Touring the Knight's Tour

Patrick May

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Abstract

This paper introduces the Knight's Tour puzzle, as well as a brief introduction to some of the graph theory necessary to mathematically model a knight's tour. We formally prove Schwenk's theorem on the existence of a closed knight's tour on $m \times n$ chessboards, and survey other potential generalizations of the knight's tour puzzle.

1 Introduction

Given a knight on an empty chessboard, can it be moved to land on every square only once? Will the knight return to its original starting point? This is a common chess puzzle given to enthusiasts, and can be solved with a small amount of ingenuity. Finding a "Knight's Tour", a path in which a knight moves on every square of a chessboard and returns to where is begins is a limited problem in scope. We present Fig. 1, a potential knight's tour on an 8 by 8 chessboard. So we have demonstrated the existence of one tour on an 8×8 board. However, there are still many unanswered questions about the Knight's Tour.

There are many different areas of the Knight's tour to explore. How many possible tours are there? Clearly there exists some solution(s), but is there an exact number? More interestingly, what about placing the knight on a chessboard of another size? Can it still complete a tour on a long, thin rectangular board? Determining existence, and given a knight's tour exists, demonstrating one on a chessboard of any size is Schwenk's Theorem.

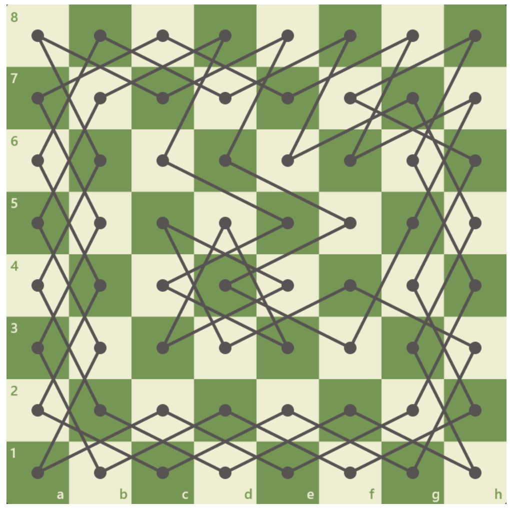


Figure 1: A possible knight's tour

2 Schwenk's Theorem

2.1 Background

The Knight's Tour can be modeled well as a problem in graph theory. Before we can understand Schwenk's theorem, we first must introduce some of the terminology and conventions we will use within the proof.

Definition 1. A **graph** is a diagram consisting of points (called vertices or nodes) and lines connecting between points (called edges). [1].

A knight's tour can be modeled as a graph, with each vertex being a square the knight has landed on, and each edge connecting to another square as a legal knight's move. Additionally, every white square is a white node and every black square is a black node.

Definition 2. A vertex is **incident** to any edge connected to it, and vice versa. [1]

Definition 3. A vertex's **degree** is the number edges incident to that vertex. [1]

As seen in Figure 1, we can see that every node in a closed knight's tour graph must have degree 2, as if any vertex had a higher degree, it would be landed on more than once, and any vertex of degree 1 would force the knight to stop or backtrack, contradicting the goal to complete a cycle and end where we started.

Definition 4. A **Hamiltonian cycle** is a spanning cycle of a graph, meaning that it is a path along the edges of a graph that visits every vertex only once and returns to where it begins [1]. Similarly, a **Hamiltonian path** is a spanning path of a graph, meaning it visits every vertex only once but does not end where it starts [1].

Not every graph is Hamiltonian, but we see a proper knight's tour is a clear example of Hamiltonian cycle. These specific tours are called *closed* knight's tours, and are the focus of Schwenk's Theorem. *Open* knight's tours are similar with the exception that they form a Hamiltonian path, and do not return to their starting point.

For the knight's tour, we define a graph $G(m,n)$ on mn vertices by taking a chessboard of m rows and n columns, or vice versa, and converting each square to a black or white node. Then we can construct an edge between every two nodes that would be a legal move of the knight. An example of this can be seen in Figure 2.

