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# Seminar Differential Games

Introduction to **Static Games**

based on Bressan, *Noncooperative Differential Games. A Tutorial*, 2010

**Patrick Finke**

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Introduction

Solution Concepts

Existence of Nash Equilibria

Randomized Strategies

Zero-Sum Games

The Cooperative-Competitive Solution

## Introduction

## Solution Concepts

## Existence of Nash Equilibria

## Randomized Strategies

## Zero-Sum Games

## The Cooperative-Competitive Solution

# Introduction

## Static Game (for two players)

- two sets of strategies  $A$  and  $B$
- payoff functions  $\Phi^A, \Phi^B : A \times B \rightarrow \mathbb{R}$
- goal of player A:  $\max_{a \in A} \Phi^A(a, b)$
- goal of player B:  $\max_{b \in B} \Phi^B(a, b)$

## Assumption

$A, B$  compact metric spaces,  $\Phi^A, \Phi^B$  continuous (A)

# Introduction

Example: Prisoner's dilemma

		Player B	
		C	N
Player A	C	-6 / -6	-8 / 0
	N	0 / -8	-1 / -1

Question: Which pair of strategies is optimal?

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# Solution Concepts

## Pareto optimality

$(a^*, b^*)$  is Pareto optimal if there exist no  $(a, b) \in A \times B$  such that

$$\Phi^A(a, b) > \Phi^A(a^*, b^*) \quad \text{and} \quad \Phi^B(a, b) \geq \Phi^B(a^*, b^*)$$

or

$$\Phi^B(a, b) > \Phi^B(a^*, b^*) \quad \text{and} \quad \Phi^A(a, b) \geq \Phi^A(a^*, b^*)$$

It is not possible to strictly increase the payoff of one player without strictly decreasing the payoff of the other.

**Note:** favourable from a social standpoint



# Solution Concepts

Example: prisoner's dilemma

		Player B	
		C	N
Player A	C	-6 / -6	-8 / 0
	N	0 / -8	-1 / -1

Pareto optimal solutions in red.

# Solution Concepts

Idea:

- A leader, B follower
- A announces  $a \in A$
- B chooses  $b^* \in R^B(a)$  say  $b^* = \beta(a)$
- goal of player A is  $\max_{a \in A} \Phi^A(a, \beta(a))$

## Stackelberg equilibrium

$(a_S, b_S)$  is Stackelberg equilibrium if

- $b_S \in R^B(a_S)$
- $\Phi^A(a, b) \leq \Phi^A(a_S, b_S) \quad \forall (a, b) \text{ with } b \in R^B(a)$

**Note:** Models asymmetry of information

(best reply map:  $R^B(a) = \{b \in B \mid \Phi^B(a, b) = \max_{\omega \in B} \Phi^B(a, \omega)\}$ )

# Solution Concepts

## Nash equilibrium

$(a^*, b^*)$  is Nash equilibrium if for every  $a \in A, b \in B$

$$\Phi^A(a, b^*) \leq \Phi^A(a^*, b^*)$$

and

$$\Phi^B(a^*, b) \leq \Phi^B(a^*, b^*)$$

No player can increase his payoff by changing his strategy as long as the other player sticks to the equilibrium strategy.

**Note:** symmetric situation, no means to cooperate, do not share information

# Solution Concepts

Example: prisoner's dilemma

		Player B	
		C	N
Player A	C	-6 / -6	-8 / 0
	N	0 / -8	-1 / -1

Nash equilibria in red.

# Solution Concepts

Some properties of Nash equilibria are:

1. Nash equilibria may not exist,
2. Nash equilibria need not be unique,
3. different Nash equilibria can yield different payoffs,
4. a Nash equilibrium may not be Pareto optimal.

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# Existence of Nash Equilibria

## Theorem

Assume  $A, B \subset \mathbb{R}^n$  compact, convex;  $\Phi^A, \Phi^B$  continuous and

$a \mapsto \Phi^A(a, b)$  is a concave function of  $a$ , for all  $b \in B$

$b \mapsto \Phi^B(a, b)$  is a concave function of  $b$ , for all  $a \in A$

Then the non-cooperative game admits a Nash equilibrium.

# Existence of Nash Equilibria

Recall:  $R^A(b) = \{a \in A \mid \Phi^A(a, b) = \max_{\omega \in A} \Phi^A(\omega, b)\}$   
 $R^B(a) = \{b \in B \mid \Phi^B(a, b) = \max_{\omega \in B} \Phi^B(a, \omega)\}$

Nash equilibrium as fixed point

$(a^*, b^*)$  is Nash equilibrium iff

$$a^* \in R^A(b^*) \text{ and } b^* \in R^B(a^*).$$

Since

$$\begin{aligned} \Phi^A(a, b^*) &\leq \Phi^A(a^*, b^*) \quad \forall a \in A \\ \Leftrightarrow \Phi^A(a^*, b^*) &= \max_{a \in A} \Phi^A(a, b^*) \\ \Leftrightarrow a^* &\in R^A(b^*) \end{aligned}$$



# Existence of Nash Equilibria

## upper semi-continuous

$Y$  compact,  $F : X \rightrightarrows Y$  a multifunction with nonempty, compact values. Then the following are equivalent:

- $F$  is upper semicontinuous (i.e.  $\forall x \in X \forall \epsilon > 0 \exists \delta = \delta(\epsilon) > 0$  such that  $F(x') \subseteq B_\epsilon(F(x))$  for all  $d(x', x) < \delta$ )
- $\text{graph}(F) = \{(x, y) \mid y \in F(x)\}$  is closed

