1 Weak factorization systems

In this section, we will briefly review the notions of functorial factorization, weak factorization system, and algebraic weak factorization system.

Arrow Categories

Let $\mathcal{C}$ be a category. Its arrow category $\mathcal{C}^2$ is the category whose objects are arrows in $\mathcal{C}$ and whose morphisms are commutative squares. The arrow category comes with two functors $\mathrm{dom}, \mathrm{cod} \colon \mathcal{C}^2 \to \mathcal{C}$, along with a natural transformation $\kappa \colon \mathrm{dom} \Rightarrow \mathrm{cod}$. The component of κ at an object f of $\mathcal{C}^2$ is simply $f \colon \mathrm{dom}\, f \to \mathrm{cod}\, f$. Moreover, $\mathcal{C}^2$ satisfies a universal property: there is an equivalence of categories

$$\mathrm{Fun}(2, \mathrm{Fun}(\mathcal{X}, \mathcal{C})) \simeq \mathrm{Fun}(\mathcal{X}, \mathcal{C}^2) \tag{1}$$

given by composition with κ. Here, 2 is the ordinal, i.e. the category with two objects and a single non-identity arrow. In other words, $\mathcal{C}^2$ is the cotensor of $\mathcal{C}$ with the category 2 in the 2-category $\mathcal{C}\mathrm{at}$.

We will make this universal property more explicit in the next lemma:

Lemma 1.1. *Let $\mathcal{C}$ be a category.*

i) For any category $\mathcal{X}$, pair of functors $F, G \colon \mathcal{X} \to \mathcal{C}$, and natural transformation $\alpha \colon F \Rightarrow G$, there is a unique functor $\hat{\alpha} \colon \mathcal{X} \to \mathcal{C}^2$ such that $\mathrm{dom}\, \hat{\alpha} = F$, $\mathrm{cod}\, \hat{\alpha} = G$, and

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\hat{\alpha}"] & \mathcal{C}^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\kappa"] & \mathcal{C} \arrow[r, phantom, "="] & \mathcal{X} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{C}. \end{tikzcd} \tag{2}$$

ii) For any functors $F, F', G, G' \colon \mathcal{X} \to \mathcal{C}$ and a commutative square of natural transformations

$$\begin{tikzcd} F \arrow[r, Rightarrow, "\gamma"] \arrow[d, Rightarrow, "\alpha"'] & F' \arrow[d, Rightarrow, "\beta"] \\ G \arrow[r, Rightarrow, "\phi"'] & G', \end{tikzcd}$$

there is a unique natural transformation $\eta \colon \hat{\alpha} \to \hat{\beta}$ such that $\mathrm{dom}\, \eta = \gamma$ and $\mathrm{cod}\, \eta = \phi$, hence

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left=50, "F"] \arrow[r, "G" description] \arrow[r, bend right=50, "G'"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\phi"] & \mathcal{C} \arrow[r, phantom, "="] & \mathcal{X} \arrow[r, bend left, "\hat{\alpha}"] \arrow[r, bend right, "\hat{\beta}"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{C}^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\kappa"] & \mathcal{C} \arrow[r, phantom, "="] & \mathcal{X} \arrow[r, bend left=50, "F"] \arrow[r, "F'" description] \arrow[r, bend right=50, "G'"'] \arrow[r, phantom, shift left=3, "\Downarrow\gamma"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \mathcal{C}. \end{tikzcd}$$

Definition 1.2. Let $\mathcal{D}$ be any 2-category. For any object A in $\mathcal{D}$, the *arrow object* of A, if it exists, is an object A^2 satisfying the universal property (1). If every object has an arrow object, i.e. if $\mathcal{D}$ has cotensors by 2, we will say $\mathcal{D}$ *has arrow objects*.

Functorial Factorizations

Definition 1.3. A functorial factorization on a category $\mathcal{C}$ consists of a functor E and two natural transformations η and ϵ which factor κ, as in

$$\begin{tikzcd}
\mathcal{C}^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\kappa"] & \mathcal{C} & = & \mathcal{C}^2 \arrow[r, "E"] \arrow[r, bend left=60, "\mathrm{dom}"] \arrow[r, bend right=60, "\mathrm{cod}"'] \arrow[r, phantom, shift left=2ex, "\Downarrow\eta"] \arrow[r, phantom, shift right=2ex, "\Downarrow\epsilon"] & \mathcal{C}.
\end{tikzcd}$$

This determines for any arrow f in $\mathcal{C}$ a factorization $f = \epsilon_f \circ \eta_f$. The factorization is natural, meaning that for any morphism $(u,v)\colon f \Rightarrow g$ in $\mathcal{C}^2$ (i.e. commutative square in $\mathcal{C}$), the two squares in

$$\begin{tikzcd}
\cdot \arrow[r, "u"] \arrow[d, "\eta_f"'] & \cdot \arrow[d, "\eta_g"] \\
\cdot \arrow[r, "E(u,v)"] \arrow[d, "\epsilon_f"'] & \cdot \arrow[d, "\epsilon_g"] \\
\cdot \arrow[r, "v"'] & \cdot
\end{tikzcd}$$

commute.

A functorial factorization also determines two functors $L, R\colon \mathcal{C}^2 \to \mathcal{C}^2$ such that $\mathrm{dom}\, L = \mathrm{dom}$, $\mathrm{cod}\, R = \mathrm{cod}$, $\mathrm{cod}\, L = \mathrm{dom}\, R = E$, $\kappa L = \eta$, and $\kappa R = \epsilon$, by the universal property of $\mathcal{C}^2$. The components of the factorization of f can then also be referred to as Lf and Rf, now thought of as objects in $\mathcal{C}^2$. There are also two canonical natural transformations, $\vec{\eta}\colon \mathrm{id} \Rightarrow R$ and $\vec{\epsilon}\colon L \Rightarrow \mathrm{id}$, determined by the commuting squares

$$\begin{tikzcd}
\mathrm{dom} \arrow[r, Rightarrow, "\eta"] \arrow[d, Rightarrow, "\kappa"'] & E \arrow[d, Rightarrow, "\epsilon"] & & \mathrm{dom} \arrow[r, Rightarrow, "\mathrm{id}"] \arrow[d, Rightarrow, "\eta"'] & \mathrm{dom} \arrow[d, Rightarrow, "\kappa"] \\
\mathrm{cod} \arrow[r, Rightarrow, "\mathrm{id}"'] & \mathrm{cod} & \text{and} & E \arrow[r, Rightarrow, "\epsilon"'] & \mathrm{cod}
\end{tikzcd}$$

respectively. These make L and R into (co)pointed endofunctors of $\mathcal{C}^2$.

