ALGEBRAIC WEAK FACTORIZATION SYSTEMS
IN DOUBLE CATEGORIES

by

PATRICK SCHULTZ

A DISSERTATION

Presented to the Department of Mathematics
and the Graduate School of the University of Oregon
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

June 2014

"Algebraic Weak Factorization Systems in Double Categories," a dissertation prepared by Patrick Schultz in partial fulfillment of the requirements for the Doctor of Philosophy degree in the Department of Mathematics. This dissertation has been approved and accepted by:

Dr. Daniel Dugger, Chair of the Examining Committee

Date

Committee in charge: Dr. Daniel Dugger, Chair
Dr. Boris Botvinnik
Dr. Dev Sinha
Dr. Arkady Vaintrob
Dr. Peter Lambert

Accepted by:

Dean of the Graduate School

©2014, Patrick Schultz.

An Abstract of the Dissertation of

Patrick Schultz for the degree of Doctor of Philosophy

in the Department of Mathematics to be taken June 2014

Title: ALGEBRAIC WEAK FACTORIZATION SYSTEMS IN DOUBLE CATEGORIES

Approved: ____

Dr. Daniel Dugger

CURRICULUM VITAE

NAME OF AUTHOR: Patrick Schultz

PLACE OF BIRTH: Oak Park, Illinois, United States

DATE OF BIRTH: September 17, 1982

GRADUATE AND UNDERGRADUATE SCHOOLS ATTENDED:

University of Oregon, Eugene, OR
University of California, Santa Cruz, CA

DEGREES AWARDED:

Doctor of Philosophy, University of Oregon, 2014
Master of Science, University of California, Santa Cruz, 2009
Bachelor of Science, University of California, Santa Cruz, 2005

AREAS OF SPECIAL INTEREST:

Category Theory, Categorical methods in Homotopy Theory,
Categorical Logic and Type Theory

PROFESSIONAL EXPERIENCE:

Graduate Teaching Fellow, University of Oregon, 2009–2014
Teaching Assistant, University of California, Santa Cruz, 2005–2009
Teaching Assistant, COSMOS, UCSC Campus, 2003–2009

ACKNOWLEDGMENTS

TABLE OF CONTENTS

Chapter Page

I INTRODUCTION 1

II WEAK FACTORIZATION SYSTEMS 6

II.1 Arrow Categories 6
II.2 Functorial Factorizations 7
II.3 Algebraic Weak Factorization Systems 9

III DOUBLE CATEGORIES 14

III.1 Review of Double Categories 15
III.2 Arrow Objects in a Double Category 18
III.3 Monads 20
III.4 Double Functors 21

IV 2-FOLD DOUBLE CATEGORIES 24

IV.1 2-Fold Double Categories 26
IV.2 Monads in 2-Fold Double Categories 28

V CYCLIC 2-FOLD DOUBLE CATEGORIES 31

VI FUNCTORIAL FACTORIZATIONS 35

VII ALGEBRAIC WEAK FACTORIZATION SYSTEMS 45

VIII $\mathbb{R}$-ALG AND $\mathbb{L}$-COALG 49

IX COMPOSITION OF $\mathbb{L}$-COALGEBRAS 52

X A UNIVERSAL PROPERTY FOR THE PUSHOUT PRODUCT 64

X.1 Review of Cyclic Double Multicategories 65
X.2 The Universal Property 65

XI CYCLIC 2-FOLD DOUBLE MULTICATEGORIES 71

XI.1 Multimorphisms of Bimonads 72
XI.2 The Cyclic 2-Fold Double Multicategory of Functorial Factorizations 74

CHAPTER I

INTRODUCTION

The theory of model categories has a long history, and has proven itself indispensable to several recent advances in mathematics, such as higher category theory, so-called spectral algebraic geometry, even finding applications in computer science and the foundations of mathematics with homotopy type theory.

In the modern treatment, a model category is defined to consist of two *weak factorization systems* on a category $\mathscr{C}$ (e.g. [MP12]). A weak factorization system is a structure which consists of two classes of morphisms of $\mathscr{C}$, call them $\mathcal{L}$ and $\mathcal{R}$, such that solutions to certain lifting problems involving one morphism from each class always exist, plus an axiom that every morphism of $\mathscr{C}$ factors as a morphism from $\mathcal{L}$ followed by a morphism from $\mathcal{R}$. In the past 20 or so years, most authors have added the requirement that this factorization can be chosen in a natural/functorial way.

Taking this one step further, in [GT06] the category theorists Marco Grandis and Walter Tholen proposed a strengthening of weak factorization systems which they called *natural* weak factorization systems, today most often referred to as *algebraic* weak factorization systems, or awfs for short. An awfs strengthens the structure in a way which provides a canonical *choice* of solution to every lifting problem, in such a way that these choices are coherent or natural in a precise sense. The structure of an awfs consists of a monad and a comonad on the category of arrows satisfying some axioms, and the categories of algebras and coalgebras for these respectively provide an algebraic analogue of the right and left classes of maps of the factorization system.

It at first seems as though this extra structure is *too* strict, and that examples would be hard to find. But in [Gar07] and [Gar09], the category theorist Richard Garner provided a modification of Quillen's small object argument which generates algebraic weak factorization systems, and which furthermore has much nicer convergence properties than Quillen's original construction, and often generates a smaller and easier to understand factorization. Best of all, Garner's small object argument operates under almost identical assumptions as Quillen's, so that in practice any cofibrantly generated weak factorization system can be strengthened to an algebraic one.

In her Ph.D. thesis, [Rie11] and [Rie13], Emily Riehl began the project of developing a full-fledged theory of *algebraic model structures*, built out of two awfs analogously to an ordinary model structure. Since then, she and her collaborators have continued to develop and find applications of this theory, e.g. [CGR12], [BR13], and [BMR13]. Of particular interest for us, she gives the first definition of algebraic Quillen functors.

If we define a lax functor of weak factorization systems to be a functor between categories each equipped with a wfs which takes morphisms in the right class of the first to morphisms in the right class of the second, then a right Quillen functor between model categories is simply a functor which is a lax functor with respect to both weak factorization systems making up the model structures. Likewise, a colax functor of wfs preserves the left classes, and a left Quillen functor is colax with respect to both wfs. It is a basic fact from model category theory that given an adjunction between weak factorization systems, the left adjoint is colax if and only if the right adjoint is lax.

An algebraic version of Quillen functors should continue to have this property, as the definition Riehl gives does, but making this precise requires some pieces of classical category theory: the mates correspondence, and double categories. The mates correspondence is a natural bijection between natural transformations involving two pairs of adjoint functors which generalizes the hom-set bijection of an adjunction. The naturality of the mates correspondence is best formulated using double categories, and for this reason double categories play a central role in this thesis. Double categories are a kind of two-dimensional categorical structure, similar to a 2-category but with separate classes of vertical and horizontal morphisms, and with square shaped 2-cells which can be composed both vertically and horizontally. Double categories were first defined by Ehresmann in the '60's and then largely ignored, but have recently enjoyed a resurgence of interest, see e.g. [Shu08], [DPP10], [TF10].

In [Gar07] and [Gar09], Garner proves as a technical tool that algebraic weak factorization systems can be seen as bialgebras in a category of functorial factorizations, supporting the intuition that an awfs is given by a functorial factorization equipped with (co)algebraic structure. The category of functorial factorizations he constructs is not a symmetric or braided monoidal category, but a so-called two-fold monoidal category, which is a generalization of braided monoidal category having two compatible monoidal structures, and in which the definition of bialgebra still makes sense.

We find this a very nice conceptual way of understanding algebraic weak factorization systems, but it has the shortcoming of being unable to say anything about functors between awfs on different categories. It is one of our primary goals of this thesis to extend this awfs-as-bialgebras perspective to include the (co)lax morphisms of awfs defined in [Rie11]. To do this, we have had to find a common generalization of double categories, used to formalize the mates correspondance and the duality relating lax and colax morphisms, and two-fold monoidal categories, in which the notion of bialgebra makes sense. We call this common generalization a *two-fold double category*.

We show that a kind of bialgebra can be defined in any two-fold double category, which we call bimonads, and that the natural generalization of bialgebra morphism bifurcates into lax and colax morphisms of bimonads. One main result of this thesis is that there is a two-fold double category of functorial factorizations (in any 2-category), and that bimonads and (co)lax morphisms of bimonads in this two-fold double category correspond precisely to awfs and (co)lax morphisms of awfs.

In the second part of her thesis, published as [Rie13], Riehl develops a theory of monoidal algebraic model categories, ultimately based on an algebraic strengthening of the notion of two-variable Quillen adjunction. Classically, a 2-variable Quillen adjunction is a functor of two variables with both adjoints (one in each variable), such that the induced pushout-product of two maps in the left classes is again in the left class. The primary motivation for this definition is to be able to define monoidal model categories, in which the tensor product is part of a 2-variable Quillen adjunction. To give an algebraic version of this definition, Riehl had to extend the mates correspondence to multivariable adjunctions, which she does with her coauthors in [CGR12]. The mates correspondence for multivariable adjunctions is most easily understood in terms of cyclic double multicategories, a kind of structure defined in [CGR12] which generalizes double categories to allow for morphisms with multiple inputs, with a cyclic action which formalizes the mates correspondence.

In order to incorporate multivariable morphisms into the bialgebraic view of awfs, we have developed a common generalization of two-fold double categories and cyclic double multicategories. Another main result of this thesis is that the pushout product—central to the definition of Quillen bifunctor, and hence to monoidal model categories, simplicial model categories, etc.—satisfies a universal property in the framework of cyclic double multicategories. The author is particularly pleased with this result, as the need for the pushout product in the axioms of monoidal model categories and simplicial model categories had always seemed slightly mysterious and ad hoc. This universal property provides a conceptual explanation: the pushout product defines the universal way of lifting a multivariable adjunction to arrow categories. This also allows us to define multivariable morphisms of bimonads in a cyclic two-fold double multicategory, generalizing the multivariable morphisms of awfs given in [Rie13].

A primary motivation for this work was to develop the theory of awfs at a high level of generality. In particular, all of the constructions and theorems of this thesis work just as well in any 2-category satisfying minor completeness conditions as they do in the 2-category of categories. For example, in [BMR13] the authors make use of *enriched* algebraic weak factorization systems, in which stronger enriched lifting properties are required. (Note that this is different than enriched model categories in the sense of, e.g., simplicial model categories.) This thesis provides a framework in which the core theory of awfs can be developed in great generality, saving the effort of reproving results for enriched awfs and any other variations of awfs yet to be considered, and it makes a start of proving the most important results in this greater generality.

Overview

In chapter II, we review the definitions of algebraic weak factorization system and morphisms of algebraic weak factorization systems, trying to lead up to the (abstract) definitions in a natural way. Then in chapter III we review the definition of double category, as well recording some constructions which will be needed later on. Of these, the definitions of arrow objects in a double category and of fully-faithful lax double functors are (to the best of our knowledge) original.

In chapter IV we introduce a definition of two-fold double categories. Generalizing bialgebras in a two-fold monoidal category, we define bimonads and (co)lax morphisms of bimonads in a two-fold double category.

In [CGR12], the authors show that the mates correspondence can be conveniently expressed as the existence of a cyclic action on a double category of adjunctions. In chapter V, we show how to generalize the cyclic action as in [CGR12] to the two-fold double categories defined in chapter IV, defining what we call a cyclic two-fold double category. We show that a cyclic action interacts well with bimonads in a two-fold double category, extending to a cyclic action on the category of bimonads. This cyclic action is the abstract form of the fact that an algebraic Quillen adjunction can be specified *either* by a lax stucture on the right adjoint *or* a colax structure on the left adjoint.

In chapter VI, we begin the core work of this thesis, constructing a cyclic two-fold double category of functorial factorizations in an arbitrary double category which has all arrow objects. Then in chapter VII we show that given any 2-category $\mathcal{D}$ with arrow objects, bimonads in the cyclic two-fold double category of adjunctions in $\mathcal{D}$ are precisely algebraic weak factorization systems in $\mathcal{D}$.

Garner proves in [Gar07] and [Gar09] that instead of specifying both the monad and comonad halves of the awfs structure, it is equivalent to define the comonad, plus a functorial composition on the category of coalgebras. This generalizes the classical fact that the left (and right) class of maps is closed under composition, but more importantly provides a convenient technical tool for constructing algebraic weak factorization systems. Similarly, in [Rie11] Riehl proves that an equivalent definition of colax morphism of awfs is a functor which lifts to the categories of coalgebras for the comonads, and which also preserves the composition of coalgebras. She uses both of these theorems repeatedly throughout the paper.

In chapter VIII we lay the groundwork towards proving a generalization of these theorems in the framework of cyclic two-fold double categories by reviewing the standard universal property for Eilenberg-Mac Lane categories for monads and comonads, first given in [Str72], and showing the particular form this universal property takes in the special case of comonads arising from an awfs. Then in chapter IX, we give the (surprisingly difficult and technical) proofs that the results mentioned above about composition of coalgebras continue to hold at our higher level of generality.

In chapter X we show that a natural generalization of the universal property for arrow objects in a double category (defined in section III.2) to cyclic double multicategories in fact uniquely characterizes the pushout/pullback product. This allows us to abstract away the the pushout/pullback product, isolating precisely the properties which are necessary to make the theory of multivariable Quillen adjunctions work, and providing a conceptual explanation for the appearance of pushout/pullback products in the definitions of monoidal model category, simplicial model category, etc.

In chapter XI we define a common generalization of the cyclic two-fold double categories of chapter V and the cyclic double multicategories of [CGR12], which we call a cyclic two-fold double multicategory. We give a definition of multivariable morphisms of bimonads, showing that this definition is stable under the cyclic action. We then generalize our construction of a cyclic two-fold double category of functorial factorizations from chapter VI to a cyclic two-fold double multicategory, and show that multivariable morphisms of bimonads recover the definition of multivariable adjunction of awfs given in [Rie13]. The fact that these multivariable morphisms are stable under the cyclic action generalizes the classical fact that if a functor which is part of a 2-variable adjunction preserves the left classes, in that the pushout product of morphisms in the left classes is in the left class, then each of the two adjoints satisfy similar properties involving a mix of left and right classes.

CHAPTER II

WEAK FACTORIZATION SYSTEMS

We will begin by briefly reviewing the notions of functorial factorization, weak factorization system, and algebraic weak factorization system.

II.1 Arrow Categories

Let $\mathscr{C}$ be a category. Its arrow category $\mathscr{C}^2$ is the category whose objects are arrows in $\mathscr{C}$ and whose morphisms are commutative squares. The arrow category comes with two functors $\mathrm{dom}, \mathrm{cod} \colon \mathscr{C}^2 \to \mathscr{C}$, along with a natural transformation $\kappa \colon \mathrm{dom} \Rightarrow \mathrm{cod}$. The component of κ at an object f of $\mathscr{C}^2$ is simply $f \colon \mathrm{dom}\, f \to \mathrm{cod}\, f$. Moreover, $\mathscr{C}^2$ satisfies a universal property: there is an equivalence of categories

$$\mathrm{Fun}(2, \mathrm{Fun}(\mathscr{X}, \mathscr{C})) \simeq \mathrm{Fun}(\mathscr{X}, \mathscr{C}^2) \tag{II.1}$$

given by composition with κ. Here, 2 is the ordinal, i.e. the category with two objects and a single non-identity arrow. In other words, $\mathscr{C}^2$ is the cotensor of $\mathscr{C}$ with the category 2 in the 2-category $\mathcal{C}\mathrm{at}$.

We will make this universal property more explicit in the next lemma, separating out the 1-dimensional and the 2-dimensional parts of (II.1):

Lemma II.1. *Let $\mathscr{C}$ be a category.*

i) For any category $\mathscr{X}$, pair of functors $F, G \colon \mathscr{X} \to \mathscr{C}$, and natural transformation $\alpha \colon F \Rightarrow G$, there is a unique functor $\hat{\alpha} \colon \mathscr{X} \to \mathscr{C}^2$ such that $\mathrm{dom}\, \hat{\alpha} = F$, $\mathrm{cod}\, \hat{\alpha} = G$, *and*

$$\begin{tikzcd} \mathscr{X} \arrow[r, "\hat{\alpha}"] & \mathscr{C}^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\kappa"] & \mathscr{C} \end{tikzcd} = \begin{tikzcd} \mathscr{X} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathscr{C}. \end{tikzcd} \tag{II.2}$$

ii) For any functors $F, F', G, G' \colon \mathscr{X} \to \mathscr{C}$ and a commutative square of natural transforma-

tions

$$\begin{tikzcd} F \arrow[r, Rightarrow, "\gamma"] \arrow[d, Rightarrow, "\alpha"'] & F' \arrow[d, Rightarrow, "\beta"] \\ G \arrow[r, Rightarrow, "\phi"'] & G', \end{tikzcd}$$

there is a unique natural transformation $\eta\colon \hat{\alpha} \to \hat{\beta}$ *such that* $\operatorname{dom}\eta = \gamma$ *and* $\operatorname{cod}\eta = \phi$*, hence*

$$\begin{tikzcd} \mathscr{X} \arrow[r, bend left=50, "F"] \arrow[r, "G" description] \arrow[r, bend right=50, "G'"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\phi"] & \mathscr{C} \end{tikzcd} = \begin{tikzcd} \mathscr{X} \arrow[r, bend left, "\hat{\alpha}"] \arrow[r, bend right, "\hat{\beta}"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathscr{C}^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\kappa"] & \mathscr{C} \end{tikzcd} = \begin{tikzcd} \mathscr{X} \arrow[r, bend left=50, "F"] \arrow[r, "F'" description] \arrow[r, bend right=50, "G'"'] \arrow[r, phantom, shift left=3, "\Downarrow\gamma"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \mathscr{C}. \end{tikzcd} \tag{II.3}$$

Definition II.2. Let $\mathscr{D}$ be any 2-category. For any object A in $\mathscr{D}$, the *arrow object* of A, if it exists, is an object A^2 satisfying the universal property (II.1). If every object has an arrow object, i.e. if $\mathscr{D}$ has cotensors by 2, we will say *$\mathscr{D}$ has arrow objects*.

In practice, we will work with arrow objects in a 2-category using the two parts of lemma II.1.

Finally, we will record here a simple proposition which will be needed later.

Proposition II.3. *Any arrow object A^2 has an internal category structure*

$$\begin{tikzcd} A^3 \arrow[r, "c"] & A^2 \arrow[r, shift left=2, "\mathrm{dom}"] \arrow[r, shift right=2, "\mathrm{cod}"'] & A \arrow[l, "i" description] \end{tikzcd}$$

where A^3 is the pullback of the span

$$A^2 \xrightarrow{\mathrm{cod}} A \xleftarrow{\mathrm{dom}} A^2$$

Proof. Using the universal property, we define i and c by the equations $\operatorname{dom} i = \mathrm{id}$, $\operatorname{cod} i = \mathrm{id}$, $\kappa i = \mathrm{id}_{\mathrm{id}}$, $\operatorname{dom} c = \operatorname{dom} p_1$, $\operatorname{cod} c = \operatorname{cod} p_2$, and $\kappa c = \kappa p_2 \circ \kappa p_1$, where p_1 and p_2 are the projections of the pullback. □

II.2 Functorial Factorizations

Definition II.4. A functorial factorization on a category $\mathscr{C}$ consists of a functor E and two natural transformations η and ϵ which factor κ, as in

$$\begin{tikzcd} C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\kappa"] & C \end{tikzcd} = \begin{tikzcd} C^2 \arrow[r, bend left=50, "\mathrm{dom}"] \arrow[r, "E" description] \arrow[r, bend right=50, "\mathrm{cod}"'] \arrow[r, phantom, shift left=3, "\Downarrow\eta"] \arrow[r, phantom, shift right=3, "\Downarrow\epsilon"] & C. \end{tikzcd}$$

This determines for any arrow f in $\mathscr{C}$ a factorization $f = \epsilon_f \circ \eta_f$. The factorization is natural, meaning that for any morphism $(u,v)\colon f \Rightarrow g$ in $\mathscr{C}^2$ (i.e. commutative square in $\mathscr{C}$), the two squares in

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "\eta_f"'] & \cdot \arrow[d, "\eta_g"] \\ \cdot \arrow[r, "E(u,v)"] \arrow[d, "\epsilon_f"'] & \cdot \arrow[d, "\epsilon_g"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd}$$

commute.

A functorial factorization also determines two functors $L, R\colon C^2 \to C^2$ such that $\operatorname{dom} L = \operatorname{dom}$, $\operatorname{cod} R = \operatorname{cod}$, $\operatorname{cod} L = \operatorname{dom} R = E$, $\kappa L = \eta$, and $\kappa R = \epsilon$, by the universal property of C^2. The components of the factorization of f can then also be referred to as Lf and Rf, now thought of as objects in $\mathscr{C}^2$. There are also two canonical natural transformations, $\vec{\eta}\colon \mathrm{id} \Rightarrow R$ and $\vec{\epsilon}\colon L \Rightarrow \mathrm{id}$, determined by the commuting squares

$$\begin{tikzcd} \mathrm{dom} \arrow[r, Rightarrow, "\eta"] \arrow[d, Rightarrow, "\kappa"'] & E \arrow[d, Rightarrow, "\epsilon"] \\ \mathrm{cod} \arrow[r, Rightarrow, "\mathrm{id}"'] & \mathrm{cod} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{dom} \arrow[r, Rightarrow, "\mathrm{id}"] \arrow[d, Rightarrow, "\eta"'] & \mathrm{dom} \arrow[d, Rightarrow, "\kappa"] \\ E \arrow[r, Rightarrow, "\epsilon"'] & \mathrm{cod} \end{tikzcd}$$

respectively. These make L and R into (co)pointed endofunctors of $\mathscr{C}^2$.

An algebra for the pointed endofunctor R is an object f in $\mathscr{C}^2$ equipped with a morphism $\vec{t}\colon Rf \Rightarrow f$, such that $\vec{t} \circ \vec{\eta}_f = \mathrm{id}_f$. Similarly, a coalgebra for the copointed endofunctor L is an f equipped with a morphism $\vec{s}\colon f \Rightarrow Lf$, such that $\vec{\epsilon}_f \circ \vec{s} = \mathrm{id}_f$.

Lemma II.5. *Let $f\colon X \to Y$ be a morphism in $\mathscr{C}$. An R-algebra structure on $f \in \mathscr{C}^2$ is precisely a choice of lift t in the square*

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "Lf"'] & X \arrow[d, "f"] \\ Ef \arrow[r, "Rf"'] \arrow[ur, dashed, "t"] & Y. \end{tikzcd} \tag{II.4}$$

Dually, an L-coalgebra structure on f is precisely a choice of lift s in the square

$$\begin{tikzcd} X \arrow[r, "Lf"] \arrow[d, "f"'] & Ef \arrow[d, "Rf"] \\ Y \arrow[r, equal] \arrow[ur, dashed, "s"] & Y. \end{tikzcd} \tag{II.5}$$

Moreover, a morphsim $(u,v)\colon f \Rightarrow g$ in $\mathscr{C}^2$ is a morphism of R-algebras if it commutes *with the lifts t and t', that is, if in the diagram*

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "Lf"'] & \cdot \arrow[d, "Lg"] \\ \cdot \arrow[r, "E(u,v)"] \arrow[d, shift right, "Rf"'] & \cdot \arrow[d, shift left, "Rg"] \\ \cdot \arrow[r, "v"'] \arrow[u, shift right, "t"'] & \cdot \arrow[u, shift left, "t'"] \end{tikzcd}$$

we have $t'v = E(u,v)t$.

II.3 Algebraic Weak Factorization Systems

To simplify the discussion of weak factorization systems, we will start by introducing a notation. For any two morphisms l and r in $\mathscr{C}$, write $l \boxslash r$ to mean that for every commutative square

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "l"'] & \cdot \arrow[d, "r"] \\ \cdot \arrow[r, "v"'] \arrow[ur, dashed, "w"] & \cdot \end{tikzcd} \tag{II.6}$$

there exists a lift w. In this case, we will say that l has the *left lifting property* with respect to r, and that r has the *right lifting property* with respect to l. Similarly, for two classes of morphisms $\mathcal{L}$ and $\mathcal{R}$, we will say $\mathcal{L} \boxslash \mathcal{R}$ if $l \boxslash r$ for every $l \in \mathcal{L}$ and $r \in \mathcal{R}$. Finally, we will write $\mathcal{L}^{\boxslash}$ for the class of morphisms having the right lifting property with respect to every morphism of $\mathcal{L}$, and ${}^{\boxslash}\mathcal{R}$ for the class of morphisms having the left lifting property with respect to every morphism of $\mathcal{R}$.

Definition II.6. A *functorial weak factorization system* on a category $\mathscr{C}$ consists of a functorial factorization on $\mathscr{C}$ and two classes $\mathcal{L}$ and $\mathcal{R}$ of morphisms in $\mathscr{C}$, such that

- for every morphism f in $\mathscr{C}$, $Lf \in \mathcal{L}$ and $Rf \in \mathcal{R}$,
- $\mathcal{L}^{\boxslash} = \mathcal{R}$ and ${}^{\boxslash}\mathcal{R} = \mathcal{L}$.

It a simple and standard proof that the lifting property condition can be replaced by two simpler conditions:

Lemma II.7. *A functorial weak factorization system can equivalently be defined to be a functorial factorization on $\mathscr{C}$ and two classes $\mathcal{L}$ and $\mathcal{R}$ of morphisms in $\mathscr{C}$, such that*

- *for every morphism f in $\mathscr{C}$, $Lf \in \mathcal{L}$ and $Rf \in \mathcal{R}$,*
- $\mathcal{L} \boxslash \mathcal{R}$,

- *$\mathcal{L}$ and $\mathcal{R}$ are both closed under retracts.*

In fact, the functorial factorization by itself already determines the two classes of morphisms, with $\mathcal{L}$ the class of morphisms admitting an L-coalgebra structure, and $\mathcal{R}$ the class of morphisms admitting an R-algebra structure. The lifting properties also follow directly from the functorial factorization, as the next lemma shows.

Lemma II.8. *For any L-coalgebra (l, s) and any R-algebra (r, t), there is a canonical choice of lift in the square* (II.6). *Any morphism of R-algebras $(u_1, v_1)\colon (r, t) \Rightarrow (r', t')$ and any morphism of L-coalgebras $(u_2, v_2)\colon (l', s') \Rightarrow (l, s)$ preserves these canonical choices of lifts.*

Proof. The construction is shown in the diagram

$$\begin{tikzcd}
\cdot \arrow[r, "u"] \arrow[d, "Ll"'] & \cdot \arrow[d, shift left, "Lr"] \\
\cdot \arrow[r, dashed, "E(u,v)"] \arrow[d, shift right, "Rl"'] & \cdot \arrow[u, dashed, shift left, "t"] \arrow[d, "Rr"] \\
\cdot \arrow[u, dashed, shift right, "s"'] \arrow[r, "v"'] & \cdot
\end{tikzcd} \tag{II.7}$$

Commutativity of (II.6) follows immediately from (II.4) and (II.5).

That a morphism of R-algebras preserves these canonical lifts can be seen in the diagram

$$\begin{tikzcd}
\cdot \arrow[r, "u"] \arrow[d, "Ll"'] & \cdot \arrow[r, "u'"] \arrow[d, shift left, "Lr"] & \cdot \arrow[d, shift left, "Lr'"] \\
\cdot \arrow[r, "E(u,v)"] \arrow[d, shift right, "Rl"'] & \cdot \arrow[u, shift left, "t"] \arrow[r, "E(u',v')"] \arrow[d, "Rr"] & \cdot \arrow[u, shift left, "t'"] \arrow[d, "Rr'"] \\
\cdot \arrow[u, shift right, "s"'] \arrow[r, "v"'] & \cdot \arrow[r, "v'"'] & \cdot
\end{tikzcd}$$

noting that $u'tE(u, v)s = t'E(u', v')E(u, v)s = t'E(u'u, v'v)s$. □

This, together with the classical fact that the class of objects admitting a (co)algebra structure for a (co)pointed endofunctor is closed under retracts, gives a third equivalent definition of a functorial weak factorization system.

Lemma II.9. *A functorial weak factorization system can equivalently be defined to be a functorial factorization on $\mathscr{C}$ such that*

- *for every morphism f in $\mathscr{C}$, Lf admits an L-coalgebra structure, and Rf admits an R-algebra structure.*

An R-algebra structure on Rf consists of a morphism $\vec{\mu}_f\colon R^2 f \to Rf$ in $\mathscr{C}^2$ such that $\vec{\mu}_f \circ \vec{\eta}_{Rf} = \mathrm{id}_{Rf}$, while an L-coalgebra structure on Lf consists of a morphism

$\vec{\delta}_f\colon Lf \to L^2 f$ such that $\vec{\epsilon}_{Lf} \circ \vec{\delta}_f = \mathrm{id}_{Lf}$. We might hope that it is possible to choose these structures for all f in a natural way, such that they form the components of natural transformations $\vec{\mu}\colon R^2 \Rightarrow R$ and $\vec{\delta}\colon L \Rightarrow L^2$.

If we want these choices of lifts to be fully coherent, we should also ask that for any R-algebra (f,t), the lift constructed as in (II.7) for the square (II.4) is equal to t, and similarly for L-coalgebras and (II.5). Lastly, we should ask that the components $\vec{\mu}_f$ and $\vec{\delta}_f$ are (co)algebra morphisms. These conditions, plus one more ensuring that there is an unambiguous notion of a morphism with both L-algebra and R-coalgebra structures, lead to the definition of an *algebraic weak factorization system*, first given in [GT06] (there called *natural* weak factorization systems), and further refined in [Gar07] and [Gar09].

Definition II.10. An *algebraic weak factorization system* on a category $\mathscr{C}$ consists of a functorial factorization $(L,\vec{\epsilon},R,\vec{\eta})$ together with natural transformations $\vec{\mu}\colon R^2 f \Rightarrow Rf$ and $\vec{\delta}\colon L \Rightarrow L^2$, such that

- $\mathbb{R} = (R,\vec{\eta},\vec{\mu})$ is a monad and $\mathbb{L} = (L,\vec{\epsilon},\vec{\delta})$ a comonad on $\mathscr{C}^2$, and
- the natural transformation $\Delta = (\delta,\mu)\colon LR \Rightarrow RL$ determined by the equation $\epsilon L \circ \delta = \mu \circ \eta R \,(= \mathrm{id}_E)$ as in lemma II.1 is a distributive law, which in this case reduces to the single condition $\delta \circ \mu = \mu L \circ E\Delta \circ \delta R$.

Just as we saw that a functorial factorization already determines the left and right classes of morphisms, there is a condition we can place on a functor F between categories equipped with functorial factorizations which implies that F preserves the left class.

