1 Introduction

The theory of model categories has a long history, and has proven itself indespensible to several recent advances in mathematics, such as higher category theory, so-called spectral algebraic geometry, even finding applications in computer science and the foundations of mathematics with homotopy type theory.

In the modern treatment, a model category is defined to consist of two *weak factorization systems* on a category $\mathscr{C}$ (e.g. [MP12]). A weak factorization system is a structure which consists of two classes of morphisms of $\mathscr{C}$, call them $\mathcal{L}$ and $\mathcal{R}$, such that solutions to certain lifting problems involving one morphism from each class always exist, plus an axiom that every morphism of $\mathscr{C}$ factors as a morphism from $\mathcal{L}$ followed by a morphism from $\mathcal{R}$. In the past 20 or so years, most authors have added the requirement that this factorization can be chosen in a natural/functorial way.

Taking this one step further, in [GT06] the category theorists Marco Grandis and Walter Tholen proposed a strengthening of weak factorization systems which they called *natural* weak factorization systems, today most often referred to as *algebraic* weak factorization systems, or awfs for short. An awfs strengthens the structure in a way which provides a canonical *choice* of solution to every lifting problem, in such a way that these choices are coherent or natural in a precise sense.

It at first seems as though this extra structure is *too* strict, and that examples would be hard to find. But in [Gar07] and [Gar09], the category theorist Richard Garner provided a modification of Quillen's small object argument which generates algebraic weak factorization systems, and which furthermore has much nicer convergence properties that Quillen's original construction, and often generates a smaller and easier to understand factorization. Best of all, Garner's small object argument operates under almost identical assumptions as Quillen's, so that in practice any cofibrantly generated weak factorization system can be strengthened to an algebraic one.

In her Ph.D. thesis, [Rie11] and [Rie13], Emily Riehl began the project of developing a full-fledged theory of *algebraic model structures*, built out of two awfs analgously to an ordinary model structure, in the second part extending this to a theory of multivariable Quillen adjunctions and monoidal algebraic model structures. Since then, she and her collaborators have continued to develop and find applications of this theory, e.g. [CGR12], [BR13], and [BMR13].

One appealing aspect of the theory of algebraic weak factorization systems, and by extension algebraic model categories, is that it is possible to express the definition entirely in terms of functors and natural transformations, opening the door to generalizing the theory by carrying out the same construction in other contexts with analogues of functors and natural transformations. The simplest and most familiar such context is a 2-category, but we have found that the most natural setting for the general theory of awfs—including the appropiate generalization of Quillen adjunctions—is a double category, a less familiar structure which has been attracting more attention in recent years.

In this paper, we aim to build a general setting in which the structure of an algebraic weak factorization system makes sense, a structure we call a *cyclic two-fold double category*, and then to translate some of the most important results from the existing theory to this larger context. We well then give an extension of this structure, called a *cyclic two-fold double multi-category*, a modification of the cyclic double multi-categories of [CGR12], in which multivariable morphisms of awfs also make sense. Finally, we will give an example application of the increased generality by showing that the theory of enriched categories has the necessary structure needed to form a special case of our theory. We expect that other examples of algebraic model structures on structures other than plain categories will be found useful, and hope that this general framework will help to quickly translate the existing theory to new settings.

2 Weak factorization systems

We will begin by briefly reviewing the notions of functorial factorization, weak factorization system, and algebraic weak factorization system.

2.1 Arrow Categories

Let $\mathscr{C}$ be a category. Its arrow category $\mathscr{C}^2$ is the category whose objects are arrows in $\mathscr{C}$ and whose morphisms are commutative squares. The arrow category comes with two functors $\mathrm{dom}, \mathrm{cod}\colon \mathscr{C}^2 \to \mathscr{C}$, along with a natural transformation $\kappa\colon \mathrm{dom} \Rightarrow \mathrm{cod}$. The component of κ at an object f of $\mathscr{C}^2$ is simply $f\colon \mathrm{dom}\, f \to \mathrm{cod}\, f$. Moreover, $\mathscr{C}^2$ satisfies a universal property: there is an equivalence of categories

$$\mathrm{Fun}(2, \mathrm{Fun}(\mathscr{X}, \mathscr{C})) \simeq \mathrm{Fun}(\mathscr{X}, \mathscr{C}^2) \tag{1}$$

given by composition with κ. Here, 2 is the ordinal, i.e. the category with two objects and a single non-identity arrow. In other words, $\mathscr{C}^2$ is the cotensor of $\mathscr{C}$ with the category 2 in the 2-category $\mathcal{C}\mathrm{at}$.

We will make this universal property more explicit in the next lemma, separating out the 1-dimensional and the 2-dimensional parts of (1):

Lemma 2.1. *Let $\mathscr{C}$ be a category.*

i) For any category $\mathscr{X}$, pair of functors $F, G\colon \mathscr{X} \to \mathscr{C}$, and natural transformation $\alpha\colon F \Rightarrow G$, there is a unique functor $\hat{\alpha}\colon \mathscr{X} \to \mathscr{C}^2$ such that $\mathrm{dom}\, \hat{\alpha} = F$, $\mathrm{cod}\, \hat{\alpha} = G$, and

$$\begin{tikzcd} \mathscr{X} \arrow[r, "\hat{\alpha}"] & \mathscr{C}^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\kappa"] & \mathscr{C} \arrow[r, phantom, "="] & \mathscr{X} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathscr{C}. \end{tikzcd} \tag{2}$$

ii) For any functors $F, F', G, G'\colon \mathscr{X} \to \mathscr{C}$ and a commutative square of natural transformations

$$\begin{tikzcd} F \arrow[r, Rightarrow, "\gamma"] \arrow[d, Rightarrow, "\alpha"'] & F' \arrow[d, Rightarrow, "\beta"] \\ G \arrow[r, Rightarrow, "\phi"'] & G', \end{tikzcd}$$

there is a unique natural transformation $\eta\colon \hat{\alpha} \to \hat{\beta}$ such that $\mathrm{dom}\, \eta = \gamma$ and $\mathrm{cod}\, \eta = \phi$, hence

$$\begin{tikzcd} \mathscr{X} \arrow[r, bend left=50, "F"] \arrow[r, dashed, "G" description] \arrow[r, bend right=50, "G'"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\phi"] & \mathscr{C} \arrow[r, phantom, "="] & \mathscr{X} \arrow[r, bend left, "\hat{\alpha}"] \arrow[r, bend right, "\hat{\beta}"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathscr{C}^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\kappa"] & \mathscr{C} \arrow[r, phantom, "="] & \mathscr{X} \arrow[r, bend left=50, "F"] \arrow[r, dashed, "F'" description] \arrow[r, bend right=50, "G'"'] \arrow[r, phantom, shift left=3, "\Downarrow\gamma"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \mathscr{C}. \end{tikzcd} \tag{3}$$

Definition 2.2. Let $\mathscr{D}$ be any 2-category. For any object A in $\mathscr{D}$, the *arrow object* of A, if it exists, is an object A^2 satisfying the universal property (1). If every object has an arrow object, i.e. if $\mathscr{D}$ has cotensors by 2, we will say $\mathscr{D}$ *has arrow objects*.

In practice, we will work with arrow objects in a 2-category using the two parts of Lemma 2.1.

Finally, we will record here a simple proposition which will be needed later.

Proposition 2.3. *Any arrow object A^2 has an internal category structure*

$$\begin{tikzcd} A^3 \arrow[r, "c"] & A^2 \arrow[r, shift left=2, "\mathrm{dom}"] \arrow[r, shift right=2, "\mathrm{cod}"'] & A \arrow[l, "i" description] \end{tikzcd}$$

where A^3 is the pullback of the span

$$\begin{tikzcd} A^2 \arrow[r, "\mathrm{cod}"] & A & A^2 \arrow[l, "\mathrm{dom}"'] \end{tikzcd}$$

Proof. Using the universal property, we define i and c by the equations $\mathrm{dom}\, i = \mathrm{id}$, $\mathrm{cod}\, i = \mathrm{id}$, $\kappa i = \mathrm{id}_{\mathrm{id}}$, $\mathrm{dom}\, c = \mathrm{dom}\, p_1$, $\mathrm{cod}\, c = \mathrm{cod}\, p_2$, and $\kappa c = \kappa p_2 \circ \kappa p_1$, where p_1 and p_2 are the projections of the pullback. ☐

2.2 Functorial Factorizations

Definition 2.4. A functorial factorization on a category $\mathscr{C}$ consists of a functor E and two natural transformations η and ϵ which factor κ, as in

$$\begin{tikzcd} C^2 \arrow[r, bend left=40, "\mathrm{dom}"{name=U}] \arrow[r, bend right=40, "\mathrm{cod}"'{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow\kappa"] & C \arrow[r, phantom, "="] & C^2 \arrow[r, dashed, "E" description] \arrow[r, bend left=60, "\mathrm{dom}", ""'{name=A}] \arrow[r, bend right=60, "\mathrm{cod}"', ""{name=B}] & C. \end{tikzcd}$$

with $\Downarrow\eta$ between dom and E, and $\Downarrow\epsilon$ between E and cod.

This determines for any arrow f in $\mathscr{C}$ a factorization $f = \epsilon_f \circ \eta_f$. The factorization is natural, meaning that for any morphism $(u,v)\colon f \Rightarrow g$ in $\mathscr{C}^2$ (i.e. commutative square in $\mathscr{C}$), the two squares in

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "\eta_f"'] & \cdot \arrow[d, "\eta_g"] \\ \cdot \arrow[r, "E(u,v)"] \arrow[d, "\epsilon_f"'] & \cdot \arrow[d, "\epsilon_g"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd}$$

commute.

A functorial factorization also determines two functors $L, R\colon C^2 \to C^2$ such that $\operatorname{dom} L = \operatorname{dom}$, $\operatorname{cod} R = \operatorname{cod}$, $\operatorname{cod} L = \operatorname{dom} R = E$, $\kappa L = \eta$, and $\kappa R = \epsilon$, by the universal property of C^2. The components of the factorization of f can then also be referred to as Lf and Rf, now thought of as objects in $\mathscr{C}^2$. There are also two canonical natural transformations, $\vec{\eta}\colon \mathrm{id} \Rightarrow R$ and $\vec{\epsilon}\colon L \Rightarrow \mathrm{id}$, determined by the commuting squares

$$\begin{tikzcd}
\mathrm{dom} \arrow[r, Rightarrow, "\eta"] \arrow[d, Rightarrow, "\kappa"'] & E \arrow[d, Rightarrow, "\epsilon"] \\
\mathrm{cod} \arrow[r, Rightarrow, "\mathrm{id}"'] & \mathrm{cod}
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
\mathrm{dom} \arrow[r, Rightarrow, "\mathrm{id}"] \arrow[d, Rightarrow, "\eta"'] & \mathrm{dom} \arrow[d, Rightarrow, "\kappa"] \\
E \arrow[r, Rightarrow, "\epsilon"'] & \mathrm{cod}
\end{tikzcd}$$

respectively. These make L and R into (co)pointed endofunctors of $\mathscr{C}^2$.

An algebra for the pointed endofunctor R is an object f in $\mathscr{C}^2$ equipped with a morphism $\vec{t}\colon Rf \Rightarrow f$, such that $\vec{t} \circ \vec{\eta}_f = \mathrm{id}_f$. Similarly, a coalgebra for the copointed endofunctor L is an f equipped with a morphism $\vec{s}\colon f \Rightarrow Lf$, such that $\vec{\epsilon}_f \circ \vec{s} = \mathrm{id}_f$.

Lemma 2.5. *Let $f\colon X \to Y$ be a morphism in $\mathscr{C}$. An R-algebra structure on $f \in \mathscr{C}^2$ is precisely a choice of lift t in the square*

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "Lf"'] & X \arrow[d, "f"] \\
Ef \arrow[r, "Rf"'] \arrow[ur, dashed, "t"] & Y.
\end{tikzcd} \tag{4}$$

Dually, an L-coalgebra structure on f is precisely a choice of lift s in the square

$$\begin{tikzcd}
X \arrow[r, "Lf"] \arrow[d, "f"'] & Ef \arrow[d, "Rf"] \\
Y \arrow[r, equal] \arrow[ur, dashed, "s"] & Y.
\end{tikzcd} \tag{5}$$

Moreover, a morphsim $(u, v)\colon f \Rightarrow g$ in $\mathscr{C}^2$ is a morphism of R-algebras if it commutes with the lifts t and t', that is, if in the diagram

$$\begin{tikzcd}
\cdot \arrow[r, "u"] \arrow[d, "Lf"'] & \cdot \arrow[d, "Lg"] \\
\cdot \arrow[r, "E(u,v)"] \arrow[d, shift right, "Rf"'] & \cdot \arrow[d, shift left, "Rg"] \\
\cdot \arrow[r, "v"'] \arrow[u, shift right, "t"'] & \cdot \arrow[u, shift left, "t'"]
\end{tikzcd}$$

we have $t'v = E(u, v)t$.

2.3 Algebraic Weak Factorization Systems

To simplify the discussion of weak factorization systems, we will start by introducing a notation. For any two morphisms l and r in $\mathscr{C}$, write $l \boxslash r$ to mean that for every commutative square

$$\begin{tikzcd}
\cdot \arrow[r, "u"] \arrow[d, "l"'] & \cdot \arrow[d, "r"] \\
\cdot \arrow[r, "v"'] \arrow[ur, dashed, "w"] & \cdot
\end{tikzcd} \tag{6}$$

there exists a lift w. In this case, we will say that l has the *left lifting property* with respect to r, and that r has the *right lifting property* with respect to l. Similarly, for two classes of morphisms $\mathcal{L}$ and $\mathcal{R}$, we will say $\mathcal{L} \boxslash \mathcal{R}$ if $l \boxslash r$ for every $l \in \mathcal{L}$ and $r \in \mathcal{R}$. Finally, we will write $\mathcal{L}^{\boxslash}$ for the class of morphisms having the right lifting property with respect to every morphism of $\mathcal{L}$, and ${}^{\boxslash}\mathcal{R}$ for the class of morphisms having the left lifting property with respect to every morphism of $\mathcal{R}$.

Definition 2.6. A *functorial weak factorization system* on a category $\mathscr{C}$ consists of a functorial factorization on $\mathscr{C}$ and two classes $\mathcal{L}$ and $\mathcal{R}$ of morphisms in $\mathscr{C}$, such that

- for every morphism f in $\mathscr{C}$, $Lf \in \mathcal{L}$ and $Rf \in \mathcal{R}$,
- $\mathcal{L}^{\boxslash} = \mathcal{R}$ and ${}^{\boxslash}\mathcal{R} = \mathcal{L}$.

It a simple and standard proof that the lifting property condition can be replaced by two simpler conditions:

Lemma 2.7. *A functorial weak factorization system can equivalently be defined to be a functorial factorization on $\mathscr{C}$ and two classes $\mathcal{L}$ and $\mathcal{R}$ of morphisms in $\mathscr{C}$, such that*

- *for every morphism f in $\mathscr{C}$, $Lf \in \mathcal{L}$ and $Rf \in \mathcal{R}$,*
- *$\mathcal{L} \boxslash \mathcal{R}$,*
- *$\mathcal{L}$ and $\mathcal{R}$ are both closed under retracts.*

In fact, the functorial factorization by itself already determines the two classes of morphisms, with $\mathcal{L}$ the class of morphisms admitting an L-coalgebra structure, and $\mathcal{R}$ the class of morphisms admitting an R-algebra structure. The lifting properties also follow directly from the functorial factorization, as the next lemma shows.

Lemma 2.8. *For any L-coalgebra (l, s) and any R-algebra (r, t), there is a canonical choice of lift in the square* (6)*. Any morphism of R-algebras $(u_1, v_1)\colon (r, t) \Rightarrow (r', t')$ and any morphism of L-coalgebras $(u_2, v_2)\colon (l', s') \Rightarrow (l, s)$ preserves these canonical choices of lifts.*

Proof. The construction is shown in the diagram

$$\begin{tikzcd}
\cdot \arrow[r, "u"] \arrow[d, "Ll"'] & \cdot \arrow[d, shift left, "Lr"] \\
\cdot \arrow[r, dashed, "E(u,v)"] \arrow[d, shift right, "Rl"'] & \cdot \arrow[u, dashed, shift left, "t"] \arrow[d, "Rr"] \\
\cdot \arrow[r, "v"'] \arrow[u, dashed, shift right, "s"'] & \cdot
\end{tikzcd} \tag{7}$$

Commutativity of (6) follows immediately from (4) and (5).

That a morphism of R-algebras preserves these canonical lifts can be seen in the diagram

$$\begin{tikzcd}
\cdot \arrow[r, "u"] \arrow[d, "Ll"'] & \cdot \arrow[r, "u'"] \arrow[d, shift left, "Lr"] & \cdot \arrow[d, shift left, "Lr'"] \\
\cdot \arrow[r, "E(u,v)"] \arrow[d, shift right, "Rl"'] & \cdot \arrow[r, "E(u',v')"] \arrow[u, shift left, "t"] \arrow[d, "Rr"] & \cdot \arrow[u, shift left, "t'"] \arrow[d, "Rr'"] \\
\cdot \arrow[r, "v"'] \arrow[u, shift right, "s"'] & \cdot \arrow[r, "v'"'] & \cdot
\end{tikzcd}$$

noting that $u'tE(u,v)s = t'E(u',v')E(u,v)s = t'E(u'u,v'v)s$. $\square$

This, together with the classical fact that the class of objects admitting a (co)algebra structure for a (co)pointed endofunctor is closed under retracts, gives a third equivalent definition of a functorial weak factorization system.

Lemma 2.9. *A functorial weak factorization system can equivalently be defined to be a functorial factorization on $\mathscr{C}$ such that*

- *for every morphism f in $\mathscr{C}$, Lf admits an L-coalgebra structure, and Rf admits an R-algebra structure.*

An R-algebra structure on Rf consists of a morphism $\vec{\mu}_f \colon R^2f \to Rf$ in $\mathscr{C}^2$ such that $\vec{\mu}_f \circ \vec{\eta}_{Rf} = \mathrm{id}_{Rf}$, while an L-coalgebra structure on Lf consists of a morphism $\vec{\delta}_f \colon Lf \to L^2f$ such that $\vec{\epsilon}_{Lf} \circ \vec{\delta}_f = \mathrm{id}_{Lf}$. We might hope that it is possible to choose these structures for all f in a natural way, such that they form the components of natural transformations $\vec{\mu} \colon R^2 \Rightarrow R$ and $\vec{\delta} \colon L \Rightarrow L^2$.

If we want these choices of lifts to be fully coherent, we should also ask that for any R-algebra (f, t), the lift constructed as in (7) for the square (4) is equal to t, and similarly for L-coalgebras and (5). Lastly, we should ask that the components $\vec{\mu}_f$ and $\vec{\delta}_f$ are (co)algebra morphisms. These conditions, plus one more ensuring that there is an unambiguous notion of a morphism with both L-algebra and R-coalgebra structures, lead to the definition of an *algebraic weak factorization system*, first given in [GT06] (there called *natural* weak factorization systems), and further refined in [Gar07] and [Gar09].