## Kakutani

Let  $K \subset \mathbb{R}^n$  compact, convex. Let  $F : K \mapsto \mathbb{R}^n$  be an upper semicontinuous multifunction with compact, convex values such that  $F(x) \subset K$  for all  $x \in K$ . Then there exists  $\bar{x} \in K$  such that

$$\bar{x} \in F(\bar{x}).$$

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# Randomized Strategies

## Randomized Strategies

- strategy for player A is probability distribution  $\mu \in P(A)$
- strategy for player B is probability distribution  $\nu \in P(B)$
- payoff functions

$$\tilde{\Phi}^A(\mu, \nu) = \int_{A \times B} \Phi^A(a, b) d(\mu, \nu)$$

$$\tilde{\Phi}^B(\mu, \nu) = \int_{A \times B} \Phi^B(a, b) d(\mu, \nu)$$

**Note:**  $\tilde{\Phi}^A, \tilde{\Phi}^B$  are expected values of the payoffs, if  $\mu$  and  $\nu$  are chosen independently

**Note:**  $\{\text{pure strategies}\} \subseteq \{\text{randomized strategies}\}$

# Randomized Strategies

existence of Nash equilibria for randomized strategies

Let (A) hold. There exist  $\mu^* \in P(A)$  and  $\nu^* \in P(B)$  such that

$$\tilde{\Phi}^A(\mu, \nu^*) \leq \tilde{\Phi}^A(\mu^*, \nu^*) \text{ for all } \mu \in P(A)$$

and

$$\tilde{\Phi}^B(\mu^*, \nu) \leq \tilde{\Phi}^B(\mu^*, \nu^*) \text{ for all } \nu \in P(B)$$

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# Zero-Sum Games

## Zero-Sum Game

- consider again static game:  $A, B, \Phi^A, \Phi^B$
- special case  $\Phi^A + \Phi^B = 0$
- single payoff function sufficient

$$\Phi := \Phi^A = -\Phi^B$$

- goal of player A is  $\max_{a \in A} \Phi(a, b)$   
goal of player B is  $\min_{b \in B} \Phi(a, b)$

## Assumption

$A, B$  compact metric spaces,  $\Phi$  continuous (A')

# Zero-Sum Games

Suppose advantage of information:

- player B chooses  $b \in B$
- player A chooses  $\alpha(b) \in A$  such that

$$\Phi(\alpha(b), b) = \max_{a \in A} \Phi(a, b)$$

- goal of player B is  $\min_{b \in B} \Phi(\alpha(b), b)$

This yields:

$$V^- := \max_{a \in A} \min_{b \in B} \Phi(a, b) \leq \min_{b \in B} \max_{a \in A} \Phi(a, b) =: V^+$$

## Value of the Game

If  $V^- = V^+ =: V$ , then  $V$  is called value of the game.

# Zero-Sum Games

## value and Nash equilibrium

Let (A') hold. The zero-sum game has a value  $V$  iff a Nash equilibrium  $(a^*, b^*)$  exists. In this case

$$V = V^- = V^+ = \Phi(a^*, b^*).$$

**Note:** In Zero-Sum Games all Nash equilibria yield the same payoff.

**Note:** The value of the game is well defined (at least for randomized strategies)



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# The Cooperative-Competitive Solution

Setting:

- 2 player static game (no zero-sum)

$$\Phi^A, \Phi^B : A \times B \rightarrow \mathbb{R}$$

- goal: adopt  $(a^\#, b^\#)$  such that

$$V^\# := \Phi^A(a^\#, b^\#) + \Phi^B(a^\#, b^\#) = \max_{(a,b) \in A \times B} \Phi^A(a, b) + \Phi^B(a, b)$$

- may need a side payment if e.g.  $\Phi^B(a^\#, b^\#) \ll \Phi^A(a^\#, b^\#)$

# The Cooperative-Competitive Solution

Split the game

$$\Phi^{\#}(a, b) := \frac{\Phi^A(a, b) + \Phi^B(a, b)}{2}$$

$$\Phi^{\flat}(a, b) := \frac{\Phi^A(a, b) - \Phi^B(a, b)}{2}$$

such that

$$\Phi^A = \Phi^{\#} + \Phi^{\flat}$$

$$\Phi^B = \Phi^{\#} - \Phi^{\flat}$$

**Note:**  $\Phi^{\#} \rightarrow \frac{V^{\#}}{2}$ ,  $\Phi^{\flat} \rightarrow V^{\flat}$  (as value of the game)

# The Cooperative-Competitive Solution

co-co value

$$\left( \frac{V^\#}{2} + V^b, \frac{V^\#}{2} - V^b \right)$$

co-co solution

strategies  $(a^\#, b^\#) \in A \times B$ , side payment  $b \in \mathbb{R}$  from player B to player A such that

$$\Phi^A(a^\#, b^\#) + p = \frac{V^\#}{2} + V^b$$

$$\Phi^B(a^\#, b^\#) - p = \frac{V^\#}{2} - V^b$$

# The Cooperative-Competitive Solution

Example: prisoner's dilemma

		Player B	
		C	N
Player A	C	-6 / -6	-8 / 0
	N	0 / -8	-1 / -1

co-co solution in red;  $b = 0$

		Player B	
		C	N
Player A	C	-6	-4
	N	-4	-1

		Player B	
		C	N
Player A	C	0	4
	N	-4	0

$$\frac{V^\#}{2} = -1$$

$$V^b = 0$$

Thank you for your kind attention!