Definition II.11. Let $\mathscr{C}$ and $\mathscr{D}$ be categories equipped with functorial factorizations (E_1,η_1,ϵ_1) and (E_2,η_2,ϵ_2) respectively. A *colax morphism of functorial factorizations* is a pair (F,ϕ) where $F\colon C \to D$ is a functor and ϕ is a natural transformation

$$\begin{tikzcd} C^2 \arrow[r, "E_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[d, "F"] \\ D^2 \arrow[r, "E_2"'] & D \end{tikzcd}$$

such that

$$
\begin{tikzcd}
C^2 \arrow[r, "E_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "E_2"] \arrow[r, bend right=60, "\mathrm{cod}"'] \arrow[r, phantom, shift right=3, "\Downarrow\epsilon_2"] & D
\end{tikzcd}
=
\begin{tikzcd}
C^2 \arrow[r, bend left=60, "E_1"] \arrow[r, phantom, shift left=3, "\Downarrow\epsilon_1"] \arrow[r, "\mathrm{cod}"'] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "\mathrm{cod}"'] & D
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
C^2 \arrow[r, bend left=60, "\mathrm{dom}"] \arrow[r, phantom, shift left=3, "\Downarrow\eta_1"] \arrow[r, "E_1"'] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "E_2"'] & D
\end{tikzcd}
=
\begin{tikzcd}
C^2 \arrow[r, "\mathrm{dom}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "\mathrm{dom}"] \arrow[r, bend right=60, "E_2"'] \arrow[r, phantom, shift right=3, "\Downarrow\eta_2"] & D.
\end{tikzcd}
$$

In components, given a morphism $f\colon X \to Y$ in C, these two equations simply say that the following diagram commutes:

$$
\begin{tikzcd}
FX \arrow[r, "L_2(Ff)"] \arrow[d, "F(L_1 f)"'] & E_2(Ff) \arrow[d, "R_2(Ff)"] \\
F(E_1 f) \arrow[r, "F(R_1 f)"'] \arrow[ur, "\phi_f"] & FY.
\end{tikzcd}
$$

Here, $\hat{F}\colon C^2 \to D^2$ is the obvious lift of F to the arrow categories, sending an object $(f\colon X \to Y)$ in C^2 to the object $(Ff\colon FX \to FY)$ in D^2.

Proposition II.12. *Let (F, ϕ) be a colax morphism of functorial factorizations as above. Then F preserves the left class of morphisms, i.e. if $f\colon X \to Y$ in C has an L_1-coalgebra structure, then Ff has an L_2-coalgebra structure.*

Proof. Let $f\colon X \to Y$ be a morphism in the left class in C, with L_1-coalgebra structure given by the lift s in

$$
\begin{tikzcd}
X \arrow[r, "Lf"] \arrow[d, "f"'] & Ef \arrow[d, "Rf"] \\
Y \arrow[r, equal] \arrow[ur, "s"] & Y.
\end{tikzcd}
$$

Then Ff has an L_2-coalgebra structure given by $\phi_f F(s)$, as shown by the commutativity of the diagram

$$\begin{tikzcd}
FX \arrow[rr, "L_2(Ff)"] \arrow[dr, "F(L_1f)"] \arrow[dd, "Ff"'] & & E_2(Ff) \arrow[dd, "R_2(Ff)"] \\
& F(E_1f) \arrow[ur, "\phi_f"] \arrow[dr, "F(R_1f)"] & \\
FY \arrow[ur, "Fs"] \arrow[rr, equal] & & FY
\end{tikzcd}$$

□

If F has a right adjoint G, then the natural transformation ϕ determines a natural transformation $\phi' \colon E_1\hat{G} \to \hat{G}E_2$ which ensures that G preserves the right class of morphisms analagously to proposition II.12, and making G what is called a *lax morphism of functorial factorizations*. The relationship between ϕ and ϕ' is what is known as the mates correspondence (see e.g. [CGR12]), and we say that ϕ' is the mate of ϕ, and vice versa.

It turns out that the mates correspondence is best understood in the context of double categories, which we review in the next section. The theory of mates underlies the definition of adjunctions of awfs, and is the reason why double categories are an essential part of our general framework.

CHAPTER III

DOUBLE CATEGORIES

Recall in definition II.11, a colax morphism of functorial factorizations is a pair (F, ϕ), where ϕ is a natural transformation

$$\begin{tikzcd} C^2 \arrow[r, "E_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[d, "F"] \\ D^2 \arrow[r, "E_2"'] & D \end{tikzcd} \tag{III.1}$$

In [Rie11] it is proven that if $F \dashv G$ is an adjunction, then specifying a natural transformation ϕ making F a colax morphism uniquely determines a natural transformation θ making G a lax morphism. The transformation θ is called the *mate* of ϕ, and is found by composing ϕ with the unit and counit of $\hat{F}$ and F respectively.

This mates correspondence (see example III.3) defines a bijection between natural transformations of the form (III.1) and natural transformations of the form

$$\begin{tikzcd} C^2 \arrow[r, "E_1"] \arrow[dr, phantom, "\Downarrow\theta"] & C \\ D^2 \arrow[r, "E_2"'] \arrow[u, "\hat{G}"] & D. \arrow[u, "G"'] \end{tikzcd}$$

The collection of square 2-cells, where the vertical 1-cells are required to be the left adjoint of an adjunction, and the horizontal 1-cells are allowed to be arbitrary functors, can be organized into a structure called a *double category*. Similarly, there is a double category where the vertical 1-cells are required to be right adjoints (with some subtlety regarding the direction vertical 1-cells and 2-cells point), and the naturality of the mates correspondence can be expressed by saying these two double categories are isomorphic.

Double categories are a fundamental structure for this thesis, primarily due to the importance of the mates correspondence to the algebraic analogue of Quillen functors: lax and colax morphisms of awfs.

In section III.1, we begin by giving an overview of double categories. Then in section III.2 we give a generalization of definition II.2, defining arrow objects in a double category by means of a universal property. This is needed to be able to define functorial factorizations and (co)lax morphisms of functorial factorizations in general double categories, which we do in chapter VI.

We will ultimately want to define an algebraic weak factorization system to be a sort of bialgebraic object in a (two-fold) double category. To prepare the way for this definition, in section III.3 we give (one possible version of) the definition of monads in a double category.

III.1 Review of Double Categories

We first give the most concise definition of a double category, which we will then break down into more concrete terms.

Definition III.1. A (strict) *double category* is an internal category object in the (large) category of categories.

So a double category $\mathbb{D}$ consists of a category $\mathbb{D}_0$ and a category $\mathbb{D}_1$, along with functors $s, t\colon \mathbb{D}_1 \to \mathbb{D}_0$, $i\colon \mathbb{D}_0 \to \mathbb{D}_1$, and $\otimes\colon \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \to \mathbb{D}_1$ satisfying the usual axioms of a category. We will call the objects of $\mathbb{D}_0$ the 0-cells (or just objects) of $\mathbb{D}$, and the morphisms of $\mathbb{D}_0$ the vertical 1-cells. Thus $\mathbb{D}_0$ forms the so-called *vertical category* of $\mathbb{D}$. We will call the objects of $\mathbb{D}_1$ the horizontal 1-cells of $\mathbb{D}$, and the morphisms of $\mathbb{D}_1$ are the 2-cells.

A morphism $\phi\colon X \to Y$ in $\mathbb{D}_1$, where $s(X) = C$, $t(X) = C'$, $s(Y) = D$, $t(Y) = D'$, $s(\phi) = f$, and $t(\phi) = g$ will be drawn as

$$\begin{tikzcd} C \arrow[r, "X", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & C' \arrow[d, "g"] \\ D \arrow[r, "Y"', "\shortmid"{marking}] & D' \end{tikzcd} \tag{III.2}$$

where the tick-mark on the horizontal 1-cells serves as a further reminder that the horizontal 1-cells are of a different nature than the vertical 1-cells. The composition in $\mathbb{D}_0$ provides a vertical composition of vertical 1-cells and 2-cells, while the composition functor $\otimes\colon \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \to \mathbb{D}_1$ provides a horizontal composition of horizontal 1-cells and 2-cells.

For any object C in $\mathbb{D}_0$, $i(C)$ is the *unit* horizontal 1-cell

$$C \xrightarrow{I_C} C$$

and acts as an identity with respect to the horizontal composition.

A 2-cell θ for which $s\theta = t\theta = \mathrm{id}$ will be called *globular*. We will sometimes draw globular 2-cells as

$$\begin{tikzcd} C \arrow[r, bend left, "X", "\shortmid"{marking}] \arrow[r, bend right, "Y"', "\shortmid"{marking}] \arrow[r, phantom, "\Downarrow\theta"] & C', \end{tikzcd}$$

to save space and help readability of diagrams.

Example III.2. For any 2-category $\mathcal{D}$, there is an associated double category $\mathbb{S}\mathrm{q}(\mathcal{D})$ of *squares* in $\mathcal{D}$, in which the vertical and horizontal 1-cells are both just 1-cells in $\mathcal{D}$, and 2-cells

$$\begin{tikzcd} C \arrow[r, "j", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & C' \arrow[d, "g"] \\ D \arrow[r, "k"', "\shortmid"{marking}] & D' \end{tikzcd}$$

are simply 2-cells $\phi\colon gj \Rightarrow kf$ in $\mathcal{D}$.

Example III.3. Given any 2-category $\mathcal{D}$, there is a double category $\mathbb{L}\mathbf{Adj}(\mathcal{D})$. The horizontal 1-cells are just 1-cells in $\mathcal{D}$, while the vertical 1-cells are fully specified adjunctions $f \dashv g$ (meaning the unit and counit have been chosen) pointing in the direction of the left adjoint. 2-cells

$$\begin{tikzcd} C \arrow[r, "j", "\shortmid"{marking}] \arrow[d, "(f\dashv g)"'] \arrow[dr, phantom, "\Downarrow\phi"] & C' \arrow[d, "(f'\dashv g')"] \\ D \arrow[r, "k"', "\shortmid"{marking}] & D' \end{tikzcd}$$

are natural transformations involving the left adjoints, $\phi\colon f'j \Rightarrow kf$.

Similarly there is a double category $\mathbb{R}\mathbf{Adj}(\mathcal{D})$ where the vertical 1-cells still point in the direction of the left adjoint, but the 2-cells are natural transformations involving the right adjoints, $\phi\colon jg \Rightarrow g'k$.

The *mates correspondence* (see e.g. [CGR12]) gives an isomorphism $\mathbb{L}\mathbf{Adj}(\mathcal{D}) \cong \mathbb{R}\mathbf{Adj}(\mathcal{D})$, which sends a 2-cell ϕ in $\mathbb{L}\mathbf{Adj}(\mathcal{D})$ to the 2-cell

$$\begin{tikzcd} & C \arrow[r, "j"] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\phi"] & C' \arrow[r, equal] \arrow[d, "f'"'] & C' \\ D \arrow[r, equal] \arrow[ur, "g"] \arrow[r, phantom, bend left, "\Downarrow\epsilon"] & D \arrow[r, "k"'] & D' \arrow[ur, "g'"'] \arrow[u, phantom, shift left=4, "\Downarrow\eta"] & \end{tikzcd}$$

given by composing with the unit/counit of the adjunctions.

Example III.4. Given any category $\mathscr{M}$, there is a pseudo double category $\mathbb{S}\mathrm{pan}(\mathscr{M})$ of *spans* in $\mathscr{M}$. The vertical category of $\mathbb{S}\mathrm{pan}(\mathscr{M})$ is just $\mathscr{M}$, while horizontal 1-cells

$$C \xrightarrow{X} D$$

are given by spans

$$C \xleftarrow{j} X \xrightarrow{k} D$$

in $\mathscr{M}$, and 2-cells

$$\begin{tikzcd} C \arrow[r, "X"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\theta"] & D \arrow[d, "g"] \\ C' \arrow[r, "Y"'] & D' \end{tikzcd}$$

are given by commutative diagrams

$$\begin{tikzcd} C \arrow[d, "f"'] & X \arrow[l, "j"'] \arrow[r, "k"] \arrow[d, "\theta"] & D \arrow[d, "g"] \\ C' & Y \arrow[l, "j'"] \arrow[r, "k'"'] & D'. \end{tikzcd}$$

The horizontal composition of spans is given by pullback. It is because this horizontal composition is only determined up to isomorphsim that this example is not a strict double category.

Definition III.5. For any double category $\mathbb{D}$, there is an associated 2-category $\mathcal{H}or(\mathbb{D})$, called the *horizontal 2-category* of $\mathbb{D}$. The objects and 1-cells of $\mathcal{H}or(\mathbb{D})$ are the objects and horizontal 1-cells of $\mathbb{D}$, while 2-cells $\phi\colon X \Rightarrow Y$ in $\mathcal{H}or(\mathbb{D})$ are the globular 2-cells in $\mathbb{D}$, i.e. those of the form

$$\begin{tikzcd} C \arrow[r, "X"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\phi"] & D \arrow[d, equal] \\ C \arrow[r, "Y"'] & D \end{tikzcd}$$

Notice that $\mathcal{H}or(\mathbb{S}\mathrm{q}(\mathcal{D}))$ is isomorphic to $\mathcal{D}$.

Definition III.6. Given a double category $\mathbb{D}$, define double categories $\mathbb{D}^{\mathrm{vop}}$ and $\mathbb{D}^{\mathrm{hop}}$, obtained by reversing the direction of the vertical and horizontal 1-cells respectively, and changing the orientation of the 2-cells as appropriate. For example, a 2-cell (III.2) in $\mathbb{D}^{\mathrm{vop}}$ is a 2-cell

$$\begin{tikzcd} D \arrow[r, "Y"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & D' \arrow[d, "g"] \\ C \arrow[r, "X"'] & C' \end{tikzcd}$$

in $\mathbb{D}$.

In terms of definition III.1, $\mathbb{D}^{\mathrm{vop}}$ is the double category obtained by replacing the categories $\mathbb{D}_0$ and $\mathbb{D}_1$ with their opposites, while $\mathbb{D}^{\mathrm{vop}}$ is the obtained by swapping the horizontal source and target functors s and t.

III.2 Arrow Objects in a Double Category

In the following we will need an extension of the universal property (II.1) to double categories. Fortunately, this is quite straightforward.

Let $\mathbb{D}$ be a double category. Given an object C of $\mathbb{D}$, the *arrow object* C^2, if it exists, is an object together with a diagram

$$\begin{tikzcd} C^2 \arrow[r, bend left, "\mathrm{dom}", "|" marking] \arrow[r, bend right, "\mathrm{cod}"', "|" marking] \arrow[r, phantom, "\Downarrow\kappa"] & C, \end{tikzcd}$$

such that any 2-cell

$$\begin{tikzcd} A \arrow[r, bend left, "d_1", "|" marking] \arrow[r, bend right, "d_0"', "|" marking] \arrow[r, phantom, "\Downarrow\alpha"] & C \end{tikzcd}$$

uniquely factors through κ, as

$$\begin{tikzcd} A \arrow[r, "\hat{\alpha}", "|" marking] & C^2 \arrow[r, bend left, "\mathrm{dom}", "|" marking] \arrow[r, bend right, "\mathrm{cod}"', "|" marking] \arrow[r, phantom, "\Downarrow\kappa"] & C. \end{tikzcd}$$

Given a vertical 1-cell $F\colon C \to D$ in $\mathbb{D}$, the *lift to arrow objects* $\hat{F}\colon C^2 \to D^2$, if it exists, is a vertical 1-cell $\hat{F}\colon C^2 \to D^2$ together with 2-cells

$$\begin{tikzcd} C^2 \arrow[r, "\mathrm{dom}", "|" marking] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_1"] & C \arrow[d, "F"] \\ D^2 \arrow[r, "\mathrm{dom}"', "|" marking] & D \end{tikzcd} \qquad \begin{tikzcd} C^2 \arrow[r, "\mathrm{cod}", "|" marking] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_0"] & C \arrow[d, "F"] \\ D^2 \arrow[r, "\mathrm{cod}"', "|" marking] & D \end{tikzcd}$$

satisfying

$$\begin{tikzcd} C^2 \arrow[r, "\mathrm{dom}", "|" marking] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_1"] & C \arrow[d, "F"] \\ D^2 \arrow[r, "\mathrm{dom}", "|" marking] \arrow[r, bend right=60, "\mathrm{cod}"'] \arrow[r, phantom, shift right=3, "\Downarrow\kappa"] & D \end{tikzcd} = \begin{tikzcd} C^2 \arrow[r, bend left=60, "\mathrm{dom}", "|" marking] \arrow[r, phantom, shift left=3, "\Downarrow\kappa"] \arrow[r, "\mathrm{cod}"', "|" marking] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_0"] & C \arrow[d, "F"] \\ D^2 \arrow[r, "\mathrm{cod}"', "|" marking] & D, \end{tikzcd}$$

such that for any 2-cells

$$\begin{tikzcd} A \arrow[r, bend left, "d_1"] \arrow[r, bend right, "d_0"'] \arrow[r, phantom, "\Downarrow\alpha"] & C \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, bend left, "d_1'"] \arrow[r, bend right, "d_0'"'] \arrow[r, phantom, "\Downarrow\alpha'"] & D \end{tikzcd}$$

and

$$\begin{tikzcd} A \arrow[r, "d_1"] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\lambda_1"] & C \arrow[d, "F"] \\ B \arrow[r, "d_1'"'] & D \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "d_0"] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\lambda_0"] & C \arrow[d, "F"] \\ B \arrow[r, "d_0'"'] & D \end{tikzcd}$$

satisfying

$$\begin{tikzcd} A \arrow[r, "d_1"] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\lambda_1"] & C \arrow[d, "F"] \\ B \arrow[r, "d_1'"] \arrow[r, bend right=60, "d_0'"'] \arrow[r, phantom, shift right=2ex, "\Downarrow\alpha'"] & D \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left=60, "d_1"] \arrow[r, phantom, shift left=2ex, "\Downarrow\alpha"] \arrow[r, "d_0"'] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\lambda_0"] & C \arrow[d, "F"] \\ B \arrow[r, "d_0'"'] & D \end{tikzcd}$$

there is a unique 2-cell

$$\begin{tikzcd} A \arrow[r, "\hat{\alpha}"] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\theta"] & C^2 \arrow[d, "\hat{F}"] \\ B \arrow[r, "\hat{\alpha}'"'] & D^2 \end{tikzcd}$$

such that the horizontal composition of θ with γ_0 and γ_1 is respectively equal to λ_0 and λ_1.

Remark III.7. Note that in most examples, the 2-cells γ_0 and γ_1 will be isomorphisms (or even identities). This just says that given any $g\colon X \to Y$ in C^2, the lift $\hat{F}(g)$ will be some $\hat{F}(g)\colon FX \to FY$. We are not aware of any examples where the γ_i are not isomorphisms, though when we generalize this universal property to the multivariable setting, they will necessarily not be isomorphisms (rather they will be projections out of a pullback).

Definition III.8. A double category $\mathbb{D}$ *has arrow objects* if for every object C of $\mathbb{D}$ there is an object C^2 and 2-cell κ, and for every vertical 1-cell F there is a vertical 1-cell $\hat{F}$ and 2-cells γ_0 and γ_1, satisfying the universal properties given above.

The intuition that this is a generalization of lemma II.1 is supported by the following two propositions, the (easy) proofs of which are left to the reader.

Proposition III.9. *If the double category $\mathbb{D}$ has arrow objects, then so does $\mathcal{H}or(\mathbb{D})$.*

Proposition III.10. *If the 2-category $\mathcal{D}$ has arrow objects, then so does $\mathbb{S}\mathrm{q}(\mathcal{D})$.*

Proof. A simple check. The 2-cells γ_0 and γ_1 will always be identities. $\square$

III.3 Monads

We will define a *monad* in a double category $\mathbb{D}$ to be a tuple (C, T, η, μ), in which C is an object, $T\colon C \to C$ is a horizontal 1-cell, and η and μ are 2-cells

$$\begin{tikzcd} C \arrow[r, "\mathrm{id}_C"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta"] & C \arrow[d, equal] \\ C \arrow[r, "T"'] & C \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "T"] \arrow[d, equal] & C \arrow[r, "T"] \arrow[d, phantom, "\Downarrow\mu"] & C \arrow[d, equal] \\ C \arrow[rr, "T"'] & & C \end{tikzcd}$$

satisfying the usual unit and associativity conditions. In other words, a monad in $\mathbb{D}$ is simply a monad in the 2-category $\mathcal{H}or(\mathbb{D})$. The non-identity vertical 1-cells come into play in the morphisms of monads.

Given two monads (C, T, η, μ) and (D, S, η', μ'), a monad morphism from (C, T) to (D, S) consists of a pair (f, ϕ), where f is a vertical 1-cell $C \to D$ and ϕ is a 2-cell

$$\begin{tikzcd} C \arrow[r, "T"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[d, "f"] \\ D \arrow[r, "S"'] & D \end{tikzcd}$$

which commutes with the unit and multiplication 2-cells in the sense of the two equations

$$\begin{tikzcd} C \arrow[r, "\mathrm{id}_C"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta"] & C \arrow[d, equal] \\ C \arrow[r, "T"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[d, "f"] \\ D \arrow[r, "S"'] & D \end{tikzcd} = \begin{tikzcd} C \arrow[r, "\mathrm{id}_C"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}_f"] & C \arrow[d, "f"] \\ D \arrow[r, "\mathrm{id}_D"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta'"] & D \arrow[d, equal] \\ D \arrow[r, "S"'] & D \end{tikzcd} \tag{III.3}$$

and

$$\begin{tikzcd} C \arrow[r, "T"] \arrow[d, equal] & C \arrow[r, "T"] \arrow[d, phantom, "\Downarrow\mu"] & C \arrow[d, equal] \\ C \arrow[rr, "T"] \arrow[d, "f"'] & \arrow[d, phantom, "\Downarrow\phi"] & C \arrow[d, "f"] \\ D \arrow[rr, "S"'] & & D \end{tikzcd} = \begin{tikzcd} C \arrow[r, "T"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[r, "T"] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[d, "f"] \\ D \arrow[r, "S"] \arrow[d, equal] & D \arrow[r, "S"] \arrow[d, phantom, "\Downarrow\mu'"] & D \arrow[d, equal] \\ D \arrow[rr, "S"'] & & D \end{tikzcd} \tag{III.4}$$

Definition III.11. Given any double category $\mathbb{D}$, we will write $\mathrm{Mon}(\mathbb{D})$ for the category of monads in $\mathbb{D}$, consisting of monads and monad morphisms as defined above. The category $\mathrm{Comon}(\mathbb{D})$ of comonads in $\mathbb{D}$ is defined to be the category $\mathrm{Mon}(\mathbb{D}^{\mathrm{op}})$ of monads in $\mathbb{D}^{\mathrm{op}}$.

Example III.12. The category $\mathrm{Mon}(\mathbb{S}\mathrm{pan}(\mathbf{Set}))$ is precisely the category of small categories. It is an easy and enlightening exercise to work this out for oneself.

Proposition III.13. *The categories of (co)monads and (co)lax morphisms in a 2-category $\mathcal{D}$ can be given in terms of (co)monads in the double category of squares as follows:*

$$
\begin{aligned}
\mathrm{Mon}_{colax}(\mathcal{D}) &= \mathrm{Mon}(\mathbb{S}\mathrm{q}(\mathcal{D})) \\
\mathrm{Comon}_{colax}(\mathcal{D}) &= \mathrm{Comon}(\mathbb{S}\mathrm{q}(\mathcal{D})) \\
\mathrm{Mon}_{lax}(\mathcal{D}) &= \mathrm{Mon}(\mathbb{S}\mathrm{q}(\mathcal{D}^{op}))^{op} \\
\mathrm{Comon}_{lax}(\mathcal{D}) &= \mathrm{Comon}(\mathbb{S}\mathrm{q}(\mathcal{D}^{op}))^{op}
\end{aligned}
$$

where by $\mathcal{D}^{op}$ we mean the 2-category obtained by reversing the direction of all 1-cells (but not 2-cells).

Proof. Immediate from the definitions. Readers unfamiliar with (co)lax morphisms of monads can take this as the definition. □

III.4 Double Functors

The natural notion of functor between double categories is a straightforward generalization of lax functors between monoidal categories. Recall that we are using the symbol $\otimes$ to denote horizontal composition.

Definition III.14. Let $\mathbb{D}$ and $\mathbb{E}$ be double categories. A *lax double functor* $F\colon \mathbb{D} \to \mathbb{E}$ consists of:

- Functors $F_0\colon \mathbb{D}_0 \to \mathbb{E}_0$ and $F_1\colon \mathbb{D}_1 \to \mathbb{E}_1$ such that $sF_1 = F_0 s$ and $tF_1 = F_0 t$
- Natural transformations with globular components $F_\otimes\colon F_1 X \otimes F_1 Y \to F_1(X \otimes Y)$ and $F_I\colon I_{F_0 C} \to F_1(I_C)$, which satisfy the usual coherence axioms for a lax monoidal functor.

A lax double functor F for which the components of F_I and $F_\otimes$ are identities will be called *strict*. For the intermediate notion where the components of F_I and $F_\otimes$ are (vertical) isomorphsims, we will simply refer to F as a double functor.

Proposition III.15. *A lax double functor $F\colon \mathbb{D} \to \mathbb{E}$ induces a functor $F\colon \mathrm{Mon}(\mathbb{D}) \to \mathrm{Mon}(\mathbb{E})$.*

Proof. This works just like the case for monoidal categories. For instance, if X is a monad in $\mathbb{D}$, FX has the multiplication

$$\begin{tikzcd} C \arrow[r, "FX", "\shortmid"{marking}] \arrow[d, equal] & C \arrow[r, "FX", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow F_\otimes"] & C \arrow[d, equal] \\ C \arrow[rr, "F(X\otimes X)", "\shortmid"{marking}] \arrow[d, equal] & {} \arrow[d, phantom, "\Downarrow F\mu"] & C \arrow[d, equal] \\ C \arrow[rr, "FX"', "\shortmid"{marking}] & {} & C \end{tikzcd}$$

The fact that F takes monad morphisms to monad morphisms can easily be checked using the naturality of F_I and $F_\otimes$. □

We will have need for a condition on a lax double functor which implies a sort of converse to proposition III.15. This condition is a slight strengthening of the notion of fully-faithful functor which makes sense for lax functors.

Definition III.16. A lax double functor $F \colon \mathbb{D} \to \mathbb{E}$ is fully-faithful on 2-cells if, for any fixed X_1, X_2, Y, f, g, the induced function from 2-cells in $\mathbb{D}$ of the shape

$$\begin{tikzcd} C_0 \arrow[r, "X_1", "\shortmid"{marking}] \arrow[d, "f"'] & C_1 \arrow[r, "X_2", "\shortmid"{marking}] & C_2 \arrow[d, "g"] \\ D_0 \arrow[rr, "Y"', "\shortmid"{marking}] & & D_1 \end{tikzcd}$$

to 2-cells in $\mathbb{E}$ of the shape

$$\begin{tikzcd} FC_0 \arrow[r, "FX_1", "\shortmid"{marking}] \arrow[d, "Ff"'] & FC_1 \arrow[r, "FX_2", "\shortmid"{marking}] & FC_2 \arrow[d, "Fg"] \\ FD_0 \arrow[rr, "FY"', "\shortmid"{marking}] & & FD_1, \end{tikzcd}$$

which takes a 2-cell θ to $F(\theta) \circ F_\otimes$, is a bijection, and if similarly the induced function from 2-cells of the shape

$$\begin{tikzcd} C \arrow[r, "I_C", "\shortmid"{marking}] \arrow[d, "f"'] & C \arrow[d, "g"] \\ D_0 \arrow[r, "Y"', "\shortmid"{marking}] & D_1 \end{tikzcd} \quad \text{to} \quad \begin{tikzcd} FC \arrow[r, "I_{FC}", "\shortmid"{marking}] \arrow[d, "Ff"'] & FC \arrow[d, "Fg"] \\ FD_0 \arrow[r, "FY"', "\shortmid"{marking}] & FD_1 \end{tikzcd}$$

taking a 2-cell ϕ to $F(\phi) \circ F_I$ is a bijection.

Remark III.17. Definition III.16 implies the function on 2-cells θ with a single horizontal

1-cell in the domain is also bijective. We leave the details of the (simple) proof to the reader.

Proposition III.18. *Let $F\colon \mathbb{D} \to \mathbb{E}$ be a fully-faithful lax double functor. Given any horizontal 1-cell X in $\mathbb{D}$, a monoid structure on FX in $\mathbb{E}$ lifts uniquely to a monoid structure on X such that the induced functor $F\colon \mathrm{Mon}(\mathbb{D}) \to \mathrm{Mon}(\mathbb{E})$ takes X to FX.*

Similarly, a vertical 1-cell $f\colon X \to Y$ for which Ff is a monoid morphism must also be a monoid morphism.

In other words, the diagram of categories is a pullback:

$$\begin{tikzcd}
\mathrm{Mon}(\mathbb{D}) \arrow[r, "F"] \arrow[d, "U"'] & \mathrm{Mon}(\mathbb{E}) \arrow[d, "U"] \\
\mathbb{D} \arrow[r, "F"'] & \mathbb{E}.
\end{tikzcd}$$

Proof. Simply use the surjectivity of the fully-faithful functor to lift the unit and multiplication 2-cells from FX to X, then use the injectivity to show that the unitary and associativity equations on FX imply those on X. □

CHAPTER IV

2-FOLD DOUBLE CATEGORIES

It is well known that the notion of bialgebra or bimonoid—an object with both monoid and comonoid structures which are compatible in a certain sense—makes sense not only in a symmetric monoidal category, but also in more general *braided* monoidal categories. A bimonoid in a braided monoidal category $\mathscr{C}$ can be defined to be a monoid in the category of comonoids in $\mathscr{C}$, or equivalently as a comonoid in the category of monoids in $\mathscr{C}$. The braiding is necessary to ensure that the monoidal structure in $\mathscr{C}$ lifts to a product in $\mathrm{Mon}(\mathscr{C})$ and $\mathrm{Comon}(\mathscr{C})$.

Less well known is the fact that the definition of bimonoid works just as well in a more general context still: the so-called 2-fold monoidal categories. A 2-fold monoidal category has two different monoidal structures, call them $(\otimes, I)$ and $(\odot, \perp)$, which are themselves compatible in certain sense. This compatibility can be stated in a way analogous to the definition of bimonoid given in the previous paragraph: a (strict) 2-fold monoidal category is a monoid object in the category $\mathbf{StrMonCat}_l$ of strict monoidal categories and lax functors, or equivalently a monoid object in the category $\mathbf{StrMonCat}_c$ of strict monoidal categories and colax functors. Notice that monoid objects in the category of strict monoidal categories and *strong* monoidal functors (in which the components of the lax structure are isomorphisms) are precisely (strict) braided monoidal categories.