Definition 2.10. An *algebraic weak factorization system* on a category $\mathscr{C}$ consists of a functorial factorization $(L, \vec{\epsilon}, R, \vec{\eta})$ together with natural transformations $\vec{\mu} \colon R^2f \Rightarrow Rf$ and $\vec{\delta} \colon L \Rightarrow L^2$, such that

- $\mathbb{R} = (R, \vec{\eta}, \vec{\mu})$ is a monad and $\mathbb{L} = (L, \vec{\epsilon}, \vec{\delta})$ a comonad on $\mathscr{C}^2$, and
- the natural transformation $\Delta = (\delta, \mu) \colon LR \Rightarrow RL$ determined by the equation $\epsilon L \circ \delta = \mu \circ \eta R \, (= \mathrm{id}_E)$ as in Lemma 2.1 is a distributive law, which in this case reduces to the single condition $\delta \circ \mu = \mu L \circ E\Delta \circ \delta R$.

3 Double Categories

In this section, we will give an overview of double categories, as well as (one possible version of) the definition of monads in a double category

A (strict) double category is a two-dimensional categorical structure, similar to a 2-category. Unlike a 2-category, a double category has two types of 1-cells, called *vertical* and *horizontal*, and 2-cells all have a square shape, with domain and codomain horizontal 1-cells as well as domain and codomain vertical 1-cells.

We will first give the most concise definition of a double category, which we will then break down into more concrete terms.

Definition 3.1. A (strict) *double category* is an internal category object in the (large) category of categories.

So a double category $\mathbb{D}$ consists of a category $\mathbb{D}_0$ and a category $\mathbb{D}_1$, along with functors $s, t\colon \mathbb{D}_1 \to \mathbb{D}_0$, $i\colon \mathbb{D}_0 \to \mathbb{D}_1$, and $\otimes\colon \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \to \mathbb{D}_1$ satisfying the usual axioms of a category. We will call the objects of $\mathbb{D}_0$ the 0-cells of $\mathbb{D}$, and the morphisms of $\mathbb{D}_0$ the vertical 1-cells. Thus $\mathbb{D}_0$ forms the so-called *vertical category* of $\mathbb{D}$. We will call the objects of $\mathbb{D}_1$ the horizontal 1-cells of $\mathbb{D}$, and the morphisms of $\mathbb{D}_1$ are the 2-cells.

A morphism $\phi\colon X \to Y$ in $\mathbb{D}_1$, where $s(X) = C$, $t(X) = C'$, $s(Y) = D$, $t(Y) = D'$, $s(\phi) = f$, and $t(\phi) = g$ will be drawn as

$$\begin{tikzcd}
C \arrow[r, "X", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & C' \arrow[d, "g"] \\
D \arrow[r, "Y"', "|" marking] & D'
\end{tikzcd} \tag{8}$$

where the tick-mark on the horizontal 1-cells serves as a further reminder that the horizontal 1-cells are of a different nature than the vertical 1-cells. The composition in $\mathbb{D}_0$ provides a vertical composition of vertical 1-cells and 2-cells, while the composition functor $\otimes\colon \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \to \mathbb{D}_1$ provides a horizontal composition of horizontal 1-cells and 2-cells.

For any object C in $\mathbb{D}_0$, $i(C)$ is the *unit* horizontal 1-cell

$$\begin{tikzcd}
C \arrow[r, "I_C", "|" marking] & C
\end{tikzcd}$$

and acts as an identity with respect to the horizontal composition.

A 2-cell θ for which $s\theta = t\theta = \mathrm{id}$ will be called *globular*. We will sometimes draw globular 2-cells as

$$\begin{tikzcd}
C \arrow[r, bend left, "X", "|" marking] \arrow[r, bend right, "Y"', "|" marking] \arrow[r, phantom, "\Downarrow\theta"] & C',
\end{tikzcd}$$

to save space and help readability of diagrams.

Example 3.2. For any 2-category $\mathcal{D}$, there is an associated double category $\mathbb{S}\mathrm{q}(\mathcal{D})$ of *squares* in $\mathcal{D}$, in which the vertical and horizontal 1-cells are both just 1-cells in $\mathcal{D}$, and 2-cells

$$\begin{tikzcd}
C \arrow[r, "j", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & C' \arrow[d, "g"] \\
D \arrow[r, "k"', "|" marking] & D'
\end{tikzcd}$$

are simply 2-cells $\phi\colon gj \Rightarrow kf$ in $\mathcal{D}$.

Example 3.3. Given any category M, there is a pseudo double category $\mathbb{S}\mathrm{pan}(M)$ of *spans* in M. The vertical category of $\mathbb{S}\mathrm{pan}(M)$ is just M, while horizontal 1-cells

$$\begin{tikzcd}
C \arrow[r, "X", "|" marking] & D
\end{tikzcd}$$

are given by spans

$$\begin{tikzcd}
C & X \arrow[l, "j"'] \arrow[r, "k"] & D
\end{tikzcd}$$

in M, and 2-cells

$$\begin{tikzcd}
C \arrow[r, "X", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\theta"] & D \arrow[d, "g"] \\
C' \arrow[r, "Y"', "|" marking] & D'
\end{tikzcd}$$

are given by commutative diagrams

$$\begin{tikzcd}
C \arrow[d, "f"'] & X \arrow[l, "j"'] \arrow[r, "k"] \arrow[d, "\theta"] & D \arrow[d, "g"] \\
C' & Y \arrow[l, "j'"] \arrow[r, "k'"'] & D.'
\end{tikzcd}$$

The horizontal composition of spans is given by pullback. It is because this horizontal composition is only determined up to isomorphsim that this example is not a strict double category.

Definition 3.4. For any double category $\mathbb{D}$, there is an associated 2-category $\mathcal{H}or(\mathbb{D})$, called the *horizontal 2-category* of $\mathbb{D}$. The objects and 1-cells of $\mathcal{H}or(\mathbb{D})$ are the objects and horizontal 1-cells of $\mathbb{D}$, while 2-cells $\phi\colon X \Rightarrow Y$ in $\mathcal{H}or(\mathbb{D})$ are the globular 2-cells in $\mathbb{D}$, i.e. those of the form

$$\begin{tikzcd}
C \arrow[r, "X", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\phi"] & D \arrow[d, equal] \\
C \arrow[r, "Y"', "|" marking] & D
\end{tikzcd}$$

Notice that $\mathcal{H}or(\mathbb{S}\mathrm{q}(\mathcal{D}))$ is isomorphic to $\mathcal{D}$.

Definition 3.5. Given a double category $\mathbb{D}$, define double categories $\mathbb{D}^{\mathrm{vop}}$ and $\mathbb{D}^{\mathrm{hop}}$, obtained by reversing the direction of the vertical and horizontal 1-cells respectively, and changing the orientation of the 2-cells as appropriate. For example, a 2-cell (8) in $\mathbb{D}^{\mathrm{vop}}$ is a 2-cell

$$\begin{tikzcd}
D \arrow[r, "Y", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & D' \arrow[d, "g"] \\
C \arrow[r, "X"', "\shortmid"{marking}] & C'
\end{tikzcd}$$

in $\mathbb{D}$.

In terms of Definition 3.1, $\mathbb{D}^{\mathrm{vop}}$ is the double category obtained by replacing the categories $\mathbb{D}_0$ and $\mathbb{D}_1$ with their opposites, while $\mathbb{D}^{\mathrm{vop}}$ is the obtained by swapping the horizontal source and target functors s and t.

3.1 Arrow Objects in a Double Category

In the following we will need an extension of the universal property (1) to double categories. Fortunately, this is quite straightforward.

Let $\mathbb{D}$ be a double category. Given an object C of $\mathbb{D}$, the *arrow object* C^2, if it exists, is an object together with a diagram

$$\begin{tikzcd}
C^2 \arrow[r, bend left, "\mathrm{dom}", "\shortmid"{marking}] \arrow[r, bend right, "\mathrm{cod}"', "\shortmid"{marking}] \arrow[r, phantom, "\Downarrow\kappa"] & C,
\end{tikzcd}$$

such that any 2-cell

$$\begin{tikzcd}
A \arrow[r, bend left, "d_1", "\shortmid"{marking}] \arrow[r, bend right, "d_0"', "\shortmid"{marking}] \arrow[r, phantom, "\Downarrow\alpha"] & C
\end{tikzcd}$$

uniquely factors through κ, as

$$\begin{tikzcd}
A \arrow[r, "\hat{\alpha}", "\shortmid"{marking}] & C^2 \arrow[r, bend left, "\mathrm{dom}", "\shortmid"{marking}] \arrow[r, bend right, "\mathrm{cod}"', "\shortmid"{marking}] \arrow[r, phantom, "\Downarrow\kappa"] & C.
\end{tikzcd}$$

Given a vertical 1-cell $F\colon C \to D$ in $\mathbb{D}$, the *lift to arrow objects* $\hat{F}\colon C^2 \to D^2$, if it exists, is a vertical 1-cell $\hat{F}\colon C^2 \to D^2$ together with 2-cells

$$\begin{tikzcd}
C^2 \arrow[r, "\mathrm{dom}", "\shortmid"{marking}] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_1"] & C \arrow[d, "F"] & & C^2 \arrow[r, "\mathrm{cod}", "\shortmid"{marking}] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_0"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "\mathrm{dom}"', "\shortmid"{marking}] & D & & D^2 \arrow[r, "\mathrm{cod}"', "\shortmid"{marking}] & D
\end{tikzcd}$$

satisfying

$$\begin{tikzcd}
C^2 \arrow[r, "\mathrm{dom}", "\shortmid"{marking}] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_1"] & C \arrow[d, "F"] & = & C^2 \arrow[r, "\mathrm{cod}"', "\shortmid"{marking}] \arrow[r, bend left=50, "\mathrm{dom}", "\shortmid"{marking}] \arrow[r, phantom, bend left=25, "\Downarrow\kappa"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_0"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "\mathrm{dom}", "\shortmid"{marking}] \arrow[r, bend right=50, "\mathrm{cod}"', "\shortmid"{marking}] \arrow[r, phantom, bend right=25, "\Downarrow\kappa"] & D & & D^2 \arrow[r, "\mathrm{cod}"', "\shortmid"{marking}] & D,
\end{tikzcd}$$

such that for any 2-cells

$$\begin{tikzcd}
A \arrow[r, bend left=40, "d_1", "\shortmid"{marking}] \arrow[r, bend right=40, "d_0"', "\shortmid"{marking}] \arrow[r, phantom, "\Downarrow\alpha"] & C & B \arrow[r, bend left=40, "d'_1", "\shortmid"{marking}] \arrow[r, bend right=40, "d'_0"', "\shortmid"{marking}] \arrow[r, phantom, "\Downarrow\alpha'"] & D
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \arrow[r, "d_1", "\shortmid"{marking}] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\lambda_1"] & C \arrow[d, "F"] & A \arrow[r, "d_0", "\shortmid"{marking}] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\lambda_0"] & C \arrow[d, "F"] \\
B \arrow[r, "d'_1"', "\shortmid"{marking}] & D & B \arrow[r, "d'_0"', "\shortmid"{marking}] & D
\end{tikzcd}$$

satisfying

$$\begin{tikzcd}
A \arrow[r, "d_1", "\shortmid"{marking}] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\lambda_1"] & C \arrow[d, "F"] & = & A \arrow[r, "d_0"', "\shortmid"{marking}] \arrow[r, bend left=50, "d_1", "\shortmid"{marking}] \arrow[r, phantom, bend left=25, "\Downarrow\alpha"] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\lambda_0"] & C \arrow[d, "F"] \\
B \arrow[r, "d'_1", "\shortmid"{marking}] \arrow[r, bend right=50, "d'_0"', "\shortmid"{marking}] \arrow[r, phantom, bend right=25, "\Downarrow\alpha'"] & D & & B \arrow[r, "d'_0"', "\shortmid"{marking}] & D
\end{tikzcd}$$

there is a unique 2-cell

$$\begin{tikzcd}
A \arrow[r, "\hat{\alpha}", "\shortmid"{marking}] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\theta"] & C^2 \arrow[d, "\hat{F}"] \\
B \arrow[r, "\hat{\alpha}'"', "\shortmid"{marking}] & D^2
\end{tikzcd}$$

such that the horizontal composition of θ with γ_0 and γ_1 is respectively equal to λ_0 and λ_1.

Definition 3.6. A double category $\mathbb{D}$ *has arrow objects* if for every object C of $\mathbb{D}$ there is an object C^2 and 2-cell κ, and for every vertical 1-cell F there is a vertical 1-cell $\hat{F}$ and 2-cells γ_0 and γ_1, satisfying the universal properties given above.

The intuition that this is a generalization of Lemma 2.1 is supported by the following two propositions, the (easy) proofs of which are left to the reader.

Proposition 3.7. *If the double category* $\mathbb{D}$ *has arrow objects, then so does* $\mathcal{H}or(\mathbb{D})$.

Proposition 3.8. *If the 2-category* $\mathcal{D}$ *has arrow objects, then so does* $\mathbb{S}\mathrm{q}(\mathcal{D})$.

Proof. A simple check. The 2-cells γ_0 and γ_1 will always be identities. □

3.2 Monads

We will define a *monad* in a double category $\mathbb{D}$ to be a tuple (C, T, η, μ), in which C is an object, $T\colon C \to C$ is a horizontal 1-cell, and η and μ are 2-cells

$$\begin{tikzcd}
C \arrow[r, "\mathrm{id}_C", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta"] & C \arrow[d, equal] \\
C \arrow[r, "T"', "|" marking] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "T", "|" marking] \arrow[d, equal] & C \arrow[r, "T", "|" marking] \arrow[d, phantom, "\Downarrow\mu"] & C \arrow[d, equal] \\
C \arrow[rr, "T"', "|" marking] & & C
\end{tikzcd}$$

satisfying the usual unit and associativity conditions.

Given two monads (C, T, η, μ) and (D, S, η', μ'), a monad morphism from (C, T) to (D, S) consists of a pair (f, ϕ), where f is a vertical 1-cell $C \to D$ and ϕ is a 2-cell

$$\begin{tikzcd}
C \arrow[r, "T", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[d, "f"] \\
D \arrow[r, "S"', "|" marking] & D
\end{tikzcd}$$

which commutes with the unit and multiplication 2-cells in the sense of the two equations

$$\begin{tikzcd}
C \arrow[r, "\mathrm{id}_C", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta"] & C \arrow[d, equal] & & C \arrow[r, "\mathrm{id}_C", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}_f"] & C \arrow[d, "f"] \\
C \arrow[r, "T", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[d, "f"] & = & D \arrow[r, "\mathrm{id}_D", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta'"] & D \arrow[d, equal] \\
D \arrow[r, "S"', "|" marking] & D & & D \arrow[r, "S"', "|" marking] & D
\end{tikzcd} \tag{9}$$

and

$$\begin{tikzcd}
C \arrow[r, "T", "|" marking] \arrow[d, equal] & C \arrow[r, "T", "|" marking] \arrow[d, phantom, "\Downarrow\mu"] & C \arrow[d, equal] & & C \arrow[r, "T", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[r, "T", "|" marking] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[d, "f"] \\
C \arrow[rr, "T", "|" marking] \arrow[d, "f"'] & \arrow[d, phantom, "\Downarrow\phi"] & C \arrow[d, "f"] & = & D \arrow[r, "S", "|" marking] \arrow[d, equal] & D \arrow[r, "S", "|" marking] \arrow[d, phantom, "\Downarrow\mu'"] & D \arrow[d, equal] \\
D \arrow[rr, "S"', "|" marking] & & D & & D \arrow[rr, "S"', "|" marking] & & D
\end{tikzcd} \tag{10}$$

Definition 3.9. Given any double category $\mathbb{D}$, we will write $\mathrm{Mon}(\mathbb{D})$ for the category of monads in $\mathbb{D}$, consisting of monads and monad morphisms as defined above. The category $\mathrm{Comon}(\mathbb{D})$ of comonads in $\mathbb{D}$ is defined to be the category $\mathrm{Mon}(\mathbb{D}^{\mathrm{op}})$ of monads in $\mathbb{D}^{\mathrm{op}}$.

Example 3.10. The category $\mathrm{Mon}(\mathbb{S}\mathrm{pan}(\mathbf{Set}))$ is precisely the category of small categories. It is an easy and enlightening exercise to work this out for oneself.

Proposition 3.11. *The categories of (co)monads and (co)lax morphisms in a 2-category $\mathcal{D}$ can be given in terms of (co)monads in the double category of squares as follows:*

$$\begin{aligned}
\mathrm{Mon}_{colax}(\mathcal{D}) &= \mathrm{Mon}(\mathbb{S}\mathrm{q}(\mathcal{D})) \\
\mathrm{Comon}_{colax}(\mathcal{D}) &= \mathrm{Comon}(\mathbb{S}\mathrm{q}(\mathcal{D})) \\
\mathrm{Mon}_{lax}(\mathcal{D}) &= \mathrm{Mon}(\mathbb{S}\mathrm{q}(\mathcal{D}^{op}))^{op} \\
\mathrm{Comon}_{lax}(\mathcal{D}) &= \mathrm{Comon}(\mathbb{S}\mathrm{q}(\mathcal{D}^{op}))^{op}
\end{aligned}$$

where by $\mathcal{D}^{op}$ we mean the 2-category obtained by reversing the direction of all 1-cells (but not 2-cells).

Proof. Immediate from the definitions. Readers unfamiliar with (co)lax morphisms of monads can take this as the definition. □

3.3 Double Functors

The natural notion of functor between double categories is a straightforward generalization of lax functors between monoidal categories. Recall that we are using the symbol $\otimes$ to denote horizontal composition.

Definition 3.12. Let $\mathbb{D}$ and $\mathbb{E}$ be double categories. A *lax double functor* $F\colon \mathbb{D} \to \mathbb{E}$ consists of:

- Functors $F_0\colon \mathbb{D}_0 \to \mathbb{E}_0$ and $F_1\colon \mathbb{D}_1 \to \mathbb{E}_1$ such that $sF_1 = F_0 s$ and $tF_1 = F_0 t$
- Natural transformations with globular components $F_\otimes\colon F_1 X \otimes F_1 Y \to F_1(X \otimes Y)$ and $F_I\colon I_{F_0 C} \to F_1(I_C)$, which satisfy the usual coherence axioms for a lax monoidal functor.

A lax double functor F for which the components of F_I and $F_\otimes$ are identities will be called *strict*. For the intermediate notion where the components of F_I and $F_\otimes$ are (vertical) isomorphsims, we will simply refer to F as a double functor.

