1 Composition of L-coalgebras

For this section, we will continue to let $\mathbb{D} = \mathbb{S}q(\mathcal{D})$ be the double category of squares in a 2-category \mathcal{D} with arrow objects.

In an algebraic weak factorization system, the categories \mathbb{L} -Coalg and \mathbb{R} -Alg respectively play the roles of the left and right classes of morphisms of the weak factorization system. In an ordinary weak factorization system, these two classes of morphisms are closed under composition. In [Gar09], this is strengthened to a composition functor

$$\mathbb{L}$$
-Coalg $\Pi_{\mathbb{C}} \mathbb{L}$ -Coalg $\to \mathbb{L}$ -Coalg

and in [Rie11], it is shown that colax morphisms of awfs preserve this composition. Similarly, there is a composition functor on \mathbb{R} -Alg which is preserved by lax morphisms of awfs.

In this section, we will generalize these results to the setting of bimonads in $\mathbb{F}F(\mathbb{D})$.

First, recall from [Str72] the following proposition.

Proposition 1.1. Let C be a category, and $\mathbb{L} = (L, \epsilon, \delta)$ be a comonad on C. The category of coalgebras \mathbb{L} -Coalg has a universal property as follows:

- There is a forgetful functor $U: \mathbb{L}\text{-Coalg} \to C$ and a natural transformtion $\alpha: U \Rightarrow LU$, satisfying $\epsilon U \circ \alpha = \mathrm{id}_U$ and $\delta U \circ \alpha = L\alpha \circ \alpha$.
- (U, α) is universal among such pairs satisfying such equations. Given another such pair (F, β) , where $F: X \to C$, there exists a unique functor $\hat{F}: X \to \mathbb{L}$ -Coalg such that $U\hat{F} = F$ and $\alpha \hat{F} = \beta$.

Any colax morphism of comonads (F,ϕ) : $(C,L_1,\epsilon_1,\delta_1) \to (D,L_2,\epsilon_2,\delta_2)$ induces a functor \tilde{F} : \mathbb{L}_1 -Coalg $\to \mathbb{L}_2$ -Coalg such that $U\tilde{F} = FU$.

For the rest of this section, assume that \mathcal{D} has EM-objects for comonads, i.e. for every comonad \mathbb{L} in \mathcal{D} there is an object \mathbb{L} -Coalg satisfying the universal property above.

The main goal of this section will be to prove the following theorem:

Theorem 1.2. *There is a lax double functor*

Coalg: Comon(
$$\mathbb{F}F(\mathbb{S}q(\mathcal{D}))$$
) \rightarrow Span(\mathcal{D}_0)

where \mathcal{D}_0 is the ordinary category underlying the (strict) 2-category \mathcal{D} , which is the identity on the vertical categories, and which takes a comonad $(E, \eta, \epsilon, \delta)$ in $\mathbb{F}F(Sq(\mathcal{D}))$ to the span

$$C \xleftarrow{\operatorname{dom} U} \mathbb{L}$$
-Coalg $\xrightarrow{\operatorname{cod} U} C$.

Before we get to the proof of Theorem 1.2, we will need to establish several technical lemmas.

Consider a comonad in $\mathbb{F}F(\mathbb{D})$ on an object C, i.e. a functorial factorization with half of the awfs structure. We can combine the universal properties of EM-objects and arrow objects into a universal property for \mathbb{L} -Coalg, where now \mathbb{L} is the comonad in \mathcal{D} arising from the comonad in $\mathbb{F}F(\mathbb{D})$.

Lemma 1.3. Let $(E, \eta, \epsilon, \delta)$ be a comonad in $\mathbb{F}F(\mathbb{D})$ on an object C. There is a 2-cell

$$\begin{array}{cccc}
U & C^2 & \text{cod} \\
\mathbb{L}\text{-Coalg} & & \downarrow \alpha & C \\
& & \downarrow U & C^2 & E
\end{array}$$

satisfying equations

$$\mathbb{L}\text{-Coalg} \xrightarrow{U} C^{2} \xrightarrow{\downarrow \kappa} C = \mathbb{L}\text{-Coalg} \xrightarrow{U} C^{2} \xrightarrow{\downarrow \eta} C \qquad (1)$$

$$\mathbb{L}\text{-Coalg} \xrightarrow{U} \overset{C^2}{\underset{\text{cod}}{\longleftarrow}} \overset{\text{cod}}{\underset{\text{cod}}{\longleftarrow}} C = X \xrightarrow{U} C^2 \xrightarrow{\text{cod}} C$$
 (2)

where $\vec{\alpha}$ is the unique 2-cell such that $\dim \vec{\alpha} = \mathrm{id}_{\dim U}$ and $\cot \vec{\alpha} = \alpha$, the existence of which is implied by Equation 1.

