1 Composition of $\mathbb{L}$-coalgebras

For this section, we will continue to let $\mathbb{D} = \mathbb{S}\mathrm{q}(\mathcal{D})$ be the double category of squares in a 2-category $\mathcal{D}$ with arrow objects.

In an algebraic weak factorization system, the categories $\mathbb{L}$-Coalg and $\mathbb{R}$-Alg respectively play the roles of the left and right classes of morphisms of the weak factorization system. In an ordinary weak factorization system, these two classes of morphisms are closed under composition. In [Gar09], this is strengthened to a composition functor

$$\mathbb{L}\text{-Coalg} \sqcap_C \mathbb{L}\text{-Coalg} \to \mathbb{L}\text{-Coalg}$$

and in [Rie11], it is shown that colax morphisms of awfs preserve this composition. Similarly, there is a composition functor on $\mathbb{R}$-Alg which is preserved by lax morphisms of awfs.

In this section, we will generalize these results to the setting of bimonads in $\mathbb{FF}(\mathbb{D})$.

First, recall from [Str72] the following proposition.

Proposition 1.1. *Let C be a category, and $\mathbb{L} = (L, \epsilon, \delta)$ be a comonad on C. The category of coalgebras $\mathbb{L}$-Coalg has a universal property as follows:*

- *There is a forgetful functor $U\colon \mathbb{L}\text{-Coalg} \to C$ and a natural transformtion $\alpha\colon U \Rightarrow LU$, satisfying $\epsilon U \circ \alpha = \mathrm{id}_U$ and $\delta U \circ \alpha = L\alpha \circ \alpha$.*
- *(U, α) is universal among such pairs satisfying such equations. Given another such pair (F, β), where $F\colon X \to C$, there exists a unique functor $\hat{F}\colon X \to \mathbb{L}\text{-Coalg}$ such that $U\hat{F} = F$ and $\alpha\hat{F} = \beta$.*

Any colax morphism of comonads $(F, \phi)\colon (C, L_1, \epsilon_1, \delta_1) \to (D, L_2, \epsilon_2, \delta_2)$ induces a functor $\tilde{F}\colon \mathbb{L}_1\text{-Coalg} \to \mathbb{L}_2\text{-Coalg}$ such that $U\tilde{F} = FU$.

For the rest of this section, assume that $\mathcal{D}$ *has EM-objects for comonads*, i.e. for every comonad $\mathbb{L}$ in $\mathcal{D}$ there is an object $\mathbb{L}$-Coalg satisfying the universal property above.

The main goal of this section will be to prove the following theorem:

Theorem 1.2. *There is a lax double functor*

$$\mathrm{Coalg}\colon \mathbb{C}\mathrm{omon}(\mathbb{FF}(\mathbb{S}\mathrm{q}(\mathcal{D}))) \to \mathbb{S}\mathrm{pan}(\mathcal{D}_0)$$

where $\mathcal{D}_0$ is the ordinary category underlying the (strict) 2-category $\mathcal{D}$, which is the identity on the vertical categories, and which takes a comonad $(E, \eta, \epsilon, \delta)$ in $\mathbb{FF}(\mathbb{S}\mathrm{q}(\mathcal{D}))$ to the span

$$C \xleftarrow{\mathrm{dom}\,U} \mathbb{L}\text{-Coalg} \xrightarrow{\mathrm{cod}\,U} C.$$

Before we get to the proof of Theorem 1.2, we will need to establish several technical lemmas.

Consider a comonad in $\mathbb{FF}(\mathbb{D})$ on an object C, i.e. a functorial factorization with half of the awfs structure. We can combine the universal properties of EM-objects and arrow objects into a universal property for $\mathbb{L}$-Coalg, where now $\mathbb{L}$ is the comonad in $\mathcal{D}$ arising from the comonad in $\mathbb{FF}(\mathbb{D})$.

