1 Double Categories

In this section, we will give an overview of double categories, as well as (one possible version of) the definition of monads in a double category

A (strict) double category is a two-dimensional categorical structure, similar to a 2-category. Unlike a 2-category, a double category has two types of 1-cells, called *vertical* and *horizontal*, and 2-cells all have a square shape, with domain and codomain horizontal 1-cells as well as domain and codomain vertical 1-cells.

We will first give the most concise definition of a double category, which we will then break down into more concrete terms.

Definition 1.1. A (strict) *double category* is an internal category object in the (large) category of categories.

So a double category $\mathbb{D}$ consists of a category $\mathbb{D}_0$ and a category $\mathbb{D}_1$, along with functors $s, t\colon \mathbb{D}_1 \to \mathbb{D}_0$, $i\colon \mathbb{D}_0 \to \mathbb{D}_1$, and $\otimes\colon \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \to \mathbb{D}_1$ satisfying the usual axioms of a category. We will call the objects of $\mathbb{D}_0$ the 0-cells of $\mathbb{D}$, and the morphisms of $\mathbb{D}_0$ the vertical 1-cells. Thus $\mathbb{D}_0$ forms the so-called *vertical category* of $\mathbb{D}$. We will call the objects of $\mathbb{D}_1$ the horizontal 1-cells of $\mathbb{D}$, and the morphisms of $\mathbb{D}_1$ are the 2-cells.

A morphism $\phi\colon X \to Y$ in $\mathbb{D}_1$, where $s(X) = C$, $t(X) = C'$, $s(Y) = D$, $t(Y) = D'$, $s(\phi) = f$, and $t(\phi) = g$ will be drawn as

$$\begin{tikzcd} C \arrow[r, "X", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & C' \arrow[d, "g"] \\ D \arrow[r, "Y"', "\shortmid"{marking}] & D' \end{tikzcd} \tag{1}$$

where the tick-mark on the horizontal 1-cells serves as a further reminder that the horizontal 1-cells are of a different nature than the vertical 1-cells. The composition in $\mathbb{D}_0$ provides a vertical composition of vertical 1-cells and 2-cells, while the composition functor $\otimes\colon \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \to \mathbb{D}_1$ provides a horizontal composition of horizontal 1-cells and 2-cells.

For any object C in $\mathbb{D}_0$, $i(C)$ is the *unit* horizontal 1-cell

$$C \xrightarrow{I_C} C$$

and acts as an identity with respect to the horizontal composition.

A 2-cell θ for which $s\theta = t\theta = \mathrm{id}$ will be called *globular*. We will sometimes draw globular 2-cells as

$$\begin{tikzcd} C \arrow[r, bend left, "X", "\shortmid"{marking}] \arrow[r, bend right, "Y"', "\shortmid"{marking}] \arrow[r, phantom, "\Downarrow\theta"] & C', \end{tikzcd}$$

to save space and help readability of diagrams.

Example 1.2. For any 2-category $\mathcal{D}$, there is an associated double category $\mathbb{S}\mathrm{q}(\mathcal{D})$ of *squares* in $\mathcal{D}$, in which the vertical and horizontal 1-cells are both just 1-cells in $\mathcal{D}$, and 2-cells

$$\begin{tikzcd} C \arrow[r, "j", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & C' \arrow[d, "g"] \\ D \arrow[r, "k"', "\shortmid" marking] & D' \end{tikzcd}$$

are simply 2-cells $\phi\colon gj \Rightarrow kf$ in $\mathcal{D}$.

Example 1.3. Given any category M, there is a pseudo double category $\mathbb{S}\mathrm{pan}(M)$ of *spans* in M. The vertical category of $\mathbb{S}\mathrm{pan}(M)$ is just M, while horizontal 1-cells

$$\begin{tikzcd} C \arrow[r, "X", "\shortmid" marking] & D \end{tikzcd}$$

are given by spans

$$\begin{tikzcd} C & X \arrow[l, "j"'] \arrow[r, "k"] & D \end{tikzcd}$$

in M, and 2-cells

$$\begin{tikzcd} C \arrow[r, "X", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\theta"] & D \arrow[d, "g"] \\ C' \arrow[r, "Y"', "\shortmid" marking] & D' \end{tikzcd}$$

are given by commutative diagrams

$$\begin{tikzcd} C \arrow[d, "f"'] & X \arrow[l, "j"'] \arrow[r, "k"] \arrow[d, "\theta"] & D \arrow[d, "g"] \\ C' & Y \arrow[l, "j'"] \arrow[r, "k'"'] & D.' \end{tikzcd}$$

The horizontal composition of spans is given by pullback. It is because this horizontal composition is only determined up to isomorphsim that this example is not a strict double category.