An algebra for the pointed endofunctor R is an object f in $\mathcal{C}^2$ equipped with a morphism $\vec{t}\colon Rf \Rightarrow f$, such that $\vec{t} \circ \vec{\eta}_f = \mathrm{id}_f$. Similarly, a coalgebra for the copointed endofunctor L is an f equipped with a morphism $\vec{s}\colon f \Rightarrow Lf$, such that $\vec{\epsilon}_f \circ \vec{s} = \mathrm{id}_f$.

Lemma 1.4. *Let $f\colon X \to Y$ be a morphism in $\mathcal{C}$. An R-algebra structure on $f \in \mathcal{C}^2$ is precisely a choice of lift t in the square*

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "Lf"'] & X \arrow[d, "f"] \\
Ef \arrow[r, "Rf"'] \arrow[ur, dashed, "t"] & Y.
\end{tikzcd} \tag{3}$$

Dually, an L-coalgebra structure on f is precisely a choice of lift s in the square

$$\begin{tikzcd}
X \arrow[r, "Lf"] \arrow[d, "f"'] & Ef \arrow[d, "Rf"] \\
Y \arrow[r, equal] \arrow[ur, dashed, "s"] & Y.
\end{tikzcd} \tag{4}$$

Algebraic Weak Factorization Systems

To simplify the discussion of weak factorization systems, we will start by introducing a notation. For any two morphisms l and r in $\mathcal{C}$, write $l \boxslash r$ to mean that for every commutative square

$$\begin{tikzcd}
\cdot \arrow[r, "u"] \arrow[d, "l"'] & \cdot \arrow[d, "r"] \\
\cdot \arrow[r, "v"'] \arrow[ur, dashed, "w"] & \cdot
\end{tikzcd} \tag{5}$$

there exists a lift w. In this case, we will say that l has the *left lifting property* with respect to r, and that r has the *right lifting property* with respect to l. Similarly, for two classes of morphisms $\mathcal{L}$ and $\mathcal{R}$, we will say $\mathcal{L} \boxslash \mathcal{R}$ if $l \boxslash r$ for every $l \in \mathcal{L}$ and $r \in \mathcal{R}$. Finally, we will write $\mathcal{L}^{\boxslash}$ for the class of morphisms having the right lifting property with respect to every morphism of $\mathcal{L}$, and ${}^{\boxslash}\mathcal{R}$ for the class of morphisms having the left lifting property with respect to every morphism of $\mathcal{R}$.

Definition 1.5. A *functorial weak factorization system* on a category $\mathcal{C}$ consists of a functorial factorization on $\mathcal{C}$ and two classes $\mathcal{L}$ and $\mathcal{R}$ of morphisms in $\mathcal{C}$, such that

- for every morphism f in $\mathcal{C}$, $Lf \in \mathcal{L}$ and $Rf \in \mathcal{R}$,
- $\mathcal{L}^{\boxslash} = \mathcal{R}$ and ${}^{\boxslash}\mathcal{R} = \mathcal{L}$.

It a simple and standard proof that the lifting property condition can be replaced by two simpler conditions:

Lemma 1.6. *A functorial weak factorization system can equivalently be defined to be a functorial factorization on $\mathcal{C}$ and two classes $\mathcal{L}$ and $\mathcal{R}$ of morphisms in $\mathcal{C}$, such that*

- *for every morphism f in $\mathcal{C}$, $Lf \in \mathcal{L}$ and $Rf \in \mathcal{R}$,*
- *$\mathcal{L} \boxslash \mathcal{R}$,*
- *$\mathcal{L}$ and $\mathcal{R}$ are both closed under retracts.*

In fact, the functorial factorization by itself already determines the two classes of morphisms, with $\mathcal{L}$ the class of morphisms admitting an L-coalgebra structure, and $\mathcal{R}$ the class of morphisms admitting an R-algebra structure. The lifting properties also follow directly from the functorial factorization, as the next lemma shows.

Lemma 1.7. *For any L-coalgebra (l, s) and any R-algebra (r, t), there is a canonical choice of lift in the square* (5).

Proof. The construction is shown in the diagram

$$\begin{tikzcd}
\cdot \arrow[r, "u"] \arrow[d, "Ll"'] & \cdot \arrow[d, "Lr"] \\
\cdot \arrow[r, dashed, "E(u,v)"] \arrow[d, shift right, "Rl"'] & \cdot \arrow[u, dashed, shift left, "t"] \arrow[d, "Rr"] \\
\cdot \arrow[u, dashed, shift right, "s"'] \arrow[r, "v"'] & \cdot
\end{tikzcd} \tag{6}$$

Commutativity of (5) follows immediately from (3) and (4). □

This, together with the classical fact that the class of objects admitting a (co)algebra structure for a (co)pointed endofunctor is closed under retracts, gives a third equivalent definition of a functorial weak factorization system.

Lemma 1.8. *A functorial weak factorization system can equivalently be defined to be a functorial factorization on $\mathcal{C}$ such that*

- *for every morphism f in $\mathcal{C}$, Lf admits an L-coalgebra structure, and Rf admits an R-algebra structure.*

An R-algebra structure on Rf consists of a morphism $\vec{\mu}_f\colon R^2 f \to Rf$ in $\mathcal{C}^2$ such that $\vec{\mu}_f \circ \vec{\eta}_f = \mathrm{id}_f$, while an L-coalgebra structure on Lf consists of a morphism $\vec{\delta}_f\colon Lf \to L^2 f$ such that $\vec{\epsilon}_f \circ \vec{\delta}_f = \mathrm{id}_f$. We might hope that it is possible to choose these structures for all f in a natural way, such that they form the components of natural transformations $\vec{\mu}\colon R^2 \Rightarrow R$ and $\vec{\delta}\colon L \Rightarrow L^2$.

If we want these choices of lifts to be fully coherent, we should also ask that for any R-algebra (f, t), the lift constructed as in (6) for the square (3) is equal to t, and similarly for L-coalgebras and (4). Lastly, we should ask that the components $\vec{\mu}_f$ and $\vec{\delta}_f$ are (co)algebra morphisms. With these conditions made, we have the definition of an *algebraic weak factorization system*, first given in [GT06] (there called *natural* weak factorization systems), and further refined in [Gar07] and [Gar09].

Definition 1.9. An *algebraic weak factorization system* on a category $\mathcal{C}$ consists of a functorial factorization $(L, \vec{\epsilon}, R, \vec{\eta})$ together with natural transformations $\vec{\mu}\colon R^2 f \Rightarrow Rf$ and $\vec{\delta}\colon L \Rightarrow L^2$, such that

- $\mathbb{R} = (R, \vec{\eta}, \vec{\mu})$ is a monad and $\mathbb{L} = (L, \vec{\epsilon}, \vec{\delta})$ a comonad on $\mathcal{C}^2$, and
- the natural transformation $\Delta = (\delta, \mu)\colon LR \Rightarrow RL$ determined by the equation $\epsilon L \circ \delta = \mu \circ \eta R \, (= \mathrm{id}_E)$ as in 1.1 is a distributive law, which in this case reduces to the single condition $\delta \circ \mu = \mu L \circ E\Delta \circ \delta R$.

