1 Introduction

The theory of model categories has a long history, and has proven itself indespensible to several recent advances in mathematics, such as higher category theory, so-called spectral algebraic geometry, even finding applications in computer science and the foundations of mathematics with homotopy type theory.

In the modern treatment, a model category is defined to consist of two *weak factorization systems* on a category $\mathscr{C}$ (e.g. [MP12]). A weak factorization system is a structure which consists of two classes of morphisms of $\mathscr{C}$, call them $\mathcal{L}$ and $\mathcal{R}$, such that solutions to certain lifting problems involving one morphism from each class always exist, plus an axiom that every morphism of $\mathscr{C}$ factors as a morphism from $\mathcal{L}$ followed by a morphism from $\mathcal{R}$. In the past 20 or so years, most authors have added the requirement that this factorization can be chosen in a natural/functorial way.

Taking this one step further, in [GT06] the category theorists Marco Grandis and Walter Tholen proposed a strengthening of weak factorization systems which they called *natural* weak factorization systems, today most often referred to as *algebraic* weak factorization systems, or awfs for short. An awfs strengthens the structure in a way which provides a canonical *choice* of solution to every lifting problem, in such a way that these choices are coherent or natural in a precise sense.

It at first seems as though this extra structure is *too* strict, and that examples would be hard to find. But in [Gar07] and [Gar09], the category theorist Richard Garner provided a modification of Quillen's small object argument which generates algebraic weak factorization systems, and which furthermore has much nicer convergence properties that Quillen's original construction, and often generates a smaller and easier to understand factorization. Best of all, Garner's small object argument operates under almost identical assumptions as Quillen's, so that in practice any cofibrantly generated weak factorization system can be strengthened to an algebraic one.

In her Ph.D. thesis, [Rie11] and [Rie13], Emily Riehl began the project of developing a full-fledged theory of *algebraic model structures*, built out of two awfs analgously to an ordinary model structure, in the second part extending this to a theory of multivariable Quillen adjunctions and monoidal algebraic model structures. Since then, she and her collaborators have continued to develop and find applications of this theory, e.g. [CGR12], [BR13], and [BMR13].

One appealing aspect of the theory of algebraic weak factorization systems, and by extension algebraic model categories, is that it is possible to express the definition entirely in terms of functors and natural transformations, opening the door to generalizing the theory by carrying out the same construction in other contexts with analogues of functors and natural transformations. The simplest and most familiar such context is a 2-category, but

we have found that the most natural setting for the general theory of awfs—including the appropiate generalization of Quillen adjunctions—is a double category, a less familiar structure which has been attracting more attention in recent years.

In this paper, we aim to build a general setting in which the structure of an algebraic weak factorization system makes sense, a structure we call a *cyclic two-fold double category*, and then to translate some of the most important results from the existing theory to this larger context. We well then give an extension of this structure, called a *cyclic two-fold double multi-category*, a modification of the cyclic double multi-categories of [CGR12], in which multivariable morphisms of awfs also make sense. Finally, we will give an example application of the increased generality by showing that the theory of enriched categories has the necessary structure needed to form a special case of our theory. We expect that other examples of algebraic model structures on structures other than plain categories will be found useful, and hope that this general framework will help to quickly translate the existing theory to new settings.

2   Weak factorization systems

We will begin by briefly reviewing the notions of functorial factorization, weak factorization system, and algebraic weak factorization system.

2.1   Arrow Categories

Let $\mathscr{C}$ be a category. Its arrow category $\mathscr{C}^2$ is the category whose objects are arrows in $\mathscr{C}$ and whose morphisms are commutative squares. The arrow category comes with two functors $\mathrm{dom}, \mathrm{cod}\colon \mathscr{C}^2 \to \mathscr{C}$, along with a natural transformation $\kappa\colon \mathrm{dom} \Rightarrow \mathrm{cod}$. The component of κ at an object f of $\mathscr{C}^2$ is simply $f\colon \mathrm{dom}\, f \to \mathrm{cod}\, f$. Moreover, $\mathscr{C}^2$ satisfies a universal property: there is an equivalence of categories

$$\mathrm{Fun}(2, \mathrm{Fun}(\mathscr{X}, \mathscr{C})) \simeq \mathrm{Fun}(\mathscr{X}, \mathscr{C}^2) \tag{1}$$

given by composition with κ. Here, 2 is the ordinal, i.e. the category with two objects and a single non-identity arrow. In other words, $\mathscr{C}^2$ is the cotensor of $\mathscr{C}$ with the category 2 in the 2-category $\mathcal{C}$at.

We will make this universal property more explicit in the next lemma, separating out the 1-dimensional and the 2-dimensional parts of (1):

Lemma 2.1. *Let $\mathscr{C}$ be a category.*

i) *For any category $\mathscr{X}$, pair of functors $F, G\colon \mathscr{X} \to \mathscr{C}$, and natural transformation $\alpha\colon F \Rightarrow G$, there is a unique functor $\hat{\alpha}\colon \mathscr{X} \to \mathscr{C}^2$ such that $\mathrm{dom}\, \hat{\alpha} = F$, $\mathrm{cod}\, \hat{\alpha} = G$, and*

$$\mathscr{X} \xrightarrow{\ \hat{\alpha}\ } \mathscr{C}^2 \overset{\mathrm{dom}}{\underset{\mathrm{cod}}{\Downarrow\kappa}} \mathscr{C} \quad = \quad \mathscr{X} \overset{F}{\underset{G}{\Downarrow\alpha}} \mathscr{C}. \tag{2}$$

ii) *For any functors $F, F', G, G'\colon \mathscr{X} \to \mathscr{C}$ and a commutative square of natural transformations*

$$
\begin{array}{ccc}
F & \overset{\gamma}{\Longrightarrow} & F' \\
\alpha \Vert & & \Vert \beta \\
G & \underset{\phi}{\Longrightarrow} & G',
\end{array}
$$

there is a unique natural transformation $\eta\colon \hat{\alpha} \to \hat{\beta}$ such that $\mathrm{dom}\, \eta = \gamma$ and $\mathrm{cod}\, \eta = \phi$, hence

$$\mathscr{X} \underset{\Downarrow\phi}{\overset{\Downarrow\alpha}{-\,G\,\to}} \mathscr{C} \quad = \quad \mathscr{X} \overset{\hat{\alpha}}{\underset{\hat{\beta}}{\Downarrow\eta}} \mathscr{C}^2 \overset{\mathrm{dom}}{\underset{\mathrm{cod}}{\Downarrow\kappa}} \mathscr{C} \quad = \quad \mathscr{X} \underset{\Downarrow\beta}{\overset{\Downarrow\gamma}{-\,F'\,\to}} \mathscr{C}. \tag{3}$$

Definition 2.2. Let $\mathscr{D}$ be any 2-category. For any object A in $\mathscr{D}$, the *arrow object* of A, if it exists, is an object A^2 satisfying the universal property (1). If every object has an arrow object, i.e. if $\mathscr{D}$ has cotensors by 2, we will say $\mathscr{D}$ *has arrow objects*.

In practice, we will work with arrow objects in a 2-category using the two parts of Lemma 2.1.

Finally, we will record here a simple proposition which will be needed later.

Proposition 2.3. *Any arrow object A^2 has an internal category structure*

$$A^3 \xrightarrow{\ c\ } A^2 \overset{\xrightarrow{\mathrm{dom}}}{\underset{\xrightarrow{\mathrm{cod}}}{\leftarrow i -}} A$$

where A^3 is the pullback of the span

$$A^2 \xrightarrow{\ \mathrm{cod}\ } A \xleftarrow{\ \mathrm{dom}\ } A^2$$

Proof. Using the universal property, we define i and c by the equations $\mathrm{dom}\, i = $ ■ id, $\mathrm{cod}\, i = \mathrm{id}$, $\kappa i = \mathrm{id}_{\mathrm{id}}$, $\mathrm{dom}\, c = \mathrm{dom}\, p_1$, $\mathrm{cod}\, c = \mathrm{cod}\, p_2$, and $\kappa c = \kappa p_2 \circ \kappa p_1$, where p_1 and p_2 are the projections of the pullback. □

2.2 Functorial Factorizations

Definition 2.4. A functorial factorization on a category $\mathscr{C}$ consists of a functor E and two natural transformations η and ϵ which factor κ, as in

$$C^2 \overset{\overset{\mathrm{dom}}{\Downarrow\kappa}}{\underset{\mathrm{cod}}{\rightrightarrows}} C \quad = \quad C^2 \overset{\overset{\mathrm{dom}}{\overbrace{\Downarrow\eta}}}{\underset{\underset{\mathrm{cod}}{\underbrace{\Downarrow\epsilon}}}{- E \to}} C.$$

This determines for any arrow f in $\mathscr{C}$ a factorization $f = \epsilon_f \circ \eta_f$. The factorization is natural, meaning that for any morphism $(u, v)\colon f \Rightarrow g$ in $\mathscr{C}^2$ (i.e. commutative square in $\mathscr{C}$), the two squares in

$$\begin{array}{ccc}
\cdot & \xrightarrow{\ u\ } & \cdot \\
{\scriptstyle \eta_f}\downarrow & & \downarrow{\scriptstyle \eta_g} \\
\cdot & \xrightarrow{E(u,v)} & \cdot \\
{\scriptstyle \epsilon_f}\downarrow & & \downarrow{\scriptstyle \epsilon_g} \\
\cdot & \xrightarrow{\ v\ } & \cdot
\end{array}$$

commute.

A functorial factorization also determines two functors $L, R: C^2 \to C^2$ such that $\operatorname{dom} L = \operatorname{dom}$, $\operatorname{cod} R = \operatorname{cod}$, $\operatorname{cod} L = \operatorname{dom} R = E$, $\kappa L = \eta$, and $\kappa R = \epsilon$, by the universal property of C^2. The components of the factorization of f can then also be referred to as Lf and Rf, now thought of as objects in $\mathscr{C}^2$. There are also two canonical natural transformations, $\vec{\eta}: \operatorname{id} \Rightarrow R$ and $\vec{\epsilon}: L \Rightarrow \operatorname{id}$, determined by the commuting squares

$$
\begin{array}{ccc}
\operatorname{dom} \xrightarrow{\ \eta\ } E & & \operatorname{dom} \xrightarrow{\ \operatorname{id}\ } \operatorname{dom} \\
\kappa \big\Vert \quad \big\Vert \epsilon & \text{and} & \eta \big\Vert \quad \big\Vert \kappa \\
\operatorname{cod} \xrightarrow[\ \operatorname{id}\ ]{} \operatorname{cod} & & E \xrightarrow[\ \epsilon\ ]{} \operatorname{cod}
\end{array}
$$

respectively. These make L and R into (co)pointed endofunctors of $\mathscr{C}^2$.

An algebra for the pointed endofunctor R is an object f in $\mathscr{C}^2$ equipped with a morphism $\vec{t}: Rf \Rightarrow f$, such that $\vec{t} \circ \vec{\eta}_f = \operatorname{id}_f$. Similarly, a coalgebra for the copointed endofunctor L is an f equipped with a morphism $\vec{s}: f \Rightarrow Lf$, such that $\vec{\epsilon}_f \circ \vec{s} = \operatorname{id}_f$.

Lemma 2.5. *Let $f: X \to Y$ be a morphism in $\mathscr{C}$. An R-algebra structure on $f \in \mathscr{C}^2$ is precisely a choice of lift t in the square*

$$
\begin{array}{ccc}
X & =\!=\!= & X \\
Lf \downarrow & {\scriptstyle t}\ \nearrow & \downarrow f \\
Ef & \xrightarrow[Rf]{} & Y.
\end{array}
\tag{4}
$$

Dually, an L-coalgebra structure on f is precisely a choice of lift s in the square

$$
\begin{array}{ccc}
X & \xrightarrow{\ Lf\ } & Ef \\
f \downarrow & {\scriptstyle s}\ \nearrow & \downarrow Rf \\
Y & =\!=\!= & Y.
\end{array}
\tag{5}
$$

Moreover, a morphsim $(u,v): f \Rightarrow g$ in $\mathscr{C}^2$ is a morphism of R-algebras if it commutes with the lifts t and t', that is, if in the diagram

$$
\begin{array}{ccc}
\cdot & \xrightarrow{\ u\ } & \cdot \\
Lf \downarrow & & \downarrow Lg \\
\cdot & \xrightarrow{\ E(u,v)\ } & \cdot \\
Rf \big\Vert {\scriptstyle t} & & {\scriptstyle t'}\ \big\Vert Rg \\
\cdot & \xrightarrow{\ v\ } & \cdot
\end{array}
$$

we have $t'v = E(u,v)t$.

2.3 Algebraic Weak Factorization Systems

To simplify the discussion of weak factorization systems, we will start by introducing a notation. For any two morphisms l and r in $\mathscr{C}$, write $l \boxempty r$ to mean that for every commutative square

$$
\begin{array}{ccc}
\cdot & \xrightarrow{\;u\;} & \cdot \\
{\scriptstyle l}\big\downarrow & {\scriptstyle w}\;\nearrow & \big\downarrow{\scriptstyle r} \\
\cdot & \xrightarrow[\;v\;]{} & \cdot
\end{array}
\tag{6}
$$

there exists a lift w. In this case, we will say that l has the *left lifting property* with respect to r, and that r has the *right lifting property* with respect to l. Similarly, for two classes of morphisms $\mathcal{L}$ and $\mathcal{R}$, we will say $\mathcal{L} \boxempty \mathcal{R}$ if $l \boxempty r$ for every $l \in \mathcal{L}$ and $r \in \mathcal{R}$. Finally, we will write $\mathcal{L}^{\boxempty}$ for the class of morphisms having the right lifting property with respect to every morphism of $\mathcal{L}$, and $^{\boxempty}\mathcal{R}$ for the class of morphisms having the left lifting property with respect to every morphism of $\mathcal{R}$.

Definition 2.6. A *functorial weak factorization system* on a category $\mathscr{C}$ consists of a functorial factorization on $\mathscr{C}$ and two classes $\mathcal{L}$ and $\mathcal{R}$ of morphisms in $\mathscr{C}$, such that

- for every morphism f in $\mathscr{C}$, $Lf \in \mathcal{L}$ and $Rf \in \mathcal{R}$,
- $\mathcal{L}^{\boxempty} = \mathcal{R}$ and $^{\boxempty}\mathcal{R} = \mathcal{L}$.

It a simple and standard proof that the lifting property condition can be replaced by two simpler conditions:

Lemma 2.7. *A functorial weak factorization system can equivalently be defined to be a functorial factorization on $\mathscr{C}$ and two classes $\mathcal{L}$ and $\mathcal{R}$ of morphisms in $\mathscr{C}$, such that*

- *for every morphism f in $\mathscr{C}$, $Lf \in \mathcal{L}$ and $Rf \in \mathcal{R}$,*
- *$\mathcal{L} \boxempty \mathcal{R}$,*
- *$\mathcal{L}$ and $\mathcal{R}$ are both closed under retracts.*

In fact, the functorial factorization by itself already determines the two classes of morphisms, with $\mathcal{L}$ the class of morphisms admitting an L-coalgebra structure, and $\mathcal{R}$ the class of morphisms admitting an R-algebra structure. The lifting properties also follow directly from the functorial factorization, as the next lemma shows.

Lemma 2.8. *For any L-coalgebra (l, s) and any R-algebra (r, t), there is a canonical choice of lift in the square (6). Any morphism of R-algebras $(u_1, v_1) \colon (r, t) \Rightarrow (r', t')$ and any morphism of L-coalgebras $(u_2, v_2) \colon (l', s') \Rightarrow (l, s)$ preserves these canonical choices of lifts.*

Proof. The construction is shown in the diagram

$$(7)$$

Commutativity of (6) follows immediately from (4) and (5).

That a morphism of R-algebras preserves these canonical lifts can be seen in the diagram

noting that $u'tE(u,v)s = t'E(u',v')E(u,v)s = t'E(u'u, v'v)s$. □

This, together with the classical fact that the class of objects admitting a (co)algebra structure for a (co)pointed endofunctor is closed under retracts, gives a third equivalent definition of a functorial weak factorization system.

Lemma 2.9. *A functorial weak factorization system can equivalently be defined to be a functorial factorization on $\mathscr{C}$ such that*

- *for every morphism f in $\mathscr{C}$, Lf admits an L-coalgebra structure, and Rf admits an R-algebra structure.*

An R-algebra structure on Rf consists of a morphism $\vec{\mu}_f : R^2 f \to Rf$ in $\mathscr{C}^2$ such that $\vec{\mu}_f \circ \vec{\eta}_{Rf} = \mathrm{id}_{Rf}$, while an L-coalgebra structure on Lf consists of a morphism $\vec{\delta}_f : Lf \to L^2 f$ such that $\vec{\epsilon}_{Lf} \circ \vec{\delta}_f = \mathrm{id}_{Lf}$. We might hope that it is possible to choose these structures for all f in a natural way, such that they form the components of natural transformations $\vec{\mu} : R^2 \Rightarrow R$ and $\vec{\delta} : L \Rightarrow L^2$.

If we want these choices of lifts to be fully coherent, we should also ask that for any R-algebra (f, t), the lift constructed as in (7) for the square (4) is equal to t, and similarly for L-coalgebras and (5). Lastly, we should ask that the components $\vec{\mu}_f$ and $\vec{\delta}_f$ are (co)algebra morphisms. These conditions, plus one more ensuring that there is an unambiguous notion of a morphism with both L-algebra and R-coalgebra structures, lead to the definition of an *algebraic weak factorization system*, first given in [GT06] (there called *natural weak factorization systems*), and further refined in [Gar07] and [Gar09].

Definition 2.10. An *algebraic weak factorization system* on a category $\mathscr{C}$ consists of a functorial factorization $(L, \vec{\epsilon}, R, \vec{\eta})$ together with natural transformations $\vec{\mu} \colon R^2 f \Rightarrow Rf$ and $\vec{\delta} \colon L \Rightarrow L^2$, such that

- $\mathbb{R} = (R, \vec{\eta}, \vec{\mu})$ is a monad and $\mathbb{L} = (L, \vec{\epsilon}, \vec{\delta})$ a comonad on $\mathscr{C}^2$, and

- the natural transformation $\Delta = (\delta, \mu) \colon LR \Rightarrow RL$ determined by the equation $\epsilon L \circ \delta = \mu \circ \eta R\, (= \mathrm{id}_E)$ as in Lemma 2.1 is a distributive law, which in this case reduces to the single condition $\delta \circ \mu = \mu L \circ E\Delta \circ \delta R$.

3   Double Categories

In this section, we will give an overview of double categories, as well as (one possible version of) the definition of monads in a double category

A (strict) double category is a two-dimensional categorical structure, similar to a 2-category. Unlike a 2-category, a double category has two types of 1-cells, called *vertical* and *horizontal*, and 2-cells all have a square shape, with domain and codomain horizontal 1-cells as well as domain and codomain vertical 1-cells.

We will first give the most concise definition of a double category, which we will then break down into more concrete terms.

Definition 3.1. A (strict) *double category* is an internal category object in the (large) category of categories.

So a double category $\mathbb{D}$ consists of a category $\mathbb{D}_0$ and a category $\mathbb{D}_1$, along with functors $s, t : \mathbb{D}_1 \to \mathbb{D}_0$, $i : \mathbb{D}_0 \to \mathbb{D}_1$, and $\otimes : \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \to \mathbb{D}_1$ satisfying the usual axioms of a category. We will call the objects of $\mathbb{D}_0$ the 0-cells of $\mathbb{D}$, and the morphisms of $\mathbb{D}_0$ the vertical 1-cells. Thus $\mathbb{D}_0$ forms the so-called *vertical category* of $\mathbb{D}$. We will call the objects of $\mathbb{D}_1$ the horizontal 1-cells of $\mathbb{D}$, and the morphisms of $\mathbb{D}_1$ are the 2-cells.

A morphism $\phi : X \to Y$ in $\mathbb{D}_1$, where $s(X) = C$, $t(X) = C'$, $s(Y) = D$, $t(Y) = D'$, $s(\phi) = f$, and $t(\phi) = g$ will be drawn as

$$
\begin{array}{ccc}
C & \overset{X}{\longrightarrow} & C' \\
{\scriptstyle f}\big\downarrow & \Downarrow\phi & \big\downarrow{\scriptstyle g} \\
D & \underset{Y}{\longrightarrow} & D'
\end{array}
\tag{8}
$$

where the tick-mark on the horizontal 1-cells serves as a further reminder that the horizontal 1-cells are of a different nature than the vertical 1-cells. The composition in $\mathbb{D}_0$ provides a vertical composition of vertical 1-cells and 2-cells, while the composition functor $\otimes : \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \to \mathbb{D}_1$ provides a horizontal composition of horizontal 1-cells and 2-cells.