Given any object X, together with a morphism $F: X \to C^2$ and a 2-cell $\beta: \operatorname{cod} F \Rightarrow EF$ satisfying equations

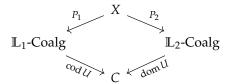
- 1. $\beta \circ \kappa F = \eta F$
- 2. $\epsilon F \circ \beta = \mathrm{id}_{\mathrm{cod}\,F}$
- 3. $\delta F \circ \beta = E \vec{\beta} \circ \beta$

where $\beta: F \Rightarrow LF$ is the unique 2-cell such that $\operatorname{dom} \beta = \operatorname{id}_{\operatorname{dom} F}$ and $\operatorname{cod} \beta = \beta$; there is a unique morphism $\hat{F}: X \to \mathbb{L}$ -Coalg such that $U\hat{F} = F$ and $\alpha \hat{F} = \beta$.

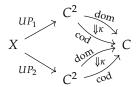
Proof. U is simply the U from proposition 1.1, while the 2-cell α there is the 2-cell $\vec{\alpha}$ here. The equation $\vec{\epsilon}U \circ \vec{\alpha} = \mathrm{id}_F$ implies that $\mathrm{dom}\,\vec{\alpha} = \mathrm{id}_{\mathrm{dom}\,U}$. With that observation, the rest of the equations follow immediately from the

universal property of C^2 and the equations $\epsilon U \circ \alpha = \mathrm{id}_U$ and $\delta U \circ \alpha = L\alpha \circ \alpha$ from Proposition 1.1.

We will now prove a couple of simple lemmas to establish the existence of certain 2-cells in \mathcal{D} using the arrow object universal property. For each of these lemmas, let $(E_1, \eta_1, \epsilon_1, \delta_1)$ and $(E_2, \eta_2, \epsilon_2, \delta_2)$ be two comonads in $\mathbb{F}F(Sq(\mathcal{D}))$, both on the same object C; let X be the pullback



let m be the 2-cell



and let \vec{m} : $X \to C^2$ be the corresponding 1-cell with $\kappa \vec{m} = m$.

Lemma 1.4. *There is a 2-cell*

$$X \xrightarrow{P_1} \mathbb{L}_1\text{-Coalg} \xrightarrow{U} C^2$$

such that dom ζ = id *and*

$$X \xrightarrow{P_1} \mathbb{L}_1\text{-Coalg} \xrightarrow{U} C^2 \xrightarrow{\text{cod}} C = X \xrightarrow{\text{dom}} C$$

$$UP_1 \xrightarrow{C^2} C \xrightarrow{\text{cod}} C$$

$$UP_2 \xrightarrow{\text{dom}} C$$

Proof. Equation (??) becomes

$$X \xrightarrow{\vec{m}} C^2 \xrightarrow{\text{dom}} C = X \xrightarrow{\text{cod}} C^2 \xrightarrow{\text{dom}} C$$

$$UP_1 \xrightarrow{\text{cod}} C \xrightarrow{\text{dom}} C$$

$$UP_2 \xrightarrow{\text{dom}} C$$

which is simply the definition of \vec{m} .

Lemma 1.5. *There is a 2-cell*

$$\begin{array}{cccc}
P_2 & \mathbb{L}_2\text{-Coalg} & U \\
X & & \downarrow \nu & C^2 \\
& \stackrel{\vec{m}}{\longrightarrow} & C^2 & R_1
\end{array}$$

such that cod v = id *and*

Proof. We just need to verify Equation (??):

where the first equation follows from (2), and the second by reducing $\cot \zeta$ using Lemma 1.4.

Proof of Theorem 1.2. For notational convenience, let G = Coalg be the lax double functor we need to establish. Both the double categories $\mathbb{C}\text{omon}(\mathbb{F}F(\mathbb{S}q(\mathscr{D})))$ and $\mathbb{S}\text{pan}(\mathscr{D}_0)$ have \mathscr{D}_0 as vertical category, and G_0 is simply the identity. From the statement of the theorem, G takes an object in $\mathbb{C}\text{omon}(\mathbb{F}F(\mathbb{S}q(\mathscr{D})))$ to the span

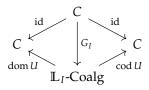
$$C \xleftarrow{\operatorname{dom} U} \mathbb{L}$$
-Coalg $\xrightarrow{\operatorname{cod} U} C$.