More concretely, the compatibility between the monoidal structures amounts to the existence of maps

$$m\colon \perp \otimes \perp \to \perp, \qquad c\colon I \to I \odot I, \qquad j\colon I \to \perp,$$

making $(\perp, j, m)$ a $\otimes$-monoid and (I, j, c) a $\odot$-comonoid, and a natural family of maps

$$z_{A,B,C,D}\colon (A \odot B) \otimes (C \odot D) \to (A \otimes C) \odot (B \otimes D)$$

satisfying some coherence axioms.

Example IV.1.

- Any braided monoidal category can be made into a 2-fold monoidal category in which the two monoidal structures coincide.

- Any monoidal category $(\mathscr{C}, \otimes, I)$ with finite products has a 2-fold monoidal structure with $(\odot, \perp)$ given by the product and terminal object. Dually, a monoidal category $(\mathscr{C}, \odot, \perp)$ with finite coproducts has a 2-fold monoidal structure with $(\otimes, I)$ given by the coproduct and initial object.

Because the $\odot$-monoidal structure is lax monoidal with respect to the $\otimes$-monoidal structure, it lifts to the category $\mathrm{Mon}_{\otimes}(\mathscr{C})$ of $\otimes$-monoids in $\mathscr{C}$. Dually, the $\otimes$-monoidal structure lifts to the category $\mathrm{Comon}_{\odot}(\mathscr{C})$ of $\odot$-comonoids in $\mathscr{C}$. Thus, we could define the category of bimonoids in $\mathscr{C}$ to be either $\mathrm{Comon}_{\odot}(\mathrm{Mon}_{\otimes}(\mathscr{C}))$ or $\mathrm{Mon}_{\otimes}(\mathrm{Comon}_{\odot}(\mathscr{C}))$, and it turns out that these are canonically isomorphic. In either case, a bimonoid is an object A with a $\otimes$-monoid structure (η, μ) and a $\odot$-comonoid structure (ϵ, δ), such that the following four diagrams commute:

$$
\begin{tikzcd}
I \arrow[r, "\eta"] \arrow[d, "c"'] & A \arrow[d, "\delta"] \\
I \odot I \arrow[r, "\eta\odot\eta"'] & A \odot A,
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A \arrow[r, "\mu"] \arrow[d, "\epsilon\otimes\epsilon"'] & A \arrow[d, "\epsilon"] \\
\perp \otimes \perp \arrow[r, "m"'] & \perp,
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \arrow[dr, "\epsilon"] & \\
I \arrow[ur, "\eta"] \arrow[rr, "j"'] & & \perp,
\end{tikzcd}
$$
$$
\begin{tikzcd}
A \otimes A \arrow[rr, "\mu"] \arrow[d, "\delta\otimes\delta"'] & & A \arrow[d, "\delta"] \\
(A \odot A) \otimes (A \odot A) \arrow[r, "z_{A,A,A,A}"'] & (A \otimes A) \odot (A \otimes A) \arrow[r, "\mu\odot\mu"'] & A \odot A.
\end{tikzcd}
\tag{IV.1}
$$

In [Gar07] and [Gar09], Garner proves that given any category $\mathscr{C}$, there is a 2-fold monoidal category of functorial factorizations on $\mathscr{C}$. Given two functorial factorizations (E_1, L_1, R_1) and (E_2, L_2, R_2), the factorization $E_1 \otimes E_2$ factors an arrow $f\colon X \to Y$ as

$$X \xrightarrow{L_2R_1f \circ L_1f} E_2R_1f \xrightarrow{R_2R_1f} Y$$

while the factorization $E_1 \odot E_2$ factors f as

$$X \xrightarrow{L_2L_1f} E_2L_1f \xrightarrow{R_1f \circ R_2L_1f} Y.$$

Garner shows that bimonoids in this 2-fold monoidal category are equivalent to algebraic weak factorization systems on $\mathscr{C}$. In other words, a bialgebra structure on a functorial factorization (E, η, ϵ) is precisely a choice of monad and comonad making E an awfs. However, as this structure only contains functorial factorizations on a fixed category $\mathscr{C}$, it can say nothing about morphisms between factorization systems on different categories.

To address this shortcoming, we will generalize this 2-fold monoidal category definition to double categories, where there are two different horizontal compositions which are compatible in a way analogous to the two monoidal structures in a 2-fold monoidal category. In chapter VI we will construct a 2-fold double category of functorial factorizations, generalizing Garner's 2-fold monoidal category, and in chapter VII we will see that bimonads and bimonad morphisms in this 2-fold double category are exactly awfs and colax morphisms of awfs.

IV.1 2-Fold Double Categories

We will start with a concise formal definition, and then expand on the definition more concretely.

Definition IV.2. A *2-fold double category* $\mathbb{D}$ with vertical category $\mathrm{Vert}(\mathbb{D}) = \mathscr{D}_0$ is a 2-fold monoid object in the 2-category $\mathcal{C}at/\mathscr{D}_0$ of categories over $\mathscr{D}_0$.

Breaking this down, we have a category $\mathscr{D}_1$, a functor $p\colon \mathscr{D}_1 \to \mathscr{D}_0$, two functors $\otimes, \odot\colon \mathscr{D}_1 \times_{\mathscr{D}_0} \mathscr{D}_1 \to \mathscr{D}_1$ commuting with p, and two functors $I, \perp\colon \mathscr{D}_0 \to \mathscr{D}_1$ which are sections of p, such that $\otimes$, $\odot$, I, and $\perp$ satisfy all the axioms of a 2-fold monoidal category. In particular, each fiber of p has a 2-fold monoidal structure.

A monoid object in $\mathcal{C}at/\mathbb{D}_0$ is equivalently a double category where the source and target functors $s, t\colon \mathbb{D}_1 \to \mathbb{D}_0$ are equal, and with the vertical category $\mathbb{D}_0$. Conversely, any double category $\mathbb{D}$ in which all horizontal 1-cells have equal domain and codomain, and all 2-cells have equal vertical 1-cells as domain and codomain, is equivalently a monoid object in $\mathcal{C}at/\mathbb{D}_0$. We will alternate between these two descriptions as convenient.

Using this shift of perspective, $\mathbb{D}$ has two underlying double categories, both with vertical category $\mathscr{D}_0$ and with source and target functors both equal to $p\colon \mathscr{D}_1 \to \mathscr{D}_0$. The double category $\mathbb{D}_\otimes$ has the rest of the double category structure given by the functors I and $\otimes$, while the double category $\mathbb{D}_\odot$ uses the functors $\perp$ and $\odot$.

Using this double category interpretation, we will find it convenient to think of a 2-fold double category as a double category with two different but interacting horizontal compositions. Notice that from this perspective, all horizontal 1-cells are endomorphisms.

Remark IV.3. It may seem somewhat ad hoc to force a 2-fold monoid object in a slice of $\mathcal{C}$ into a double category mold, with the odd looking restriction to having only endomorphisms in the horizontal direction. We will make essential use of double functors from $\mathbb{D}_\odot$ and $\mathbb{D}_\otimes$ to genuine double categories (without the endomorphism restriction), and it is mostly for this reason that we have found the double categorical perspective useful, if perhaps only psychologically.

We did give some thought to how one might define a 2-fold double category with non-endomorphism horizontal 1-cells and 2-cells, and while it seems like there might be a workable definition, it would require a very large increase in complexity. As we are mostly interested in the monads and comonads in a 2-fold double category, which are structures on endomorphism horizontal 1-cells, this restriction was of no concern to this work.

Now let us explicitly look at the 2-fold monoidal structure from the double categorical perspective. For any object C there are 2-cells

$$\begin{tikzcd} C \arrow[r, "\perp_C \otimes \perp_C", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow m"] & C \arrow[d, equal] \\ C \arrow[r, "\perp_C"', "\shortmid"{marking}] & C \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "I_C", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow c"] & C \arrow[d, equal] \\ C \arrow[r, "I_C \odot I_C"', "\shortmid"{marking}] & C \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "I_C", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow j"] & C \arrow[d, equal] \\ C \arrow[r, "\perp_C"', "\shortmid"{marking}] & C \end{tikzcd} \tag{IV.2}$$

and for any four horizontal morphisms $W, X, Y, Z\colon C \nrightarrow C$ there is a 2-cell

$$\begin{tikzcd} C \arrow[r, "(W \odot X) \otimes (Y \odot Z)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow z"] & C \arrow[d, equal] \\ C \arrow[r, "(W \otimes Y) \odot (X \otimes Z)"', "\shortmid"{marking}] & C. \end{tikzcd} \tag{IV.3}$$

These are natural in the sense that, for any vertical morphism $f\colon C \to D$ we have an equality

$$\begin{tikzcd} C \arrow[r, "\perp_C \otimes \perp_C", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow m"] & C \arrow[d, equal] \\ C \arrow[r, "\perp_C", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \perp_f"] & C \arrow[d, "f"] \\ D \arrow[r, "\perp_D"', "\shortmid"{marking}] & D \end{tikzcd} = \begin{tikzcd} C \arrow[r, "\perp_C \otimes \perp_C", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \perp_f \otimes \perp_f"] & C \arrow[d, "f"] \\ D \arrow[r, "\perp_D \otimes \perp_D"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow m"] & D \arrow[d, equal] \\ D \arrow[r, "\perp_D"', "\shortmid"{marking}] & D \end{tikzcd}$$

and similarly for c and j, and for any four 2-cells $\theta_1, \ldots, \theta_4$ of the appropriate form, we have an equality

$$
\begin{tikzcd}
C \arrow[r, "(W\odot X)\otimes(Y\odot Z)"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow z"] & C \arrow[d, equal] \\
C \arrow[r, "(W\otimes Y)\odot(X\otimes Z)"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow(\theta_1\otimes\theta_3)\odot(\theta_2\otimes\theta_4)"] & C \arrow[d, "f"] \\
D \arrow[r, "(W'\otimes Y')\odot(X'\otimes Z')"'] & D
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[r, "(W\odot X)\otimes(Y\odot Z)"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow(\theta_1\odot\theta_2)\otimes(\theta_3\odot\theta_4)"] & C \arrow[d, "f"] \\
C \arrow[r, "(W'\odot X')\otimes(Y'\odot Z')"'] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow z"] & C \arrow[d, equal] \\
D \arrow[r, "(W'\otimes Y')\odot(X'\otimes Z')"'] & D
\end{tikzcd}
$$

IV.2 Monads in 2-Fold Double Categories

Definition IV.4. A monad in a 2-fold double category $\mathbb{D}$ is a monad in $\mathbb{D}_\otimes$; a comonad in $\mathbb{D}$ is a comonad in $\mathbb{D}_\odot$. Furthermore, we define the categories $\mathrm{Mon}(\mathbb{D}) = \mathrm{Mon}(\mathbb{D}_\otimes)$ and $\mathrm{Comon}(\mathbb{D}) = \mathrm{Comon}(\mathbb{D}_\odot)$.

So a monad X and a comonad Y in $\mathbb{D}$ are given by 2-cells

$$
\begin{tikzcd}
C \arrow[r, "I_C"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta"] & C \arrow[d, equal] \\
C \arrow[r, "X"'] & C
\end{tikzcd}
\quad
\begin{tikzcd}
C \arrow[r, "X\otimes X"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mu"] & C \arrow[d, equal] \\
C \arrow[r, "X"'] & C
\end{tikzcd}
\quad
\begin{tikzcd}
C \arrow[r, "X"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\epsilon"] & C \arrow[d, equal] \\
C \arrow[r, "\perp_C"'] & C
\end{tikzcd}
\quad
\begin{tikzcd}
C \arrow[r, "X"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\delta"] & C \arrow[d, equal] \\
C \arrow[r, "X\odot X"'] & C.
\end{tikzcd}
$$

The categories $\mathrm{Mon}(\mathbb{D})$ and $\mathrm{Comon}(\mathbb{D})$ come naturally equipped with functors to $\mathcal{D}_0$, defined on objects and morphisms simply by applying p to the underlying 1-cells and 2-cells respectively. It turns out that the interaction between the $\otimes$ and $\odot$ compositions in the 2-fold double category structure is precisely what is needed to lift $\odot$ to $\mathrm{Mon}(\mathbb{D})$ and to lift $\otimes$ to $\mathrm{Comon}(\mathbb{D})$. In this way, we can define double categories $\mathbb{M}\mathrm{on}(\mathbb{D})$ and $\mathbb{C}\mathrm{omon}(\mathbb{D})$, both having $\mathcal{D}_0$ as vertical category.

These lifted compositions are defined as follows: Given two monads (C, X, η, μ) and (C, Y, η', μ') in $\mathbb{D}$, the horizontal composition

$$
C \xrightarrow{(X,\eta,\mu)} C \xrightarrow{(Y,\eta',\mu')} C
$$

is the monoid with underlying horizontal 1-cell $X \odot Y$ and unit and multiplication 2-cells

$$
\begin{tikzcd}
C \arrow[r, "I_C", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow c"] & C \arrow[d, equal] \\
C \arrow[r, "I_C \odot I_C", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \eta \odot \eta'"] & C \arrow[d, equal] \\
C \arrow[r, "X \odot Y"', "|" marking] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "(X \odot Y) \otimes (X \odot Y)", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow z"] & C \arrow[d, equal] \\
C \arrow[r, "(X \otimes X) \odot (Y \otimes Y)", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \mu \odot \mu'"] & C \arrow[d, equal] \\
C \arrow[r, "X \odot Y"', "|" marking] & C.
\end{tikzcd}
$$

The unit for this composition is I_C, given the trivial monad structure with $\eta = \mu = \mathrm{id}_{I_C}$.

Similarly, the horizontal composition of two 2-cells in $\mathbb{M}\mathrm{on}(\mathbb{D})$ is the $\odot$ product of the underlying 2-cells in $\mathbb{D}$. The fact that this commutes with the unit and multiplication defined above follows from the naturality of c and z.

In this same way, we can define the horizontal composition of two 1-cells (X, ϵ, δ) and (Y, ϵ', δ') in $\mathbb{C}\mathrm{omon}(\mathbb{D})$ to be a comonad with underlying horizontal 1-cell $X \otimes Y$, with horizontal unit $\perp$ with the trivial comonad structure.

This allows us to define (ordinary) categories $\mathrm{Mon}(\mathbb{C}\mathrm{omon}(\mathbb{D}))$ and $\mathrm{Comon}(\mathbb{M}\mathrm{on}(\mathbb{D}))$. Furthermore, these two categories are equivalent, leading to the next definition.

Definition IV.5. A *bimonad* in a 2-fold double category $\mathbb{D}$ is a monad in $\mathbb{C}\mathrm{omon}(\mathbb{D})$, or equivalently a comonad in $\mathbb{M}\mathrm{on}(\mathbb{D})$. We can define a category of bimonads in $\mathbb{D}$ as

$$\mathrm{Bimon}(\mathbb{D}) := \mathrm{Mon}(\mathbb{C}\mathrm{omon}(\mathbb{D})) \simeq \mathrm{Comon}(\mathbb{M}\mathrm{on}(\mathbb{D}))$$

Concretely, a bimonad in $\mathbb{D}$ is a tuple $(X, \eta, \mu, \epsilon, \delta)$ where X is a horizontal 1-cell,

(X, η, μ) is a monad and (X, ϵ, δ) is a comonad as above, such that four equations hold:

$$
\begin{tikzcd}
C \arrow[r, "I_C", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta"] & C \arrow[d, equal] \\
C \arrow[r, "X", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\delta"] & C \arrow[d, equal] \\
C \arrow[r, "X\odot X"', "\shortmid"{marking}] & C
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[r, "I_C", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow c"] & C \arrow[d, equal] \\
C \arrow[r, "I_C\odot I_C", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta\odot\eta"] & C \arrow[d, equal] \\
C \arrow[r, "X\odot X"', "\shortmid"{marking}] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "X\otimes X", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mu"] & C \arrow[d, equal] \\
C \arrow[r, "X", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\epsilon"] & C \arrow[d, equal] \\
C \arrow[r, "\perp_C"', "\shortmid"{marking}] & C
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[r, "X\otimes X", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\epsilon\otimes\epsilon"] & C \arrow[d, equal] \\
C \arrow[r, "\perp_C\otimes\perp_C", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow m"] & C \arrow[d, equal] \\
C \arrow[r, "\perp_C"', "\shortmid"{marking}] & C
\end{tikzcd}
$$

$$
\begin{tikzcd}
C \arrow[r, "I_C", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta"] & C \arrow[d, equal] \\
C \arrow[r, "X", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\epsilon"] & C \arrow[d, equal] \\
C \arrow[r, "\perp_C"', "\shortmid"{marking}] & C
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[r, "I_C", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow j"] & C \arrow[d, equal] \\
C \arrow[r, "\perp_C"', "\shortmid"{marking}] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "X\otimes X", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\delta\otimes\delta"] & C \arrow[d, equal] \\
C \arrow[r, "(X\odot X)\otimes(X\odot X)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow z"] & C \arrow[d, equal] \\
C \arrow[r, "(X\otimes X)\odot(X\otimes X)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mu\odot\mu"] & C \arrow[d, equal] \\
C \arrow[r, "X\odot X"', "\shortmid"{marking}] & C
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[r, "X\otimes X", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mu"] & C \arrow[d, equal] \\
C \arrow[r, "X", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\delta"] & C \arrow[d, equal] \\
C \arrow[r, "X\odot X"', "\shortmid"{marking}] & C
\end{tikzcd}
\tag{IV.4}
$$

A bimonoid morphism is simply a 2-cell which is simultaneously a monoid morphism and a comonoid morphism.

CHAPTER V

CYCLIC 2-FOLD DOUBLE CATEGORIES

Recall the notion of a cyclic double category from [CGR12]. A cyclic double category $\mathbb{D}$ is a double category with an extra involutive operation. On objects and horizontal 1-cells $X\colon C \nrightarrow C$, this operation is written

$$
\begin{tikzcd}
C_1^\bullet \arrow[r, "X^\bullet", "\shortmid"{marking}] & C_2^\bullet
\end{tikzcd}
$$

and respects horizontal identities and composition. The involution takes any vertical 1-cell $f\colon C \to D$ to some $\sigma f\colon D^\bullet \to C^\bullet$, and any 2-cell

$$
\begin{tikzcd}
C_1 \arrow[r, "X", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\theta"] & C_2 \arrow[d, "g"] \\
D_1 \arrow[r, "Y"', "\shortmid"{marking}] & D_2
\end{tikzcd}
\quad \text{to} \quad
\begin{tikzcd}
D_1^\bullet \arrow[r, "Y^\bullet", "\shortmid"{marking}] \arrow[d, "\sigma f"'] \arrow[dr, phantom, "\Downarrow\sigma\theta"] & D_2^\bullet \arrow[d, "\sigma g"] \\
C_1^\bullet \arrow[r, "X^\bullet"', "\shortmid"{marking}] & C_2^\bullet
\end{tikzcd}
$$

respecting vertical identities and composition.

The next example is the fundamental example of a cyclic double category.

Example V.1. Recall the double categories $\mathbb{L}\mathbf{Adj}(\mathcal{D})$ and $\mathbb{R}\mathbf{Adj}(\mathcal{D})$ from example III.3. If the 2-category $\mathcal{D}$ has an involution $(-)^\bullet\colon \mathcal{D}^{\mathrm{co}} \to \mathcal{D}$, such as $\mathcal{C}at$ with $(-)^{\mathrm{op}}$, then the double category $\mathbb{L}\mathbf{Adj}(\mathcal{D})$ has a natural cyclic action: on vertical 1-cells $\sigma(f \dashv g) = (g^\bullet \dashv f^\bullet)$, and if ϕ is a 2-cell

$$
\begin{tikzcd}
C \arrow[r, "j", "\shortmid"{marking}] \arrow[d, "(f\dashv g)"'] \arrow[dr, phantom, "\Downarrow\phi"] & C' \arrow[d, "(f'\dashv g')"] \\
D \arrow[r, "k"', "\shortmid"{marking}] & D'
\end{tikzcd}
$$

with mate θ then $\sigma\phi = \theta^\bullet$:

$$
\begin{tikzcd}
D^\bullet \arrow[r, "k^\bullet", "\shortmid"{marking}] \arrow[d, "(g^\bullet\dashv f^\bullet)"'] \arrow[dr, phantom, "\Downarrow\theta^\bullet"] & D'^\bullet \arrow[d, "(g'^\bullet\dashv f'^\bullet)"] \\
C^\bullet \arrow[r, "j^\bullet"', "\shortmid"{marking}] & C'^\bullet
\end{tikzcd}
$$

This cyclic action encodes the naturality of the mates correspondence using only a single double category, and is a convenient alternative to the isomorphism $\mathbb{L}\mathbf{Adj}(\mathcal{D}) \cong \mathbb{R}\mathbf{Adj}(\mathcal{D})$. This simplification will be even more important when we need the multivariable mates correspondence in chapters X and XI.

For a clear summary of the mates correspondence and the cyclic action on $\mathbb{L}\mathbf{Adj}$, see [CGR12] Section 1.

Proposition V.2. *Let $\mathbb{D}$ be a cyclic double category with arrow objects. For any object C, $(C^\bullet)^2 = (C^2)^\bullet$, as witnessed by*

$$\begin{tikzcd} (C^2)^\bullet \arrow[r, bend left, "\mathrm{cod}^\bullet", ""{name=U, below}] \arrow[r, bend right, "\mathrm{dom}^\bullet"', ""{name=D, above}] & C^\bullet \arrow[from=U, to=D, phantom, "\Downarrow\sigma\kappa"] \end{tikzcd}$$

For any vertical 1-cell F, the lift to arrow objects of $F^\bullet$ is $(\hat{F})^\bullet$, as witnessed by the 2-cells

$$\begin{tikzcd} (D^2)^\bullet \arrow[r, "\mathrm{cod}^\bullet"] \arrow[d, "(\hat{F})^\bullet"'] \arrow[dr, phantom, "\Downarrow\sigma\gamma_0"] & D^\bullet \arrow[d, "F^\bullet"] \\ (C^2)^\bullet \arrow[r, "\mathrm{cod}^\bullet"'] & C^\bullet \end{tikzcd} \qquad \begin{tikzcd} (D^2)^\bullet \arrow[r, "\mathrm{dom}^\bullet"] \arrow[d, "(\hat{F})^\bullet"'] \arrow[dr, phantom, "\Downarrow\sigma\gamma_1"] & D^\bullet \arrow[d, "F^\bullet"] \\ (C^2)^\bullet \arrow[r, "\mathrm{dom}^\bullet"'] & C^\bullet \end{tikzcd}$$

Proof. It is a very simple matter to verify the universal properties of section III.2 □

We will generalize this to a cyclic action on a 2-fold double category. Suppose that $\mathbb{D}$ is a 2-fold double category. A cyclic action, written as above, must satisfy the following:

- For every object C,

$$I_{C^\bullet} = (\perp_C)^\bullet \qquad \text{and} \qquad \perp_{C^\bullet} = (I_C)^\bullet.$$

- For every composable pair of horizontal 1-cells $X, Y\colon C \nrightarrow C$,

$$(X \otimes Y)^\bullet = X^\bullet \odot Y^\bullet \qquad \text{and} \qquad (X \odot Y)^\bullet = X^\bullet \otimes Y^\bullet$$

- For every vertical 1-cell $f\colon C \to D$, there are equalities

$$\begin{tikzcd} D^\bullet \arrow[r, "I_{D^\bullet}"] \arrow[d, "\sigma f"'] \arrow[dr, phantom, "\Downarrow I_{\sigma f}"] & D^\bullet \arrow[d, "\sigma f"] \\ C^\bullet \arrow[r, "I_{C^\bullet}"'] & C^\bullet \end{tikzcd} = \begin{tikzcd} D^\bullet \arrow[r, "(\perp_D)^\bullet"] \arrow[d, "\sigma f"'] \arrow[dr, phantom, "\Downarrow\sigma\perp_f"] & D^\bullet \arrow[d, "\sigma f"] \\ C^\bullet \arrow[r, "(\perp_C)^\bullet"'] & C^\bullet \end{tikzcd}$$

$$\begin{tikzcd} D^\bullet \arrow[r, "\perp_{D^\bullet}", "|"{marking}] \arrow[d, "\sigma f"'] \arrow[dr, phantom, "\Downarrow\perp_{\sigma f}"] & D^\bullet \arrow[d, "\sigma f"] \\ C^\bullet \arrow[r, "\perp_{C^\bullet}"', "|"{marking}] & C^\bullet \end{tikzcd} = \begin{tikzcd} D^\bullet \arrow[r, "(I_D)^\bullet", "|"{marking}] \arrow[d, "\sigma f"'] \arrow[dr, phantom, "\Downarrow\sigma I_f"] & D^\bullet \arrow[d, "\sigma f"] \\ C^\bullet \arrow[r, "(I_C)^\bullet"', "|"{marking}] & C^\bullet \end{tikzcd}$$

- For every horizontally composable pair of 2-cells

$$\begin{tikzcd} C \arrow[r, "X", "|"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\theta"] & C \arrow[r, "Y", "|"{marking}] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[d, "f"] \\ D \arrow[r, "X'"', "|"{marking}] & D \arrow[r, "Y'"', "|"{marking}] & D \end{tikzcd}$$

  there are equalities

$$\begin{tikzcd} D^\bullet \arrow[r, "(X'\otimes Y')^\bullet", "|"{marking}] \arrow[d, "\sigma f"'] \arrow[dr, phantom, "\Downarrow\sigma(\theta\otimes\phi)"] & D^\bullet \arrow[d, "\sigma f"] \\ C^\bullet \arrow[r, "(X\otimes Y)^\bullet"', "|"{marking}] & C^\bullet \end{tikzcd} = \begin{tikzcd} D^\bullet \arrow[r, "X'^\bullet\odot Y'^\bullet", "|"{marking}] \arrow[d, "\sigma f"'] \arrow[dr, phantom, "\Downarrow\sigma(\theta)\odot\sigma(\phi)"] & D^\bullet \arrow[d, "\sigma f"] \\ C^\bullet \arrow[r, "X^\bullet\odot Y^\bullet"', "|"{marking}] & C^\bullet \end{tikzcd}$$

$$\begin{tikzcd} D^\bullet \arrow[r, "(X'\odot Y')^\bullet", "|"{marking}] \arrow[d, "\sigma f"'] \arrow[dr, phantom, "\Downarrow\sigma(\theta\odot\phi)"] & D^\bullet \arrow[d, "\sigma f"] \\ C^\bullet \arrow[r, "(X\odot Y)^\bullet"', "|"{marking}] & C^\bullet \end{tikzcd} = \begin{tikzcd} D^\bullet \arrow[r, "X'^\bullet\otimes Y'^\bullet", "|"{marking}] \arrow[d, "\sigma f"'] \arrow[dr, phantom, "\Downarrow\sigma(\theta)\otimes\sigma(\phi)"] & D^\bullet \arrow[d, "\sigma f"] \\ C^\bullet \arrow[r, "X^\bullet\otimes Y^\bullet"', "|"{marking}] & C^\bullet \end{tikzcd}$$

One nice consequence of this definition is that a cyclic action on a 2-fold double category $\mathbb{D}$ induces a cyclic action on the category of bimonoids $\mathrm{Bimon}(\mathbb{D})$.

Proposition V.3. *Suppose $\mathbb{D}$ is a cyclic 2-fold double category. Then the category* $\mathrm{Bimon}(\mathbb{D})$ *of bimonoids in $\mathbb{D}$ carries a natural cyclic action (contravariant isomorphism).*

Proof. The involution $(-)^\bullet$ gives an isomorphism of double categories $\mathbb{D}_\otimes \cong \mathbb{D}_\odot^{\mathrm{op}}$. Therefore it also induces an isomorphism

$$\mathbb{M}\mathrm{on}(\mathbb{D}) = \mathbb{M}\mathrm{on}(\mathbb{D}_\otimes) \cong \mathbb{M}\mathrm{on}(\mathbb{D}_\odot^{\mathrm{op}}) \cong \mathbb{C}\mathrm{omon}(\mathbb{D}_\odot)^{\mathrm{op}} = \mathbb{C}\mathrm{omon}(\mathbb{D})^{\mathrm{op}}$$

as well as an isomorphism

$$\begin{aligned} \mathrm{Bimon}(\mathbb{D}) = \mathrm{Comon}(\mathbb{M}\mathrm{on}(\mathbb{D})) &\cong \mathrm{Comon}(\mathbb{C}\mathrm{omon}(\mathbb{D})^{\mathrm{op}}) \\ &\cong \mathrm{Mon}(\mathbb{C}\mathrm{omon}(\mathbb{D}))^{\mathrm{op}} = \mathrm{Bimon}(\mathbb{D})^{\mathrm{op}}. \end{aligned}$$

□

In more concrete terms, the involution takes a bimonoid $(X,\eta,\mu,\epsilon,\delta)$ to $(X,\eta,\mu,\epsilon,\delta)^\bullet = (X^\bullet,\epsilon^\bullet,\delta^\bullet,\eta^\bullet,\delta^\bullet)$, swapping the monoid and comonoid structures. This is again a bimonoid, as the top two equations of (IV.4) are interchanged under the involution, while the bottom two equations are self-dual.

The action of the involution on bimonoid morphisms can be broken down as in the following lemma.

Lemma V.4. *Let $(X,\eta,\mu,\epsilon,\delta)$ and $(Y,\eta',\mu',\epsilon',\delta')$ be bimonoids in a cyclic 2-fold double category $\mathbb{D}$, and let ϕ be a 2-cell in $\mathbb{D}$*

$$\begin{tikzcd}
C \arrow[r, "X", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[d, "f"] \\
D \arrow[r, "Y"', "\shortmid"{marking}] & D.
\end{tikzcd}$$

Then (f,ϕ) is a monoid morphism $X \to Y$ if and only if $(\sigma f,\phi^\bullet)$ is a comonoid morphism $Y^\bullet \to X^\bullet$. Dually, ϕ is a comonoid morphism $X \to Y$ if and only if $\phi^\bullet$ is a monoid morphism $Y^\bullet \to X^\bullet$.

Proof. Simply notice that the involution takes equations (III.3) and (III.4) to the equations defining a comonad morphism in $\mathbb{D}$. $\square$

This immediately implies a useful characterization of bimonoid morphisms.

Corollary V.5. *Given bimonoids $(X,\eta,\mu,\epsilon,\delta)$ and $(Y,\eta',\mu',\epsilon',\delta')$ in a cyclic 2-fold double category $\mathbb{D}$, a bimonoid morphism $X \to Y$ consists of:*

- Either *a monoid morphism $X \to Y$* or *a comonoid morphism $Y^\bullet \to X^\bullet$*, and
- Either *a comonoid morphism $X \to Y$* or *a monoid morphism $Y^\bullet \to X^\bullet$*.