Proposition 3.13. *A lax double functor $F\colon \mathbb{D} \to \mathbb{E}$ induces a functor $F\colon \mathrm{Mon}(\mathbb{D}) \to \mathrm{Mon}(\mathbb{E})$.*

Proof. This works just like the case for monoidal categories. For instance, if X is a monad in $\mathbb{D}$, FX has the multiplication

$$\begin{tikzcd}
C \arrow[r, "FX", "\shortmid"{marking}] \arrow[d, equal] & C \arrow[r, "FX", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow F_{\otimes}"] & C \arrow[d, equal] \\
C \arrow[rr, "F(X \otimes X)", "\shortmid"{marking}] \arrow[d, equal] & \arrow[d, phantom, "\Downarrow F\mu"] & C \arrow[d, equal] \\
C \arrow[rr, "FX"', "\shortmid"{marking}] & & C
\end{tikzcd}$$

The fact that F takes monad morphisms to monad morphisms can easily be checked using the naturality of F_I and $F_{\otimes}$. □

4 2-Fold Double Categories

It is well known that the notion of bialgebra or bimonoid—an object with both monoid and comonoid structures which are compatible in a certain sense—makes sense not only in a symmetric monoidal category, but also in more general *braided* monoidal categories. A bimonoid in a braided monoidal category $\mathscr{C}$ can be defined to be a monoid in the category of comonoids in $\mathscr{C}$, or equivalently as a comonoid in the category of monoids in $\mathscr{C}$. The braiding is necessary to ensure that the monoidal structure in $\mathscr{C}$ lifts to a product in $\mathrm{Mon}(\mathscr{C})$ and $\mathrm{Comon}(\mathscr{C})$.

Less well known is the fact that the definition of bimonoid works just as well in a more general context still: so-called 2-fold monoidal categories. A 2-fold monoidal category has two different monoidal structures, call them $(\otimes, I)$ and $(\odot, \perp)$, which are themselves compatible in certain sense. This compatibility can be stated in a way analagous to the definition of bimonoid given in the previous paragraph: a (strict) 2-fold monoidal category is a monoid object in the category $\mathbf{StrMonCat}_l$ of strict monoidal categories and lax functors, or equivalently a monoid object in the category $\mathbf{StrMonCat}_c$ of strict monoidal categories and colax functors. Notice that monoid objects in the category of strict monoidal categories and *strong* monoidal functors (in which the components of the lax structure are isomorphisms) are precisely (strict) braided monoidal categories.

More concretely, the compatibility between the monoidal structures amounts to the existence of maps

$$m\colon \perp \otimes \perp \to \perp, \qquad c\colon I \to I \odot I, \qquad j\colon I \to \perp,$$

making $(\perp, j, m)$ a $\otimes$-monoid and (I, j, c) a $\odot$-comonoid, and a natural family of maps

$$z_{A,B,C,D}\colon (A \odot B) \otimes (C \odot D) \to (A \otimes C) \odot (B \otimes D)$$

satisfying some coherence axioms.

Example 4.1.

- Any braided monoidal category can be made into a 2-fold monoidal category in which the two monoidal structures coincide.
- Any monoidal category $(\mathscr{C}, \otimes, I)$ with finite products has a 2-fold monoidal structure with $(\odot, \perp)$ given by the product and terminal object. Dually, a monoidal category $(\mathscr{C}, \odot, \perp)$ with finite coproducts has a 2-fold monoidal structure with $(\otimes, I)$ given by the coproduct and initial object.

Because the $\odot$-monoidal structure is lax monoidal with respect to the $\otimes$-monoidal structure, it lifts to the category $\mathrm{Mon}_\otimes(\mathscr{C})$ of $\otimes$-monoids in $\mathscr{C}$. Dually, the $\otimes$-monoidal structure lifts to the category $\mathrm{Comon}_\odot(\mathscr{C})$ of $\odot$-comonoids in $\mathscr{C}$. Thus, we could define the category of bimonoids in $\mathscr{C}$ to be either $\mathrm{Comon}_\odot(\mathrm{Mon}_\otimes(\mathscr{C}))$ or $\mathrm{Mon}_\otimes(\mathrm{Comon}_\odot(\mathscr{C}))$, and it turns out that these are canonically isomorphic. In either case, a bimonoid is an object A with a $\otimes$-monoid structure (η, μ) and a $\odot$-comonoid structure (ϵ, δ), such that the following four diagrams commute:

$$\begin{tikzcd}
I \arrow[r, "\eta"] \arrow[d, "c"'] & A \arrow[d, "\delta"] \\
I \odot I \arrow[r, "\eta\odot\eta"'] & A \odot A,
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A \arrow[r, "\mu"] \arrow[d, "\epsilon\otimes\epsilon"'] & A \arrow[d, "\epsilon"] \\
\bot \otimes \bot \arrow[r, "m"'] & \bot,
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \arrow[dr, "\epsilon"] & \\
I \arrow[ur, "\eta"] \arrow[rr, "j"'] & & \bot,
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes A \arrow[rr, "\mu"] \arrow[d, "\delta\otimes\delta"'] & & A \arrow[d, "\delta"] \\
(A \odot A) \otimes (A \odot A) \arrow[r, "z_{A,A,A,A}"'] & (A \otimes A) \odot (A \otimes A) \arrow[r, "\mu\odot\mu"'] & A \odot A.
\end{tikzcd} \tag{11}$$

We would now like to generalize this 2-fold monoidal category definition to double categories, where there are two different horizontal compositions which are compatible in a way analagous to the two monoidal structures in a 2-fold monoidal category. We will start with a concise formal definition, and then expand on the definition more concretely.

Definition 4.2. A *2-fold double category* $\mathbb{D}$ with vertical category $\mathrm{Vert}(\mathbb{D}) = \mathscr{D}_0$ is a 2-fold monoid object in the 2-category $\mathcal{C}at/\mathscr{D}_0$ of categories over $\mathscr{D}_0$.

Breaking this down, we have a category $\mathscr{D}_1$, a functor $p\colon \mathscr{D}_1 \to \mathscr{D}_0$, two functors $\otimes, \odot\colon \mathscr{D}_1 \times_{\mathscr{D}_0} \mathscr{D}_1 \to \mathscr{D}_1$ commuting with p, and two functors $I, \bot\colon \mathscr{D}_0 \to \mathscr{D}_1$ which are sections of p, such that $\otimes$, $\odot$, I, and $\bot$ satisfy all the axioms of a 2-fold monoidal category. In particular, each fiber of p has a 2-fold monoidal structure.

A monoid object in $\mathcal{C}at/\mathbb{D}_0$ is equivalently a double category where the source and target functors $s, t\colon \mathbb{D}_1 \to \mathbb{D}_0$ are equal, and with the vertical category $\mathbb{D}_0$. Conversely, any double category $\mathbb{D}$ in which all horizontal 1-cells have equal domain and codomain, and all 2-cells have equal vertical 1-cells as domain and codomain, is equivalently a monoid object in $\mathcal{C}at/\mathbb{D}_0$. We will alternate between these two descriptions as convenient.

Using this shift of perspective, $\mathbb{D}$ has two underlying double categories, both with vertical category $\mathscr{D}_0$ and with source and target functors both equal to $p\colon \mathscr{D}_1 \to \mathscr{D}_0$. The double category $\mathbb{D}_\otimes$ has the rest of the double category structure given by the functors I and $\otimes$, while the double category $\mathbb{D}_\odot$ uses the functors $\bot$ and $\odot$.

Using this double category interpretation, we will find it convenient to think of a 2-fold double category as a double category with two different but interacting horizontal compositions. Notice that from this perspective, all horizontal 1-cells are endomorphisms.

Remark 4.3. It may seem somewhat ad hoc to force a 2-fold monoid object in a slice of $\mathcal{C}$ into a double category mold, with the odd looking restriction to having only endomorphisms in the horizontal direction. We will make essential use of double functors from $\mathbb{D}_\odot$ and $\mathbb{D}_\otimes$ to genuine double categories (without the endomorphism restriction), and it is mostly for this reason that we have found the double categorical perspective useful, if perhaps only psychologically.

We did give some thought to how one might define a 2-fold double category with non-endomorphism horizontal 1-cells and 2-cells, and while it seems like there might be a workable definition, it would require a very large increase in complexity. As we are mostly interested in the monads and comonads in a 2-fold double category, which are structures on endomorphism horizontal 1-cells, this restriction was of no concern to this work.

Now let us explicitly look at the 2-fold monoidal structure from the double categorical perspective. For any object C there are 2-cells

$$\begin{tikzcd}
C \arrow[r, "\bot_C\otimes\bot_C", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow m"] & C \arrow[d, equal] & C \arrow[r, "I_C", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow c"] & C \arrow[d, equal] & C \arrow[r, "I_C", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow j"] & C \arrow[d, equal] \\
C \arrow[r, "\bot_C"', "\shortmid" marking] & C & C \arrow[r, "I_C\odot I_C"', "\shortmid" marking] & C & C \arrow[r, "\bot_C"', "\shortmid" marking] & C
\end{tikzcd} \tag{12}$$

and for any four horizontal morphisms $W, X, Y, Z \colon C \nrightarrow C$ there is a 2-cell

$$\begin{tikzcd}
C \arrow[r, "(W\odot X)\otimes(Y\odot Z)", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow z"] & C \arrow[d, equal] \\
C \arrow[r, "(W\otimes Y)\odot(X\otimes Z)"', "\shortmid" marking] & C.
\end{tikzcd} \tag{13}$$

These are natural in the sense that, for any vertical morphism $f \colon C \to D$ we have an equality

$$\begin{tikzcd}
C \arrow[r, "\bot_C\otimes\bot_C", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow m"] & C \arrow[d, equal] & & C \arrow[r, "\bot_C\otimes\bot_C", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \bot_f\otimes\bot_f"] & C \arrow[d, "f"] \\
C \arrow[r, "\bot_C", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \bot_f"] & C \arrow[d, "f"] & = & D \arrow[r, "\bot_D\otimes\bot_D"', "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow m"] & D \arrow[d, equal] \\
D \arrow[r, "\bot_D"', "\shortmid" marking] & D & & D \arrow[r, "\bot_D"', "\shortmid" marking] & D
\end{tikzcd}$$

and similarly for c and j, and for any four 2-cells $\theta_1, \ldots, \theta_4$ of the apropriate form, we have an equality

$$\begin{tikzcd}
C \arrow[rr, "(W\odot X)\otimes(Y\odot Z)", "\shortmid"{marking}] \arrow[dd, equal] & & C \arrow[dd, equal] & & C \arrow[rr, "(W\odot X)\otimes(Y\odot Z)", "\shortmid"{marking}] \arrow[dd, "f"'] & & C \arrow[dd, "f"] \\
& \Downarrow z & & & & \Downarrow(\theta_1\odot\theta_2)\otimes(\theta_3\odot\theta_4) & \\
C \arrow[rr, "(W\otimes Y)\odot(X\otimes Z)", "\shortmid"{marking}] \arrow[dd, "f"'] & & C \arrow[dd, "f"] & = & C \arrow[rr, "(W'\odot X')\otimes(Y'\odot Z')"', "\shortmid"{marking}] \arrow[dd, equal] & & C \arrow[dd, equal] \\
& \Downarrow(\theta_1\otimes\theta_3)\odot(\theta_2\otimes\theta_4) & & & & \Downarrow z & \\
D \arrow[rr, "(W'\otimes Y')\odot(X'\otimes Z')"', "\shortmid"{marking}] & & D & & D \arrow[rr, "(W'\otimes Y')\odot(X'\otimes Z')"', "\shortmid"{marking}] & & D
\end{tikzcd}$$

Definition 4.4. A monad in a 2-fold double category $\mathbb{D}$ is a monad in $\mathbb{D}_\otimes$; a comonad in $\mathbb{D}$ is a comonad in $\mathbb{D}_\odot$. Furthermore, we define the categories $\mathrm{Mon}(\mathbb{D}) = \mathrm{Mon}(\mathbb{D}_\otimes)$ and $\mathrm{Comon}(\mathbb{D}) = \mathrm{Comon}(\mathbb{D}_\odot)$.

So a monad X and a comonad Y in $\mathbb{D}$ are given by 2-cells

$$\begin{tikzcd}
C \arrow[rr, "I_C", "\shortmid"{marking}] \arrow[dd, equal] & & C \arrow[dd, equal] & C \arrow[rr, "X\otimes X", "\shortmid"{marking}] \arrow[dd, equal] & & C \arrow[dd, equal] & C \arrow[rr, "X", "\shortmid"{marking}] \arrow[dd, equal] & & C \arrow[dd, equal] & C \arrow[rr, "X", "\shortmid"{marking}] \arrow[dd, equal] & & C \arrow[dd, equal] \\
& \Downarrow\eta & & & \Downarrow\mu & & & \Downarrow\epsilon & & & \Downarrow\delta & \\
C \arrow[rr, "X"', "\shortmid"{marking}] & & C & C \arrow[rr, "X"', "\shortmid"{marking}] & & C & C \arrow[rr, "\bot_C"', "\shortmid"{marking}] & & C & C \arrow[rr, "X\odot X"', "\shortmid"{marking}] & & C.
\end{tikzcd}$$

The categories $\mathrm{Mon}(\mathbb{D})$ and $\mathrm{Comon}(\mathbb{D})$ come naturally equipped with functors to $\mathcal{D}_0$, defined on objects and morphisms simply by applying p to the underlying 1-cells and 2-cells respectively. It turns out that the interaction between the $\otimes$ and $\odot$ compositions in the 2-fold double category structure is precisely what is needed to lift $\odot$ to $\mathrm{Mon}(\mathbb{D})$ and to lift $\otimes$ to $\mathrm{Comon}(\mathbb{D})$. In this way, we can define double categories $\mathbb{M}\mathrm{on}(\mathbb{D})$ and $\mathbb{C}\mathrm{omon}(\mathbb{D})$, both having $\mathcal{D}_0$ as vertical category.

These lifted compositions are defined as follows: Given two monads (C, X, η, μ) and (C, Y, η', μ') in $\mathbb{D}$, the horizontal composition

$$\begin{tikzcd}
C \arrow[r, "{(X,\eta,\mu)}", "\shortmid"{marking}] & C \arrow[r, "{(Y,\eta',\mu')}", "\shortmid"{marking}] & C
\end{tikzcd}$$

is the monoid with underlying horizontal 1-cell $X \odot Y$ and unit and multiplication 2-cells

$$\begin{tikzcd}
C \arrow[rr, "I_C", "\shortmid"{marking}] \arrow[dd, equal] & & C \arrow[dd, equal] & C \arrow[rr, "(X\odot Y)\otimes(X\odot Y)", "\shortmid"{marking}] \arrow[dd, equal] & & C \arrow[dd, equal] \\
& \Downarrow c & & & \Downarrow z & \\
C \arrow[rr, "I_C\odot I_C", "\shortmid"{marking}] \arrow[dd, equal] & & C \arrow[dd, equal] & C \arrow[rr, "(X\otimes X)\odot(Y\otimes Y)", "\shortmid"{marking}] \arrow[dd, equal] & & C \arrow[dd, equal] \\
& \Downarrow\eta\odot\eta' & & & \Downarrow\mu\odot\mu' & \\
C \arrow[rr, "X\odot Y"', "\shortmid"{marking}] & & C & C \arrow[rr, "X\odot Y"', "\shortmid"{marking}] & & C.
\end{tikzcd}$$

The unit for this composition is I_C, given the trivial monad structure with $\eta = \mu = \mathrm{id}_{I_C}$.

Similarly, the horizontal composition of two 2-cells in $\mathbb{M}\mathrm{on}(\mathbb{D})$ is the $\odot$ product of the underlying 2-cells in $\mathbb{D}$. The fact that this commutes with the unit and multiplication defined above follows from the naturality of c and z.

In this same way, we can define the horizontal composition of two 1-cells (X,ϵ,δ) and (Y,ϵ',δ') in $\mathbb{C}\mathrm{omon}(\mathbb{D})$ to be a comonad with underlying horizontal 1-cell $X\otimes Y$, with horizontal unit $\perp$ with the trivial comonad structure.

This allows us to define (ordinary) categories $\mathrm{Mon}(\mathbb{C}\mathrm{omon}(\mathbb{D}))$ and $\mathrm{Comon}(\mathbb{M}\mathrm{on}(\mathbb{D}))$. Furthermore, these two categories are equivalent, leading to the next definition.

Definition 4.5. A *bimonad* in a 2-fold double category $\mathbb{D}$ is a monad in $\mathbb{C}\mathrm{omon}(\mathbb{D})$, or equivalently a comonad in $\mathbb{M}\mathrm{on}(\mathbb{D})$. We can define a category of bimonads in $\mathbb{D}$ as

$$\mathrm{Bimon}(\mathbb{D}) := \mathrm{Mon}(\mathbb{C}\mathrm{omon}(\mathbb{D})) \simeq \mathrm{Comon}(\mathbb{M}\mathrm{on}(\mathbb{D}))$$

Concretely, a bimonad in $\mathbb{D}$ is a tuple $(X,\eta,\mu,\epsilon,\delta)$ where X is a horizontal 1-cell, (X,η,μ) is a monad and (X,ϵ,δ) is a comonad as above, such that four equations hold:

$$\begin{tikzcd} C \arrow[r, "I_C", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta"] & C \arrow[d, equal] \\ C \arrow[r, "X", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\delta"] & C \arrow[d, equal] \\ C \arrow[r, "X\odot X"', "\shortmid"{marking}] & C \end{tikzcd} = \begin{tikzcd} C \arrow[r, "I_C", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow c"] & C \arrow[d, equal] \\ C \arrow[r, "I_C\odot I_C", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta\odot\eta"] & C \arrow[d, equal] \\ C \arrow[r, "X\odot X"', "\shortmid"{marking}] & C \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "X\otimes X", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mu"] & C \arrow[d, equal] \\ C \arrow[r, "X", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\epsilon"] & C \arrow[d, equal] \\ C \arrow[r, "\perp_C"', "\shortmid"{marking}] & C \end{tikzcd} = \begin{tikzcd} C \arrow[r, "X\otimes X", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\epsilon\otimes\epsilon"] & C \arrow[d, equal] \\ C \arrow[r, "\perp_C\otimes\perp_C", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow m"] & C \arrow[d, equal] \\ C \arrow[r, "\perp_C"', "\shortmid"{marking}] & C \end{tikzcd}$$

$$\begin{tikzcd} C \arrow[r, "I_C", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta"] & C \arrow[d, equal] \\ C \arrow[r, "X", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\epsilon"] & C \arrow[d, equal] \\ C \arrow[r, "\perp_C"', "\shortmid"{marking}] & C \end{tikzcd} = \begin{tikzcd} C \arrow[r, "I_C", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow j"] & C \arrow[d, equal] \\ C \arrow[r, "\perp_C"', "\shortmid"{marking}] & C \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "X\otimes X", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\delta\otimes\delta"] & C \arrow[d, equal] \\ C \arrow[r, "(X\odot X)\otimes(X\odot X)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow z"] & C \arrow[d, equal] \\ C \arrow[r, "(X\otimes X)\odot(X\otimes X)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mu\odot\mu"] & C \arrow[d, equal] \\ C \arrow[r, "X\odot X"', "\shortmid"{marking}] & C \end{tikzcd} = \begin{tikzcd} C \arrow[r, "X\otimes X", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mu"] & C \arrow[d, equal] \\ C \arrow[r, "X", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\delta"] & C \arrow[d, equal] \\ C \arrow[r, "X\odot X"', "\shortmid"{marking}] & C \end{tikzcd} \tag{14}$$

A bimonoid morphism is simply a 2-cell which is simultaineously a monoid morphism and a comonoid morphism.