Lemma 1.3. *Let $(E, \eta, \epsilon, \delta)$ be a comonad in $\mathbb{FF}(\mathbb{D})$ on an object C. There is a 2-cell*

$$\begin{tikzcd} & C^2 \arrow[dr, "\mathrm{cod}"] \arrow[dd, phantom, "\Downarrow\alpha"] & \\ \mathbb{L}\text{-Coalg} \arrow[ur, bend left, "U"] \arrow[dr, bend right, "U"'] & & C \\ & C^2 \arrow[ur, "E"'] & \end{tikzcd}$$

satisfying equations

$$\begin{tikzcd} \mathbb{L}\text{-Coalg} \arrow[r, "U"] \arrow[dr, bend right, "U"'] & C^2 \arrow[r, bend left=60, "\mathrm{dom}"] \arrow[r, phantom, shift left=3, "\Downarrow\kappa"] \arrow[r, "\mathrm{cod}"'] \arrow[d, phantom, "\Downarrow\alpha"] & C \\ & C^2 \arrow[ur, "E"'] & \end{tikzcd} = \begin{tikzcd} \mathbb{L}\text{-Coalg} \arrow[r, "U"] & C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, phantom, "\Downarrow\eta"] \arrow[r, bend right, "E"'] & C \end{tikzcd} \tag{1}$$

$$\begin{tikzcd} & C^2 \arrow[dr, "\mathrm{cod}"] \arrow[d, phantom, "\Downarrow\alpha"] & \\ \mathbb{L}\text{-Coalg} \arrow[ur, bend left, "U"] \arrow[r, "U"'] & C^2 \arrow[r, "E"] \arrow[r, phantom, shift right=3, "\Downarrow\epsilon"] \arrow[r, bend right=60, "\mathrm{cod}"'] & C \end{tikzcd} = \begin{tikzcd} X \arrow[r, "U"] & C^2 \arrow[r, "\mathrm{cod}"] & C \end{tikzcd} \tag{2}$$

$$\begin{tikzcd} & C^2 \arrow[dr, "\mathrm{cod}"] \arrow[d, phantom, "\Downarrow\alpha"] & \\ \mathbb{L}\text{-Coalg} \arrow[ur, bend left, "U"] \arrow[r, "U"'] & C^2 \arrow[r, "E"] \arrow[dr, "L"'] \arrow[r, phantom, shift right=4, "\Downarrow\delta"] & C \\ & & C^2 \arrow[u, "E"'] \end{tikzcd} = \begin{tikzcd} & C^2 \arrow[dr, "\mathrm{cod}"] \arrow[d, phantom, "\Downarrow\alpha"] & \\ \mathbb{L}\text{-Coalg} \arrow[ur, bend left, "U"] \arrow[r, "U"] \arrow[dr, bend right, "U"'] & C^2 \arrow[r, "E"'] \arrow[d, phantom, "\Downarrow\vec{\alpha}"] & C. \\ & C^2 \arrow[u, "L"'] & \end{tikzcd} \tag{3}$$

where $\vec{\alpha}$ is the unique 2-cell such that $\mathrm{dom}\,\vec{\alpha} = \mathrm{id}_{\mathrm{dom}\,U}$ and $\mathrm{cod}\,\vec{\alpha} = \alpha$, the existence of which is implied by Equation 1.

Given any object X, together with a morphism $F\colon X \to C^2$ and a 2-cell $\beta\colon \mathrm{cod}\,F \Rightarrow EF$ satisfying equations

1. $\beta \circ \kappa F = \eta F$
2. $\epsilon F \circ \beta = \mathrm{id}_{\mathrm{cod}\,F}$
3. $\delta F \circ \beta = E\vec{\beta} \circ \beta$

where $\vec{\beta}\colon F \Rightarrow LF$ is the unique 2-cell such that $\mathrm{dom}\,\vec{\beta} = \mathrm{id}_{\mathrm{dom}\,F}$ and $\mathrm{cod}\,\vec{\beta} = \beta$; there is a unique morphism $\hat{F}\colon X \to \mathbb{L}\text{-Coalg}$ such that $U\hat{F} = F$ and $\alpha\hat{F} = \vec{\beta}$.