CHAPTER VI

FUNCTORIAL FACTORIZATIONS

Let $\mathbb{D}$ be a cyclic double category, and assume it has arrow objects in the sense of section III.2. In this section, we will define a 2-fold double category $\mathbb{FF}(\mathbb{D})$ of functorial factorizations in $\mathbb{D}$, as follows:

- The objects and vertical 1-cells are the same as in $\mathbb{D}$.
- Horizontal 1-cells $C \to\!\!\!\!\!+ \; C$ in $\mathbb{FF}(\mathbb{D})$ are tuples (E, η, ϵ), where $E \colon C^2 \to C$ is a horizontal 1-cell in $\mathbb{D}$, and

$$\begin{tikzcd} C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "E"'] \arrow[r, phantom, "\Downarrow\eta"] & C \end{tikzcd} \qquad \begin{tikzcd} C^2 \arrow[r, bend left, "E"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\epsilon"] & C \end{tikzcd}$$

are 2-cells in $\mathbb{D}$ such that

$$\begin{tikzcd} C^2 \arrow[r, bend left=60, "\mathrm{dom}"] \arrow[r, "E" description] \arrow[r, bend right=60, "\mathrm{cod}"'] \arrow[r, phantom, shift left=3, "\Downarrow\eta"] \arrow[r, phantom, shift right=3, "\Downarrow\epsilon"] & C \end{tikzcd} = \begin{tikzcd} C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\kappa"] & C. \end{tikzcd}$$

By the universal property of C^2, this also determines horizontal 1-cells $L, R \colon C^2 \to C^2$ such that $\mathrm{dom} \circ L = \mathrm{dom}$, $\mathrm{cod} \circ R = \mathrm{cod}$, $\mathrm{cod} \circ L = \mathrm{dom} \circ R = E$, $\kappa \circ L = \eta$, and $\kappa \circ R = \epsilon$, and 2-cells

$$\begin{tikzcd} C^2 \arrow[r, bend left, "L"] \arrow[r, bend right, "\mathrm{id}"'] \arrow[r, phantom, "\Downarrow\vec{\epsilon}"] & C^2. \end{tikzcd} \qquad \begin{tikzcd} C^2 \arrow[r, bend left, "\mathrm{id}"] \arrow[r, bend right, "R"'] \arrow[r, phantom, "\Downarrow\vec{\eta}"] & C^2. \end{tikzcd}$$

such that $\mathrm{dom} \circ \vec{\epsilon} = \mathrm{id}_{\mathrm{dom}}$, $\mathrm{cod} \circ \vec{\epsilon} = \epsilon$, $\mathrm{dom} \circ \vec{\eta} = \eta$, and $\mathrm{cod} \circ \vec{\eta} = \mathrm{id}_{\mathrm{cod}}$.

- The horizontal composition $(E_1, \eta_1, \epsilon_1) \otimes (E_2, \eta_2, \epsilon_2)$ of two horizontal 1-cells

$$C \xrightarrow{(E_1, \eta_1, \epsilon_1)} C \xrightarrow{(E_2, \eta_2, \epsilon_2)} C$$

in $\mathbb{FF}(\mathbb{D})$ is a horizontal 1-cell $(E_{1\otimes 2}, \eta_{1\otimes 2}, \epsilon_{1\otimes 2})$, where

$$E_{1\otimes 2} = \quad C^2 \xrightarrow{R_1} C^2 \xrightarrow{E_2} C$$

$$\eta_{1\otimes 2} = \begin{tikzcd} C^2 \arrow[r, bend left, "\mathrm{id}"] \arrow[r, bend right, "R_1"'] \arrow[r, phantom, "\Downarrow\vec{\eta}_1"] & C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "E_2"'] \arrow[r, phantom, "\Downarrow\eta_2"] & C \end{tikzcd}$$

$$\epsilon_{1\otimes 2} = \begin{tikzcd} C^2 \arrow[r, "R_1"] & C^2 \arrow[r, bend left, "E_2"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\epsilon_2"] & C \end{tikzcd}$$

which also determines that $R_{1\otimes 2} = R_2 \circ R_1$.

- The horizontal unit I_C for $\otimes$ is $(\mathrm{dom}, \mathrm{id}, \kappa)$.
- The second horizontal composition $(E_1, \eta_1, \epsilon_1) \odot (E_2, \eta_2, \epsilon_2)$ is a horizontal 1-cell $(E_{1\odot 2}, \eta_{1\odot 2}, \epsilon_{1\odot 2})$, where

$$E_{1\odot 2} = \quad C^2 \xrightarrow{L_1} C^2 \xrightarrow{E_2} C$$

$$\eta_{1\odot 2} = \begin{tikzcd} C^2 \arrow[r, "L_1"] & C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "E_2"'] \arrow[r, phantom, "\Downarrow\eta_2"] & C \end{tikzcd}$$

$$\epsilon_{1\odot 2} = \begin{tikzcd} C^2 \arrow[r, bend left, "L_1"] \arrow[r, bend right, "\mathrm{id}"'] \arrow[r, phantom, "\Downarrow\vec{\epsilon}_1"] & C^2 \arrow[r, bend left, "E_2"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\epsilon_2"] & C \end{tikzcd}$$

which also determines that $L_{1\odot 2} = L_2 \circ L_1$.

- The horizontal unit $\perp_C$ for $\odot$ is $(\mathrm{cod}, \kappa, \mathrm{id})$.
- 2-cells

$$\begin{tikzcd} C \arrow[r, "(E_1,\eta_1,\epsilon_1)", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\theta"] & C \arrow[d, "F"] \\ D \arrow[r, "(E_2,\eta_2,\epsilon_2)"', "\shortmid"{marking}] & D \end{tikzcd}$$

in $\mathbb{FF}(\mathbb{D})$ are given by 2-cells

$$\begin{tikzcd} C^2 \arrow[r, "E_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta"] & C \arrow[d, "F"] \\ D^2 \arrow[r, "E_2"'] & D \end{tikzcd}$$

in $\mathbb{D}$ such that

$$\begin{tikzcd} C^2 \arrow[r, "E_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta"] & C \arrow[d, "F"] \\ D^2 \arrow[r, "E_2"] \arrow[r, bend right, "\mathrm{cod}"', ""{name=B}] \arrow[r, phantom, shift right=3, "\Downarrow\epsilon_2"] & D \end{tikzcd} = \begin{tikzcd} C^2 \arrow[r, bend left, "E_1"] \arrow[r, phantom, shift left=3, "\Downarrow\epsilon_1"] \arrow[r, "\mathrm{cod}"'] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_0"] & C \arrow[d, "F"] \\ D^2 \arrow[r, "\mathrm{cod}"'] & D \end{tikzcd} \tag{VI.1}$$

and

$$\begin{tikzcd} C^2 \arrow[r, "\mathrm{dom}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_1"] & C \arrow[d, "F"] \\ D^2 \arrow[r, "\mathrm{dom}"] \arrow[r, bend right, "E_2"'] \arrow[r, phantom, shift right=3, "\Downarrow\eta_2"] & D \end{tikzcd} = \begin{tikzcd} C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, phantom, shift left=3, "\Downarrow\eta_1"] \arrow[r, "E_1"'] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta"] & C \arrow[d, "F"] \\ D^2 \arrow[r, "E_2"'] & D \end{tikzcd} \tag{VI.2}$$

This also determines unique 2-cells

$$\begin{tikzcd} C^2 \arrow[r, "R_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta^R"] & C^2 \arrow[d, "\hat{F}"] \\ D^2 \arrow[r, "R_2"'] & D^2 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C^2 \arrow[r, "L_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta^L"] & C^2 \arrow[d, "\hat{F}"] \\ D^2 \arrow[r, "L_2"'] & D^2 \end{tikzcd}$$

such that composing horizontally with γ_0 or γ_1 gives γ_0, γ_1, or θ as appropriate. For instance:

$$\begin{tikzcd} C^2 \arrow[r, "R_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta^R"] & C^2 \arrow[r, "\mathrm{dom}"] \arrow[d, "\hat{F}"] \arrow[dr, phantom, "\Downarrow\gamma_1"] & C \arrow[d, "F"] \\ D^2 \arrow[r, "R_2"'] & D^2 \arrow[r, "\mathrm{dom}"'] & D \end{tikzcd} = \begin{tikzcd} C^2 \arrow[r, "E_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta"] & C \arrow[d, "F"] \\ D^2 \arrow[r, "E_2"'] & D \end{tikzcd}$$

- Given a pair of composable 2-cells in $\mathbb{FF}(\mathbb{D})$ as in

$$\begin{tikzcd} C \arrow[r, "{(E_1,\eta_1,\epsilon_1)}", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\theta_1"] & C \arrow[r, "{(E_2,\eta_2,\epsilon_2)}", "\shortmid"{marking}] \arrow[d, "F"] \arrow[dr, phantom, "\Downarrow\theta_2"] & C \arrow[d, "F"] \\ D \arrow[r, "{(E'_1,\eta'_1,\epsilon'_1)}"', "\shortmid"{marking}] & D \arrow[r, "{(E'_2,\eta'_2,\epsilon'_2)}"', "\shortmid"{marking}] & D \end{tikzcd}$$

the composite $\theta_1 \otimes \theta_2$ is given by

$$\begin{tikzcd}
C^2 \arrow[r, "R_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta_1^R"] & C^2 \arrow[r, "E_2"] \arrow[d, "\hat{F}"] \arrow[dr, phantom, "\Downarrow\theta_2"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "R_1'"'] & D^2 \arrow[r, "E_2'"'] & D
\end{tikzcd}$$

while the composite $\theta_1 \odot \theta_2$ is given by

$$\begin{tikzcd}
C^2 \arrow[r, "L_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta_1^L"] & C^2 \arrow[r, "E_2"] \arrow[d, "\hat{F}"] \arrow[dr, phantom, "\Downarrow\theta_2"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "L_1'"'] & D^2 \arrow[r, "E_2'"'] & D
\end{tikzcd}$$

It is a straightforward exercise to check that these definitions satisfy equations (VI.1) and (VI.2). To illustrate, we will demonstrate that $\theta_1 \otimes \theta_2$ satisfies (VI.1):

$$\begin{tikzcd}
C^2 \arrow[r, "E_{1\otimes 2}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta_1\otimes\theta_2"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "E_{1'\otimes 2'}"] \arrow[r, bend right=50, "\mathrm{cod}"', "\Downarrow\epsilon_{1'\otimes 2'}"] & D
\end{tikzcd}
=
\begin{tikzcd}
C^2 \arrow[r, "R_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta_1^R"] & C^2 \arrow[r, "E_2"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta_2"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "R_1'"'] & D^2 \arrow[r, "E_2'"] \arrow[r, bend right=50, "\mathrm{cod}"', "\Downarrow\epsilon_2'"] & D
\end{tikzcd}$$

$$=
\begin{tikzcd}
C^2 \arrow[r, "R_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta_1^R"] & C^2 \arrow[r, bend left=50, "E_2", "\Downarrow\epsilon_2"'] \arrow[r, "\mathrm{cod}"'] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_0"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "R_1'"'] & D^2 \arrow[r, "\mathrm{cod}"'] & D
\end{tikzcd}$$

$$=
\begin{tikzcd}
C^2 \arrow[r, bend left=50, "E_{1\otimes 2}", "\Downarrow\epsilon_{1\otimes 2}"'] \arrow[r, "\mathrm{cod}"'] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_0"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "\mathrm{cod}"'] & D
\end{tikzcd}$$

Example VI.1. Functorial factorizations in the double category $\mathbb{D} = \mathbb{S}\mathrm{q}(\mathcal{C}\mathrm{at})$ of squares in the 2-category of categories are precisely functorial factorizations as defined in section II.2.

It is straightforward to check that $\otimes$ and $\odot$ are each associative and unital. It takes more work to provide the compatibility between $\otimes$ and $\odot$, which is the content of the proof of the next proposition.

Proposition VI.2. $\mathbb{FF}(\mathbb{D})$ *has the structure of a 2-fold double category.*

Proof. The primary structure of $\mathbb{FF}(\mathbb{D})$ was given in the first part of this section. What is left is to provide the coherence data (IV.2) and (IV.3).

First, note that I_C is initial in the sense that, given any vertical morphism $F: C \to D$ and any functorial factorization (E, η, ϵ) on D, there is a unique 2-cell

$$\begin{tikzcd} C \arrow[r, "I_C", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow"] & C \arrow[d, "F"] \\ D \arrow[r, "(E,\eta,\epsilon)"', "\shortmid"{marking}] & D \end{tikzcd}$$

given by

$$\begin{tikzcd} C^2 \arrow[r, "\mathrm{dom}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_1"] & C \arrow[d, "F"] \\ D^2 \arrow[r, "\mathrm{dom}"] \arrow[r, bend right, "E"', "\Downarrow\eta"] & D. \end{tikzcd}$$

Similarly, $\perp_C$ is terminal. Thus there is only one possible way to define the 2-cells m, c, and j, and naturality and all other coherence equations follows immediately from this uniqueness.

We still need to construct the 2-cell z, which will take some work. We begin by defining 2-cells

$$\begin{tikzcd} C \arrow[r, "E_1 \odot E_2", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow p_{E_1,E_2}"] & C \arrow[d, equal] \\ C \arrow[r, "E_1"', "\shortmid"{marking}] & C \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C \arrow[r, "E_1", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow i_{E_1,E_2}"] & C \arrow[d, equal] \\ C \arrow[r, "E_1 \otimes E_2"', "\shortmid"{marking}] & C. \end{tikzcd}$$

for any pair of functorial factorizations. The 2-cell p is given by the underlying 2-cell in $\mathbb{D}$

$$\begin{tikzcd} C^2 \arrow[r, "L_1"] & C^2 \arrow[r, bend left, "E_2"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\epsilon_2"] & C \end{tikzcd}$$

and i is given by

$$\begin{tikzcd} C^2 \arrow[r, "R_1"] & C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "E_2"'] \arrow[r, phantom, "\Downarrow\eta_2"] & C. \end{tikzcd}$$

To illustrate the verification that these give well-defined 2-cells in $\mathbb{FF}(\mathbb{D})$, we will show that i satisfies (VI.1) (keep in mind that when F is an identity, γ_0 and γ_1 are also identities):

$$\begin{tikzcd} C^2 \arrow[r, "L_1"] & C^2 \arrow[r, bend left=50, "\mathrm{dom}"] \arrow[r, "E_2" description] \arrow[r, bend right=50, "\mathrm{cod}"'] \arrow[r, phantom, shift left=3, "\Downarrow\eta_2"] \arrow[r, phantom, shift right=3, "\Downarrow\epsilon_2"] & C \end{tikzcd} = \begin{tikzcd} C^2 \arrow[r, "L_1"] & C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\kappa"] & C \end{tikzcd}$$

$$= \begin{tikzcd} C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "E_1"'] \arrow[r, phantom, "\Downarrow\eta_1"] & C. \end{tikzcd}$$

Moreover, it is straightforward to check that i and p are natural families of 2-cells. Specifically, for any pair of 2-cells θ_1 and θ_2

$$\begin{tikzcd} C \arrow[r, "E_1\odot E_2"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow p_{E_1,E_2}"] & C \arrow[d, equal] \\ C \arrow[r, "E_1"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\theta_1"] & C \arrow[d, "F"] \\ D \arrow[r, "E_1'"'] & D \end{tikzcd} = \begin{tikzcd} C \arrow[r, "E_1\odot E_2"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\theta_1\odot\theta_2"] & C \arrow[d, "F"] \\ D \arrow[r, "E_1'\odot E_2'"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow p_{E_1',E_2'}"] & D \arrow[d, equal] \\ D \arrow[r, "E_1'"'] & D \end{tikzcd}$$

$$\begin{tikzcd} C \arrow[r, "E_1"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow i_{E_1,E_2}"] & C \arrow[d, equal] \\ C \arrow[r, "E_1\otimes E_2"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\theta_1\otimes\theta_2"] & C \arrow[d, "F"] \\ D \arrow[r, "E_1'\otimes E_2'"'] & D \end{tikzcd} = \begin{tikzcd} C \arrow[r, "E_1"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\theta_1"] & C \arrow[d, "F"] \\ D \arrow[r, "E_1'"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow i_{E_1',E_2'}"] & D \arrow[d, equal] \\ D \arrow[r, "E_1'\otimes E_2'"'] & D \end{tikzcd}$$

As with any 2-cell in $\mathbb{FF}(\mathbb{D})$, p and i induce 2-cells in $\mathbb{D}$

$$\begin{tikzcd} C^2 \arrow[r, bend left, "R_{1\odot 2}"] \arrow[r, bend right, "R_1"'] \arrow[r, phantom, "\Downarrow p^R"] & C^2 \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} C^2 \arrow[r, bend left, "L_1"] \arrow[r, bend right, "L_{1\otimes 2}"'] \arrow[r, phantom, "\Downarrow i^L"] & C^2. \end{tikzcd}$$

such that

$$\begin{tikzcd} C^2 \arrow[r, bend left, "R_{1\odot 2}"] \arrow[r, bend right, "R_1"'] \arrow[r, phantom, "\Downarrow p^R"] & C^2 \arrow[r, "\mathrm{dom}"] & C \end{tikzcd} = \begin{tikzcd} C^2 \arrow[r, "L_1"] & C^2 \arrow[r, bend left, "E_2"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\epsilon_2"] & C \end{tikzcd} \tag{VI.3}$$

$$\begin{tikzcd} C^2 \arrow[r, bend left, "L_1"] \arrow[r, bend right, "L_{1\otimes 2}"'] \arrow[r, phantom, "\Downarrow i^L"] & C^2 \arrow[r, "\mathrm{cod}"] & C \end{tikzcd} = \begin{tikzcd} C^2 \arrow[r, "R_1"] & C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "E_2"'] \arrow[r, phantom, "\Downarrow\eta_2"] & C \end{tikzcd} \tag{VI.4}$$

Now suppose given three functorial factorizations E_1, E_2, E_3 on an object C. We define a 2-cell in $\mathbb{D}$

$$\begin{tikzcd}
& C^2 \arrow[dr, "L_3"] \arrow[dd, phantom, "\Downarrow w"] & \\
C^2 \arrow[ur, "R_{1\odot 2}"] \arrow[dr, "L_{1\otimes 3}"'] & & C^2 \\
& C^2 \arrow[ur, "R_2"'] &
\end{tikzcd}$$

such that

$$\begin{tikzcd}
& C^2 \arrow[dr, "L_3"] \arrow[dd, phantom, "\Downarrow w"] & & & & & & \\
C^2 \arrow[ur, "R_{1\odot 2}"] \arrow[dr, "L_{1\otimes 3}"'] & & C^2 \arrow[r, "\mathrm{dom}"] & C \arrow[r, phantom, "="] & C^2 \arrow[r, bend left, "L_1"] \arrow[r, bend right, "L_{1\otimes 3}"'] \arrow[r, phantom, "\Downarrow i^L"] & C^2 \arrow[r, "E_2"] & C \\
& C^2 \arrow[ur, "R_2"'] & & & & &
\end{tikzcd} \tag{VI.5}$$

$$\begin{tikzcd}
& C^2 \arrow[dr, "L_3"] \arrow[dd, phantom, "\Downarrow w"] & & & & & & \\
C^2 \arrow[ur, "R_{1\odot 2}"] \arrow[dr, "L_{1\otimes 3}"'] & & C^2 \arrow[r, "\mathrm{cod}"] & C \arrow[r, phantom, "="] & C^2 \arrow[r, bend left, "R_{1\odot 2}"] \arrow[r, bend right, "R_1"'] \arrow[r, phantom, "\Downarrow p^R"] & C^2 \arrow[r, "E_3"] & C. \\
& C^2 \arrow[ur, "R_2"'] & & & & &
\end{tikzcd} \tag{VI.6}$$

Using the universal property for C^2, it suffices to check that

$$\begin{tikzcd}
C^2 \arrow[r, "L_{1\otimes 3}"'] \arrow[r, bend left=60, "L_1"] \arrow[r, phantom, bend left=30, "\Downarrow i^L"] & C^2 \arrow[r, "E_2"] \arrow[r, bend right=60, "\mathrm{cod}"'] \arrow[r, phantom, bend right=30, "\Downarrow \epsilon_2"] & C \arrow[r, phantom, "="] & C^2 \arrow[r, "R_{1\odot 2}"] \arrow[r, bend right=60, "R_1"'] \arrow[r, phantom, bend right=30, "\Downarrow p^R"] & C^2 \arrow[r, "E_3"'] \arrow[r, bend left=60, "\mathrm{dom}"] \arrow[r, phantom, bend left=30, "\Downarrow \eta_3"] & C
\end{tikzcd}$$

and a quick check using equations (VI.3) and (VI.4) shows that both are equal to

$$\begin{tikzcd}
& C^2 \arrow[dr, "\mathrm{cod}"'] \arrow[dr, bend left=60, "E_2"] \arrow[dr, phantom, bend left=30, "\Downarrow \epsilon_2"] & \\
C^2 \arrow[ur, "L_1"] \arrow[dr, "R_1"'] & & C \\
& C^2 \arrow[ur, "\mathrm{dom}"] \arrow[ur, bend right=60, "E_3"'] \arrow[ur, phantom, bend right=30, "\Downarrow \eta_3"] &
\end{tikzcd}$$

where the inner diamond is the equality $\mathrm{cod}\, L_1 = \mathrm{dom}\, R_1 = E_1$.

We also check that w is natural with respect to 2-cells in $\mathbb{FF}(\mathbb{D})$ in the following sense: given three 2-cells θ_1, θ_2, and θ_3, there is an equality

$$
\begin{tikzcd}
 & C^2 \arrow[dr, "L_3"] \arrow[d, phantom, "\Downarrow w"] & \\
C^2 \arrow[ur, "R_{1\odot 2}"] \arrow[dr, "L_{1\otimes 3}"] \arrow[dd, "\hat{F}"'] & {} & C^2 \arrow[dd, "\hat{F}"] \\
 & C^2 \arrow[ur, "R_2"] \arrow[dd, "\hat{F}"] \arrow[l, phantom, "\Downarrow(\theta_1\otimes\theta_3)^L"'] \arrow[r, phantom, "\Downarrow\theta_2^R"] & {} \\
D^2 \arrow[dr, "L'_{1\otimes 3}"'] & & D^2 \\
 & D^2 \arrow[ur, "R'_2"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & C^2 \arrow[dr, "L_3"] \arrow[dd, "\hat{F}"] & \\
C^2 \arrow[ur, "R_{1\odot 2}"] \arrow[dd, "\hat{F}"'] \arrow[r, phantom, "\Downarrow(\theta_1\odot\theta_2)^R"] & {} \arrow[r, phantom, "\Downarrow\theta_3^L"] & C^2 \arrow[dd, "\hat{F}"] \\
 & D^2 \arrow[dr, "L'_3"'] \arrow[dd, phantom, "\Downarrow w"] & \\
D^2 \arrow[ur, "R'_{1\odot 2}"'] \arrow[dr, "L'_{1\otimes 3}"'] & & D^2. \\
 & D^2 \arrow[ur, "R'_2"'] &
\end{tikzcd}
$$

To verify this equation, it suffices to check equality upon right composition with γ_0 and γ_1. We will illustrate the γ_1 case, making use of the naturality of i:

$$
\begin{tikzcd}
 & C^2 \arrow[dr, "L_3"] \arrow[d, phantom, "\Downarrow w"] & & \\
C^2 \arrow[ur, "R_{1\odot 2}"] \arrow[dr, "L_{1\otimes 3}"] \arrow[dd, "\hat{F}"'] & {} & C^2 \arrow[dd, "\hat{F}"] \arrow[r, "\mathrm{dom}"] \arrow[ddr, phantom, "\Downarrow\gamma_1"] & C \arrow[dd, "F"] \\
 & C^2 \arrow[ur, "R_2"] \arrow[dd, "\hat{F}"] \arrow[l, phantom, "\Downarrow(\theta_1\otimes\theta_3)^L"'] \arrow[r, phantom, "\Downarrow\theta_2^R"] & {} & \\
D^2 \arrow[dr, "L'_{1\otimes 3}"'] & & D^2 \arrow[r, "\mathrm{dom}"'] & D \\
 & D^2 \arrow[ur, "R'_2"'] & &
\end{tikzcd}
=
\begin{tikzcd}
C^2 \arrow[r, "L_{1\otimes 3}"'] \arrow[r, bend left=50, "L_1"] \arrow[r, phantom, bend left=25, "\Downarrow i^L"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow(\theta_1\otimes\theta_3)^L"] & C^2 \arrow[r, "E_2"] \arrow[d, "\hat{F}"] \arrow[dr, phantom, "\Downarrow\theta_2"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "L_{1'\otimes 3'}"'] & D^2 \arrow[r, "E'_2"'] & D
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
C^2 \arrow[r, "L_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta_1^L"] & C^2 \arrow[r, "E_2"] \arrow[d, "\hat{F}"] \arrow[dr, phantom, "\Downarrow\theta_2"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "L'_1"] \arrow[r, bend right=50, "L'_{1\otimes 3}"'] \arrow[r, phantom, bend right=25, "\Downarrow i^L"] & D^2 \arrow[r, "E'_2"'] & D
\end{tikzcd}
=
\begin{tikzcd}
 & C^2 \arrow[dr, "L_3"] \arrow[dd, "\hat{F}"] & & \\
C^2 \arrow[ur, "R_{1\odot 2}"] \arrow[dd, "\hat{F}"'] \arrow[r, phantom, "\Downarrow(\theta_1\odot\theta_2)^R"] & {} \arrow[r, phantom, "\Downarrow\theta_3^L"] & C^2 \arrow[dd, "\hat{F}"] \arrow[r, "\mathrm{dom}"] \arrow[ddr, phantom, "\Downarrow\gamma_1"] & C \arrow[dd, "F"] \\
 & D^2 \arrow[dr, "L'_3"'] \arrow[dd, phantom, "\Downarrow w"] & & \\
D^2 \arrow[ur, "R'_{1\odot 2}"'] \arrow[dr, "L'_{1\otimes 3}"'] & & D^2 \arrow[r, "\mathrm{dom}"'] & D. \\
 & D^2 \arrow[ur, "R'_2"'] & &
\end{tikzcd}
$$

Finally, given four functorial factorizations E_1, E_2, E_3, E_4 on an object C, we define the 2-cell

$$
\begin{tikzcd}
C \arrow[r, "(1\odot 2)\otimes(3\odot 4)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow z_{1,2,3,4}"] & C \arrow[d, equal] \\
C \arrow[r, "(1\otimes 3)\odot(2\otimes 4)"', "\shortmid"{marking}] & C
\end{tikzcd}
$$

in $\mathbb{FF}(\mathbb{D})$, where $(1 \odot 2)$ is shorthand for $(E_1, \eta_1, \epsilon_1) \odot (E_2, \eta_2, \epsilon_2)$, to have the underlying 2-cell in $\mathbb{D}$

$$
\begin{tikzcd}
 & C^2 \arrow[dr, "L_3"] \arrow[dd, phantom, "\Downarrow w"] & & \\
C^2 \arrow[ur, "R_{1\odot 2}"] \arrow[dr, "L_{1\otimes 3}"'] & & C^2 \arrow[r, "E_4"] & C. \\
 & C^2 \arrow[ur, "R_2"'] & &
\end{tikzcd}
$$

The naturality of z follows immediately from that of w, but we still need to check that this satisfies equations (VI.1) and (VI.2). We will leave the details to the reader, but note that (VI.2) comes down to the verification of the equality

$$
\begin{tikzcd}
 & C^2 \arrow[dr, "L_3"] \arrow[dd, phantom, "\Downarrow w"] & & & & & & & & \\
C^2 \arrow[ur, bend left=70, "\mathrm{id}"] \arrow[ur, "R_{1\odot 2}"'] \arrow[ur, phantom, bend left=35, "\Downarrow\vec{\eta}_{1\odot 2}"] \arrow[dr, "L_{1\otimes 3}"'] & & C^2 \arrow[r, bend left=50, "\mathrm{dom}"] \arrow[r, "E_4"'] \arrow[r, phantom, bend left=25, "\Downarrow\eta_4"] & C & = & C^2 \arrow[r, "L_{1\otimes 3}"] & C^2 \arrow[r, bend left, "\mathrm{id}"] \arrow[r, bend right, "R_2"'] \arrow[r, phantom, "\Downarrow\vec{\eta}_2"] & C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "E_4"'] \arrow[r, phantom, "\Downarrow\eta_4"] & C, \\
 & C^2 \arrow[ur, "R_2"'] & & & & & & & &
\end{tikzcd}
$$

which follows from equation (VI.5) and the fact that $\mathrm{dom} \circ i^L = \mathrm{id}_{\mathrm{dom}}$. □

Lemma VI.3. *There is a strict double functor $R\colon \mathbb{FF}(\mathbb{D})_{\otimes} \to \mathbb{D}$ whose behavior on 2-cells is*

$$
\begin{tikzcd}
C \arrow[r, "{(E_1,\eta_1,\epsilon_1)}", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\theta"] & C \arrow[d, "F"] \\
D \arrow[r, "{(E_2,\eta_2,\epsilon_2)}"', "\shortmid"{marking}] & D
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
C^2 \arrow[r, "R_1", "\shortmid"{marking}] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta^R"] & C^2 \arrow[d, "\hat{F}"] \\
D^2 \arrow[r, "R_2"', "\shortmid"{marking}] & D^2
\end{tikzcd}
$$

and a double functor $L\colon \mathbb{FF}(\mathbb{D})_{\odot} \to \mathbb{D}$ whose behavior on 2-cells is

$$
\begin{tikzcd}
C \arrow[r, "{(E_1,\eta_1,\epsilon_1)}", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\theta"] & C \arrow[d, "F"] \\
D \arrow[r, "{(E_2,\eta_2,\epsilon_2)}"', "\shortmid"{marking}] & D
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
C^2 \arrow[r, "L_1", "\shortmid"{marking}] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta^L"] & C^2 \arrow[d, "\hat{F}"] \\
D^2 \arrow[r, "L_2"', "\shortmid"{marking}] & D^2
\end{tikzcd}
$$

Corollary VI.4. *R and L respectively induce functors*

$$\mathrm{Mon}(\mathbb{FF}(\mathbb{D})) \to \mathrm{Mon}(\mathbb{D}) \qquad \textit{and} \qquad \mathrm{Comon}(\mathbb{FF}(\mathbb{D})) \to \mathrm{Comon}(\mathbb{D}).$$

Up to this point, we have demonstrated that given any double category $\mathbb{D}$ having arrow objects, there is a 2-fold double category $\mathbb{FF}(\mathbb{D})$ of functorial factorizations in $\mathbb{D}$. The last thing we want to say about this construction is that a cyclic action on $\mathbb{D}$ lifts to one on $\mathbb{FF}(\mathbb{D})$, and hence also to one on $\mathrm{Bimon}(\mathbb{FF}(\mathbb{D}))$.