2 Double Categories

In this section, we will give an overview of double categories, as well as (one possible version of) the definition of monads in a double category

A (strict) double category is a two-dimensional categorical structure, similar to a 2-category. Unlike a 2-category, a double category has two types of 1-cells, called *vertical* and *horizontal*, and 2-cells all have a square shape, with domain and codomain horizontal 1-cells as well as domain and codomain vertical 1-cells.

We will first give the most concise definition of a double category, which we will then break down into more concrete terms.

Definition 2.1. A *double category* is an internal category object in the (large) category of categories.

So a double category $\mathbb{D}$ consists of a category D_O and a category D_A, along with functors $s, t\colon D_A \to D_O$, $1\colon D_O \to D_A$, and $\circ\colon D_A \times_{D_O} D_A \to D_A$ satisfying the usual axioms of a category. We will call the objects of D_O the 0-cells of $\mathbb{D}$, and the morphisms of D_O the vertical 1-cells. Thus D_O forms the so-called *vertical category* of $\mathbb{D}$. We will call the objects of D_A the horizontal 1-cells of $\mathbb{D}$, and the morphisms of D_A are the 2-cells.

A morphism $\phi\colon X \to Y$ in D_A, where $s(X) = C$, $t(X) = C'$, $s(Y) = D$, $t(Y) = D'$, $s(\phi) = f$, and $t(\phi) = g$ will be drawn as

$$\begin{tikzcd} C \arrow[r, "X", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & C' \arrow[d, "g"] \\ D \arrow[r, "Y"', "\shortmid"{marking}] & D' \end{tikzcd} \tag{7}$$

where the tick-mark on the horizontal 1-cells serves as a further reminder that the horizontal 1-cells are of a different nature than the vertical 1-cells. The composition in D_O provides a vertical composition of vertical 1-cells and 2-cells, while the composition functor $\circ\colon D_A \times_{D_O} D_A \to D_A$ provides a horizontal composition of horizontal 1-cells and 2-cells.

Example 2.2. For any 2-category $\mathcal{D}$, there is an associated double category $\mathrm{Sq}(\mathcal{D})$ of *squares* in $\mathcal{D}$, in which the vertical and horizontal 1-cells are both just 1-cells in $\mathcal{D}$, and 2-cells

$$\begin{tikzcd} C \arrow[r, "j", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & C' \arrow[d, "g"] \\ D \arrow[r, "k"', "\shortmid"{marking}] & D' \end{tikzcd}$$

are simply 2-cells $\phi\colon gj \Rightarrow kf$ in $\mathcal{D}$.

Example 2.3. For any double category $\mathbb{D}$, there is an associated 2-category $\mathcal{V}\mathrm{ert}(\mathbb{D})$, called the *vertical 2-category* of $\mathbb{D}$. The objects and 1-cells of $\mathcal{V}\mathrm{ert}(\mathbb{D})$ are the objects and vertical 1-cells of $\mathbb{D}$, while 2-cells $\phi\colon g \Rightarrow f$ in $\mathcal{V}\mathrm{ert}(\mathbb{D})$ are 2-cells in $\mathbb{D}$ of the form

$$\begin{tikzcd}
C \arrow[r, "\mathrm{id}_C", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[d, "g"] \\
D \arrow[r, "\mathrm{id}_D"', "|" marking] & D
\end{tikzcd}$$

Notice that $\mathcal{V}\mathrm{ert}(\mathbb{S}\mathrm{q}(\mathcal{D}))$ is isomorphic to $\mathcal{D}$.

We will define a *monad* in a double category $\mathbb{D}$ to be a tuple (C, T, η, μ), in which C is an object, $T\colon C \to C$ is a horizontal 1-cell, and η and μ are 2-cells

$$\begin{tikzcd}
C \arrow[r, "\mathrm{id}_C", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta"] & C \arrow[d, equal] \\
C \arrow[r, "T"', "|" marking] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "T", "|" marking] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\mu"] & C \arrow[r, "T", "|" marking] & C \arrow[d, equal] \\
C \arrow[rr, "T"', "|" marking] & & C
\end{tikzcd}$$

satisfying the usual unit and associativity conditions.

Given two monads (C, T, η, μ) and (D, S, η', μ'), a monad morphism from (C, T) to (D, S) consists of a pair (f, ϕ), where f is a vertical 1-cell $C \to D$ and ϕ is a 2-cell

$$\begin{tikzcd}
C \arrow[r, "T", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[d, "f"] \\
D \arrow[r, "S"', "|" marking] & D
\end{tikzcd}$$

which commutes with the unit and multiplication 2-cells in the sense of the two equations

$$\begin{tikzcd}
C \arrow[r, "\mathrm{id}_C", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta"] & C \arrow[d, equal] & & C \arrow[r, "\mathrm{id}_C", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}_f"] & C \arrow[d, "f"] \\
C \arrow[r, "T", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[d, "f"] & = & D \arrow[r, "\mathrm{id}_D", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta'"] & D \arrow[d, equal] \\
D \arrow[r, "S"', "|" marking] & D & & D \arrow[r, "S"', "|" marking] & D
\end{tikzcd} \tag{8}$$

and

$$\begin{tikzcd}
C \arrow[r, "T", "\shortmid" marking] \arrow[d, equal] & C \arrow[r, "T", "\shortmid" marking] \arrow[d, phantom, "\Downarrow\mu"] & C \arrow[d, equal] & & C \arrow[r, "T", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[r, "T", "\shortmid" marking] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[d, "f"] \\
C \arrow[rr, "T", "\shortmid" marking] \arrow[d, "f"'] & {} \arrow[d, phantom, "\Downarrow\phi"] & C \arrow[d, "f"] & = & D \arrow[r, "S", "\shortmid" marking] \arrow[d, equal] & D \arrow[r, "S", "\shortmid" marking] \arrow[d, phantom, "\Downarrow\mu'"] & D \arrow[d, equal] \\
D \arrow[rr, "S"', "\shortmid" marking] & {} & D & & D \arrow[rr, "S"', "\shortmid" marking] & {} & D
\end{tikzcd} \tag{9}$$

Definition 2.4. Given any double category $\mathbb{D}$, we will write $\mathrm{Mon}(\mathbb{D})$ for the category of monads in $\mathbb{D}$, consisting of monads and monad morphisms as defined above.