Proof. U is simply the U from proposition 1.1, while the 2-cell α there is the 2-cell $\vec{\alpha}$ here. The equation $\vec{\epsilon}U \circ \vec{\alpha} = \mathrm{id}_F$ implies that $\mathrm{dom}\,\vec{\alpha} = \mathrm{id}_{\mathrm{dom}\,U}$. With that observation, the rest of the equations follow immediately from the universal property of C^2 and the equations $\epsilon U \circ \alpha = \mathrm{id}_U$ and $\delta U \circ \alpha = L\alpha \circ \alpha$ from Proposition 1.1. ☐

We will now prove a couple of simple lemmas to establish the existence of certain 2-cells in $\mathcal{D}$ using the arrow object universal property. For each of these lemmas, let $(E_1, \eta_1, \epsilon_1, \delta_1)$ and $(E_2, \eta_2, \epsilon_2, \delta_2)$ be two comonads in $\mathbb{FF}(\mathbb{S}\mathrm{q}(\mathcal{D}))$, both on the same object C; let X be the pullback

$$\begin{tikzcd}
& X \arrow[dl, "P_1"'] \arrow[dr, "P_2"] & \\
\mathbb{L}_1\text{-Coalg} \arrow[dr, "\operatorname{cod} U"'] & & \mathbb{L}_2\text{-Coalg} \arrow[dl, "\operatorname{dom} U"] \\
& C &
\end{tikzcd}$$

let m be the 2-cell

$$\begin{tikzcd}
& C^2 \arrow[dr, bend left, "\operatorname{dom}"] \arrow[dr, bend right, "\operatorname{cod}"'] \arrow[dr, phantom, "\Downarrow\kappa"] & \\
X \arrow[ur, "UP_1"] \arrow[dr, "UP_2"'] & & C \\
& C^2 \arrow[ur, bend left, "\operatorname{dom}"] \arrow[ur, bend right, "\operatorname{cod}"'] \arrow[ur, phantom, "\Downarrow\kappa"] &
\end{tikzcd}$$

and let $\vec{m}\colon X \to C^2$ be the corresponding 1-cell with $\kappa\vec{m} = m$.

Lemma 1.4. *There is a 2-cell*

$$\begin{tikzcd}
X \arrow[r, bend left, "P_1"] \arrow[rr, bend right, "\vec{m}"'] & \mathbb{L}_1\text{-Coalg} \arrow[r, bend left, "U"] \arrow[d, phantom, "\Downarrow\zeta"] & C^2 \\
& \, &
\end{tikzcd}$$

such that $\operatorname{dom}\zeta = \mathrm{id}$ *and*

$$\begin{tikzcd}
X \arrow[r, bend left, "P_1"] \arrow[rr, bend right, "\vec{m}"'] & \mathbb{L}_1\text{-Coalg} \arrow[r, bend left, "U"] \arrow[d, phantom, "\Downarrow\zeta"] & C^2 \arrow[r, "\operatorname{cod}"] & C \\
& \, & &
\end{tikzcd}
=
\begin{tikzcd}
& C^2 \arrow[dr, "\operatorname{cod}"] & \\
X \arrow[ur, "UP_1"] \arrow[dr, "UP_2"'] & & C \\
& C^2 \arrow[ur, bend left, "\operatorname{dom}"] \arrow[ur, bend right, "\operatorname{cod}"'] \arrow[ur, phantom, "\Downarrow\kappa"] &
\end{tikzcd}$$

Proof. Equation (**??**) becomes

$$\begin{tikzcd}
X \arrow[r, "\vec{m}"] & C^2 \arrow[r, bend left, "\operatorname{dom}"] \arrow[r, bend right, "\operatorname{cod}"'] \arrow[r, phantom, "\Downarrow\kappa"] & C
\end{tikzcd}
=
\begin{tikzcd}
& C^2 \arrow[dr, bend left, "\operatorname{dom}"] \arrow[dr, bend right, "\operatorname{cod}"'] \arrow[dr, phantom, "\Downarrow\kappa"] & \\
X \arrow[ur, "UP_1"] \arrow[dr, "UP_2"'] & & C \\
& C^2 \arrow[ur, bend left, "\operatorname{dom}"] \arrow[ur, bend right, "\operatorname{cod}"'] \arrow[ur, phantom, "\Downarrow\kappa"] &
\end{tikzcd}$$