For any object C in $\mathbb{D}_0$, $i(C)$ is the *unit* horizontal 1-cell

$$
C \overset{I_C}{\longrightarrow} C
$$

and acts as an identity with respect to the horizontal composition.

A 2-cell θ for which $s\theta = t\theta = \mathrm{id}$ will be called *globular*. We will sometimes draw globular 2-cells as

$$
C \underset{Y}{\overset{X}{\Rightarrow}}{\Downarrow\theta}\, C',
$$

to save space and help readability of diagrams.

Example 3.2. For any 2-category $\mathcal{D}$, there is an associated double category $\mathbb{S}q(\mathcal{D})$ of *squares* in $\mathcal{D}$, in which the vertical and horizontal 1-cells are both just 1-cells in $\mathcal{D}$, and 2-cells

$$
\begin{array}{ccc}
C & \xrightarrow{j} & C' \\
f \downarrow & \Downarrow \phi & \downarrow g \\
D & \xrightarrow[k]{} & D'
\end{array}
$$

are simply 2-cells $\phi: gj \Rightarrow kf$ in $\mathcal{D}$.

Example 3.3. Given any category M, there is a pseudo double category $\mathbb{S}\text{pan}(M)$ of *spans* in M. The vertical category of $\mathbb{S}\text{pan}(M)$ is just M, while horizontal 1-cells

$$ C \xrightarrow{X} D $$

are given by spans

$$ C \xleftarrow{j} X \xrightarrow{k} D $$

in M, and 2-cells

$$
\begin{array}{ccc}
C & \xrightarrow{X} & D \\
f \downarrow & \Downarrow \theta & \downarrow g \\
C' & \xrightarrow[Y]{} & D'
\end{array}
$$

are given by commutative diagrams

$$
\begin{array}{ccccc}
C & \xleftarrow{j} & X & \xrightarrow{k} & D \\
f \downarrow & & \downarrow \theta & & \downarrow g \\
C' & \xleftarrow[j']{} & Y & \xrightarrow[k']{} & D.'
\end{array}
$$

The horizontal composition of spans is given by pullback. It is because this horizontal composition is only determined up to isomorphsim that this example is not a strict double category.

Definition 3.4. For any double category $\mathbb{D}$, there is an associated 2-category $\mathcal{H}or(\mathbb{D})$, called the *horizontal 2-category* of $\mathbb{D}$. The objects and 1-cells of $\mathcal{H}or(\mathbb{D})$ are the objects and horizontal 1-cells of $\mathbb{D}$, while 2-cells $\phi: X \Rightarrow Y$ in $\mathcal{H}or(\mathbb{D})$ are the globular 2-cells in $\mathbb{D}$, i.e. those of the form

$$
\begin{array}{ccc}
C & \xrightarrow{X} & D \\
\| & \Downarrow \phi & \| \\
C & \xrightarrow[Y]{} & D
\end{array}
$$

Notice that $\mathcal{H}or(\mathbb{S}q(\mathcal{D}))$ is isomorphic to $\mathcal{D}$.

Definition 3.5. Given a double category $\mathbb{D}$, define double categories $\mathbb{D}^{vop}$ and $\mathbb{D}^{hop}$, obtained by reversing the direction of the vertical and horizontal 1-cells respectively, and changing the orientation of the 2-cells as appropriate. For example, a 2-cell (8) in $\mathbb{D}^{vop}$ is a 2-cell

$$
\begin{array}{ccc}
D & \xrightarrow{Y} & D' \\
f \downarrow & \Downarrow\phi & \downarrow g \\
C & \xrightarrow[X]{} & C'
\end{array}
$$

in $\mathbb{D}$.

In terms of Definition 3.1, $\mathbb{D}^{vop}$ is the double category obtained by replacing the categories $\mathbb{D}_0$ and $\mathbb{D}_1$ with their opposites, while $\mathbb{D}^{vop}$ is the obtained by swapping the horizontal source and target functors s and t.

3.1 Arrow Objects in a Double Category

In the following we will need an extension of the universal property (1) to double categories. Fortunately, this is quite straightforward.

Let $\mathbb{D}$ be a double category. Given an object C of $\mathbb{D}$, the *arrow object* C^2, if it exists, is an object together with a diagram

$$
C^2 \underset{\text{cod}}{\overset{\text{dom}}{\Rrightarrow\kappa}} C,
$$

such that any 2-cell

$$
A \underset{d_0}{\overset{d_1}{\Rrightarrow\alpha}} C
$$

uniquely factors through κ, as

$$
A \xrightarrow{\hat{\alpha}} C^2 \underset{\text{cod}}{\overset{\text{dom}}{\Rrightarrow\kappa}} C.
$$

Given a vertical 1-cell $F\colon C \to D$ in $\mathbb{D}$, the *lift to arrow objects* $\hat{F}\colon C^2 \to D^2$, if it exists, is a vertical 1-cell $\hat{F}\colon C^2 \to D^2$ together with 2-cells

$$
\begin{array}{ccc}
C^2 & \xrightarrow{\text{dom}} & C \\
\hat{F}\downarrow & \Downarrow\gamma_1 & \downarrow F \\
D^2 & \xrightarrow[\text{dom}]{} & D
\end{array}
\qquad
\begin{array}{ccc}
C^2 & \xrightarrow{\text{cod}} & C \\
\hat{F}\downarrow & \Downarrow\gamma_0 & \downarrow F \\
D^2 & \xrightarrow[\text{cod}]{} & D
\end{array}
$$

satisfying

$$
\begin{array}{ccc}
C^2 \xrightarrow{\text{dom}} C & & C^2 \xrightarrow[\text{cod}]{\text{dom}} C \\
\hat{F}\downarrow \quad \Downarrow\gamma_1 \quad \downarrow F & = & \hat{F}\downarrow \quad \Downarrow\gamma_0 \quad \downarrow F \\
D^2 \xrightarrow[\text{cod}]{\text{dom}} D & & D^2 \xrightarrow[\text{cod}]{} D,
\end{array}
$$

such that for any 2-cells

$$
A \underset{d_0}{\overset{d_1}{\Rightarrow\alpha}} C \qquad B \underset{d_0'}{\overset{d_1'}{\Rightarrow\alpha'}} D
$$

and

$$
\begin{array}{ccc}
A \xrightarrow{d_1} C & \qquad & A \xrightarrow{d_0} C \\
G\downarrow \quad \Downarrow\lambda_1 \quad \downarrow F & & G\downarrow \quad \Downarrow\lambda_0 \quad \downarrow F \\
B \xrightarrow{d_1'} D & & B \xrightarrow{d_0'} D
\end{array}
$$

satisfying

$$
\begin{array}{ccc}
A \xrightarrow{d_1} C & & A \xrightarrow{\overset{d_1}{\Downarrow\alpha}}_{d_0} C \\
G\downarrow \quad \Downarrow\lambda_1 \quad \downarrow F & = & G\downarrow \quad \Downarrow\lambda_0 \quad \downarrow F \\
B \xrightarrow[\Downarrow\alpha']{d_1'} D & & B \xrightarrow{d_0'} D \\
\quad\ {}_{d_0'} & &
\end{array}
$$

there is a unique 2-cell

$$
\begin{array}{ccc}
A \xrightarrow{\hat{\alpha}} C^2 \\
G\downarrow \quad \Downarrow\theta \quad \downarrow \hat{F} \\
B \xrightarrow{\hat{\alpha}'} D^2
\end{array}
$$

such that the horizontal composition of θ with γ_0 and γ_1 is respectively equal to λ_0 and λ_1.

Definition 3.6. A double category $\mathbb{D}$ *has arrow objects* if for every object C of $\mathbb{D}$ there is an object C^2 and 2-cell κ, and for every vertical 1-cell F there is a vertical 1-cell $\hat{F}$ and 2-cells γ_0 and γ_1, satisfying the universal properties given above.

The intuition that this is a generalization of Lemma 2.1 is supported by the following two propositions, the (easy) proofs of which are left to the reader.

Proposition 3.7. *If the double category $\mathbb{D}$ has arrow objects, then so does $\mathcal{H}or(\mathbb{D})$.*

Proposition 3.8. *If the 2-category $\mathcal{D}$ has arrow objects, then so does $\mathsf{Sq}(\mathcal{D})$.*

Proof. A simple check. The 2-cells γ_0 and γ_1 will always be identities. $\square$

3.2 Monads

We will define a *monad* in a double category $\mathbb{D}$ to be a tuple (C, T, η, μ), in which C is an object, $T: C \to C$ is a horizontal 1-cell, and η and μ are 2-cells

$$
\begin{array}{ccc}
C \xrightarrow{\text{id}_C} C & \qquad & C \xrightarrow{T} C \xrightarrow{T} C \\
\Big\| \quad \Downarrow\eta \quad \Big\| & & \Big\| \qquad \Downarrow\mu \qquad \Big\| \\
C \xrightarrow[T]{} C & & C \xrightarrow[T]{} C
\end{array}
$$

satisfying the usual unit and associativity conditions.

Given two monads (C, T, η, μ) and (D, S, η', μ'), a monad morphism from (C, T) to (D, S) consists of a pair (f, ϕ), where f is a vertical 1-cell $C \to D$ and ϕ is a 2-cell

$$
\begin{array}{ccc}
C & \xrightarrow{T} & C \\
f\Big\downarrow & \Downarrow\phi & \Big\downarrow f \\
D & \xrightarrow[S]{} & D
\end{array}
$$

which commutes with the unit and multiplication 2-cells in the sense of the two equations

$$
\begin{array}{ccccc}
C \xrightarrow{\text{id}_C} C & & & C \xrightarrow{\text{id}_C} C & \\
\Big\| \quad \Downarrow\eta \quad \Big\| & & & f\Big\downarrow \quad \Downarrow\text{id}_f \quad \Big\downarrow f & \\
C \xrightarrow{T} C & = & & D \xrightarrow{\text{id}_D} D & \\
f\Big\downarrow \quad \Downarrow\phi \quad \Big\downarrow f & & & \Big\| \quad \Downarrow\eta' \quad \Big\| & \\
D \xrightarrow[S]{} D & & & D \xrightarrow[S]{} D &
\end{array}
\qquad (9)
$$

and

$$
\begin{array}{ccc}
C \xrightarrow{T} C \xrightarrow{T} C & \quad C \xrightarrow{T} C \xrightarrow{T} C \\
\Vert \quad \Downarrow\mu \quad \Vert & \quad f\downarrow \;\; \Downarrow\phi \;\; f\downarrow \;\; \Downarrow\phi \;\; f\downarrow \\
C \xrightarrow{T} C \;\; = \;\; D \xrightarrow{S} D \xrightarrow{S} D \\
f\downarrow \;\; \Downarrow\phi \;\; f\downarrow & \quad \Vert \quad \Downarrow\mu' \quad \Vert \\
D \xrightarrow{S} D & \quad D \xrightarrow{S} D
\end{array}
\tag{10}
$$

Definition 3.9. Given any double category $\mathbb{D}$, we will write $\mathrm{Mon}(\mathbb{D})$ for the category of monads in $\mathbb{D}$, consisting of monads and monad morphisms as defined above. The category $\mathrm{Comon}(\mathbb{D})$ of comonads in $\mathbb{D}$ is defined to be the category $\mathrm{Mon}(\mathbb{D}^{\mathrm{op}})$ of monads in $\mathbb{D}^{\mathrm{op}}$.

Example 3.10. The category $\mathrm{Mon}(\mathbb{S}\mathrm{pan}(\mathbf{Set}))$ is precisely the category of small ▮ categories. It is an easy and enlightening exercise to work this out for oneself.

Proposition 3.11. *The categories of (co)monads and (co)lax morphisms in a 2-category $\mathcal{D}$ can be given in terms of (co)monads in the double category of squares as follows:*

$$
\begin{aligned}
\mathrm{Mon}_{colax}(\mathcal{D}) &= \mathrm{Mon}(\mathbb{S}\mathrm{q}(\mathcal{D})) \\
\mathrm{Comon}_{colax}(\mathcal{D}) &= \mathrm{Comon}(\mathbb{S}\mathrm{q}(\mathcal{D})) \\
\mathrm{Mon}_{lax}(\mathcal{D}) &= \mathrm{Mon}(\mathbb{S}\mathrm{q}(\mathcal{D}^{op}))^{op} \\
\mathrm{Comon}_{lax}(\mathcal{D}) &= \mathrm{Comon}(\mathbb{S}\mathrm{q}(\mathcal{D}^{op}))^{op}
\end{aligned}
$$

where by $\mathcal{D}^{op}$ we mean the 2-category obtained by reversing the direction of all 1-cells (but not 2-cells).

Proof. Immediate from the definitions. Readers unfamiliar with (co)lax morphisms of monads can take this as the definition. $\qquad\square$

3.3 Double Functors

The natural notion of functor between double categories is a straightforward generalization of lax functors between monoidal categories. Recall that we are using the symbol $\otimes$ to denote horizontal composition.

Definition 3.12. Let $\mathbb{D}$ and $\mathbb{E}$ be double categories. A *lax double functor $F\colon \mathbb{D} \to \mathbb{E}$* consists of:
  • Functors $F_0\colon \mathbb{D}_0 \to \mathbb{E}_0$ and $F_1\colon \mathbb{D}_1 \to \mathbb{E}_1$ such that $sF_1 = F_0 s$ and $tF_1 = F_0 t$
  • Natural transformations with globular components $F_\otimes\colon F_1 X \otimes F_1 Y \to F_1(X \otimes Y)$ and $F_I\colon I_{F_0 C} \to F_1(I_C)$, which satisfy the usual coherence axioms for a lax monoidal functor.

A lax double functor F for which the components of F_I and $F_\otimes$ are identities will be called *strict*. For the intermediate notion where the components of F_I and $F_\otimes$ are (vertical) isomorphsims, we will simply refer to F as a double functor.

Proposition 3.13. *A lax double functor $F\colon \mathbb{D} \to \mathbb{E}$ induces a functor $F\colon \mathrm{Mon}(\mathbb{D}) \to \mathrm{Mon}(\mathbb{E})$.*

Proof. This works just like the case for monoidal categories. For instance, if X is a monad in $\mathbb{D}$, FX has the multiplication

$$
\begin{array}{ccccc}
C & \xrightarrow{FX} & C & \xrightarrow{FX} & C \\
\| & & \Downarrow F_\otimes & & \| \\
C & \xrightarrow{F(X\otimes X)} & & & C \\
\| & & \Downarrow F\mu & & \| \\
C & \xrightarrow{FX} & & & C
\end{array}
$$

The fact that F takes monad morphisms to monad morphisms can easily be checked using the naturality of F_I and $F_\otimes$. $\qquad\square$

We will have need for a condition on a lax double functor which implies a sort of converse to Proposition 3.13. This condition is a slight strengthening of the notion of fully-faithful functor which makes sense for lax functors.

Definition 3.14. A lax double functor $F\colon \mathbb{D} \to \mathbb{E}$ is fully-faithful on 2-cells if, for any fixed X_1, X_2, Y, f, g, the induced function from 2-cells in $\mathbb{D}$ of the shape

$$
\begin{array}{ccccc}
C_0 & \xrightarrow{X_1} & C_1 & \xrightarrow{X_2} & C_2 \\
f\downarrow & & & & \downarrow g \\
D_0 & \xrightarrow{Y} & & & D_1
\end{array}
$$

to 2-cells in $\mathbb{E}$ of the shape

$$
\begin{array}{ccccc}
FC_0 & \xrightarrow{FX_1} & FC_1 & \xrightarrow{FX_2} & FC_2 \\
Ff\downarrow & & & & \downarrow Fg \\
FD_0 & \xrightarrow{FY} & & & FD_1,
\end{array}
$$

which takes a 2-cell θ to $F(\theta) \circ F_\otimes$, is a bijection, and if similarly the induced function from 2-cells of the shape

$$
\begin{array}{ccc}
C & \xrightarrow{I_C} & C \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
D_0 & \xrightarrow[Y]{} & D_1
\end{array}
\qquad \text{to} \qquad
\begin{array}{ccc}
FC & \xrightarrow{I_{FC}} & FC \\
{\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Fg} \\
FD_0 & \xrightarrow[FY]{} & FD_1
\end{array}
$$

taking a 2-cell ϕ to $F(\phi) \circ F_I$ is a bijection.

Remark 3.15. Definition 3.14 implies the function on 2-cells θ with a single horizontal 1-cell in the domain is also bijective. We leave the details of the (simple) proof to the reader.

Proposition 3.16. *Let $F: \mathbb{D} \to \mathbb{E}$ be a fully-faithful lax double functor. Given any horizontal 1-cell X in $\mathbb{D}$, a monoid structure on FX in $\mathbb{E}$ lifts uniquely to a monoid structure on X such that the induced functor $F: \mathrm{Mon}(\mathbb{D}) \to \mathrm{Mon}(\mathbb{E})$ takes X to FX.*

Similarly, a vertical 1-cell $f: X \to Y$ for which Ff is a monoid morphism must also be a monoid morphism.

In other words, the diagram of categories is a pullback:

$$
\begin{array}{ccc}
\mathrm{Mon}(\mathbb{D}) & \xrightarrow{F} & \mathrm{Mon}\mathbb{E} \\
{\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u} \\
\mathbb{D} & \xrightarrow[F]{} & \mathbb{E}.
\end{array}
$$

Proof. Simply use the surjectivity of the fully-faithful functor to lift the unit and multiplication 2-cells from FX to X, then use the injectivity to show that the unitary and associativity equations on FX imply those on X. $\square$

4   2-Fold Double Categories

It is well known that the notion of bialgebra or bimonoid—an object with both monoid and comonoid structures which are compatible in a certain sense—makes sense not only in a symmetric monoidal category, but also in more general *braided* monoidal categories. A bimonoid in a braided monoidal category $\mathscr{C}$ can be defined to be a monoid in the category of comonoids in $\mathscr{C}$, or equivalently as a comonoid in the category of monoids in $\mathscr{C}$. The braiding is necessary to ensure that the monoidal structure in $\mathscr{C}$ lifts to a product in $\mathrm{Mon}(\mathscr{C})$ and $\mathrm{Comon}(\mathscr{C})$.

Less well known is the fact that the definition of bimonoid works just as well in a more general context still: so-called 2-fold monoidal categories. A 2-fold monoidal category has two different monoidal structures, call them $(\otimes, I)$ and $(\odot, \perp)$, which are themselves compatible in certain sense. This compatibility can be stated in a way analogous to the definition of bimonoid given in the previous paragraph: a (strict) 2-fold monoidal category is a monoid object in the category **StrMonCat**$_l$ of strict monoidal categories and lax functors, or equivalently a monoid object in the category **StrMonCat**$_c$ of strict monoidal categories and colax functors. Notice that monoid objects in the category of strict monoidal categories and *strong* monoidal functors (in which the components of the lax structure are isomorphisms) are precisely (strict) braided monoidal categories.

More concretely, the compatibility between the monoidal structures amounts to the existence of maps

$$m: \perp \otimes \perp \to \perp, \qquad c: I \to I \odot I, \qquad j: I \to \perp,$$

making $(\perp, j, m)$ a $\otimes$-monoid and (I, j, c) a $\odot$-comonoid, and a natural family of maps

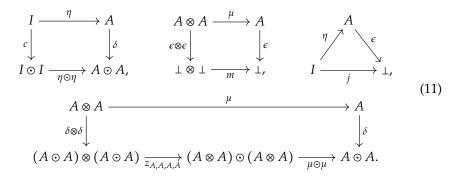$$z_{A,B,C,D}: (A \odot B) \otimes (C \odot D) \to (A \otimes C) \odot (B \otimes D)$$

satisfying some coherence axioms.

Example 4.1.
- Any braided monoidal category can be made into a 2-fold monoidal category in which the two monoidal structures coincide.
- Any monoidal category $(\mathscr{C}, \otimes, I)$ with finite products has a 2-fold monoidal structure with $(\odot, \perp)$ given by the product and terminal object. Dually, a monoidal category $(\mathscr{C}, \odot, \perp)$ with finite coproducts has a 2-fold monoidal structure with $(\otimes, I)$ given by the coproduct and initial object.

