Transition to Advanced Mathematics

Fall 2021

Practically Perfect Proof

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Question 6.

Theorem 1. *There do not exist prime numbers a, b, c such that $a^3 + b^3 = c^3$.*

Proof. Proceeding by proof of contradiction, assume there exist prime numbers a, b, c such that $a^3 + b^3 = c^3$.

Consider 3 cases, when $a = b = 2$, when a, b are both odd primes, and when a or b is 2 and the other is an odd prime.

Case 1. Let $a = b = 2$. It follows,

$$\begin{aligned} & a^3 + b^3 = c^3 \\ & \implies 2^3 + 2^3 = c^3 \\ & \implies 16 = c^3 \\ & \implies \sqrt[3]{16} = c \end{aligned}$$

c is a prime number, but $c = \sqrt[3]{16}$, which is not a prime. This is a contradiction.

Case 2. Let a, b both be odd primes.

Then $a = 2m+1, b = 2n+1$ for some $m, n \in \mathbb{Z}$.

It follows,

$$\begin{aligned}
&a^3 + b^3 = c^3 \\
&\implies (2m+1)^3 + (2n+1)^3 = c^3 \\
&\implies 8m^3 + 12m^2 + 6m + 1 + 8n^3 + 12n^2 + 6n + 1 = c^3 \\
&\implies 2(4m^3 + 4n^3 + 6m^2 + 6n^2 + 3m + 3n + 1) = c^3
\end{aligned}$$

Note that $a^3 = 2(4m^2 + 6m^2 + 3m) + 1$, which means that the cube of an odd number is still odd.

Since $4m^3 + 4n^3 + 6m^2 + 6n^2 + 3m + 3n + 1$ is an integer by closure, c^3 is even. Using the contrapositive of our note above, we know that c is even.

However, for c to be even and prime, it must be 2, but since a, b are odd primes, and thus greater than 2, $c \neq 2$. This is a contradiction.

Case 3. Without loss of generality, let a be an odd prime and b be the only even prime number 2.

$$a^3 + b^3 = c^3 \tag{1}$$

$$\implies b^3 = c^3 - a^3 \tag{2}$$

$$\implies 2^3 = (c-a)(c^2 + ca + a^2) \tag{3}$$

$$\implies 8 = (c-a)(c^2 + ca + a^2) \tag{4}$$

Observe 4. 8 can only be factored into the pairs $1 * 8$ and $2 * 4$.

Since a, c are primes and therefore positive, $(c^2 + ca + a^2)$ must at least be 3.

So for 4 to be valid,

$$c^2 + ca + a^2 = 4 \text{ and } c - a = 2$$

or

$$c^2 + ca + a^2 = 8 \text{ and } c - a = 1$$

Proceeding into 2 cases of the two possible factorizations of 8 above.

Subcase (i): Let $c - a = 2$ and $c^2 + ca + a^2 = 4$.

Then $c = 2 + a$.

It follows:

$$c^2 + ca + a^2 = 4 \tag{5}$$

$$\implies (2+a)^2 + (2+a)a + a^2 = 4 \tag{6}$$

$$\implies 4 + 4a + a^2 + 2a + a^2 + a^2 = 4 \tag{7}$$

$$\implies 3a^2 + 6a + 4 = 4 \tag{8}$$

$$\implies 3a^2 + 6a = 0 \tag{9}$$

$$\implies 3a(a+2) = 0 \tag{10}$$

However, a is an odd prime number and therefore positive. So $a \neq 0$, but to satisfy 10, $a = 0$. This is a contradiction.

Subcase (ii): Let $c - a = 1$ and $c^2 + ca + a^2 = 8$.

Then $c = 1 + a$.

It follows:

$$
\begin{aligned}
& c^2 + ca + a^2 = 8 && (11) \\
\implies & (1+a)^2 + (1+a)a + a^2 = 8 && (12) \\
\implies & 1 + 2a + a^2 + a + a^2 + a^2 = 8 && (13) \\
\implies & 3a^2 + 3a + 1 = 8 && (14) \\
\implies & 3a^2 + 3a = 7 && (15) \\
& && (16)
\end{aligned}
$$

However, a is an odd prime number and therefore $a \geq 3$.

Then $3a^2 + 3a \geq 33$.

However, to satisfy 15, $3a^2 + 3a = 7$.

Then $7 \geq 33$. This is a contradiction.

Hence, since all possible cases produce a contradiction when one assumes there do exist prime numbers a, b, c such that $a^3 + b^3 = c^3$, the original theorem must be true: there do not exist prime numbers a, b, c such that $a^3 + b^3 = c^3$. □