Transition to Advanced Mathematics

Fall 2021

Practically Perfect Proof

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Question 7.

Theorem 1. *Let n be a natural number. Prove that there exist unique natural numbers a and b such that $n = 2^{a-1}(2b-1)$.*

Proof. Existence of n is proven in two cases, n is odd and n is even.

Case 1. n is odd.

Let $a = 1$, and $b = \frac{n+1}{2}$. Then a is clearly within $\mathbb{N}$, and since n is odd, $n+1$ is even, so $\frac{n+1}{2} \in \mathbb{N}$ as well.

Case 2. n is even.

Using the Fundamental Theorem of Arithmetic, and since $n \geq 2$, we know n can be written as a unique product of primes:

$$n = p_1^{z_1} * p_2^{z_2} * p_3^{z_3} * \ldots * p_k^{z_k} \tag{1}$$

where $p_1 < p_2 < \ldots < p_k$ are distinct prime numbers and $z_1, z_2, \ldots, z_3 \in \mathbb{N}$ or $\{0\}$.

Note that $p_1 = 2$, and $p_2, ..., p_k$ are all odd numbers.

Let $a = z_1 + 1$ and $b = \dfrac{(p_2^{z_2} * ... * p_k^{z_k}) + 1}{2}$.

Since $z_1 \geq 0$, we know $a = z_1 + 1 \in \mathbb{N}$.

Similar to the case when n is odd, $p_2 * ... * p_k$ is odd, since all primes besides 2 are odd, the product of any number of odd numbers is odd. Therefore $p_2 * ... * p_k + 1$ is even, so $b = \frac{p_2 * ... * p_k + 1}{2} \in \mathbb{N}$.

Uniqueness. To prove uniqueness, let $k = 2^{a-1}(2b-1) = 2^{m-1}(2n-1)$ for some $k, m, n \in \mathbb{N}$.

Proceeding in two cases, when k is odd, and k is even.

Case 3. k is odd.

Then $a = 1$ and $b = \frac{n+1}{2}$ from existence case (1). Then a and b are clearly unique.

Case 4. k is even.

Let $m > a$.

It follows:

$$\begin{aligned} 2^{m-1}(2n-1) &= 2^{a-1}(2b-1) \\ \implies \frac{2^{m-1}}{2^{a-1}}(2n-1) &= (2b-1) \\ \implies 2^{m-a}(2n-1) &= (2b-1) \end{aligned}$$

We know $m > a$, so 2^{m-a} is even, as 2 raised to any natural number greater than or equal to 1 is even.

Then $2^{m-a}(2n-1)$ is an even number, and $(2b-1)$ is odd. But $2^{m-a}(2n-1) = (2b-1)$. This is impossible, as no even number is an odd number, so $m \not> a$.

Let $m < a$.

It follows:

$$
\begin{aligned}
2^{m-1}(2n-1) &= 2^{a-1}(2b-1) \\
\implies (2n-1) &= \frac{2^{a-1}}{2^{m-1}}(2b-1) \\
\implies (2n-1) &= 2^{a-m}(2b-1)
\end{aligned}
$$

We know $a > m$, so 2^{a-m} is even, as 2 raised to any natural number greater than or equal to 1 is even.

Then $2^{a-m}(2b-1)$ is an even number, and $(2n-1)$ is odd. But $2^{m-a}(2n-1) = (2b-1)$. This is impossible, as no even number is an odd number, so $a \not> m$.

When $m = a$, $2^{a-m} = 2^0 = 1$, so $(2b-1) = (2n-1)$, which is true when $n = b$.

Hence $m = a, n = b$

Therefore a, b are unique natural numbers such that $n = 2^{a-1}(2b-1)$.

$\square$