The cyclic action on objects and vertical morphisms is given directly by that on $\mathbb{D}$. Given a horizontal 1-cell (E,η,ϵ) on an object C, we define the 1-cell $(E,\eta,\epsilon)^\bullet$ on $C^\bullet$ to be $(E^\bullet,\epsilon^\bullet,\eta^\bullet)$. This also implies that the cyclic action swaps L and R for any given functorial factorization.

A quick look at the definitions of the two horizontal compositions is now enough to see that for any two functorial factorizations E_1 and E_2, we have

$$(E_1 \otimes E_2)^\bullet = E_1^\bullet \odot E_2^\bullet \qquad \text{and} \qquad (E_1 \odot E_2)^\bullet = E_1^\bullet \otimes E_2^\bullet$$

Similarly, the cyclic action on 2-cells in $\mathbb{FF}(\mathbb{D})$ is given by the cyclic action in $\mathbb{D}$ on the underlying 2-cell. This gives a valid 2-cell in $\mathbb{FF}(\mathbb{D})$ since the cyclic action simply swaps the equations (VI.1) and (VI.2).

CHAPTER VII

ALGEBRAIC WEAK FACTORIZATION SYSTEMS

For this section, let $\mathbb{D} = \mathbb{S}\mathrm{q}(\mathcal{D})$ be the double category of squares in a 2-category $\mathcal{D}$. We will show that bimonoids in $\mathbb{FF}(\mathbb{D})$ are precisely algebraic weak factorization systems, and more generally that the morphisms in $\mathrm{Bimon}(\mathbb{FF}(\mathbb{D}))$ are given by (co)lax morphisms of algebraic weak factorization systems.

Suppose that $E = (E, \eta, \epsilon)$ is a functorial factorization on a category $\mathscr{C}$, and consider a monoid structure on E. As I_C is initial, the unit of the monoid is forced, and is simply η. The multiplication is given by a natural transformation $\mu\colon ER \Rightarrow E$ satisfying equations (VI.1) and (VI.2), which now take the form $\epsilon \circ \mu = \epsilon R$ and $\mu \circ (\eta \cdot \vec{\eta}) = \eta$.

The unit axioms for the monoid give the equations $\mu \circ E\vec{\eta} = \mathrm{id}_E = \mu \circ \eta R$, which together imply the equation $\mu \circ (\eta \cdot \vec{\eta}) = \eta$ above. And finally, writing $\vec{\mu} = \mu^R\colon R^2 \to R$ for the natural transformation induced by the 2-cell μ, the associativity axiom gives the equation $\mu \circ E\vec{\mu} = \mu \circ \mu R$.

Proposition VII.1. *A monoid structure on an object (E, η, ϵ) in $\mathbb{FF}(\mathbb{D})$ is given by a natural transformation $\mu\colon ER \Rightarrow E$, satisfying equations*

$$\epsilon \circ \mu = \epsilon R \qquad \mu \circ E\vec{\eta} = \mathrm{id}_E = \mu \circ \eta R \qquad \mu \circ E\vec{\mu} = \mu \circ \mu R. \tag{VII.1}$$

This determines a monad $\mathbb{R} = (R, \vec{\eta}, \vec{\mu})$, such that $\mathrm{dom}\,\vec{\mu} = \mu$ *and* $\mathrm{cod}\,\vec{\mu} = \mathrm{id}_{\mathrm{cod}}$.

Similarly, a comonoid structure on (E, η, ϵ) is given by a natural transformation $\delta\colon E \Rightarrow EL$, satisfying equations

$$\delta \circ \eta = \eta L \qquad E\vec{\epsilon} \circ \delta = \mathrm{id}_E = \epsilon L \circ \delta \qquad E\vec{\delta} \circ \delta = \delta L \circ \delta, \tag{VII.2}$$

which determines a comonad $\mathbb{L} = (L, \vec{\epsilon}, \vec{\delta})$, such that $\mathrm{dom}\,\vec{\delta} = \mathrm{id}_{\mathrm{dom}}$ *and* $\mathrm{cod}\,\vec{\delta} = \delta$.

Hence a functorial factorization which simultaineously has a monoid structure and a comonoid structure in $\mathbb{FF}(\mathbb{D})$ is precisely an algebraic weak factorization system, missing only the second bullet of definition II.10: the distributive law condition. This is not surprising, as it is the only condition requiring a compatability between the monad and comonad structures. We will see that a bialgebra in $\mathbb{FF}(\mathbb{D})$ adds precisely this compatibility.

Proposition VII.2. *A bimonoid structure on a horizontal morphism $(E, \eta, \epsilon)\colon C \to C$ in $\mathbb{FF}(\mathbb{D})$ is precisely an algebraic weak factorization system on C with underlying functorial factorization system (E, η, ϵ).*

Proof. We have already shown how the monoid an comonoid structures give rise to the monod and comonad of the awfs. All that remains is to show that the equations (IV.4) amount to just the distributive law, i.e. the equation

$$
\begin{tikzcd}
 & C^2 \arrow[dr, "L"] \arrow[drr, bend left, "E"] \arrow[dr, phantom, shift left=4ex, "\Downarrow\delta"] & & \\
C^2 \arrow[ur, "R"] \arrow[dr, "L"'] \arrow[rr, phantom, "\Downarrow\Delta"] & & C^2 \arrow[r, "E"] & C \\
 & C^2 \arrow[ur, "R"'] \arrow[urr, bend right, "E"'] \arrow[ur, phantom, shift right=4ex, "\Downarrow\mu"] & &
\end{tikzcd}
=
\begin{tikzcd}
 & C^2 \arrow[dr, bend left, "E"] \arrow[d, phantom, "\Downarrow\mu"] & \\
C^2 \arrow[ur, bend left, "R"] \arrow[dr, bend right, "L"'] \arrow[rr, "E"] & {} \arrow[d, phantom, "\Downarrow\delta"] & C. \\
 & C^2 \arrow[ur, bend right, "E"'] &
\end{tikzcd}
\tag{VII.3}
$$

First of all, notice that the first three equations of (IV.4) follow trivially from the initiality of I_C and the terminality of $\perp_C$ in $\mathbb{FF}(\mathbb{D})$, hence they do not impose any further conditions.

The fourth equation here takes the form

$$
\begin{tikzcd}
C^2 \arrow[r, "R"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\delta^R"] & C^2 \arrow[rr, "E"] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\delta"] & & C \arrow[d, equal] \\
C^2 \arrow[r, "R_{E\odot E}"'] \arrow[d, equal] & C^2 \arrow[r, "L"] \arrow[d, phantom, "\Downarrow w"] & C^2 \arrow[r, "E"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_E"] & C \arrow[d, equal] \\
C^2 \arrow[r, "L_{E\otimes E}"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mu^L"] & C^2 \arrow[r, "R"'] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\mu"] & C^2 \arrow[r, "E"'] & C \arrow[d, equal] \\
C^2 \arrow[r, "L"'] & C^2 \arrow[rr, "E"'] & & C
\end{tikzcd}
=
\begin{tikzcd}
C^2 \arrow[r, "R"] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\mu"] & C^2 \arrow[r, "E"] & C \arrow[d, equal] \\
C^2 \arrow[rr, "E"] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\delta"] & & C \arrow[d, equal] \\
C^2 \arrow[r, "L"'] & C^2 \arrow[r, "E"'] & C,
\end{tikzcd}
$$

and so to prove (VII.3), it suffices to show that

$$
\begin{tikzcd}
 & C^2 \arrow[dr, "L"] & \\
C^2 \arrow[ur, bend left=60, "R"] \arrow[ur, "R_{E\odot E}"'] \arrow[dr, "L_{E\otimes E}"] \arrow[dr, bend right=60, "L"'] \arrow[r, phantom, "\Downarrow w"] & {} & C^2 \\
 & C^2 \arrow[ur, "R"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & C^2 \arrow[dr, "L"] & \\
C^2 \arrow[ur, "R"] \arrow[dr, "L"'] \arrow[rr, phantom, "\Downarrow\Delta"] & & C^2. \\
 & C^2 \arrow[ur, "R"'] &
\end{tikzcd}
$$

We can check this using the universal property of C^2 by composing with dom and cod.

First, use (VI.5) and (VI.6) to check that

$$
\begin{tikzcd}
 & C^2 \arrow[dr, "L"] & & \\
C^2 \arrow[ur, bend left=70, "R"] \arrow[ur, "R_{E\odot E}"'] \arrow[ur, phantom, bend left=35, "\Downarrow\delta^R"] \arrow[dr, "L_{E\otimes E}"] \arrow[dr, bend right=70, "L"'] \arrow[dr, phantom, bend right=35, "\Downarrow\mu^L"] \arrow[rr, phantom, "\Downarrow w"] & & C^2 \arrow[r, "\mathrm{dom}"] & C \\
 & C^2 \arrow[ur, "R"'] & &
\end{tikzcd}
=
\begin{tikzcd}
C^2 \arrow[r, bend left, "L"] \arrow[r, "L_{E\otimes E}"' description] \arrow[r, phantom, "\Downarrow i^L"] \arrow[r, bend right=80, "L"'] \arrow[rr, bend left=70, "E"] & C^2 \arrow[r, "E"'] \arrow[u, phantom, "\Downarrow\delta"] \arrow[d, phantom, "\Downarrow\mu^L"] & C
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & C^2 \arrow[dr, "L"] & & \\
C^2 \arrow[ur, bend left=70, "R"] \arrow[ur, "R_{E\odot E}"'] \arrow[ur, phantom, bend left=35, "\Downarrow\delta^R"] \arrow[dr, "L_{E\otimes E}"] \arrow[dr, bend right=70, "L"'] \arrow[dr, phantom, bend right=35, "\Downarrow\mu^L"] \arrow[rr, phantom, "\Downarrow w"] & & C^2 \arrow[r, "\mathrm{cod}"] & C \\
 & C^2 \arrow[ur, "R"'] & &
\end{tikzcd}
=
\begin{tikzcd}
C^2 \arrow[r, bend left=80, "R"] \arrow[r, "R_{E\odot E}" description] \arrow[r, phantom, "\Downarrow p^R"] \arrow[r, bend right, "R"'] \arrow[rr, bend right=70, "E"'] & C^2 \arrow[r, "E"] \arrow[u, phantom, "\Downarrow\delta^R"] \arrow[d, phantom, "\Downarrow\mu"] & C.
\end{tikzcd}
$$

Then use the definitions of i and p to check that $\mu \circ i = \mu \circ \eta R = \mathrm{id}_E$ and $p \circ \delta = \epsilon L \circ \delta = \mathrm{id}_E$, so that the first row above just equals δ, and the second row equals μ. Since Δ also (by definition) satisfies $\mathrm{dom}\,\Delta = \delta$ and $\mathrm{cod}\,\Delta = \mu$, we are done. $\square$

The appropriate notion of morphism between awfs, analagous to left and right Quillen functors and Quillen adjunctions, is (to our knowledge) first given in [Rie11].

Definition VII.3. Suppose that $(E_1, \eta_1, \mu_1, \epsilon_1, \delta_1)$ and $(E_2, \eta_2, \mu_2, \epsilon_2, \delta_2)$ are awfs on $\mathscr{C}$ and $\mathscr{D}$ respectively.

- A *lax morphism of awfs* $(G, \rho): E_1 \to E_2$ consists of a functor $G: \mathscr{C} \to \mathscr{D}$ and a natural transformation $\rho: E_2\hat{G} \Rightarrow GE_1$, such that $(1, \rho): L_2\hat{G} \Rightarrow GL_1$ is a lax morphism of comonads and $(\rho, 1): R_2\hat{G} \Rightarrow GR_1$ is a lax morphism of monads.

- A *colax morphism of awfs* $(F, \lambda): E_1 \to E_2$ consists of a functor $F: \mathscr{C} \to \mathscr{D}$ and a natural transformation $\lambda: FE_1 \Rightarrow E_2\hat{F}$, such that $(1, \lambda): FL_1 \Rightarrow L_2\hat{F}$ is a colax morphism of comonads and $(\lambda, 1): FR_1 \Rightarrow R_2\hat{F}$ is a colax morphism of monads.

Notice that a lax morphism of awfs induces a lift of the functor $\hat{G}$ to a functor $\mathbb{R}_1\mathrm{Alg} \to \mathbb{R}_2\mathrm{Alg}$. In that sense, G "preserves the right class," so is analagous to a right Quillen functor. Similarly, a colax morphism of awfs induces a lift of $\hat{F}$ to $\mathbb{L}_1\mathrm{Coalg} \to \mathbb{L}_2\mathrm{Coalg}$, so is analagous to a left Quillen functor.

By proposition V.3, there is a cyclic action on $\mathrm{Bimon}(\mathbb{FF}(\mathbb{D}))$ induced by the cyclic action on $\mathbb{FF}(\mathbb{D})$. This action is given on awfs by

$$(E, \eta, \mu, \epsilon, \delta)^\bullet = (E^\bullet, \epsilon^\bullet, \delta^\bullet, \eta^\bullet, \mu^\bullet)$$

swapping the monad and comonad structures. This cyclic action allows us to capture both types of morphism of awfs in the same structure.

Proposition VII.4. *Morphisms in* Bimon($\mathbb{FF}(\mathbb{D})$) *are precisely the colax morphisms of awfs. A colax morphism*

$$(E_2, \eta_2, \mu_2, \epsilon_2, \delta_2)^\bullet \to (E_1, \eta_1, \mu_1, \epsilon_1, \delta_1)^\bullet$$

is equivalent to a lax morphism of awfs

$$(E_1, \eta_1, \mu_1, \epsilon_1, \delta_1) \to (E_2, \eta_2, \mu_2, \epsilon_2, \delta_2)$$

Proof. As above, let $(E_1, \eta_1, \mu_1, \epsilon_1, \delta_1)$ and $(E_2, \eta_2, \mu_2, \epsilon_2, \delta_2)$ be awfs on $\mathscr{C}$ and $\mathscr{D}$ respectively. A morphism of bimonoids is given by a 2-cell

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "{(E_1,\eta_1,\epsilon_1)}", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\lambda"] & \mathscr{C} \arrow[d, "F"] \\
\mathscr{D} \arrow[r, "{(E_2,\eta_2,\epsilon_2)}"', "\shortmid"{marking}] & \mathscr{D}
\end{tikzcd}$$

which commutes with the monoid and comonoid structures. It is straightforward to check that this implies the natural transformations

$$\begin{tikzcd}
\mathscr{C}^2 \arrow[r, "L_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\lambda^L"] & \mathscr{C}^2 \arrow[d, "\hat{F}"] \\
\mathscr{D}^2 \arrow[r, "L_2"'] & \mathscr{D}^2
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{C}^2 \arrow[r, "R_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\lambda^R"] & \mathscr{C}^2 \arrow[d, "\hat{F}"] \\
\mathscr{D}^2 \arrow[r, "R_2"'] & \mathscr{D}^2
\end{tikzcd}$$

are colax morphisms of comonads and monads respectively. □

CHAPTER VIII

$\mathbb{R}$-ALG AND $\mathbb{L}$-COALG

For this section, we will continue to let $\mathbb{D} = \mathbb{S}\mathrm{q}(\mathcal{D})$ be the double category of squares in a 2-category $\mathcal{D}$ with arrow objects.

A weak factorization system on a category C is defined by two classes of morphisms, $\mathcal{L}$ and $\mathcal{R}$. In an algebraic weak factorization system, these classes of morphisms are replaced by categories $\mathbb{L}$-Coalg and $\mathbb{R}$-Alg equipped with functors to C^2. In this section, we will discuss the universal property satisfied by these categories, allowing us to define analagous objects in other 2-categories, and record several technical lemmas which we will need in the next section. We will focus on comonads, but there are dual results for monads which we leave to the reader.

Recall from [Str72] the following proposition.

Proposition VIII.1. *Let C be a category, and $\mathbb{L} = (L, \epsilon, \delta)$ be a comonad on C. The category of coalgebras* $\mathbb{L}$-Coalg *has a universal property as follows:*

- *There is a forgetful functor $U\colon \mathbb{L}\text{-Coalg} \to C$ and a natural transformtion $\alpha\colon U \Rightarrow LU$, satisfying $\epsilon U \circ \alpha = \mathrm{id}_U$ and $\delta U \circ \alpha = L\alpha \circ \alpha$.*
- *(U, α) is universal among such pairs satisfying such equations. Given another such pair (F, β), where $F\colon X \to C$, there exists a unique functor $\hat{F}\colon X \to \mathbb{L}\text{-Coalg}$ such that $U\hat{F} = F$ and $\alpha\hat{F} = \beta$.*

Any colax morphism of comonads $(F, \phi)\colon (C, L_1, \epsilon_1, \delta_1) \to (D, L_2, \epsilon_2, \delta_2)$ induces a functor $\tilde{F}\colon \mathbb{L}_1\text{-Coalg} \to \mathbb{L}_2\text{-Coalg}$ such that $U_2\tilde{F} = FU_1$ and $\alpha_2\tilde{F} = \phi U_1 \circ F\alpha_1$.

A natural transformation

$$\begin{tikzcd} X \arrow[r, bend left, "\hat{F}_1"] \arrow[r, bend right, "\hat{F}_2"'] \arrow[r, phantom, "\Downarrow\hat{\theta}"] & \mathbb{L}\text{-Coalg} \end{tikzcd}$$

is uniquely determined by the functors $F_1 = U\hat{F}_1$ and $F_2 = U\hat{F}_2$ and natural transformations $\beta_1 = \alpha\hat{F}_1$ and $\beta_2 = \alpha\hat{F}_2$, and the natural transformation $\theta = U\hat{\theta}\colon F_1 \Rightarrow F_2$, satisfying $L\theta \circ \beta_1 = \beta_2 \circ \theta$.

For the rest of this section, assume that $\mathcal{D}$ *has EM-objects for comonads*, i.e. for every comonad $\mathbb{L}$ in $\mathcal{D}$ there is an object $\mathbb{L}\text{-Coalg}$ satisfying the universal property above.

It is not too hard to use this universal property to construct the free/forgetful adjunction:

Proposition VIII.2. *For any comonad $\mathbb{L}$ on an object C in $\mathcal{D}$, the 1-cell $U\colon \mathbb{L}\text{-Coalg} \to C$ has a right adjoint $\hat{L}$ with $U\hat{L} = L$ and $\alpha\hat{L} = \delta$. The counit of this adjunction is simply the counit of $\mathbb{L}$, $\epsilon\colon U\hat{L} \Rightarrow \mathrm{id}_C$, while the unit is a 2-cell $\hat{\alpha}\colon \mathrm{id}_{\mathbb{L}\text{-Coalg}} \Rightarrow \hat{L}U$ satisfying $U\hat{\alpha} = \alpha$.*

Proof. By proposition VIII.1, to prove the existence of the 1-cell $\hat{L}$, it suffices to verify the equations $\epsilon L \circ \delta = \mathrm{id}_L$ and $\delta L \circ \delta = L\delta \circ \delta$, which are simply two of the comonad axioms.

Using the 2-dimensional part of proposition VIII.1, the existence of the 2-cell $\hat{\alpha}$ follows from the equation $L\alpha \circ \alpha = \delta U \circ \alpha$, which is the remaining comonad axiom.

We leave the verification of the triangle identities for the adjunction to the reader. □

As our interest is in (co)monads in $\mathbb{FF}(\mathbb{D})$, which induce (co)monads on arrow objects, it will be useful to record the universal property that results from the interaction of the EM-object and arrow object universal properties.

Consider a comonad in $\mathbb{FF}(\mathbb{D})$ on an object C, i.e. a functorial factorization with half of the awfs structure. We can combine the universal properties of EM-objects and arrow objects into a universal property for $\mathbb{L}\text{-Coalg}$, where now $\mathbb{L}$ is the comonad in $\mathcal{D}$ arising from the comonad in $\mathbb{FF}(\mathbb{D})$.

Lemma VIII.3. *Let $(E, \eta, \epsilon, \delta)$ be a comonad in $\mathbb{FF}(\mathbb{D})$ on an object C. There is a 2-cell*

$$\begin{tikzcd}
 & C^2 \arrow[dr, "\mathrm{cod}"] \arrow[dd, phantom, "\Downarrow\alpha"] & \\
\mathbb{L}\text{-Coalg} \arrow[ur, bend left, "U"] \arrow[dr, bend right, "U"'] & & C \\
 & C^2 \arrow[ur, "E"'] &
\end{tikzcd}$$

satisfying equations

$$
\begin{tikzcd}
\mathbb{L}\text{-Coalg} \arrow[r, "U"] \arrow[dr, bend right, "U"'] & C^2 \arrow[r, bend left=60, "\mathrm{dom}"] \arrow[r, "\mathrm{cod}"'] \arrow[r, phantom, bend left=30, "\Downarrow\kappa"] \arrow[d, phantom, "\Downarrow\alpha"] & C \\
& C^2 \arrow[ur, bend right, "E"'] &
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{L}\text{-Coalg} \arrow[r, "U"] & C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "E"'] \arrow[r, phantom, "\Downarrow\eta"] & C
\end{tikzcd}
\tag{VIII.1}
$$

$$
\begin{tikzcd}
& C^2 \arrow[dr, bend left, "\mathrm{cod}"] \arrow[d, phantom, "\Downarrow\alpha"] & \\
\mathbb{L}\text{-Coalg} \arrow[ur, bend left, "U"] \arrow[r, "U"'] & C^2 \arrow[r, "E"] \arrow[r, bend right=60, "\mathrm{cod}"'] \arrow[r, phantom, bend right=30, "\Downarrow\epsilon"] & C
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "U"] & C^2 \arrow[r, "\mathrm{cod}"] & C
\end{tikzcd}
\tag{VIII.2}
$$

$$
\begin{tikzcd}
& C^2 \arrow[dr, bend left, "\mathrm{cod}"] \arrow[d, phantom, "\Downarrow\alpha"] & \\
\mathbb{L}\text{-Coalg} \arrow[ur, bend left, "U"] \arrow[r, "U"'] & C^2 \arrow[r, "E"] \arrow[dr, bend right, "L"'] \arrow[dr, phantom, "\Downarrow\delta"] & C \\
& & C^2 \arrow[u, bend right, "E"']
\end{tikzcd}
=
\begin{tikzcd}
& C^2 \arrow[dr, bend left, "\mathrm{cod}"] \arrow[d, phantom, "\Downarrow\alpha"] & \\
\mathbb{L}\text{-Coalg} \arrow[ur, bend left, "U"] \arrow[r, "U"] \arrow[dr, bend right, "U"'] \arrow[dr, phantom, "\Downarrow\vec{\alpha}"] & C^2 \arrow[r, "E"'] & C. \\
& C^2 \arrow[u, bend right, "L"'] &
\end{tikzcd}
\tag{VIII.3}
$$

where $\vec{\alpha}$ *is the unique 2-cell such that* $\mathrm{dom}\,\vec{\alpha} = \mathrm{id}_{\mathrm{dom}\,U}$ *and* $\mathrm{cod}\,\vec{\alpha} = \alpha$*, the existence of which is implied by equation* (VIII.1)*.*

Given any object X*, together with a morphism* $F\colon X \to C^2$ *and a 2-cell* $\beta\colon \mathrm{cod}\,F \Rightarrow EF$ *satisfying equations*

1. $\beta \circ \kappa F = \eta F$
2. $\epsilon F \circ \beta = \mathrm{id}_{\mathrm{cod}\,F}$
3. $\delta F \circ \beta = E\vec{\beta} \circ \beta$

where $\vec{\beta}\colon F \Rightarrow LF$ *is the unique 2-cell such that* $\mathrm{dom}\,\vec{\beta} = \mathrm{id}_{\mathrm{dom}\,F}$ *and* $\mathrm{cod}\,\vec{\beta} = \beta$*; there is a unique morphism* $\hat{F}\colon X \to \mathbb{L}\text{-Coalg}$ *such that* $U\hat{F} = F$ *and* $\alpha\hat{F} = \beta$*.*

Given any pair of morphisms $\hat{F}_1, \hat{F}_2\colon X \to \mathbb{L}\text{-Coalg}$ *and a 2-cell* $\vec{\theta}\colon F_2 \Rightarrow F_2$ *such that*

$$E\vec{\theta} \circ \beta_1 = \beta_2 \circ \mathrm{cod}\,\vec{\theta}$$

(where $F_i = U\hat{F}_i$ *and* $\beta_i = \mathrm{cod}\,\alpha\hat{F}_i$ *as in the previous paragraph), there is a unique 2-cell* $\hat{\theta}\colon \hat{F}_1 \Rightarrow \hat{F}_2$ *such that* $U\hat{\theta} = \vec{\theta}$*.*

Proof. U is simply the U from proposition VIII.1, while the 2-cell α there is the 2-cell $\vec{\alpha}$ here. The equation $\vec{\epsilon}U \circ \vec{\alpha} = \mathrm{id}_F$ implies that $\mathrm{dom}\,\vec{\alpha} = \mathrm{id}_{\mathrm{dom}\,U}$. With that observation, the rest of the equations follow immediately from the universal property of C^2 and the equations $\epsilon U \circ \alpha = \mathrm{id}_U$ and $\delta U \circ \alpha = L\alpha \circ \alpha$ from proposition VIII.1. □

CHAPTER IX

COMPOSITION OF $\mathbb{L}$-COALGEBRAS

In an algebraic weak factorization system, the categories $\mathbb{L}$-Coalg and $\mathbb{R}$-Alg respectively play the roles of the left and right classes of morphisms of the weak factorization system. In an ordinary weak factorization system, these two classes of morphisms are closed under composition. In [Gar09], this is strengthened to a composition functor

$$\mathbb{L}\text{-Coalg} \sqcap_C \mathbb{L}\text{-Coalg} \to \mathbb{L}\text{-Coalg}$$

and in [Rie11], it is shown that colax morphisms of awfs preserve this composition. Similarly, there is a composition functor on $\mathbb{R}$-Alg which is preserved by lax morphisms of awfs.

In this section, we will generalize these results to the setting of bimonads in $\mathbb{FF}(\mathbb{S}\mathrm{q}(\mathcal{D}))$. In fact we will prove the following more general theorem, from which the desired results will follow as corollaries using proposition III.15.

Theorem IX.1. *Let $\mathcal{D}$ be a 2-category with arrow objects and with EM-objects for comonads. There is a lax double functor*

$$\mathrm{Coalg}\colon \mathbb{C}\mathrm{omon}(\mathbb{FF}(\mathbb{S}\mathrm{q}(\mathcal{D}))) \to \mathbb{S}\mathrm{pan}(\mathcal{D}_0)_{/(-)^2}$$

where $\mathcal{D}_0$ is the ordinary category underlying the (strict) 2-category $\mathcal{D}$, which is the identity on the vertical categories, and which takes a comonad $(E, \eta, \epsilon, \delta)$ in $\mathbb{FF}(\mathbb{S}\mathrm{q}(\mathcal{D}))$ to the span

$$C \xleftarrow{\mathrm{dom}\, U} \mathbb{L}\text{-Coalg} \xrightarrow{\mathrm{cod}\, U} C.$$

In [Gar09] it is further shown that given a functorial factorization with only the comonad half of the awfs structure, a composition functor on $\mathbb{L}$-Coalg uniquely determines the monad half of the structure. The paper [Rie11] makes much use of this fact, and also extends it to morphisms of awfs. In our framework, these results will follow from proposition III.18 and the theorem:

Theorem IX.2. *The lax double functor Coalg is fully-faithful.*

First we should explain the notation $\mathbb{S}\mathrm{pan}(\mathcal{D}_0)_{/(-)^2}$. There is a natural family of monads in $\mathbb{S}\mathrm{pan}(\mathcal{D}_0)$, given for each object C by the span

$$C \xleftarrow{\mathrm{dom}} C^2 \xrightarrow{\mathrm{cod}} C$$

with multiplication given by the composition of the internal category structure of C^2 given in proposition II.3. That this is a natural family means that for any morphism $f\colon C \to D$ in $\mathcal{D}_0$ there is a morphism of spans

$$\begin{tikzcd}
C \arrow[d, "f"'] & C^2 \arrow[l, "\mathrm{dom}"'] \arrow[r, "\mathrm{cod}"] \arrow[d, "f^2"] & C \arrow[d, "f"] \\
D & D^2 \arrow[l, "\mathrm{dom}"'] \arrow[r, "\mathrm{cod}"] & D.
\end{tikzcd}$$

That this morphism of spans commutes with the multiplications follows easily from the universal property of arrow objects.

The double category $\mathbb{S}\mathrm{pan}(\mathcal{D}_0)_{/(-)^2}$ has the same vertical category as $\mathbb{S}\mathrm{pan}(\mathcal{D}_0)$—namely $\mathcal{D}_0$—with horizontal 1-cells $C \to C$ given by spans S equipped with a (globular) morphism $S \Rightarrow C^2$, i.e. a commuting diagram

$$\begin{tikzcd}
 & S \arrow[dl, "u"'] \arrow[dr, "v"] \arrow[dd, "p"] & \\
C & & C \\
 & C^2 \arrow[ul, "\mathrm{dom}"] \arrow[ur, "\mathrm{cod}"'] &
\end{tikzcd}$$

and with 2-cells given by 2-cells in $\mathbb{S}\mathrm{pan}(\mathcal{D}_0)$ which commute with these structure maps, i.e. by pairs (f, θ) such that

$$\begin{tikzcd}
C \arrow[d, "f"'] & S \arrow[l, "u"'] \arrow[r, "v"] \arrow[d, "\theta"] & C \arrow[d, "f"] \\
D & S' \arrow[l, "u'"'] \arrow[r, "v'"] \arrow[d, "p'"] & D \\
 & D^2 \arrow[ul, "\mathrm{dom}"] \arrow[ur, "\mathrm{cod}"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & S \arrow[dl, "u"'] \arrow[dr, "v"] \arrow[d, "p"] & \\
C \arrow[d, "f"'] & C^2 \arrow[l, "\mathrm{dom}"] \arrow[r, "\mathrm{cod}"'] \arrow[d, "f^2"] & C \arrow[d, "f"] \\
D & D^2 \arrow[l, "\mathrm{dom}"] \arrow[r, "\mathrm{cod}"'] & D.
\end{tikzcd}$$

The composition of two horizontal 1-cells in $\mathbb{S}\mathrm{pan}(\mathcal{D}_0)_{/(-)^2}$

$$\begin{tikzcd} & S_1 \arrow[dl, "u_1"'] \arrow[dr, "v_1"] \arrow[d, "p_1"] & & S_2 \arrow[dl, "u_2"'] \arrow[dr, "v_2"] \arrow[d, "p_2"] & \\ C & C^2 \arrow[l, "\mathrm{dom}"] \arrow[r, "\mathrm{cod}"'] & C & C^2 \arrow[l, "\mathrm{dom}"] \arrow[r, "\mathrm{cod}"'] & C \end{tikzcd}$$

is given by their horizontal composition in $\mathbb{S}\mathrm{pan}(\mathcal{D}_0)$, and the structure map to C^2 is given by the horizontal composition of the p_1 and p_2 composed with the multiplication of C^2, i.e.

$$S_1 \prod_C S_2 \xrightarrow{(p_1,p_2)} C^2 \prod_C C^2 = C^3 \xrightarrow{c} C.^2$$

The identity for the horizontal composition is

$$\begin{tikzcd} & C \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] \arrow[d, "i"] & \\ C & C^2 \arrow[l, "\mathrm{dom}"] \arrow[r, "\mathrm{cod}"'] & C, \end{tikzcd}$$

where $i\colon C \to C^2$ is the identity of the internal category structure on C^2 from proposition II.3.