Because the $\odot$-monoidal structure is lax monoidal with respect to the $\otimes$-monoidal structure, it lifts to the category $\mathrm{Mon}_\otimes(\mathscr{C})$ of $\otimes$-monoids in $\mathscr{C}$. Dually, the $\otimes$-monoidal structure lifts to the category $\mathrm{Comon}_\odot(\mathscr{C})$ of $\odot$-comonoids in $\mathscr{C}$. Thus, we could define the category of bimonoids in $\mathscr{C}$ to be either $\mathrm{Comon}_\odot(\mathrm{Mon}_\otimes(\mathscr{C}))$ or $\mathrm{Mon}_\otimes(\mathrm{Comon}_\odot(\mathscr{C}))$, and it turns out

that these are canonically isomorphic. In either case, a bimonoid is an object A with a $\otimes$-monoid structure (η, μ) and a $\odot$-comonoid structure (ϵ, δ), such that the following four diagrams commute:

$$
\begin{array}{ccc}
I \xrightarrow{\ \eta\ } A & A \otimes A \xrightarrow{\ \mu\ } A & A \\
\downarrow{c} \quad\quad \downarrow{\delta} & \downarrow{\epsilon\otimes\epsilon} \quad\quad \downarrow{\epsilon} & \eta \nearrow \quad \searrow \epsilon \\
I \odot I \xrightarrow[\eta\odot\eta]{} A \odot A, & \bot \otimes \bot \xrightarrow[m]{} \bot, & I \xrightarrow[j]{} \bot,
\end{array}
$$

$$
\begin{array}{ccc}
A \otimes A & \xrightarrow{\quad\quad\quad\quad \mu \quad\quad\quad\quad} & A \\
\downarrow{\delta\otimes\delta} & & \downarrow{\delta} \\
(A \odot A) \otimes (A \odot A) \xrightarrow[z_{A,A,A,A}]{} (A \otimes A) \odot (A \otimes A) \xrightarrow[\mu\odot\mu]{} & & A \odot A.
\end{array}
$$

$$(11)$$

We would now like to generalize this 2-fold monoidal category definition to double categories, where there are two different horizontal compositions which are compatible in a way analagous to the two monoidal structures in a 2-fold monoidal category. We will start with a concise formal definition, and then expand on the definition more concretely.

Definition 4.2. A *2-fold double category* $\mathbb{D}$ with vertical category $\mathrm{Vert}(\mathbb{D}) = \mathscr{D}_0$ is a 2-fold monoid object in the 2-category $\mathcal{C}at/\mathscr{D}_0$ of categories over $\mathscr{D}_0$.

Breaking this down, we have a category $\mathscr{D}_1$, a functor $p\colon \mathscr{D}_1 \to \mathscr{D}_0$, two functors $\otimes, \odot\colon \mathscr{D}_1 \times_{\mathscr{D}_0} \mathscr{D}_1 \to \mathscr{D}_1$ commuting with p, and two functors $I, \bot\colon \mathscr{D}_0 \to \mathscr{D}_1$ which are sections of p, such that $\otimes$, $\odot$, I, and $\bot$ satisfy all the axioms of a 2-fold monoidal category. In particular, each fiber of p has a 2-fold monoidal structure.

A monoid object in $\mathcal{C}at/\mathbb{D}_0$ is equivalently a double category where the source and target functors $s, t\colon \mathbb{D}_1 \to \mathbb{D}_0$ are equal, and with the vertical category $\mathbb{D}_0$. Conversely, any double category $\mathbb{D}$ in which all horizontal 1-cells have equal domain and codomain, and all 2-cells have equal vertical 1-cells as domain and codomain, is equivalently a monoid object in $\mathcal{C}at/\mathbb{D}_0$. We will alternate between these two descriptions as convenient.

Using this shift of perspective, $\mathbb{D}$ has two underlying double categories, both with vertical category $\mathscr{D}_0$ and with source and target functors both equal to $p\colon \mathscr{D}_1 \to \mathscr{D}_0$. The double category $\mathbb{D}_{\otimes}$ has the rest of the double category structure given by the functors I and $\otimes$, while the double category $\mathbb{D}_{\odot}$ uses the functors $\bot$ and $\odot$.

Using this double category interpretation, we will find it convenient to think of a 2-fold double category as a double category with two different but interacting horizontal compositions. Notice that from this perspective, all horizontal 1-cells are endomorphisms.

Remark 4.3. It may seem somewhat ad hoc to force a 2-fold monoid object in a slice of $\mathcal{C}$ into a double category mold, with the odd looking restriction to having only endomorphisms in the horizontal direction. We will make essential use of double functors from $\mathbb{D}_\odot$ and $\mathbb{D}_\otimes$ to genuine double categories (without the endomorphism restriction), and it is mostly for this reason that we have found the double categorical perspective useful, if perhaps only psychologically.

We did give some thought to how one might define a 2-fold double category with non-endomorphism horizontal 1-cells and 2-cells, and while it seems like there might be a workable definition, it would require a very large increase in complexity. As we are mostly interested in the monads and comonads in a 2-fold double category, which are structures on endomorphism horizontal 1-cells, this restriction was of no concern to this work.

Now let us explicitly look at the 2-fold monoidal structure from the double categorical perspective. For any object C there are 2-cells

$$
\begin{array}{ccc}
C \xrightarrow{\perp_C \otimes \perp_C} C & C \xrightarrow{I_C} C & C \xrightarrow{I_C} C \\
\Big\| \quad \Downarrow m \quad \Big\| & \Big\| \quad \Downarrow c \quad \Big\| & \Big\| \quad \Downarrow j \quad \Big\| \\
C \xrightarrow{\perp_C} C & C \xrightarrow{I_C \odot I_C} C & C \xrightarrow{\perp_C} C
\end{array}
\tag{12}
$$

and for any four horizontal morphisms $W, X, Y, Z : C \nrightarrow C$ there is a 2-cell

$$
\begin{array}{c}
C \xrightarrow{(W \odot X) \otimes (Y \odot Z)} C \\
\Big\| \quad \Downarrow z \quad \Big\| \\
C \xrightarrow[(W \otimes Y) \odot (X \otimes Z)]{} C.
\end{array}
\tag{13}
$$

These are natural in the sense that, for any vertical morphism $f : C \to D$ we have an equality

$$
\begin{array}{ccc}
C \xrightarrow{\perp_C \otimes \perp_C} C & & C \xrightarrow{\perp_C \otimes \perp_C} C \\
\Big\| \quad \Downarrow m \quad \Big\| & & f\Big\downarrow \quad \Downarrow \perp_f \otimes \perp_f \quad \Big\downarrow f \\
C \xrightarrow{\perp_C} C & = & D \xrightarrow{\perp_D \otimes \perp_D} D \\
f\Big\downarrow \quad \Downarrow \perp_f \quad \Big\downarrow f & & \Big\| \quad \Downarrow m \quad \Big\| \\
D \xrightarrow{\perp_D} D & & D \xrightarrow{\perp_D} D
\end{array}
$$

and similarly for c and j, and for any four 2-cells $\theta_1, \ldots, \theta_4$ of the apropriate

form, we have an equality

$$
\begin{array}{ccc}
\begin{array}{ccc}
C & \xrightarrow{(W\odot X)\otimes(Y\odot Z)} & C \\
\| & \Downarrow z & \| \\
C & \xrightarrow{(W\otimes Y)\odot(X\otimes Z)} & C \\
f \downarrow & \Downarrow (\theta_1\otimes\theta_3)\odot(\theta_2\otimes\theta_4) & \downarrow f \\
D & \xrightarrow{(W'\otimes Y')\odot(X'\otimes Z')} & D
\end{array}
& = &
\begin{array}{ccc}
C & \xrightarrow{(W\odot X)\otimes(Y\odot Z)} & C \\
f\downarrow & \Downarrow (\theta_1\odot\theta_2)\otimes(\theta_3\odot\theta_4) & \downarrow f \\
C & \xrightarrow{(W'\odot X')\otimes(Y'\odot Z')} & C \\
\| & \Downarrow z & \| \\
D & \xrightarrow{(W'\otimes Y')\odot(X'\otimes Z')} & D
\end{array}
\end{array}
$$

Definition 4.4. A monad in a 2-fold double category $\mathbb{D}$ is a monad in $\mathbb{D}_\otimes$; a comonad in $\mathbb{D}$ is a comonad in $\mathbb{D}_\odot$. Furthermore, we define the categories $\mathrm{Mon}(\mathbb{D}) = \mathrm{Mon}(\mathbb{D}_\otimes)$ and $\mathrm{Comon}(\mathbb{D}) = \mathrm{Comon}(\mathbb{D}_\odot)$.

So a monad X and a comonad Y in $\mathbb{D}$ are given by 2-cells

$$
\begin{array}{ccc}
C & \xrightarrow{I_C} & C \\
\| & \Downarrow \eta & \| \\
C & \xrightarrow{X} & C
\end{array}
\qquad
\begin{array}{ccc}
C & \xrightarrow{X\otimes X} & C \\
\| & \Downarrow \mu & \| \\
C & \xrightarrow{X} & C
\end{array}
\qquad
\begin{array}{ccc}
C & \xrightarrow{X} & C \\
\| & \Downarrow \epsilon & \| \\
C & \xrightarrow{\perp_C} & C
\end{array}
\qquad
\begin{array}{ccc}
C & \xrightarrow{X} & C \\
\| & \Downarrow \delta & \| \\
C & \xrightarrow{X\odot X} & C.
\end{array}
$$

The categories $\mathrm{Mon}(\mathbb{D})$ and $\mathrm{Comon}(\mathbb{D})$ come naturally equipped with functors to $\mathcal{D}_0$, defined on objects and morphisms simply by applying p to the underlying 1-cells and 2-cells respectively. It turns out that the interaction between the $\otimes$ and $\odot$ compositions in the 2-fold double category structure is precisely what is needed to lift $\odot$ to $\mathrm{Mon}(\mathbb{D})$ and to lift $\otimes$ to $\mathrm{Comon}(\mathbb{D})$. In this way, we can define double categories $\mathbb{Mon}(\mathbb{D})$ and $\mathbb{Comon}(\mathbb{D})$, both having $\mathcal{D}_0$ as vertical category.

These lifted compositions are defined as follows: Given two monads (C, X, η, μ) and (C, Y, η', μ') in $\mathbb{D}$, the horizontal composition

$$
C \xrightarrow{(X,\eta,\mu)} C \xrightarrow{(Y,\eta',\mu')} C
$$

is the monoid with underlying horizontal 1-cell $X \odot Y$ and unit and multiplication 2-cells

$$
\begin{array}{ccc}
C & \xrightarrow{I_C} & C \\
\| & \Downarrow c & \| \\
C & \xrightarrow{I_C\odot I_C} & C \\
\| & \Downarrow \eta\odot\eta' & \| \\
C & \xrightarrow{X\odot Y} & C
\end{array}
\qquad
\begin{array}{ccc}
C & \xrightarrow{(X\odot Y)\otimes(X\odot Y)} & C \\
\| & \Downarrow z & \| \\
C & \xrightarrow{(X\otimes X)\odot(Y\otimes Y)} & C \\
\| & \Downarrow \mu\odot\mu' & \| \\
C & \xrightarrow{X\odot Y} & C.
\end{array}
$$

The unit for this composition is I_C, given the trivial monad structure with $\eta = \mu = \mathrm{id}_{I_C}$.

Similarly, the horizontal composition of two 2-cells in $\mathbb{Mon}(\mathbb{D})$ is the $\odot$ product of the underlying 2-cells in $\mathbb{D}$. The fact that this commutes with the unit and multiplication defined above follows from the naturality of c and z.

In this same way, we can define the horizontal composition of two 1-cells (X, ϵ, δ) and (Y, ϵ', δ') in $\mathbb{Comon}(\mathbb{D})$ to be a comonad with underlying horizontal 1-cell $X \otimes Y$, with horizontal unit $\perp$ with the trivial comonad structure.

This allows us to define (ordinary) categories $\mathrm{Mon}(\mathbb{Comon}(\mathbb{D}))$ and $\mathrm{Comon}(\mathbb{Mon}(\mathbb{D}))$. Furthermore, these two categories are equivalent, leading to the next definition.

Definition 4.5. A *bimonad* in a 2-fold double category $\mathbb{D}$ is a monad in $\mathbb{Comon}(\mathbb{D})$, or equivalently a comonad in $\mathbb{Mon}(\mathbb{D})$. We can define a category of bimonads in $\mathbb{D}$ as

$$\mathrm{Bimon}(\mathbb{D}) := \mathrm{Mon}(\mathbb{Comon}(\mathbb{D})) \simeq \mathrm{Comon}(\mathbb{Mon}(\mathbb{D}))$$

Concretely, a bimonad in $\mathbb{D}$ is a tuple $(X, \eta, \mu, \epsilon, \delta)$ where X is a horizontal 1-cell, (X, η, μ) is a monad and (X, ϵ, δ) is a comonad as above, such that four equations hold:

$$(14)$$

A bimonoid morphism is simply a 2-cell which is simultaneously a monoid morphism and a comonoid morphism.

5 Cyclic 2-fold Double Categories

Recall the notion of a cyclic double category from [CGR12]. A cyclic double category $\mathbb{D}$ is a double category with an extra involutive operation. On objects and horizontal 1-cells $X\colon C \nrightarrow C$, this operation is written

$$C_1^\bullet \xrightarrow{\;X^\bullet\;} C_2^\bullet$$

and respects horizontal identities and composition. The involution takes any vertical 1-cell $f\colon C \to D$ to some $\sigma f\colon D^\bullet \to C^\bullet$, and any 2-cell

$$
\begin{array}{ccc}
C_1 \xrightarrow{\;X\;} C_2 & & D_1^\bullet \xrightarrow{\;Y^\bullet\;} D_2^\bullet \\
f\downarrow \quad \Downarrow\theta \quad \downarrow g & \text{to} & \sigma f\downarrow \quad \Downarrow\sigma\theta \quad \downarrow \sigma g \\
D_1 \xrightarrow[\;Y\;]{} D_2 & & C_1^\bullet \xrightarrow[\;X^\bullet\;]{} C_2^\bullet
\end{array}
$$

respecting vertical identities and composition.

Example 5.1. Let $\mathcal{D}$ be any 2-category with an involution $(-)^\bullet\colon\mathcal{D}^{\mathrm{co}} \to \mathcal{D}$, such as $\mathcal{C}at$ with $(-)^{\mathrm{op}}$. The double category $\mathbb{S}q(\mathcal{D})$ has a sub-double category $\mathbb{L}\mathbf{Adj}(\mathcal{D})$, where the vertical 1-cells are restricted to the adjunctions—pointing in the direction of the left adjoint, say—but with no other restrictions. This double category has a natural cyclic action: if $F \dashv G$ is an adjunction, then $\sigma F = G^\bullet$, the left adjoint of the adjunction $G^\bullet \dashv F^\bullet$, and if θ is a 2-cell with mate ϕ then $\sigma\theta = \phi^\bullet$. For a clear summary of the mates correspondence and the cyclic action on $\mathbb{L}\mathbf{Adj}$, see [CGR12] Section 1.

Proposition 5.2. *Let* $\mathbb{D}$ *be a cyclic double category with arrow objects. For any object C, $(C^\bullet)^2 = (C^2)^\bullet$, as witnessed by*

$$
(C^2)^\bullet \quad \underset{\mathrm{dom}^\bullet}{\overset{\mathrm{cod}^\bullet}{\rightrightarrows}} \quad \Downarrow\sigma\kappa \quad C^\bullet
$$

For any vertical 1-cell F, the lift to arrow objects of $F^\bullet$ is $(\hat{F})^\bullet$, as witnessed by the 2-cells

$$
\begin{array}{ccc}
(D^2)^\bullet \xrightarrow{\;\mathrm{cod}^\bullet\;} D^\bullet & \qquad & (D^2)^\bullet \xrightarrow{\;\mathrm{dom}^\bullet\;} D^\bullet \\
(\hat{F})^\bullet\downarrow \quad \Downarrow\sigma\gamma_0 \quad \downarrow F^\bullet & & (\hat{F})^\bullet\downarrow \quad \Downarrow\sigma\gamma_1 \quad \downarrow F^\bullet \\
(C^2)^\bullet \xrightarrow[\;\mathrm{cod}^\bullet\;]{} C^\bullet & & (C^2)^\bullet \xrightarrow[\;\mathrm{dom}^\bullet\;]{} C^\bullet
\end{array}
$$

Proof. It is a very simple matter to verify the universal properties of Section 3.1 $\qquad\square$

We will generalize this to a cyclic action on a 2-fold double category. Suppose that $\mathbb{D}$ is a 2-fold double category. A cyclic action, written as above, must satisfy the following:

- For every object C,

$$I_{C^\bullet} = (\bot_C)^\bullet \qquad \text{and} \qquad \bot_{C^\bullet} = (I_C)^\bullet.$$

- For every composable pair of horizontal 1-cells $\quad X, Y: C \nrightarrow C \quad$,

$$(X \otimes Y)^\bullet = X^\bullet \odot Y^\bullet \qquad \text{and} \qquad (X \odot Y)^\bullet = X^\bullet \otimes Y^\bullet$$

- For every vertical 1-cell $f: C \to D$, there are equalities

$$
\begin{array}{ccc}
D^\bullet \xrightarrow{I_{D^\bullet}} D^\bullet & & D^\bullet \xrightarrow{(\bot_D)^\bullet} D^\bullet \\
\sigma f \downarrow \quad \Downarrow I_{\sigma f} \quad \downarrow \sigma f & = & \sigma f \downarrow \quad \Downarrow \sigma \bot_f \quad \downarrow \sigma f \\
C^\bullet \xrightarrow[I_{C^\bullet}]{} C^\bullet & & C^\bullet \xrightarrow[(\bot_C)^\bullet]{} C^\bullet
\end{array}
$$

$$
\begin{array}{ccc}
D^\bullet \xrightarrow{\bot_{D^\bullet}} D^\bullet & & D^\bullet \xrightarrow{(I_D)^\bullet} D^\bullet \\
\sigma f \downarrow \quad \Downarrow \bot_{\sigma f} \quad \downarrow \sigma f & = & \sigma f \downarrow \quad \Downarrow \sigma I_f \quad \downarrow \sigma f \\
C^\bullet \xrightarrow[\bot_{C^\bullet}]{} C^\bullet & & C^\bullet \xrightarrow[(I_C)^\bullet]{} C^\bullet
\end{array}
$$

- For every horizontally composable pair of 2-cells

$$
\begin{array}{ccccc}
C & \xrightarrow{X} & C & \xrightarrow{Y} & C \\
f \downarrow & \Downarrow \theta & \downarrow f & \Downarrow \phi & \downarrow f \\
D & \xrightarrow[X']{} & D & \xrightarrow[Y']{} & D
\end{array}
$$

there are equalities

$$
\begin{array}{ccc}
D^\bullet \xrightarrow{(X' \otimes Y')^\bullet} D^\bullet & & D^\bullet \xrightarrow{X'^\bullet \odot Y'^\bullet} D^\bullet \\
\sigma f \downarrow \quad \Downarrow \sigma(\theta \otimes \phi) \quad \downarrow \sigma f & = & \sigma f \downarrow \quad \Downarrow \sigma(\theta) \odot \sigma(\phi) \quad \downarrow \sigma f \\
C^\bullet \xrightarrow[(X \otimes Y)^\bullet]{} C^\bullet & & C^\bullet \xrightarrow[X^\bullet \odot Y^\bullet]{} C^\bullet
\end{array}
$$

$$
\begin{array}{ccc}
D^\bullet \xrightarrow{(X' \odot Y')^\bullet} D^\bullet & & D^\bullet \xrightarrow{X'^\bullet \otimes Y'^\bullet} D^\bullet \\
\sigma f \downarrow \quad \Downarrow \sigma(\theta \odot \phi) \quad \downarrow \sigma f & = & \sigma f \downarrow \quad \Downarrow \sigma(\theta) \otimes \sigma(\phi) \quad \downarrow \sigma f \\
C^\bullet \xrightarrow[(X \odot Y)^\bullet]{} C^\bullet & & C^\bullet \xrightarrow[X^\bullet \otimes Y^\bullet]{} C^\bullet
\end{array}
$$

One nice consequence of this definition is that a cyclic action on a 2-fold double category $\mathbb{D}$ induces a cyclic action on the category of bimonoids $\mathrm{Bimon}(\mathbb{D})$.