Definition 1.4. For any double category $\mathbb{D}$, there is an associated 2-category $\mathcal{H}or(\mathbb{D})$, called the *horizontal 2-category* of $\mathbb{D}$. The objects and 1-cells of $\mathcal{H}or(\mathbb{D})$ are the objects and horizontal 1-cells of $\mathbb{D}$, while 2-cells $\phi\colon X \Rightarrow Y$ in $\mathcal{H}or(\mathbb{D})$ are the globular 2-cells in $\mathbb{D}$, i.e. those of the form

$$\begin{tikzcd} C \arrow[r, "X", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\phi"] & D \arrow[d, equal] \\ C \arrow[r, "Y"', "\shortmid" marking] & D \end{tikzcd}$$

Notice that $\mathcal{H}or(\mathbb{S}\mathrm{q}(\mathcal{D}))$ is isomorphic to $\mathcal{D}$.

Definition 1.5. Given a double category $\mathbb{D}$, define double categories $\mathbb{D}^{\mathrm{vop}}$ and $\mathbb{D}^{\mathrm{hop}}$, obtained by reversing the direction of the vertical and horizontal 1-cells respectively, and changing the orientation of the 2-cells as appropriate. For example, a 2-cell (1) in $\mathbb{D}^{\mathrm{vop}}$ is a 2-cell

$$\begin{tikzcd} D \arrow[r, "Y", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & D' \arrow[d, "g"] \\ C \arrow[r, "X"', "\shortmid"{marking}] & C' \end{tikzcd}$$

in $\mathbb{D}$.

In terms of Definition 1.1, $\mathbb{D}^{\mathrm{vop}}$ is the double category obtained by replacing the categories $\mathbb{D}_0$ and $\mathbb{D}_1$ with their opposites, while $\mathbb{D}^{\mathrm{vop}}$ is the obtained by swapping the horizontal source and target functors s and t.

1.1 Arrow Objects in a Double Category

In the following we will need an extension of the universal property (**??**) to double categories. Fortunately, this is quite straightforward.

Let $\mathbb{D}$ be a double category. Given an object C of $\mathbb{D}$, the *arrow object* C^2, if it exists, is an object together with a diagram

$$\begin{tikzcd} C^2 \arrow[r, bend left, "\mathrm{dom}", "\shortmid"{marking}] \arrow[r, bend right, "\mathrm{cod}"', "\shortmid"{marking}] \arrow[r, phantom, "\Downarrow\kappa"] & C, \end{tikzcd}$$

such that any 2-cell

$$\begin{tikzcd} A \arrow[r, bend left, "d_1", "\shortmid"{marking}] \arrow[r, bend right, "d_0"', "\shortmid"{marking}] \arrow[r, phantom, "\Downarrow\alpha"] & C \end{tikzcd}$$

uniquely factors through κ, as

$$\begin{tikzcd} A \arrow[r, "\hat{\alpha}", "\shortmid"{marking}] & C^2 \arrow[r, bend left, "\mathrm{dom}", "\shortmid"{marking}] \arrow[r, bend right, "\mathrm{cod}"', "\shortmid"{marking}] \arrow[r, phantom, "\Downarrow\kappa"] & C. \end{tikzcd}$$

Given a vertical 1-cell $F\colon C \to D$ in $\mathbb{D}$, the *lift to arrow objects* $\hat{F}\colon C^2 \to D^2$, if it exists, is a vertical 1-cell $\hat{F}\colon C^2 \to D^2$ together with 2-cells

$$\begin{tikzcd} C^2 \arrow[r, "\mathrm{dom}", "\shortmid"{marking}] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_1"] & C \arrow[d, "F"] \\ D^2 \arrow[r, "\mathrm{dom}"', "\shortmid"{marking}] & D \end{tikzcd} \qquad \begin{tikzcd} C^2 \arrow[r, "\mathrm{cod}", "\shortmid"{marking}] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_0"] & C \arrow[d, "F"] \\ D^2 \arrow[r, "\mathrm{cod}"', "\shortmid"{marking}] & D \end{tikzcd}$$

satisfying

$$
\begin{tikzcd}
C^2 \arrow[r, "\mathrm{dom}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_1"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "\mathrm{dom}"] \arrow[r, bend right=50, "\mathrm{cod}"'] \arrow[r, phantom, shift right=3, "\Downarrow\kappa"] & D
\end{tikzcd}
=
\begin{tikzcd}
C^2 \arrow[r, bend left=50, "\mathrm{dom}"] \arrow[r, phantom, shift left=3, "\Downarrow\kappa"] \arrow[r, "\mathrm{cod}"'] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_0"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "\mathrm{cod}"'] & D,
\end{tikzcd}
$$