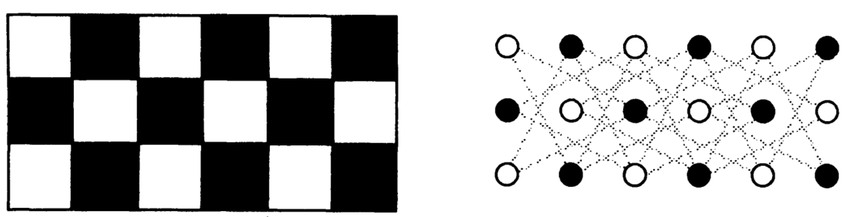


Figure 2: Converting a 3×6 board to $G(3,6)$ [5].

Any node can be referenced in (i,j) form, where i is the row and j is the column of the node counting from the top left. Any edge can be referenced in $(i,j)-(a,b)$ form, where (i,j) and (a,b) are vertices that share the desired edge.

Now we are able to rigorously define the question Schwenk's theorem answers:

On which $m \times n$ boards can a knight make successive legal moves, visit every cell once, and return to its starting cell? [5]

Schwenk answers this question in proof by first showing the cases when a knight's tour does not exist. Then he proves a lemma that allows for a pre-existing knight's tour to be expanded, and concludes by inducting on provided knight's tours to cover all proper $m \times n$.

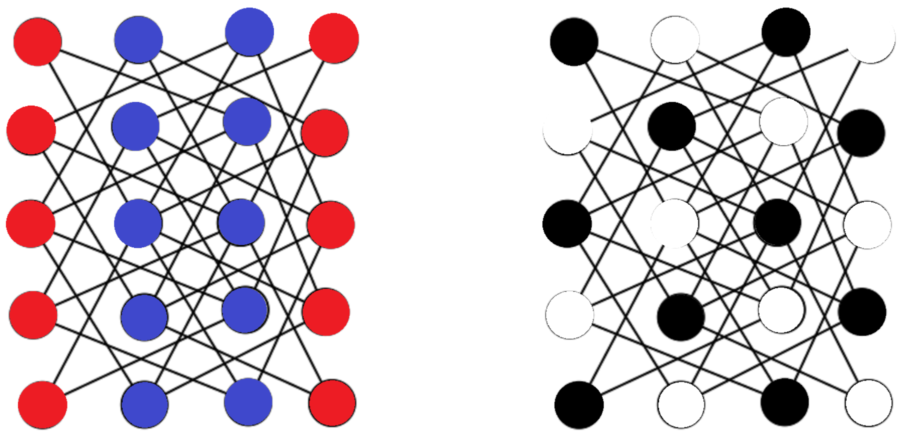
2.2 Proof of Schwenk's Theorem

Theorem 1 (from [5])**.** *An $m \times n$ chessboard with $m \leq n$ has a knight's tour unless one of these three conditions holds:*

- *m and n are both odd*
- *$m = 1$, 2, or 4; or*
- *$m = 3$ and $n = 4$, 6, or 8.*

Proof. First, if m and n are both odd, the number of vertices mn, which is our knight's tour, is odd. However, every other vertex in a knight's tour graph alternates between black and white squares on a chessboard. If the number of vertices is odd, then the tour will be odd and the knight will end on a different colored square, meaning a tour does not exist [5].

If $m = 1$ or 2, the board is not wide enough for a tour to exist. A board 1 square wide does not allow a knight to make a single legal move. A board 2 squares wide is likewise impossible as the board is clearly not large enough to allow the knight to orient itself to touch every square [5]. The problem when $m = 4$ was found by Louis Posa as a teenager [5]. Contrary to what we are trying to prove, first assume we find a Hamiltonian cycle $v_1, v_2, \ldots, v_{4n}, v_1$. Then re-color vertices red and blue, with every vertex in rows 1 and 4 red and every vertex rows 2 and 3 blue. An example of this color scheme (rotated to vertical), along with the regular chessboard scheme can be seen in Figure 3.



(a) Red/Blue (inner/outer) (b) White/Black (chessboard pattern)

Figure 3: Different Colorings of a 5×4 board.