which is simply the definition of $\vec{m}$. ☐

Lemma 1.5. *There is a 2-cell*

$$\begin{tikzcd}
 & \mathbb{L}_2\text{-Coalg} \arrow[dr, bend left, "U"] & \\
X \arrow[ur, bend left, "P_2"] \arrow[dr, bend right, "\vec{m}"'] & \Downarrow\nu & C^2 \\
 & C^2 \arrow[ur, bend right, "R_1"'] &
\end{tikzcd}$$

such that $\operatorname{cod}\nu = \mathrm{id}$ *and*

$$\begin{tikzcd}
 & \mathbb{L}_2\text{-Coalg} \arrow[dr, bend left, "U"] & & \\
X \arrow[ur, bend left, "P_2"] \arrow[dr, bend right, "\vec{m}"'] & \Downarrow\nu & C^2 \arrow[r, "\mathrm{dom}"] & C \\
 & C^2 \arrow[ur, bend right, "R_1"'] & &
\end{tikzcd}
=
\begin{tikzcd}
 & \mathbb{L}_2\text{-Coalg} \arrow[r, "U"] & C^2 \arrow[dr, bend left, "\mathrm{dom}"] & \\
X \arrow[ur, bend left, "P_2"] \arrow[r, "P_1"] \arrow[drr, bend right, "\vec{m}"'] & \mathbb{L}_1\text{-Coalg} \arrow[r, "U"] \arrow[dr, "U"'] \arrow[d, phantom, "\Downarrow\zeta"] & C^2 \arrow[r, "\mathrm{cod}"] \arrow[d, phantom, "\Downarrow\alpha_1"] & C \\
 & {} & C^2 \arrow[ur, bend right, "E_1"'] &
\end{tikzcd}$$

Proof. We just need to verify Equation (**??**):

$$\begin{tikzcd}
 & \mathbb{L}_2\text{-Coalg} \arrow[r, "U"] & C^2 \arrow[dr, bend left, "\mathrm{dom}"] & \\
X \arrow[ur, bend left, "P_2"] \arrow[r, "P_1"] \arrow[drr, bend right, "\vec{m}"'] & \mathbb{L}_1\text{-Coalg} \arrow[r, "U"] \arrow[dr, "U"'] \arrow[d, phantom, "\Downarrow\zeta"] & C^2 \arrow[r, "\mathrm{cod}"] \arrow[d, phantom, "\Downarrow\alpha_1"] & C \\
 & {} & C^2 \arrow[ur, "E_1"] \arrow[ur, bend right, "\mathrm{cod}"'] \arrow[ur, phantom, "\Downarrow\epsilon_1"] &
\end{tikzcd}$$

$$=\begin{tikzcd}
 & \mathbb{L}_2\text{-Coalg} \arrow[r, "U"] & C^2 \arrow[dr, bend left, "\mathrm{dom}"] & \\
X \arrow[ur, bend left, "P_2"] \arrow[r, "P_1"] \arrow[rr, bend right=60, "\vec{m}"'] & \mathbb{L}_1\text{-Coalg} \arrow[r, "U"] \arrow[d, phantom, "\Downarrow\zeta"] & C^2 \arrow[r, "\mathrm{cod}"] & C \\
 & {} & &
\end{tikzcd}$$

$$= \begin{tikzcd}
X \arrow[r, "P_2"] & \mathbb{L}_1\text{-Coalg} \arrow[r, "U"] & C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\kappa"] & C
\end{tikzcd}$$

where the first equation follows from (2), and the second by reducing $\operatorname{cod}\zeta$ using Lemma 1.4. □

Proof of Theorem 1.2. For notational convenience, let $G = \mathrm{Coalg}$ be the lax double functor we need to establish. Both the double categories $\mathbb{C}\mathrm{omon}(\mathbb{FF}(\mathbb{S}\mathrm{q}(\mathscr{D})))$ and $\mathbb{S}\mathrm{pan}(\mathscr{D}_0)$ have $\mathscr{D}_0$ as vertical category, and G_0 is simply the identity. From the statement of the theorem, G takes an object in $\mathbb{C}\mathrm{omon}(\mathbb{FF}(\mathbb{S}\mathrm{q}(\mathscr{D})))$ to the span

$$C \xleftarrow{\mathrm{dom}\,U} \mathbb{L}\text{-Coalg} \xrightarrow{\mathrm{cod}\,U} C.$$