5 Cyclic 2-fold Double Categories

Recall the notion of a cyclic double category from [CGR12]. A cyclic double category $\mathbb{D}$ is a double category with an extra involutive operation. On objects and horizontal 1-cells $X\colon C \nrightarrow C$, this operation is written

$$\begin{tikzcd} C_1^\bullet \arrow[r, "X^\bullet", "|" marking] & C_2^\bullet \end{tikzcd}$$

and respects horizontal identities and composition. The involution takes any vertical 1-cell $f\colon C \to D$ to some $\sigma f\colon D^\bullet \to C^\bullet$, and any 2-cell

$$\begin{tikzcd} C_1 \arrow[r, "X", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\theta"] & C_2 \arrow[d, "g"] \\ D_1 \arrow[r, "Y"', "|" marking] & D_2 \end{tikzcd} \quad \text{to} \quad \begin{tikzcd} D_1^\bullet \arrow[r, "Y^\bullet", "|" marking] \arrow[d, "\sigma f"'] \arrow[dr, phantom, "\Downarrow\sigma\theta"] & D_2^\bullet \arrow[d, "\sigma g"] \\ C_1^\bullet \arrow[r, "X^\bullet"', "|" marking] & C_2^\bullet \end{tikzcd}$$

respecting vertical identities and composition.

Example 5.1. Let $\mathcal{D}$ be any 2-category with an involution $(-)^\bullet\colon \mathcal{D}^{\mathrm{co}} \to \mathcal{D}$, such as $\mathcal{C}at$ with $(-)^{\mathrm{op}}$. The double category $\mathbb{S}\mathrm{q}(\mathcal{D})$ has a sub-double category $\mathbb{L}\mathbf{Adj}(\mathcal{D})$, where the vertical 1-cells are restricted to the adjunctions—pointing in the direction of the left adjoint, say—but with no other restrictions. This double category has a natural cyclic action: if $F \dashv G$ is an adjunction, then $\sigma F = G^\bullet$, the left adjoint of the adjunction $G^\bullet \dashv F^\bullet$, and if θ is a 2-cell with mate ϕ then $\sigma\theta = \phi^\bullet$. For a clear summary of the mates correspondence and the cyclic action on $\mathbb{L}\mathbf{Adj}$, see [CGR12] Section 1.

Proposition 5.2. *Let $\mathbb{D}$ be a cyclic double category with arrow objects. For any object C, $(C^\bullet)^2 = (C^2)^\bullet$, as witnessed by*

$$\begin{tikzcd} (C^2)^\bullet \arrow[r, bend left, "\mathrm{cod}^\bullet", "|" marking] \arrow[r, bend right, "\mathrm{dom}^\bullet"', "|" marking] \arrow[r, phantom, "\Downarrow\sigma\kappa"] & C^\bullet \end{tikzcd}$$

For any vertical 1-cell F, the lift to arrow objects of $F^\bullet$ is $(\hat{F})^\bullet$, as witnessed by the 2-cells

$$\begin{tikzcd} (D^2)^\bullet \arrow[r, "\mathrm{cod}^\bullet", "|" marking] \arrow[d, "(\hat{F})^\bullet"'] \arrow[dr, phantom, "\Downarrow\sigma\gamma_0"] & D^\bullet \arrow[d, "F^\bullet"] \\ (C^2)^\bullet \arrow[r, "\mathrm{cod}^\bullet"', "|" marking] & C^\bullet \end{tikzcd} \qquad \begin{tikzcd} (D^2)^\bullet \arrow[r, "\mathrm{dom}^\bullet", "|" marking] \arrow[d, "(\hat{F})^\bullet"'] \arrow[dr, phantom, "\Downarrow\sigma\gamma_1"] & D^\bullet \arrow[d, "F^\bullet"] \\ (C^2)^\bullet \arrow[r, "\mathrm{dom}^\bullet"', "|" marking] & C^\bullet \end{tikzcd}$$

Proof. It is a very simple matter to verify the universal properties of Section 3.1 ☐

We will generalize this to a cyclic action on a 2-fold double category. Suppose that $\mathbb{D}$ is a 2-fold double category. A cyclic action, written as above, must satisfy the following:

- For every object C,

$$I_{C^\bullet} = (\perp_C)^\bullet \qquad \text{and} \qquad \perp_{C^\bullet} = (I_C)^\bullet .$$

- For every composable pair of horizontal 1-cells $X, Y\colon C \relbar\joinrel\mapstochar\joinrel\rightarrow C$,

$$(X \otimes Y)^\bullet = X^\bullet \odot Y^\bullet \qquad \text{and} \qquad (X \odot Y)^\bullet = X^\bullet \otimes Y^\bullet$$

- For every vertical 1-cell $f\colon C \to D$, there are equalities

$$\begin{tikzcd}
D^\bullet \arrow[r, "I_{D^\bullet}", "\shortmid"{marking}] \arrow[d, "\sigma f"'] \arrow[dr, phantom, "\Downarrow I_{\sigma f}"] & D^\bullet \arrow[d, "\sigma f"] \arrow[dr, phantom, "="] & D^\bullet \arrow[r, "(\perp_D)^\bullet", "\shortmid"{marking}] \arrow[d, "\sigma f"'] \arrow[dr, phantom, "\Downarrow \sigma \perp_f"] & D^\bullet \arrow[d, "\sigma f"] \\
C^\bullet \arrow[r, "I_{C^\bullet}"', "\shortmid"{marking}] & C^\bullet & C^\bullet \arrow[r, "(\perp_C)^\bullet"', "\shortmid"{marking}] & C^\bullet
\end{tikzcd}$$

$$\begin{tikzcd}
D^\bullet \arrow[r, "\perp_{D^\bullet}", "\shortmid"{marking}] \arrow[d, "\sigma f"'] \arrow[dr, phantom, "\Downarrow \perp_{\sigma f}"] & D^\bullet \arrow[d, "\sigma f"] \arrow[dr, phantom, "="] & D^\bullet \arrow[r, "(I_D)^\bullet", "\shortmid"{marking}] \arrow[d, "\sigma f"'] \arrow[dr, phantom, "\Downarrow \sigma I_f"] & D^\bullet \arrow[d, "\sigma f"] \\
C^\bullet \arrow[r, "\perp_{C^\bullet}"', "\shortmid"{marking}] & C^\bullet & C^\bullet \arrow[r, "(I_C)^\bullet"', "\shortmid"{marking}] & C^\bullet
\end{tikzcd}$$

- For every horizontally composable pair of 2-cells

$$\begin{tikzcd}
C \arrow[r, "X", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \theta"] & C \arrow[r, "Y", "\shortmid"{marking}] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow \phi"] & C \arrow[d, "f"] \\
D \arrow[r, "X'"', "\shortmid"{marking}] & D \arrow[r, "Y'"', "\shortmid"{marking}] & D
\end{tikzcd}$$

there are equalities

$$\begin{tikzcd}
D^\bullet \arrow[r, "(X' \otimes Y')^\bullet", "\shortmid"{marking}] \arrow[d, "\sigma f"'] \arrow[dr, phantom, "\Downarrow \sigma(\theta \otimes \phi)"] & D^\bullet \arrow[d, "\sigma f"] \arrow[dr, phantom, "="] & D^\bullet \arrow[r, "X'^\bullet \odot Y'^\bullet", "\shortmid"{marking}] \arrow[d, "\sigma f"'] \arrow[dr, phantom, "\Downarrow \sigma(\theta) \odot \sigma(\phi)"] & D^\bullet \arrow[d, "\sigma f"] \\
C^\bullet \arrow[r, "(X \otimes Y)^\bullet"', "\shortmid"{marking}] & C^\bullet & C^\bullet \arrow[r, "X^\bullet \odot Y^\bullet"', "\shortmid"{marking}] & C^\bullet
\end{tikzcd}$$

$$\begin{tikzcd}
D^\bullet \arrow[r, "(X' \odot Y')^\bullet", "\shortmid"{marking}] \arrow[d, "\sigma f"'] \arrow[dr, phantom, "\Downarrow \sigma(\theta \odot \phi)"] & D^\bullet \arrow[d, "\sigma f"] \arrow[dr, phantom, "="] & D^\bullet \arrow[r, "X'^\bullet \otimes Y'^\bullet", "\shortmid"{marking}] \arrow[d, "\sigma f"'] \arrow[dr, phantom, "\Downarrow \sigma(\theta) \otimes \sigma(\phi)"] & D^\bullet \arrow[d, "\sigma f"] \\
C^\bullet \arrow[r, "(X \odot Y)^\bullet"', "\shortmid"{marking}] & C^\bullet & C^\bullet \arrow[r, "X^\bullet \otimes Y^\bullet"', "\shortmid"{marking}] & C^\bullet
\end{tikzcd}$$

One nice consequence of this definition is that a cyclic action on a 2-fold double category $\mathbb{D}$ induces a cyclic action on the category of bimonoids $\mathrm{Bimon}(\mathbb{D})$.

Proposition 5.3. *Suppose $\mathbb{D}$ is a cyclic 2-fold double category. Then the category $\mathrm{Bimon}(\mathbb{D})$ of bimonoids in $\mathbb{D}$ carries a natural cyclic action (contravariant isomorphism).*

Proof. The involution $(-)^\bullet$ gives an isomorphism of double categories $\mathbb{D}_\otimes \cong \mathbb{D}_\odot^{\mathrm{op}}$. Therefore it also induces an isomorphism

$$\mathbb{M}\mathrm{on}(\mathbb{D}) = \mathbb{M}\mathrm{on}(\mathbb{D}_\otimes) \cong \mathbb{M}\mathrm{on}(\mathbb{D}_\odot^{\mathrm{op}}) \cong \mathbb{C}\mathrm{omon}(\mathbb{D}_\odot)^{\mathrm{op}} = \mathbb{C}\mathrm{omon}(\mathbb{D})^{\mathrm{op}}$$

as well as an isomorphism

$$\begin{aligned}\mathrm{Bimon}(\mathbb{D}) = \mathrm{Comon}(\mathbb{M}\mathrm{on}(\mathbb{D})) &\cong \mathrm{Comon}(\mathbb{C}\mathrm{omon}(\mathbb{D})^{\mathrm{op}}) \\ &\cong \mathrm{Mon}(\mathbb{C}\mathrm{omon}(\mathbb{D}))^{\mathrm{op}} = \mathrm{Bimon}(\mathbb{D})^{\mathrm{op}}.\end{aligned}$$

□

In more concrete terms, the involution takes a bimonoid $(X,\eta,\mu,\epsilon,\delta)$ to $(X,\eta,\mu,\epsilon,\delta)^\bullet = (X^\bullet,\epsilon^\bullet,\delta^\bullet,\eta^\bullet,\delta^\bullet)$, swapping the monoid and comonoid structures. This is again a bimonoid, as the top two equations of (14) are interchanged under the involution, while the bottom two equations are self-dual.

The action of the involution on bimonoid morphisms can be broken down as in the following lemma.

Lemma 5.4. *Let $(X,\eta,\mu,\epsilon,\delta)$ and $(Y,\eta',\mu',\epsilon',\delta')$ be bimonoids in a cyclic 2-fold double category $\mathbb{D}$, and let ϕ be a 2-cell in $\mathbb{D}$*

$$\begin{tikzcd} C \arrow[r, "X", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[d, "f"] \\ D \arrow[r, "Y"', "\shortmid"{marking}] & D. \end{tikzcd}$$

Then (f,ϕ) is a monoid morphism $X \to Y$ if and only if $(\sigma f,\phi^\bullet)$ is a comonoid morphism $Y^\bullet \to X^\bullet$. Dually, ϕ is a comonoid morphism $X \to Y$ if and only if $\phi^\bullet$ is a monoid morphism $Y^\bullet \to X^\bullet$.

Proof. Simply notice that the involution takes equations (9) and (10) to the equations defining a comonad morphism in $\mathbb{D}$. □

This immediately implies a useful characterization of bimonoid morphisms.

Corollary 5.5. *Given bimonoids $(X,\eta,\mu,\epsilon,\delta)$ and $(Y,\eta',\mu',\epsilon',\delta')$ in a cyclic 2-fold double category $\mathbb{D}$, a bimonoid morphism $X \to Y$ consists of:*

- Either *a monoid morphism $X \to Y$* or *a comonoid morphism $Y^\bullet \to X^\bullet$*, and
- Either *a comonoid morphism $X \to Y$* or *a monoid morphism $Y^\bullet \to X^\bullet$*.

6 Functorial Factorizations

Let $\mathbb{D}$ be a cyclic double category, and assume it has arrow objects in the sense of Section 3.1. In this section, we will define a 2-fold double category $\mathbb{FF}(\mathbb{D})$ of functorial factorizations in $\mathbb{D}$, as follows:

- The objects and vertical 1-cells are the same as in $\mathbb{D}$.
- Horizontal 1-cells $C \mathrel{\mapstochar\rightarrow} C$ in $\mathbb{FF}(\mathbb{D})$ are tuples (E, η, ϵ), where $E\colon C^2 \to C$ is a horizontal 1-cell in $\mathbb{D}$, and

  $$\begin{tikzcd} C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "E"'] \arrow[r, phantom, "\Downarrow\eta"] & C \end{tikzcd} \qquad \begin{tikzcd} C^2 \arrow[r, bend left, "E"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\epsilon"] & C \end{tikzcd}$$

  are 2-cells in $\mathbb{D}$ such that

  $$\begin{tikzcd} C^2 \arrow[r, bend left=60, "\mathrm{dom}"] \arrow[r, "E" description] \arrow[r, bend right=60, "\mathrm{cod}"'] \arrow[r, phantom, shift left=3, "\Downarrow\eta"] \arrow[r, phantom, shift right=3, "\Downarrow\epsilon"] & C \end{tikzcd} = \begin{tikzcd} C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\kappa"] & C. \end{tikzcd}$$

  By the universal property of C^2, this also determines horizontal 1-cells $L, R\colon C^2 \to C^2$ such that $\mathrm{dom} \circ L = \mathrm{dom}$, $\mathrm{cod} \circ R = \mathrm{cod}$, $\mathrm{cod} \circ L = \mathrm{dom} \circ R = E$, $\kappa \circ L = \eta$, and $\kappa \circ R = \epsilon$, and 2-cells

  $$\begin{tikzcd} C^2 \arrow[r, bend left, "L"] \arrow[r, bend right, "\mathrm{id}"'] \arrow[r, phantom, "\Downarrow\vec{\epsilon}"] & C^2. \end{tikzcd} \qquad \begin{tikzcd} C^2 \arrow[r, bend left, "\mathrm{id}"] \arrow[r, bend right, "R"'] \arrow[r, phantom, "\Downarrow\vec{\eta}"] & C^2. \end{tikzcd}$$

  such that $\mathrm{dom} \circ \vec{\epsilon} = \mathrm{id}_{\mathrm{dom}}$, $\mathrm{cod} \circ \vec{\epsilon} = \epsilon$, $\mathrm{dom} \circ \vec{\eta} = \eta$, and $\mathrm{cod} \circ \vec{\eta} = \mathrm{id}_{\mathrm{cod}}$.
- The horizontal composition $(E_1, \eta_1, \epsilon_1) \otimes (E_2, \eta_2, \epsilon_2)$ of two horizontal 1-cells

  $$\begin{tikzcd} C \arrow[r, "{(E_1,\eta_1,\epsilon_1)}", "\shortmid" marking] & C \arrow[r, "{(E_2,\eta_2,\epsilon_2)}", "\shortmid" marking] & C \end{tikzcd}$$

  in $\mathbb{FF}(\mathbb{D})$ is a horizontal 1-cell $(E_{1\otimes2}, \eta_{1\otimes2}, \epsilon_{1\otimes2})$, where

  $$E_{1\otimes2} = \begin{tikzcd} C^2 \arrow[r, "R_1"] & C^2 \arrow[r, "E_2"] & C \end{tikzcd}$$

  $$\eta_{1\otimes2} = \begin{tikzcd} C^2 \arrow[r, bend left, "\mathrm{id}"] \arrow[r, bend right, "R_1"'] \arrow[r, phantom, "\Downarrow\vec{\eta}_1"] & C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "E_2"'] \arrow[r, phantom, "\Downarrow\eta_2"] & C \end{tikzcd}$$

  $$\epsilon_{1\otimes2} = \begin{tikzcd} C^2 \arrow[r, "R_1"] & C^2 \arrow[r, bend left, "E_2"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\epsilon_2"] & C \end{tikzcd}$$

  which also determines that $R_{1\otimes2} = R_2 \circ R_1$.