Proposition 5.3. *Suppose $\mathbb{D}$ is a cyclic 2-fold double category. Then the category* $\mathrm{Bimon}(\mathbb{D})$ *of bimonoids in $\mathbb{D}$ carries a natural cyclic action (contravariant isomorphism).*

Proof. The involution $(-)^\bullet$ gives an isomorphism of double categories $\mathbb{D}_\otimes \cong \mathbb{D}_\odot^{\mathrm{op}}$. Therefore it also induces an isomorphism

$$\mathbb{M}\mathrm{on}(\mathbb{D}) = \mathbb{M}\mathrm{on}(\mathbb{D}_\otimes) \cong \mathbb{M}\mathrm{on}(\mathbb{D}_\odot^{\mathrm{op}}) \cong \mathbb{C}\mathrm{omon}(\mathbb{D}_\odot)^{\mathrm{op}} = \mathbb{C}\mathrm{omon}(\mathbb{D})^{\mathrm{op}}$$

as well as an isomorphism

$$\mathrm{Bimon}(\mathbb{D}) = \mathbb{C}\mathrm{omon}(\mathbb{M}\mathrm{on}(\mathbb{D})) \cong \mathbb{C}\mathrm{omon}(\mathbb{C}\mathrm{omon}(\mathbb{D})^{\mathrm{op}})$$
$$\cong \mathbb{M}\mathrm{on}(\mathbb{C}\mathrm{omon}(\mathbb{D}))^{\mathrm{op}} = \mathrm{Bimon}(\mathbb{D})^{\mathrm{op}}.$$

$\square$

In more concrete terms, the involution takes a bimonoid $(X, \eta, \mu, \epsilon, \delta)$ to $(X, \eta, \mu, \epsilon, \delta)^\bullet = (X^\bullet, \epsilon^\bullet, \delta^\bullet, \eta^\bullet, \delta^\bullet)$, swapping the monoid and comonoid structures. This is again a bimonoid, as the top two equations of (14) are interchanged under the involution, while the bottom two equations are self-dual.

The action of the involution on bimonoid morphisms can be broken down as in the following lemma.

Lemma 5.4. *Let $(X, \eta, \mu, \epsilon, \delta)$ and $(Y, \eta', \mu', \epsilon', \delta')$ be bimonoids in a cyclic 2-fold double category $\mathbb{D}$, and let ϕ be a 2-cell in $\mathbb{D}$*

$$
\begin{array}{ccc}
C & \xrightarrow{\ X\ } & C \\
f \downarrow & \Downarrow \phi & \downarrow f \\
D & \xrightarrow[\ Y\ ]{} & D.
\end{array}
$$

Then (f, ϕ) is a monoid morphism $X \to Y$ if and only if $(\sigma f, \phi^\bullet)$ is a comonoid morphism $Y^\bullet \to X^\bullet$. Dually, ϕ is a comonoid morphism $X \to Y$ if and only if $\phi^\bullet$ is a monoid morphism $Y^\bullet \to X^\bullet$.

Proof. Simply notice that the involution takes equations (9) and (10) to the equations defining a comonad morphism in $\mathbb{D}$. $\square$

This immediately implies a useful characterization of bimonoid morphisms.

Corollary 5.5. *Given bimonoids $(X, \eta, \mu, \epsilon, \delta)$ and $(Y, \eta', \mu', \epsilon', \delta')$ in a cyclic 2-fold double category $\mathbb{D}$, a bimonoid morphism $X \to Y$ consists of:*
- *Either a monoid morphism $X \to Y$ or a comonoid morphism $Y^\bullet \to X^\bullet$, and*
- *Either a comonoid morphism $X \to Y$ or a monoid morphism $Y^\bullet \to X^\bullet$.*

6 Functorial Factorizations

Let $\mathbb{D}$ be a cyclic double category, and assume it has arrow objects in the sense of Section 3.1. In this section, we will define a 2-fold double category $\mathbb{FF}(\mathbb{D})$ of functorial factorizations in $\mathbb{D}$, as follows:

- The objects and vertical 1-cells are the same as in $\mathbb{D}$.

- Horizontal 1-cells $\quad C \rightarrowtail C \quad$ in $\mathbb{FF}(\mathbb{D})$ are tuples (E, η, ϵ), where $E \colon C^2 \to$ ▮
  C is a horizontal 1-cell in $\mathbb{D}$, and

$$C^2 \underset{E}{\overset{\mathrm{dom}}{\Downarrow \eta}} C \qquad\qquad C^2 \underset{\mathrm{cod}}{\overset{E}{\Downarrow \epsilon}} C$$

  are 2-cells in $\mathbb{D}$ such that

$$C^2 \overset{\Downarrow \eta}{\underset{\Downarrow \epsilon}{- E \to}} C \;=\; C^2 \underset{\mathrm{cod}}{\overset{\mathrm{dom}}{\Downarrow \kappa}} C.$$

  By the universal property of C^2, this also determines horizontal 1-cells $L, R \colon C^2 \to C^2$ such that $\mathrm{dom} \circ L = \mathrm{dom}$, $\mathrm{cod} \circ R = \mathrm{cod}$, $\mathrm{cod} \circ L = \mathrm{dom} \circ R = E$, $\kappa \circ L = \eta$, and $\kappa \circ R = \epsilon$, and 2-cells

$$C^2 \underset{\mathrm{id}}{\overset{L}{\Downarrow \vec{\epsilon}}} C^2. \qquad\qquad C^2 \underset{R}{\overset{\mathrm{id}}{\Downarrow \vec{\eta}}} C^2.$$

  such that $\mathrm{dom} \circ \vec{\epsilon} = \mathrm{id}_{\mathrm{dom}}$, $\mathrm{cod} \circ \vec{\epsilon} = \epsilon$, $\mathrm{dom} \circ \vec{\eta} = \eta$, and $\mathrm{cod} \circ \vec{\eta} = \mathrm{id}_{\mathrm{cod}}$.

- The horizontal composition $(E_1, \eta_1, \epsilon_1) \otimes (E_2, \eta_2, \epsilon_2)$ of two horizontal 1-cells

$$C \xrightarrow{(E_1, \eta_1, \epsilon_1)} C \xrightarrow{(E_2, \eta_2, \epsilon_2)} C$$

  in $\mathbb{FF}(\mathbb{D})$ is a horizontal 1-cell $(E_{1 \otimes 2}, \eta_{1 \otimes 2}, \epsilon_{1 \otimes 2})$, where

$$E_{1 \otimes 2} = \quad C^2 \xrightarrow{R_1} C^2 \xrightarrow{E_2} C$$

$$\eta_{1 \otimes 2} = \quad C^2 \underset{R_1}{\overset{\mathrm{id}}{\Downarrow \vec{\eta_1}}} C^2 \underset{E_2}{\overset{\mathrm{dom}}{\Downarrow \eta_2}} C$$

$$\epsilon_{1 \otimes 2} = \quad C^2 \xrightarrow{R_1} C^2 \underset{\mathrm{cod}}{\overset{E_2}{\Downarrow \epsilon_2}} C$$

  which also determines that $R_{1 \otimes 2} = R_2 \circ R_1$.

- The horizontal unit I_C for $\otimes$ is $(\mathrm{dom}, \mathrm{id}, \kappa)$.
- The second horizontal composition $(E_1, \eta_1, \epsilon_1) \odot (E_2, \eta_2, \epsilon_2)$ is a horizontal 1-cell $(E_{1 \odot 2}, \eta_{1 \odot 2}, \epsilon_{1 \odot 2})$, where

$$E_{1 \odot 2} = \quad C^2 \xrightarrow{L_1} C^2 \xrightarrow{E_2} C$$

$$\eta_{1 \odot 2} = \quad C^2 \xrightarrow{L_1} C^2 \overset{\mathrm{dom}}{\underset{E_2}{\Downarrow \eta_2}} C$$

$$\epsilon_{1 \odot 2} = \quad C^2 \overset{L_1}{\underset{\mathrm{id}}{\Downarrow \vec{\epsilon_1}}} C^2 \overset{E_2}{\underset{\mathrm{cod}}{\Downarrow \epsilon_2}} C$$

which also determines that $L_{1 \odot 2} = L_2 \circ L_1$.

- The horizontal unit $\perp_C$ for $\odot$ is $(\mathrm{cod}, \kappa, \mathrm{id})$.
- 2-cells

$$
\begin{array}{ccc}
C & \xrightarrow{(E_1, \eta_1, \epsilon_1)} & C \\
F \downarrow & \Downarrow \theta & \downarrow F \\
D & \xrightarrow[(E_2, \eta_2, \epsilon_2)]{} & D
\end{array}
$$

in $\mathbb{FF}(\mathbb{D})$ are given by 2-cells

$$
\begin{array}{ccc}
C^2 & \xrightarrow{E_1} & C \\
\hat{F} \downarrow & \Downarrow \theta & \downarrow F \\
D^2 & \xrightarrow[E_2]{} & D
\end{array}
$$

in $\mathbb{D}$ such that

$$
\begin{array}{ccc}
C^2 \xrightarrow{E_1} C & & C^2 \overset{E_1}{\underset{\mathrm{cod}}{\Downarrow \epsilon_1}} C \\
\hat{F} \downarrow \ \Downarrow \theta \ \downarrow F & = & \hat{F} \downarrow \quad \Downarrow \gamma_0 \quad \downarrow F \\
D^2 \underset{\mathrm{cod}}{\overset{E_2}{\underset{\Downarrow \epsilon_2}{\longrightarrow}}} D & & D^2 \xrightarrow[\mathrm{cod}]{} D
\end{array}
\tag{15}
$$

and

$$
\begin{array}{ccc}
C^2 \xrightarrow{\ \text{dom}\ } C & & C^2 \xrightarrow{\ E_1\ } C \\
\ \Big\downarrow{\scriptstyle\hat{F}} \quad \Downarrow\gamma_1 \quad \Big\downarrow{\scriptstyle F} & = & \ \Big\downarrow{\scriptstyle\hat{F}} \quad \Downarrow\theta \quad \Big\downarrow{\scriptstyle F} \\
D^2 \xrightarrow[\ \text{dom}\ ]{} D & & D^2 \xrightarrow[\ E_2\ ]{} D
\end{array}
\qquad (16)
$$

This also determines unique 2-cells

$$
\begin{array}{ccc}
C^2 \xrightarrow{\ R_1\ } C^2 & & C^2 \xrightarrow{\ L_1\ } C^2 \\
\Big\downarrow{\scriptstyle\hat{F}} \ \ \Downarrow\theta^R \ \ \Big\downarrow{\scriptstyle\hat{F}} & \text{and} & \Big\downarrow{\scriptstyle\hat{F}} \ \ \Downarrow\theta^L \ \ \Big\downarrow{\scriptstyle\hat{F}} \\
D^2 \xrightarrow[\ R_2\ ]{} D^2 & & D^2 \xrightarrow[\ L_2\ ]{} D^2
\end{array}
$$

such that composing horizontally with γ_0 or γ_1 gives γ_0, γ_1, or θ as appropriate. For instance:

$$
\begin{array}{ccccc}
C^2 \xrightarrow{\ R_1\ } C^2 \xrightarrow{\ \text{dom}\ } C & & & C^2 \xrightarrow{\ E_1\ } C \\
\Big\downarrow{\scriptstyle\hat{F}} \ \Downarrow\theta^R \ \Big\downarrow{\scriptstyle\hat{F}} \ \Downarrow\gamma_1 \ \Big\downarrow{\scriptstyle F} & = & & \Big\downarrow{\scriptstyle\hat{F}} \ \ \Downarrow\theta \ \ \Big\downarrow{\scriptstyle F} \\
D^2 \xrightarrow[\ R_2\ ]{} D^2 \xrightarrow[\ \text{dom}\ ]{} D & & & D^2 \xrightarrow[\ E_2\ ]{} D
\end{array}
$$

- Given a pair of composable 2-cells in $\mathbb{FF}(\mathbb{D})$ as in

$$
\begin{array}{ccc}
C \xrightarrow{\ (E_1,\eta_1,\epsilon_1)\ } C \xrightarrow{\ (E_2,\eta_2,\epsilon_2)\ } C \\
\Big\downarrow{\scriptstyle F} \ \ \Downarrow\theta_1 \ \ \Big\downarrow{\scriptstyle F} \ \ \Downarrow\theta_2 \ \ \Big\downarrow{\scriptstyle F} \\
D \xrightarrow[\ (E_1',\eta_1',\epsilon_1')\ ]{} D \xrightarrow[\ (E_2',\eta_2',\epsilon_2')\ ]{} D
\end{array}
$$

the composite $\theta_1 \otimes \theta_2$ is given by

$$
\begin{array}{ccc}
C^2 \xrightarrow{\ R_1\ } C^2 \xrightarrow{\ E_2\ } C \\
\Big\downarrow{\scriptstyle\hat{F}} \ \ \Downarrow\theta_1^R \ \ \Big\downarrow{\scriptstyle\hat{F}} \ \ \Downarrow\theta_2 \ \ \Big\downarrow{\scriptstyle F} \\
D^2 \xrightarrow[\ R_1'\ ]{} D^2 \xrightarrow[\ E_2'\ ]{} D
\end{array}
$$

while the composite $\theta_1 \odot \theta_2$ is given by

$$
\begin{array}{ccccc}
C^2 & \xrightarrow{L_1} & C^2 & \xrightarrow{E_2} & C \\
\hat{F} \downarrow & \Downarrow\theta_1^L & \hat{F} \downarrow & \Downarrow\theta_2 & \downarrow F \\
D^2 & \xrightarrow{L_1'} & D^2 & \xrightarrow{E_2'} & D
\end{array}
$$

It is a straightforward exercise to check that these definitions satisfy equations (15) and (16). To illustrate, we will demonstrate that $\theta_1 \otimes \theta_2$ satisfies (15):

$$
\begin{array}{cc}
\begin{array}{ccc}
C^2 & \xrightarrow{E_{1\otimes 2}} & C \\
\hat{F}\downarrow & \Downarrow\theta_1\otimes\theta_2 & \downarrow F \\
D^2 & \xrightarrow{E_{1'\otimes 2'}} & D \\
& \underset{\Downarrow\epsilon_{1'\otimes 2'}}{\xrightarrow{\quad}} & \\
& \text{cod} &
\end{array}
&
\begin{array}{ccccc}
C^2 & \xrightarrow{R_1} & C^2 & \xrightarrow{E_2} & C \\
\hat{F}\downarrow & \Downarrow\theta_1^R & \hat{F}\downarrow & \Downarrow\theta_2 & \downarrow F \\
D^2 & \xrightarrow{R_1'} & D^2 & \xrightarrow{E_2'} & D \\
& & & \underset{\Downarrow\epsilon_2'}{\xrightarrow{\quad}} & \\
& & & \text{cod} &
\end{array}
\end{array}
$$

$$
=
\begin{array}{ccccc}
 & & & \xrightarrow{E_2} & \\
C^2 & \xrightarrow{R_1} & C^2 & \Downarrow\epsilon_2 & C \\
 & & & \xrightarrow{\text{cod}} & \\
\hat{F}\downarrow & \Downarrow\theta_1^R & \hat{F}\downarrow & \Downarrow\gamma_0 & \downarrow F \\
D^2 & \xrightarrow{R_1'} & D^2 & \xrightarrow{\text{cod}} & D
\end{array}
$$

$$
=
\begin{array}{ccc}
 & \xrightarrow{E_{1\otimes 2}} & \\
C^2 & \Downarrow\epsilon_{1\otimes 2} & C \\
 & \xrightarrow{\text{cod}} & \\
\hat{F}\downarrow & \Downarrow\gamma_0 & \downarrow F \\
D^2 & \xrightarrow{\text{cod}} & D
\end{array}
$$

Example 6.1. Functorial factorizations in the double category $\mathbb{D} = \mathcal{S}\mathrm{q}(\mathcal{C}\mathrm{at})$ of squares in the 2-category of categories are precisely functorial factorizations as defined in Section 2.2.

It is straightforward to check that $\otimes$ and $\odot$ are each associative and unital. It takes more work to provide the compatibility between $\otimes$ and $\odot$, which is the content of the proof of the next proposition.

Proposition 6.2. $\mathbb{FF}(\mathbb{D})$ *has the structure of a 2-fold double category.*

Proof. The primary structure of $\mathbb{FF}(\mathbb{D})$ was given in the first part of this section. What is left is to provide the coherence data (12) and (13).

First, note that I_C is initial in the sense that, given any vertical morphism $F: C \to D$ and any functorial factorization (E, η, ϵ) on D, there is a unique 2-cell

$$
\begin{array}{ccc}
C & \xrightarrow{I_C} & C \\
F\downarrow & \Downarrow & \downarrow F \\
D & \xrightarrow[(E,\eta,\epsilon)]{} & D
\end{array}
$$

given by

$$
\begin{array}{ccc}
C^2 & \xrightarrow{\mathrm{dom}} & C \\
\hat{F}\downarrow & \Downarrow\gamma_1 & \downarrow F \\
D^2 & \xrightarrow{\mathrm{dom}} & D. \\
& \underset{E}{\searrow\ \Downarrow\eta\ \nearrow} &
\end{array}
$$

Similarly, $\perp_C$ is terminal. Thus there is only one possible way to define the 2-cells m, c, and j, and naturality and all other coherence equations follows immediately from this uniqueness.