To define the behavior of G on 2-cells, consider a 2-cell in $\mathbb{C}\mathrm{omon}(\mathbb{FF}(\mathbb{S}\mathrm{q}(\mathscr{D})))$:

$$\begin{tikzcd} C \arrow[r, "{(E_1,\eta_1,\epsilon_1,\delta_1)}", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[d, "F"] \\ D \arrow[r, "{(E_2,\eta_2,\epsilon_2,\delta_2)}"', "\shortmid"{marking}] & D. \end{tikzcd}$$

By Corollary **??**, ϕ induces a colax morphism of comonads from L_1 to L_2, hence by Proposition 1.1 there is an induced morphim $\tilde{\phi}$ between the EM-objects such that $U\tilde{\phi} = F^2U$. We can then define $G\phi$ to be the morphism of spans

$$\begin{tikzcd} C \arrow[d, "F"'] & C^2 \arrow[l, "\mathrm{dom}"'] \arrow[d, "F^2"] & \mathbb{L}_1\text{-Coalg} \arrow[l, "U"'] \arrow[r, "U"] \arrow[d, "\tilde{\phi}"] & C^2 \arrow[r, "\mathrm{cod}"] \arrow[d, "F^2"] & C \arrow[d, "F"] \\ D & D^2 \arrow[l, "\mathrm{dom}"] & \mathbb{L}_2\text{-Coalg} \arrow[l, "U"] \arrow[r, "U"'] & D^2 \arrow[r, "\mathrm{cod}"'] & D. \end{tikzcd}$$

Next we must define the coherence data G_I and $G_\otimes$. We will define G_I to be the morphims of spans

$$\begin{tikzcd} & C \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] \arrow[d, "G_I"] & \\ C & \mathbb{L}_I\text{-Coalg} \arrow[l, "\mathrm{dom}\,U"] \arrow[r, "\mathrm{cod}\,U"'] & C \end{tikzcd}$$

defined via Lemma 1.3 by the equations $UG_I = i\colon C \to C^2$ and $\alpha_I G_I$ is the identity on $\mathrm{dom}\, i = \mathrm{cod}\, i$. The conditions of the lemma are trivially satisfied.

We will similarly use Lemma 1.3 to define $G_\otimes$. Let X, $\vec{m}$, ζ, and ν be as defined earlier in the section. $G_\otimes$ is a morphism of spans

$$\begin{tikzcd} & X \arrow[dl, "\mathrm{dom}\,UP_1"'] \arrow[dr, "\mathrm{cod}\,UP_2"] \arrow[d, "G_\otimes"] & \\ C & \mathbb{L}_{1\otimes 2}\text{-Coalg} \arrow[l, "\mathrm{dom}\,U"] \arrow[r, "\mathrm{cod}\,U"'] & C. \end{tikzcd}$$

We will define $G_\otimes$ to be the 1-cell such that $UG_\otimes = \vec{m}$ and

$$\begin{tikzcd} & & C^2 \arrow[dr, "\mathrm{cod}"] & \\ X \arrow[r, "G_\otimes"] & \mathbb{L}\text{-Coalg} \arrow[ur, bend left, "U"] \arrow[dr, bend right, "U"'] \arrow[r, phantom, "\Downarrow\alpha_{1\otimes 2}"] & & C \\ & & C^2 \arrow[ur, "E_{1\otimes 2}"'] & \end{tikzcd} = \begin{tikzcd} & \mathbb{L}_2\text{-Coalg} \arrow[r, "U"] \arrow[dr, "U"] & C^2 \arrow[dr, bend left, "\mathrm{cod}"] & \\ X \arrow[ur, bend left, "P_2"] \arrow[dr, bend right, "\vec{m}"'] \arrow[r, phantom, "\Downarrow\nu"] & & \Downarrow\alpha_2 & C \\ & C^2 \arrow[r, "R_1"'] & C^2 \arrow[ur, bend right, "E_2"'] & \end{tikzcd}$$

In other words, in the notation of Lemma 1.3 let $F = \vec{m}$ and $\beta = E_2\nu \circ \alpha_2 P_2$, and define $G_\otimes = \hat{F}$.

