_{x_2 \in X_2} \Phi(x_1^*, x_2) = \Phi(x_1^*, x_2^*) = \max_{x_1 \in X_1} \Phi(x_1, x_2^*), \tag{5}$$

then we call the pair (x_1^*, x_2^*) a saddle point of the game Γ_0.

First, a remark concerning the interpretation of a saddle point.

Remark 6.6. Calling the common value in (5) W, there holds the following

1. If player 1 adopts strategy x_1^*, she is guaranteed to receive a payoff no less than W, i.e.

$$\Phi(x_1^*, x_2) \geqslant \min_{\gamma \in X_2} \Phi(x_1^*, \gamma) = W \quad \forall x_2 \in X_2.$$

2. Conversely, if player 2 adopts the strategy x_2^*, she is guaranteed to lose no more than W, i.e.

$$\Phi(x_1, x_2^*) \leqslant \max_{\gamma \in X_1} \Phi(\gamma, x_2^*) = W \quad \forall x_1 \in X_1.$$

The following corollary describes the connection between saddle points and Nash equilibria for 2 player zero-sum games.

Corollary 6.7. *For a 2-player zero-sum game Γ_0 the concept of a saddle point is equivalent to that of a Nash equilibrium.*

Proof.

1. In remark 6.6 we also have $W = \Phi(x_1^*, x_2^*)$ which yields that every saddle point is a Nash equilibrium.

2. Let (x_1^*, x_2^*) be a Nash equilibrium for the zero-sum game, i.e.

$$\begin{aligned} \Phi(x_1, x_2^*) &\leqslant \Phi(x_1^*, x_2^*) \quad \forall x_1 \in X_1 \\ \Phi(x_1^*, x_2) &\geqslant \Phi(x_1^*, x_2^*) \quad \forall x_2 \in X_2. \end{aligned}$$

Because $x_1^* \in X_1$ and $x_2^* \in X_2$, this yields

$$\begin{aligned} \Phi(x_1^*, x_2^*) &= \max_{x_1 \in X_1} \Phi(x_1, x_2^*) \\ \Phi(x_1^*, x_2^*) &= \min_{x_2 \in X_2} \Phi(x_1^*, x_2), \end{aligned}$$

which is the definition of a saddle point.

□

Let us now also connect saddle points with the value of the game.

Theorem 6.8. *Consider a 2 player zero-sum game $\Gamma_0 = (X_1, X_2, \Phi)$. Assume X_1, X_2 to be compact matrix spaces and Φ to be continuous. Then, Γ_0 has a value V iff a saddle point (x_1^*, x_2^*) exists. In this case, there holds*

$$V = V^- = V^+ = \Phi(x_1^*, x_2^*).$$

Proof.

1. Assume that a saddle point (x_1^*, x_2^*) exists. Then

$$\begin{aligned} V^- &= \max_{x_1 \in X_1} \min_{x_2 \in X_2} \Phi(x_1, x_2) \\ &\geqslant \min_{x_2 \in X_2} \Phi(x_1^*, x_2) \\ &= \Phi(x_1^*, x_2^*) \\ &= \max_{x_1 \in X_1} \Phi(x_1, x_2^*) \\ &\geqslant \min_{x_2 \in X_2} \max_{x_1 \in X_1} \Phi(x_1, x_2) = V^+. \end{aligned}$$

By lemma 6.3 there holds $V^- \leqslant V^+$ and thus, equality.

2. Assume that the game has a value. Let $x_1 \mapsto \beta(x_1) \in S_2(x_1)$ be the best reply of player 2 to the strategy x_1. As in the proof of lemma 6.3, for all $\varepsilon > 0$ there exists some $x_\varepsilon \in X_1$, such that

$$\Phi(x_\varepsilon, \beta(x_\varepsilon)) > V^- - \varepsilon. \tag{6}$$

Since both X_1 and X_2 are compact, we can chose a subsequence $\varepsilon_n \to 0$ $(n \to \infty)$ such that the strategies converge, i.e.

$$x_{\varepsilon_n} \to x_1^* \quad \text{and} \quad \beta(x_{\varepsilon_n}) \to x_2^*$$

for some $x_1^* \in X_1$ and $x_2^* \in X_2$. We now show that (x_1^*, x_2^*) is a saddle point. Observe

$$\begin{aligned} V^- - \varepsilon_n &< \Phi(x_{\varepsilon_n}, \beta(x_{\varepsilon_n})) \\ &\leqslant \sup_{x_1 \in X_1} \Phi(x_1, \beta(x_1)) = V^+ \end{aligned}$$

by (6) and the definition of β. Letting $n \to \infty$ (i.e. $\varepsilon_n \to 0$) we get by continuity of Φ

$$V^- \leqslant \lim_{n\to\infty} \Phi(x_{\varepsilon_n}, \beta(x_{\varepsilon_n})) = \Phi(x_1^*, x_2^*) \leqslant V^+.$$

Since $V^- = V^+$ by assumption, we have

$$V^- = \Phi(x_1^*, x_2^*) = V^+.$$

□

Remark 6.9. Corrolary 6.7 and theorem 6.8 imply the following: If a Nash equilibrium exists in a 2 player zero-sum game, then all Nash equilibria yield the same payoff, i.e. the value of the game.