We still need to construct the 2-cell z, which will take some work. We begin by defining 2-cells

$$
\begin{array}{ccc}
C & \xrightarrow{E_1\odot E_2} & C \\
\| & \Downarrow p_{E_1,E_2} & \| \\
C & \xrightarrow[E_1]{} & C
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
C & \xrightarrow{E_1} & C \\
\| & \Downarrow i_{E_1,E_2} & \| \\
C & \xrightarrow[E_1\otimes E_2]{} & C.
\end{array}
$$

for any pair of functorial factorizations. The 2-cell p is given by the underlying 2-cell in $\mathbb{D}$

$$
C^2 \xrightarrow{L_1} C^2 \overset{E_2}{\underset{\mathrm{cod}}{\Downarrow\epsilon_2}} C
$$

and i is given by

$$
C^2 \xrightarrow{R_1} C^2 \overset{\mathrm{dom}}{\underset{E_2}{\Downarrow\eta_2}} C.
$$

To illustrate the verification that these give well-defined 2-cells in $\mathbb{FF}(\mathbb{D})$, we will show that i satisfies (15) (keep in mind that when F is an identity, γ_0 and

γ_1 are also identities):

$$
C^2 \xrightarrow{L_1} C^2 \overset{\text{dom}}{\underset{\text{cod}}{\overset{\Downarrow\eta_2}{\underset{\Downarrow\epsilon_2}{E_2}}}} C \;\; = \;\; C^2 \xrightarrow{L_1} C^2 \overset{\text{dom}}{\underset{\text{cod}}{\Downarrow\kappa}} C
$$

$$
= \;\; C^2 \overset{\text{dom}}{\underset{E_1}{\Downarrow\eta_1}} C.
$$

Moreover, it is straightforward to check that i and p are natural families of 2-cells. Specifically, for any pair of 2-cells θ_1 and θ_2

$$
\begin{array}{ccc}
C \xrightarrow{E_1 \odot E_2} C & & C \xrightarrow{E_1 \odot E_2} C \\
\| \quad \Downarrow p_{E_1, E_2} \quad \| & & F \downarrow \quad \Downarrow \theta_1 \odot \theta_2 \quad \downarrow F \\
C \xrightarrow{E_1} C & & D \xrightarrow{E_1' \odot E_2'} D \\
F \downarrow \quad \Downarrow \theta_1 \quad \downarrow F & = & \| \quad \Downarrow p_{E_1', E_2'} \quad \| \\
D \xrightarrow{E_1'} D & & D \xrightarrow{E_1'} D
\end{array}
$$

$$
\begin{array}{ccc}
C \xrightarrow{E_1} C & & C \xrightarrow{E_1} C \\
\| \quad \Downarrow i_{E_1, E_2} \quad \| & & F \downarrow \quad \Downarrow \theta_1 \quad \downarrow F \\
C \xrightarrow{E_1 \otimes E_2} C & & D \xrightarrow{E_1'} D \\
F \downarrow \quad \Downarrow \theta_1 \otimes \theta_2 \quad \downarrow F & = & \| \quad \Downarrow i_{E_1', E_2'} \quad \| \\
D \xrightarrow{E_1' \otimes E_2'} D & & D \xrightarrow{E_1' \otimes E_2'} D
\end{array}
$$

As with any 2-cell in $\mathbb{FF}(\mathbb{D})$, p and i induce 2-cells in $\mathbb{D}$

$$
C^2 \overset{R_{1\odot 2}}{\underset{R_1}{\Downarrow p^R}} C^2 \qquad \text{and} \qquad C^2 \overset{L_1}{\underset{L_{1\otimes 2}}{\Downarrow i^L}} C^2.
$$

such that

$$
C^2 \overset{R_{1\odot 2}}{\underset{R_1}{\Downarrow p^R}} C^2 \xrightarrow{\text{dom}} C \;\; = \;\; C^2 \xrightarrow{L_1} C^2 \overset{E_2}{\underset{\text{cod}}{\Downarrow\epsilon_2}} C \qquad (17)
$$

$$
C^2 \overset{L_1}{\underset{L_{1\otimes 2}}{\Downarrow i^L}} C^2 \xrightarrow{\text{cod}} C \;\; = \;\; C^2 \xrightarrow{R_1} C^2 \overset{\text{dom}}{\underset{E_2}{\Downarrow\eta_2}} C \qquad (18)
$$

Now suppose given three functorial factorizations E_1, E_2, E_3 on an object C. We define a 2-cell in $\mathbb{D}$

$$
\begin{array}{c}
C^2 \xrightarrow{R_{1\odot 2}} C^2 \xrightarrow{L_3} C^2 \\
\Downarrow w \\
C^2 \xrightarrow{L_{1\otimes 3}} C^2 \xrightarrow{R_2} C^2
\end{array}
$$

such that

$$
\begin{array}{c}
C^2 \xrightarrow{R_{1\odot 2}} C^2 \xrightarrow{L_3} C^2 \xrightarrow{\mathrm{dom}} C \\
\Downarrow w \\
C^2 \xrightarrow{L_{1\otimes 3}} C^2 \xrightarrow{R_2}
\end{array}
\quad = \quad
C^2 \underset{L_{1\otimes 3}}{\overset{L_1}{\Downarrow i^L}} C^2 \xrightarrow{E_2} C
\qquad (19)
$$

$$
\begin{array}{c}
C^2 \xrightarrow{R_{1\odot 2}} C^2 \xrightarrow{L_3} C^2 \xrightarrow{\mathrm{cod}} C \\
\Downarrow w \\
C^2 \xrightarrow{L_{1\otimes 3}} C^2 \xrightarrow{R_2}
\end{array}
\quad = \quad
C^2 \underset{R_1}{\overset{R_{1\odot 2}}{\Downarrow p^R}} C^2 \xrightarrow{E_3} C.
\qquad (20)
$$

Using the universal property for C^2, it suffices to check that

$$
C^2 \underset{L_{1\otimes 3}}{\overset{L_1}{\Downarrow i^L}} C^2 \underset{\mathrm{cod}}{\overset{E_2}{\Downarrow \epsilon_2}} C
\quad = \quad
C^2 \xrightarrow{R_{1\odot 2}} C^2 \underset{E_3}{\overset{\mathrm{dom}}{\Downarrow \eta_3}} C
$$

and a quick check using equations (17) and (18) shows that both are equal to

$$
\begin{array}{c}
C^2 \xrightarrow{L_1} C^2 \overset{E_2}{\underset{\mathrm{cod}}{\Downarrow \epsilon_2}} \\
C^2 \qquad \qquad C \\
R_1 \qquad C^2 \underset{E_3}{\overset{\mathrm{dom}}{\Downarrow \eta_3}}
\end{array}
$$

where the inner diamond is the equality $\mathrm{cod}\, L_1 = \mathrm{dom}\, R_1 = E_1$.

We also check that w is natural with respect to 2-cells in $\mathbb{FF}(\mathbb{D})$ in the following sense: given three 2-cells θ_1, θ_2, and θ_3, there is an equality

$$
\begin{array}{c}
C^2 \xrightarrow{R_{1\odot 2}} C^2 \xrightarrow{L_3} C^2 \\
\Downarrow w \quad R_2 \\
\hat{F} \downarrow \quad L_{1\otimes 3} \quad C^2 \qquad \hat{F} \downarrow \\
\Downarrow (\theta_1\otimes\theta_3)^L \quad \hat{F} \quad \Downarrow \theta_2^R \\
D^2 \xrightarrow{L'_{1\otimes 3}} D^2 \xrightarrow{R'_2} D^2
\end{array}
\quad = \quad
\begin{array}{c}
C^2 \xrightarrow{R_{1\odot 2}} C^2 \xrightarrow{L_3} C^2 \\
\hat{F} \uparrow \quad \hat{F} \downarrow \quad \Downarrow \theta_3^L \quad \hat{F} \downarrow \\
\hat{F} \downarrow \Downarrow (\theta_1\odot\theta_2)^R \quad D^2 \quad L'_3 \\
D^2 \xrightarrow{R'_{1\odot 2}} \Downarrow w \\
D^2 \xrightarrow{L'_{1\otimes 3}} D^2 \xrightarrow{R'_2} D^2.
\end{array}
$$

To verify this equation, it suffices to check equality upon right composition with γ_0 and γ_1. We will illustrate the γ_1 case, making use of the naturality of i:

$$
\begin{array}{c}
\text{(diagram)}
\end{array}
$$

Finally, given four functorial factorizations E_1, E_2, E_3, E_4 on an object C, we define the 2-cell

$$
\begin{array}{ccc}
C & \xrightarrow{(1\odot 2)\otimes(3\odot 4)} & C \\
\| & \Downarrow z_{1,2,3,4} & \| \\
C & \xrightarrow{(1\otimes 3)\odot(2\otimes 4)} & C
\end{array}
$$

in $\mathbb{FF}(\mathbb{D})$, where $(1 \odot 2)$ is shorthand for $(E_1, \eta_1, \epsilon_1) \odot (E_2, \eta_2, \epsilon_2)$, to have the underlying 2-cell in $\mathbb{D}$

$$
\begin{array}{c}
C^2 \xrightarrow{R_{1\odot 2}} C^2 \xrightarrow{L_3} \\
\Downarrow w \quad C^2 \xrightarrow{E_4} C. \\
L_{1\otimes 3} \searrow C^2 \nearrow R_2
\end{array}
$$

The naturality of z follows immediately from that of w, but we still need to check that this satisfies equations (15) and (16). We will leave the details to the reader, but note that (16) comes down to the verification of the equality

$$
\begin{array}{c}
\text{(diagram)}
\end{array}
$$

which follows from equation (19) and the fact that $\mathrm{dom} \circ i^L = \mathrm{id}_{\mathrm{dom}}$. □

Lemma 6.3. *There is a strict double functor* $R\colon \mathbb{FF}(\mathbb{D})_\otimes \to \mathbb{D}$ *whose behavior on 2-cells is*

$$
\begin{array}{ccc}
C \xrightarrow{(E_1,\eta_1,\epsilon_1)} C & & C^2 \xrightarrow{R_1} C^2 \\
F\downarrow \quad \Downarrow\theta \quad \downarrow F & \mapsto & \hat{F}\downarrow \quad \Downarrow\theta^R \quad \downarrow\hat{F} \\
D \xrightarrow[(E_2,\eta_2,\epsilon_2)]{} D & & D^2 \xrightarrow[R_2]{} D^2
\end{array}
$$

and a double functor $L\colon \mathbb{FF}(\mathbb{D})_\odot \to \mathbb{D}$ *whose behavior on 2-cells is*

$$
\begin{array}{ccc}
C \xrightarrow{(E_1,\eta_1,\epsilon_1)} C & & C^2 \xrightarrow{L_1} C^2 \\
F\downarrow \quad \Downarrow\theta \quad \downarrow F & \mapsto & \hat{F}\downarrow \quad \Downarrow\theta^L \quad \downarrow\hat{F} \\
D \xrightarrow[(E_2,\eta_2,\epsilon_2)]{} D & & D^2 \xrightarrow[L_2]{} D^2
\end{array}
$$

Corollary 6.4. *R and L respectively induce functors* $\mathrm{Mon}(\mathbb{FF}(\mathbb{D})) \to \mathrm{Mon}(\mathbb{D})$ *and* $\mathrm{Comon}(\mathbb{FF}(\mathbb{D})) \to \mathrm{Comon}(\mathbb{D})$.

Up to this point, we have demonstrated that given any double category $\mathbb{D}$ having arrow objects, there is a 2-fold double category $\mathbb{FF}(\mathbb{D})$ of functorial factorizations in $\mathbb{D}$. The last thing we want to say about this construction is that a cyclic action on $\mathbb{D}$ lifts to one on $\mathbb{FF}(\mathbb{D})$, and hence also to one on $\mathrm{Bimon}(\mathbb{FF}(\mathbb{D}))$.

The cyclic action on objects and vertical morphisms is given directly by that on $\mathbb{D}$. Given a horizontal 1-cell (E,η,ϵ) on an object C, we define the 1-cell $(E,\eta,\epsilon)^\bullet$ on $C^\bullet$ to be $(E^\bullet,\epsilon^\bullet,\eta^\bullet)$. This also implies that the cyclic action swaps L and R for any given functorial factorization.

A quick look at the definitions of the two horizontal compositions is now enough to see that for any two functorial factorizations E_1 and E_2, we have

$$(E_1 \otimes E_2)^\bullet = E_1^\bullet \odot E_2^\bullet \qquad \text{and} \qquad (E_1 \odot E_2)^\bullet = E_1^\bullet \otimes E_2^\bullet$$

Similarly, the cyclic action on 2-cells in $\mathbb{FF}(\mathbb{D})$ is given by the cyclic action in $\mathbb{D}$ on the underlying 2-cell. This gives a valid 2-cell in $\mathbb{FF}(\mathbb{D})$ since the cyclic action simply swaps the equations (15) and (16).

7 Algebraic Weak Factorization Systems

For this section, let $\mathbb{D} = \mathbb{Sq}(\mathcal{D})$ be the double category of squares in a 2-category $\mathcal{D}$. We will show that bimonoids in $\mathbb{FF}(\mathbb{D})$ are precisely algebraic weak factorization systems, and more generally that the morphisms in $\mathrm{Bimon}(\mathbb{FF}(\mathbb{D}))$ are given by (co)lax morphisms of algebraic weak factorization systems.

Suppose that $E = (E, \eta, \epsilon)$ is a functorial factorization on a category $\mathscr{C}$, and consider a monoid structure on E. As I_C is initial, the unit of the monoid is forced, and is simply η. The multiplication is given by a natural transformation $\mu: ER \Rightarrow E$ satisfying equations (15) and (16), which now take the form $\epsilon \circ \mu = \epsilon R$ and $\mu \circ (\eta \cdot \vec{\eta}) = \eta$.

The unit axioms for the monoid give the equations $\mu \circ E\vec{\eta} = \mathrm{id}_E = \mu \circ \eta R$, which together imply the equation $\mu \circ (\eta \cdot \vec{\eta}) = \eta$ above. And finally, writing $\vec{\mu} = \mu^R: R^2 \to R$ for the natural transformation induced by the 2-cell μ, the associativity axiom gives the equation $\mu \circ E\vec{\mu} = \mu \circ \mu R$.

Proposition 7.1. *A monoid structure on an object* (E, η, ϵ) *in* $\mathbb{FF}(\mathbb{D})$ *is given by a natural transformation* $\mu: ER \Rightarrow E$, *satisfying equations*

$$\epsilon \circ \mu = \epsilon R \qquad \mu \circ E\vec{\eta} = \mathrm{id}_E = \mu \circ \eta R \qquad \mu \circ E\vec{\mu} = \mu \circ \mu R. \tag{21}$$

This determines a monad $\mathbb{R} = (R, \vec{\eta}, \vec{\mu})$, *such that* $\mathrm{dom}\,\vec{\mu} = \mu$ *and* $\mathrm{cod}\,\vec{\mu} = \mathrm{id}_{\mathrm{cod}}$.

Similarly, a comonoid structure on (E, η, ϵ) *is given by a natural transformation* $\delta: E \Rightarrow EL$, *satisfying equations*

$$\delta \circ \eta = \eta L \qquad E\vec{\epsilon} \circ \delta = \mathrm{id}_E = \epsilon L \circ \delta \qquad E\vec{\delta} \circ \delta = \delta L \circ \delta, \tag{22}$$

which determines a comonad $\mathbb{L} = (L, \vec{\epsilon}, \vec{\delta})$, *such that* $\mathrm{dom}\,\vec{\delta} = \mathrm{id}_{\mathrm{dom}}$ *and* $\mathrm{cod}\,\vec{\delta} = \delta$.

Hence a functorial factorization which simultaneously has a monoid structure and a comonoid structure in $\mathbb{FF}(\mathbb{D})$ is precisely a weak factorization system, missing only the second bullet of Definition 2.10, the distributive law condition. This is not surprising, as it is the only condition requiring a compatability between the monad and comonad structures. We will see that a bialgebra in $\mathbb{FF}(\mathbb{D})$ adds precisely this compatibility.

Proposition 7.2. *A bimonoid structure on a horizontal morphism* $(E, \eta, \epsilon): C \to C$ *in* $\mathbb{FF}(\mathbb{D})$ *is precisely an algebraic weak factorization system on C with underlying functorial factorization system* (E, η, ϵ).

Proof. We have already shown how the monoid an comonoid structures give rise to the monod and comonad of the awfs. All that remains is to show that

the equations (14) amount to just the distributive law, i.e. the equation

$$
\begin{array}{c}
(23)
\end{array}
$$

First of all, notice that the first three equations of (14) follow trivially from the initiality of I_C and the terminality of $\perp_C$ in $\mathbb{FF}(\mathbb{D})$, hence they do not impose any further conditions.

The fourth equation here takes the form

and so to prove (23), it suffices to show that

We can check this using the universal property of C^2 by composing with dom

and cod. First, use (19) and (20) to check that

Then use the definitions of i and p to check that $\mu \circ i = \mu \circ \eta R = \mathrm{id}_E$ and $p \circ \delta = \epsilon L \circ \delta = \mathrm{id}_E$, so that the first row above just equals δ, and the second row equals μ. Since Δ also (by definition) satisfies $\mathrm{dom}\,\Delta = \delta$ and $\mathrm{cod}\,\Delta = \mu$, we are done. $\square$

The appropriate notion of morphism between awfs, analagous to left/right Quillen functors and Quillen adjunctions, is (to our knowledge) first given in [Rie11].

Definition 7.3. Suppose that $(E_1, \eta_1, \mu_1, \epsilon_1, \delta_1)$ and $(E_2, \eta_2, \mu_2, \epsilon_2, \delta_2)$ are awfs on $\mathscr{C}$ and $\mathscr{D}$ respectively.

- A *lax morphism of awfs* $(G, \rho): E_1 \to E_2$ consists of a functor $G: \mathscr{C} \to \mathscr{D}$ and a natural transformation $\rho: E_2 \hat{G} \Rightarrow G E_1$, such that $(1, \rho): L_2 \hat{G} \Rightarrow G L_1$ is a lax morphism of comonads and $(\rho, 1): R_2 \hat{G} \Rightarrow G R_1$ is a lax morphism of monads.

- A *colax morphism of awfs* $(F, \lambda): E_1 \to E_2$ consists of a functor $F: \mathscr{C} \to \mathscr{D}$ and a natural transformation $\lambda: F E_1 \Rightarrow E_2 \hat{F}$, such that $(1, \lambda): F L_1 \Rightarrow L_2 \hat{F}$ is a colax morphism of comonads and $(\lambda, 1): F R_1 \Rightarrow R_2 \hat{F}$ is a colax morphism of monads.

Notice that a lax morphism of awfs induces a lift of the functor $\hat{G}$ to a functor $\mathbb{R}_1 \mathrm{Alg} \to \mathbb{R}_2 \mathrm{Alg}$. In that sense, G "preserves the right class," so is analogous to a right Quillen functor. Similarly, a colax morphism of awfs induces a lift of $\hat{F}$ to $\mathbb{L}_1 \mathrm{Coalg} \to \mathbb{L}_2 \mathrm{Coalg}$, so is analogous to a left Quillen functor.

By Proposition 5.3, the cyclic action on $\mathbb{FF}(\mathbb{D})$ induces a cyclic action on $\mathrm{Bimon}(\mathbb{FF}(\mathbb{D}))$. This action is given on awfs by

$$(E, \eta, \mu, \epsilon, \delta)^\bullet = (E^\bullet, \epsilon^\bullet, \delta^\bullet, \eta^\bullet, \mu^\bullet)$$

swapping the monad and comonad structures. This cyclic action allows us to capture both types of morphism of awfs in the same structure.

Proposition 7.4. *Morphisms in* $\mathrm{Bimon}(\mathbb{FF}(\mathbb{D}))$ *are precisely the colax morphisms of awfs. A colax morphism*

$$(E_2, \eta_2, \mu_2, \epsilon_2, \delta_2)^{\bullet} \to (E_1, \eta_1, \mu_1, \epsilon_1, \delta_1)^{\bullet}$$

is equivalent to a lax morphism of awfs

$$(E_1, \eta_1, \mu_1, \epsilon_1, \delta_1) \to (E_2, \eta_2, \mu_2, \epsilon_2, \delta_2)$$

Proof. As above, let $(E_1, \eta_1, \mu_1, \epsilon_1, \delta_1)$ and $(E_2, \eta_2, \mu_2, \epsilon_2, \delta_2)$ be awfs on $\mathscr{C}$ and $\mathscr{D}$ respectively. A morphism of bimonoids is given by a 2-cell

$$
\begin{array}{ccc}
\mathscr{C} & \xrightarrow{(E_1, \eta_1, \epsilon_1)} & \mathscr{C} \\
{\scriptstyle F}\downarrow & \Downarrow\lambda & \downarrow{\scriptstyle F} \\
\mathscr{D} & \xrightarrow[(E_2, \eta_2, \epsilon_2)]{} & \mathscr{D}
\end{array}
$$

which commutes with the monoid and comonoid structures. It is straightforward to check that this implies the natural transformations

$$
\begin{array}{ccc}
\mathscr{C}^2 & \xrightarrow{L_1} & \mathscr{C}^2 \\
{\scriptstyle \hat{F}}\downarrow & \Downarrow\lambda^L & \downarrow{\scriptstyle \hat{F}} \\
\mathscr{D}^2 & \xrightarrow[L_2]{} & \mathscr{D}^2
\end{array}
\qquad
\begin{array}{ccc}
\mathscr{C}^2 & \xrightarrow{R_1} & \mathscr{C}^2 \\
{\scriptstyle \hat{F}}\downarrow & \Downarrow\lambda^R & \downarrow{\scriptstyle \hat{F}} \\
\mathscr{D}^2 & \xrightarrow[R_2]{} & \mathscr{D}^2
\end{array}
$$

are colax morphisms of comonads and monads respectively. □

8   ℝ**-Alg and** 𝕃**-Coalg**

For this section, we will continue to let $\mathbb{D} = \mathbb{S}\mathrm{q}(\mathcal{D})$ be the double category of squares in a 2-category $\mathcal{D}$ with arrow objects.

A weak factorization system on a category C is defined by two classes of morphisms, $\mathcal{L}$ and $\mathcal{R}$. In an algebraic weak factorization system, these classes of morphisms are replaced by categories 𝕃-Coalg and ℝ-Alg equipped with functors to C^2. In this section, we will discuss the universal property satisfied by these categories, allowing us to define analagous objects in other 2-categories, and record several technical lemmas which we will need in the next section. We will focus on comonads, but there are dual results for monads which we leave to the reader.
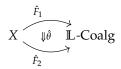
Recall from [Str72] the following proposition.