We now need to check equations 1-3 of Lemma 1.3 to verify that $G_\otimes$ is well defined. We will check these equationally to save space, but the reader may want to draw out the diagrams for themselves to follow along. For the first equation:

$$\begin{aligned}
&E_2\nu \circ \alpha_2 P_2 \circ \kappa\vec{m} \\
&\quad = E_2\nu \circ \alpha_2 P_2 \circ \kappa U P_2 \circ \kappa U P_1 && \text{Def of } \vec{m} \\
&\quad = E_2\nu \circ (\alpha_2 \circ \kappa U) P_2 \circ \kappa U P_1 \\
&\quad = E_2\nu \circ \eta_2 U P_2 \circ \kappa U P_1 && \text{Eq (1)} \\
&\quad = \eta_2 R_1 \vec{m} \circ \operatorname{dom}\nu \circ \kappa U P_1 && \text{Interchange} \\
&\quad = \eta_2 R_1 \vec{m} \circ E_1\zeta \circ \alpha_1 P_1 \circ \kappa U P_1 && \text{Def of } \nu \\
&\quad = \eta_2 R_1 \vec{m} \circ E_1\zeta \circ (\alpha_1 \circ \kappa U) P_1 \\
&\quad = \eta_2 R_1 \vec{m} \circ E_1\zeta \circ \eta_1 U P_1 && \text{Eq (1)} \\
&\quad = \eta_{1\otimes 2}\vec{m} \circ \operatorname{dom}\zeta && \text{Interchange; Def of } \eta_{1\otimes 2} \\
&\quad = \eta_{1\otimes 2}\vec{m} && \operatorname{dom}\zeta = \mathrm{id}
\end{aligned}$$

and the second:

$$\begin{aligned}
&\epsilon_{1\otimes 2}\vec{m} \circ E_2\nu \circ \alpha_2 P_2 \\
&\quad = \epsilon_2 R_1 \vec{m} \circ E_2\nu \circ \alpha_2 P_2 && \text{Def of } \epsilon_{1\otimes 2} \\
&\quad = \operatorname{cod}\nu \circ (\epsilon_2 U \circ \alpha_2) P_2 && \text{Interchange} \\
&\quad = \mathrm{id}_{\operatorname{cod}\vec{m}}\,. && \text{Eq (2); } \operatorname{cod}\nu = \mathrm{id}
\end{aligned}$$

The third equation is a bit trickier to prove. We will need to prove two intermediate equations first, using the arrow object universal property.

Lemma.

$$i^L\vec{m} \circ L_1\zeta \circ \vec{\alpha}_1 P_1 = \vec{\beta} \circ \zeta \tag{4}$$

Proof. We must show the 2-cells become equal upon composition with dom and cod:

$$\operatorname{dom}(i^L\vec{m} \circ L_1\zeta \circ \vec{\alpha}_1 P_1) = \mathrm{id}_{\operatorname{dom}\vec{m}} = \operatorname{dom}(\vec{\beta} \circ \zeta)$$

and

$$\begin{aligned}
&\operatorname{cod}(i^L\vec{m}\circ L_1\zeta\circ\vec{\alpha}_1P_1)\\
&\quad=\operatorname{cod} i^L\vec{m}\circ E_1\zeta\circ\operatorname{cod}\vec{\alpha}_1P_1\\
&\quad=\eta_2R_1\vec{m}\circ E_1\zeta\circ\alpha_1P_1 && \text{Def of } i^L,\ \vec{\alpha}\\
&\quad=\eta_2R_1\vec{m}\circ\operatorname{dom}\nu && \text{Def of } \nu\\
&\quad=E_2\nu\circ\eta_2UP_2 && \text{Interchange}\\
&\quad=E_2\nu\circ(\alpha_2\circ\kappa U)P_2 && \text{Eq (1)}\\
&\quad=(E_2\nu\circ\alpha_2P_2)\circ\kappa UP_2\\
&\quad=\operatorname{cod}\vec{\beta}\circ\operatorname{cod}\zeta && \text{Def of } \vec{\beta},\ \zeta\\
&\quad=\operatorname{cod}(\vec{\beta}\circ\zeta).
\end{aligned}$$

