Transition to Advanced Mathematics

Fall 2021

Practically Perfect Proof

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Question 12.

Proposition 1. *Let $A = a_1, a_2, \dots, a_n$ be an n-element set and let R be an equivalence relation defined on A.*

$$\sum_{i=1}^{n} \|[a_i]\|$$

is even if and only if n is even.

Proof. Suppose there are k equivalence classes. Let $c_1, c_2, c_3, \dots, c_k$ denote the cardinalities of each equivalence class.

Recall that any equivalence class can be indexed by any element of said equivalence class.

It follows that every equivalence class occurs a number of times equal to its cardinality in the sum $\sum_{i=1}^{n} \|[a_i]\|$.

Then

$$\sum_{i=1}^{n} \|[a_i]\| = c_1^2 + c_2^2 + \dots + c_k^2 \tag{1}$$

.

Additionally recall that the sum of all cardinalities of equivalence classes is equal to the cardinality of A, or n.

$$\sum_{i=1}^{k} c_i = |A| = n \tag{2}$$

Finally, note that since squaring a number preserves parity, there are the same number of even numbers and odd numbers in (1) and (2).

Assume n is even.

Let c_{even} be the sum of all even cardinalities and c_{odd} all odd cardinalities.

We know that $c_{even} + c_{odd} = n$, so c_{even} and c_{odd} must both have the same parity.

c_{even} is the sum of all even cardinalities and is even. Then c_{odd} must be even as well.

Then there is an even number of odd equivalence classes, as c_{odd} is composed of all odd sized equivalence classes, so the only way to get an even number from q odd numbers is if q is even.

Then there is an even number of odd equivalence classes and an even number of even equivalence classes.

Then 1 is a sum of even squares and an even number of odd squares, which is overall even.

Then equation (1) is even.

Conversely, assume $\sum_{i=1}^{n} |[a_i]|$ is even.

An equivalence class of even cardinality means there is an even number of elements within that partition of A.

If an odd cardinality class exists without a disjoint pair, $\sum_{i=1}^{n} |[a_i]| = c_1^2 + c_2^2 + ... + c_k^2$ will be odd, as it will be a sum of even squares plus an odd squared, which is odd, therefore there exist another disjoint equivalence class of odd cardinality so that the sum of equivalence classes is even.

Then an equivalence class of odd cardinality has an odd number of elements in A, but due to the existence of another disjoint equivalence class of odd cardinality, the additional odd number of elements of A mean A has an even number of elements.

Then n is even.

$\square$