We now state two direct implications of theorem 4.3 and theorem 5.3. Note that the goal of player 2 is to maximize $\Phi_2 = -\Phi$ and Φ being concave is equivalent to $-\Phi$ being convex by definition.

The following implication of theorm 4.3 is also called the Von Neumann Minimax Theorem [4].

Corollary 6.10. *Consider a 2-player zero-sum game $\Gamma_0 = (X_1, X_2, \Phi)$. Assume X_1, X_2 are compact, convex metric spaces and Φ is continuous. Moreover let*

$$\begin{aligned} &x_1 \mapsto \Phi(x_1, x_2) \text{ be concave for all } x_2 \in X_2 \\ &x_2 \mapsto \Phi(x_1, x_2) \text{ be convex for all } x_1 \in X_1. \end{aligned}$$

Then, the game Γ_0 admits a Nash equilibrium.

In the case of randomized strategies we get the following, as an implication of theorem 5.3.

Corollary 6.11. *Consider a 2-player zero-sum game $\Gamma_0 = (X_1, X_2, \Phi)$. Assume X_1, X_2 are compact metric spaces and Φ is continuous. Then Γ_0 has a value and a Nash equilibrium (i.e. saddle point) in the class of randomized strategies. By definition of a saddle point, there exist $(\mu_1^*, \mu_2^*) \in \mathcal{P}(X_1) \times \mathcal{P}(X_2)$ such that*

$$\int_{\mathcal{X}} \Phi(x_1, x_2)\ \mathrm{d}(\mu_1, \mu_2^*) \leqslant \int_{\mathcal{X}} \Phi(x_1, x_2)\ \mathrm{d}(\mu_1^*, \mu_2^*) \leqslant \int_{\mathcal{X}} \Phi(x_1, x_2)\ \mathrm{d}(\mu_1^*, \mu_2)$$

for all $\mu_i \in \mathcal{P}(X_i)$, where $\mathcal{X} = X_1 \times X_2$.

Remark 6.12. As the class of pure strategies is a subset of the class of randomized strategies (see remark 5.2), the following holds: If the game Γ_0 already has a value in the class of pure strategies, this value and the one from theorem 6.11 above are the same.

7 The Cooperative-Competitive Solution

The setting here is a bit different than in the sections before. We want to look at a 2-player game $\Gamma = \{X_i, \Phi_i\}_{i=1,2}$ where X_1, X_2 are compact metric spaces and Φ_1, Φ_2 are continuous. In contrary to definition 2.2 the goal of the players shall now be to maximize the sum of the payoffs, i.e. adopt strategies $(x_1^\#, x_2^\#) \in X_1 \times X_2$ such that

$$\Phi_1(x_1^\#, x_2^\#) + \Phi(x_1^\#, x_2^\#) = \max_{(x_1,x_2)\in X_1\times X_2} \Phi_1(x_1, x_2) + \Phi_2(x_1, x_2) =: V^\#.$$

We now face the following problem: Suppose that $\Phi_1(x_1^\#, x_2^\#) \ll \Phi_2(x_1^\#, x_2^\#)$. As Φ_i are the payoffs the individual players earn, the adoption of $(x_1^\#, x_2^\#)$ may maximize the total payoff of the group, but may not be agreeable to player 1. In a case like this, it may be wise to introduct a side payment $p \in \mathbb{R}$ from player 2 to player 1, say. After each player earns her reward Φ_i, player 2 pays the side payment p to player 1. If the side payment is chosen appropriately, this may convince player 1 to adopt $(x_1^\#, x_2^\#)$ and thus maximize the group payoff. Note that in case $\Phi_2 \ll \Phi_1$, the side payment p should of course be negative.

To dertermine the appropriate value of such a side payment, let us split the game Γ into two subgames $\Gamma^\#$ and $\Gamma^\flat$ in the following way:

$$\Phi^\#(x_1, x_2 := \frac{\Phi_1(x_1, x_2) + \Phi_2(x_1, x_2)}{2}$$
$$\Phi^\flat(x_1, x_2) := \frac{\Phi_1(x_1, x_2) - \Phi_2(x_1, x_2)}{2}$$

such that

$$\Phi_1 = \Phi^\# + \Phi^\flat \quad \text{and} \quad \Phi_2 = \Phi^\# - \Phi^\flat.$$

This splits the original game Γ into the sum of the subgames

$$\Gamma^\# = \{(X_1, \Phi^\#), (X_1, \Phi^\#)\}$$

which is purely cooperative and the goal is to maximize the group payoff, and

$$\Gamma^\flat = \{(X_1, \Phi^\flat), (X_2, -\Phi^\flat)\}$$

which is a zero-sum game and thus purely competitive. We shall use the value $V^\flat$ of the zero-sum game $\Gamma^\flat$ to dertermine the relative strength of each player and thus the side payment p. Note that $V^\flat$ is well-defined (at least in terms of randomized strategies) by theorem 5.3.