such that for any 2-cells

$$
\begin{tikzcd}
A \arrow[r, bend left, "d_1"] \arrow[r, bend right, "d_0"'] \arrow[r, phantom, "\Downarrow\alpha"] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, bend left, "d_1'"] \arrow[r, bend right, "d_0'"'] \arrow[r, phantom, "\Downarrow\alpha'"] & D
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
A \arrow[r, "d_1"] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\lambda_1"] & C \arrow[d, "F"] \\
B \arrow[r, "d_1'"'] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "d_0"] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\lambda_0"] & C \arrow[d, "F"] \\
B \arrow[r, "d_0'"'] & D
\end{tikzcd}
$$

satisfying

$$
\begin{tikzcd}
A \arrow[r, "d_1"] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\lambda_1"] & C \arrow[d, "F"] \\
B \arrow[r, "d_1'"] \arrow[r, bend right=50, "d_0'"'] \arrow[r, phantom, shift right=3, "\Downarrow\alpha'"] & D
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, bend left=50, "d_1"] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, "d_0"'] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\lambda_0"] & C \arrow[d, "F"] \\
B \arrow[r, "d_0'"'] & D
\end{tikzcd}
$$

there is a unique 2-cell

$$
\begin{tikzcd}
A \arrow[r, "\hat{\alpha}"] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\theta"] & C^2 \arrow[d, "\hat{F}"] \\
B \arrow[r, "\hat{\alpha}'"'] & D^2
\end{tikzcd}
$$

such that the horizontal composition of θ with γ_0 and γ_1 is respectively equal to λ_0 and λ_1.

Definition 1.6. A double category $\mathbb{D}$ *has arrow objects* if for every object C of $\mathbb{D}$ there is an object C^2 and 2-cell κ, and for every vertical 1-cell F there is a vertical 1-cell $\hat{F}$ and 2-cells γ_0 and γ_1, satisfying the universal properties given above.

The intuition that this is a generalization of Lemma **??** is supported by the following two propositions, the (easy) proofs of which are left to the reader.

Proposition 1.7. *If the double category $\mathbb{D}$ has arrow objects, then so does $\mathcal{H}or(\mathbb{D})$.*

Proposition 1.8. *If the 2-category $\mathcal{D}$ has arrow objects, then so does $\mathbb{S}\mathrm{q}(\mathcal{D})$.*

Proof. A simple check. The 2-cells γ_0 and γ_1 will always be identities. □

1.2 Monads

We will define a *monad* in a double category $\mathbb{D}$ to be a tuple (C, T, η, μ), in which C is an object, $T\colon C \to C$ is a horizontal 1-cell, and η and μ are 2-cells

$$
\begin{tikzcd}
C \arrow[r, "\mathrm{id}_C"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta"] & C \arrow[d, equal] \\
C \arrow[r, "T"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "T"] \arrow[d, equal] & C \arrow[r, "T"] \arrow[d, phantom, "\Downarrow\mu"] & C \arrow[d, equal] \\
C \arrow[rr, "T"'] & & C
\end{tikzcd}
$$

satisfying the usual unit and associativity conditions.

Given two monads (C, T, η, μ) and (D, S, η', μ'), a monad morphism from (C, T) to (D, S) consists of a pair (f, ϕ), where f is a vertical 1-cell $C \to D$ and ϕ is a 2-cell

$$
\begin{tikzcd}
C \arrow[r, "T"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[d, "f"] \\
D \arrow[r, "S"'] & D
\end{tikzcd}
$$

which commutes with the unit and multiplication 2-cells in the sense of the two equations

$$
\begin{tikzcd}
C \arrow[r, "\mathrm{id}_C"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta"] & C \arrow[d, equal] \\
C \arrow[r, "T"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[d, "f"] \\
D \arrow[r, "S"'] & D
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[r, "\mathrm{id}_C"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}_f"] & C \arrow[d, "f"] \\
D \arrow[r, "\mathrm{id}_D"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta'"] & D \arrow[d, equal] \\
D \arrow[r, "S"'] & D
\end{tikzcd}
\tag{2}
$$

and

$$
\begin{tikzcd}
C \arrow[r, "T"] \arrow[d, equal] & C \arrow[r, "T"] \arrow[d, phantom, "\Downarrow\mu"] & C \arrow[d, equal] \\
C \arrow[rr, "T"] \arrow[d, "f"'] & \arrow[d, phantom, "\Downarrow\phi"] & C \arrow[d, "f"] \\
D \arrow[rr, "S"'] & & D
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[r, "T"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[r, "T"] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[d, "f"] \\
D \arrow[r, "S"] \arrow[d, equal] & D \arrow[r, "S"] \arrow[d, phantom, "\Downarrow\mu'"] & D \arrow[d, equal] \\
D \arrow[rr, "S"'] & & D
\end{tikzcd}
\tag{3}
$$