To define the behavior of G on 2-cells, consider a 2-cell in $Comon(\mathbb{F}F(Sq(\mathcal{D})))$:

$$\begin{array}{ccc}
C & \xrightarrow{(E_1, \eta_1, \epsilon_1, \delta_1)} & C \\
F \downarrow & & \downarrow \phi & \downarrow F \\
D & \xrightarrow{(E_2, \eta_2, \epsilon_2, \delta_2)} & D.
\end{array}$$

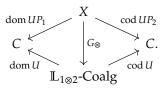
By Corollary ??, ϕ induces a colax morphism of comonads from L_1 to L_2 , hence by Proposition 1.1 there is an induced morphim $\tilde{\phi}$ between the EM-objects such that $U\tilde{\phi} = F^2U$. We can then define $G\phi$ to be the morphism of spans

Next we must define the coherence data G_I and G_{\otimes} . We will define G_I to be the morphims of spans



defined via Lemma 1.3 by the equations $UG_I = i: C \to C^2$ and $\alpha_I G_I$ is the identity on dom $i = \operatorname{cod} i$. The conditions of the lemma are trivially satisfied.

We will similarly use Lemma 1.3 to define G_{\otimes} . Let X, \vec{m} , ζ , and ν be as defined earlier in the section. G_{\otimes} is a morphism of spans



We will define G_{\otimes} to be the 1-cell such that $UG_{\otimes} = \vec{m}$ and

$$X \xrightarrow{G_{\otimes}} \mathbb{L}\text{-Coalg} \xrightarrow{U} C^{2} \xrightarrow{\operatorname{cod}} C = X \xrightarrow{P_{2}} \mathbb{L}_{2}\text{-Coalg} \xrightarrow{U} C^{2} \xrightarrow{\operatorname{cod}} C$$

$$U \xrightarrow{G_{\otimes}} \mathbb{L}^{2}\text{-Coalg} \xrightarrow{U} C^{2} \xrightarrow{V} C^{2} \xrightarrow{U} C^{2} \xrightarrow{C} C^{2} \xrightarrow{R_{1}} C^{2} \xrightarrow{E_{2}} C^{2}$$

In other words, in the notation of Lemma 1.3 let $F = \vec{m}$ and $\beta = E_2 \nu \circ \alpha_2 P_2$, and define $G_{\otimes} = \hat{F}$.

We now need to check equations 1-3 of Lemma 1.3 to verify that G_{\otimes} is well defined. We will check these equationally to save space, but the reader may want to draw out the diagrams for themselves to follow along. For the first equation:

$$\begin{split} E_{2}\nu \circ \alpha_{2}P_{2} \circ \kappa \vec{m} &= E_{2}\nu \circ \alpha_{2}P_{2} \circ \kappa UP_{2} \circ \kappa UP_{1} & \text{Def of } \vec{m} \\ &= E_{2}\nu \circ (\alpha_{2} \circ \kappa U)P_{2} \circ \kappa UP_{1} & \text{Eq (1)} \\ &= E_{2}\nu \circ \eta_{2}UP_{2} \circ \kappa UP_{1} & \text{Eq (1)} \\ &= \eta_{2}R_{1}\vec{m} \circ \text{dom } \nu \circ \kappa UP_{1} & \text{Interchange} \\ &= \eta_{2}R_{1}\vec{m} \circ E_{1}\zeta \circ \alpha_{1}P_{1} \circ \kappa UP_{1} & \text{Def of } \nu \\ &= \eta_{2}R_{1}\vec{m} \circ E_{1}\zeta \circ (\alpha_{1} \circ \kappa U)P_{1} & \text{Eq (1)} \\ &= \eta_{2}R_{1}\vec{m} \circ E_{1}\zeta \circ \eta_{1}UP_{1} & \text{Eq (1)} \\ &= \eta_{1\otimes 2}\vec{m} \circ \text{dom } \zeta & \text{Interchange; Def of } \eta_{1\otimes 2} \\ &= \eta_{1\otimes 2}\vec{m} & \text{dom } \zeta & \text{dom } \zeta = \text{id} \end{split}$$

and the second:

$$\begin{split} & \epsilon_{1\otimes 2}\vec{m} \circ E_2\nu \circ \alpha_2 P_2 \\ & = \epsilon_2 R_1 \vec{m} \circ E_2\nu \circ \alpha_2 P_2 \\ & = \cot\nu \circ (\epsilon_2 U \circ \alpha_2) P_2 \\ & = \operatorname{id}_{\operatorname{cod}\vec{m}}. \end{split} \qquad \text{Def of } \epsilon_{1\otimes 2} \\ & = \operatorname{Eq}(2); \operatorname{cod}\nu = \operatorname{id}_{\operatorname{cod}\vec{m}}. \end{split}$$

The third equation is a bit trickier to prove. We will need to prove two intermediate equations first, using the arrow object universal property.