- The horizontal unit I_C for $\otimes$ is $(\mathrm{dom}, \mathrm{id}, \kappa)$.
- The second horizontal composition $(E_1, \eta_1, \epsilon_1) \odot (E_2, \eta_2, \epsilon_2)$ is a horizontal 1-cell $(E_{1\odot 2}, \eta_{1\odot 2}, \epsilon_{1\odot 2})$, where

$$E_{1\odot 2} = \begin{tikzcd} C^2 \arrow[r, "L_1"] & C^2 \arrow[r, "E_2"] & C \end{tikzcd}$$

$$\eta_{1\odot 2} = \begin{tikzcd} C^2 \arrow[r, "L_1"] & C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "E_2"'] \arrow[r, phantom, "\Downarrow\eta_2"] & C \end{tikzcd}$$

$$\epsilon_{1\odot 2} = \begin{tikzcd} C^2 \arrow[r, bend left, "L_1"] \arrow[r, bend right, "\mathrm{id}"'] \arrow[r, phantom, "\Downarrow\vec{\epsilon_1}"] & C^2 \arrow[r, bend left, "E_2"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\epsilon_2"] & C \end{tikzcd}$$

  which also determines that $L_{1\odot 2} = L_2 \circ L_1$.
- The horizontal unit $\perp_C$ for $\odot$ is $(\mathrm{cod}, \kappa, \mathrm{id})$.
- 2-cells

$$\begin{tikzcd} C \arrow[r, "{(E_1,\eta_1,\epsilon_1)}", "\mid" marking] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\theta"] & C \arrow[d, "F"] \\ D \arrow[r, "{(E_2,\eta_2,\epsilon_2)}"', "\mid" marking] & D \end{tikzcd}$$

  in $\mathbb{FF}(\mathbb{D})$ are given by 2-cells

$$\begin{tikzcd} C^2 \arrow[r, "E_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta"] & C \arrow[d, "F"] \\ D^2 \arrow[r, "E_2"'] & D \end{tikzcd}$$

  in $\mathbb{D}$ such that

$$\begin{tikzcd} C^2 \arrow[r, "E_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta"] & C \arrow[d, "F"] & {} \arrow[d, phantom, "="] & C^2 \arrow[r, bend left=60, "E_1"] \arrow[r, phantom, bend left=30, "\Downarrow\epsilon_1"] \arrow[r, "\mathrm{cod}"'] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_0"] & C \arrow[d, "F"] \\ D^2 \arrow[r, "E_2"] \arrow[r, bend right=60, "\mathrm{cod}"'] \arrow[r, phantom, bend right=30, "\Downarrow\epsilon_2"] & D & {} & D^2 \arrow[r, "\mathrm{cod}"'] & D \end{tikzcd} \tag{15}$$

  and

$$\begin{tikzcd} C^2 \arrow[r, "\mathrm{dom}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_1"] & C \arrow[d, "F"] & {} \arrow[d, phantom, "="] & C^2 \arrow[r, bend left=60, "\mathrm{dom}"] \arrow[r, phantom, bend left=30, "\Downarrow\eta_1"] \arrow[r, "E_1"'] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta"] & C \arrow[d, "F"] \\ D^2 \arrow[r, "\mathrm{dom}"] \arrow[r, bend right=60, "E_2"'] \arrow[r, phantom, bend right=30, "\Downarrow\eta_2"] & D & {} & D^2 \arrow[r, "E_2"'] & D \end{tikzcd} \tag{16}$$

This also determines unique 2-cells

$$\begin{tikzcd}
C^2 \arrow[r, "R_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow \theta^R"] & C^2 \arrow[d, "\hat{F}"] \\
D^2 \arrow[r, "R_2"'] & D^2
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
C^2 \arrow[r, "L_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow \theta^L"] & C^2 \arrow[d, "\hat{F}"] \\
D^2 \arrow[r, "L_2"'] & D^2
\end{tikzcd}$$

such that composing horizontally with γ_0 or γ_1 gives γ_0, γ_1, or θ as appropriate. For instance:

$$\begin{tikzcd}
C^2 \arrow[r, "R_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow \theta^R"] & C^2 \arrow[r, "\mathrm{dom}"] \arrow[d, "\hat{F}"] \arrow[dr, phantom, "\Downarrow \gamma_1"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "R_2"'] & D^2 \arrow[r, "\mathrm{dom}"'] & D
\end{tikzcd} = \begin{tikzcd}
C^2 \arrow[r, "E_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow \theta"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "E_2"'] & D
\end{tikzcd}$$

- Given a pair of composable 2-cells in $\mathbb{FF}(\mathbb{D})$ as in

$$\begin{tikzcd}
C \arrow[r, "{(E_1,\eta_1,\epsilon_1)}", "|" marking] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow \theta_1"] & C \arrow[r, "{(E_2,\eta_2,\epsilon_2)}", "|" marking] \arrow[d, "F"] \arrow[dr, phantom, "\Downarrow \theta_2"] & C \arrow[d, "F"] \\
D \arrow[r, "{(E_1',\eta_1',\epsilon_1')}"', "|" marking] & D \arrow[r, "{(E_2',\eta_2',\epsilon_2')}"', "|" marking] & D
\end{tikzcd}$$

the composite $\theta_1 \otimes \theta_2$ is given by

$$\begin{tikzcd}
C^2 \arrow[r, "R_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow \theta_1^R"] & C^2 \arrow[r, "E_2"] \arrow[d, "\hat{F}"] \arrow[dr, phantom, "\Downarrow \theta_2"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "R_1'"'] & D^2 \arrow[r, "E_2'"'] & D
\end{tikzcd}$$

while the composite $\theta_1 \odot \theta_2$ is given by

$$\begin{tikzcd}
C^2 \arrow[r, "L_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow \theta_1^L"] & C^2 \arrow[r, "E_2"] \arrow[d, "\hat{F}"] \arrow[dr, phantom, "\Downarrow \theta_2"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "L_1'"'] & D^2 \arrow[r, "E_2'"'] & D
\end{tikzcd}$$

It is a straightforward exercise to check that these definitions satisfy equations (15) and (16). To illustrate, we will demonstrate that $\theta_1 \otimes \theta_2$

satisfies (15):

$$\begin{tikzcd}
C^2 \arrow[r, "E_{1\otimes 2}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta_1\otimes\theta_2"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "E_{1'\otimes 2'}"] \arrow[r, bend right=50, "\mathrm{cod}"'] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\epsilon_{1'\otimes 2'}"] & D
\end{tikzcd}
=
\begin{tikzcd}
C^2 \arrow[r, "R_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta_1^R"] & C^2 \arrow[r, "E_2"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta_2"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "R_1'"'] & D^2 \arrow[r, "E_2'"] \arrow[r, bend right=50, "\mathrm{cod}"'] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\epsilon_2'"] & D
\end{tikzcd}$$

$$=
\begin{tikzcd}
C^2 \arrow[r, "R_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta_1^R"] & C^2 \arrow[r, "\mathrm{cod}"'] \arrow[r, bend left=50, "E_2"] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\epsilon_2"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_0"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "R_1'"'] & D^2 \arrow[r, "\mathrm{cod}"'] & D
\end{tikzcd}$$

$$=
\begin{tikzcd}
C^2 \arrow[r, "\mathrm{cod}"'] \arrow[r, bend left=50, "E_{1\otimes 2}"] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\epsilon_{1\otimes 2}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_0"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "\mathrm{cod}"'] & D
\end{tikzcd}$$

Example 6.1. Functorial factorizations in the double category $\mathbb{D} = \mathbb{S}\mathrm{q}(\mathcal{C}\mathrm{at})$ of squares in the 2-category of categories are precisely functorial factorizations as defined in Section 2.2.

It is straightforward to check that $\otimes$ and $\odot$ are each associative and unital. It takes more work to provide the compatibility between $\otimes$ and $\odot$, which is the content of the proof of the next proposition.

Proposition 6.2. $\mathbb{FF}(\mathbb{D})$ *has the structure of a 2-fold double category.*

Proof. The primary structure of $\mathbb{FF}(\mathbb{D})$ was given in the first part of this section. What is left is to provide the coherence data (12) and (13).

First, note that I_C is initial in the sense that, given any vertical morphism $F: C \to D$ and any functorial factorization (E, η, ϵ) on D, there is a unique 2-cell

$$\begin{tikzcd}
C \arrow[r, "I_C", "\shortmid" marking] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow"] & C \arrow[d, "F"] \\
D \arrow[r, "{(E,\eta,\epsilon)}"', "\shortmid" marking] & D
\end{tikzcd}$$

given by

$$\begin{tikzcd}
C^2 \arrow[r, "\mathrm{dom}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_1"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "\mathrm{dom}"] \arrow[r, bend right=50, "E"'] \arrow[r, phantom, bend right=25, "\Downarrow\eta"] & D.
\end{tikzcd}$$

Similarly, $\perp_C$ is terminal. Thus there is only one possible way to define the 2-cells m, c, and j, and naturality and all other coherence equations follows immediately from this uniqueness.

We still need to construct the 2-cell z, which will take some work. We begin by defining 2-cells

$$\begin{tikzcd}
C \arrow[r, "E_1 \odot E_2", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow p_{E_1,E_2}"] & C \arrow[d, equal] \\
C \arrow[r, "E_1"', "\shortmid" marking] & C
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
C \arrow[r, "E_1", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow i_{E_1,E_2}"] & C \arrow[d, equal] \\
C \arrow[r, "E_1 \otimes E_2"', "\shortmid" marking] & C.
\end{tikzcd}$$

for any pair of functorial factorizations. The 2-cell p is given by the underlying 2-cell in $\mathbb{D}$

$$\begin{tikzcd}
C^2 \arrow[r, "L_1"] & C^2 \arrow[r, bend left, "E_2"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\epsilon_2"] & C
\end{tikzcd}$$

and i is given by

$$\begin{tikzcd}
C^2 \arrow[r, "R_1"] & C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "E_2"'] \arrow[r, phantom, "\Downarrow\eta_2"] & C.
\end{tikzcd}$$

To illustrate the verification that these give well-defined 2-cells in $\mathbb{FF}(\mathbb{D})$, we will show that i satisfies (15) (keep in mind that when F is an identity, γ_0 and γ_1 are also identities):

$$\begin{tikzcd}
C^2 \arrow[r, "L_1"] & C^2 \arrow[r, bend left=70, "\mathrm{dom}"] \arrow[r, "E_2" description] \arrow[r, bend right=70, "\mathrm{cod}"'] \arrow[r, phantom, shift left=4, "\Downarrow\eta_2"] \arrow[r, phantom, shift right=4, "\Downarrow\epsilon_2"] & C & = & C^2 \arrow[r, "L_1"] & C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\kappa"] & C \\
& & & = & C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "E_1"'] \arrow[r, phantom, "\Downarrow\eta_1"] & C. &
\end{tikzcd}$$

Moreover, it is straightforward to check that i and p are natural families of 2-cells. Specifically, for any pair of 2-cells θ_1 and θ_2

$$\begin{tikzcd}
C \arrow[rr, "E_1\odot E_2", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow p_{E_1,E_2}"] & & C \arrow[d, equal] & & C \arrow[rr, "E_1\odot E_2", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[drr, phantom, "\Downarrow \theta_1\odot\theta_2"] & & C \arrow[d, "F"] \\
C \arrow[rr, "E_1", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[drr, phantom, "\Downarrow \theta_1"] & & C \arrow[d, "F"] & = & D \arrow[rr, "E_1'\odot E_2'", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow p_{E_1',E_2'}"] & & D \arrow[d, equal] \\
D \arrow[rr, "E_1'"', "\shortmid"{marking}] & & D & & D \arrow[rr, "E_1'"', "\shortmid"{marking}] & & D
\end{tikzcd}$$

$$\begin{tikzcd}
C \arrow[rr, "E_1", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow i_{E_1,E_2}"] & & C \arrow[d, equal] & & C \arrow[rr, "E_1", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[drr, phantom, "\Downarrow \theta_1"] & & C \arrow[d, "F"] \\
C \arrow[rr, "E_1\otimes E_2", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[drr, phantom, "\Downarrow \theta_1\otimes\theta_2"] & & C \arrow[d, "F"] & = & D \arrow[rr, "E_1'", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow i_{E_1',E_2'}"] & & D \arrow[d, equal] \\
D \arrow[rr, "E_1'\otimes E_2'"', "\shortmid"{marking}] & & D & & D \arrow[rr, "E_1'\otimes E_2'"', "\shortmid"{marking}] & & D
\end{tikzcd}$$

As with any 2-cell in $\mathbb{FF}(\mathbb{D})$, p and i induce 2-cells in $\mathbb{D}$

$$\begin{tikzcd}
C^2 \arrow[r, bend left, "R_{1\odot 2}"] \arrow[r, bend right, "R_1"'] \arrow[r, phantom, "\Downarrow p^R"] & C^2 & \text{and} & C^2 \arrow[r, bend left, "L_1"] \arrow[r, bend right, "L_{1\otimes 2}"'] \arrow[r, phantom, "\Downarrow i^L"] & C^2.
\end{tikzcd}$$

such that

$$\begin{tikzcd}
C^2 \arrow[r, bend left, "R_{1\odot 2}"] \arrow[r, bend right, "R_1"'] \arrow[r, phantom, "\Downarrow p^R"] & C^2 \arrow[r, "\mathrm{dom}"] & C & = & C^2 \arrow[r, "L_1"] & C^2 \arrow[r, bend left, "E_2"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow \epsilon_2"] & C
\end{tikzcd} \tag{17}$$

$$\begin{tikzcd}
C^2 \arrow[r, bend left, "L_1"] \arrow[r, bend right, "L_{1\otimes 2}"'] \arrow[r, phantom, "\Downarrow i^L"] & C^2 \arrow[r, "\mathrm{cod}"] & C & = & C^2 \arrow[r, "R_1"] & C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "E_2"'] \arrow[r, phantom, "\Downarrow \eta_2"] & C
\end{tikzcd} \tag{18}$$

Now suppose given three functorial factorizations E_1, E_2, E_3 on an object C. We define a 2-cell in $\mathbb{D}$

$$\begin{tikzcd}
 & C^2 \arrow[dr, "L_3"] & \\
C^2 \arrow[ur, "R_{1\odot 2}"] \arrow[dr, "L_{1\otimes 3}"'] & \Downarrow w & C^2 \\
 & C^2 \arrow[ur, "R_2"'] &
\end{tikzcd}$$

such that

$$\begin{tikzcd}
 & C^2 \arrow[dr, "L_3"] \arrow[dd, phantom, "\Downarrow w"] & & & & & & \\
C^2 \arrow[ur, "R_{1\odot 2}"] \arrow[dr, "L_{1\otimes 3}"'] & & C^2 \arrow[r, "\mathrm{dom}"] & C & = & C^2 \arrow[r, bend left, "L_1"] \arrow[r, bend right, "L_{1\otimes 3}"'] \arrow[r, phantom, "\Downarrow i^L"] & C^2 \arrow[r, "E_2"] & C \\
 & C^2 \arrow[ur, "R_2"'] & & & & & &
\end{tikzcd} \tag{19}$$

$$\begin{tikzcd}
 & C^2 \arrow[dr, "L_3"] \arrow[dd, phantom, "\Downarrow w"] & & & & & & \\
C^2 \arrow[ur, "R_{1\odot 2}"] \arrow[dr, "L_{1\otimes 3}"'] & & C^2 \arrow[r, "\mathrm{cod}"] & C & = & C^2 \arrow[r, bend left, "R_{1\odot 2}"] \arrow[r, bend right, "R_1"'] \arrow[r, phantom, "\Downarrow p^R"] & C^2 \arrow[r, "E_3"] & C. \\
 & C^2 \arrow[ur, "R_2"'] & & & & & &
\end{tikzcd} \tag{20}$$

Using the universal property for C^2, it suffices to check that

$$\begin{tikzcd}
C^2 \arrow[r, bend left=60, "L_1"] \arrow[r, "L_{1\otimes 3}"'] \arrow[r, phantom, bend left=30, "\Downarrow i^L"] & C^2 \arrow[r, "E_2"] \arrow[r, bend right=60, "\mathrm{cod}"'] \arrow[r, phantom, bend right=30, "\Downarrow \epsilon_2"] & C & = & C^2 \arrow[r, "R_{1\odot 2}"] \arrow[r, bend right=60, "R_1"'] \arrow[r, phantom, bend right=30, "\Downarrow p^R"] & C^2 \arrow[r, "E_3"'] \arrow[r, bend left=60, "\mathrm{dom}"] \arrow[r, phantom, bend left=30, "\Downarrow \eta_3"] & C
\end{tikzcd}$$

and a quick check using equations (17) and (18) shows that both are equal to

$$\begin{tikzcd}
 & C^2 \arrow[dr, bend left=60, "E_2"] \arrow[dr, "\mathrm{cod}"'] \arrow[dr, phantom, bend left=30, "\Downarrow \epsilon_2"] & \\
C^2 \arrow[ur, "L_1"] \arrow[dr, "R_1"'] & & C \\
 & C^2 \arrow[ur, "\mathrm{dom}"] \arrow[ur, bend right=60, "E_3"'] \arrow[ur, phantom, bend right=30, "\Downarrow \eta_3"] &
\end{tikzcd}$$

where the inner diamond is the equality $\mathrm{cod}\, L_1 = \mathrm{dom}\, R_1 = E_1$.

We also check that w is natural with respect to 2-cells in $\mathbb{FF}(\mathbb{D})$ in the following sense: given three 2-cells θ_1, θ_2, and θ_3, there is an equality

$$\begin{tikzcd}
 & C^2 \arrow[dr, "L_3"] \arrow[dd, phantom, "\Downarrow w"] & & & & C^2 \arrow[dr, "L_3"] \arrow[dd, "\hat{F}"] & \\
C^2 \arrow[ur, "R_{1\odot 2}"] \arrow[dr, "L_{1\otimes 3}"] \arrow[dd, "\hat{F}"'] \arrow[ddr, phantom, "\Downarrow(\theta_1\otimes\theta_2)^L"] & & C^2 \arrow[dd, "\hat{F}"] & & C^2 \arrow[ur, "R_{1\odot 2}"] \arrow[dd, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow(\theta_1\odot\theta_2)^R"] & & C^2 \arrow[dd, "\hat{F}"] \arrow[dl, phantom, "\Downarrow\theta_3^L"] \\
 & C^2 \arrow[ur, "R_2"] \arrow[dd, "\hat{F}"] \arrow[dr, phantom, "\Downarrow\theta_2^R"] & & = & & D^2 \arrow[dr, "L'_3"] \arrow[dd, phantom, "\Downarrow w"] & \\
D^2 \arrow[dr, "L'_{1\otimes 3}"'] & & D^2 & & D^2 \arrow[ur, "R'_{1\odot 2}"'] \arrow[dr, "L'_{1\otimes 3}"'] & & D^2. \\
 & D^2 \arrow[ur, "R'_2"'] & & & & D^2 \arrow[ur, "R'_2"'] &
\end{tikzcd}$$