Proposition 8.1. *Let C be a category, and $\mathbb{L} = (L, \epsilon, \delta)$ be a comonad on C. The category of coalgebras 𝕃-Coalg has a universal property as follows:*

- *There is a forgetful functor $U\colon\mathbb{L}\text{-Coalg} \to C$ and a natural transformtion $\alpha\colon U \Rightarrow LU$, satisfying $\epsilon U \circ \alpha = \mathrm{id}_U$ and $\delta U \circ \alpha = L\alpha \circ \alpha$.*

- *(U, α) is universal among such pairs satisfying such equations. Given another such pair (F, β), where $F\colon X \to C$, there exists a unique functor $\hat{F}\colon X \to \mathbb{L}\text{-Coalg}$ such that $U\hat{F} = F$ and $\alpha\hat{F} = \beta$.*

Any colax morphism of comonads $(F, \phi)\colon (C, L_1, \epsilon_1, \delta_1) \to (D, L_2, \epsilon_2, \delta_2)$ induces a functor $\tilde{F}\colon \mathbb{L}_1\text{-Coalg} \to \mathbb{L}_2\text{-Coalg}$ such that $U_2\tilde{F} = FU_1$ and $\alpha_2\tilde{F} = \phi U_1 \circ F\alpha_1$.

A natural transformation

$$
X \underset{\hat{F}_2}{\overset{\hat{F}_1}{\Longrightarrow}} \mathbb{L}\text{-Coalg} \quad \Downarrow\hat{\theta}
$$

is uniquely determined by the functors $F_1 = U\hat{F}_1$ and $F_2 = U\hat{F}_2$ and natural transformations $\beta_1 = \alpha\hat{F}_1$ and $\beta_2 = \alpha\hat{F}_2$, and the natural transformation $\theta = U\hat{\theta}\colon F_1 \Rightarrow F_2$, satisfying $L\theta \circ \beta_1 = \beta_2 \circ \theta$.

For the rest of this section, assume that $\mathcal{D}$ *has EM-objects for comonads*, i.e. for every comonad 𝕃 in $\mathcal{D}$ there is an object 𝕃-Coalg satisfying the universal property above.

It is not too hard to use this universal property to construct the free/forgetful▪ adjunction:

Proposition 8.2. *For any comonad 𝕃 on an object C in $\mathcal{D}$, the 1-cell $U\colon\mathbb{L}\text{-Coalg} \to C$ has a right adjoint $\hat{L}$ with $U\hat{L} = L$ and $\alpha\hat{L} = \delta$. The counit of this adjunction is simply the counit of 𝕃, $\epsilon\colon U\hat{L} \Rightarrow \mathrm{id}_C$, while the unit is a 2-cell $\hat{\alpha}\colon\mathrm{id}_{\mathbb{L}\text{-Coalg}} \Rightarrow \hat{L}U$ satisfying $U\hat{\alpha} = \alpha$.*

Proof. By Proposition 8.1, to prove the existence of the 1-cell $\hat{L}$, it suffices to verify the equations $\epsilon L \circ \delta = \mathrm{id}_L$ and $\delta L \circ \delta = L\delta \circ \delta$, which are simply two of the comonad axioms.

Using the 2-dimensional part of Proposition 8.1, the existence of the 2-cell $\hat{\alpha}$ follows from the equation $L\alpha \circ \alpha = \delta U \circ \alpha$, which is the remaining comonad axiom.

We leave the verification of the triangle identities for the adjunction to the reader.   □

As our interest is in (co)monads in $\mathbb{FF}(\mathbb{D})$, which induce (co)monads on arrow objects, it will be useful to record the universal property that results from the interaction of the EM-object and arrow object universal properties.

Consider a comonad in $\mathbb{FF}(\mathbb{D})$ on an object C, i.e. a functorial factorization with half of the awfs structure. We can combine the universal properties of EM-objects and arrow objects into a universal property for 𝕃-Coalg, where now 𝕃 is the comonad in $\mathcal{D}$ arising from the comonad in $\mathbb{FF}(\mathbb{D})$.

Lemma 8.3. *Let* $(E, \eta, \epsilon, \delta)$ *be a comonad in* $\mathbb{FF}(\mathbb{D})$ *on an object C. There is a 2-cell*

satisfying equations

$$(24)$$

$$(25)$$

$$(26)$$

where $\vec{\alpha}$ is the unique 2-cell such that $\mathrm{dom}\,\vec{\alpha} = \mathrm{id}_{\mathrm{dom}\,U}$ *and* $\mathrm{cod}\,\vec{\alpha} = \alpha$*, the existence of which is implied by Equation 24.*

Given any object X, together with a morphism $F: X \to C^2$ and a 2-cell $\beta: \operatorname{cod} F \Rightarrow EF$ satisfying equations

1. $\beta \circ \kappa F = \eta F$
2. $\epsilon F \circ \beta = \operatorname{id}_{\operatorname{cod} F}$
3. $\delta F \circ \beta = E\vec{\beta} \circ \beta$

where $\vec{\beta}: F \Rightarrow LF$ is the unique 2-cell such that $\operatorname{dom} \vec{\beta} = \operatorname{id}_{\operatorname{dom} F}$ and $\operatorname{cod} \vec{\beta} = \beta$; there is a unique morphism $\hat{F}: X \to \mathbb{L}\text{-Coalg}$ such that $U\hat{F} = F$ and $\alpha \hat{F} = \beta$.

Given any pair of morphisms $\hat{F}_1, \hat{F}_2: X \to \mathbb{L}\text{-Coalg}$ and a 2-cell $\vec{\theta}: F_2 \Rightarrow F_2$ such that

$$E\vec{\theta} \circ \beta_1 = \beta_2 \circ \operatorname{cod} \vec{\theta}$$

(where $F_i = U\hat{F}_i$ and $\beta_i = \operatorname{cod} \alpha \hat{F}_i$ as in the previous paragraph), there is a unique 2-cell $\hat{\theta}: \hat{F}_1 \Rightarrow \hat{F}_2$ such that $U\hat{\theta} = \vec{\theta}$.

Proof. U is simply the U from proposition 8.1, while the 2-cell α there is the 2-cell $\vec{\alpha}$ here. The equation $\vec{\epsilon}U \circ \vec{\alpha} = \operatorname{id}_F$ implies that $\operatorname{dom} \vec{\alpha} = \operatorname{id}_{\operatorname{dom} U}$. With that observation, the rest of the equations follow immediately from the universal property of C^2 and the equations $\epsilon U \circ \alpha = \operatorname{id}_U$ and $\delta U \circ \alpha = L\alpha \circ \alpha$ from Proposition 8.1. □

9 Composition of $\mathbb{L}$-coalgebras

In an algebraic weak factorization system, the categories $\mathbb{L}$-Coalg and $\mathbb{R}$-Alg respectively play the roles of the left and right classes of morphisms of the weak factorization system. In an ordinary weak factorization system, these two classes of morphisms are closed under composition. In [Gar09], this is strengthened to a composition functor

$$\mathbb{L}\text{-Coalg}\,\Pi_C\,\mathbb{L}\text{-Coalg} \to \mathbb{L}\text{-Coalg}$$

and in [Rie11], it is shown that colax morphisms of awfs preserve this composition. Similarly, there is a composition functor on $\mathbb{R}$-Alg which is preserved by lax morphisms of awfs.

In this section, we will generalize these results to the setting of bimonads in $\mathbb{FF}(\mathbb{Sq}(\mathcal{D}))$. In fact we will prove the following more general theorem, from which the desired results will follow as corollaries using Proposition 3.13.

Theorem 9.1. *Let $\mathcal{D}$ be a 2-category with arrow objects and with EM-objects for comonads. There is a lax double functor*

$$\text{Coalg:}\,\mathbb{Comon}(\mathbb{FF}(\mathbb{Sq}(\mathcal{D}))) \to \mathbb{Span}(\mathcal{D}_0)_{/(-)^2}$$

where $\mathcal{D}_0$ is the ordinary category underlying the (strict) 2-category $\mathcal{D}$, which is the identity on the vertical categories, and which takes a comonad $(E, \eta, \epsilon, \delta)$ in $\mathbb{FF}(\mathbb{Sq}(\mathcal{D}))$ to the span

$$C \xleftarrow{\;\text{dom}\,U\;} \mathbb{L}\text{-Coalg} \xrightarrow{\;\text{cod}\,U\;} C.$$

In [Gar09] it is further shown that given a functorial factorization with only the comonad half of the awfs structure, a composition functor on $\mathbb{L}$-Coalg uniquely determines the monad half of the structure. The paper [Rie11] makes much use of this fact, and also extends it to morphisms of awfs. In our framework, these results will follow from Proposition 3.16 and the theorem:

Theorem 9.2. *The lax double functor Coalg is fully-faithful.*

First we should explain the notation $\mathbb{Span}(\mathcal{D}_0)_{/(-)^2}$. There is a natural family of monads in $\mathbb{Span}(\mathcal{D}_0)$, given for each object C by the span

$$C \xleftarrow{\;\text{dom}\;} C^2 \xrightarrow{\;\text{cod}\;} C$$

with multiplication given by the composition of the internal category structure of C^2 given in Proposition 2.3. That this is a natural family means that

for any morphism $f\colon C \to D$ in $\mathcal{D}_0$ there is a morphism of spans

$$
\begin{array}{ccccc}
C & \xleftarrow{\text{dom}} & C^2 & \xrightarrow{\text{cod}} & C \\
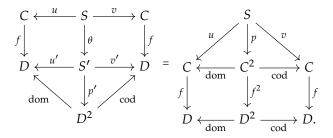f\downarrow & & \downarrow f^2 & & \downarrow f \\
D & \xleftarrow{\text{dom}} & D^2 & \xrightarrow{\text{cod}} & D.
\end{array}
$$

That this morphism of spans commutes with the multiplications follows easily from the universal property of arrow objects.

The double category $\mathbb{S}\text{pan}(\mathcal{D}_0)_{/(-)^2}$ has the same vertical category as $\mathbb{S}\text{pan}(\mathcal{D}_0)$—namely $\mathcal{D}_0$—with horizontal 1-cells $C \to C$ given by spans S equipped with a (globular) morphism $S \Rightarrow C^2$, i.e. a commuting diagram

$$
\begin{array}{ccc}
 & S & \\
u \swarrow & \downarrow p & \searrow v \\
C & & C \\
\text{dom} \nwarrow & \nearrow \text{cod} & \\
 & C^2 &
\end{array}
$$

and with 2-cells given by 2-cells in $\mathbb{S}\text{pan}(\mathcal{D}_0)$ which commute with these structure maps, i.e. by pairs (f, θ) such that

$$
\begin{array}{ccc}
C \xleftarrow{u} S \xrightarrow{v} C & & S \\
f\downarrow \quad \downarrow\theta \quad \downarrow f & & {}^{u}\swarrow \downarrow p \searrow^{v} \\
D \xleftarrow{u'} S' \xrightarrow{v'} D & = & C \xleftarrow{\text{dom}} C^2 \xrightarrow{\text{cod}} C \\
\text{dom}\searrow \downarrow p' \nearrow \text{cod} & & f\downarrow \quad \downarrow f^2 \quad \downarrow f \\
D^2 & & D \xleftarrow{\text{dom}} D^2 \xrightarrow{\text{cod}} D.
\end{array}
$$

The composition of two horizontal 1-cells in $\mathbb{S}\text{pan}(\mathcal{D}_0)_{/(-)^2}$

$$
\begin{array}{ccccc}
 & S_1 & & S_2 & \\
{}^{u_1}\swarrow \downarrow p_1 \searrow^{v_1} & & {}^{u_2}\swarrow \downarrow p_2 \searrow^{v_2} & & \\
C & & C & & C \\
\text{dom}\nwarrow \nearrow\text{cod} & & \text{dom}\nwarrow \nearrow\text{cod} & &
\end{array}
$$

is given by their horizontal composition in $\mathbb{S}\text{pan}(\mathcal{D}_0)$, and the structure map to C^2 is given by the horizontal composition of the p_1 and p_2 composed with the multiplication of C^2, i.e.
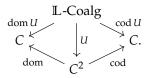
$$
S_1 \textstyle\prod_C S_2 \xrightarrow{(p_1, p_2)} C^2 \textstyle\prod_C C^2 = C^3 \xrightarrow{c} C^2.
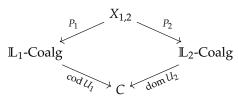$$

The identity for the horizontal composition is

$$
\begin{array}{ccc}
 & C & \\
\text{id} \swarrow & \downarrow i & \searrow \text{id} \\
C & & C, \\
\text{dom} \searrow & C^2 & \nearrow \text{cod}
\end{array}
$$

where $i: C \to C^2$ is the identity of the internal category structure on C^2 from Proposition 2.3.

We will now prove a couple of simple lemmas to establish the existence of certain 2-cells in $\mathcal{D}$ using the arrow object universal property. First, notice that any comonad $(E, \eta, \epsilon, \delta)$ in $\mathbb{FF}(\mathbb{Sq}(\mathcal{D}))$ gives rise to the horizontal 1-cell in $\mathrm{Span}(\mathcal{D}_0)_{/(-)^2}$

$$
\begin{array}{ccc}
 & \mathbb{L}\text{-Coalg} & \\
\text{dom}\, U \swarrow & \downarrow U & \searrow \text{cod}\, U \\
C & & C. \\
\text{dom} \searrow & C^2 & \nearrow \text{cod}
\end{array}
$$

For each of the following lemmas, let $(E_1, \eta_1, \epsilon_1, \delta_1)$ and $(E_2, \eta_2, \epsilon_2, \delta_2)$ be two comonads in $\mathbb{FF}(\mathbb{Sq}(\mathcal{D}))$, both on the same object C, and let $X_{1,2}$ be the pullback

$$
\begin{array}{ccccc}
 & & X_{1,2} & & \\
 & P_1 \swarrow & & \searrow P_2 & \\
\mathbb{L}_1\text{-Coalg} & & & & \mathbb{L}_2\text{-Coalg} \\
 & \searrow^{\text{cod}\, U_1} & C & \swarrow^{\text{dom}\, U_2} &
\end{array}
$$

with structure map $U_{1,2}: X \to C^2$ given by the composition

$$
X_{1,2} \xrightarrow{\langle U_1, U_2 \rangle} C^3 \xrightarrow{c} C.^2
$$

Recall from Proposition 2.3 that c by definition satisfies $\mathrm{dom}\, c = \mathrm{dom}\, P_1$, $\mathrm{cod}\, c = \mathrm{cod}\, P_2$, $\kappa c = \kappa P_2 \circ \kappa P_1$. We also record for later reference:

$$
\mathrm{dom}\, U_{1,2} = \mathrm{dom}\, c(U_1, U_2) = \mathrm{dom}\, P_1(U_1, U_2) = \mathrm{dom}\, U_1 P_1 \tag{27}
$$

$$
\mathrm{cod}\, U_{1,2} = \mathrm{cod}\, c(U_1, U_2) = \mathrm{cod}\, P_2(U_1, U_2) = \mathrm{cod}\, U_2 P_2 \tag{28}
$$

$$
\kappa U_{1,2} = \kappa c(U_1, U_2) = (\kappa P_2 \circ \kappa P_1)(U_1, U_2) = \kappa U_2 P_2 \circ \kappa U_1 P_1 \tag{29}
$$

Lemma 9.3. *There is a 2-cell*

$$
\begin{array}{ccccc}
 & \xrightarrow{P_1} & \mathbb{L}_1\text{-Coalg} & \xrightarrow{U_1} & \\
X_{1,2} & & \Downarrow \zeta & & C^2 \\
 & & \xrightarrow[U_{1,2}]{} & &
\end{array}
$$

such that dom ζ = id *and*

Proof. Equation (3) becomes

which is just Equation (29) □

Lemma 9.4. *There is a 2-cell*

such that cod ν = id *and*

Proof. We just need to verify Equation (3):

$$
\begin{array}{c}
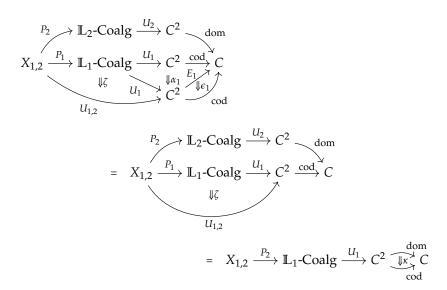\mathbb{L}_2\text{-Coalg} \xrightarrow{U_2} C^2 \xrightarrow{\text{dom}} \\
X_{1,2} \xrightarrow{P_1} \mathbb{L}_1\text{-Coalg} \xrightarrow{U_1} C^2 \xrightarrow{\text{cod}} C \\
\Downarrow\zeta \quad U_1 \searrow C^2 \quad \Downarrow\epsilon_1 \\
U_{1,2}
\end{array}
$$

with P_2, α_1, E_1, cod labels

$$
= \quad
\begin{array}{c}
\mathbb{L}_2\text{-Coalg} \xrightarrow{U_2} C^2 \xrightarrow{\text{dom}} \\
X_{1,2} \xrightarrow{P_1} \mathbb{L}_1\text{-Coalg} \xrightarrow{U_1} C^2 \xrightarrow{\text{cod}} C \\
\Downarrow\zeta \\
U_{1,2}
\end{array}
$$

with P_2 label

$$
= \quad X_{1,2} \xrightarrow{P_2} \mathbb{L}_1\text{-Coalg} \xrightarrow{U_1} C^2 \underset{\text{cod}}{\overset{\text{dom}}{\rightrightarrows}} C \quad \Downarrow\kappa
$$

where the first equation follows from (25), and the second by reducing $\text{cod}\,\zeta$ using Lemma 9.3. $\qquad\square$

Proof of Theorem 9.1. For notational convenience, let $G = \text{Coalg}$ be the lax double functor we need to establish. Both the double categories $\mathbb{C}\text{omon}(\mathbb{F}F(\mathbb{S}q(\mathscr{D})))$ ∎ and $\mathbb{S}\text{pan}(\mathcal{D}_0)_{/(-)^2}$ have $\mathcal{D}_0$ as vertical category, and G_0 (the component of G on vertical categories) is simply the identity. From the statement of the theorem, G takes an object in $\mathbb{C}\text{omon}(\mathbb{F}F(\mathbb{S}q(\mathscr{D})))$ to the span and structure map

$$
\begin{array}{ccc}
& \mathbb{L}\text{-Coalg} & \\
\text{dom}\,U \swarrow & \downarrow U & \searrow \text{cod}\,U \\
C \ \nwarrow & & \nearrow\ C. \\
\text{dom} \searrow & C^2 & \swarrow \text{cod}
\end{array}
$$

To define the behavior of G on 2-cells, consider a 2-cell in $\mathbb{C}\text{omon}(\mathbb{F}F(\mathbb{S}q(\mathscr{D})))$: ∎

$$
\begin{array}{ccc}
C & \xrightarrow{(E_1,\eta_1,\epsilon_1,\delta_1)} & C \\
F \downarrow & \Downarrow\phi & \downarrow F \\
D & \xrightarrow[(E_2,\eta_2,\epsilon_2,\delta_2)]{} & D.
\end{array}
$$

By Corollary 6.4, ϕ induces a colax morphism of comonads from L_1 to L_2, hence by Proposition 8.1 there is an induced morphism $\tilde{\phi}$ between the EM-objects such that $U_2\tilde{\phi} = F^2 U_1$. We can then define $G\phi$ to be the morphism of
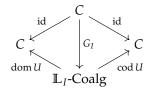
spans

$$C \xleftarrow{\text{dom}} C^2 \xleftarrow{U_1} \mathbb{L}_1\text{-Coalg} \xrightarrow{U_1} C^2 \xrightarrow{\text{cod}} C$$
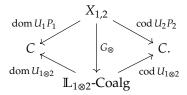
That $\tilde{\phi}$ commutes with the structure maps is simply the commutativity of the square $U_2\tilde{\phi} = F^2 U_1$.

Next we must define the coherence data G_I and $G_\otimes$. We will define G_I to be the morphims of spans

defined via Lemma 9 by the equations $UG_I = i\colon C \to C^2$ and $\alpha_I G_I$ is the identity on $\text{dom}\, i = \text{cod}\, i$. The conditions of the lemma are trivially satisfied.

We will similarly use Lemma 9 to define $G_\otimes$. Let $X_{1,2}$, $U_{1,2}$, ζ, and ν be as defined earlier in the section. $G_\otimes$ is a morphism of spans

We will define $G_\otimes$ to be the 1-cell such that $U_{1\otimes 2}G_\otimes = U_{1,2}$ and

In other words, in the notation of Lemma 9 let $F = U_{1,2}$ and $\beta = E_2\nu \circ \alpha_2 P_2$, and define $G_\otimes = \hat{F}$.