Lemma.

$$i^{L}\vec{m} \circ L_{1}\zeta \circ \vec{\alpha}_{1}P_{1} = \vec{\beta} \circ \zeta \tag{4}$$

Proof. We must show the 2-cells become equal upon composition with dom and cod:

$$\operatorname{dom}(i^{L}\vec{m}\circ L_{1}\zeta\circ\vec{\alpha}_{1}P_{1})=\operatorname{id}_{\operatorname{dom}\vec{m}}=\operatorname{dom}(\vec{\beta}\circ\zeta)$$

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and

$$cod(i^{L}\vec{m} \circ L_{1}\zeta \circ \vec{\alpha}_{1}P_{1})$$

$$= cod i^{L}\vec{m} \circ E_{1}\zeta \circ cod \vec{\alpha}_{1}P_{1}$$

$$= \eta_{2}R_{1}\vec{m} \circ E_{1}\zeta \circ \alpha_{1}P_{1} \qquad Def of i^{L}, \vec{\alpha}$$

$$= \eta_{2}R_{1}\vec{m} \circ dom \nu \qquad Def of \nu$$

$$= E_{2}\nu \circ \eta_{2}UP_{2} \qquad Interchange$$

$$= E_{2}\nu \circ (\alpha_{2} \circ \kappa U)P_{2} \qquad Eq (1)$$

$$= (E_{2}\nu \circ \alpha_{2}P_{2}) \circ \kappa UP_{2}$$

$$= cod \vec{\beta} \circ cod \zeta \qquad Def of \vec{\beta}, \zeta$$

$$= cod(\vec{\beta} \circ \zeta).$$

Lemma.

$$R_1 \vec{\beta} \circ \nu = w \vec{m} \circ L_2 \delta_1^R \vec{m} \circ L_2 \nu \circ \vec{\alpha}_2 P_2 \tag{5}$$

Proof. Again we must prove equality after composing with dom and cod:

 $\begin{aligned} &\operatorname{dom}(R_{1}\vec{\beta}\circ\nu)\\ &=E_{1}\vec{\beta}\circ\operatorname{dom}\nu\\ &=E_{1}\vec{\beta}\circ E_{1}\zeta\circ\alpha_{1}P_{1} & \operatorname{Def} \text{ of } \nu\\ &=E_{1}(\vec{\beta}\circ\zeta)\circ\alpha_{1}P_{1} & \operatorname{Eq} (4)\\ &=E_{1}(i^{L}\vec{m}\circ L_{1}\zeta\circ\vec{\alpha}_{1}P_{1})\circ\alpha_{1}P_{1} & \operatorname{Eq} (4)\\ &=E_{1}i^{L}\vec{m}\circ E_{1}L_{1}\zeta\circ(E_{1}\vec{\alpha}_{1}\circ\alpha_{1})P_{1} & \operatorname{Eq} (3)\\ &=E_{1}i^{L}\vec{m}\circ E_{1}L_{1}\zeta\circ(\delta_{1}U\circ\alpha_{1})P_{1} & \operatorname{Eq} (3)\\ &=E_{1}i^{L}\vec{m}\circ\delta_{1}\vec{m}\circ E_{1}\zeta\circ\alpha_{1}P_{1} & \operatorname{Interchange}\\ &=\operatorname{dom} w\vec{m}\circ\operatorname{dom}\delta_{1}^{R}\vec{m}\circ\operatorname{dom}\nu\circ\operatorname{dom}\vec{\alpha}_{2}P_{2} & \operatorname{Defs} \text{ of } w,\delta^{R},\nu,\vec{\alpha}\\ &=\operatorname{dom}(w\vec{m}\circ L_{2}\delta_{1}^{R}\vec{m}\circ L_{2}\nu\circ\vec{\alpha}_{2}P_{2}) \end{aligned}$

and

 $cod(R_1\vec{\beta} \circ \nu)$ $= cod \vec{\beta} \circ cod \nu$ $= E_2 \nu \circ \alpha_2 P_2 \qquad Defs of \vec{\beta}, \nu$ $= E_2(p^R \circ \delta_1^R) \vec{m} \circ E_2 \nu \circ \alpha_2 P_2 \qquad p^R \circ \delta^R = id$ $= E_2 p^R \vec{m} \circ E_2 \delta_1^R \vec{m} \circ E_2 \nu \circ \alpha_2 P_2$ $= cod w \vec{m} \circ cod L_2 \delta_1^R \vec{m} \circ cod L_2 \nu \circ cod \vec{\alpha}_2 P_2 \qquad Defs of w, L, \vec{\alpha}$ $= cod(w \vec{m} \circ L_2 \delta_1^R \vec{m} \circ L_2 \nu \circ \vec{\alpha}_2 P_2)$