□

Lemma.

$$R_1\vec{\beta}\circ\nu=w\vec{m}\circ L_2\delta_1^R\vec{m}\circ L_2\nu\circ\vec{\alpha}_2P_2 \tag{5}$$

Proof. Again we must prove equality after composing with dom and cod:

$$\begin{aligned}
&\operatorname{dom}(R_1\vec{\beta}\circ\nu)\\
&\quad=E_1\vec{\beta}\circ\operatorname{dom}\nu\\
&\quad=E_1\vec{\beta}\circ E_1\zeta\circ\alpha_1P_1 && \text{Def of } \nu\\
&\quad=E_1(\vec{\beta}\circ\zeta)\circ\alpha_1P_1\\
&\quad=E_1(i^L\vec{m}\circ L_1\zeta\circ\vec{\alpha}_1P_1)\circ\alpha_1P_1 && \text{Eq (4)}\\
&\quad=E_1i^L\vec{m}\circ E_1L_1\zeta\circ(E_1\vec{\alpha}_1\circ\alpha_1)P_1\\
&\quad=E_1i^L\vec{m}\circ E_1L_1\zeta\circ(\delta_1U\circ\alpha_1)P_1 && \text{Eq (3)}\\
&\quad=E_1i^L\vec{m}\circ\delta_1\vec{m}\circ E_1\zeta\circ\alpha_1P_1 && \text{Interchange}\\
&\quad=\operatorname{dom} w\vec{m}\circ\operatorname{dom}\delta_1^R\vec{m}\circ\operatorname{dom}\nu\circ\operatorname{dom}\vec{\alpha}_2P_2 && \text{Defs of } w,\ \delta^R,\ \nu,\ \vec{\alpha}\\
&\quad=\operatorname{dom}(w\vec{m}\circ L_2\delta_1^R\vec{m}\circ L_2\nu\circ\vec{\alpha}_2P_2)
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{cod}(R_1\vec{\beta}\circ\nu)\\
&\quad=\operatorname{cod}\vec{\beta}\circ\operatorname{cod}\nu\\
&\quad=E_2\nu\circ\alpha_2P_2 && \text{Defs of } \vec{\beta},\ \nu\\
&\quad=E_2(p^R\circ\delta_1^R)\vec{m}\circ E_2\nu\circ\alpha_2P_2 && p^R\circ\delta^R=\mathrm{id}\\
&\quad=E_2p^R\vec{m}\circ E_2\delta_1^R\vec{m}\circ E_2\nu\circ\alpha_2P_2\\
&\quad=\operatorname{cod} w\vec{m}\circ\operatorname{cod} L_2\delta_1^R\vec{m}\circ\operatorname{cod} L_2\nu\circ\operatorname{cod}\vec{\alpha}_2P_2 && \text{Defs of } w,\ L,\ \vec{\alpha}\\
&\quad=\operatorname{cod}(w\vec{m}\circ L_2\delta_1^R\vec{m}\circ L_2\nu\circ\vec{\alpha}_2P_2)
\end{aligned}$$

$\square$

Now we are prepared to prove the third equation of Lemma 1.3 validating our definition of $G_\otimes$:

$$
\begin{aligned}
&\delta_{1\otimes 2}\vec{m} \circ E_2\nu \circ \alpha_2 P_2 \\
&\quad = (E_2 w \circ \delta_2 R_{1\odot 1} \circ E_2\delta_1^R)\vec{m} \circ E_2\nu \circ \alpha_2 P_2 && \text{Def of } \delta_{1\otimes 2} \\
&\quad = E_2(w\vec{m} \circ L_2\delta_1^R\vec{m} \circ L_2\nu) \circ (\delta_2 U \circ \alpha_2)P_2 && \text{Interchange} \\
&\quad = E_2(w\vec{m} \circ L_2\delta_1^R\vec{m} \circ L_2\nu) \circ (E_2\vec{\alpha}_2 \circ \alpha_2)P_2 && \text{Eq (3)} \\
&\quad = E_2(w\vec{m} \circ L_2\delta_1^R\vec{m} \circ L_2\nu \circ \vec{\alpha}_2 P_2) \circ \alpha_2 P_2 && \\
&\quad = E_2(R_1\vec{\beta} \circ \nu) \circ \alpha_2 P_2 && \text{Eq (5)} \\
&\quad = E_{1\otimes 2}\vec{\beta} \circ E_2\nu \circ \alpha_2 P_2 && \text{Def of } E_{1\otimes 2}
\end{aligned}
$$

The verification that the definitions of G_I and $G_\otimes$ form natural families, and of the coherence axioms for a lax double functor, is tedious, but follows from what we have presented here without requiring any new ideas or ingenuity. $\square$

Corollary 1.6. *For any awfs $(E, \eta, \mu, \epsilon, \delta)$ on an object C in $\mathcal{D}$, the multiplication μ induces a composition functor on $\mathbb{L}$-Coalg, and the functor between EM-objects induced by any colax morphism of awfs preserves this composition.*

Proof. Any awfs $(E, \eta, \mu, \epsilon, \delta)$ has an underlying object in $\mathbb{C}\mathrm{omon}(\mathbb{FF}(\mathbb{S}\mathrm{q}(\mathcal{D})))$, by simply forgetting μ. The lax double-functor Coalg takes this to a span

$$C \xleftarrow{\operatorname{dom} U} \mathbb{L}\text{-Coalg} \xrightarrow{\operatorname{cod} U} C.$$

The multiplication μ provides this object in $\mathbb{C}\mathrm{omon}(\mathbb{FF}(\mathbb{S}\mathrm{q}(\mathcal{D})))$ with a monad structure, and lax double-functors preserve monads, so μ induces a monad structure on this span. A multiplication on this span is a morphism π:

$$
\begin{tikzcd}
 & X \arrow[dl, "\operatorname{dom} UP_1"'] \arrow[dr, "\operatorname{cod} UP_2"] \arrow[d, "\pi"] & \\
C & \mathbb{L}\text{-Coalg} \arrow[l, "\operatorname{dom} U"] \arrow[r, "\operatorname{cod} U"'] & C
\end{tikzcd}
$$

where X is the pullback in the composite span

$$
\begin{tikzcd}
 & & X \arrow[dl, "P_1"'] \arrow[dr, "P_2"] & & \\
 & \mathbb{L}\text{-Coalg} \arrow[dl, "\operatorname{dom} U"'] \arrow[dr, "\operatorname{cod} U"] & & \mathbb{L}\text{-Coalg} \arrow[dl, "\operatorname{dom} U"'] \arrow[dr, "\operatorname{cod} U"] & \\
C & & C & & C.
\end{tikzcd}
$$

The morphism π is the composition structure that we want. If $\mathcal{D} = \mathrm{Cat}$ is the 2-category of small categories, then an object (f, g) in X is a pair of morphisms in C equipped with coalgebra structures, such that $\operatorname{cod} f = \operatorname{dom} g$, and $\pi(f, g)$ is a morphism equipped with a coalgebra structure, with $\operatorname{dom} \pi(f, g) = \operatorname{dom} f$ and $\operatorname{cod} \pi(f, g) = \operatorname{cod} g$.

Of course, what we really want is that the morphism underlying the coalgebra $\pi(f, g)$ is the composition $g \circ f$. To see that this is the case, notice that the composition π is given by $G(\vec{\mu}) \circ G_{\otimes}$, where $G(\vec{\mu})$ is the 1-cell between EM-objects induced by the colax morphism of comonads $\vec{\mu}$. Using the fact that $\vec{\mu}$ is a globular 2-cell in $\mathbb{FF}(\mathbb{S}\mathrm{q}(\mathcal{D}))$, we have $U\pi = UG(\vec{\mu})G_{\otimes} = UG_{\otimes} = \vec{m}$. $\square$

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