To verify this equation, it suffices to check equality upon right composition with γ_0 and γ_1. We will illustrate the γ_1 case, making use of the naturality of i:

$$\begin{tikzcd}
& C^2 \arrow[rd, bend left, "L_3"] \arrow[dd, phantom, "\Downarrow w"] & & \\
C^2 \arrow[ru, bend left, "R_{1\odot 2}"] \arrow[rd, bend right, "L_{1\otimes 3}"] \arrow[ddd, "\hat{F}"'] \arrow[rddd, phantom, "\Downarrow(\theta_1\otimes\theta_3)^L"] & & C^2 \arrow[r, "\mathrm{dom}"] \arrow[ddd, "\hat{F}"] \arrow[rddd, phantom, "\Downarrow\gamma_1"] & C \arrow[ddd, "F"] \\
& C^2 \arrow[ru, bend right, "R_2"] \arrow[ddd, "\hat{F}"] \arrow[r, phantom, "\Downarrow\theta_2^R"] & & \\
& & & \\
D^2 \arrow[rd, bend right, "L'_{1\otimes 3}"'] & & D^2 \arrow[r, "\mathrm{dom}"'] & D \\
& D^2 \arrow[ru, bend right, "R'_2"'] & &
\end{tikzcd}
=
\begin{tikzcd}
C^2 \arrow[r, bend left=50, "L_1"] \arrow[r, "L_{1\otimes 3}"'] \arrow[r, phantom, bend left=25, "\Downarrow i^L"] \arrow[d, "\hat{F}"'] \arrow[rd, phantom, "\Downarrow(\theta_1\otimes\theta_3)^L"] & C^2 \arrow[r, "E_2"] \arrow[d, "\hat{F}"] \arrow[rd, phantom, "\Downarrow\theta_2"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "L_{1'\otimes 3'}"'] & D^2 \arrow[r, "E'_2"'] & D
\end{tikzcd}$$

$$=
\begin{tikzcd}
C^2 \arrow[r, "L_1"] \arrow[d, "\hat{F}"'] \arrow[rd, phantom, "\Downarrow\theta_1^L"] & C^2 \arrow[r, "E_2"] \arrow[d, "\hat{F}"] \arrow[rd, phantom, "\Downarrow\theta_2"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "L'_1"] \arrow[r, bend right=50, "L'_{1\otimes 3}"'] \arrow[r, phantom, bend right=25, "\Downarrow i^L"] & D^2 \arrow[r, "E'_2"'] & D
\end{tikzcd}
=
\begin{tikzcd}
& C^2 \arrow[rd, bend left, "L_3"] \arrow[dd, "\hat{F}"] \arrow[rdd, phantom, "\Downarrow\theta_3^L"] & & \\
C^2 \arrow[ru, bend left, "R_{1\odot 2}"] \arrow[dd, "\hat{F}"'] \arrow[rd, phantom, "\Downarrow(\theta_1\odot\theta_2)^R"] & & C^2 \arrow[r, "\mathrm{dom}"] \arrow[dd, "\hat{F}"] \arrow[rdd, phantom, "\Downarrow\gamma_1"] & C \arrow[dd, "F"] \\
& D^2 \arrow[rd, bend left, "L'_3"] \arrow[dd, phantom, "\Downarrow w"] & & \\
D^2 \arrow[ru, bend left, "R'_{1\odot 2}"] \arrow[rd, bend right, "L'_{1\otimes 3}"'] & & D^2 \arrow[r, "\mathrm{dom}"'] & D. \\
& D^2 \arrow[ru, bend right, "R'_2"'] & &
\end{tikzcd}$$

Finally, given four functorial factorizations E_1, E_2, E_3, E_4 on an object C, we define the 2-cell

$$\begin{tikzcd}
C \arrow[r, "(1\odot 2)\otimes(3\odot 4)", "\shortmid"{marking}] \arrow[d, equals] \arrow[rd, phantom, "\Downarrow z_{1,2,3,4}"] & C \arrow[d, equals] \\
C \arrow[r, "(1\otimes 3)\odot(2\otimes 4)"', "\shortmid"{marking}] & C
\end{tikzcd}$$

in $\mathbb{FF}(\mathbb{D})$, where $(1\odot 2)$ is shorthand for $(E_1,\eta_1,\epsilon_1)\odot(E_2,\eta_2,\epsilon_2)$, to have the underlying 2-cell in $\mathbb{D}$

$$\begin{tikzcd}
& C^2 \arrow[rd, bend left, "L_3"] \arrow[dd, phantom, "\Downarrow w"] & & \\
C^2 \arrow[ru, bend left, "R_{1\odot 2}"] \arrow[rd, bend right, "L_{1\otimes 3}"'] & & C^2 \arrow[r, "E_4"] & C. \\
& C^2 \arrow[ru, bend right, "R_2"'] & &
\end{tikzcd}$$

The naturality of z follows immediately from that of w, but we still need to check that this satisfies equations (15) and (16). We will leave the details to the reader, but note that (16) comes down to the verification of the equality

$$\begin{tikzcd}
& C^2 \arrow[rd, bend left, "L_3"] \arrow[dd, phantom, "\Downarrow w"] & & \\
C^2 \arrow[ru, bend left=70, "\mathrm{id}"] \arrow[ru, "R_{1\odot 2}"'] \arrow[ru, phantom, bend left=35, "\Downarrow\vec{\eta}_{1\odot 2}"] \arrow[rd, bend right, "L_{1\otimes 3}"'] & & C^2 \arrow[r, bend left=50, "\mathrm{dom}"] \arrow[r, "E_4"'] \arrow[r, phantom, bend left=25, "\Downarrow\eta_4"] & C \\
& C^2 \arrow[ru, bend right, "R_2"'] & &
\end{tikzcd}
=
\begin{tikzcd}
C^2 \arrow[r, "L_{1\otimes 3}"] & C^2 \arrow[r, bend left, "\mathrm{id}"] \arrow[r, bend right, "R_2"'] \arrow[r, phantom, "\Downarrow\vec{\eta}_2"] & C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "E_4"'] \arrow[r, phantom, "\Downarrow\eta_4"] & C,
\end{tikzcd}$$

which follows from equation (19) and the fact that $\mathrm{dom}\circ i^L = \mathrm{id}_{\mathrm{dom}}$. □

Lemma 6.3. *There is a strict double functor* $R\colon \mathbb{FF}(\mathbb{D})_{\otimes} \to \mathbb{D}$ *whose behavior on 2-cells is*

$$\begin{tikzcd}
C \arrow[r, "{(E_1,\eta_1,\epsilon_1)}", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\theta"] & C \arrow[d, "F"] & & C^2 \arrow[r, "R_1", "\shortmid"{marking}] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta^R"] & C^2 \arrow[d, "\hat{F}"] \\
D \arrow[r, "{(E_2,\eta_2,\epsilon_2)}"', "\shortmid"{marking}] & D & \mapsto & D^2 \arrow[r, "R_2"', "\shortmid"{marking}] & D^2
\end{tikzcd}$$

and a double functor $L\colon \mathbb{FF}(\mathbb{D})_{\odot} \to \mathbb{D}$ *whose behavior on 2-cells is*

$$\begin{tikzcd}
C \arrow[r, "{(E_1,\eta_1,\epsilon_1)}", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\theta"] & C \arrow[d, "F"] & & C^2 \arrow[r, "L_1", "\shortmid"{marking}] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta^L"] & C^2 \arrow[d, "\hat{F}"] \\
D \arrow[r, "{(E_2,\eta_2,\epsilon_2)}"', "\shortmid"{marking}] & D & \mapsto & D^2 \arrow[r, "L_2"', "\shortmid"{marking}] & D^2
\end{tikzcd}$$

Corollary 6.4. *R and L respectively induce functors* $\mathrm{Mon}(\mathbb{FF}(\mathbb{D})) \to \mathrm{Mon}(\mathbb{D})$ *and* $\mathrm{Comon}(\mathbb{FF}(\mathbb{D})) \to \mathrm{Comon}(\mathbb{D})$.

Up to this point, we have demonstrated that given any double category $\mathbb{D}$ having arrow objects, there is a 2-fold double category $\mathbb{FF}(\mathbb{D})$ of functorial factorizations in $\mathbb{D}$. The last thing we want to say about this construction is that a cyclic action on $\mathbb{D}$ lifts to one on $\mathbb{FF}(\mathbb{D})$, and hence also to one on $\mathrm{Bimon}(\mathbb{FF}(\mathbb{D}))$.

The cyclic action on objects and vertical morphisms is given directly by that on $\mathbb{D}$. Given a horizontal 1-cell (E,η,ϵ) on an object C, we define the 1-cell $(E,\eta,\epsilon)^\bullet$ on $C^\bullet$ to be $(E^\bullet,\epsilon^\bullet,\eta^\bullet)$. This also implies that the cyclic action swaps L and R for any given functorial factorization.

A quick look at the definitions of the two horizontal compositions is now enough to see that for any two functorial factorizations E_1 and E_2, we have

$$(E_1 \otimes E_2)^\bullet = E_1^\bullet \odot E_2^\bullet \qquad \text{and} \qquad (E_1 \odot E_2)^\bullet = E_1^\bullet \otimes E_2^\bullet$$

Similarly, the cyclic action on 2-cells in $\mathbb{FF}(\mathbb{D})$ is given by the cyclic action in $\mathbb{D}$ on the underlying 2-cell. This gives a valid 2-cell in $\mathbb{FF}(\mathbb{D})$ since the cyclic action simply swaps the equations (15) and (16).

7 Algebraic Weak Factorization Systems

For this section, let $\mathbb{D} = \mathbb{S}\mathrm{q}(\mathcal{D})$ be the double category of squares in a 2-category $\mathcal{D}$. We will show that bimonoids in $\mathbb{FF}(\mathbb{D})$ are precisely algebraic weak factorization systems, and more generally that the morphisms in $\mathrm{Bimon}(\mathbb{FF}(\mathbb{D}))$ are given by (co)lax morphisms of algebraic weak factorization systems.

Suppose that $E = (E, \eta, \epsilon)$ is a functorial factorization on a category $\mathscr{C}$, and consider a monoid structure on E. As I_C is initial, the unit of the monoid is forced, and is simply η. The multiplication is given by a natural transformation $\mu \colon ER \Rightarrow E$ satisfying equations (15) and (16), which now take the form $\epsilon \circ \mu = \epsilon R$ and $\mu \circ (\eta \cdot \vec{\eta}) = \eta$.

The unit axioms for the monoid give the equations $\mu \circ E\vec{\eta} = \mathrm{id}_E = \mu \circ \eta R$, which together imply the equation $\mu \circ (\eta \cdot \vec{\eta}) = \eta$ above. And finally, writing $\vec{\mu} = \mu^R \colon R^2 \to R$ for the natural transformation induced by the 2-cell μ, the associativity axiom gives the equation $\mu \circ E\vec{\mu} = \mu \circ \mu R$.

Proposition 7.1. *A monoid structure on an object* (E, η, ϵ) *in* $\mathbb{FF}(\mathbb{D})$ *is given by a natural transformation* $\mu \colon ER \Rightarrow E$*, satisfying equations*

$$\epsilon \circ \mu = \epsilon R \qquad \mu \circ E\vec{\eta} = \mathrm{id}_E = \mu \circ \eta R \qquad \mu \circ E\vec{\mu} = \mu \circ \mu R. \tag{21}$$

This determines a monad $\mathbb{R} = (R, \vec{\eta}, \vec{\mu})$*, such that* $\mathrm{dom}\,\vec{\mu} = \mu$ *and* $\mathrm{cod}\,\vec{\mu} = \mathrm{id}_{\mathrm{cod}}$.

Similarly, a comonoid structure on (E, η, ϵ) *is given by a natural transformation* $\delta \colon E \Rightarrow EL$*, satisfying equations*

$$\delta \circ \eta = \eta L \qquad E\vec{\epsilon} \circ \delta = \mathrm{id}_E = \epsilon L \circ \delta \qquad E\vec{\delta} \circ \delta = \delta L \circ \delta, \tag{22}$$

which determines a comonad $\mathbb{L} = (L, \vec{\epsilon}, \vec{\delta})$*, such that* $\mathrm{dom}\,\vec{\delta} = \mathrm{id}_{\mathrm{dom}}$ *and* $\mathrm{cod}\,\vec{\delta} = \delta$.

Hence a functorial factorization which simultaineously has a monoid structure and a comonoid structure in $\mathbb{FF}(\mathbb{D})$ is precisely a weak factorization system, missing only the second bullet of Definition 2.10, the distributive law condition. This is not surprising, as it is the only condition requiring a compatability between the monad and comonad structures. We will see that a bialgebra in $\mathbb{FF}(\mathbb{D})$ adds precisely this compatibility.

Proposition 7.2. *A bimonoid structure on a horizontal morphism* $(E, \eta, \epsilon) \colon C \to C$ *in* $\mathbb{FF}(\mathbb{D})$ *is precisely an algebraic weak factorization system on* C *with underlying functorial factorization system* (E, η, ϵ).

Proof. We have already shown how the monoid an comonoid structures give rise to the monod and comonad of the awfs. All that remains is to show that the equations (14) amount to just the distributive law, i.e. the equation

$$\begin{tikzcd}
 & C^2 \arrow[dr, "L"'] \arrow[drr, bend left, "E"] \arrow[d, phantom, "\Downarrow\Delta"] \arrow[dr, phantom, "\Downarrow\delta" near end] & & & & C^2 \arrow[dr, bend left, "E"] \arrow[d, phantom, "\Downarrow\mu"] & \\
C^2 \arrow[ur, "R"] \arrow[dr, "L"'] & C^2 \arrow[r, "R"] & C^2 \arrow[r, "E"] & C \arrow[r, phantom, "="] & C^2 \arrow[ur, bend left, "R"] \arrow[dr, bend right, "L"'] \arrow[rr, "E"] & \Downarrow\delta & C. \\
 & C^2 \arrow[ur, "R"'] \arrow[urr, bend right, "E"'] \arrow[ur, phantom, "\Downarrow\mu" near end] & & & & C^2 \arrow[ur, bend right, "E"'] &
\end{tikzcd} \tag{23}$$

First of all, notice that the first three equations of (14) follow trivially from the initiality of I_C and the terminality of $\bot_C$ in $\mathbb{FF}(\mathbb{D})$, hence they do not impose any further conditions.

The fourth equation here takes the form

$$\begin{tikzcd}
C^2 \arrow[r, "R"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\delta^R"] & C^2 \arrow[rr, "E"] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\delta"] & & C \arrow[d, equal] & & & \\
C^2 \arrow[r, "R_{E\odot E}"'] \arrow[d, equal] & C^2 \arrow[r, "L"] \arrow[d, phantom, "\Downarrow w"] & C^2 \arrow[r, "E"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_E"] & C \arrow[d, equal] & C^2 \arrow[r, "R"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mu"] & C^2 \arrow[r, "E"] & C \arrow[d, equal] \\
C^2 \arrow[r, "L_{E\otimes E}"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mu^L"] & C^2 \arrow[r, "R"'] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\mu"] & C^2 \arrow[r, "E"'] & C \arrow[d, equal] \arrow[r, phantom, "="] & C^2 \arrow[rr, "E"] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\delta"] & & C \arrow[d, equal] \\
C^2 \arrow[r, "L"'] & C^2 \arrow[rr, "E"'] & & C & C^2 \arrow[r, "L"'] & C^2 \arrow[r, "E"'] & C,
\end{tikzcd}$$

and so to prove (23), it suffices to show that

$$\begin{tikzcd}
 & C^2 \arrow[dr, bend left, "L"] & & & C^2 \arrow[dr, bend left, "L"] & \\
C^2 \arrow[ur, bend left=60, "R"] \arrow[ur, "R_{E\odot E}"'] \arrow[dr, "L_{E\otimes E}"] \arrow[dr, bend right=60, "L"'] \arrow[ur, phantom, "\Downarrow\delta^R", near start, shift left=3] \arrow[dr, phantom, "\Downarrow\mu^L", near start, shift right=3] & \Downarrow w & C^2 \arrow[r, phantom, "="] & C^2 \arrow[ur, bend left, "R"] \arrow[dr, bend right, "L"'] & \Downarrow\Delta & C^2. \\
 & C^2 \arrow[ur, bend right, "R"'] & & & C^2 \arrow[ur, bend right, "R"'] &
\end{tikzcd}$$

We can check this using the universal property of C^2 by composing with dom and cod. First, use (19) and (20) to check that

$$\begin{tikzcd}
 & C^2 \arrow[dr, bend left, "L"] & & & & & \\
C^2 \arrow[ur, bend left=50, "R"] \arrow[ur, "\Downarrow\delta^R", "R_{E\odot E}"'] \arrow[dr, "L_{E\otimes E}", "\Downarrow\mu^L"'] \arrow[dr, bend right=50, "L"'] & \Downarrow w & C^2 \arrow[r, "\mathrm{dom}"] & C \arrow[r, phantom, "="] & C^2 \arrow[r, bend left, "L", "\Downarrow i^L"'] \arrow[r, bend right, "L_{E\otimes E}"'] \arrow[r, bend right=90, "\Downarrow\mu^L", "L"'] \arrow[rr, bend left=60, "E", "\Downarrow\delta"'] & C^2 \arrow[r, "E"'] & C \\
 & C^2 \arrow[ur, bend right, "R"'] & & & & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & C^2 \arrow[dr, bend left, "L"] & & & & & \\
C^2 \arrow[ur, bend left=50, "R"] \arrow[ur, "\Downarrow\delta^R", "R_{E\odot E}"'] \arrow[dr, "L_{E\otimes E}", "\Downarrow\mu^L"'] \arrow[dr, bend right=50, "L"'] & \Downarrow w & C^2 \arrow[r, "\mathrm{cod}"] & C \arrow[r, phantom, "="] & C^2 \arrow[r, bend left=90, "R", "\Downarrow\delta^R"'] \arrow[r, bend left, "R_{E\odot E}", "\Downarrow p^R"'] \arrow[r, bend right, "R"'] \arrow[rr, bend right=50, "\Downarrow\mu", "E"'] & C^2 \arrow[r, "E"] & C. \\
 & C^2 \arrow[ur, bend right, "R"'] & & & & &
\end{tikzcd}$$

Then use the definitions of i and p to check that $\mu \circ i = \mu \circ \eta R = \mathrm{id}_E$ and $p \circ \delta = \epsilon L \circ \delta = \mathrm{id}_E$, so that the first row above just equals δ, and the second row equals μ. Since Δ also (by definition) satisfies $\mathrm{dom}\,\Delta = \delta$ and $\mathrm{cod}\,\Delta = \mu$, we are done. □

The appropriate notion of morphism between awfs, analagous to left/right Quillen functors and Quillen adjunctions, is (to our knowledge) first given in [Rie11].

Definition 7.3. Suppose that $(E_1, \eta_1, \mu_1, \epsilon_1, \delta_1)$ and $(E_2, \eta_2, \mu_2, \epsilon_2, \delta_2)$ are awfs on $\mathscr{C}$ and $\mathscr{D}$ respectively.

- A *lax morphism of awfs* $(G, \rho) \colon E_1 \to E_2$ consists of a functor $G \colon \mathscr{C} \to \mathscr{D}$ and a natural transformation $\rho \colon E_2 \hat{G} \Rightarrow G E_1$, such that $(1, \rho) \colon L_2 \hat{G} \Rightarrow G L_1$ is a lax morphism of comonads and $(\rho, 1) \colon R_2 \hat{G} \Rightarrow G R_1$ is a lax morphism of monads.
- A *colax morphism of awfs* $(F, \lambda) \colon E_1 \to E_2$ consists of a functor $F \colon \mathscr{C} \to \mathscr{D}$ and a natural transformation $\lambda \colon F E_1 \Rightarrow E_2 \hat{F}$, such that $(1, \lambda) \colon F L_1 \Rightarrow L_2 \hat{F}$ is a colax morphism of comonads and $(\lambda, 1) \colon F R_1 \Rightarrow R_2 \hat{F}$ is a colax morphism of monads.

Notice that a lax morphism of awfs induces a lift of the functor $\hat{G}$ to a functor $\mathbb{R}_1\mathrm{Alg} \to \mathbb{R}_2\mathrm{Alg}$. In that sense, G "preserves the right class," so is analagous to a right Quillen functor. Similarly, a colax morphism of awfs induces a lift of $\hat{F}$ to $\mathbb{L}_1\mathrm{Coalg} \to \mathbb{L}_2\mathrm{Coalg}$, so is analagous to a left Quillen functor.