We now need to check equations 1-3 of Lemma 9 to verify that $G_\otimes$ is well defined. We will check these equationally to save space, but the reader may want to draw out the diagrams for themselves to follow along. For the first

equation:

$$E_2\nu \circ \alpha_2 P_2 \circ \kappa U_{1,2}$$

$$\begin{aligned}
&= E_2\nu \circ \alpha_2 P_2 \circ \kappa U_2 P_2 \circ \kappa U_1 P_1 && \text{Eq (29)} \\
&= E_2\nu \circ (\alpha_2 \circ \kappa U_2)P_2 \circ \kappa U_1 P_1 && \\
&= E_2\nu \circ \eta_2 U_2 P_2 \circ \kappa U_1 P_1 && \text{Eq (24)} \\
&= \eta_2 R_1 U_{1,2} \circ \text{dom}\,\nu \circ \kappa U_1 P_1 && \text{Interchange} \\
&= \eta_2 R_1 U_{1,2} \circ E_1\zeta \circ \alpha_1 P_1 \circ \kappa U_1 P_1 && \text{Def of } \nu \\
&= \eta_2 R_1 U_{1,2} \circ E_1\zeta \circ (\alpha_1 \circ \kappa U_1)P_1 && \\
&= \eta_2 R_1 U_{1,2} \circ E_1\zeta \circ \eta_1 U_1 P_1 && \text{Eq (24)} \\
&= \eta_{1\otimes2} U_{1,2} \circ \text{dom}\,\zeta && \text{Interchange; Def of } \eta_{1\otimes2} \\
&= \eta_{1\otimes2} U_{1,2} && \text{dom}\,\zeta = \text{id}
\end{aligned}$$

and the second:

$$\epsilon_{1\otimes2} U_{1,2} \circ E_2\nu \circ \alpha_2 P_2$$

$$\begin{aligned}
&= \epsilon_2 R_1 U_{1,2} \circ E_2\nu \circ \alpha_2 P_2 && \text{Def of } \epsilon_{1\otimes2} \\
&= \text{cod}\,\nu \circ (\epsilon_2 U_2 \circ \alpha_2)P_2 && \text{Interchange} \\
&= \text{id}_{\text{cod}\,U_{1,2}}. && \text{Eq (25); cod}\,\nu = \text{id}
\end{aligned}$$

The third equation is a bit trickier to prove. We will need to prove two intermediate equations first, using the arrow object universal property.

Lemma.

$$i^L U_{1,2} \circ L_1\zeta \circ \vec{\alpha}_1 P_1 = \vec{\beta} \circ \zeta \tag{30}$$

Proof. We must show the 2-cells become equal upon composition with dom and cod:

$$\text{dom}(i^L U_{1,2} \circ L_1\zeta \circ \vec{\alpha}_1 P_1) = \text{id}_{\text{dom}\,U_{1,2}} = \text{dom}(\vec{\beta} \circ \zeta)$$

and

$$\text{cod}(i^L U_{1,2} \circ L_1\zeta \circ \vec{\alpha}_1 P_1)$$

$$\begin{aligned}
&= \text{cod}\,i^L U_{1,2} \circ E_1\zeta \circ \text{cod}\,\vec{\alpha}_1 P_1 && \\
&= \eta_2 R_1 U_{1,2} \circ E_1\zeta \circ \alpha_1 P_1 && \text{Def of } i^L, \vec{\alpha} \\
&= \eta_2 R_1 U_{1,2} \circ \text{dom}\,\nu && \text{Def of } \nu \\
&= E_2\nu \circ \eta_2 U_2 P_2 && \text{Interchange} \\
&= E_2\nu \circ (\alpha_2 \circ \kappa U_2)P_2 && \text{Eq (24)} \\
&= (E_2\nu \circ \alpha_2 P_2) \circ \kappa U_2 P_2 && \\
&= \text{cod}\,\vec{\beta} \circ \text{cod}\,\zeta && \text{Def of } \vec{\beta}, \zeta \\
&= \text{cod}(\vec{\beta} \circ \zeta).
\end{aligned}$$

$\square$

Lemma.

$$R_1\vec{\beta} \circ \nu = wU_{1,2} \circ L_2\delta_1^R U_{1,2} \circ L_2\nu \circ \vec{\alpha}_2 P_2 \tag{31}$$

Proof. Again we must prove equality after composing with dom and cod:

$\text{dom}(R_1\vec{\beta} \circ \nu)$

$\quad = E_1\vec{\beta} \circ \text{dom}\,\nu$

$\quad = E_1\vec{\beta} \circ E_1\zeta \circ \alpha_1 P_1$ $\qquad\qquad\qquad\qquad\qquad$ Def of ν

$\quad = E_1(\vec{\beta} \circ \zeta) \circ \alpha_1 P_1$

$\quad = E_1(i^L U_{1,2} \circ L_1\zeta \circ \vec{\alpha}_1 P_1) \circ \alpha_1 P_1$ $\qquad\qquad$ Eq (30)

$\quad = E_1 i^L U_{1,2} \circ E_1 L_1\zeta \circ (E_1\vec{\alpha}_1 \circ \alpha_1)P_1$

$\quad = E_1 i^L U_{1,2} \circ E_1 L_1\zeta \circ (\delta_1 U_1 \circ \alpha_1)P_1$ $\qquad$ Eq (26)

$\quad = E_1 i^L U_{1,2} \circ \delta_1 U_{1,2} \circ E_1\zeta \circ \alpha_1 P_1$ $\qquad\qquad$ Interchange

$\quad = \text{dom}\,wU_{1,2} \circ \text{dom}\,\delta_1^R U_{1,2} \circ \text{dom}\,\nu \circ \text{dom}\,\vec{\alpha}_2 P_2$ $\quad$ Defs of w, δ^R, ν, $\vec{\alpha}$

$\quad = \text{dom}(wU_{1,2} \circ L_2\delta_1^R U_{1,2} \circ L_2\nu \circ \vec{\alpha}_2 P_2)$

and

$\text{cod}(R_1\vec{\beta} \circ \nu)$

$\quad = \text{cod}\,\vec{\beta} \circ \text{cod}\,\nu$

$\quad = E_2\nu \circ \alpha_2 P_2$ $\qquad\qquad\qquad\qquad\qquad\qquad$ Defs of $\vec{\beta}$, ν

$\quad = E_2(p^R \circ \delta_1^R)U_{1,2} \circ E_2\nu \circ \alpha_2 P_2$ $\qquad\qquad$ $p^R \circ \delta^R = \text{id}$

$\quad = E_2 p^R U_{1,2} \circ E_2\delta_1^R U_{1,2} \circ E_2\nu \circ \alpha_2 P_2$

$\quad = \text{cod}\,wU_{1,2} \circ \text{cod}\,L_2\delta_1^R U_{1,2} \circ \text{cod}\,L_2\nu \circ \text{cod}\,\vec{\alpha}_2 P_2$ $\quad$ Defs of w, L, $\vec{\alpha}$

$\quad = \text{cod}(wU_{1,2} \circ L_2\delta_1^R U_{1,2} \circ L_2\nu \circ \vec{\alpha}_2 P_2)$

$\hfill \square$

Now we are prepared to prove the third equation of Lemma 9 validating our definition of $G_\otimes$:

$\delta_{1\otimes2}U_{1,2} \circ E_2\nu \circ \alpha_2 P_2$

$\quad = (E_2 w \circ \delta_2 R_{1\odot1} \circ E_2\delta_1^R)U_{1,2} \circ E_2\nu \circ \alpha_2 P_2$ $\qquad$ Def of $\delta_{1\otimes2}$

$\quad = E_2(wU_{1,2} \circ L_2\delta_1^R U_{1,2} \circ L_2\nu) \circ (\delta_2 U_2 \circ \alpha_2)P_2$ $\qquad$ Interchange

$\quad = E_2(wU_{1,2} \circ L_2\delta_1^R U_{1,2} \circ L_2\nu) \circ (E_2\vec{\alpha}_2 \circ \alpha_2)P_2$ $\qquad$ Eq (26)

$\quad = E_2(wU_{1,2} \circ L_2\delta_1^R U_{1,2} \circ L_2\nu \circ \vec{\alpha}_2 P_2) \circ \alpha_2 P_2$

$\quad = E_2(R_1\vec{\beta} \circ \nu) \circ \alpha_2 P_2$ $\qquad\qquad\qquad\qquad\qquad$ Eq (31)

$\quad = E_{1\otimes2}\vec{\beta} \circ E_2\nu \circ \alpha_2 P_2$ $\qquad\qquad\qquad\qquad\quad$ Def of $E_{1\otimes2}$
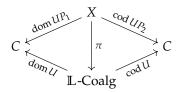
The verification that the definitions of G_I and $G_\otimes$ form natural families, and of the coherence axioms for a lax double functor, is tedious, but follows from what we have presented here without requiring any new ideas or ingenuity. □

Corollary 9.5. *For any awfs* $(E, \eta, \mu, \epsilon, \delta)$ *on an object C in $\mathcal{D}$, the multiplication μ induces a composition functor on $\mathbb{L}$-Coalg, and the functor between EM-objects induced by any colax morphism of awfs preserves this composition.*

Proof. Any awfs $(E, \eta, \mu, \epsilon, \delta)$ has an underlying object in $\mathbb{C}\mathrm{omon}(\mathbb{FF}(\mathbb{S}\mathrm{q}(\mathcal{D})))$, by simply forgetting μ. The lax double-functor Coalg takes this to a span

$$C \xleftarrow{\ \mathrm{dom}\, U\ } \mathbb{L}\text{-Coalg} \xrightarrow{\ \mathrm{cod}\, U\ } C.$$

The multiplication μ provides this object in $\mathbb{C}\mathrm{omon}(\mathbb{FF}(\mathbb{S}\mathrm{q}(\mathcal{D})))$ with a monad structure, and lax double-functors preserve monads, so μ induces a monad structure on this span. A multiplication on this span is a morphism π:

where X is the pullback in the composite span

The morphism π is the composition structure that we want. If $\mathcal{D} = \mathrm{Cat}$ is the 2-category of small categories, then an object (f, g) in X is a pair of morphisms in C equipped with coalgebra structures, such that $\mathrm{cod}\, f = \mathrm{dom}\, g$, and $\pi(f, g)$ is a morphism equipped with a coalgebra structure, with $\mathrm{dom}\, \pi(f, g) = \mathrm{dom}\, f$ and $\mathrm{cod}\, \pi(f, g) = \mathrm{cod}\, g$.

Of course, what we really want is that the morphism underlying the coalgebra $\pi(f, g)$ is the composition $g \circ f$. But this is simply the fact that π defines a 2-cell in $\mathbb{S}\mathrm{pan}(\mathcal{D}_0)_{/(-)^2}$, hence commutes with the structure maps to C^2. Recall that the structure map for the horizontal composite X is defined using $c \colon C^3 \to C^2$, hence $U\pi(f, g) = c(Uf, Ug)$. □

Now we will continue on to the proof of Theorem 9.2. The proof is surprisingly difficult and tedious—we will outline the main steps but leave many of the routine verifications to the reader.

Proof of Theorem 9.2. The bijectivity of Coalg acting on 2-cells with domain I is simple to check, since I is initial in $\mathbb{C}\text{omon}(\mathbb{FF}(\mathbb{Sq}(\mathcal{D})))$, and from Lemma it■ is easy to see that there is a unique morphism $!\colon C \to \mathbb{L}\text{-Coalg}$ satisfying $U! = i$, with $\alpha! = \eta i$.

Now let $(E_i, \eta_i, \epsilon_i, \delta_i)$, $i \in \{1, 2, 3\}$, be three comonads in $\mathbb{FF}(\mathbb{Sq}(\mathcal{D}))$, with $E_1, E_2\colon C^2 \to C$ and $E_3\colon D^2 \to D$, and let $F\colon C \to D$ be a morphism. Given a morphism $X_{1,2} \to \mathbb{L}_3\text{-Coalg}$ such that $U_3\theta = F^2 U_{1,2}$, we need to prove the unique existence of a 2-cell

$$
\begin{array}{ccc}
C \xrightarrow{(E_1,\eta_1,\epsilon_1,\delta_1)} C \xrightarrow{(E_2,\eta_2,\epsilon_2,\delta_2)} C \\
F \downarrow \qquad \Downarrow\phi \qquad \downarrow F \\
D \xrightarrow{\qquad (E_3,\eta_3,\epsilon_3,\delta_3) \qquad} D
\end{array}
$$

such that $\tilde{\phi}G_\otimes = \theta$.

Outline of proof:

- Define a morphism $\check{L}_{1\otimes2}\colon C^2 \to X_{1,2}$ such that

$$P_1\check{L}_{1\otimes2} = \hat{L}_1 \qquad \text{and} \qquad P_2\check{L}_{1\otimes2} = \hat{L}_2 R_1.$$

  Show that $U_{1,2}\check{L}_{1\otimes2} = L_{1\otimes2}$.

- Define a 2-cell

$$\psi\colon U_3\theta\check{L}_{1\otimes2} \Rightarrow F^2$$

  by $\psi = F^2\vec{\epsilon}_{1\otimes2}$, noting that $U_3\theta\check{L}_{1\otimes2} = F^2 L_{1\otimes2}$.

- Let the 2-cell $\psi' = \hat{L}_3\psi \circ \hat{\alpha}_3\theta\check{L}_{1\otimes2}$,

$$
\begin{array}{ccc}
C^2 \xrightarrow{\check{L}_{1\otimes2}} X_{1,2} \xrightarrow{\theta} \mathbb{L}_3\text{-Coalg} \\
\| \qquad \Downarrow\psi' \qquad \uparrow \hat{L}_3 \\
C^2 \xrightarrow{\qquad F^2 \qquad} D^2
\end{array}
$$

  be the mate of ψ under the adjunction $U_3 \dashv \hat{L}_3$.

- Define the desired 2-cell ϕ to be the codomain component of ψ':

$$\phi = \text{cod } U_3\psi' = E_3\psi \circ \alpha_3\theta\check{L}_{1\otimes2}\colon FE_2R_1 \to E_3F^2$$

- First we must verify that ϕ defines a valid 2-cell in $\mathbb{FF}(\mathbb{Sq}(\mathcal{D}))$ by checking equations (15) and (16). Equation (15) is simple to show directly, while (16) follows from the well definedness of $U_3\psi'$. In fact, we have

$$\phi^L = U_3\psi'\colon F^2 L_{1\otimes2} \Rightarrow L_3F^2$$

- Next we must verify that ϕ defines a valid 2-cell in $\mathbb{C}\mathrm{omon}(\mathbb{FF}(\mathbb{Sq}(\mathcal{D})))$, which means showing that it commutes with the comultipication 2-cells:

$$
\begin{array}{ccc}
\begin{array}{ccccc}
C^2 & \xrightarrow{R_1} & C^2 & \xrightarrow{E_2} & C \\
\| & & \Downarrow\delta_{1\otimes2} & & \| \\
C^2 & \xrightarrow{L_{1\otimes2}} & C^2 & \xrightarrow{E_2R_1} & C \\
\Big\downarrow F^2 & \Downarrow\phi^L & \Big\downarrow F^2 & \Downarrow\phi & \Big\downarrow F \\
D^2 & \xrightarrow{L_3} & D^2 & \xrightarrow{E_3} & D
\end{array}
& = &
\begin{array}{ccccc}
C^2 & \xrightarrow{R_1} & C^2 & \xrightarrow{E_2} & C \\
\Big\downarrow F^2 & & \Downarrow\phi & & \Big\downarrow F \\
D^2 & & \xrightarrow{E_3} & & D \\
\| & & \Downarrow\delta_3 & & \| \\
D & \xrightarrow{L_3} & D^2 & \xrightarrow{E_3} & D.
\end{array}
\end{array}
$$

To do this, first verify the existence of a 2-cell $\check{\delta}_{1\otimes2}\colon \check{L}_{1\otimes2} \Rightarrow \check{L}_{1\otimes2}\check{L}_{1\otimes2}$ satisfying

$$P_1\check{\delta}_{1\otimes2} = \hat{L}_1 i^L \circ \hat{\delta}_1 \quad \text{and} \quad P_2\check{\delta}_{1\otimes2} = \hat{L}_2 w \circ \hat{L}_2 L_2 \delta_1^R \circ \hat{\delta}_2 R_1 \tag{32}$$

where $\hat{\delta}_i$ is the unique 2-cell with $U_i\hat{\delta}_i = \vec{\delta}_i$. Show that $U_{1,2}\check{\delta}_{1\otimes2} = \vec{\delta}_{1\otimes2}$. Define

$$\tau_1 = \hat{L}_3\phi^L \circ \psi' L_{1\otimes2} \circ \theta\check{\delta}_{1\otimes2} \quad \text{and} \quad \tau_2 = \hat{\delta}_3 F^2 \circ \psi'$$

and check that $\operatorname{cod} U_3\tau_1 = E_3\phi^L \circ \phi L_{1\otimes2} \circ F\delta_{1\otimes2}$ and $\operatorname{cod} U_3\tau_2 = \delta_3 F^2 \circ \phi$. Hence to prove (32) it suffices to show $\tau_1 = \tau_2$. To do this, show that the mates of each are equal to ϕ^L.

- We have defined a 2-cell ϕ in $\mathbb{C}\mathrm{omon}(\mathbb{FF}(\mathbb{Sq}(\mathcal{D})))$, now we need to show that the lax functor Coalg takes this ϕ to the 2-cell θ we began with, i.e. that $\theta = \tilde{\phi}G_\otimes$. It is easy to see that

$$U_3\tilde{\phi}G_\otimes = F^2 U_{1\otimes2}G_\otimes = F^2 U_{1,2} = U_3\theta,$$

so it only remains to show that $\alpha_3\tilde{\phi}G_\otimes = \alpha_3\theta$.
Begin by verifying the existence of a 2-cell $\rho\colon \mathrm{id}_{X_{1,2}} \Rightarrow \check{L}_{1\otimes2}U_{1,2}$ such that

$$P_1\rho = \hat{L}_1\zeta \circ \hat{\alpha}_1 P_1 \quad \text{and} \quad P_2\rho = \hat{L}_2\nu \circ \hat{\alpha}_2 P_2,$$

and show that $U_{1,2}\rho = \vec{\alpha}_{1\otimes2}G_\otimes$.
Finally, show that

$$\psi U_{1,2} \circ F^2\vec{\alpha}_{1\otimes2}G_\otimes = \mathrm{id}_{F^2 U_{1,2}},$$

and use this to show that $\vec{\alpha}_3\tilde{\phi}G_\otimes = \vec{\alpha}_3\theta$. Thus we have shown the existance of the 2-cell ϕ such that $\tilde{\phi}G_\otimes = \theta$, and the uniqueness follows by a very similar computation.

$\square$

Combining this with Proposition 3.13 immediately implies:

Corollary 9.6. *Suppose $(E_1, \eta_1, \epsilon_1, \delta_1)$ is a comonad in $\mathbb{FF}(\mathbb{Sq}(\mathcal{D}))$. A composition on $\mathbb{L}$-Coalg is equivalent to completing E_1 to an awfs.*

10 A universal property for the pushout product

We would now like to generalize the framework of cyclic 2-fold double categories to cyclic 2-fold double multicategories, in order to incorporate the multivariable adjunctions of awfs defined in [Rie13].

[TODO: Review cyclic multicategories/double multicategories]

First, we will need to find a generalization of the universal property for arrow objects to (cyclic) double multicategories. In the theory of multivariable Quillen adjunctions, the lift of a multivariable adjunction to arrow categories is provided by the pushout/pullback product, so we will identify a universal property satisfied by this construction.

Define a cyclic double multicategory $\mathbb{J}$ as follows. The objects are A_i, B_i, for $i \in \{0, 1, 2\}$, and their duals. The horizontal 1-cells are $d_0^i, d_1^i \colon B_i \to A_i$. The vertical 1-cells are $F_i \colon (A_{i-1}, A_{i+1}) \to A_i^{\bullet}$ and $G_i \colon (B_{i-1}, B_{i+1}) \to B_i^{\bullet}$, which form two orbits under the cyclic action.

There are two types of 2-cells. There are

$$
\begin{array}{ccc}
B_i & \xrightarrow{\;d_1^i\;} & A_i \\
\text{id} \downarrow & \quad\Downarrow \alpha_i & \downarrow \text{id} \\
B_i & \xrightarrow[d_0^i]{} & A_i
\end{array}
$$

for each i. We will often draw these 2-cells globularly.