Since red nodes are only connected to blue nodes and there are the same number of red and blue nodes, a proper Hamiltonian cycle must alternate between red and blue nodes. However, the original chessboard pattern of white and black nodes must hold true as well, with each white node connecting to a black node and vice versa. Then the white and black node coloring must be the same as the red and blue, but it clearly is not. Hence, there is a contradiction and no Hamiltonian cycle exists on a $4 \times n$ board.

The third condition is the least trivial. The 3×4 board has already been excluded as we could instead say it is a board where $m = 4$, something for which the existence of a Hamiltonian path has already been disproven by the paragraph above.

Given some Knight's Tour as a graph G, we can remove a vertex v, and therefore remove all edges incident to v. For any G having a Hamiltonian cycle, it is clear that removing any set of k vertices can leave at most k connected components. In G(3, 6), if we remove (1, 3) and (3, 3) leaves three components, we must conclude that no Hamiltonian cycle exists for the 3×6 board [5].

When $m = 3$ and $n = 8$, our graph G(3,8) has vertices (1, 1), (2, 1), (3, 1), (2, 2), (1, 8), (2, 8), (3, 8), and (2, 7) all have degree two, meaning that any legal knight's tour must include those sixteen edges. These subgraphs of vertices of degree two along with the unconnected (2,4) and (2,5) can be seen in 4. Now, we can construct a new graph $G'(3,8)$, where each of the six required paths and the two unconnected vertices form new nodes, and edges being legal moves a knight could make to connect two paths. A Hamiltonian cycle present in $G(3,8)$ means there must exist a Hamiltonian cycle in $G'(3,8)$, however we can clearly see that there are two nodes in $G'(3,8)$ of degree 3, meaning a Hamiltonian cycle does not exist in $G'(3,8)$ and thus $G(3,8)$ [5].

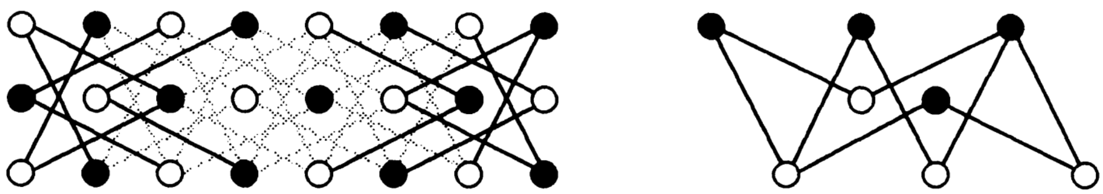


Figure 4: Sixteen required edges for a closed tour on G(3, 8) and the derived graph G'(3, 8) [5]

Now we have shown when a closed knight's tour does not exist, but how do we show the existence of a tour for all other board sizes? We propose that we can use strong induction upon an existing tour to add 4 rows or columns provided the requirements of the following lemma are met, and thus show a tour exists for any board size not excluded by the requirements we provided above.

Lemma 2 (from [5]). *If* $G(m,n)$ *has a Hamiltonian cycle that includes 10 edges seen in below in Figure 5:*

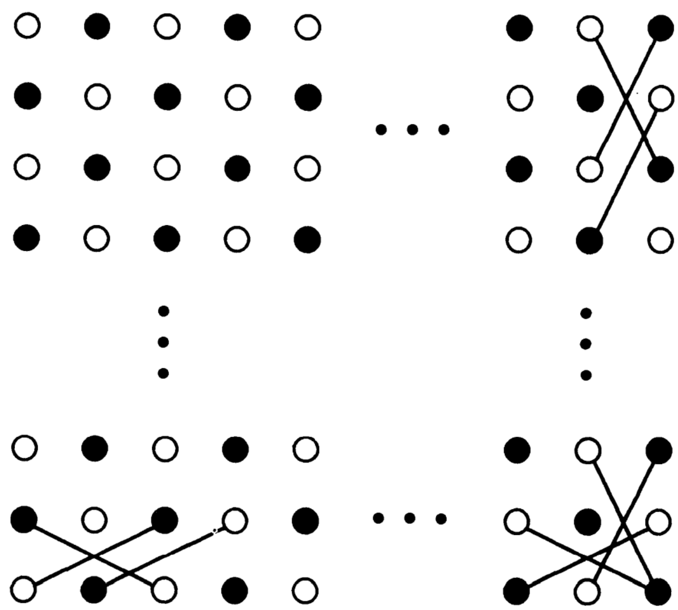


Figure 5: The 10 edges upon which additional columns and rows can be constructed [5]

then a graph with 4 more columns, $\mathrm{G}(m, n+4)$*, also has a Hamiltonian cycle including the corresponding 10 edges seen in Figure 5.*