By Proposition 5.3, the cyclic action on $\mathbb{FF}(\mathbb{D})$ induces a cyclic action on $\mathrm{Bimon}(\mathbb{FF}(\mathbb{D}))$. This action is given on awfs by

$$(E, \eta, \mu, \epsilon, \delta)^\bullet = (E^\bullet, \epsilon^\bullet, \delta^\bullet, \eta^\bullet, \mu^\bullet)$$

swapping the monad and comonad structures. This cyclic action allows us to capture both types of morphism of awfs in the same structure.

Proposition 7.4. *Morphisms in* Bimon($\mathbb{FF}(\mathbb{D})$) *are precisely the colax morphisms of awfs. A colax morphism*

$$(E_2, \eta_2, \mu_2, \epsilon_2, \delta_2)^\bullet \to (E_1, \eta_1, \mu_1, \epsilon_1, \delta_1)^\bullet$$

is equivalent to a lax morphism of awfs

$$(E_1, \eta_1, \mu_1, \epsilon_1, \delta_1) \to (E_2, \eta_2, \mu_2, \epsilon_2, \delta_2)$$

Proof. As above, let $(E_1, \eta_1, \mu_1, \epsilon_1, \delta_1)$ and $(E_2, \eta_2, \mu_2, \epsilon_2, \delta_2)$ be awfs on $\mathscr{C}$ and $\mathscr{D}$ respectively. A morphism of bimonoids is given by a 2-cell

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "{(E_1,\eta_1,\epsilon_1)}", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\lambda"] & \mathscr{C} \arrow[d, "F"] \\
\mathscr{D} \arrow[r, "{(E_2,\eta_2,\epsilon_2)}"', "\shortmid"{marking}] & \mathscr{D}
\end{tikzcd}$$

which commutes with the monoid and comonoid structures. It is straightforward to check that this implies the natural transformations

$$\begin{tikzcd}
\mathscr{C}^2 \arrow[r, "L_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\lambda^L"] & \mathscr{C}^2 \arrow[d, "\hat{F}"] & \mathscr{C}^2 \arrow[r, "R_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\lambda^R"] & \mathscr{C}^2 \arrow[d, "\hat{F}"] \\
\mathscr{D}^2 \arrow[r, "L_2"'] & \mathscr{D}^2 & \mathscr{D}^2 \arrow[r, "R_2"'] & \mathscr{D}^2
\end{tikzcd}$$

are colax morphisms of comonads and monads respectively. □

8 Composition of $\mathbb{L}$-coalgebras

For this section, we will continue to let $\mathbb{D} = \mathbb{S}\mathrm{q}(\mathcal{D})$ be the double category of squares in a 2-category $\mathcal{D}$ with arrow objects.

In an algebraic weak factorization system, the categories $\mathbb{L}$-Coalg and $\mathbb{R}$-Alg respectively play the roles of the left and right classes of morphisms of the weak factorization system. In an ordinary weak factorization system, these two classes of morphisms are closed under composition. In [Gar09], this is strengthened to a composition functor

$$\mathbb{L}\text{-Coalg} \sqcap_C \mathbb{L}\text{-Coalg} \to \mathbb{L}\text{-Coalg}$$

and in [Rie11], it is shown that colax morphisms of awfs preserve this composition. Similarly, there is a composition functor on $\mathbb{R}$-Alg which is preserved by lax morphisms of awfs.

In this section, we will generalize these results to the setting of bimonads in $\mathbb{FF}(\mathbb{D})$.

First, recall from [Str72] the following proposition.

Proposition 8.1. *Let C be a category, and $\mathbb{L} = (L, \epsilon, \delta)$ be a comonad on C. The category of coalgebras $\mathbb{L}$-Coalg has a universal property as follows:*

- *There is a forgetful functor $U \colon \mathbb{L}\text{-Coalg} \to C$ and a natural transformtion $\alpha \colon U \Rightarrow LU$, satisfying $\epsilon U \circ \alpha = \mathrm{id}_U$ and $\delta U \circ \alpha = L\alpha \circ \alpha$.*
- *(U, α) is universal among such pairs satisfying such equations. Given another such pair (F, β), where $F \colon X \to C$, there exists a unique functor $\hat{F} \colon X \to \mathbb{L}\text{-Coalg}$ such that $U\hat{F} = F$ and $\alpha\hat{F} = \beta$.*

Any colax morphism of comonads $(F, \phi) \colon (C, L_1, \epsilon_1, \delta_1) \to (D, L_2, \epsilon_2, \delta_2)$ induces a functor $\tilde{F} \colon \mathbb{L}_1\text{-Coalg} \to \mathbb{L}_2\text{-Coalg}$ such that $U\tilde{F} = FU$.

For the rest of this section, assume that *$\mathcal{D}$ has EM-objects for comonads*, i.e. for every comonad $\mathbb{L}$ in $\mathcal{D}$ there is an object $\mathbb{L}$-Coalg satisfying the universal property above.

The main goal of this section will be to prove the following theorem:

Theorem 8.2. *There is a lax double functor*

$$\mathrm{Coalg} \colon \mathbb{C}\mathrm{omon}(\mathbb{FF}(\mathbb{S}\mathrm{q}(\mathcal{D}))) \to \mathbb{S}\mathrm{pan}(\mathcal{D}_0)$$

where $\mathcal{D}_0$ is the ordinary category underlying the (strict) 2-category $\mathcal{D}$, which is the identity on the vertical categories, and which takes a comonad $(E, \eta, \epsilon, \delta)$ in $\mathbb{FF}(\mathbb{S}\mathrm{q}(\mathcal{D}))$ to the span

$$C \xleftarrow{\mathrm{dom}\, U} \mathbb{L}\text{-Coalg} \xrightarrow{\mathrm{cod}\, U} C.$$

Before we get to the proof of Theorem 8.2, we will need to establish several technical lemmas.

Consider a comonad in $\mathbb{FF}(\mathbb{D})$ on an object C, i.e. a functorial factorization with half of the awfs structure. We can combine the universal properties of EM-objects and arrow objects into a universal property for $\mathbb{L}$-Coalg, where now $\mathbb{L}$ is the comonad in $\mathcal{D}$ arising from the comonad in $\mathbb{FF}(\mathbb{D})$.

Lemma 8.3. *Let $(E, \eta, \epsilon, \delta)$ be a comonad in $\mathbb{FF}(\mathbb{D})$ on an object C. There is a 2-cell*

$$\begin{tikzcd}
& C^2 \arrow[dr, "\mathrm{cod}"] & \\
\mathbb{L}\text{-Coalg} \arrow[ur, bend left, "U"] \arrow[dr, bend right, "U"'] & \Downarrow\alpha & C \\
& C^2 \arrow[ur, "E"'] &
\end{tikzcd}$$

satisfying equations

$$\begin{tikzcd}
\mathbb{L}\text{-Coalg} \arrow[r, "U"] \arrow[dr, bend right, "U"'] & C^2 \arrow[r, "\mathrm{cod}"'] \arrow[r, bend left=60, "\mathrm{dom}", ""'{name=D}] \arrow[d, phantom, "\Downarrow\alpha"] & C \\
& C^2 \arrow[ur, "E"'] &
\end{tikzcd} \overset{\Downarrow\kappa}{} = \begin{tikzcd}
\mathbb{L}\text{-Coalg} \arrow[r, "U"] & C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "E"'] \arrow[r, phantom, "\Downarrow\eta"] & C
\end{tikzcd} \tag{24}$$

$$\begin{tikzcd}
& C^2 \arrow[dr, "\mathrm{cod}"] & \\
\mathbb{L}\text{-Coalg} \arrow[ur, bend left, "U"] \arrow[r, "U"'] & C^2 \arrow[u, phantom, "\Downarrow\alpha"] \arrow[r, "E"] \arrow[r, bend right=60, "\mathrm{cod}"'] \arrow[r, phantom, shift right=3, "\Downarrow\epsilon"] & C
\end{tikzcd} = \begin{tikzcd}
X \arrow[r, "U"] & C^2 \arrow[r, "\mathrm{cod}"] & C
\end{tikzcd} \tag{25}$$

$$\begin{tikzcd}
& C^2 \arrow[dr, "\mathrm{cod}"] & \\
\mathbb{L}\text{-Coalg} \arrow[ur, bend left, "U"] \arrow[r, "U"'] & C^2 \arrow[u, phantom, "\Downarrow\alpha"] \arrow[r, "E"] \arrow[d, "L"'] \arrow[dr, phantom, "\Downarrow\delta"] & C \\
& C^2 \arrow[ur, "E"'] &
\end{tikzcd} = \begin{tikzcd}
& C^2 \arrow[dr, "\mathrm{cod}"] & \\
\mathbb{L}\text{-Coalg} \arrow[ur, bend left, "U"] \arrow[r, "U"] \arrow[dr, bend right, "U"'] & C^2 \arrow[u, phantom, "\Downarrow\alpha"] \arrow[r, "E"'] & C. \\
& C^2 \arrow[u, "L"'] \arrow[u, phantom, shift right=6, "\Downarrow\vec{\alpha}"] &
\end{tikzcd} \tag{26}$$

where $\vec{\alpha}$ is the unique 2-cell such that $\mathrm{dom}\,\vec{\alpha} = \mathrm{id}_{\mathrm{dom}\,U}$ *and* $\mathrm{cod}\,\vec{\alpha} = \alpha$*, the existence of which is implied by Equation 24.*

Given any object X, together with a morphism $F \colon X \to C^2$ and a 2-cell $\beta \colon \mathrm{cod}\,F \Rightarrow EF$ satisfying equations

1. $\beta \circ \kappa F = \eta F$
2. $\epsilon F \circ \beta = \mathrm{id}_{\mathrm{cod}\,F}$
3. $\delta F \circ \beta = E\vec{\beta} \circ \beta$

where $\vec{\beta} \colon F \Rightarrow LF$ is the unique 2-cell such that $\mathrm{dom}\,\vec{\beta} = \mathrm{id}_{\mathrm{dom}\,F}$ *and* $\mathrm{cod}\,\vec{\beta} = \beta$*; there is a unique morphism $\hat{F} \colon X \to$* $\mathbb{L}$-Coalg *such that $U\hat{F} = F$ and $\alpha\hat{F} = \vec{\beta}$.*

Proof. U is simply the U from proposition 8.1, while the 2-cell α there is the 2-cell $\vec{\alpha}$ here. The equation $\vec{\epsilon}U \circ \vec{\alpha} = \mathrm{id}_F$ implies that $\mathrm{dom}\,\vec{\alpha} = \mathrm{id}_{\mathrm{dom}\,U}$. With that observation, the rest of the equations follow immediately from the universal property of C^2 and the equations $\epsilon U \circ \alpha = \mathrm{id}_U$ and $\delta U \circ \alpha = L\alpha \circ \alpha$ from Proposition 8.1. ☐

We will now prove a couple of simple lemmas to establish the existence of certain 2-cells in $\mathcal{D}$ using the arrow object universal property. For each of these lemmas, let $(E_1, \eta_1, \epsilon_1, \delta_1)$ and $(E_2, \eta_2, \epsilon_2, \delta_2)$ be two comonads in $\mathbb{FF}(\mathbb{S}\mathrm{q}(\mathcal{D}))$, both on the same object C; let X be the pullback

$$\begin{tikzcd}
& X \arrow[dl, "P_1"'] \arrow[dr, "P_2"] & \\
\mathbb{L}_1\text{-Coalg} \arrow[dr, "\mathrm{cod}\, U"'] & & \mathbb{L}_2\text{-Coalg} \arrow[dl, "\mathrm{dom}\, U"] \\
& C &
\end{tikzcd}$$

let m be the 2-cell

$$\begin{tikzcd}
& C^2 \arrow[dr, bend left, "\mathrm{dom}"] \arrow[dr, bend right, "\mathrm{cod}"'] \arrow[dr, phantom, "\Downarrow\kappa"] & \\
X \arrow[ur, "UP_1"] \arrow[dr, "UP_2"'] & & C \\
& C^2 \arrow[ur, bend left, "\mathrm{dom}"] \arrow[ur, bend right, "\mathrm{cod}"'] \arrow[ur, phantom, "\Downarrow\kappa"] &
\end{tikzcd}$$

and let $\vec{m}\colon X \to C^2$ be the corresponding 1-cell with $\kappa\vec{m} = m$.

Lemma 8.4. *There is a 2-cell*

$$\begin{tikzcd}
& \mathbb{L}_1\text{-Coalg} \arrow[dr, bend left, "U"] \arrow[d, phantom, "\Downarrow\zeta"] & \\
X \arrow[ur, bend left, "P_1"] \arrow[rr, bend right, "\vec{m}"'] & {} & C^2
\end{tikzcd}$$

such that $\mathrm{dom}\, \zeta = \mathrm{id}$ *and*

$$\begin{tikzcd}
& \mathbb{L}_1\text{-Coalg} \arrow[dr, bend left, "U"] \arrow[d, phantom, "\Downarrow\zeta"] & & & & & C^2 \arrow[dr, "\mathrm{cod}"] & \\
X \arrow[ur, bend left, "P_1"] \arrow[rr, bend right, "\vec{m}"'] & {} & C^2 \arrow[r, "\mathrm{cod}"] & C & = & X \arrow[ur, "UP_1"] \arrow[dr, "UP_2"'] & & C \\
& & & & & & C^2 \arrow[ur, bend left, "\mathrm{dom}"] \arrow[ur, bend right, "\mathrm{cod}"'] \arrow[ur, phantom, "\Downarrow\kappa"] &
\end{tikzcd}$$

Proof. Equation (3) becomes

$$\begin{tikzcd}
& & & & & C^2 \arrow[dr, bend left, "\mathrm{dom}"] \arrow[dr, bend right, "\mathrm{cod}"'] \arrow[dr, phantom, "\Downarrow\kappa"] & \\
X \arrow[r, "\vec{m}"] & C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\kappa"] & C & = & X \arrow[ur, "UP_1"] \arrow[dr, "UP_2"'] & & C \\
& & & & & C^2 \arrow[ur, bend left, "\mathrm{dom}"] \arrow[ur, bend right, "\mathrm{cod}"'] \arrow[ur, phantom, "\Downarrow\kappa"] &
\end{tikzcd}$$

which is simply the definition of $\vec{m}$. ☐

Lemma 8.5. *There is a 2-cell*

$$\begin{tikzcd}
& \mathbb{L}_2\text{-Coalg} \arrow[dr, "U"] & \\
X \arrow[ur, bend left, "P_2"] \arrow[dr, bend right, "\vec{m}"'] & \Downarrow\nu & C^2 \\
& C^2 \arrow[ur, bend right, "R_1"'] &
\end{tikzcd}$$

such that $\mathrm{cod}\,\nu = \mathrm{id}$ *and*

$$\begin{tikzcd}
& \mathbb{L}_2\text{-Coalg} \arrow[dr, "U"] & & \\
X \arrow[ur, bend left, "P_2"] \arrow[dr, bend right, "\vec{m}"'] & \Downarrow\nu & C^2 \arrow[r, "\mathrm{dom}"] & C \\
& C^2 \arrow[ur, bend right, "R_1"'] & &
\end{tikzcd}
=
\begin{tikzcd}
& \mathbb{L}_2\text{-Coalg} \arrow[r, "U"] & C^2 \arrow[dr, bend left, "\mathrm{dom}"] & \\
X \arrow[ur, bend left, "P_2"] \arrow[r, "P_1"] \arrow[drr, bend right, "\vec{m}"'] & \mathbb{L}_1\text{-Coalg} \arrow[r, "U"] \arrow[dr, "U"'] \arrow[d, phantom, "\Downarrow\zeta"] & C^2 \arrow[r, "\mathrm{cod}"] \arrow[d, phantom, "\Downarrow\alpha_1"] & C \\
& {} & C^2 \arrow[ur, bend right, "E_1"'] &
\end{tikzcd}$$

Proof. We just need to verify Equation (3):

$$\begin{tikzcd}
& \mathbb{L}_2\text{-Coalg} \arrow[r, "U"] & C^2 \arrow[dr, bend left, "\mathrm{dom}"] & \\
X \arrow[ur, bend left, "P_2"] \arrow[r, "P_1"] \arrow[drr, bend right, "\vec{m}"'] & \mathbb{L}_1\text{-Coalg} \arrow[r, "U"] \arrow[dr, "U"'] \arrow[d, phantom, "\Downarrow\zeta"] & C^2 \arrow[r, "\mathrm{cod}"] \arrow[d, phantom, "\Downarrow\alpha_1"] & C \arrow[d, phantom, "\Downarrow\epsilon_1"] \\
& {} & C^2 \arrow[ur, "E_1"] \arrow[ur, bend right, "\mathrm{cod}"'] & {}
\end{tikzcd}$$

$$= \begin{tikzcd}
& \mathbb{L}_2\text{-Coalg} \arrow[r, "U"] & C^2 \arrow[dr, bend left, "\mathrm{dom}"] & \\
X \arrow[ur, bend left, "P_2"] \arrow[r, "P_1"] \arrow[rr, bend right=60, "\vec{m}"'] & \mathbb{L}_1\text{-Coalg} \arrow[r, "U"] \arrow[d, phantom, "\Downarrow\zeta"] & C^2 \arrow[r, "\mathrm{cod}"] & C \\
& {} & &
\end{tikzcd}$$

$$= \begin{tikzcd}
X \arrow[r, "P_2"] & \mathbb{L}_1\text{-Coalg} \arrow[r, "U"] & C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\kappa"] & C
\end{tikzcd}$$

where the first equation follows from (25), and the second by reducing $\mathrm{cod}\,\zeta$ using Lemma 8.4. □

Proof of Theorem 8.2. For notational convenience, let $G = \mathrm{Coalg}$ be the lax double functor we need to establish. Both the double categories $\mathbb{C}\mathrm{omon}(\mathbb{FF}(\mathbb{S}\mathrm{q}(\mathscr{D})))$ and $\mathbb{S}\mathrm{pan}(\mathscr{D}_0)$ have $\mathscr{D}_0$ as vertical category, and G_0 is simply the identity. From the statement of the theorem, G takes an object in $\mathbb{C}\mathrm{omon}(\mathbb{FF}(\mathbb{S}\mathrm{q}(\mathscr{D})))$ to the span

$$C \xleftarrow{\mathrm{dom}\,U} \mathbb{L}\text{-Coalg} \xrightarrow{\mathrm{cod}\,U} C.$$

To define the behavior of G on 2-cells, consider a 2-cell in $\mathbb{C}\mathrm{omon}(\mathbb{FF}(\mathbb{S}\mathrm{q}(\mathscr{D})))$:

$$\begin{tikzcd}
C \arrow[r, "{(E_1,\eta_1,\epsilon_1,\delta_1)}", "\shortmid" marking] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[d, "F"] \\
D \arrow[r, "{(E_2,\eta_2,\epsilon_2,\delta_2)}"', "\shortmid" marking] & D.
\end{tikzcd}$$

By Corollary 6.4, ϕ induces a colax morphism of comonads from L_1 to L_2, hence by Proposition 8.1 there is an induced morphim $\tilde{\phi}$ between the EM-objects such that $U\tilde{\phi} = F^2U$. We can then define $G\phi$ to be the morphism of spans

$$\begin{tikzcd}
C \arrow[d, "F"'] & C^2 \arrow[l, "\mathrm{dom}"'] \arrow[d, "F^2"] & \mathbb{L}_1\text{-Coalg} \arrow[l, "U"'] \arrow[r, "U"] \arrow[d, "\tilde{\phi}"] & C^2 \arrow[r, "\mathrm{cod}"] \arrow[d, "F^2"] & C \arrow[d, "F"] \\
D & D^2 \arrow[l, "\mathrm{dom}"] & \mathbb{L}_2\text{-Coalg} \arrow[l, "U"] \arrow[r, "U"'] & D^2 \arrow[r, "\mathrm{cod}"'] & D.
\end{tikzcd}$$

Next we must define the coherence data G_I and $G_\otimes$. We will define G_I to be the morphims of spans

$$\begin{tikzcd}
& C \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] \arrow[dd, "G_I"] & \\
C & & C \\
& \mathbb{L}_I\text{-Coalg} \arrow[ul, "\mathrm{dom}\,U"] \arrow[ur, "\mathrm{cod}\,U"'] &
\end{tikzcd}$$

defined via Lemma 8.3 by the equations $UG_I = i\colon C \to C^2$ and $\alpha_I G_I$ is the identity on $\mathrm{dom}\, i = \mathrm{cod}\, i$. The conditions of the lemma are trivially satisfied.

