Transition to Advanced Mathematics

Fall 2021

Practically Perfect Proof

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Question 2.

Theorem 1. *Let $a, b, c \in \mathbb{Z}$. If $a^2 + b^2 = c^2$, then abc is even.*

Proof. Assume $a^2 + b^2 = c^2$.

Recall **Proposition 2.3.50:**

$$\text{If } n \text{ is odd, } n^2 \text{ is odd.} \tag{1}$$

$$\implies \text{If } n^2 \text{ is even, } n \text{ is even.} \tag{2}$$

With this knowledge, we proceed into following cases:

Case 1. Without loss of generality, assume a is even and b is odd.

Then $a = 2m$ for some $m \in \mathbb{Z}$.

Then $abc = 2mbc = 2(mbc)$.

Since $mbc \in \mathbb{Z}$ by closure, abc is even.

Therefore when a and b have opposite parity, abc is even.

Case 2. a and b are both even.

Then $a = 2m$ for some $m \in \mathbb{Z}$.

Then $abc = 2mbc = 2(mbc)$.

Since $mbc \in \mathbb{Z}$ by closure, abc is even.

Therefore, when a and b are both even, abc is even.

Case 3. a and b are both odd.

Then by **Proposition 2.3.50**, a^2 and b^2 are odd.

Then $a^2 = 2d + 1$ and $b^2 = 2e + 1$ for some $d, e \in \mathbb{Z}$.

Then $c^2 = (2d + 1) + (2e + 1) = 2(d + e + 1)$.

Since $d + e + 1$ is an integer by closure, we know c^2 is even.

Using the contrapositive of **Proposition 2.3.50**, we know that c is even.

Then $c = 2m$ for some $m \in \mathbb{Z}$.

Then $abc = ab(2m) = 2(abm)$.

Since $abm \in \mathbb{Z}$ by closure, abc is even.

Therefore, when a and b are both odd, abc is even.

$\square$