Definition 1.9. Given any double category $\mathbb{D}$, we will write $\mathrm{Mon}(\mathbb{D})$ for the category of monads in $\mathbb{D}$, consisting of monads and monad morphisms as defined above. The category $\mathrm{Comon}(\mathbb{D})$ of comonads in $\mathbb{D}$ is defined to be the category $\mathrm{Mon}(\mathbb{D}^{\mathrm{op}})$ of monads in $\mathbb{D}^{\mathrm{op}}$.

Example 1.10. The category $\mathrm{Mon}(\mathbb{S}\mathrm{pan}(\mathbf{Set}))$ is precisely the category of small categories. It is an easy and enlightening exercise to work this out for oneself.

Proposition 1.11. *The categories of (co)monads and (co)lax morphisms in a 2-category $\mathcal{D}$ can be given in terms of (co)monads in the double category of squares as follows:*

$$\begin{aligned}
\mathrm{Mon}_{colax}(\mathcal{D}) &= \mathrm{Mon}(\mathbb{S}\mathrm{q}(\mathcal{D})) \\
\mathrm{Comon}_{colax}(\mathcal{D}) &= \mathrm{Comon}(\mathbb{S}\mathrm{q}(\mathcal{D})) \\
\mathrm{Mon}_{lax}(\mathcal{D}) &= \mathrm{Mon}(\mathbb{S}\mathrm{q}(\mathcal{D}^{op}))^{op} \\
\mathrm{Comon}_{lax}(\mathcal{D}) &= \mathrm{Comon}(\mathbb{S}\mathrm{q}(\mathcal{D}^{op}))^{op}
\end{aligned}$$

where by $\mathcal{D}^{op}$ we mean the 2-category obtained by reversing the direction of all 1-cells (but not 2-cells).

Proof. Immediate from the definitions. Readers unfamiliar with (co)lax morphisms of monads can take this as the definition. □

1.3 Double Functors

The natural notion of functor between double categories is a straightforward generalization of lax functors between monoidal categories. Recall that we are using the symbol $\otimes$ to denote horizontal composition.

Definition 1.12. Let $\mathbb{D}$ and $\mathbb{E}$ be double categories. A *lax double functor* $F\colon \mathbb{D} \to \mathbb{E}$ consists of:

- Functors $F_0\colon \mathbb{D}_0 \to \mathbb{E}_0$ and $F_1\colon \mathbb{D}_1 \to \mathbb{E}_1$ such that $sF_1 = F_0 s$ and $tF_1 = F_0 t$
- Natural transformations with globular components $F_\otimes\colon F_1 X \otimes F_1 Y \to F_1(X \otimes Y)$ and $F_I\colon I_{F_0 C} \to F_1(I_C)$, which satisfy the usual coherence axioms for a lax monoidal functor.

A lax double functor F for which the components of F_I and $F_\otimes$ are identities will be called *strict*. For the intermediate notion where the components of F_I and $F_\otimes$ are (vertical) isomorphsims, we will simply refer to F as a double functor.

Proposition 1.13. *A lax double functor $F\colon \mathbb{D} \to \mathbb{E}$ induces a functor $F\colon \mathrm{Mon}(\mathbb{D}) \to \mathrm{Mon}(\mathbb{E})$.*

Proof. This works just like the case for monoidal categories. For instance, if X is a monad in $\mathbb{D}$, FX has the multiplication

$$
\begin{tikzcd}
C \arrow[r, "FX"] \arrow[d, equal] & C \arrow[r, "FX"] \arrow[d, phantom, "\Downarrow F_{\otimes}"] & C \arrow[d, equal] \\
C \arrow[rr, "F(X\otimes X)"] \arrow[d, equal] & \arrow[d, phantom, "\Downarrow F\mu"] & C \arrow[d, equal] \\
C \arrow[rr, "FX"'] & & C
\end{tikzcd}
$$

The fact that F takes monad morphisms to monad morphisms can easily be checked using the naturality of F_I and $F_{\otimes}$. $\square$