Proposition 2.5. *A monad in $\mathbb{S}\mathrm{q}(\mathcal{D})$ is a monad in the 2-category $\mathcal{D}$ in the usual sense. A monad morphism in $\mathbb{S}\mathrm{q}(\mathcal{D})$ is what is known as a colax morphism of monads in $\mathbb{D}$. A monad morphism in $\mathbb{S}\mathrm{q}(\mathcal{D}^{co})$, where we have reversed the direction of all 2-cells, is a lax morphism of monads.*

Proof. Immediate from the definitions. Readers unfamiliar with (co)lax morphisms of monads can take this as the definition. ☐

3 2-Fold Double Categories

In this section we will propose a generalization of the 2-fold monoidal categories as used in [Gar09].

A 2-fold double category $\mathbb{D}$ is a structure which has two different underlying double categories, both of which have the same vertical category $\mathrm{Vert}(\mathbb{D})$. We will start with a concise formal definition, and then expand on the definition more concretely.

Definition 3.1. A *2-fold double category* $\mathbb{D}$ with vertical category $\mathrm{Vert}(\mathbb{D}) = \mathcal{D}_0$ is a 2-fold monoid object in the 2-category $\mathit{Cat}/\mathcal{D}_0$ of categories over $\mathcal{D}_0$.

Breaking this down, we have a category $\mathcal{D}$, a functor $p\colon \mathcal{D} \to \mathcal{D}_0$, two functors $\otimes, \odot\colon \mathcal{D} \times_{\mathcal{D}_0} \mathcal{D} \to \mathcal{D}$ commuting with p, and two functors $I, \bot\colon \mathcal{D}_0 \to \mathcal{D}$ which are sections of p, such that $\otimes$, $\odot$, I, and $\bot$ satisfy all the axioms of a 2-fold monoidal category. In particular, each fiber of p has a 2-fold monoidal structure.

We will find it convenient to present this structure in the form of a double category $\mathbb{D}$, as follows:

- The objects and vertical morphisms of $\mathbb{D}$ are those of $\mathcal{D}_0$, so that $\mathrm{Vert}(\mathbb{D}) = \mathcal{D}_0$.
- The horizontal morphisms of $\mathbb{D}$ are the objects X of $\mathcal{D}$, with $p(X)$ as both domain and codomain. We will draw these as marked arrows

$$\begin{tikzcd} p(X) \arrow[r, "X", "\shortmid"{marking}] & p(X). \end{tikzcd}$$

- The 2-cells are the morphisms of $\mathcal{D}$. So a morphism $\phi\colon X \to Y$ in $\mathcal{D}$ with $p(\phi) = f\colon C \to D$ would be drawn as

$$\begin{tikzcd} C \arrow[r, "X", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[d, "f"] \\ D \arrow[r, "Y"', "\shortmid"{marking}] & D \end{tikzcd}$$

The two tensor products of $\mathcal{D}$ provide two different horizontal compositions for $\mathbb{D}$. For any object C there are 2-cells

$$\begin{tikzcd} C \arrow[r, "\bot_C\otimes\bot_C", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow m"] & C \arrow[d, equal] \\ C \arrow[r, "\bot_C"', "\shortmid"{marking}] & C \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "I_C", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow c"] & C \arrow[d, equal] \\ C \arrow[r, "I_C\odot I_C"', "\shortmid"{marking}] & C \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "I_C", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow j"] & C \arrow[d, equal] \\ C \arrow[r, "\bot_C"', "\shortmid"{marking}] & C \end{tikzcd} \tag{10}$$

and for any four horizontal morphisms $W, X, Y, Z\colon C \rightarrow\mapsfromchar C$ there is a 2-cell

$$\begin{tikzcd}
C \arrow[r, "(W\odot X)\otimes(Y\odot Z)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow z"] & C \arrow[d, equal] \\
C \arrow[r, "(W\otimes Y)\odot(X\otimes Z)"', "\shortmid"{marking}] & C.
\end{tikzcd} \tag{11}$$

These are natural in the sense that, for any vertical morphism $f\colon C \to D$ we have an equality

$$\begin{tikzcd}
C \arrow[r, "\perp_C\otimes\perp_C", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow m"] & C \arrow[d, equal] \\
C \arrow[r, "\perp_C", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \perp_f"] & C \arrow[d, "f"] \\
D \arrow[r, "\perp_D"', "\shortmid"{marking}] & D
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[r, "\perp_C\otimes\perp_C", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \perp_f\otimes\perp_f"] & C \arrow[d, "f"] \\
D \arrow[r, "\perp_D\otimes\perp_D"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow m"] & D \arrow[d, equal] \\
D \arrow[r, "\perp_D"', "\shortmid"{marking}] & D
\end{tikzcd}$$

and similarly for c and j, and for any four 2-cells $\theta_1, \ldots, \theta_4$ of the apropriate form, we have an equality

$$\begin{tikzcd}
C \arrow[r, "(W\odot X)\otimes(Y\odot Z)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow z"] & C \arrow[d, equal] \\
C \arrow[r, "(W\otimes Y)\odot(X\otimes Z)", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow(\theta_1\otimes\theta_3)\odot(\theta_2\otimes\theta_4)"] & C \arrow[d, "f"] \\
D \arrow[r, "(W'\otimes Y')\odot(X'\otimes Z')"', "\shortmid"{marking}] & D
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[r, "(W\odot X)\otimes(Y\odot Z)", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow(\theta_1\odot\theta_2)\otimes(\theta_3\odot\theta_4)"] & C \arrow[d, "f"] \\
C \arrow[r, "(W'\odot X')\otimes(Y'\odot Z')"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow z"] & C \arrow[d, equal] \\
D \arrow[r, "(W'\otimes Y')\odot(X'\otimes Z')"', "\shortmid"{marking}] & D
\end{tikzcd}$$

A 2-fold double category $\mathbb{D}$ has two underlying ordinary double categories: $\mathbb{D}_\otimes$ using $\otimes$ for the horizontal composition, and $\mathbb{D}_\odot$ using $\odot$.

4 Monoids

There are several possible ways to define (co)monoids/(co)monads in a double category, but there is one way in particular which interacts nicely with the 2-fold double category structure.

Definition 4.1. A *monoid* in a 2-fold double category $\mathbb{D}$ is a monoid in $\mathbb{D}_\otimes$. Specifically, it is a horizontal 1-cell $X\colon C \mathrel{\mapstochar\rightarrow} C$, together with unit and multiplication 2-cells

$$\begin{tikzcd}
C \arrow[r, "I_C", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta"] & C \arrow[d, equal] \\
C \arrow[r, "X"', "\shortmid"{marking}] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "X\otimes X", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mu"] & C \arrow[d, equal] \\
C \arrow[r, "X"', "\shortmid"{marking}] & C
\end{tikzcd}$$

satisfying the usual unit and associativity conditions.

