Transition to Advanced Mathematics

Fall 2021

Practically Perfect Proof

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Question 11.

Theorem 1.

$$\bigcup_{n\in\mathbb{N}}[3+\frac{1}{n},5-\frac{1}{n}]=(3,5) \tag{1}$$

Proof. First, let $x \in \bigcup_{n\in\mathbb{N}}[3+\frac{1}{n},5-\frac{1}{n}]$.

Then $3+\frac{1}{n} \leq x \leq 5-\frac{1}{n}$ for some $n \in \mathbb{N}$.

Observe that $3+\frac{1}{n} > 3$ for any $n \in \mathbb{N}$, so then since $x \geq 3+\frac{1}{n}$, we know $x > 3$.

Similarly, observe that $5-\frac{1}{n} < 5$ for any $n \in \mathbb{N}$, so then since $x \leq 5-\frac{1}{n}$, we know $x < 5$.

Then $3 < x < 5$.

Hence $x \in (3,5)$.

Conversely, let $x \in (3,5)$.

Then $3 < x < 5$.

Let $a = x-3$. $a \in \mathbb{R}$ by closure, and $a > 0$ as $x > 3$.

Note that since $a \neq 0$, there exists $b \in \mathbb{R}$ that satisfies $b = \frac{1}{a}$.

By the Archimedian Property of Natural numbers, we know there exists some $g \in \mathbb{N}$ such that $g > b$.

Additionally, if b is a natural number, $g = b$.

It follows:

$$\begin{aligned}
&g \geq b \\
&\implies \frac{1}{g} \leq \frac{1}{b} \\
&\implies \frac{1}{g} \leq \frac{1}{\frac{1}{a}} \\
&\implies \frac{1}{g} \leq a
\end{aligned}$$

Then there is some fraction of the form $\frac{1}{g}$, where $g \in \mathbb{N}$ such that $\frac{1}{g} \leq a$, where $a \in \mathbb{R}$.

We know $3 < x$ and $\frac{1}{g} < a$, then $3 + \frac{1}{g} < 3 + a$, hence $3 + \frac{1}{g} \leq x$.

Similarly, let $c = 5 - x$. We know $c \in \mathbb{R}$ by closure, and $c > 2$ since $x > 3$.

Then, since $c \neq 0$, we know there exists some $d \in \mathbb{N}$ such that $c = \frac{1}{d}$.

By the Archimedian Property of Natural numbers, we know there exists some natural number h such that $h > d$.

If d is also a natural number, let $h = d$.

Then $h \geq d$.

It follows:

$$\begin{aligned}
&h \geq d \\
&\implies \frac{1}{h} \leq \frac{1}{d} \\
&\implies \frac{1}{h} \leq \frac{1}{\frac{1}{c}} \\
&\implies \frac{1}{h} \leq c
\end{aligned}$$

Then,

$$
\begin{aligned}
&c \geq \frac{1}{h} \\
&\implies 5 - c \leq 5 - \frac{1}{h} \\
&\implies x \leq 5 - \frac{1}{h}
\end{aligned}
$$

If $h \geq g$, let $k = h$. Otherwise, let $k = g$.

Then $3 + \frac{1}{k} \leq x \leq 5 - \frac{1}{k}$ for some $k \in \mathbb{N}$.

Hence $x \in \bigcup_{k \in \mathbb{N}} [3 + \frac{1}{k}, 5 - \frac{1}{k}]$.

$\square$