We interpret a positive value of $V^\flat$ as "player 1 is stronger than player 2" and a negative value as "player 2 is stronger than player 1". The magnitude of $V^\flat$ is a measure of just how much stronger this stronger player is.

Based on this, we define

Definition 7.1. We call

$$\left(\frac{V^\#}{2} + V^\flat, \frac{V^\#}{2} - V^\flat\right)$$

the cooperative-competitive value (co-co value).

The co-co value is what each player should earn, considering her relative strength. Note that each player gets half of the playoff of the cooperative game $\Gamma^{\#}$ and player 1 earns an extra $V^{\flat}$ (if $V^{\flat}$ is positive, i.e. player 1 is stronger).

Definition 7.2. The cooperative-competitive solution (co-co solution) is a set of strategies $(x_1^{\#}, x_2^{\#}) \in X_1 \times X_2$ and a side payment $p \in \mathbb{R}$ from player 2 to player 1, such that

$$\Phi_1(x_1^{\#}, x_2^{\#}) + p = \frac{V^{\#}}{2} + V^{\flat}$$
$$\Phi_2(x_1^{\#}, x_2^{\#}) - p = \frac{V^{\#}}{2} - V^{\flat},$$

i.e. the final payoff of each player (the payoff after the side payment) corresponds to their respective co-co values.

Appendices

A quick summary of the other talks from the seminar follows. If not otherwise noted, everything that follows is stated in [3]. In particular, we will use the notation from [3] and symbols that are already defined above may be redefined. Note, that because of space regulations, not all definitions can be mentioned and some assumptions may be skipped. For more details we refer to the original work [3].

In contrary to this report where we analysed static games (see definition 2.2), what follows will be about dynamic games, i.e. there is a dependence on time. The following (prose) definition is given in [3]:

A game is said to be dynamic if at least one player can use a strategy which conditions his single-period action at any instant of time on the actions taken previously in the game.

A Markovian equilibria with simultaneous play

This is based on Chapter 4 of [3].

We consider a game with extends over the intervall $[0, T\rangle$. The state of the game is defined by a vector $x(t) \in X$, $t \in [0, T\rangle$ where we call $X \subset \mathbb{R}^n$ the state space. The initial state of the game is given by some $x_0 \in X$.

There are $N \in \mathbb{N}$ players. At time $t \in [0, T\rangle$ player i chooses a control variable $u^i(t)$ out of the set of feasible controls $U^i(x(t), t) \subset \mathbb{R}^{m_i}$.

The state $x(t)$ of the game evolves according to a differential equation

$$\dot{x}(t) = f(x(t), u^1(t), \ldots, u^N(t), t), \quad x(0) = x_0$$

where f is called the system dynamic. Thus, the game we consider is a so called differential game.

The goal of each player i is to maximize her objective functional J^i given by

$$J^i(u^i(\cdot)) = \int_0^T e^{-r_i t} F^i(x(t), u^1(t), \ldots, u^N(t), t) \, \mathrm{d}t + e^{r_i T} S^i(x(T))$$

where we call r_i the rate of time preference, F_i the utility function (which are comparable to the payoff functions from definition 2.2) and S^i the scrap value function. When $T = \infty$ we set $S^i(x) = 0$ for all $x \in X$.

Consider player i. If all her opponents use Markovian strategies, i.e.

$$u^j(t) = \Phi^j(x(t), t) \text{ for all } j \neq i$$

then one can show that player i faces a control problem, see (4.1) of [3].

For Markovian strategies there is a notion of Nash equilibrium: $(\Phi^i)_{i=1}^N$ is called Markovian Nash equilibrium iff for all i there exists an optimal control path u^i such that $u^i(t) = \Phi^i(x(t), t)$.

Theorem 4.1 and 4.2 of [3] give sufficient conditions for Markovian Nash equilibria. They are discussed with several tricks concerning their application.

In the last part of the chapter, two important properties of equilibria are introduced: The first one is called time consistency. A Markovian Nash equilibrium $(\Phi^i)_{i=1}^N$ is time consitent if, for each $t \in [0, T\rangle$, the subgame $\Gamma(x(t), t)$ admits a Markovian Nash equilibrium that is the same as $(\Phi^i)_{i=1}^N$ from t onward. Here $x(\cdot)$ is the unique state trajectory generated by the original game. One can show that all Markovian Nash equilibria are time consistent.

The second property is called subgame perfect. The difference to time consistency is that not only $\Gamma(x(t), t)$ must yield the same equilibrius but $\Gamma(x, t)$ for all $x \in X$. However there may be configurations $(x, t) \in X \cap [0, T\rangle$ that cannot be reached. If we only consider the reachable configurations we get the property called weakly subgame perfect.

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