Definition 4.2. Given two monoids (C, X, η, μ) and (D, Y, η', μ') in $\mathbb{D}$, a morphism of monoids consists of a vertical morphism $f\colon C \to D$ and a 2-cell

$$\begin{tikzcd}
C \arrow[r, "X", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[d, "f"] \\
D \arrow[r, "Y"', "\shortmid"{marking}] & D
\end{tikzcd}$$

which preserves the unit and multiplication, in that

$$\begin{tikzcd}
C \arrow[r, "I_C", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta"] & C \arrow[d, equal] \\
C \arrow[r, "X"', "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[d, "f"] \\
D \arrow[r, "Y"', "\shortmid"{marking}] & D
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[r, "I_C", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow I_f"] & C \arrow[d, "f"] \\
D \arrow[r, "I_D"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta'"] & D \arrow[d, equal] \\
D \arrow[r, "Y"', "\shortmid"{marking}] & D
\end{tikzcd}
\tag{12}$$

and

$$\begin{tikzcd}
C \arrow[r, "X\otimes X", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mu"] & C \arrow[d, equal] \\
C \arrow[r, "X"', "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[d, "f"] \\
D \arrow[r, "Y"', "\shortmid"{marking}] & D
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[r, "X\otimes X", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi\otimes\phi"] & C \arrow[d, "f"] \\
D \arrow[r, "Y\otimes Y"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mu'"] & D \arrow[d, equal] \\
D \arrow[r, "Y"', "\shortmid"{marking}] & D.
\end{tikzcd}
\tag{13}$$

Definition 4.3. A comonoid in a 2-fold double category $\mathbb{D}$ is a comonoid in $\mathbb{D}_\odot$. Specifically, it is a horizontal 1-cell $X\colon C \nrightarrow C$, together with counit and comultiplication 2-cells

$$\begin{tikzcd}
C \arrow[r, "X", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\epsilon"] & C \arrow[d, equal] \\
C \arrow[r, "\perp_C"', "\shortmid"{marking}] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "X", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\delta"] & C \arrow[d, equal] \\
C \arrow[r, "X\odot X"', "\shortmid"{marking}] & C
\end{tikzcd}$$

satisfying the usual unit and associativity conditions.

Definition 4.4. Given two comonoids (C,X,ϵ,δ) and (D,Y,ϵ',δ') in $\mathbb{D}$, a morphism of comonoids consists of a vertical morphism $f\colon C \to D$ and a 2-cell

$$\begin{tikzcd}
C \arrow[r, "X", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[d, "f"] \\
D \arrow[r, "Y"', "\shortmid"{marking}] & D
\end{tikzcd}$$

which preserves the counit and comultiplication, in that

$$\begin{tikzcd}
C \arrow[r, "X", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\epsilon"] & C \arrow[d, equal] & & C \arrow[r, "X", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[d, "f"] \\
C \arrow[r, "\perp_C"', "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\perp_f"] & C \arrow[d, "f"] & = & D \arrow[r, "Y"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\epsilon'"] & D \arrow[d, equal] \\
D \arrow[r, "\perp_D"', "\shortmid"{marking}] & D & & D \arrow[r, "\perp_D"', "\shortmid"{marking}] & D
\end{tikzcd} \tag{14}$$

and

$$\begin{tikzcd}
C \arrow[r, "X", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\delta"] & C \arrow[d, equal] & & C \arrow[r, "X", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[d, "f"] \\
C \arrow[r, "X\odot X"', "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi\odot\phi"] & C \arrow[d, "f"] & = & D \arrow[r, "Y"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\delta'"] & D \arrow[d, equal] \\
D \arrow[r, "Y\odot Y"', "\shortmid"{marking}] & D & & D \arrow[r, "Y\odot Y"', "\shortmid"{marking}] & D.
\end{tikzcd} \tag{15}$$

The 2-fold double category structure on $\mathbb{D}$ allows us to form double categories $\mathbb{M}\mathrm{on}(\mathbb{D})$ and $\mathbb{C}\mathrm{omon}(\mathbb{D})$ of (co)monoids in $\mathbb{D}$, in which the objects and vertical morphisms are the same as in $\mathbb{D}$, and the horizontal 1-cells and 2-cells are (co)monoids and (co)monoid morphisms in $\mathbb{D}$. The only thing we still have to provide is a horizontal composition.

Given two monoids (C, X, η, μ) and (C, Y, η', μ') in $\mathbb{D}$, thought of as horizontal 1-cells $C \to\!\!\!\!\!\shortmid\; C$ in $\mathbb{M}\mathrm{on}(\mathbb{D})$, the horizontal composition

$$\begin{tikzcd}
C \arrow[r, "{(X,\eta,\mu)}", "\shortmid"{marking}] & C \arrow[r, "{(Y,\eta',\mu')}", "\shortmid"{marking}] & C
\end{tikzcd}$$

is the monoid with underlying horizontal 1-cell $X \odot Y$ and unit and multiplication 2-cells

$$\begin{tikzcd}
C \arrow[r, "I_C", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow c"] & C \arrow[d, equal] & & C \arrow[r, "(X\odot Y)\otimes(X\odot Y)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow z"] & C \arrow[d, equal] \\
C \arrow[r, "I_C\odot I_C", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \eta\odot\eta'"] & C \arrow[d, equal] & & C \arrow[r, "(X\otimes X)\odot(Y\otimes Y)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \mu\odot\mu'"] & C \arrow[d, equal] \\
C \arrow[r, "X\odot Y"', "\shortmid"{marking}] & C & & C \arrow[r, "X\odot Y"', "\shortmid"{marking}] & C.
\end{tikzcd}$$

Similarly, the horizontal composition of two 2-cells in $\mathbb{M}\mathrm{on}(\mathbb{D})$ is the $\odot$ product of the underlying 2-cells in $\mathbb{D}$. The fact that this commutes with the unit and multiplication defined above follows from the naturality of c and z.

In this same way, we can define the horizontal composition of two 1-cells (X, ϵ, δ) and (Y, ϵ', δ') in $\mathbb{C}\mathrm{omon}(\mathbb{D})$ to be a comonoid with underlying horizontal 1-cell $X \otimes Y$.

This allows us to define (ordinary) categories $\mathrm{Mon}(\mathbb{C}\mathrm{omon}(\mathbb{D}))$ and $\mathrm{Comon}(\mathbb{M}\mathrm{on}(\mathbb{D}))$. Furthermore, these two categories are equivalent, leading to the next definition.

