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Seminar Optimization

Introduction to **Game Theory and Static Games**
based on Bressan, *Noncooperative Differential Games. A Tutorial, 2010*

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Introduction

Solution Concepts

Existence of Nash Equilibria

Randomized Strategies

Zero-Sum Games

The Cooperative-Competitive Solution

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Introduction

Static Game (for two players)

- two sets of strategies A and B
- payoff functions $\Phi^A, \Phi^B : A \times B \to \mathbb{R}$
- goal of player A: $\max_{a \in A} \Phi^A(a, b)$
- goal of player B: $\max_{b \in B} \Phi^B(a, b)$

Assumption

$$A, B \text{ compact metric spaces, } \Phi^A, \Phi^B \text{ continuous} \tag{A}$$

Introduction

Example: Prisoner's dilemma

		Player B	
		C	N
Player A	C	-6 \ -6	0 \ -8
	N	-8 \ 0	-1 \ -1

Question: Which pair of strategies is optimal?

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Solution Concepts

Pareto optimality

(a^*, b^*) is Pareto optimal if there exist no $(a, b) \in A \times B$ such that

$$\Phi^A(a,b) > \Phi^A(a^*,b^*) \quad \text{and} \quad \Phi^B(a,b) \geq \Phi^B(a^*,b^*)$$

or

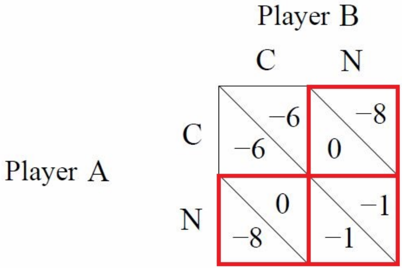
$$\Phi^B(a,b) > \Phi^B(a^*,b^*) \quad \text{and} \quad \Phi^A(a,b) \geq \Phi^A(a^*,b^*)$$

It is not possible to strictly increase the payoff of one player without strictly decreasing the payoff of the other.

Note: favourable from a social standpoint

Solution Concepts

Example: prisoner's dilemma



Pareto optimal solutions in red.

Solution Concepts

Idea:

- A leader, B follower
- A announces $a \in A$
- B chooses $b^* \in R^B(a)$ say $b^* = \beta(a)$
- goal of player A is $\max_{a \in A} \Phi^A(a, \beta(a))$

Stackelberg equilibrium

(a_S, b_S) is Stackelberg equilibrium if

- $b_S \in R^B(a_S)$
- $\Phi^A(a, b) \leq \Phi^A(a_S, b_S) \quad \forall (a, b) \text{ with } b \in R^B(a)$

Note: Models asymmetry of information
(best reply map: $R^B(a) = \{b \in B \mid \Phi^B(a, b) = \max_{\omega \in B} \Phi^B(a, \omega)\}$)

Solution Concepts

Nash equilibrium

(a^*, b^*) is Nash equilibrium if for every $a \in A, b \in B$

$$\Phi^A(a, b^*) \leq \Phi^A(a^*, b^*)$$

and

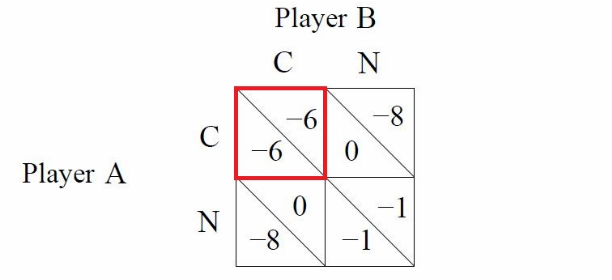
$$\Phi^B(a^*, b) \leq \Phi^B(a^*, b^*)$$

No player can increase his payoff by changing his strategy as long as the other player sticks to the equilibrium strategy.

Note: symmetric situation, no means to cooperate, do not share information

Solution Concepts

Example: prisoner's dilemma



Nash equilibria in red.

Solution Concepts

Some properties of Nash equilibria are:

1. Nash equilibria may not exist,
2. Nash equilibria need not be unique,
3. different Nash equilibria can yield different payoffs,
4. a Nash equilibrium may not be Pareto optimal.

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Existence of Nash Equilibria

Theorem

Assume $A, B \subset \mathbb{R}^n$ compact, convex; Φ^A, Φ^B continuous and

$$a \mapsto \Phi^A(a, b) \text{ is a concave function of } a, \text{ for all } b \in B$$

$$b \mapsto \Phi^B(a, b) \text{ is a concave function of } b, \text{ for all } a \in A$$

Then the non-cooperative game admits a Nash equilibrium.

Existence of Nash Equilibria

Recall:
$$R^A(b) = \{a \in A \mid \Phi^A(a,b) = \max_{\omega \in A} \Phi^A(\omega, b)\}$$
$$R^B(a) = \{b \in B \mid \Phi^B(a,b) = \max_{\omega \in B} \Phi^B(a, \omega)\}$$

Nash equilibrium as fixed point

(a^*, b^*) is Nash equilibrium iff

$$a^* \in R^A(b^*) \text{ and } b^* \in R^B(a^*).$$

Since

$$\begin{aligned} &\Phi^A(a, b^*) \leq \Phi^A(a^*, b^*) \ \forall a \in A \\ \Leftrightarrow\ &\Phi^A(a^*, b^*) = \max_{a \in A} \Phi^A(a, b^*) \\ \Leftrightarrow\ &a^* \in R^A(b^*) \end{aligned}$$

Existence of Nash Equilibria

upper semi-continuous

Y compact, $F : X \rightrightarrows Y$ a multifunction with nonempty, compact values. Then the following are equivalent:

- F is upper semicontinuous (i.e. $\forall x \in X\, \forall \epsilon > 0\, \exists \delta = \delta(\epsilon) > 0$ such that $F(x') \subseteq B_\epsilon(F(x))$ for all $d(x', x) < \delta$)
- $graph(F) = \{(x, y) \mid y \in F(x)\}$ is closed

Kakutani

Let $K \subset \mathbb{R}^n$ compact, convex. Let $F : K \mapsto \mathbb{R}^n$ be an upper semicontinuous multifunction with compact, convex values such that $F(x) \subset K$ for all $x \in K$. Then there exists $\bar{x} \in K$ such that

$$\bar{x} \in F(\bar{x}).$$

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Randomized Strategies

- strategy for player A is probability distribution $\mu \in P(A)$
- strategy for player B is probability distribution $\nu \in P(B)$
- payoff functions

$$\tilde{\Phi}^A(\mu, \nu) = \int_{A \times B} \Phi^A(a, b)\, d(\mu, \nu)$$

$$\tilde{\Phi}^B(\mu, \nu) = \int_{A \times B} \Phi^B(a, b)\, d(\mu, \nu)$$

Note: $\tilde{\Phi}^A, \tilde{\Phi}^B$ are expected values of the payoffs, if μ and ν are chosen independently

Note: $\{\text{pure strategies}\} \subseteq \{\text{randomized strategies}\}$

Randomized Strategies

existence of Nash equilibria for randomized strategies

Let (A) hold. There exist $\mu^* \in P(A)$ and $\nu^* \in P(B)$ such that

$$\tilde{\Phi}^A(\mu, \nu^*) \leq \tilde{\Phi}^A(\mu^*, \nu^*) \text{ for all } \mu \in P(A)$$

and

$$\tilde{\Phi}^B(\mu^*, \nu) \leq \tilde{\Phi}^B(\mu^*, \nu^*) \text{ for all } \nu \in P(B)$$

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Zero-Sum Game

- consider again static game: A, B, Φ^A, Φ^B
- spacial case $\Phi^A + \Phi^B = 0$
- single payoff function sufficient

$$\Phi := \Phi^A = -\Phi^B$$

- goal of player A is $\max_{a \in A} \Phi(a, b)$
  goal of player B is $\min_{b \in B} \Phi(a, b)$

Assumption

$$A, B \text{ compact metric spaces, } \Phi \text{ continuous} \tag{A'}$$

Zero-Sum Games

Suppose advantage of information:

- player B chooses $b \in B$
- player A chooses $\alpha(b) \in A$ such that

$$\Phi(\alpha(b), b) = \max_{a \in A} \Phi(a, b)$$

- goal of player B is $\min_{b \in B} \Phi(\alpha(b), b)$

This yields:

$$V^- := \max_{a \in A} \min_{b \in B} \Phi(a, b) \leq \min_{b \in B} \max_{a \in A} \Phi(a, b) =: V^+$$

Value of the Game

If $V^- = V^+ =: V$, then V is called value of the game.

Zero-Sum Games

value and Nash equilibrium

Let (A') hold. The zero-sum game has a value V iff a Nash equilibrium (a^*, b^*) exists. In this case

$$V = V^- = V^+ = \Phi(a^*, b^*).$$

Note: In Zero-Sum Games all Nash equilibria yield the same payoff.

Note: The value of the game is well defined (at least for randomized strategies)

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The Cooperative-Competitive Solution

Setting:

- 2 player static game (no zero-sum)

$$\Phi^A, \Phi^B : A \times B \to \mathbb{R}$$

- goal: adopt $(a^\#, b^\#)$ such that

$$V^\# := \Phi^A(a^\#, b^\#) + \Phi^B(a^\#, b^\#) = \max_{(a,b)\in A\times B} \Phi^A(a, b) + \Phi^B(a, b)$$

- may need a side payment if e.g. $\Phi^B(a^\#, b^\#) \ll \Phi^A(a^\#, b^\#)$

The Cooperative-Competitive Solution

Split the game

$$\Phi^{\#}(a,b) := \frac{\Phi^A(a,b) + \Phi^B(a,b)}{2}$$

$$\Phi^{\flat}(a,b) := \frac{\Phi^A(a,b) - \Phi^B(a,b)}{2}$$

such that

$$\Phi^A = \Phi^{\#} + \Phi^{\flat}$$

$$\Phi^B = \Phi^{\#} - \Phi^{\flat}$$

Note: $\Phi^{\#} \to \frac{V^{\#}}{2}$, $\Phi^{\flat} \to V^{\flat}$ (as value of the game)

The Cooperative-Competitive Solution

co-co value

$$\left(\frac{V^{\#}}{2} + V^{\flat}, \frac{V^{\#}}{2} - V^{\flat}\right)$$

co-co solution

strategies $(a^{\#}, b^{\#}) \in A \times B$, side payment $b \in \mathbb{R}$ from player B to player A such that

$$\Phi^{A}(a^{\#}, b^{\#}) + p = \frac{V^{\#}}{2} + V^{\flat}$$

$$\Phi^{B}(a^{\#}, b^{\#}) - p = \frac{V^{\#}}{2} - V^{\flat}$$

The Cooperative-Competitive Solution

Example: prisoner's dilemma

		Player B	
		C	N
Player A	C	−6, −6	0, −8
	N	−8, 0	−1, −1

co-co solution in red; $b = 0$

		Player B	
		C	N
Player A	C	-6	-4
	N	-4	-1

		Player B	
		C	N
Player A	C	0	4
	N	-4	0

$$\frac{V^{\#}}{2} = -1$$

$$V^{\flat} = 0$$

Thank you for your kind attention!