We will similarly use Lemma 8.3 to define $G_\otimes$. Let X, $\vec{m}$, ζ, and ν be as defined earlier in the section. $G_\otimes$ is a morphism of spans

$$\begin{tikzcd}
& X \arrow[dl, "\mathrm{dom}\,UP_1"'] \arrow[dr, "\mathrm{cod}\,UP_2"] \arrow[dd, "G_\otimes"] & \\
C & & C. \\
& \mathbb{L}_{1\otimes 2}\text{-Coalg} \arrow[ul, "\mathrm{dom}\,U"] \arrow[ur, "\mathrm{cod}\,U"'] &
\end{tikzcd}$$

We will define $G_\otimes$ to be the 1-cell such that $UG_\otimes = \vec{m}$ and

$$\begin{tikzcd}
& & C^2 \arrow[dr, bend left, "\mathrm{cod}"] & & & & \mathbb{L}_2\text{-Coalg} \arrow[r, "U"] \arrow[ddr, "U"] & C^2 \arrow[dr, bend left, "\mathrm{cod}"] & \\
X \arrow[r, "G_\otimes"] & \mathbb{L}\text{-Coalg} \arrow[ur, bend left, "U"] \arrow[dr, bend right, "U"'] & \Downarrow\alpha_{1\otimes 2} & C & = & X \arrow[ur, bend left, "P_2"] \arrow[dr, bend right, "\vec{m}"'] & \Downarrow\nu & \Downarrow\alpha_2 & C \\
& & C^2 \arrow[ur, bend right, "E_{1\otimes 2}"'] & & & & C^2 \arrow[r, "R_1"'] & C^2 \arrow[ur, bend right, "E_2"'] &
\end{tikzcd}$$

In other words, in the notation of Lemma 8.3 let $F = \vec{m}$ and $\beta = E_2\nu \circ \alpha_2 P_2$, and define $G_\otimes = \hat{F}$.

We now need to check equations 1-3 of Lemma 8.3 to verify that $G_\otimes$ is well defined. We will check these equationally to save space, but the reader may want to draw out the diagrams for themselves to follow along. For the first equation:

$$\begin{aligned}
& E_2\nu \circ \alpha_2 P_2 \circ \kappa\vec{m} \\
&\quad = E_2\nu \circ \alpha_2 P_2 \circ \kappa U P_2 \circ \kappa U P_1 && \text{Def of } \vec{m} \\
&\quad = E_2\nu \circ (\alpha_2 \circ \kappa U) P_2 \circ \kappa U P_1 \\
&\quad = E_2\nu \circ \eta_2 U P_2 \circ \kappa U P_1 && \text{Eq (24)} \\
&\quad = \eta_2 R_1 \vec{m} \circ \operatorname{dom}\nu \circ \kappa U P_1 && \text{Interchange} \\
&\quad = \eta_2 R_1 \vec{m} \circ E_1\zeta \circ \alpha_1 P_1 \circ \kappa U P_1 && \text{Def of } \nu \\
&\quad = \eta_2 R_1 \vec{m} \circ E_1\zeta \circ (\alpha_1 \circ \kappa U) P_1 \\
&\quad = \eta_2 R_1 \vec{m} \circ E_1\zeta \circ \eta_1 U P_1 && \text{Eq (24)} \\
&\quad = \eta_{1\otimes 2}\vec{m} \circ \operatorname{dom}\zeta && \text{Interchange; Def of } \eta_{1\otimes 2} \\
&\quad = \eta_{1\otimes 2}\vec{m} && \operatorname{dom}\zeta = \mathrm{id}
\end{aligned}$$

and the second:

$$\begin{aligned}
& \epsilon_{1\otimes 2}\vec{m} \circ E_2\nu \circ \alpha_2 P_2 \\
&\quad = \epsilon_2 R_1 \vec{m} \circ E_2\nu \circ \alpha_2 P_2 && \text{Def of } \epsilon_{1\otimes 2} \\
&\quad = \operatorname{cod}\nu \circ (\epsilon_2 U \circ \alpha_2) P_2 && \text{Interchange} \\
&\quad = \mathrm{id}_{\operatorname{cod}\vec{m}}\,. && \text{Eq (25); } \operatorname{cod}\nu = \mathrm{id}
\end{aligned}$$

The third equation is a bit trickier to prove. We will need to prove two intermediate equations first, using the arrow object universal property.

Lemma.

$$i^L\vec{m} \circ L_1\zeta \circ \vec{\alpha}_1 P_1 = \vec{\beta} \circ \zeta \tag{27}$$

Proof. We must show the 2-cells become equal upon composition with dom and cod:

$$\operatorname{dom}(i^L\vec{m} \circ L_1\zeta \circ \vec{\alpha}_1 P_1) = \mathrm{id}_{\operatorname{dom}\vec{m}} = \operatorname{dom}(\vec{\beta} \circ \zeta)$$

and

$$\begin{aligned}
&\operatorname{cod}(i^L\vec{m} \circ L_1\zeta \circ \vec{\alpha}_1 P_1)\\
&\quad = \operatorname{cod} i^L\vec{m} \circ E_1\zeta \circ \operatorname{cod}\vec{\alpha}_1 P_1\\
&\quad = \eta_2 R_1\vec{m} \circ E_1\zeta \circ \alpha_1 P_1 && \text{Def of } i^L,\ \vec{\alpha}\\
&\quad = \eta_2 R_1\vec{m} \circ \operatorname{dom}\nu && \text{Def of } \nu\\
&\quad = E_2\nu \circ \eta_2 U P_2 && \text{Interchange}\\
&\quad = E_2\nu \circ (\alpha_2 \circ \kappa U) P_2 && \text{Eq (24)}\\
&\quad = (E_2\nu \circ \alpha_2 P_2) \circ \kappa U P_2\\
&\quad = \operatorname{cod}\vec{\beta} \circ \operatorname{cod}\zeta && \text{Def of } \vec{\beta},\ \zeta\\
&\quad = \operatorname{cod}(\vec{\beta} \circ \zeta).
\end{aligned}$$

□

Lemma.

$$R_1\vec{\beta} \circ \nu = w\vec{m} \circ L_2\delta_1^R\vec{m} \circ L_2\nu \circ \vec{\alpha}_2 P_2 \tag{28}$$

Proof. Again we must prove equality after composing with dom and cod:

$$\begin{aligned}
&\operatorname{dom}(R_1\vec{\beta} \circ \nu)\\
&\quad = E_1\vec{\beta} \circ \operatorname{dom}\nu\\
&\quad = E_1\vec{\beta} \circ E_1\zeta \circ \alpha_1 P_1 && \text{Def of } \nu\\
&\quad = E_1(\vec{\beta} \circ \zeta) \circ \alpha_1 P_1\\
&\quad = E_1(i^L\vec{m} \circ L_1\zeta \circ \vec{\alpha}_1 P_1) \circ \alpha_1 P_1 && \text{Eq (27)}\\
&\quad = E_1 i^L\vec{m} \circ E_1 L_1\zeta \circ (E_1\vec{\alpha}_1 \circ \alpha_1) P_1\\
&\quad = E_1 i^L\vec{m} \circ E_1 L_1\zeta \circ (\delta_1 U \circ \alpha_1) P_1 && \text{Eq (26)}\\
&\quad = E_1 i^L\vec{m} \circ \delta_1\vec{m} \circ E_1\zeta \circ \alpha_1 P_1 && \text{Interchange}\\
&\quad = \operatorname{dom} w\vec{m} \circ \operatorname{dom}\delta_1^R\vec{m} \circ \operatorname{dom}\nu \circ \operatorname{dom}\vec{\alpha}_2 P_2 && \text{Defs of } w,\ \delta^R,\ \nu,\ \vec{\alpha}\\
&\quad = \operatorname{dom}(w\vec{m} \circ L_2\delta_1^R\vec{m} \circ L_2\nu \circ \vec{\alpha}_2 P_2)
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{cod}(R_1\vec{\beta} \circ \nu)\\
&\quad = \operatorname{cod}\vec{\beta} \circ \operatorname{cod}\nu\\
&\quad = E_2\nu \circ \alpha_2 P_2 && \text{Defs of } \vec{\beta},\ \nu\\
&\quad = E_2(p^R \circ \delta_1^R)\vec{m} \circ E_2\nu \circ \alpha_2 P_2 && p^R \circ \delta^R = \text{id}\\
&\quad = E_2 p^R\vec{m} \circ E_2\delta_1^R\vec{m} \circ E_2\nu \circ \alpha_2 P_2\\
&\quad = \operatorname{cod} w\vec{m} \circ \operatorname{cod} L_2\delta_1^R\vec{m} \circ \operatorname{cod} L_2\nu \circ \operatorname{cod}\vec{\alpha}_2 P_2 && \text{Defs of } w,\ L,\ \vec{\alpha}\\
&\quad = \operatorname{cod}(w\vec{m} \circ L_2\delta_1^R\vec{m} \circ L_2\nu \circ \vec{\alpha}_2 P_2)
\end{aligned}$$

□

Now we are prepared to prove the third equation of Lemma 8.3 validating our definition of $G_\otimes$:

$$\begin{aligned}
&\delta_{1\otimes 2}\vec{m} \circ E_2\nu \circ \alpha_2 P_2 \\
&\quad = (E_2 w \circ \delta_2 R_{1\odot 1} \circ E_2\delta_1^R)\vec{m} \circ E_2\nu \circ \alpha_2 P_2 && \text{Def of } \delta_{1\otimes 2} \\
&\quad = E_2(w\vec{m} \circ L_2\delta_1^R\vec{m} \circ L_2\nu) \circ (\delta_2 U \circ \alpha_2)P_2 && \text{Interchange} \\
&\quad = E_2(w\vec{m} \circ L_2\delta_1^R\vec{m} \circ L_2\nu) \circ (E_2\vec{\alpha}_2 \circ \alpha_2)P_2 && \text{Eq (26)} \\
&\quad = E_2(w\vec{m} \circ L_2\delta_1^R\vec{m} \circ L_2\nu \circ \vec{\alpha}_2 P_2) \circ \alpha_2 P_2 \\
&\quad = E_2(R_1\vec{\beta} \circ \nu) \circ \alpha_2 P_2 && \text{Eq (28)} \\
&\quad = E_{1\otimes 2}\vec{\beta} \circ E_2\nu \circ \alpha_2 P_2 && \text{Def of } E_{1\otimes 2}
\end{aligned}$$

The verification that the definitions of G_I and $G_\otimes$ form natural families, and of the coherence axioms for a lax double functor, is tedious, but follows from what we have presented here without requiring any new ideas or ingenuity. □

Corollary 8.6. *For any awfs $(E, \eta, \mu, \epsilon, \delta)$ on an object C in $\mathcal{D}$, the multiplication μ induces a composition functor on $\mathbb{L}$-Coalg, and the functor between EM-objects induced by any colax morphism of awfs preserves this composition.*

Proof. Any awfs $(E, \eta, \mu, \epsilon, \delta)$ has an underlying object in $\mathbb{C}\text{omon}(\mathbb{FF}(\mathbb{S}\text{q}(\mathcal{D})))$, by simply forgetting μ. The lax double-functor Coalg takes this to a span

$$C \xleftarrow{\operatorname{dom} U} \mathbb{L}\text{-Coalg} \xrightarrow{\operatorname{cod} U} C.$$

The multiplication μ provides this object in $\mathbb{C}\text{omon}(\mathbb{FF}(\mathbb{S}\text{q}(\mathcal{D})))$ with a monad structure, and lax double-functors preserve monads, so μ induces a monad structure on this span. A multiplication on this span is a morphism π:

$$\begin{tikzcd}
 & X \arrow[dl, "\operatorname{dom} UP_1"'] \arrow[dr, "\operatorname{cod} UP_2"] \arrow[dd, "\pi"] & \\
C & & C \\
 & \mathbb{L}\text{-Coalg} \arrow[ul, "\operatorname{dom} U"] \arrow[ur, "\operatorname{cod} U"'] &
\end{tikzcd}$$

where X is the pullback in the composite span

$$\begin{tikzcd}
 & & X \arrow[dl, "P_1"'] \arrow[dr, "P_2"] & & \\
 & \mathbb{L}\text{-Coalg} \arrow[dl, "\operatorname{dom} U"'] \arrow[dr, "\operatorname{cod} U"] & & \mathbb{L}\text{-Coalg} \arrow[dl, "\operatorname{dom} U"'] \arrow[dr, "\operatorname{cod} U"] & \\
C & & C & & C.
\end{tikzcd}$$

The morphism π is the composition structure that we want. If $\mathcal{D} = \mathrm{Cat}$ is the 2-category of small categories, then an object (f, g) in X is a pair of morphisms in C equipped with coalgebra structures, such that $\operatorname{cod} f = \operatorname{dom} g$, and $\pi(f, g)$ is a morphism equipped with a coalgebra structure, with $\operatorname{dom} \pi(f, g) = \operatorname{dom} f$ and $\operatorname{cod} \pi(f, g) = \operatorname{cod} g$.

Of course, what we really want is that the morphism underlying the coalgebra $\pi(f, g)$ is the composition $g \circ f$. To see that this is the case, notice that the composition π is given by $G(\vec{\mu}) \circ G_{\otimes}$, where $G(\vec{\mu})$ is the 1-cell between EM-objects induced by the colax morphism of comonads $\vec{\mu}$. Using the fact that $\vec{\mu}$ is a globular 2-cell in $\mathbb{FF}(\mathbb{S}\mathrm{q}(\mathcal{D}))$, we have $U\pi = UG(\vec{\mu})G_{\otimes} = UG_{\otimes} = \vec{m}$. Looking at the definition of $\vec{m}\colon X \to C^2$, it is clear that it is precisely the functor that sends the pair (f, g) to the composition gf, ignoring the coalgebra structures. □

Corollary 8.6 actually has a converse, which was used extensively in [Rie11]. We can prove this converse in our generalized framework as well.

Proposition 8.7. *Let $(E, \eta, \epsilon, \delta)$ be a comonad in $\mathbb{FF}(\mathbb{D})$. A multiplication map on the span* Coalg*E determines a monad structure on E.*

Furthermore, given two bimonads $(E_1, \eta_1, \mu_1, \epsilon_1, \delta_1)$ and $(E_2, \eta_2, \mu_2, \epsilon_2, \delta_2)$, and a morphism $\theta\colon E_1 \to E_2$ in Comon($\mathbb{FF}(\mathbb{D})$)*, if the induced map $\tilde{\theta}\colon \mathbb{L}_1$*-Coalg $\to$ $\mathbb{L}_2$-Coalg *commutes with the multiplication, then θ is in fact a bimonad morphism.*

Proof. We will again need to start with some preliminary constructions.

Let $\hat{L}\colon C^2 \to \mathbb{L}$-Coalg be the morphism determined by $U\hat{L} = L$ and $\alpha\hat{L} = \delta$.

Let $(\hat{L}, \hat{L}R)$ be the map into the pushout X as in the diagram

$$\begin{tikzcd}
 & & \mathbb{L}\text{-Coalg} \arrow[dr, "\operatorname{cod} U"] & \\
C^2 \arrow[urr, bend left, "\hat{L}"] \arrow[r, "(\hat{L}{,}\hat{L}R)"] \arrow[drr, bend right, "\hat{L}R"'] & X \arrow[ur, "P_1"] \arrow[dr, "P_2"'] & & C. \\
 & & \mathbb{L}\text{-Coalg} \arrow[ur, "\operatorname{dom} U"'] &
\end{tikzcd}$$

This is well defined since $\operatorname{cod} U\hat{L} = E = \operatorname{dom} U\hat{L}R$.

Consider the morphism $\vec{m}(\hat{L}, \hat{L}R)\colon C^2 \to C^2$. We compute

$$\operatorname{dom} \vec{m}(\hat{L}, \hat{L}R) = \operatorname{dom} UP_1(\hat{L}, \hat{L}R) = \operatorname{dom} U\hat{L} = \operatorname{dom} L = \operatorname{dom}$$

and

$$\operatorname{cod} \vec{m}(\hat{L}, \hat{L}R) = \operatorname{cod} UP_2(\hat{L}, \hat{L}R) = \operatorname{cod} U\hat{L}R = \operatorname{cod} LR = ER.$$

There is a 2-cell $\psi\colon \vec{m}(\hat{L}, \hat{L}R) \Rightarrow \mathrm{id}_{C^2}$ with $\operatorname{dom} \psi = \mathrm{id}_{\mathrm{dom}}$ and $\operatorname{cod} \psi = \epsilon R$. That ψ is well defined comes down to the computation

$$\operatorname{cod} \psi \circ \kappa\vec{m}(\hat{L}, \hat{L}R) = \epsilon R \circ \eta R \circ \operatorname{dom} \vec{\eta} = \kappa R \circ \eta = \epsilon \circ \eta = \kappa.$$

We will define the monad multiplication $\mu\colon ER \Rightarrow E$ to be the composition

$$\mu = E\psi \circ \alpha\pi(\hat{L}, \hat{L}R)$$

which makes sense since $\operatorname{cod}\alpha\pi(\hat{L}, \hat{L}R) = EU\pi(\hat{L}, \hat{L}R) = E\vec{m}(\hat{L}, \hat{L}R)$.

[I've gotten stuck on proving the monad axioms, to the point that I'm not sure if this converse holds at this level of generality.] ☐

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