Definition 4.5. A *bimonoid* in a 2-fold double category $\mathbb{D}$ is a monoid in $\mathbb{C}\mathrm{omon}(\mathbb{D})$, or equivalently a comonoid in $\mathbb{M}\mathrm{on}(\mathbb{D})$. We can define a category of bimonoids in $\mathbb{D}$ as

$$\mathrm{Bimon}(\mathbb{D}) := \mathrm{Mon}(\mathbb{C}\mathrm{omon}(\mathbb{D})) \simeq \mathrm{Comon}(\mathbb{M}\mathrm{on}(\mathbb{D}))$$

Concretely, a bimonoid in $\mathbb{D}$ is a tuple $(X, \eta, \mu, \epsilon, \delta)$ where X is a horizontal 1-cell, (X, η, μ) is a monoid and (X, ϵ, δ) is a comonoid as above, such that four equations hold:

$$\begin{tikzcd}
C \arrow[r, "I_C", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta"] & C \arrow[d, equal] & & C \arrow[r, "I_C", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow c"] & C \arrow[d, equal] \\
C \arrow[r, "X", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\delta"] & C \arrow[d, equal] & = & C \arrow[r, "I_C\odot I_C", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta\odot\eta"] & C \arrow[d, equal] \\
C \arrow[r, "X\odot X"', "\shortmid"{marking}] & C & & C \arrow[r, "X\odot X"', "\shortmid"{marking}] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "X\otimes X", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mu"] & C \arrow[d, equal] & & C \arrow[r, "X\otimes X", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\epsilon\otimes\epsilon"] & C \arrow[d, equal] \\
C \arrow[r, "X", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\epsilon"] & C \arrow[d, equal] & = & C \arrow[r, "\perp_C\otimes\perp_C", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow m"] & C \arrow[d, equal] \\
C \arrow[r, "\perp_C"', "\shortmid"{marking}] & C & & C \arrow[r, "\perp_C"', "\shortmid"{marking}] & C
\end{tikzcd}$$

$$\begin{tikzcd}
C \arrow[r, "I_C", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta"] & C \arrow[d, equal] \\
C \arrow[r, "X", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\epsilon"] & C \arrow[d, equal] \\
C \arrow[r, "\perp_C"', "\shortmid"{marking}] & C
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[r, "I_C", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow j"] & C \arrow[d, equal] \\
C \arrow[r, "\perp_C"', "\shortmid"{marking}] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "X\otimes X", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\delta\otimes\delta"] & C \arrow[d, equal] \\
C \arrow[r, "(X\odot X)\otimes(X\odot X)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow z"] & C \arrow[d, equal] \\
C \arrow[r, "(X\otimes X)\odot(X\otimes X)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mu\odot\mu"] & C \arrow[d, equal] \\
C \arrow[r, "X\odot X"', "\shortmid"{marking}] & C
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[r, "X\otimes X", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mu"] & C \arrow[d, equal] \\
C \arrow[r, "X", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\delta"] & C \arrow[d, equal] \\
C \arrow[r, "X\odot X"', "\shortmid"{marking}] & C
\end{tikzcd}$$

(16)

A bimonoid morphism is simply a 2-cell which is simultaineously a monoid morphism and a comonoid morphism.

5 Cyclic 2-fold Double Categories

Recall the notion of a cyclic double category from [GR12]. A cyclic double category $\mathbb{D}$ is a double category with an extra involutive operation. On objects and horizontal 1-cells $X\colon C \nrightarrow C$, this operation is written

$$\begin{tikzcd} C^\bullet \arrow[r, "X^\bullet", "\shortmid"{marking}] & C^\bullet \end{tikzcd}$$

and respects horizontal identities and composition. The involution takes any vertical 1-cell $f\colon C \to D$ to some $\sigma f\colon D^\bullet \to C^\bullet$, and any 2-cell

$$\begin{tikzcd} C \arrow[r, "X", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\theta"] & C \arrow[d, "f"] \\ D \arrow[r, "Y"', "\shortmid"{marking}] & D \end{tikzcd} \quad \text{to} \quad \begin{tikzcd} D^\bullet \arrow[r, "Y^\bullet", "\shortmid"{marking}] \arrow[d, "\sigma f"'] \arrow[dr, phantom, "\Downarrow\sigma\theta"] & D^\bullet \arrow[d, "\sigma f"] \\ C^\bullet \arrow[r, "X^\bullet"', "\shortmid"{marking}] & C^\bullet \end{tikzcd}$$

respecting vertical identities and composition.

We will generalize this to a cyclic action on a 2-fold double category. Suppose that $\mathbb{D}$ is a 2-fold double category. A cyclic action, written as above, must satisfy the following:

- For every object C,

$$I_{C^\bullet} = (\bot_C)^\bullet \qquad \text{and} \qquad \bot_{C^\bullet} = (I_C)^\bullet.$$

- For every composable pair of horizontal 1-cells $X, Y\colon C \nrightarrow C$,

$$(X \otimes Y)^\bullet = X^\bullet \odot Y^\bullet \qquad \text{and} \qquad (X \odot Y)^\bullet = X^\bullet \otimes Y^\bullet$$

- For every vertical 1-cell $f\colon C \to D$, there are equalities

$$\begin{tikzcd} D^\bullet \arrow[r, "I_{D^\bullet}", "\shortmid"{marking}] \arrow[d, "\sigma f"'] \arrow[dr, phantom, "\Downarrow I_{\sigma f}"] & D^\bullet \arrow[d, "\sigma f"] \\ C^\bullet \arrow[r, "I_{C^\bullet}"', "\shortmid"{marking}] & C^\bullet \end{tikzcd} = \begin{tikzcd} D^\bullet \arrow[r, "(\bot_D)^\bullet", "\shortmid"{marking}] \arrow[d, "\sigma f"'] \arrow[dr, phantom, "\Downarrow\sigma\bot_f"] & D^\bullet \arrow[d, "\sigma f"] \\ C^\bullet \arrow[r, "(\bot_C)^\bullet"', "\shortmid"{marking}] & C^\bullet \end{tikzcd}$$

$$\begin{tikzcd} D^\bullet \arrow[r, "\bot_{D^\bullet}", "\shortmid"{marking}] \arrow[d, "\sigma f"'] \arrow[dr, phantom, "\Downarrow\bot_{\sigma f}"] & D^\bullet \arrow[d, "\sigma f"] \\ C^\bullet \arrow[r, "\bot_{C^\bullet}"', "\shortmid"{marking}] & C^\bullet \end{tikzcd} = \begin{tikzcd} D^\bullet \arrow[r, "(I_D)^\bullet", "\shortmid"{marking}] \arrow[d, "\sigma f"'] \arrow[dr, phantom, "\Downarrow\sigma I_f"] & D^\bullet \arrow[d, "\sigma f"] \\ C^\bullet \arrow[r, "(I_C)^\bullet"', "\shortmid"{marking}] & C^\bullet \end{tikzcd}$$