Proof. If $m = 3$, then the 10 necessary edges becomes 7 [5]. To add four columns to any Hamiltonian cycle in $\mathrm{G}(3, n)$ that contains the critical seven edges, we can place a 3×4 graph next to the existing tour $\mathrm{G}(3, n)$, delete the edge $(1, n-1) - (3, n)$ from the existing cycle, and insert edges $(1, n-1) - (2, n+1)$ and $(3, n) - (1, n+1)$ to "link" the path into the cycle [5], demonstrating a Hamiltonian cycle on $\mathrm{G}(3, n+4)$. This process can be seen in Fig. 6. Note that the new Hamiltonian path contains the same edges, shifted over 4 columns, that were used to expand $\mathrm{G}(3, n)$ initially, which means repeated construction of tours 4 columns larger than a prior existing tour is possible.

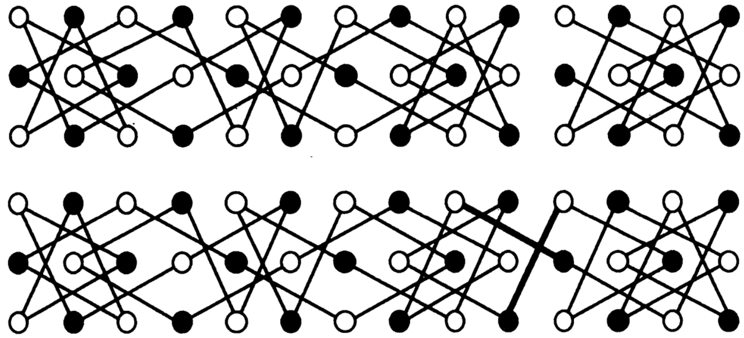
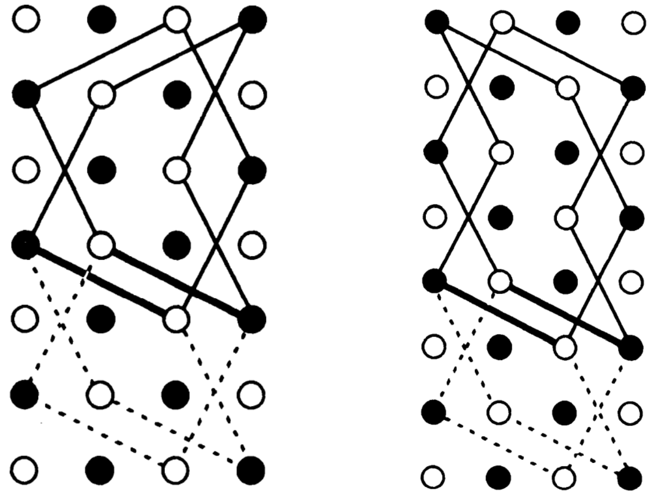


Figure 6: Extension of a $G(3,n)$ to $G(3,n+4)$ demonstrated on $n = 10$. [5]

For any $m \geq 5$, the tour on $G(m,n)$ will contain all 10 edges highlighted in 5. To begin, we start from $G(m,4)$ and derive a $m \times 4$ array $H(m,4)$ by deleting all edges between columns two and three and all edges joining vertices two columns apart except those joining vertices in rows 1 and 2 and those joining vertices in rows $m-1$ and m. Then $H(m,4)$ has only vertices of degree 2. We observe that $H(m,4)$ has a pair of $2m$ length cycles when m is odd and four m length cycles when m is even. Figure 7 presents half-examples of $H(m,4)$ for both odd and even cases. The other cycles on $H(m,4)$ are obtained through reflecting the path(s) across the vertical axis of the chessboard [5]. Additionally, note that the extension in dotted lines contains the same two edges that we remove to extend it, seen in Figure 7, and thus can be repeated to extend $H(m,4)$ to arbitrary lengths, with even and odd size m having different patterns within $H(m,4)$.

