Transition to Advanced Mathematics

Fall 2021

Practically Perfect Proof

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Question 8.

Theorem 1. *The sum of the cubes of any three consecutive natural numbers is a multiple of 9.*

Proof. Let n be a natural number.

Let $S(n) = n^3 + (n+1)^3 + (n+2)^3$.

We will prove (1) by inducting on n.

Base Case. Let $n = 1$.

Then $S(1) = 1^3 + 2^3 + 3^3 = 1 + 8 + 27 = 36$.

Then $S(1)$ is a multiple of 9, as $9(4) = 36 = S(1)$.

Induction Step. Assume $9 \mid S(k)$ for some $k \in \mathbb{N}$.

Then $9a = k^3 + (k+1)^3 + (k+2)^3$ for some $a \in \mathbb{N}$.

It follows:

$$
\begin{aligned}
&9a = k^3 + (k+1)^3 + (k+2)^3 \\
&\implies 9a + (k+3)^3 = k^3 + (k+1)^3 + (k+2)^3 + (k+3)^3 \\
&\implies 9a + k^3 + 9k^2 + 27k + 27 = k^3 + (k+1)^3 + (k+2)^3 + (k+3)^3 \\
&\implies 9a + 9k^2 + 27k + 27 = (k+1)^3 + (k+2)^3 + (k+3)^3 \\
&\implies 9(a + k^2 + 3k + 3) = \mathrm{S}(k+1)
\end{aligned}
$$

Since $a + k^2 + 3k + 3 \in \mathbb{N}$, $9 \mid \mathrm{S}(k+1)$.

Hence, $9 \mid \mathrm{S}(k)$ implies $9 \mid \mathrm{S}(k+1)$.

Thus, by the Principle of Mathematical Induction, Theorem (1) holds true for all $n \in \mathbb{N}$. □