- For every horizontally composable pair of 2-cells

$$\begin{tikzcd}
C \arrow[r, "X", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\theta"] & C \arrow[r, "Y", "\shortmid"{marking}] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[d, "f"] \\
D \arrow[r, "X'"', "\shortmid"{marking}] & D \arrow[r, "Y'"', "\shortmid"{marking}] & D
\end{tikzcd}$$

  there are equalities

$$\begin{tikzcd}
D^\bullet \arrow[r, "(X'\otimes Y')^\bullet", "\shortmid"{marking}] \arrow[d, "\sigma f"'] \arrow[dr, phantom, "\Downarrow\sigma(\theta\otimes\phi)"] & D^\bullet \arrow[d, "\sigma f"] \arrow[r, phantom, "="] & D^\bullet \arrow[r, "X'^\bullet\odot Y'^\bullet", "\shortmid"{marking}] \arrow[d, "\sigma f"'] \arrow[dr, phantom, "\Downarrow\sigma(\theta)\odot\sigma(\phi)"] & D^\bullet \arrow[d, "\sigma f"] \\
C^\bullet \arrow[r, "(X\otimes Y)^\bullet"', "\shortmid"{marking}] & C^\bullet & C^\bullet \arrow[r, "X^\bullet\odot Y^\bullet"', "\shortmid"{marking}] & C^\bullet
\end{tikzcd}$$

$$\begin{tikzcd}
D^\bullet \arrow[r, "(X'\odot Y')^\bullet", "\shortmid"{marking}] \arrow[d, "\sigma f"'] \arrow[dr, phantom, "\Downarrow\sigma(\theta\odot\phi)"] & D^\bullet \arrow[d, "\sigma f"] \arrow[r, phantom, "="] & D^\bullet \arrow[r, "X'^\bullet\otimes Y'^\bullet", "\shortmid"{marking}] \arrow[d, "\sigma f"'] \arrow[dr, phantom, "\Downarrow\sigma(\theta)\otimes\sigma(\phi)"] & D^\bullet \arrow[d, "\sigma f"] \\
C^\bullet \arrow[r, "(X\odot Y)^\bullet"', "\shortmid"{marking}] & C^\bullet & C^\bullet \arrow[r, "X^\bullet\otimes Y^\bullet"', "\shortmid"{marking}] & C^\bullet
\end{tikzcd}$$

One nice consequence of this definition is that a cyclic action on a 2-fold double category $\mathbb{D}$ induces a cyclic action on the category of bimonoids $\mathrm{Bimon}(\mathbb{D})$.

Proposition 5.1. *Suppose $\mathbb{D}$ is a cyclic 2-fold double category. Then the category* $\mathrm{Bimon}(\mathbb{D})$ *of bimonoids in $\mathbb{D}$ carries a natural cyclic action (contravariant isomorphism).*

Proof. The involution $(-)^\bullet$ gives an isomorphism of double categories $\mathbb{D}_\otimes \cong \mathbb{D}_\odot^{\mathrm{op}}$. Therefore it also induces an isomorphism

$$\mathbb{M}\mathrm{on}(\mathbb{D}) = \mathbb{M}\mathrm{on}(\mathbb{D}_\otimes) \cong \mathbb{M}\mathrm{on}(\mathbb{D}_\odot^{\mathrm{op}}) \cong \mathbb{C}\mathrm{omon}(\mathbb{D}_\odot)^{\mathrm{op}} = \mathbb{C}\mathrm{omon}(\mathbb{D})^{\mathrm{op}}$$

as well as an isomorphism

$$\begin{aligned}\mathrm{Bimon}(\mathbb{D}) = \mathrm{Comon}(\mathbb{M}\mathrm{on}(\mathbb{D})) &\cong \mathrm{Comon}(\mathbb{C}\mathrm{omon}(\mathbb{D})^{\mathrm{op}}) \\ &\cong \mathrm{Mon}(\mathbb{C}\mathrm{omon}(\mathbb{D}))^{\mathrm{op}} = \mathrm{Bimon}(\mathbb{D})^{\mathrm{op}}.\end{aligned}$$

□

In more concrete terms, the involution takes a bimonoid $(X,\eta,\mu,\epsilon,\delta)$ to $(X,\eta,\mu,\epsilon,\delta)^\bullet = (X^\bullet,\epsilon^\bullet,\delta^\bullet,\eta^\bullet,\delta^\bullet)$, swapping the monoid and comonoid structures. This is again a bimonoid, as the top two equations of (16) are interchanged under the involution, while the bottom two equations are self-dual.

The action of the involution on bimonoid morphisms can be broken down as in the following lemma.

Lemma 5.2. *Let* $(X,\eta,\mu,\epsilon,\delta)$ *and* $(Y,\eta',\mu',\epsilon',\delta')$ *be bimonoids in a cyclic 2-fold double category* $\mathbb{D}$*, and let* ϕ *be a 2-cell in* $\mathbb{D}$

$$\begin{tikzcd}
C \arrow[r, "X", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & C \arrow[d, "f"] \\
D \arrow[r, "Y"', "\shortmid"{marking}] & D.
\end{tikzcd}$$

Then (f,ϕ) *is a monoid morphism* $X \to Y$ *if and only if* $(\sigma f,\phi^\bullet)$ *is a comonoid morphism* $Y^\bullet \to X^\bullet$*. Dually,* ϕ *is a comonoid morphism* $X \to Y$ *if and only if* $\phi^\bullet$ *is a monoid morphism* $Y^\bullet \to X^\bullet$*.*

Proof. Simply notice that the involution interchanges equations (12) and (13) with (14) and (15). □

This immediately implies a useful characterization of bimonoid morphisms.

Corollary 5.3. *Given bimonoids* $(X,\eta,\mu,\epsilon,\delta)$ *and* $(Y,\eta',\mu',\epsilon',\delta')$ *in a cyclic 2-fold double category* $\mathbb{D}$*, a bimonoid morphism* $X \to Y$ *consists of:*

- Either *a monoid morphism* $X \to Y$ or *a comonoid morphism* $Y^\bullet \to X^\bullet$, and
- Either *a comonoid morphism* $X \to Y$ or *a monoid morphism* $Y^\bullet \to X^\bullet$.