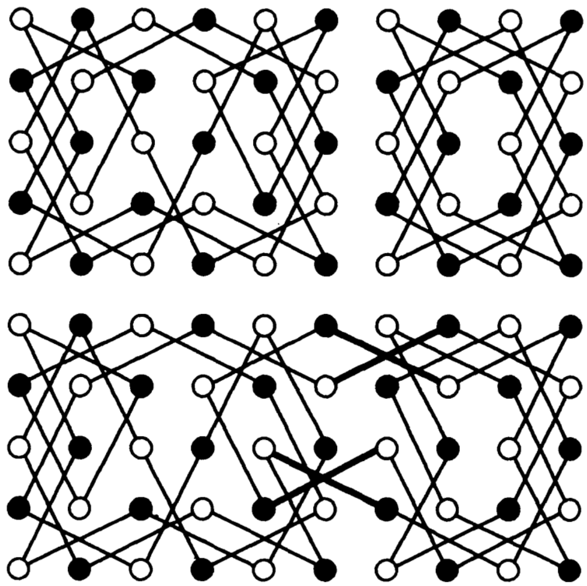


(a) Odd $\mathrm{H}(m,4)$ for $m = 5$ to $m = 7$

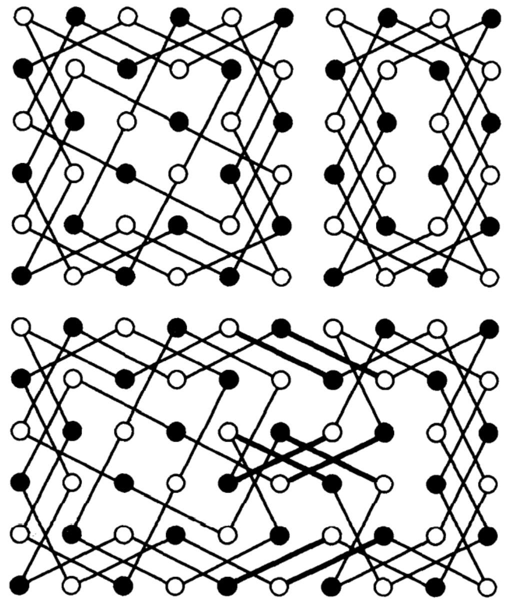
(b) Even $\mathrm{H}(m,4)$ for $m = 6$ to $m = 8$

Figure 7: Construction process of $\mathrm{H}(m,4)$ for $m \geq 5$ [5]

With $\mathrm{H}(m,4)$ created, we can now "link" it to our current graph. When m is odd and thus $\mathrm{H}(m,4)$ has two separate cycles, we start by removing the two edges $(1,n)-(3,n-1)$ and $(2,n)-(4,n-1)$. Then we place $\mathrm{H}(m,4)$ next to $\mathrm{G}(m,n)$ and insert the 4 edges seen in bold in Figure 8a. When m is even, and thus $\mathrm{H}(m,4)$ has four separate cycles, we first remove the four edges $(1,n-1)-(3,n),(1,n)-(3,n-1),(m-2,n-1)-(m,n),(m-3,n+2)-(m,n-1)$ from $\mathrm{G}(m,n)$. Then we place $\mathrm{H}(m,4)$ next to $\mathrm{G}(m,n)$ and insert the 8 edges seen in bold in Figure 8b.



(a) Extension of G(5, 6) to G(5, 6 + 4)



(b) Extension of G(6, 6) to G(6, 6 + 4)

Figure 8: Example "Linking" of $G(m, n)$ with $H(m, 4)$ for both odd and even m[5]

We have now successfully expanded $G(m, n)$ to $G(m, n+4)$ [5]. An important note is that $H(m, 4)$ can be constructed sideways as $H(4, n)$ as 4 additional rows instead of 4 additional columns, and can still be linked to $G(m, n)$. This is possible as the edges needed to add 4 rows to the bottom are the same as the edges needed to add 4 columns to the right, just flipped over the diagonal. Then we can also conclude that $G(m, n)$ can be expanded to $G(m + 4, n)$.