There are also 2-cells

$$
\begin{array}{ccc}
B_{i+1}, B_{i-1} & \xrightarrow{d_{k_{i+1}}^{i+1}, d_{k_{i-1}}^{i-1}} & A_{i+1}, A_{i-1} \\
G_i \downarrow & \quad\Downarrow \lambda_{k_{i+1}, k_{i-1}, k_i}^{i} & \downarrow F_i \\
B_i^{\bullet} & \xrightarrow[d_{k_i}^{i\bullet}]{} & A_i^{\bullet}
\end{array}
$$

for all choices of $(k_0, k_1, k_2) \in \{0, 1\}^3$ except $(0, 0, 0)$.

Notice that there is at most one element of every hom-set, so all compositions and cyclic actions are uniquely defined. From now on, we will omit indices whenever doing so is unambiguous.

Remark 10.1. The cyclic double multicategory $\mathbb{J}$ is generated under composition by the α_i and the $\lambda_{k_{i+1}, k_{i-1}, k_i}^{i}$ with exactly one of k_0, k_1, k_2 equal to 1. These nine λ generators are further generated under the cyclic action by only three, though there are many choices of which three. These generators satisfy the relations

$$
\begin{array}{ccc}
B_1, B_2 \xrightarrow{d_1, d_0} A_1, A_2 & & B_1, B_2 \xrightarrow[d_0, d_0]{\overset{d_1, d_0}{\Downarrow \alpha, \mathrm{id}}} A_1, A_2 \\
\Big\downarrow G_0 \quad \Downarrow \lambda \quad \Big\downarrow F_0 & = & \Big\downarrow G_0 \quad \Downarrow \lambda \quad \Big\downarrow F_0 \\
B_0^{\bullet} \xrightarrow{d_0^{\bullet}} A_0^{\bullet} & & B_0^{\bullet} \xrightarrow{d_1^{\bullet}} A_0^{\bullet} \\
\quad \Downarrow \alpha^{\bullet} \\
\quad d_1^{\bullet}
\end{array}
$$

$$
\begin{array}{ccc}
B_1, B_2 \xrightarrow{d_0, d_1} A_1, A_2 & & B_1, B_2 \xrightarrow[d_0, d_0]{\overset{d_0, d_1}{\Downarrow \mathrm{id}, \alpha}} A_1, A_2 \\
\Big\downarrow G_0 \quad \Downarrow \lambda \quad \Big\downarrow F_0 & = & \Big\downarrow G_0 \quad \Downarrow \lambda \quad \Big\downarrow F_0 \\
B_0^{\bullet} \xrightarrow{d_0^{\bullet}} A_0^{\bullet} & & B_0^{\bullet} \xrightarrow{d_1^{\bullet}} A_0^{\bullet} \\
\quad \Downarrow \alpha^{\bullet} \\
\quad d_1^{\bullet}
\end{array}
$$

$$
\begin{array}{ccc}
B_1, B_2 \xrightarrow[d_1, d_0]{\overset{d_1, d_1}{\Downarrow \mathrm{id}, \alpha}} A_1, A_2 & & B_1, B_2 \xrightarrow[d_0, d_1]{\overset{d_1, d_1}{\Downarrow \alpha, \mathrm{id}}} A_1, A_2 \\
\Big\downarrow G_0 \quad \Downarrow \lambda \quad \Big\downarrow F_0 & = & \Big\downarrow G_0 \quad \Downarrow \lambda \quad \Big\downarrow F_0 \\
B_0^{\bullet} \xrightarrow{d_0^{\bullet}} A_0^{\bullet} & & B_0^{\bullet} \xrightarrow{d_0^{\bullet}} A_0^{\bullet}
\end{array}
$$

and their reflections under the cyclic action.

Example 10.2. Let $\mathbb{M}\mathbf{Adj}$ be the double cyclic multicategory of categories, functors, and multivariable right adjunctions. Any multivariable right adjunction $F_0 \colon \mathscr{A}_1 \times \mathscr{A}_2 \to \mathscr{A}_0$ extends to a functor $\widehat{\mathbb{F}} \colon \mathbb{J} \to \mathbb{M}\mathbf{Adj}$ as follows.

- B_i is sent to $\mathscr{A}_i^{\mathbf{2}}$, the arrow category of $\mathscr{A}_i$.
- The d_1 are sent to the domain functors $\mathrm{dom} \colon \mathscr{A}_i^{\mathbf{2}} \to \mathscr{A}_i$ and the d_0 are sent to the codomain functors $\mathrm{cod} \colon \mathscr{A}_i^{\mathbf{2}} \to \mathscr{A}_i$.
- The α are sent to the canonical natural transformations $\mathrm{dom} \Rightarrow \mathrm{cod}$.
- The G_i are sent to functors $\hat{F}_i$. Given morphisms $f \colon A \to B \in \mathscr{A}_1$ and

$g: X \to Y \in \mathscr{A}_2$, $\hat{F}_0(f,g)$ is defined as in the diagram

$$
\begin{array}{ccc}
F_0(B,Y) & \xrightarrow{\quad F_0(1,g) \quad} & \\
& \hat{F}_0(f,g) & \\
& F_0(A,Y) \prod_{F_0(A,X)} F_0(B,X) \xrightarrow{\ p_2\ } F_0(B,X) & \\
F_0(f,1) & \quad p_1 \downarrow & \quad \downarrow F_0(f,1) \\
& F_0(A,Y) \xrightarrow{\quad F_0(1,g) \quad} F_0(A,X) &
\end{array}
\tag{33}
$$

It is a standard fact that the $\hat{F}_i$ form a two-variable adjunction between the arrow categories.

- Looking at diagram 33,

$$
\begin{aligned}
(\lambda^0_{1,0,0})_{f,g} &= p_1 \colon \operatorname{cod} \hat{F}_0(f,g) \to F_0(\operatorname{dom} f, \operatorname{cod} g) \\
(\lambda^0_{0,1,0})_{f,g} &= p_2 \colon \operatorname{cod} \hat{F}_0(f,g) \to F_0(\operatorname{cod} f, \operatorname{dom} g) \\
(\lambda^0_{0,0,1})_{f,g} &= \operatorname{id} \colon \operatorname{dom} \hat{F}_0(f,g) \to F_0(\operatorname{cod} f, \operatorname{cod} g).
\end{aligned}
$$

The three relations (1)-(3) then correspond precisely to the commutativity of the three regions in diagram (33).

Exercise 10.1. Check that the mates of the morphism p_1 in diagram 33 are p_2 and id in the two similar diagrams defining $\hat{F}_1$ and $\hat{F}_2$.

Let $\mathbb{I}$ be the sub-category of $\mathbb{J}$ consisting of just the 1-cells F_i. Let **CDMCat** denote the 2-category of cyclic double multicategories, functors, and horizontal transformations.

Theorem 10.3. *Fix a functor* $\mathbb{F} \colon \mathbb{I} \to \mathbb{MAdj}$. *Then the functor* $\hat{\mathbb{F}} \colon \mathbb{J} \to \mathbb{MAdj}$ *constructed in example 10.2 is terminal in the category* $\mathbf{CDMCat}_{\mathbb{F}}(\mathbb{J}, \mathbb{MAdj})$ *of functors on* $\mathbb{J}$ *restricting to* $\mathbb{F}$ *on* $\mathbb{I}$.

Proof. Concretely, the theorem says that given the data of a functor $\mathbb{J} \to \mathbb{MAdj}$, there is a unique 2-cell

$$
\begin{array}{ccc}
\mathscr{B}_1, \mathscr{B}_2 & \xrightarrow{\ H_1, H_2\ } & \mathscr{A}_1^2, \mathscr{A}_2^2 \\
G_0 \downarrow & \Downarrow \theta & \downarrow \hat{F}_0 \\
\mathscr{B}_0^{\bullet} & \xrightarrow[\ H_3^{\bullet}\ ]{} & \mathscr{A}_0^{\bullet 2}
\end{array}
$$

such that

$$
\begin{array}{ccc}
\mathscr{B}_1,\mathscr{B}_2 \xrightarrow{H_1,H_2} \mathscr{A}_1^{\mathbf{2}},\mathscr{A}_2^{\mathbf{2}} \xrightarrow{\text{cod,cod}} \mathscr{A}_1,\mathscr{A}_2 & & \mathscr{B}_1,\mathscr{B}_2 \xrightarrow{d_0,d_0} \mathscr{A}_1,\mathscr{A}_2 \\
G_0\downarrow \quad \Downarrow\theta \quad \hat{F}_0\downarrow \quad \Downarrow\text{id} \quad \downarrow F_0 & = & G_0\downarrow \quad \Downarrow\lambda \quad \downarrow F_0 \\
\mathscr{B}_0^\bullet \xrightarrow[H_3^\bullet]{} \mathscr{A}_0^{\bullet\mathbf{2}} \xrightarrow[\text{dom}^\bullet]{} \mathscr{A}_0^\bullet & & \mathscr{B}_0^\bullet \xrightarrow[d_1^\bullet]{} \mathscr{A}_0^\bullet
\end{array}
\tag{34}
$$

$$
\begin{array}{ccc}
\mathscr{B}_1,\mathscr{B}_2 \xrightarrow{H_1,H_2} \mathscr{A}_1^{\mathbf{2}},\mathscr{A}_2^{\mathbf{2}} \xrightarrow{\text{dom,cod}} \mathscr{A}_1,\mathscr{A}_2 & & \mathscr{B}_1,\mathscr{B}_2 \xrightarrow{d_1,d_0} \mathscr{A}_1,\mathscr{A}_2 \\
G_0\downarrow \quad \Downarrow\theta \quad \hat{F}_0\downarrow \quad \Downarrow p_1 \quad \downarrow F_0 & = & G_0\downarrow \quad \Downarrow\lambda \quad \downarrow F_0 \\
\mathscr{B}_0^\bullet \xrightarrow[H_3^\bullet]{} \mathscr{A}_0^{\bullet\mathbf{2}} \xrightarrow[\text{cod}^\bullet]{} \mathscr{A}_0^\bullet & & \mathscr{B}_0^\bullet \xrightarrow[d_0^\bullet]{} \mathscr{A}_0^\bullet
\end{array}
\tag{35}
$$

$$
\begin{array}{ccc}
\mathscr{B}_1,\mathscr{B}_2 \xrightarrow{H_1,H_2} \mathscr{A}_1^{\mathbf{2}},\mathscr{A}_2^{\mathbf{2}} \xrightarrow{\text{cod,dom}} \mathscr{A}_1,\mathscr{A}_2 & & \mathscr{B}_1,\mathscr{B}_2 \xrightarrow{d_0,d_1} \mathscr{A}_1,\mathscr{A}_2 \\
G_0\downarrow \quad \Downarrow\theta \quad \hat{F}_0\downarrow \quad \Downarrow p_2 \quad \downarrow F_0 & = & G_0\downarrow \quad \Downarrow\lambda \quad \downarrow F_0 \\
\mathscr{B}_0^\bullet \xrightarrow[H_3^\bullet]{} \mathscr{A}_0^{\bullet\mathbf{2}} \xrightarrow[\text{cod}^\bullet]{} \mathscr{A}_0^\bullet & & \mathscr{B}_0^\bullet \xrightarrow[d_0^\bullet]{} \mathscr{A}_0^\bullet
\end{array}
\tag{36}
$$

Fix objects $B_1 \in \mathscr{B}_1$, $B_2 \in \mathscr{B}_2$. The H_i are the functors sending B_i to $H_i(B_i) = \alpha_{B_i}\colon d_1 B_i \to d_0 B_i$. The component of θ at (B_1, B_2) is a square

$$
\begin{array}{ccc}
d_1 G_0(B_1, B_2) & \longrightarrow & F_0(d_0 B_1, d_0 B_2) \\
\downarrow & & \downarrow \\
d_0 G_0(B_1, B_2) & \longrightarrow & F_0(d_1 B_1, d_0 B_2) \underset{F_0(d_1 B_1, d_1 B_2)}{\prod} F_0(d_0 B_1, d_1 B_2)
\end{array}
$$

The top arrow is uniquely determined by equation (34), while the components of the bottom arrow are uniquely determined by equations (35) and (36). $\qquad\square$

Now let $\mathbb{M}$ be any double multicategory. Given an object C of $\mathbb{M}$, an arrow object C^2 is an object together with a globular 2-cell $\kappa\colon \text{dom} \Rightarrow \text{cod}$ satisfying the same universal property as in Section 3.1 (this only involves the horizontal 2-category, so carries over unchanged).

Given a vertical 1-cell $F\colon (C_1, C_2) \to C_0^\bullet$, the lift to arrow objects $\hat{F}$ is a vertical 1-cell $\hat{F}$ together with 2-cells

$$
\begin{array}{ccccc}
C_1^2, C_2^2 \xrightarrow{\text{cod,cod}} C_1, C_2 & \quad & C_1^2, C_2^2 \xrightarrow{\text{dom,cod}} C_1, C_2 & \quad & C_1^2, C_2^2 \xrightarrow{\text{cod,dom}} C_1, C_2 \\
\hat{F}\downarrow \quad \Downarrow\gamma_0 \quad \downarrow F & & \hat{F}\downarrow \quad \Downarrow\gamma_1 \quad \downarrow F & & \hat{F}\downarrow \quad \Downarrow\gamma_2 \quad \downarrow F \\
C_0^{\bullet 2} \xrightarrow[\text{dom}^\bullet]{} C_0^\bullet & & C_0^{\bullet 2} \xrightarrow[\text{cod}^\bullet]{} C_0^\bullet & & C_0^{\bullet 2} \xrightarrow[\text{cod}^\bullet]{} C_0^\bullet
\end{array}
$$

satisfying the equations

$$
\begin{array}{ccc}
C_1^2, C_2^2 \xrightarrow{\mathrm{dom,cod}} C_1, C_2 & & C_1^2, C_2^2 \;\substack{\xrightarrow{\mathrm{dom,cod}} \\ \Downarrow\kappa,\mathrm{id} \\ \xrightarrow{\mathrm{cod,cod}}}\; C_1, C_2 \\
\Big\downarrow{}^{G_0}\;\; \Downarrow\gamma_1 \;\;\Big\downarrow{}^{F_0} & = & \Big\downarrow{}^{G_0}\;\; \Downarrow\gamma_0 \;\;\Big\downarrow{}^{F_0} \\
C_0^{\bullet 2} \;\substack{\xrightarrow{\mathrm{cod}^\bullet} \\ \Downarrow\kappa^\bullet \\ \xrightarrow{\mathrm{dom}^\bullet}}\; C_0^\bullet & & C_0^{\bullet 2} \xrightarrow{\mathrm{dom}^\bullet} C_0^\bullet
\end{array}
\qquad (37)
$$

$$
\begin{array}{ccc}
C_1^2, C_2^2 \xrightarrow{\mathrm{cod,dom}} C_1, C_2 & & C_1^2, C_2^2 \;\substack{\xrightarrow{\mathrm{cod,dom}} \\ \Downarrow\mathrm{id},\kappa \\ \xrightarrow{\mathrm{cod,cod}}}\; C_1, C_2 \\
\Big\downarrow{}^{G_0}\;\; \Downarrow\gamma_2 \;\;\Big\downarrow{}^{F_0} & = & \Big\downarrow{}^{G_0}\;\; \Downarrow\gamma_0 \;\;\Big\downarrow{}^{F_0} \\
C_0^{\bullet 2} \;\substack{\xrightarrow{\mathrm{cod}^\bullet} \\ \Downarrow\kappa^\bullet \\ \xrightarrow{\mathrm{dom}^\bullet}}\; C_0^\bullet & & C_0^{\bullet 2} \xrightarrow{\mathrm{dom}^\bullet} C_0^\bullet
\end{array}
\qquad (38)
$$

$$
\begin{array}{ccc}
C_1^2, C_2^2 \;\substack{\xrightarrow{\mathrm{dom,dom}} \\ \Downarrow\mathrm{id},\kappa \\ \xrightarrow{\mathrm{dom,cod}}}\; C_1, C_2 & & C_1^2, C_2^2 \;\substack{\xrightarrow{\mathrm{dom,dom}} \\ \Downarrow\kappa,\mathrm{id} \\ \xrightarrow{\mathrm{cod,dom}}}\; C_1, C_2 \\
\Big\downarrow{}^{G_0}\;\; \Downarrow\gamma_1 \;\;\Big\downarrow{}^{F_0} & = & \Big\downarrow{}^{G_0}\;\; \Downarrow\gamma_2 \;\;\Big\downarrow{}^{F_0} \\
C_0^{\bullet 2} \xrightarrow{\mathrm{cod}^\bullet} C_0^\bullet & & C_0^{\bullet 2} \xrightarrow{\mathrm{cod}^\bullet} C_0^\bullet
\end{array}
\qquad (39)
$$

and which is universal, meaning that given any objects X_0, X_1, X_2, horizontal 1-cells $d_{i,0}, d_{i,1} \colon X_i \to C_i$, a vertical 1-cell $G \colon X_1, X_2 \to X_0^\bullet$, globular 2-cells $\alpha_i \colon d_{i,1} \Rightarrow d_{i,0}$, and 2-cells

$$
\begin{array}{ccc}
X_1, X_2 \xrightarrow{d_{1,0}, d_{2,0}} C_1, C_2 & X_1, X_2 \xrightarrow{d_{1,1}, d_{2,0}} C_1, C_2 & X_1, X_2 \xrightarrow{d_{1,0}, d_{2,1}} C_1, C_2 \\
\Big\downarrow{}^{G}\;\; \Downarrow\lambda_0 \;\;\Big\downarrow{}^{F} & \Big\downarrow{}^{G}\;\; \Downarrow\lambda_0 \;\;\Big\downarrow{}^{F} & \Big\downarrow{}^{G}\;\; \Downarrow\lambda_2 \;\;\Big\downarrow{}^{F} \\
X_0^\bullet \xrightarrow{d_{0,1}^\bullet} C_0^\bullet & X_0^\bullet \xrightarrow{d_{0,0}^\bullet} C_0^\bullet & X_0^\bullet \xrightarrow{d_{0,0}^\bullet} C_0^\bullet
\end{array}
$$

satisfying the three equations analagous to (37)–(39), there exists a unique 2-cell

$$
\begin{array}{ccc}
X_1, X_2 \xrightarrow{\hat{\alpha}_1, \hat{\alpha}_2} C_1^2, C_2^2 \\
\Big\downarrow{}^{G}\;\; \Downarrow\theta \;\;\Big\downarrow{}^{\hat{F}} \\
X_0^\bullet \xrightarrow{\hat{\alpha}_0^\bullet} C_0^{\bullet 2}
\end{array}
$$

(where $\hat{\alpha}_i$ is the 1-cell determined by α_i by the universal property of the arrow object C_i) such that

$$
\begin{array}{ccccccccc}
X_1, X_2 & \xrightarrow{\hat{\alpha}_1, \hat{\alpha}_2} & C_1^2, C_2^2 & \longrightarrow & C_1, C_2 & & X_1, X_2 & \longrightarrow & C_1, C_2 \\
\downarrow{\scriptstyle G} & \Downarrow\theta & \downarrow{\scriptstyle \hat{F}} & \Downarrow\gamma_i & \downarrow{\scriptstyle F} & = & \downarrow{\scriptstyle G} & \Downarrow\lambda_i & \downarrow{\scriptstyle F} \\
X_0^\bullet & \xrightarrow{\hat{\alpha}_0^\bullet} & C_0^{\bullet 2} & \longrightarrow & C_0^\bullet & & X_0^\bullet & \longrightarrow & C_0^\bullet
\end{array}
$$

for each $i \in \{0, 1, 2\}$.

Similarly, we define the lift of a vertical 1-cell $F: (C_1, \ldots, C_n) \to C_0^\bullet$ to arrow objects to be a vertical 1-cell $\hat{F}$ together with $(n+1)$ 2-cells γ_i satisfying $(n+1)$ equations analagous to (37)–(39) and which is universal in the analagous way.

Definition 10.4. Let $\mathbb{M}$ be a double multicategory. We say $\mathbb{M}$ *has arrow objects* if for every object C there is an arrow object C^2, and if for every vertical 1-cell $F: (C_1, \ldots, C_n) \to C_0^\bullet$ there is a lift to arrow objects $\hat{F}$.

We have given the universal property of arrow objects and lifts of vertical 1-cells in ordinary double multicategories, but it is clear from the cyclical symmetry of the construction that a cyclic action respects arrow objects. Specifically, $(C^2)^\bullet = (C^\bullet)^2$ for any object C, and $\sigma(\hat{F}) = \widehat{\sigma F}$ for any vertical 1-cell F.

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