Transition to Advanced Mathematics

Fall 2021

Practically Perfect Proof

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Question 4.

Theorem 1. *Suppose n is a natural number and let p be the smallest prime divisor of n. If $p > \sqrt[3]{n}$, then $\frac{n}{p}$ is not composite.*

Proof. Assume $p > \sqrt[3]{n}$.

Proceeding via contrapositive, assume that $\frac{n}{p}$ is composite.

Since $\frac{n}{p}$ is composite, there exists a natural number d, a divisor of $\frac{n}{p}$ that is not $\frac{n}{p}$ or 1.

Then $ad = \frac{n}{p}$ for some $a \in \mathbb{Z}$.

$$ad = \frac{n}{p} \tag{1}$$

$$\implies pad = n \tag{2}$$

$$\implies pa = \frac{n}{d} \tag{3}$$

Since pa is an integer by closure, so $d \mid n$.

Likewise, pd is an integer by closure, so $a \mid n$.

Note that since $d \in \mathbb{N}$, and $1 < d < \frac{n}{p}$, and since $ad = \frac{n}{p}$, we know $a \neq 0$, as that would mean $d = \frac{n}{p}$. Likewise, $a \neq \frac{n}{p}$ as that would mean $d = 1$. As a is a divisor of $\frac{n}{p}$ such that $a \neq 1, a \neq \frac{n}{p}$, we know $1 < a < \frac{n}{p}$.

Then since p is the smallest prime divisor of n, and because $d > 1$ and $a > 1$,

$$d \geq p \ , a \geq p \tag{4}$$

It follows from our original assumption:

$$\begin{aligned} & p > \sqrt[3]{n} && (5)\\ \implies & p^3 > n && (6)\\ \implies & p^2 > \frac{n}{p} && (7)\\ \implies & p^2 > ad && (8) \end{aligned}$$

Using 4 it follows that $ad \geq p^2$, which implies either $ad > p^2$ or $p^2 = ad$.

Case 1. Assume $p^2 = ad$.

Referring to 8, this implies $p^2 > p^2$, which is a contradiction.

Case 2. Assume $ad > p^2$.

Referring to 8, this implies $p^2 > ad$ and $ad > p^2$, which is a contradiction.

Thus, by contradiction, the original claim is true.

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