Additionally, note that this expanded graph still contains the 10 specific edges needed to perform another expansion of the tour, seen in Figure 5.

Hence we have demonstrated that we can successfully expand any Hamiltonian cycle on $G(m, n)$ containing 10 specific edges in a way such that $G(m, n + 4)$ also contains the same edges, expanded outwards appropriately. □

With the lemma (2), we can induct upon any pre-existing knight's tour to get a tour that is 4 rows or 4 columns larger. Since a tour expands in steps of 4 rows/columns, we need multiple base cases to cover all different $m \times n$ board sizes. This means we are using strong induction. Initially, it may seem like we need 16 different base cases, as that covers all possible combinations of $m \times n$ modulo 4. Since we can rotate our chessboard, we only need 10 base cases, as the others can be obtained by taking an $m \times n$ board and turning it sideways to produce a $n \times m$ knight's tour. Excluding $1 \times 1, 1 \times 3$, and 3×3 as they satisfy conditions we have shown prevent a tour from existing, we have to show a Hamiltonian cycle on chessboards of size $3\times6, 3\times8, 5\times6, 5\times8, 6\times6, 6\times8$, and 8×8. Since 3×6 and 3×8 boards cannot contain a knight's tour, we replace the 3×6 with 3×10 and 7×6 boards and the 3×8 board with 3×12 and 7×8 boards [5]. Then we only need to demonstrate 9 tours upon which any size knight's tour can be constructed. These 9 base case Hamiltonian cycles can be seen in Figure 9 [5]. Hence, using strong induction, we can demonstrate a knight's tour for any $m \times n$ board provided it satisfies all prior requirements from the beginning of this proof.

