Transition to Advanced Mathematics

Fall 2021

Practically Perfect Proof

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Question 9.

Proposition 1. *Let* $F_1, F_2, \ldots, F_n, \ldots$ *be the sequence of Fibonacci numbers and let* $L_1, L_2, \ldots, L_n, \ldots$ *be the sequence of Lucas numbers. Then for each natural number* n *with* $n \geq 3$*, we have*

$$L_n = F_{n+2} - F_{n-2} \tag{1}$$

Proof. By strong induction on n.

Base Cases. Let $n = 3$.

Then $4 = 5 - 1 = 4$, which is true.

Let $n = 4$.

Then $L_4 = F_6 - F_2 = 7 = 8 - 1 = 7$, which is true.

Induction Step. Using strong induction, assume $L_k = F_{k+2} - F_{k-2}$ for every $3 \leq k \leq i$.

By the recursive definition of Lucas numbers, $L_{k+1} = L_k + L_{k-1}$.

Then by the induction hypothesis, $L_k = F_{k+2} - F_{k-2}$, and $L_{k-1} = F_{k+1} - F_{k-3}$.

It follows:

$$
\begin{aligned}
& L_{k+1} = (F_{k+2} - F_{k-2}) + (F_{k+1} - F_{k-3}) \\
& \implies L_{k+1} = (F_{k+2} + F_{k+1}) - (F_{k-2} + F_{k-3}) \\
& \implies L_{k+1} = F_{k+3} - F_{k-1}
\end{aligned}
$$

Then $L_i = F_{i+2} - F_{i-2}$ implies $L_{i+1} = F_{(i+2)+1} - F_{(i-2)+1}$.

Hence, by the Principle of Strong Mathematical Induction, proposition (1) holds true for all $n \geq 3$. □