We will now prove a couple of simple lemmas to establish the existence of certain 2-cells in $\mathcal{D}$ using the arrow object universal property. First, notice that any comonad $(E, \eta, \epsilon, \delta)$ in $\mathbb{FF}(\mathbb{S}\mathrm{q}(\mathcal{D}))$ gives rise to the horizontal 1-cell in $\mathbb{S}\mathrm{pan}(\mathcal{D}_0)_{/(-)^2}$

$$\begin{tikzcd} & \mathbb{L}\text{-Coalg} \arrow[dl, "\mathrm{dom}\,U"'] \arrow[dr, "\mathrm{cod}\,U"] \arrow[d, "U"] & \\ C & C^2 \arrow[l, "\mathrm{dom}"] \arrow[r, "\mathrm{cod}"'] & C. \end{tikzcd}$$

For each of the following lemmas, let $(E_1, \eta_1, \epsilon_1, \delta_1)$ and $(E_2, \eta_2, \epsilon_2, \delta_2)$ be two comonads in $\mathbb{FF}(\mathbb{S}\mathrm{q}(\mathcal{D}))$, both on the same object C, and let $X_{1,2}$ be the pullback

$$\begin{tikzcd} & X_{1,2} \arrow[dl, "P_1"'] \arrow[dr, "P_2"] & \\ \mathbb{L}_1\text{-Coalg} \arrow[dr, "\mathrm{cod}\,U_1"'] & & \mathbb{L}_2\text{-Coalg} \arrow[dl, "\mathrm{dom}\,U_2"] \\ & C & \end{tikzcd}$$

with structure map $U_{1,2}\colon X \to C^2$ given by the composition

$$X_{1,2} \xrightarrow{(U_1,U_2)} C^3 \xrightarrow{\;c\;} C.^2$$

Recall from proposition II.3 that c by definition satisfies $\operatorname{dom} c = \operatorname{dom} P_1$, $\operatorname{cod} c = \operatorname{cod} P_2$, $\kappa c = \kappa P_2 \circ \kappa P_1$. We also record for later reference:

$$\operatorname{dom} U_{1,2} = \operatorname{dom} c(U_1,U_2) = \operatorname{dom} P_1(U_1,U_2) = \operatorname{dom} U_1 P_1 \tag{IX.1}$$

$$\operatorname{cod} U_{1,2} = \operatorname{cod} c(U_1,U_2) = \operatorname{cod} P_2(U_1,U_2) = \operatorname{cod} U_2 P_2 \tag{IX.2}$$

$$\kappa U_{1,2} = \kappa c(U_1,U_2) = (\kappa P_2 \circ \kappa P_1)(U_1,U_2) = \kappa U_2 P_2 \circ \kappa U_1 P_1 \tag{IX.3}$$

Lemma IX.3. *There is a 2-cell*

$$\begin{tikzcd}
 & \mathbb{L}_1\text{-Coalg} \arrow[dr, bend left, "U_1"] & \\
X_{1,2} \arrow[ur, bend left, "P_1"] \arrow[rr, bend right, "U_{1,2}"'] & \Downarrow\zeta & C^2
\end{tikzcd}$$

such that $\operatorname{dom}\zeta = \mathrm{id}$ *and*

$$\begin{tikzcd}
 & \mathbb{L}_1\text{-Coalg} \arrow[dr, bend left, "U_1"] & & \\
X_{1,2} \arrow[ur, bend left, "P_1"] \arrow[rr, bend right, "U_{1,2}"'] & \Downarrow\zeta & C^2 \arrow[r, "\operatorname{cod}"] & C
\end{tikzcd}
=
\begin{tikzcd}
 & C^2 \arrow[dr, "\operatorname{cod}"] & \\
X_{1,2} \arrow[ur, "UP_1"] \arrow[dr, "UP_2"'] & & C \\
 & C^2 \arrow[ur, bend left, "\operatorname{dom}"] \arrow[ur, bend right, "\operatorname{cod}"'] \arrow[ur, phantom, "\Downarrow\kappa"] &
\end{tikzcd}$$

Proof. Equation (II.3) becomes

$$\begin{tikzcd}
X_{1,2} \arrow[r, "U_{1,2}"] & C^2 \arrow[r, bend left, "\operatorname{dom}"] \arrow[r, bend right, "\operatorname{cod}"'] \arrow[r, phantom, "\Downarrow\kappa"] & C
\end{tikzcd}
=
\begin{tikzcd}
 & C^2 \arrow[dr, bend left, "\operatorname{dom}"] \arrow[dr, bend right, "\operatorname{cod}"'] \arrow[dr, phantom, "\Downarrow\kappa"] & \\
X_{1,2} \arrow[ur, "UP_1"] \arrow[dr, "UP_2"'] & & C \\
 & C^2 \arrow[ur, bend left, "\operatorname{dom}"] \arrow[ur, bend right, "\operatorname{cod}"'] \arrow[ur, phantom, "\Downarrow\kappa"] &
\end{tikzcd}$$

which is just equation (IX.3) $\square$

Lemma IX.4. *There is a 2-cell*

$$\begin{tikzcd}
 & \mathbb{L}_2\text{-Coalg} \arrow[dr, bend left, "U_2"] & \\
X_{1,2} \arrow[ur, bend left, "P_2"] \arrow[dr, bend right, "U_{1,2}"'] & \Downarrow\nu & C^2 \\
 & C^2 \arrow[ur, bend right, "R_1"'] &
\end{tikzcd}$$

such that $\operatorname{cod}\nu = \mathrm{id}$ *and*

$$\begin{tikzcd}
& \mathbb{L}_2\text{-Coalg} \arrow[dr, "U_2"] \arrow[d, phantom, "\Downarrow\nu"] & & & & & \mathbb{L}_2\text{-Coalg} \arrow[r, "U_2"] & C^2 \arrow[dr, bend left, "\mathrm{dom}"] & \\
X_{1,2} \arrow[ur, "P_2"] \arrow[dr, "U_{1,2}"'] & {} & C^2 \arrow[r, "\mathrm{dom}"] & C & = & X_{1,2} \arrow[ur, bend left, "P_2"] \arrow[r, "P_1"] \arrow[drr, bend right, "U_{1,2}"'] & \mathbb{L}_1\text{-Coalg} \arrow[r, "U_1"] \arrow[dr, "U_1"'] \arrow[d, phantom, "\Downarrow\zeta"] & C^2 \arrow[r, "\mathrm{cod}"] \arrow[d, phantom, "\Downarrow\alpha_1"] & C \\
& C^2 \arrow[ur, "R_1"'] & & & & & {} & C^2 \arrow[ur, bend right, "E_1"'] &
\end{tikzcd}$$

Proof. We just need to verify equation (II.3):

$$\begin{tikzcd}
& \mathbb{L}_2\text{-Coalg} \arrow[r, "U_2"] & C^2 \arrow[dr, bend left, "\mathrm{dom}"] & \\
X_{1,2} \arrow[ur, bend left, "P_2"] \arrow[r, "P_1"] \arrow[drr, bend right, "U_{1,2}"'] & \mathbb{L}_1\text{-Coalg} \arrow[r, "U_1"] \arrow[dr, "U_1"'] \arrow[d, phantom, "\Downarrow\zeta"] & C^2 \arrow[r, "\mathrm{cod}"] \arrow[d, phantom, "\Downarrow\alpha_1"] & C \\
& {} & C^2 \arrow[ur, "E_1"] \arrow[ur, bend right=60, "\mathrm{cod}"'] \arrow[ur, phantom, bend right=30, "\Downarrow\epsilon_1"] &
\end{tikzcd}$$

$$= \begin{tikzcd}
& \mathbb{L}_2\text{-Coalg} \arrow[r, "U_2"] & C^2 \arrow[dr, bend left, "\mathrm{dom}"] & \\
X_{1,2} \arrow[ur, bend left, "P_2"] \arrow[r, "P_1"] \arrow[rr, bend right=60, "U_{1,2}"'] & \mathbb{L}_1\text{-Coalg} \arrow[r, "U_1"] \arrow[d, phantom, "\Downarrow\zeta"] & C^2 \arrow[r, "\mathrm{cod}"] & C \\
& {} & &
\end{tikzcd}$$

$$= \begin{tikzcd}
X_{1,2} \arrow[r, "P_2"] & \mathbb{L}_1\text{-Coalg} \arrow[r, "U_1"] & C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\kappa"] & C
\end{tikzcd}$$

where the first equation follows from (VIII.2), and the second by reducing $\operatorname{cod}\zeta$ using lemma IX.3. □

Proof of Theorem IX.1. For notational convenience, let $G = \mathrm{Coalg}$ be the lax double functor we need to establish. Both the double categories $\mathbb{C}\mathrm{omon}(\mathbb{FF}(\mathbb{S}\mathrm{q}(\mathscr{D})))$ and $\mathbb{S}\mathrm{pan}(\mathcal{D}_0)_{/(-)^2}$ have $\mathcal{D}_0$ as vertical category, and G_0 (the component of G on vertical categories) is simply the identity. From the statement of the theorem, G takes an object in $\mathbb{C}\mathrm{omon}(\mathbb{FF}(\mathbb{S}\mathrm{q}(\mathscr{D})))$ to the span and structure map

$$\begin{tikzcd}
& \mathbb{L}\text{-Coalg} \arrow[dl, "\mathrm{dom}\, U"'] \arrow[dr, "\mathrm{cod}\, U"] \arrow[dd, "U"] & \\
C & & C. \\
& C^2 \arrow[ul, "\mathrm{dom}"] \arrow[ur, "\mathrm{cod}"'] &
\end{tikzcd}$$

To define the behavior of G on 2-cells, consider a 2-cell in $\mathbb{C}\mathrm{omon}(\mathbb{F}\mathbb{F}(\mathbb{S}\mathrm{q}(\mathscr{D})))$:

$$\begin{tikzcd} C \arrow[r, "{(E_1,\eta_1,\epsilon_1,\delta_1)}", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[d, "F"] \\ D \arrow[r, "{(E_2,\eta_2,\epsilon_2,\delta_2)}"', "\shortmid"{marking}] & D. \end{tikzcd}$$

By corollary VI.4, ϕ induces a colax morphism of comonads from L_1 to L_2, hence by proposition VIII.1 there is an induced morphism $\tilde{\phi}$ between the EM-objects such that $U_2\tilde{\phi} = F^2U_1$. We can then define $G\phi$ to be the morphism of spans

$$\begin{tikzcd} C \arrow[d, "F"'] & C^2 \arrow[l, "\mathrm{dom}"'] \arrow[d, "F^2"] & \mathbb{L}_1\text{-Coalg} \arrow[l, "U_1"'] \arrow[r, "U_1"] \arrow[d, "\tilde{\phi}"] & C^2 \arrow[r, "\mathrm{cod}"] \arrow[d, "F^2"] & C \arrow[d, "F"] \\ D & D^2 \arrow[l, "\mathrm{dom}"] & \mathbb{L}_2\text{-Coalg} \arrow[l, "U_2"] \arrow[r, "U_2"'] & D^2 \arrow[r, "\mathrm{cod}"'] & D. \end{tikzcd}$$

That $\tilde{\phi}$ commutes with the structure maps is simply the commutativity of the square $U_2\tilde{\phi} = F^2U_1$.

Next we must define the coherence data G_I and $G_\otimes$. We will define G_I to be the morphims of spans

$$\begin{tikzcd} & C \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] \arrow[dd, "G_I"] & \\ C & & C \\ & \mathbb{L}_I\text{-Coalg} \arrow[ul, "\mathrm{dom}\, U"] \arrow[ur, "\mathrm{cod}\, U"'] & \end{tikzcd}$$

defined via lemma VIII.3 by the equations $UG_I = i\colon C \to C^2$ and $\alpha_I G_I$ is the identity on $\mathrm{dom}\, i = \mathrm{cod}\, i$. The conditions of the lemma are trivially satisfied.

We will similarly use lemma VIII.3 to define $G_\otimes$. Let $X_{1,2}$, $U_{1,2}$, ζ, and ν be as defined earlier in the section. $G_\otimes$ is a morphism of spans

$$\begin{tikzcd} & X_{1,2} \arrow[dl, "\mathrm{dom}\, U_1P_1"'] \arrow[dr, "\mathrm{cod}\, U_2P_2"] \arrow[dd, "G_\otimes"] & \\ C & & C. \\ & \mathbb{L}_{1\otimes 2}\text{-Coalg} \arrow[ul, "\mathrm{dom}\, U_{1\otimes 2}"] \arrow[ur, "\mathrm{cod}\, U_{1\otimes 2}"'] & \end{tikzcd}$$

We will define $G_\otimes$ to be the 1-cell such that $U_{1\otimes2}G_\otimes = U_{1,2}$ and

$$
\begin{tikzcd}
 & & C^2 \arrow[dr, "\mathrm{cod}"] & \\
X_{1,2} \arrow[r, "G_\otimes"] & \mathbb{L}_{1\otimes2}\text{-Coalg} \arrow[ur, bend left, "U_{1\otimes2}"] \arrow[dr, bend right, "U_{1\otimes2}"'] \arrow[rr, phantom, "\Downarrow\alpha_{1\otimes2}"] & & C \\
 & & C^2 \arrow[ur, "E_{1\otimes2}"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & \mathbb{L}_2\text{-Coalg} \arrow[r, "U_2"] \arrow[ddr, "U_2"] & C^2 \arrow[dr, "\mathrm{cod}"] & \\
X \arrow[ur, bend left, "P_2"] \arrow[dr, bend right, "U_{1,2}"'] & \Downarrow\nu & \Downarrow\alpha_2 & C \\
 & C^2 \arrow[r, "R_1"'] & C^2 \arrow[ur, bend right, "E_2"'] &
\end{tikzcd}
$$

In other words, in the notation of lemma VIII.3 let $F = U_{1,2}$ and $\beta = E_2\nu \circ \alpha_2 P_2$, and define $G_\otimes = \hat{F}$.

We now need to check equations 1-3 of lemma VIII.3 to verify that $G_\otimes$ is well defined. We will check these equationally to save space, but the reader may want to draw out the diagrams for themselves to follow along. For the first equation:

$$
\begin{aligned}
& E_2\nu \circ \alpha_2 P_2 \circ \kappa U_{1,2} \\
&\quad = E_2\nu \circ \alpha_2 P_2 \circ \kappa U_2 P_2 \circ \kappa U_1 P_1 && \text{Eq. (IX.3)} \\
&\quad = E_2\nu \circ (\alpha_2 \circ \kappa U_2) P_2 \circ \kappa U_1 P_1 && \\
&\quad = E_2\nu \circ \eta_2 U_2 P_2 \circ \kappa U_1 P_1 && \text{Eq. (VIII.1)} \\
&\quad = \eta_2 R_1 U_{1,2} \circ \operatorname{dom}\nu \circ \kappa U_1 P_1 && \text{Interchange} \\
&\quad = \eta_2 R_1 U_{1,2} \circ E_1\zeta \circ \alpha_1 P_1 \circ \kappa U_1 P_1 && \text{Def of } \nu \\
&\quad = \eta_2 R_1 U_{1,2} \circ E_1\zeta \circ (\alpha_1 \circ \kappa U_1) P_1 && \\
&\quad = \eta_2 R_1 U_{1,2} \circ E_1\zeta \circ \eta_1 U_1 P_1 && \text{Eq. (VIII.1)} \\
&\quad = \eta_{1\otimes2} U_{1,2} \circ \operatorname{dom}\zeta && \text{Interchange; Def of } \eta_{1\otimes2} \\
&\quad = \eta_{1\otimes2} U_{1,2} && \operatorname{dom}\zeta = \mathrm{id}
\end{aligned}
$$

and the second:

$$
\begin{aligned}
& \epsilon_{1\otimes2} U_{1,2} \circ E_2\nu \circ \alpha_2 P_2 \\
&\quad = \epsilon_2 R_1 U_{1,2} \circ E_2\nu \circ \alpha_2 P_2 && \text{Def of } \epsilon_{1\otimes2} \\
&\quad = \operatorname{cod}\nu \circ (\epsilon_2 U_2 \circ \alpha_2) P_2 && \text{Interchange} \\
&\quad = \mathrm{id}_{\operatorname{cod} U_{1,2}}\,. && \text{Eq. (VIII.2); } \operatorname{cod}\nu = \mathrm{id}
\end{aligned}
$$

The third equation is a bit trickier to prove. We will need to prove two intermediate equations first, using the arrow object universal property.

Lemma.

$$
i^L U_{1,2} \circ L_1\zeta \circ \vec{\alpha}_1 P_1 = \vec{\beta} \circ \zeta \tag{IX.4}
$$

Proof. We must show the 2-cells become equal upon composition with dom and cod:

$$\mathrm{dom}(i^L U_{1,2} \circ L_1\zeta \circ \vec{\alpha}_1 P_1) = \mathrm{id}_{\mathrm{dom}\, U_{1,2}} = \mathrm{dom}(\vec{\beta} \circ \zeta)$$

and

$$\begin{aligned}
&\mathrm{cod}(i^L U_{1,2} \circ L_1\zeta \circ \vec{\alpha}_1 P_1) \\
&\quad = \mathrm{cod}\, i^L U_{1,2} \circ E_1\zeta \circ \mathrm{cod}\, \vec{\alpha}_1 P_1 \\
&\quad = \eta_2 R_1 U_{1,2} \circ E_1\zeta \circ \alpha_1 P_1 && \text{Def of } i^L, \vec{\alpha} \\
&\quad = \eta_2 R_1 U_{1,2} \circ \mathrm{dom}\, \nu && \text{Def of } \nu \\
&\quad = E_2\nu \circ \eta_2 U_2 P_2 && \text{Interchange} \\
&\quad = E_2\nu \circ (\alpha_2 \circ \kappa U_2) P_2 && \text{Eq. (VIII.1)} \\
&\quad = (E_2\nu \circ \alpha_2 P_2) \circ \kappa U_2 P_2 \\
&\quad = \mathrm{cod}\, \vec{\beta} \circ \mathrm{cod}\, \zeta && \text{Def of } \vec{\beta}, \zeta \\
&\quad = \mathrm{cod}(\vec{\beta} \circ \zeta).
\end{aligned}$$

∎

Lemma.

$$R_1\vec{\beta} \circ \nu = w U_{1,2} \circ L_2 \delta_1^R U_{1,2} \circ L_2\nu \circ \vec{\alpha}_2 P_2 \tag{IX.5}$$

Proof. Again we must prove equality after composing with dom and cod:

$$\begin{aligned}
&\mathrm{dom}(R_1\vec{\beta} \circ \nu) \\
&\quad = E_1\vec{\beta} \circ \mathrm{dom}\, \nu \\
&\quad = E_1\vec{\beta} \circ E_1\zeta \circ \alpha_1 P_1 && \text{Def of } \nu \\
&\quad = E_1(\vec{\beta} \circ \zeta) \circ \alpha_1 P_1 \\
&\quad = E_1(i^L U_{1,2} \circ L_1\zeta \circ \vec{\alpha}_1 P_1) \circ \alpha_1 P_1 && \text{Eq. (IX.4)} \\
&\quad = E_1 i^L U_{1,2} \circ E_1 L_1 \zeta \circ (E_1\vec{\alpha}_1 \circ \alpha_1) P_1 \\
&\quad = E_1 i^L U_{1,2} \circ E_1 L_1 \zeta \circ (\delta_1 U_1 \circ \alpha_1) P_1 && \text{Eq. (VIII.3)} \\
&\quad = E_1 i^L U_{1,2} \circ \delta_1 U_{1,2} \circ E_1\zeta \circ \alpha_1 P_1 && \text{Interchange} \\
&\quad = \mathrm{dom}\, w U_{1,2} \circ \mathrm{dom}\, \delta_1^R U_{1,2} \circ \mathrm{dom}\, \nu \circ \mathrm{dom}\, \vec{\alpha}_2 P_2 && \text{Defs of } w, \delta^R, \nu, \vec{\alpha} \\
&\quad = \mathrm{dom}(w U_{1,2} \circ L_2 \delta_1^R U_{1,2} \circ L_2\nu \circ \vec{\alpha}_2 P_2)
\end{aligned}$$

and

$$
\begin{aligned}
&\operatorname{cod}(R_1\vec{\beta}\circ\nu) \\
&\quad = \operatorname{cod}\vec{\beta}\circ\operatorname{cod}\nu \\
&\quad = E_2\nu\circ\alpha_2P_2 && \text{Defs of } \vec{\beta},\ \nu \\
&\quad = E_2(p^R\circ\delta_1^R)U_{1,2}\circ E_2\nu\circ\alpha_2P_2 && p^R\circ\delta^R = \mathrm{id} \\
&\quad = E_2p^RU_{1,2}\circ E_2\delta_1^RU_{1,2}\circ E_2\nu\circ\alpha_2P_2 \\
&\quad = \operatorname{cod} wU_{1,2}\circ\operatorname{cod} L_2\delta_1^RU_{1,2}\circ\operatorname{cod} L_2\nu\circ\operatorname{cod}\vec{\alpha}_2P_2 && \text{Defs of } w,\ L,\ \vec{\alpha} \\
&\quad = \operatorname{cod}(wU_{1,2}\circ L_2\delta_1^RU_{1,2}\circ L_2\nu\circ\vec{\alpha}_2P_2)
\end{aligned}
$$

□

Now we are prepared to prove the third equation of lemma VIII.3 validating our definition of $G_\otimes$:

$$
\begin{aligned}
&\delta_{1\otimes 2}U_{1,2}\circ E_2\nu\circ\alpha_2P_2 \\
&\quad = (E_2w\circ\delta_2R_{1\odot 1}\circ E_2\delta_1^R)U_{1,2}\circ E_2\nu\circ\alpha_2P_2 && \text{Def of } \delta_{1\otimes 2} \\
&\quad = E_2(wU_{1,2}\circ L_2\delta_1^RU_{1,2}\circ L_2\nu)\circ(\delta_2U_2\circ\alpha_2)P_2 && \text{Interchange} \\
&\quad = E_2(wU_{1,2}\circ L_2\delta_1^RU_{1,2}\circ L_2\nu)\circ(E_2\vec{\alpha}_2\circ\alpha_2)P_2 && \text{Eq. (VIII.3)} \\
&\quad = E_2(wU_{1,2}\circ L_2\delta_1^RU_{1,2}\circ L_2\nu\circ\vec{\alpha}_2P_2)\circ\alpha_2P_2 \\
&\quad = E_2(R_1\vec{\beta}\circ\nu)\circ\alpha_2P_2 && \text{Eq. (IX.5)} \\
&\quad = E_{1\otimes 2}\vec{\beta}\circ E_2\nu\circ\alpha_2P_2 && \text{Def of } E_{1\otimes 2}
\end{aligned}
$$

The verification that the definitions of G_I and $G_\otimes$ form natural families, and of the coherence axioms for a lax double functor, is tedious, but follows from what we have presented here without requiring any new ideas or ingenuity. □

Corollary IX.5. *For any awfs $(E,\eta,\mu,\epsilon,\delta)$ on an object C in $\mathcal{D}$, the multiplication μ induces a composition functor on* $\mathbb{L}$-Coalg*, and the functor between EM-objects induced by any colax morphism of awfs preserves this composition.*

Proof. Any awfs $(E,\eta,\mu,\epsilon,\delta)$ has an underlying object in $\mathbb{C}\mathrm{omon}(\mathbb{FF}(\mathbb{S}\mathrm{q}(\mathcal{D})))$, by simply forgetting μ. The lax double-functor Coalg takes this to a span

$$C \xleftarrow{\operatorname{dom} U} \mathbb{L}\text{-Coalg} \xrightarrow{\operatorname{cod} U} C.$$

The multiplication μ provides this object in $\mathbb{C}\mathrm{omon}(\mathbb{FF}(\mathbb{S}\mathrm{q}(\mathcal{D})))$ with a monad structure, and lax double-functors preserve monads, so μ induces a monad structure on this span. A multiplication on this span is a morphism π:

$$\begin{tikzcd}
 & X \arrow[dl, "\mathrm{dom}\, U P_1"'] \arrow[dr, "\mathrm{cod}\, U P_2"] \arrow[d, "\pi"] & \\
C & & C \\
 & \mathbb{L}\text{-Coalg} \arrow[ul, "\mathrm{dom}\, U"] \arrow[ur, "\mathrm{cod}\, U"'] &
\end{tikzcd}$$

where X is the pullback in the composite span

$$\begin{tikzcd}
 & & X \arrow[dl, "P_1"'] \arrow[dr, "P_2"] & & \\
 & \mathbb{L}\text{-Coalg} \arrow[dl, "\mathrm{dom}\, U"'] \arrow[dr, "\mathrm{cod}\, U"] & & \mathbb{L}\text{-Coalg} \arrow[dl, "\mathrm{dom}\, U"'] \arrow[dr, "\mathrm{cod}\, U"] & \\
C & & C & & C.
\end{tikzcd}$$

The morphism π is the composition structure that we want. If $\mathcal{D} = \mathrm{Cat}$ is the 2-category of small categories, then an object (f, g) in X is a pair of morphisms in C equipped with coalgebra structures, such that $\mathrm{cod}\, f = \mathrm{dom}\, g$, and $\pi(f, g)$ is a morphism equipped with a coalgebra structure, with $\mathrm{dom}\, \pi(f, g) = \mathrm{dom}\, f$ and $\mathrm{cod}\, \pi(f, g) = \mathrm{cod}\, g$.

Of course, what we really want is that the morphism underlying the coalgebra $\pi(f, g)$ is the composition $g \circ f$. But this is simply the fact that π defines a 2-cell in $\mathbb{S}\mathrm{pan}(\mathcal{D}_0)_{/(-)^2}$, hence commutes with the structure maps to C^2. Recall that the structure map for the horizontal composite X is defined using $c\colon C^3 \to C^2$, hence $U\pi(f, g) = c(Uf, Ug)$. □

Now we will continue on to the proof of theorem IX.2. The proof is surprisingly difficult and tedious—we will outline the main steps but leave many of the routine verifications to the reader.

Proof of Theorem IX.2. The bijectivity of Coalg acting on 2-cells with domain I is simple to check, since I is initial in $\mathbb{C}\mathrm{omon}(\mathbb{FF}(\mathbb{S}\mathrm{q}(\mathcal{D})))$, and from lemma VIII.3 it is easy to see that there is a unique morphism $!\colon C \to \mathbb{L}\text{-Coalg}$ satisfying $U! = i$, with $\alpha ! = \eta i$.

Now let $(E_i, \eta_i, \epsilon_i, \delta_i)$, $i \in \{1, 2, 3\}$, be three comonads in $\mathbb{FF}(\mathbb{S}\mathrm{q}(\mathcal{D}))$, with $E_1, E_2\colon C^2 \to C$ and $E_3\colon D^2 \to D$, and let $F\colon C \to D$ be a morphism. Given a morphism $X_{1,2} \to \mathbb{L}_3\text{-Coalg}$ such that $U_3\theta = F^2 U_{1,2}$, we need to prove the unique existence of a 2-cell

$$\begin{tikzcd}
C \arrow[r, "{(E_1,\eta_1,\epsilon_1,\delta_1)}", "\shortmid"{marking}] \arrow[d, "F"'] & C \arrow[r, "{(E_2,\eta_2,\epsilon_2,\delta_2)}", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\phi"] & C \arrow[d, "F"] \\
D \arrow[rr, "{(E_3,\eta_3,\epsilon_3,\delta_3)}"', "\shortmid"{marking}] & & D
\end{tikzcd}$$

such that $\tilde{\phi} G_\otimes = \theta$.