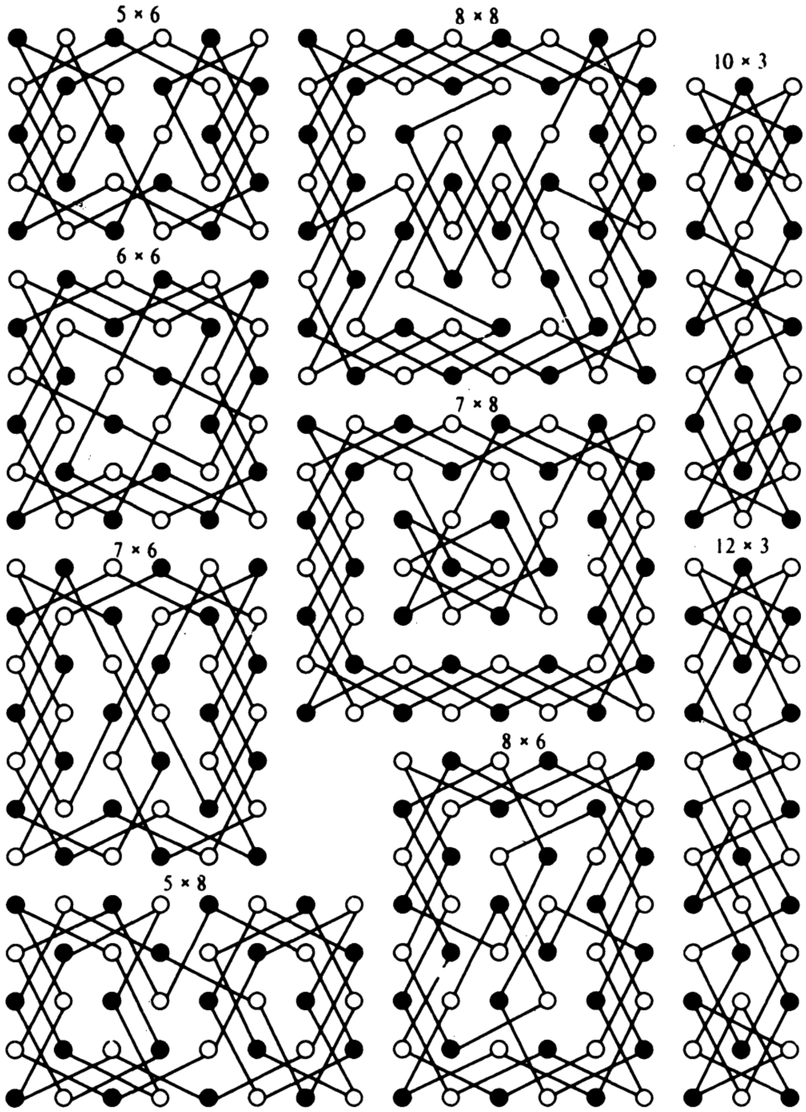


Figure 9: Base Case knight's tours needed to construct any other size knight's tour. [5]

□

3 Conclusion

With this, Schwenk has shown when a closed Knight's tour exists on a $m \times n$ board and can construct an example tour of any size given it exists. However, this conclusively solves only one possible mathematical question that arises from the original knight's tour puzzle. There are many other boundaries of the puzzle to be explored. While their proofs are outside the scope of this paper, some of these alternate explorations into the knight's tour are worth mentioning, as their areas of exploration are intriguing. McKay et. al. calculated there to be 13,267,364,410,532 undirected knight's tours on the standard 8 $\times$ 8 chessboard [4]. More recently, in 2012, Erde et. al. generalized the knight's tour to n dimensions [3]. Chia and Ong finalize a proof of open knight's tours, as Schwenk only proves closed knight tours, as well as generalize the existence of a tour based on a knights of any traversal distance $(a, 1)$ compared to the standard (2, 1) move scheme [2]. This seemingly small chess puzzle shows that almost anything can be generalized further than what is trivially apparent.

References

[1] J. A. Bondy and U. S. R. Murty. *Graph theory*. Springer, 2008.

This book on graph theory provides the necessary graph theory definitions useful in translating the chess board to a rigorous graph. Additionally, it provides definitions of Hamiltonian paths and cycles that will be useful in understanding a knight's tour from a mathematical perspective.

[2] G.L. Chia and Siew-Hui Ong. Generalized knight's tours on rectangular chessboards. *Discrete Applied Mathematics*, 150(1):80–98, 2005.

Chia and Ong present an interesting proof about a knight generalized to any movement still in the classic knight pattern of moving in two different dimensions as a legal move. Their work is outside the scope of this paper, but especially intersting given they include and add small final parts to a proof about the existence of open knight's tours.

[3] Joshua Erde, Bruno Golénia, and Sylvain Golénia. The closed knight tour problem in higher dimensions. *The Electronic Journal of Combinatorics*, 19(4), 2012.

This article is an interesting extension of the knight's tour into higher dimenions. While the explanation to Erde, Golénia, and Golénia's proof is outside the scope of this paper, its existence is someting interesting and an interesting demonstration in further generalizing a "solved" problem.

[4] Brendan D McKay et al. Knight's tours of an 8 x 8 chessboard. *Joint Computer Science Report Series*, 1997.

McKay researches how many different knight's tours there are on a standard 8x8 chessboard. He built off another paper's findings, using a backtracking algorithm and breaking a knight's tour into upper and lower halves, and concluded by having a computer calculate the end result of 13,267,364,410,532 undirected tours. McKay's findings are useful in demonstrating one possible approach to the knight's tour problem.

[5] Allen J. Schwenk. Which rectangular chessboards have a knights tour? *Mathematics Magazine*, 64(5):325–332, Dec 1991.

This is the first complete, correct proof that formally defines existence/nonexistence of knight's tours for n by m boards. Schwenk's proof begins by showing what cases a knight's tour does not exist. Then Schwenk demonstrates how smaller knights tours can be combined to form one full cycle. He finishes by providing a set of different sized boards with knights tours that allows for any other sized tour to be constructed via induction. Proving this will be a main focus of the paper.