Now we are prepared to prove the third equation of Lemma 1.3 validating our definition of G_{\otimes} :

$$\begin{split} &\delta_{1\otimes 2}\vec{m}\circ E_2\nu\circ\alpha_2P_2\\ &=(E_2w\circ\delta_2R_{1\odot 1}\circ E_2\delta_1^R)\vec{m}\circ E_2\nu\circ\alpha_2P_2 & \text{Def of }\delta_{1\otimes 2}\\ &=E_2(w\vec{m}\circ L_2\delta_1^R\vec{m}\circ L_2\nu)\circ(\delta_2U\circ\alpha_2)P_2 & \text{Interchange}\\ &=E_2(w\vec{m}\circ L_2\delta_1^R\vec{m}\circ L_2\nu)\circ(E_2\vec{\alpha}_2\circ\alpha_2)P_2 & \text{Eq (3)}\\ &=E_2(w\vec{m}\circ L_2\delta_1^R\vec{m}\circ L_2\nu\circ\vec{\alpha}_2P_2)\circ\alpha_2P_2 & \text{Eq (5)}\\ &=E_2(R_1\vec{\beta}\circ\nu)\circ\alpha_2P_2 & \text{Eq (5)}\\ &=E_{1\otimes 2}\vec{\beta}\circ E_2\nu\circ\alpha_2P_2 & \text{Def of }E_{1\otimes 2} \end{split}$$

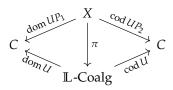
The verification that the definitions of G_I and G_{\otimes} form natural families, and of the coherence axioms for a lax double functor, is tedious, but follows from what we have presented here without requiring any new ideas or ingenuity.

Corollary 1.6. For any awfs $(E, \eta, \mu, \epsilon, \delta)$ on an object C in D, the multiplication μ induces a composition functor on \mathbb{L} -Coalg, and the functor between EM-objects induced by any colax morphism of awfs preserves this composition.

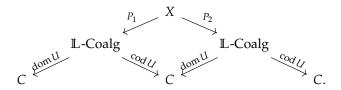
Proof. Any awfs $(E, \eta, \mu, \epsilon, \delta)$ has an underlying object in Comon($\mathbb{F}F(Sq(\mathcal{D}))$), by simply forgetting μ . The lax double-functor Coalg takes this to a span

$$C \xleftarrow{\operatorname{dom} U} \mathbb{L}\operatorname{-Coalg} \xrightarrow{\operatorname{cod} U} C.$$

The multiplication μ provides this object in $Comon(\mathbb{F}F(Sq(\mathcal{D})))$ with a monad structure, and lax double-functors preserve monads, so μ induces a monad structure on this span. A multiplication on this span is a morphism π :



where *X* is the pullback in the composite span



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The morphism π is the composition structure that we want. If $\mathcal{D}=\operatorname{Cat}$ is the 2-category of small categories, then an object (f,g) in X is a pair of morphisms in C equipped with coalgebra structures, such that $\operatorname{cod} f=\operatorname{dom} g$, and $\pi(f,g)$ is a morphism equipped with a coalgebra structure, with $\operatorname{dom} \pi(f,g)=\operatorname{dom} f$ and $\operatorname{cod} \pi(f,g)=\operatorname{cod} g$.

Of course, what we really want is that the morphism underlying the coalgebra $\pi(f,g)$ is the composition $g\circ f$. To see that this is the case, notice that the composition π is given by $G(\vec{\mu})\circ G_{\otimes}$, where $G(\vec{\mu})$ is the 1-cell between EM-objects induced by the colax morphism of comonads $\vec{\mu}$. Using the fact that $\vec{\mu}$ is a globular 2-cell in $\mathbb{FF}(\mathrm{Sq}(\mathcal{D}))$, we have $U\pi=UG(\vec{\mu})G_{\otimes}=UG_{\otimes}=\vec{m}$.

Bibliography

- [Gar09] Richard Garner. Understanding the small object argument. *Appl. Categ. Structures*, 17(3):247–285, 2009. arXiv:0712.0724.
- [Rie11] Emily Riehl. Algebraic model structures. *New York Journal of Mathematics*, 17:173–231, 2011.
- [Str72] Ross Street. The formal theory of monads. *J. Pure Appl. Algebra*, 2(2):149–168, 1972.