Outline of proof:

- Define a morphism $\check{L}_{1\otimes 2}\colon C^2 \to X_{1,2}$ such that

$$P_1 \check{L}_{1\otimes 2} = \hat{L}_1 \qquad \text{and} \qquad P_2 \check{L}_{1\otimes 2} = \hat{L}_2 R_1.$$

  Show that $U_{1,2}\check{L}_{1\otimes 2} = L_{1\otimes 2}$.
- Define a 2-cell

$$\psi\colon U_3 \theta \check{L}_{1\otimes 2} \Rightarrow F^2$$

  by $\psi = F^2 \vec{\epsilon}_{1\otimes 2}$, noting that $U_3 \theta \check{L}_{1\otimes 2} = F^2 L_{1\otimes 2}$.
- Let the 2-cell $\psi' = \hat{L}_3 \psi \circ \hat{\alpha}_3 \theta \check{L}_{1\otimes 2}$,

$$\begin{tikzcd}
C^2 \arrow[r, "\check{L}_{1\otimes 2}"] \arrow[d, equal] & X_{1,2} \arrow[r, "\theta"] \arrow[d, phantom, "\Downarrow\psi'"] & \mathbb{L}_3\text{-Coalg} \\
C^2 \arrow[rr, "F^2"'] & & D^2 \arrow[u, "\hat{L}_3"']
\end{tikzcd}$$

  be the mate of ψ under the adjunction $U_3 \dashv \hat{L}_3$.
- Define the desired 2-cell ϕ to be the codomain component of ψ':

$$\phi = \operatorname{cod} U_3 \psi' = E_3 \psi \circ \alpha_3 \theta \check{L}_{1\otimes 2}\colon F E_2 R_1 \to E_3 F^2$$

- First we must verify that ϕ defines a valid 2-cell in $\mathbb{FF}(\mathbb{S}\mathrm{q}(\mathcal{D}))$ by checking equations (VI.1) and (VI.2). Equation (VI.1) is simple to show directly, while (VI.2) follows from the well definedness of $U_3\psi'$. In fact, we have

$$\phi^L = U_3 \psi' \colon F^2 L_{1\otimes 2} \Rightarrow L_3 F^2$$

- Next we must verify that ϕ defines a valid 2-cell in $\mathbb{C}\mathrm{omon}(\mathbb{FF}(\mathbb{S}\mathrm{q}(\mathcal{D})))$, which means showing that it commutes with the comultipication 2-cells:

$$\begin{tikzcd}
C^2 \arrow[r, "R_1"] \arrow[d, equal] & C^2 \arrow[r, "E_2"] \arrow[d, phantom, "\Downarrow\delta_{1\otimes 2}"] & C \arrow[d, equal] \\
C^2 \arrow[r, "L_{1\otimes 2}"] \arrow[d, "F^2"'] & C^2 \arrow[r, "E_2 R_1"] \arrow[d, "F^2"] \arrow[l, phantom, "\Downarrow\phi^L"] \arrow[r, phantom, "\Downarrow\phi"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "L_3"'] & D^2 \arrow[r, "E_3"'] & D
\end{tikzcd}
=
\begin{tikzcd}
C^2 \arrow[r, "R_1"] \arrow[d, "F^2"'] & C^2 \arrow[r, "E_2"] \arrow[d, phantom, "\Downarrow\phi"] & C \arrow[d, "F"] \\
D^2 \arrow[rr, "E_3"] \arrow[d, equal] & \arrow[d, phantom, "\Downarrow\delta_3"] & D \arrow[d, equal] \\
D \arrow[r, "L_3"'] & D^2 \arrow[r, "E_3"'] & D.
\end{tikzcd}$$

To do this, first verify the existence of a 2-cell $\check{\delta}_{1\otimes 2}\colon \check{L}_{1\otimes 2} \Rightarrow \check{L}_{1\otimes 2}L_{1\otimes 2}$ satisfying

$$P_1\check{\delta}_{1\otimes 2} = \hat{L}_1 i^L \circ \hat{\delta}_1 \quad \text{and} \quad P_2\check{\delta}_{1\otimes 2} = \hat{L}_2 w \circ \hat{L}_2 L_2 \delta_1^R \circ \hat{\delta}_2 R_1 \tag{IX.6}$$

where $\hat{\delta}_i$ is the unique 2-cell with $U_i\hat{\delta}_i = \vec{\delta}_i$. Show that $U_{1,2}\check{\delta}_{1\otimes 2} = \vec{\delta}_{1\otimes 2}$.
Define

$$\tau_1 = \hat{L}_3\phi^L \circ \psi' L_{1\otimes 2} \circ \theta\check{\delta}_{1\otimes 2} \quad \text{and} \quad \tau_2 = \hat{\delta}_3 F^2 \circ \psi'$$

and check that $\operatorname{cod} U_3\tau_1 = E_3\phi^L \circ \phi L_{1\otimes 2} \circ F\delta_{1\otimes 2}$ and $\operatorname{cod} U_3\tau_2 = \delta_3 F^2 \circ \phi$. Hence to prove (IX.6) it suffices to show $\tau_1 = \tau_2$. To do this, show that the mates of each are equal to ϕ^L.

- We have defined a 2-cell ϕ in $\mathbb{C}\mathrm{omon}(\mathbb{FF}(\mathbb{S}\mathrm{q}(\mathcal{D})))$, now we need to show that the lax functor Coalg takes this ϕ to the 2-cell θ we began with, i.e. that $\theta = \tilde{\phi}G_\otimes$. It is easy to see that

$$U_3\tilde{\phi}G_\otimes = F^2 U_{1\otimes 2} G_\otimes = F^2 U_{1,2} = U_3\theta,$$

so it only remains to show that $\alpha_3\tilde{\phi}G_\otimes = \alpha_3\theta$.
Begin by verifying the existence of a 2-cell $\rho\colon \mathrm{id}_{X_{1,2}} \Rightarrow \check{L}_{1\otimes 2}U_{1,2}$ such that

$$P_1\rho = \hat{L}_1\zeta \circ \hat{\alpha}_1 P_1 \quad \text{and} \quad P_2\rho = \hat{L}_2\nu \circ \hat{\alpha}_2 P_2,$$

and show that $U_{1,2}\rho = \vec{\alpha}_{1\otimes 2}G_\otimes$.
Finally, show that

$$\psi U_{1,2} \circ F^2\vec{\alpha}_{1\otimes 2}G_\otimes = \mathrm{id}_{F^2 U_{1,2}},$$

and use this to show that $\vec{\alpha}_3\tilde{\phi}G_\otimes = \vec{\alpha}_3\theta$. Thus we have shown the existance of the 2-cell ϕ such that $\tilde{\phi}G_\otimes = \theta$, and the uniqueness follows by a very similar computation.

□

Combining this with proposition III.15 immediately implies:

Corollary IX.6. *Suppose $(E, \eta, \epsilon, \delta)$ is a comonad in $\mathbb{FF}(\mathbb{S}\mathrm{q}(\mathcal{D}))$. A composition on $\mathbb{L}$-Coalg is equivalent to completing E to an awfs.*

CHAPTER X

A UNIVERSAL PROPERTY FOR THE PUSHOUT PRODUCT

In this chapter we will begin the work of incorporating adjunctions of several variables into the framework given so far. These are essential to making precise the definitions of monoidal model category and of a model category enriched in a monoidal model category.

Recall that a monoidal category $\mathscr{M}$ is called *biclosed* if the tensor product has adjoints in each variable, i.e. if there are functors $\hom_l, \hom_r \colon \mathscr{M}^{\mathrm{op}} \times \mathscr{M} \to \mathscr{M}$ and isomorphisms

$$\mathscr{M}(A \otimes B, C) \cong \mathscr{M}(B, \hom_l(A,C)) \cong \mathscr{M}(A, \hom_r(B,C))$$

natural in all three variables. If $\mathscr{M}$ has a model structure, then one of the requirements for $\mathscr{M}$ to be a monoidal model category is that the three bifunctors $\otimes$, $\hom_l$, and $\hom_r$ form a *Quillen adjunction of two variables*. There are three equivalent conditions for this:

1. Given any cofibrations $i \colon A \to B$ and $j \colon J \to K$, the map $i \hat{\otimes} j$ defined by the pushout

$$\begin{tikzcd}
A \otimes J \arrow[r, "A \otimes j"] \arrow[d, "i \otimes J"'] & A \otimes K \arrow[d] \arrow[ddr, bend left, "i \otimes K"] & \\
B \otimes J \arrow[r] \arrow[drr, bend right, "B \otimes j"'] & A \otimes K \coprod_{A \otimes J} B \otimes J \arrow[dr, dashed, "i \hat{\otimes} j"] & \\
 & & B \otimes K
\end{tikzcd}$$

   is a cofibration (which is trivial if either i or j is).

2. Given any cofibration $i \colon A \to B$ and fibration $f \colon X \to Y$, the map

$$\hat{\hom}_l(i,f) \colon \hom_l(B,X) \to \hom_l(A,X) \times_{\hom_l(A,Y)} \hom_l(B,Y)$$

   is a fibration (which is trivial if either i or f is).

3. Given any cofibration $j\colon J \to K$ and fibration $f\colon X \to Y$, the map

$$\hat{\mathrm{hom}}_r(j,f)\colon \mathrm{hom}_y(K,X) \to \mathrm{hom}_r(J,X) \underset{\mathrm{hom}_r(J,Y)}{\times} \mathrm{hom}_r(K,Y)$$

is a fibration (which is trivial if either i or f is).

Proving the equivalence of these three conditions is a routine but tedious exercise in adjunctions. Another exercise in adjunctions shows that $\hat{\otimes}$, $\hat{\mathrm{hom}}_l$, and $\hat{\mathrm{hom}}_r$ in fact make up an adjunction of two variables on the arrow category $\mathscr{M}^2$.

In this chapter, we will give a universal property satisfied by the functors $\hat{\otimes}$, $\hat{\mathrm{hom}}_l$, and $\hat{\mathrm{hom}}_r$ which will trivialize these kinds of routine adjunction arguments, as well as making precise the clear symmetry involved. Then in chapter XI we will make use of this universal property in order to show that the algebraic analogue of Quillen adjunctions of two variables (defined in [Rie13]) can be recovered as multivariable morphisms of bimonads in a precise sense.

X.1 Review of Cyclic Double Multicategories

X.2 The Universal Property

Define a cyclic double multicategory $\mathbb{J}$ as follows. The objects are A_i, B_i, for $i \in \{0,1,2\}$, and their duals. The horizontal 1-cells are $d_0^i, d_1^i\colon B_i \to A_i$. The vertical 1-cells are $F_i\colon (A_{i-1}, A_{i+1}) \to A_i^\bullet$ and $G_i\colon (B_{i-1}, B_{i+1}) \to B_i^\bullet$, which form two orbits under the cyclic action.

There are two types of 2-cells. There are

$$\begin{tikzcd}
B_i \arrow[r, "d_1^i"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\Downarrow\alpha_i"] & A_i \arrow[d, "\mathrm{id}"] \\
B_i \arrow[r, "d_0^i"'] & A_i
\end{tikzcd}$$

for each i. We will often draw these 2-cells globularly.

There are also 2-cells

$$\begin{tikzcd}
B_{i+1}, B_{i-1} \arrow[r, "{d_{k_{i+1}}^{i+1}, d_{k_{i-1}}^{i-1}}"] \arrow[d, "G_i"'] \arrow[dr, phantom, "\Downarrow\lambda^i_{k_{i+1},k_{i-1},k_i}"] & A_{i+1}, A_{i-1} \arrow[d, "F_i"] \\
B_i^\bullet \arrow[r, "d_{k_i}^{i\bullet}"'] & A_i^\bullet
\end{tikzcd}$$

for all choices of $(k_0,k_1,k_2) \in \{0,1\}^3$ except $(0,0,0)$.

Notice that there is at most one element of every hom-set, so all compositions and cyclic actions are uniquely defined. From now on, we will omit indices whenever doing so is unambiguous.

Remark X.1. The cyclic double multicategory $\mathbb{J}$ is generated under composition by the α_i and the $\lambda^i_{k_{i+1},k_{i-1},k_i}$ with exactly one of k_0,k_1,k_2 equal to 1. These nine λ generators are further generated under the cyclic action by only three, though there are many choices of which three. These generators satisfy the relations

$$
\begin{tikzcd}
B_1,B_2 \arrow[r, "{d_1,d_0}"] \arrow[d, "G_0"'] \arrow[dr, phantom, "\Downarrow\lambda"] & A_1,A_2 \arrow[d, "F_0"] \\
B_0^\bullet \arrow[r, "d_0^\bullet"] \arrow[r, bend right=50, "d_1^\bullet"'] \arrow[r, phantom, shift right=3, "\Downarrow\alpha^\bullet"] & A_0^\bullet
\end{tikzcd}
=
\begin{tikzcd}
B_1,B_2 \arrow[r, bend left=50, "{d_1,d_0}"] \arrow[r, "{d_0,d_0}"'] \arrow[r, phantom, shift left=3, "{\Downarrow\alpha,\mathrm{id}}"] \arrow[d, "G_0"'] \arrow[dr, phantom, "\Downarrow\lambda"] & A_1,A_2 \arrow[d, "F_0"] \\
B_0^\bullet \arrow[r, "d_1^\bullet"'] & A_0^\bullet
\end{tikzcd}
$$

$$
\begin{tikzcd}
B_1,B_2 \arrow[r, "{d_0,d_1}"] \arrow[d, "G_0"'] \arrow[dr, phantom, "\Downarrow\lambda"] & A_1,A_2 \arrow[d, "F_0"] \\
B_0^\bullet \arrow[r, "d_0^\bullet"] \arrow[r, bend right=50, "d_1^\bullet"'] \arrow[r, phantom, shift right=3, "\Downarrow\alpha^\bullet"] & A_0^\bullet
\end{tikzcd}
=
\begin{tikzcd}
B_1,B_2 \arrow[r, bend left=50, "{d_0,d_1}"] \arrow[r, "{d_0,d_0}"'] \arrow[r, phantom, shift left=3, "{\Downarrow\mathrm{id},\alpha}"] \arrow[d, "G_0"'] \arrow[dr, phantom, "\Downarrow\lambda"] & A_1,A_2 \arrow[d, "F_0"] \\
B_0^\bullet \arrow[r, "d_1^\bullet"'] & A_0^\bullet
\end{tikzcd}
$$

$$
\begin{tikzcd}
B_1,B_2 \arrow[r, bend left=50, "{d_1,d_1}"] \arrow[r, "{d_1,d_0}"'] \arrow[r, phantom, shift left=3, "{\Downarrow\mathrm{id},\alpha}"] \arrow[d, "G_0"'] \arrow[dr, phantom, "\Downarrow\lambda"] & A_1,A_2 \arrow[d, "F_0"] \\
B_0^\bullet \arrow[r, "d_0^\bullet"'] & A_0^\bullet
\end{tikzcd}
=
\begin{tikzcd}
B_1,B_2 \arrow[r, bend left=50, "{d_1,d_1}"] \arrow[r, "{d_0,d_1}"'] \arrow[r, phantom, shift left=3, "{\Downarrow\alpha,\mathrm{id}}"] \arrow[d, "G_0"'] \arrow[dr, phantom, "\Downarrow\lambda"] & A_1,A_2 \arrow[d, "F_0"] \\
B_0^\bullet \arrow[r, "d_0^\bullet"'] & A_0^\bullet
\end{tikzcd}
$$

and their reflections under the cyclic action.

Example X.2. Let $\mathbb{M}\mathbf{Adj}$ be the cyclic double multicategory of categories, functors, and multivariable right adjunctions. Any multivariable right adjunction $F_0 \colon \mathscr{A}_1 \times \mathscr{A}_2 \to \mathscr{A}_0$ extends to a functor $\widehat{\mathbb{F}} \colon \mathbb{J} \to \mathbb{M}\mathbf{Adj}$ as follows.

- B_i is sent to $\mathscr{A}_i^{\mathbf{2}}$, the arrow category of $\mathscr{A}_i$.

- The d_1 are sent to the domain functors $\mathrm{dom} \colon \mathscr{A}_i^{\mathbf{2}} \to \mathscr{A}_i$ and the d_0 are sent to the codomain functors $\mathrm{cod} \colon \mathscr{A}_i^{\mathbf{2}} \to \mathscr{A}_i$.

- The α are sent to the canonical natural transformations dom $\Rightarrow$ cod.
- The G_i are sent to functors $\hat{F}_i$. Given morphisms $f\colon A \to B \in \mathscr{A}_1$ and $g\colon X \to Y \in \mathscr{A}_2$, $\hat{F}_0(f,g)$ is defined as in the diagram

$$\begin{tikzcd}
F_0(B,Y) \arrow[dr, dashed, "\hat{F}_0(f,g)"] \arrow[drr, bend left, "F_0(1,g)"] \arrow[ddr, bend right, "F_0(f,1)"'] & & \\
& F_0(A,Y) \underset{F_0(A,X)}{\times} F_0(B,X) \arrow[r, "p_2"] \arrow[d, "p_1"'] & F_0(B,X) \arrow[d, "F_0(f,1)"] \\
& F_0(A,Y) \arrow[r, "F_0(1,g)"'] & F_0(A,X)
\end{tikzcd} \tag{X.1}$$

  It is a standard fact that the $\hat{F}_i$ form a two-variable adjunction between the arrow categories.

- Looking at diagram (X.1),

$$\begin{aligned}
(\lambda^0_{1,0,0})_{f,g} &= p_1\colon \operatorname{cod} \hat{F}_0(f,g) \to F_0(\operatorname{dom} f, \operatorname{cod} g) \\
(\lambda^0_{0,1,0})_{f,g} &= p_2\colon \operatorname{cod} \hat{F}_0(f,g) \to F_0(\operatorname{cod} f, \operatorname{dom} g) \\
(\lambda^0_{0,0,1})_{f,g} &= \mathrm{id}\colon \operatorname{dom} \hat{F}_0(f,g) \to F_0(\operatorname{cod} f, \operatorname{cod} g).
\end{aligned}$$

  The three relations (1)-(3) then correspond precisely to the commutativity of the three regions in diagram (X.1).

Let $\mathbb{I}$ be the sub-category of $\mathbb{J}$ consisting of just the 1-cells F_i. Let **CDMCat** denote the 2-category of cyclic double multicategories, functors, and horizontal transformations.

Theorem X.3. *Fix a functor $\mathbb{F}\colon \mathbb{I} \to \mathbb{M}\mathbf{Adj}$. Then the functor $\hat{\mathbb{F}}\colon \mathbb{J} \to \mathbb{M}\mathbf{Adj}$ constructed in example X.2 is terminal in the category $\mathbf{CDMCat}_{\mathbb{F}}(\mathbb{J}, \mathbb{M}\mathbf{Adj})$ of functors on $\mathbb{J}$ restricting to $\mathbb{F}$ on $\mathbb{I}$.*

Proof. Concretely, the theorem says that given the data of a functor $\mathbb{J} \to \mathbb{M}\mathbf{Adj}$, determining 2-variable adjunctions F_i and G_i and the rest of the structure spelled out in remark X.1, there is a unique 2-cell

$$\begin{tikzcd}
\mathscr{B}_1, \mathscr{B}_2 \arrow[r, "H_1, H_2"] \arrow[d, "G_0"'] \arrow[dr, phantom, "\Downarrow\theta"] & \mathscr{A}_1^{\mathbf{2}}, \mathscr{A}_2^{\mathbf{2}} \arrow[d, "\hat{F}_0"] \\
\mathscr{B}_0^{\bullet} \arrow[r, "H_3^{\bullet}"'] & \mathscr{A}_0^{\bullet\mathbf{2}}
\end{tikzcd}$$

where $\hat{F}_i$ is the pullback product defined in (X.1), such that

$$\begin{tikzcd}
\mathscr{B}_1,\mathscr{B}_2 \arrow[r, "{H_1,H_2}"] \arrow[d, "G_0"'] \arrow[dr, phantom, "\Downarrow\theta"] & \mathscr{A}_1^{\mathbf{2}},\mathscr{A}_2^{\mathbf{2}} \arrow[r, "{\mathrm{cod},\mathrm{cod}}"] \arrow[d, "\hat{F}_0"] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & \mathscr{A}_1,\mathscr{A}_2 \arrow[d, "F_0"] \\
\mathscr{B}_0^\bullet \arrow[r, "H_3^\bullet"'] & \mathscr{A}_0^{\bullet\mathbf{2}} \arrow[r, "\mathrm{dom}^\bullet"'] & \mathscr{A}_0^\bullet
\end{tikzcd}
=
\begin{tikzcd}
\mathscr{B}_1,\mathscr{B}_2 \arrow[r, "{d_0,d_0}"] \arrow[d, "G_0"'] \arrow[dr, phantom, "\Downarrow\lambda"] & \mathscr{A}_1,\mathscr{A}_2 \arrow[d, "F_0"] \\
\mathscr{B}_0^\bullet \arrow[r, "d_1^\bullet"'] & \mathscr{A}_0^\bullet
\end{tikzcd}
\tag{X.2}$$

$$\begin{tikzcd}
\mathscr{B}_1,\mathscr{B}_2 \arrow[r, "{H_1,H_2}"] \arrow[d, "G_0"'] \arrow[dr, phantom, "\Downarrow\theta"] & \mathscr{A}_1^{\mathbf{2}},\mathscr{A}_2^{\mathbf{2}} \arrow[r, "{\mathrm{dom},\mathrm{cod}}"] \arrow[d, "\hat{F}_0"] \arrow[dr, phantom, "\Downarrow p_1"] & \mathscr{A}_1,\mathscr{A}_2 \arrow[d, "F_0"] \\
\mathscr{B}_0^\bullet \arrow[r, "H_3^\bullet"'] & \mathscr{A}_0^{\bullet\mathbf{2}} \arrow[r, "\mathrm{cod}^\bullet"'] & \mathscr{A}_0^\bullet
\end{tikzcd}
=
\begin{tikzcd}
\mathscr{B}_1,\mathscr{B}_2 \arrow[r, "{d_1,d_0}"] \arrow[d, "G_0"'] \arrow[dr, phantom, "\Downarrow\lambda"] & \mathscr{A}_1,\mathscr{A}_2 \arrow[d, "F_0"] \\
\mathscr{B}_0^\bullet \arrow[r, "d_0^\bullet"'] & \mathscr{A}_0^\bullet
\end{tikzcd}
\tag{X.3}$$

$$\begin{tikzcd}
\mathscr{B}_1,\mathscr{B}_2 \arrow[r, "{H_1,H_2}"] \arrow[d, "G_0"'] \arrow[dr, phantom, "\Downarrow\theta"] & \mathscr{A}_1^{\mathbf{2}},\mathscr{A}_2^{\mathbf{2}} \arrow[r, "{\mathrm{cod},\mathrm{dom}}"] \arrow[d, "\hat{F}_0"] \arrow[dr, phantom, "\Downarrow p_2"] & \mathscr{A}_1,\mathscr{A}_2 \arrow[d, "F_0"] \\
\mathscr{B}_0^\bullet \arrow[r, "H_3^\bullet"'] & \mathscr{A}_0^{\bullet\mathbf{2}} \arrow[r, "\mathrm{cod}^\bullet"'] & \mathscr{A}_0^\bullet
\end{tikzcd}
=
\begin{tikzcd}
\mathscr{B}_1,\mathscr{B}_2 \arrow[r, "{d_0,d_1}"] \arrow[d, "G_0"'] \arrow[dr, phantom, "\Downarrow\lambda"] & \mathscr{A}_1,\mathscr{A}_2 \arrow[d, "F_0"] \\
\mathscr{B}_0^\bullet \arrow[r, "d_0^\bullet"'] & \mathscr{A}_0^\bullet
\end{tikzcd}
\tag{X.4}$$

Fix objects $B_1 \in \mathscr{B}_1$, $B_2 \in \mathscr{B}_2$. The H_i are the functors sending B_i to $H_i(B_i) = \alpha_{B_i}\colon d_1 B_i \to d_0 B_i$. The component of θ at (B_1, B_2) is a square

$$\begin{tikzcd}
d_1 G_0(B_1,B_2) \arrow[r] \arrow[d] & F_0(d_0B_1, d_0B_2) \arrow[d] \\
d_0 G_0(B_1,B_2) \arrow[r] & F_0(d_1B_1, d_0B_2) \prod\limits_{F_0(d_1B_1,d_1B_2)} F_0(d_0B_1, d_1B_2)
\end{tikzcd}$$

The top arrow is uniquely determined by equation (X.2), while the components of the bottom arrow are uniquely determined by equations (X.3) and (X.4). □

Now let $\mathbb{M}$ be any cyclic double multicategory. We will take theorem X.3 as our definition of what it means for a general cyclic double multicategory to have arrow objects. For future reference, we will spell this out more concretely.

Given an object C of $\mathbb{M}$, an arrow object C^2 is an object together with a globular 2-cell $\kappa\colon \mathrm{dom} \Rightarrow \mathrm{cod}$ satisfying the same universal property as in section III.2 (this only involves the horizontal 2-category, so carries over unchanged).

Given a vertical 1-cell $F\colon (C_1, C_2) \to C_0^\bullet$, the lift to arrow objects $\hat{F}$ is a vertical

1-cell $\hat{F}$ together with 2-cells

$$\begin{tikzcd} C_1^2, C_2^2 \arrow[r, "{\mathrm{cod},\mathrm{cod}}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_0"] & C_1, C_2 \arrow[d, "F"] \\ C_0^{\bullet 2} \arrow[r, "\mathrm{dom}^\bullet"'] & C_0^\bullet \end{tikzcd} \qquad \begin{tikzcd} C_1^2, C_2^2 \arrow[r, "{\mathrm{dom},\mathrm{cod}}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_1"] & C_1, C_2 \arrow[d, "F"] \\ C_0^{\bullet 2} \arrow[r, "\mathrm{cod}^\bullet"'] & C_0^\bullet \end{tikzcd} \qquad \begin{tikzcd} C_1^2, C_2^2 \arrow[r, "{\mathrm{cod},\mathrm{dom}}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_2"] & C_1, C_2 \arrow[d, "F"] \\ C_0^{\bullet 2} \arrow[r, "\mathrm{cod}^\bullet"'] & C_0^\bullet \end{tikzcd}$$

satisfying the equations

$$\begin{tikzcd} C_1^2, C_2^2 \arrow[r, "{\mathrm{dom},\mathrm{cod}}"] \arrow[d, "G_0"'] \arrow[dr, phantom, "\Downarrow\gamma_1"] & C_1, C_2 \arrow[d, "F_0"] \\ C_0^{\bullet 2} \arrow[r, "\mathrm{cod}^\bullet"] \arrow[r, bend right=50, "\mathrm{dom}^\bullet"', "\Downarrow\kappa^\bullet"] & C_0^\bullet \end{tikzcd} = \begin{tikzcd} C_1^2, C_2^2 \arrow[r, "{\mathrm{cod},\mathrm{cod}}"'] \arrow[r, bend left=50, "{\mathrm{dom},\mathrm{cod}}", "{\Downarrow\kappa,\mathrm{id}}"'] \arrow[d, "G_0"'] \arrow[dr, phantom, "\Downarrow\gamma_0"] & C_1, C_2 \arrow[d, "F_0"] \\ C_0^{\bullet 2} \arrow[r, "\mathrm{dom}^\bullet"'] & C_0^\bullet \end{tikzcd} \tag{X.5}$$

$$\begin{tikzcd} C_1^2, C_2^2 \arrow[r, "{\mathrm{cod},\mathrm{dom}}"] \arrow[d, "G_0"'] \arrow[dr, phantom, "\Downarrow\gamma_2"] & C_1, C_2 \arrow[d, "F_0"] \\ C_0^{\bullet 2} \arrow[r, "\mathrm{cod}^\bullet"] \arrow[r, bend right=50, "\mathrm{dom}^\bullet"', "\Downarrow\kappa^\bullet"] & C_0^\bullet \end{tikzcd} = \begin{tikzcd} C_1^2, C_2^2 \arrow[r, "{\mathrm{cod},\mathrm{cod}}"'] \arrow[r, bend left=50, "{\mathrm{cod},\mathrm{dom}}", "{\Downarrow\mathrm{id},\kappa}"'] \arrow[d, "G_0"'] \arrow[dr, phantom, "\Downarrow\gamma_0"] & C_1, C_2 \arrow[d, "F_0"] \\ C_0^{\bullet 2} \arrow[r, "\mathrm{dom}^\bullet"'] & C_0^\bullet \end{tikzcd} \tag{X.6}$$

$$\begin{tikzcd} C_1^2, C_2^2 \arrow[r, "{\mathrm{dom},\mathrm{cod}}"'] \arrow[r, bend left=50, "{\mathrm{dom},\mathrm{dom}}", "{\Downarrow\mathrm{id},\kappa}"'] \arrow[d, "G_0"'] \arrow[dr, phantom, "\Downarrow\gamma_1"] & C_1, C_2 \arrow[d, "F_0"] \\ C_0^{\bullet 2} \arrow[r, "\mathrm{cod}^\bullet"'] & C_0^\bullet \end{tikzcd} = \begin{tikzcd} C_1^2, C_2^2 \arrow[r, "{\mathrm{cod},\mathrm{dom}}"'] \arrow[r, bend left=50, "{\mathrm{dom},\mathrm{dom}}", "{\Downarrow\kappa,\mathrm{id}}"'] \arrow[d, "G_0"'] \arrow[dr, phantom, "\Downarrow\gamma_2"] & C_1, C_2 \arrow[d, "F_0"] \\ C_0^{\bullet 2} \arrow[r, "\mathrm{cod}^\bullet"'] & C_0^\bullet \end{tikzcd} \tag{X.7}$$

and which is universal, meaning that given any objects X_0, X_1, X_2, horizontal 1-cells $d_{i,0}, d_{i,1}\colon X_i \to C_i$, a vertical 1-cell $G\colon X_1, X_2 \to X_0^\bullet$, globular 2-cells $\alpha_i\colon d_{i,1} \Rightarrow d_{i,0}$, and 2-cells

$$\begin{tikzcd} X_1, X_2 \arrow[r, "{d_{1,0},d_{2,0}}"] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\lambda_0"] & C_1, C_2 \arrow[d, "F"] \\ X_0^\bullet \arrow[r, "d_{0,1}^\bullet"'] & C_0^\bullet \end{tikzcd} \qquad \begin{tikzcd} X_1, X_2 \arrow[r, "{d_{1,1},d_{2,0}}"] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\lambda_0"] & C_1, C_2 \arrow[d, "F"] \\ X_0^\bullet \arrow[r, "d_{0,0}^\bullet"'] & C_0^\bullet \end{tikzcd} \qquad \begin{tikzcd} X_1, X_2 \arrow[r, "{d_{1,0},d_{2,1}}"] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\lambda_2"] & C_1, C_2 \arrow[d, "F"] \\ X_0^\bullet \arrow[r, "d_{0,0}^\bullet"'] & C_0^\bullet \end{tikzcd}$$

satisfying the three equations analagous to (X.5)–(X.7), there exists a unique 2-cell

$$\begin{tikzcd} X_1, X_2 \arrow[r, "\hat{\alpha}_1, \hat{\alpha}_2"] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\theta"] & C_1^2, C_2^2 \arrow[d, "\hat{F}"] \\ X_0^\bullet \arrow[r, "\hat{\alpha}_0^\bullet"'] & C_0^{\bullet 2} \end{tikzcd}$$

(where $\hat{\alpha}_i$ is the 1-cell determined by α_i by the universal property of the arrow object C_i) such that

$$\begin{tikzcd} X_1, X_2 \arrow[r, "\hat{\alpha}_1, \hat{\alpha}_2"] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\theta"] & C_1^2, C_2^2 \arrow[r] \arrow[d, "\hat{F}"] \arrow[dr, phantom, "\Downarrow\gamma_i"] & C_1, C_2 \arrow[d, "F"] \\ X_0^\bullet \arrow[r, "\hat{\alpha}_0^\bullet"'] & C_0^{\bullet 2} \arrow[r] & C_0^\bullet \end{tikzcd} = \begin{tikzcd} X_1, X_2 \arrow[r] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\lambda_i"] & C_1, C_2 \arrow[d, "F"] \\ X_0^\bullet \arrow[r] & C_0^\bullet \end{tikzcd}$$

for each $i \in \{0, 1, 2\}$.

Similarly, we define the lift of a vertical 1-cell $F\colon (C_1, \ldots, C_n) \to C_0^\bullet$ to arrow objects to be a vertical 1-cell $\hat{F}$ together with $(n+1)$ 2-cells γ_i satisfying $(n+1)$ equations analagous to (X.5)–(X.7) and which is universal in the analagous way.

Definition X.4. Let $\mathbb{M}$ be a double multicategory. We say $\mathbb{M}$ *has arrow objects* if for every object C there is an arrow object C^2, and if for every vertical 1-cell $F\colon (C_1, \ldots, C_n) \to C_0^\bullet$ there is a lift to arrow objects $\hat{F}$.

