Transition to Advanced Mathematics

Fall 2021

Practically Perfect Proof

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Question 3.

Theorem 1. *Suppose a and b are natural numbers such that $a^2 = b^3$. If $4 \mid b$, then $8 \mid a$.*

Proof. Assume $4 \mid b$.

Then $4m = b$ for some $m \in \mathbb{Z}$.

It follows:

$$4m = b \tag{1}$$

$$\implies (4m)^3 = b^3 \tag{2}$$

$$\implies 64(m^3) = a^2 \tag{3}$$

$$\implies 2(32m^3) = a^2 \tag{4}$$

In (3), since $m^3 \in \mathbb{Z}$ by closure, we have

$$64 \mid a^2 \tag{5}$$

Additionally, note that since $32m^3 \in \mathbb{Z}$, $2 \mid a^2$. From prior proofs, we know that if a^2 is even, a is even.

Then $a = 2d$ for some $d \in \mathbb{Z}$.

Rewriting (5), we have $64 \mid (2d)^2$.

Then $64e = 4d^2$ for some $e \in \mathbb{Z}$.

So $16e = 2(8e) = d^2$. Again, note that $8e \in \mathbb{Z}$, so d^2 is even, thus d is even.

Then $d = 2g$ for some $g \in \mathbb{Z}$.

Then $16e = 4g^2$, so $4e = 2(2e) = g^2$.

Again, since $2e \in \mathbb{Z}$, we know g^2 is even. Thus g is even.

Then $g = 2h$ for some $h \in \mathbb{Z}$.

Then $4e = 4g^2$, so $e = g^2$.

Recall (5), $64e = 4d^2 = a^2$. Then substituting $e = g^2$,

$$\begin{aligned} & 64g^2 = a^2 \\ \implies & \sqrt{64g^2} = \sqrt{a^2} \\ \implies & 8g = a \end{aligned}$$

Recall g is an integer, hence $8 \mid a$.

$\square$