We have given the universal property of arrow objects and lifts of vertical 1-cells in ordinary double multicategories, but it is clear from the cyclical symmetry of the construction that a cyclic action respects arrow objects. Specifically, $(C^2)^\bullet = (C^\bullet)^2$ for any object C, and $\sigma(\hat{F}) = \widehat{\sigma F}$ for any vertical 1-cell F, with $\sigma(\gamma_i) = \gamma_{i+1}$.

CHAPTER XI

CYCLIC 2-FOLD DOUBLE MULTICATEGORIES

In this section we will introduce a common generalization of the cyclic double multicategories of [CGR12] and the cyclic 2-fold double categories introduced in chapter V.

A cyclic two-fold double multicategory $\mathbb{M}$ consists of the same underlying data as a cyclic double multicategory, i.e. a vertical multicategory, horizontal 1-cells, and 2-cells of the form

$$\begin{tikzcd} C_1,\ldots,C_n \arrow[r, "X_1,\ldots,X_n"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\theta"] & C_1,\ldots,C_n \arrow[d, "F"] \\ C_0^\bullet \arrow[r, "X_0^\bullet"'] & C_0^\bullet \end{tikzcd}$$

which compose vertically in the same way as in a cyclic double multicategory, and where as in a two-fold double multicategory the horizontal 1-cells are endomorphisms. There are two composition structures on the horizontal 1-cells, $(I,\otimes)$ and $(\perp,\odot)$, such that for any object C, $(I_C)^\bullet = \perp_{C^\bullet}$ and $(\perp_C)^\bullet = I_{C^\bullet}$, and such that for any composable pair of horizontal 1-cells X, and Y, $(X\otimes Y)^\bullet = X^\bullet \odot Y^\bullet$ and $(X\odot Y)^\bullet = X^\bullet \otimes Y^\bullet$.

Perhaps surprisingly, given two composable 2-cells

$$\begin{tikzcd} C_1,\ldots,C_n \arrow[r, "X_1,\ldots,X_n"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\theta"] & C_1,\ldots,C_n \arrow[r, "Y_1,\ldots,Y_n"] \arrow[d, "F"] \arrow[dr, phantom, "\Downarrow\phi"] & C_1,\ldots,C_n \arrow[d, "F"] \\ C_0^\bullet \arrow[r, "X_0^\bullet"'] & C_0^\bullet \arrow[r, "Y_0^\bullet"'] & C_0^\bullet \end{tikzcd}$$

there are $(n+1)$ different horizontal compositions:

$$\begin{tikzcd} C_1,\ldots,C_n \arrow[r, "X_1\odot Y_1,\ldots,X_i\otimes Y_i,\ldots,X_n\odot Y_n"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\theta\otimes_i\phi"] & C_1,\ldots,C_n \arrow[d, "F"] \\ C_0^\bullet \arrow[r, "(X_0\odot Y_0)^\bullet"'] & C_0^\bullet \end{tikzcd}$$

for $i \in \{1, \ldots, n\}$, and

$$\begin{tikzcd} C_1, \ldots, C_n \arrow[r, "X_1 \odot Y_1, \ldots, X_n \odot Y_n"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow \theta \otimes_0 \phi"] & C_1, \ldots, C_n \arrow[d, "F"] \\ C_0^\bullet \arrow[r, "(X_0 \otimes Y_0)^\bullet"'] & C_0^\bullet. \end{tikzcd}$$

In all cases, there is exactly one $\otimes$ in the ith position, and the rest of the horizontal compositions are $\odot$. Notice that this pattern only holds when using the convention of dualizing everything in the codomain. Similarly, given any vertical n-ary 1-cell F, there are $(n+1)$ unit 2-cells:

$$\begin{tikzcd} C_1, \ldots, C_n \arrow[r, "\perp_{C_1}, \ldots, I_{C_i}, \ldots, \perp_{C_n}"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow I_{iF}"] & C_1, \ldots, C_n \arrow[d, "F"] \\ C_0^\bullet \arrow[r, "\perp_{C_0}^\bullet"'] & C_0^\bullet \end{tikzcd}$$

for $i \in \{1, \ldots, n\}$, and

$$\begin{tikzcd} C_1, \ldots, C_n \arrow[r, "\perp_{C_1}, \ldots, \perp_{C_n}"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow I_{0F}"] & C_1, \ldots, C_n \arrow[d, "F"] \\ C_0^\bullet \arrow[r, "I_{C_0}^\bullet"'] & C_0^\bullet \end{tikzcd}$$

The horizontal compositions and units must respect the cyclic action, such that the equations hold:

$$\sigma(\theta \otimes_i \phi) = (\sigma\theta) \otimes_{i+1} (\sigma\phi) \qquad \sigma(I_{iF}) = I_{(i+1)\sigma F}$$

We require the existence of the families of globular coherence 2-cells m, c, j, z, satisfying the same conditions as in a cyclic 2-fold double category. Notably, we only require naturality of z with respect to unary 2-cells. It is unclear whether there is any sensible compatibility between z and multivariable 2-cells that could be asked for, but such a compatibility is not needed for our purposes.

XI.1 Multimorphisms of Bimonads

The definition IV.5 of bimonads in a 2-fold double category uses only globular 2-cells, so works unchanged in a cyclic 2-fold double multicategory $\mathbb{M}$. However, using the multicategory structure of $\mathbb{M}$ we will now be able to expand the category of bimonads in $\mathbb{M}$ to a multicategory $\mathrm{Bimon}(\mathbb{M})$, and the cyclic structure of $\mathbb{M}$ will lift to $\mathrm{Bimon}(\mathbb{M})$, making it a cyclic multicategory.

Definition XI.1. Let $\mathbb{M}$ be a cyclic 2-fold double multicategory, let $(X_i, \eta_i, \mu_i, \epsilon_i, \delta_i)$, $i \in \{0,1,2\}$, be bimonads in $\mathbb{M}$, and let F and ϕ be as in the diagram

$$\begin{tikzcd} C_1, C_2 \arrow[r, "{X_1, X_2}"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\phi"] & C_1, C_2 \arrow[d, "F"] \\ C_0 \arrow[r, "X_0^\bullet"'] & C_0. \end{tikzcd}$$

Say that (F, ϕ) is a 0-colax morphism of bimonads if the following two equations are satisfied:

$$\begin{tikzcd} C_1, C_2 \arrow[r, "{X_1, X_2}"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\phi"] & C_1, C_2 \arrow[d, "F"] \\ C_0 \arrow[r, "X_0^\bullet"] \arrow[r, bend right=50, "I^\bullet"'] \arrow[r, phantom, bend right=30, "\Downarrow\eta_0^\bullet"] & C_0 \end{tikzcd} = \begin{tikzcd} C_1, C_2 \arrow[r, bend left=50, "{X_1, X_2}"] \arrow[r, phantom, bend left=30, "\Downarrow\epsilon_1, \epsilon_2"] \arrow[r, "{\perp, \perp}"'] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow I_{0F}"] & C_1, C_2 \arrow[d, "F"] \\ C_0 \arrow[r, "I^\bullet"'] & C_0 \end{tikzcd}$$

$$\begin{tikzcd} C_1, C_2 \arrow[r, "{X_1, X_2}"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\phi"] & C_1, C_2 \arrow[d, "F"] \\ C_0 \arrow[r, "X_0^\bullet"] \arrow[r, bend right=50, "(X_0 \otimes X_0)^\bullet"'] \arrow[r, phantom, bend right=30, "\Downarrow\mu_0^\bullet"] & C_0 \end{tikzcd} = \begin{tikzcd} C_1, C_2 \arrow[r, bend left=50, "{X_1, X_2}"] \arrow[r, phantom, bend left=30, "\Downarrow\delta_1, \delta_2"] \arrow[r, "{X_1 \odot X_1, X_2 \odot X_2}"'] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\phi \otimes_0 \phi"] & C_1, C_2 \arrow[d, "F"] \\ C_0 \arrow[r, "(X_0 \otimes X_0)^\bullet"'] & C_0. \end{tikzcd}$$

(F,ϕ) is a 1-colax morphism of bimonads if the two equations

$$
\begin{tikzcd}
C_1,C_2 \arrow[r, "X_1{,}X_2"'] \arrow[r, bend left=60, "I{,}X_2"] \arrow[r, phantom, bend left=30, "\Downarrow\eta_1{,}\mathrm{id}"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\phi"] & C_1,C_2 \arrow[d, "F"] \\
C_0 \arrow[r, "X_0^\bullet"'] & C_0
\end{tikzcd}
=
\begin{tikzcd}
C_1,C_2 \arrow[r, "I{,}\perp"'] \arrow[r, bend left=60, "I{,}X_2"] \arrow[r, phantom, bend left=30, "\Downarrow\mathrm{id}{,}\epsilon_2"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow I_{1F}"] & C_1,C_2 \arrow[d, "F"] \\
C_0 \arrow[r, "\perp^\bullet"] \arrow[r, bend right=60, "X_0^\bullet"'] \arrow[r, phantom, bend right=30, "\Downarrow\epsilon_0^\bullet"] & C_0
\end{tikzcd}
$$

$$
\begin{tikzcd}
C_1,C_2 \arrow[r, "X_1{,}X_2"'] \arrow[r, bend left=60, "X_1\otimes X_1{,}X_2"] \arrow[r, phantom, bend left=30, "\Downarrow\mu_1{,}\mathrm{id}"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\phi"] & C_1,C_2 \arrow[d, "F"] \\
C_0 \arrow[r, "X_0^\bullet"'] & C_0
\end{tikzcd}
=
\begin{tikzcd}
C_1,C_2 \arrow[r, "X_1\otimes X_1{,}X_2\odot X_2"'] \arrow[r, bend left=60, "X_1\otimes X_1{,}X_2"] \arrow[r, phantom, bend left=30, "\Downarrow\mathrm{id}{,}\delta_2"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\phi\otimes_1\phi"] & C_1,C_2 \arrow[d, "F"] \\
C_0 \arrow[r, "(X_0\odot X_0)^\bullet"] \arrow[r, bend right=60, "X_0^\bullet"'] \arrow[r, phantom, bend right=30, "\Downarrow\delta_0^\bullet"] & C_0
\end{tikzcd}
$$

and (F,ϕ) is 2-colax if the analogous two equations hold. We will call (F,ϕ) a colax (multi)morphism of bimonads if it is i-colax for all $i \in \{0,1,2\}$.

The definition of colax multimorphisms with arity n should be clear from the $n = 2$ case.

It is straightforward to see that multimorphisms of bimonads compose multicategorically, so we have the multicategory $\mathrm{Bimon}(\mathbb{M})$ of bimonads in $\mathbb{M}$. Furthermore, the definition of colax multimorphism is clearly symmetric with respect to the cyclic action, so that $\mathrm{Bimon}(\mathbb{M})$ inherits a cyclic action.

Definition XI.2. Let $\mathbb{M}$ be a cyclic 2-fold double multicategory. The cyclic multicategory $\mathrm{Bimon}(\mathbb{M})$ has as objects bimonads in $\mathbb{M}$, and has colax morphisms as (multi)morphisms. ∎

XI.2 The Cyclic 2-Fold Double Multicategory of Functorial Factorizations

In this section, given a cyclic double multicategory $\mathbb{M}$, we will construct a cyclic 2-fold double multicategory $\mathbb{FF}(\mathbb{M})$ of functorial factorizations in $\mathbb{M}$.

The objects and vertical multicategory of $\mathbb{FF}(\mathbb{M})$ are those of $\mathbb{M}$. The horizontal 1-cells of $\mathbb{FF}(\mathbb{M})$ are functorial factorizations in $\mathbb{M}$. As with bimonads, the definition of functorial factorization given in chapter VI involves only globular 2-cells, so no modification is necessary to define functorial factorizations in $\mathbb{M}$.

Also as with bimonads, we will give an explicit definition of 2-ary 2-cell and let the reader extend the (easy) pattern to n-ary 2-cells for arbitrary n.

Definition XI.3. Let $(E_i, \eta_i, \epsilon_i)$, $i \in \{0,1,2\}$, be functorial factorizations in $\mathbb{M}$ on objects C_i. A 2-ary 2-cell in $\mathbb{FF}(\mathbb{M})$

$$\begin{tikzcd}
C_1, C_2 \arrow[r, "E_1{,}E_2", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\theta"] & C_1, C_2 \arrow[d, "F"] \\
C_0^\bullet \arrow[r, "E_0^\bullet"', "\shortmid"{marking}] & C_0^\bullet
\end{tikzcd}$$

is given by a 2-cell

$$\begin{tikzcd}
C_1^2, C_2^2 \arrow[r, "E_1{,}E_2", "\shortmid"{marking}] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta"] & C_1, C_2 \arrow[d, "F"] \\
C_0^{\bullet 2} \arrow[r, "E_0"', "\shortmid"{marking}] & C_0^\bullet
\end{tikzcd}$$

in $\mathbb{M}$ satisfying three equations:

$$\begin{tikzcd}
C_1^2, C_2^2 \arrow[r, "E_1{,}E_2", "\shortmid"{marking}] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta"] & C_1, C_2 \arrow[d, "F"] \\
C_0^{\bullet 2} \arrow[r, "E_0^\bullet", "\shortmid"{marking}] \arrow[r, bend right=50, "\mathrm{dom}^\bullet"', "\shortmid"{marking}] \arrow[r, phantom, bend right=25, "\Downarrow\eta_0^\bullet"] & C_0^\bullet
\end{tikzcd}
=
\begin{tikzcd}
C_1^2, C_2^2 \arrow[r, bend left=50, "E_1{,}E_2", "\shortmid"{marking}] \arrow[r, phantom, bend left=25, "\Downarrow\epsilon_1{,}\epsilon_2"] \arrow[r, "\mathrm{cod}{,}\mathrm{cod}"', "\shortmid"{marking}] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_0"] & C_1, C_2 \arrow[d, "F"] \\
C_0^{\bullet 2} \arrow[r, "\mathrm{dom}^\bullet"', "\shortmid"{marking}] & C_0^\bullet
\end{tikzcd}
\tag{XI.1}$$

$$\begin{tikzcd}
C_1^2, C_2^2 \arrow[r, bend left=50, "\mathrm{dom}{,}E_2", "\shortmid"{marking}] \arrow[r, phantom, bend left=25, "\Downarrow\eta_1{,}\mathrm{id}"] \arrow[r, "E_1{,}E_2"', "\shortmid"{marking}] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta"] & C_1, C_2 \arrow[d, "F"] \\
C_0^{\bullet 2} \arrow[r, "E_0^\bullet"', "\shortmid"{marking}] & C_0^\bullet
\end{tikzcd}
=
\begin{tikzcd}
C_1^2, C_2^2 \arrow[r, bend left=50, "\mathrm{dom}{,}E_2", "\shortmid"{marking}] \arrow[r, phantom, bend left=25, "\Downarrow\mathrm{id}{,}\epsilon_2"] \arrow[r, "\mathrm{dom}{,}\mathrm{cod}"', "\shortmid"{marking}] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_1"] & C_1, C_2 \arrow[d, "F"] \\
C_0^{\bullet 2} \arrow[r, "\mathrm{cod}^\bullet", "\shortmid"{marking}] \arrow[r, bend right=50, "E_0^\bullet"', "\shortmid"{marking}] \arrow[r, phantom, bend right=25, "\Downarrow\epsilon_0^\bullet"] & C_0^\bullet
\end{tikzcd}
\tag{XI.2}$$

$$\begin{tikzcd}
C_1^2, C_2^2 \arrow[r, bend left=50, "E_1{,}\mathrm{dom}", "\shortmid"{marking}] \arrow[r, phantom, bend left=25, "\Downarrow\mathrm{id}{,}\eta_2"] \arrow[r, "E_1{,}E_2"', "\shortmid"{marking}] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta"] & C_1, C_2 \arrow[d, "F"] \\
C_0^{\bullet 2} \arrow[r, "E_0^\bullet"', "\shortmid"{marking}] & C_0^\bullet
\end{tikzcd}
=
\begin{tikzcd}
C_1^2, C_2^2 \arrow[r, bend left=50, "E_1{,}\mathrm{dom}", "\shortmid"{marking}] \arrow[r, phantom, bend left=25, "\Downarrow\epsilon_1{,}\mathrm{id}"] \arrow[r, "\mathrm{cod}{,}\mathrm{dom}"', "\shortmid"{marking}] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_2"] & C_1, C_2 \arrow[d, "F"] \\
C_0^{\bullet 2} \arrow[r, "\mathrm{cod}^\bullet", "\shortmid"{marking}] \arrow[r, bend right=50, "E_0^\bullet"', "\shortmid"{marking}] \arrow[r, phantom, bend right=25, "\Downarrow\epsilon_0^\bullet"] & C_0^\bullet
\end{tikzcd}
\tag{XI.3}$$

The cyclic action on a 2-cell in $\mathbb{FF}(\mathbb{M})$ is simply given by the cyclic action on the underlying 2-cell in $\mathbb{M}$. This is well defined since the definition of 2-cell in $\mathbb{FF}(\mathbb{M})$ is clearly stable under the cyclic action in $\mathbb{M}$.

Proposition XI.4. *Continuing the notation of the previous definition, the 2-cell θ induces 2-cells in $\mathbb{M}$*

$$
\begin{tikzcd}
C_1^2, C_2^2 \arrow[r, "{L_1, L_2}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\hat{\theta}^0"] & C_1^2, C_2^2 \arrow[d, "\hat{F}"] \\
C_0^{\bullet 2} \arrow[r, "R_0^\bullet"'] & C_0^{\bullet 2}
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1^2, C_2^2 \arrow[r, "{R_1, L_2}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\hat{\theta}^1"] & C_1^2, C_2^2 \arrow[d, "\hat{F}"] \\
C_0^{\bullet 2} \arrow[r, "L_0^\bullet"'] & C_0^{\bullet 2}
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1^2, C_2^2 \arrow[r, "{L_1, R_2}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\hat{\theta}^2"] & C_1^2, C_2^2 \arrow[d, "\hat{F}"] \\
C_0^{\bullet 2} \arrow[r, "L_0^\bullet"'] & C_0^{\bullet 2}
\end{tikzcd}
$$

satisfying

$$
\begin{tikzcd}
C_1^2, C_2^2 \arrow[r, "{L_1, R_2}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\hat{\theta}^2"] & C_1^2, C_2^2 \arrow[r, "{\mathrm{cod}, \mathrm{cod}}"] \arrow[d, "\hat{F}"] \arrow[dr, phantom, "\Downarrow\gamma_0"] & C_1, C_2 \arrow[d, "F"] \\
C_0^{\bullet 2} \arrow[r, "L_0^\bullet"'] & C_0^{\bullet 2} \arrow[r, "\mathrm{dom}^\bullet"'] & C_0^\bullet
\end{tikzcd}
=
\begin{tikzcd}
C_1^2, C_2^2 \arrow[r, bend left=60, "{E_1, \mathrm{cod}}"] \arrow[r, phantom, bend left=30, "\Downarrow\epsilon_1{,}\mathrm{id}"] \arrow[r, "{\mathrm{cod}, \mathrm{cod}}"'] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_0"] & C_1, C_2 \arrow[d, "F"] \\
C_0^{\bullet 2} \arrow[r, "\mathrm{dom}^\bullet"'] & C_0^\bullet
\end{tikzcd}
$$

$$
\begin{tikzcd}
C_1^2, C_2^2 \arrow[r, "{L_1, R_2}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\hat{\theta}^2"] & C_1^2, C_2^2 \arrow[r, "{\mathrm{dom}, \mathrm{cod}}"] \arrow[d, "\hat{F}"] \arrow[dr, phantom, "\Downarrow\gamma_1"] & C_1, C_2 \arrow[d, "F"] \\
C_0^{\bullet 2} \arrow[r, "L_0^\bullet"'] & C_0^{\bullet 2} \arrow[r, "\mathrm{cod}^\bullet"'] & C_0^\bullet
\end{tikzcd}
=
\begin{tikzcd}
C_1^2, C_2^2 \arrow[r, "{\mathrm{dom}, \mathrm{cod}}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_1"] & C_1, C_2 \arrow[d, "F"] \\
C_0^{\bullet 2} \arrow[r, "\mathrm{cod}^\bullet"] \arrow[r, phantom, bend right=30, "\Downarrow\epsilon_0^\bullet"] \arrow[r, bend right=60, "E_2^\bullet"'] & C_0^\bullet
\end{tikzcd}
$$

$$
\begin{tikzcd}
C_1^2, C_2^2 \arrow[r, "{L_1, R_2}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\hat{\theta}^2"] & C_1^2, C_2^2 \arrow[r, "{\mathrm{cod}, \mathrm{dom}}"] \arrow[d, "\hat{F}"] \arrow[dr, phantom, "\Downarrow\gamma_2"] & C_1, C_2 \arrow[d, "F"] \\
C_0^{\bullet 2} \arrow[r, "L_0^\bullet"'] & C_0^{\bullet 2} \arrow[r, "\mathrm{cod}^\bullet"'] & C_0^\bullet
\end{tikzcd}
=
\begin{tikzcd}
C_1^2, C_2^2 \arrow[r, "{E_1, E_2}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta"] & C_1, C_2 \arrow[d, "F"] \\
C_0^{\bullet 2} \arrow[r, "E_0^\bullet"'] & C_0^\bullet
\end{tikzcd}
$$

and a similar three equations for each of $\hat{\theta}^0$ and $\hat{\theta}^1$.

In general, an n-ary 2-cell θ in $\mathbb{FF}(\mathbb{M})$ induces 2-cells $\hat{\theta}^i$ in $\mathbb{M}$, $i \in \{0, \ldots, n\}$.

Proof. We will verify the existence of $\hat{\theta}^2$. The pattern extending to all other cases should be evident.

Using the universal property for arrow objects in a cyclic double multicategory, we only need to check the three equations obtained by composing each side of the equations (X.5)–(X.7) with $\hat{\theta}^2$. Equation (X.5) remains unchanged after composition with $\hat{\theta}^2$, equation (X.6) becomes (XI.1), and equation (X.7) turns into (XI.2).

Note that equation (XI.3) proves that $\hat{\theta}^2$ respects the units/counits of L_0, L_1, R_2, i.e. that the unit condition for a 2-colax morphism of bimonads holds, so that all three equations (XI.1)–(XI.3) go into establishing $\hat{\theta}^i$ for each i. $\square$

To finish the construction of the cyclic 2-fold double multicategory $\mathbb{FF}(\mathbb{M})$, we still need to define the horizontal composites and units for n-ary 2-cells. Given 2-cells

$$
\begin{tikzcd}
C_1^2, C_2^2 \arrow[r, "E_1, E_2", "|"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta"] & C_1, C_2 \arrow[d, "F"] \\
C_0^{\bullet 2} \arrow[r, "E_0^\bullet"', "|"] & C_0^\bullet
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
C_1^2, C_2^2 \arrow[r, "E_1', E_2'", "|"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\phi"] & C_1, C_2 \arrow[d, "F"] \\
C_0^{\bullet 2} \arrow[r, "E_0'^\bullet"', "|"] & C_0^\bullet
\end{tikzcd}
$$

in $\mathbb{M}$ underlying 2-cells in $\mathbb{FF}(\mathbb{M})$ (i.e. satisfying equations (XI.1)–(XI.3)), define

$$
\begin{tikzcd}
C_1^2, C_2^2 \arrow[r, "E_{1\odot 1'}, E_{2\otimes 2'}", "|"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta\otimes_2\phi"] & C_1, C_2 \arrow[d, "F"] \\
C_0^{\bullet 2} \arrow[r, "E_{0\odot 0'}^\bullet"', "|"] & C_0^\bullet
\end{tikzcd}
\quad \text{by} \quad
\begin{tikzcd}
C_1^2, C_2^2 \arrow[r, "L_1, R_2", "|"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\hat{\theta}^2"] & C_1^2, C_2^2 \arrow[r, "E_1', E_2'", "|"] \arrow[d, "\hat{F}"] \arrow[dr, phantom, "\Downarrow\phi"] & C_1, C_2 \arrow[d, "F"] \\
C_0^{\bullet 2} \arrow[r, "L_0^\bullet"', "|"] & C_0^{\bullet 2} \arrow[r, "E_0'^\bullet"', "|"] & C_0^\bullet
\end{tikzcd}
$$

and likewise for the other horizontal composites. Checking that this composite 2-cell satisfies equations (XI.1)–(XI.3) is easy but notationally cumbersome, so we will verify that $\theta \otimes_2 \phi$ satisfies (XI.2) to convey the idea:

$$
\begin{tikzcd}
C_1^2, C_2^2 \arrow[r, "{E_{1\odot 1'}, E_{2\otimes 2'}}"'] \arrow[r, bend left=50, "{\mathrm{dom}, E_{2\otimes 2'}}"] \arrow[r, phantom, bend left=25, "{\Downarrow \eta_{1\odot 1'}, \mathrm{id}}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow \theta \otimes_2 \phi"] & C_1, C_2 \arrow[d, "F"] \\
C_0^{\bullet 2} \arrow[r, "E^\bullet_{0\odot 0'}"'] & C_0^\bullet
\end{tikzcd}
=
\begin{tikzcd}
C_1^2, C_2^2 \arrow[r, "{L_1, R_2}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow \hat{\theta}^2"] & C_1^2, C_2^2 \arrow[r, "{E'_1, E'_2}"'] \arrow[r, bend left=50, "{\mathrm{dom}, E'_2}"] \arrow[r, phantom, bend left=25, "{\Downarrow \eta'_1, \mathrm{id}}"] \arrow[d, "\hat{F}"] \arrow[dr, phantom, "\Downarrow \phi"] & C_1, C_2 \arrow[d, "F"] \\
C_0^{\bullet 2} \arrow[r, "L_0^\bullet"'] & C_0^{\bullet 2} \arrow[r, "E_0'^\bullet"'] & C_0^\bullet
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
C_1^2, C_2^2 \arrow[r, "{L_1, R_2}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow \hat{\theta}^2"] & C_1^2, C_2^2 \arrow[r, "{\mathrm{dom}, \mathrm{cod}}"'] \arrow[r, bend left=50, "{\mathrm{dom}, E'_2}"] \arrow[r, phantom, bend left=25, "{\Downarrow \mathrm{id}, \epsilon'_2}"] \arrow[d, "\hat{F}"] \arrow[dr, phantom, "\Downarrow \gamma_1"] & C_1, C_2 \arrow[d, "F"] \\
C_0^{\bullet 2} \arrow[r, "L_0^\bullet"'] & C_0^{\bullet 2} \arrow[r, "\mathrm{cod}^\bullet"] \arrow[r, bend right=50, "E_0'^\bullet"'] \arrow[r, phantom, bend right=25, "\Downarrow \epsilon_0'^\bullet"] & C_0^\bullet
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
C_1^2, C_2^2 \arrow[r, "{L_1, R_2}"] \arrow[d, "\hat{F}"'] & C_1^2, C_2^2 \arrow[r, "{\mathrm{dom}, \mathrm{cod}}"'] \arrow[r, bend left=50, "{\mathrm{dom}, E'_2}"] \arrow[r, phantom, bend left=25, "{\Downarrow \mathrm{id}, \epsilon'_2}"] \arrow[d, phantom, "\Downarrow \gamma_1"] & C_1, C_2 \arrow[d, "F"] \\
C_0^{\bullet 2} \arrow[r, "\mathrm{id}"] \arrow[r, bend right=50, "L_0^\bullet"'] \arrow[r, phantom, bend right=25, "\Downarrow \vec{\epsilon}_0^\bullet"] & C_0^{\bullet 2} \arrow[r, "\mathrm{cod}^\bullet"] \arrow[r, bend right=50, "E_0'^\bullet"'] \arrow[r, phantom, bend right=25, "\Downarrow \epsilon_0'^\bullet"] & C_0^\bullet
\end{tikzcd}
=
\begin{tikzcd}
C_1^2, C_2^2 \arrow[r, "{\mathrm{dom}, \mathrm{cod}}"'] \arrow[r, bend left=50, "{\mathrm{dom}, E_{2\otimes 2'}}"] \arrow[r, phantom, bend left=25, "{\Downarrow \mathrm{id}, \epsilon'_{2\otimes 2'}}"] \arrow[d, "\hat{F}"] \arrow[dr, phantom, "\Downarrow \gamma_1"] & C_1, C_2 \arrow[d, "F"] \\
C_0^{\bullet 2} \arrow[r, "\mathrm{cod}^\bullet"] \arrow[r, bend right=50, "E^\bullet_{0\odot 0'}"'] \arrow[r, phantom, bend right=25, "\Downarrow \epsilon^\bullet_{0\odot 0'}"] & C_0^\bullet
\end{tikzcd}
\tag{XI.4}
$$

Finally, given a n-ary vertical 1-cell F, the unit 2-cells I_{iF} are simply given by γ_i, which are easily verified to define 2-cells in $\mathbb{FF}(\mathbb{M})$.

Bibliography

[BMR13] T. Barthel, J.P. May, and E. Riehl. Six model structures for dg-modules over dgas: Model category theory in homological action. 2013.

[BR13] Tobias Barthel and Emily Riehl. On the construction of functorial factorizations for model categories. *Algebraic & Geometric Topology*, 13:1089–1124, 2013.

[CGR12] E. Cheng, N. Gurski, and E. Riehl. Multivariable adjunctions and mates. *New York Journal of Mathematics*, 2012.

[DPP10] R. Dawson, R. Pare, and D. A. Pronk. The span construction. *Theory Appl. Categ.*, 24:302–377 (electronic), 2010.

[Gar07] Richard Garner. Cofibrantly generated natural weak factorisation systems. arXiv:math/0702290v1, 2007.

[Gar09] Richard Garner. Understanding the small object argument. *Appl. Categ. Structures*, 17(3):247–285, 2009. arXiv:0712.0724.

[GT06] Marco Grandis and Walter Tholen. Natural weak factorization systems. *Arch. Math. (Brno)*, 42(4):397–408, 2006.

[MP12] J. P. May and K. Ponto. *More concise algebraic topology*. Chicago lectures in mathematics. University of Chicago Press, 2012.

[Rie11] Emily Riehl. Algebraic model structures. *New York Journal of Mathematics*, 17:173–231, 2011.

[Rie13] Emily Riehl. Monoidal algebraic model structures. *Journal of Pure and Applied Algebra*, 217:1069–1104, 2013.

[Shu08] Michael Shulman. Framed bicategories and monoidal fibrations. *Theory and Applications of Categories*, 20(18):650–738, 2008.

[Str72] Ross Street. The formal theory of monads. *J. Pure Appl. Algebra*, 2(2):149–168, 1972.

[TF10] et al. T.M. Fiore. Monads in double categories. *Journal of Pure and Applied Algebra*, 215:1174–1197, 2010.