6 Functorial Factorizations

Now let $\mathbb{D}$ be a cyclic double category, and assume it has arrow objects in the sense of Section **??**. Let us spell out what this means for the single variable case:

- For every object C there is a diagram

$$\begin{tikzcd} C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\kappa"] & C. \end{tikzcd}$$

- Any 2-cell

$$\begin{tikzcd} A \arrow[r, bend left, "d_1"] \arrow[r, bend right, "d_0"'] \arrow[r, phantom, "\Downarrow\alpha"] & C \end{tikzcd}$$

  uniquely factors through κ, as

$$\begin{tikzcd} A \arrow[r, "\hat{\alpha}"] & C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\kappa"] & C. \end{tikzcd}$$

- For every vertical 1-cell $F\colon C \to D$ there is a vertical 1-cell $\hat{F}\colon C^2 \to D^2$ and 2-cells

$$\begin{tikzcd} C^2 \arrow[r, "\mathrm{dom}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_1"] & C \arrow[d, "F"] \\ D^2 \arrow[r, "\mathrm{dom}"'] & D \end{tikzcd} \qquad \begin{tikzcd} C^2 \arrow[r, "\mathrm{cod}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_0"] & C \arrow[d, "F"] \\ D^2 \arrow[r, "\mathrm{cod}"'] & D \end{tikzcd}$$

  such that

$$\begin{tikzcd} C^2 \arrow[r, "\mathrm{dom}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_1"] & C \arrow[d, "F"] \\ D^2 \arrow[r, "\mathrm{dom"] \arrow[r, bend right=50, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\kappa"', shift right=2] & D \end{tikzcd} = \begin{tikzcd} C^2 \arrow[r, bend left=50, "\mathrm{dom}"] \arrow[r, phantom, "\Downarrow\kappa", shift left=2] \arrow[r, "\mathrm{cod}"'] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_0"] & C \arrow[d, "F"] \\ D^2 \arrow[r, "\mathrm{cod}"'] & D \end{tikzcd}$$

- Given any 2-cells

$$\begin{tikzcd} A \arrow[r, bend left, "d_1"] \arrow[r, bend right, "d_0"'] \arrow[r, phantom, "\Downarrow\alpha"] & C \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, bend left, "d'_1"] \arrow[r, bend right, "d'_0"'] \arrow[r, phantom, "\Downarrow\alpha'"] & D \end{tikzcd}$$

and

$$\begin{tikzcd}
A \arrow[r, "d_1"] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\lambda_1"] & C \arrow[d, "F"] & A \arrow[r, "d_0"] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\lambda_0"] & C \arrow[d, "F"] \\
B \arrow[r, "d'_1"'] & D & B \arrow[r, "d'_0"'] & D
\end{tikzcd}$$

such that

$$\begin{tikzcd}
A \arrow[r, "d_1"] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\lambda_1"] & C \arrow[d, "F"] & & A \arrow[r, bend left=60, "d_1", "\Downarrow\alpha"'] \arrow[r, "d_0"'] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\lambda_0"] & C \arrow[d, "F"] \\
B \arrow[r, "d'_1"] \arrow[r, bend right=60, "d'_0"', "\Downarrow\alpha'"] & D & = & B \arrow[r, "d'_0"'] & D
\end{tikzcd}$$

there is a unique 2-cell

$$\begin{tikzcd}
A \arrow[r, "\hat{\alpha}"] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\theta"] & C^2 \arrow[d, "\hat{F}"] \\
B \arrow[r, "\hat{\alpha}'"'] & D^2
\end{tikzcd}$$

such that the horizontal composition of θ with γ_0 and γ_1 is respectively equal to λ_0 and λ_1.

Remark 6.1 (TODO: Remark that this generalizes the 2-dimensional part of the usual universal property in 2-categories.).

We will now define a 2-fold double category $\mathbb{FF}(\mathbb{D})$ of functorial factorizations in $\mathbb{D}$, as follows:

- The objects and vertical 1-cells are the same as in $\mathbb{D}$.
- Horizontal 1-cells $C \nrightarrow C$ in $\mathbb{FF}(\mathbb{D})$ are tuples (E, η, ϵ), where $E\colon C^2 \to C$ is a horizontal 1-cell in $\mathbb{D}$, and

$$\begin{tikzcd}
C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "E"'] \arrow[r, phantom, "\Downarrow\eta"] & C & C^2 \arrow[r, bend left, "E"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\epsilon"] & C
\end{tikzcd}$$

are 2-cells in $\mathbb{D}$ such that

$$\begin{tikzcd}
C^2 \arrow[r, "E" description] \arrow[r, bend left=70, "\mathrm{dom}", "\Downarrow\eta"'] \arrow[r, bend right=70, "\mathrm{cod}"', "\Downarrow\epsilon"] & C & = & C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\kappa"] & C.
\end{tikzcd}$$

By the universal property of C^2, this also determines horizontal 1-cells $L, R\colon C^2 \to C^2$ such that $\mathrm{dom} \circ L = \mathrm{dom}$, $\mathrm{cod} \circ R = \mathrm{cod}$, $\mathrm{cod} \circ L = \mathrm{dom} \circ R = E$, $\kappa \circ L = \eta$, and $\kappa \circ R = \epsilon$, and 2-cells

$$\begin{tikzcd}
C^2 \arrow[r, bend left, "L"{name=U}] \arrow[r, bend right, "\mathrm{id}"'{name=D}] \arrow[phantom, from=U, to=D, "\Downarrow\vec{\epsilon}"] & C^2. & C^2 \arrow[r, bend left, "\mathrm{id}"{name=U2}] \arrow[r, bend right, "R"'{name=D2}] \arrow[phantom, from=U2, to=D2, "\Downarrow\vec{\eta}"] & C^2.
\end{tikzcd}$$

such that $\mathrm{dom} \circ \vec{\epsilon} = \mathrm{id}_{\mathrm{dom}}$, $\mathrm{cod} \circ \vec{\epsilon} = \epsilon$, $\mathrm{dom} \circ \vec{\eta} = \eta$, and $\mathrm{cod} \circ \vec{\eta} = \mathrm{id}_{\mathrm{cod}}$.

- The horizontal composition $(E_1, \eta_1, \epsilon_1) \otimes (E_2, \eta_2, \epsilon_2)$ of two horizontal 1-cells

$$\begin{tikzcd}
C \arrow[r, "{(E_1,\eta_1,\epsilon_1)}", "\shortmid"{marking}] & C \arrow[r, "{(E_2,\eta_2,\epsilon_2)}", "\shortmid"{marking}] & C
\end{tikzcd}$$

  in $\mathbb{FF}(\mathbb{D})$ is a horizontal 1-cell $(E_{1\otimes 2}, \eta_{1\otimes 2}, \epsilon_{1\otimes 2})$, where

$$E_{1\otimes 2} = \begin{tikzcd} C^2 \arrow[r, "R_1"] & C^2 \arrow[r, "E_2"] & C \end{tikzcd}$$

$$\eta_{1\otimes 2} = \begin{tikzcd}
C^2 \arrow[r, bend left, "\mathrm{id}"{name=U}] \arrow[r, bend right, "R_1"'{name=D}] \arrow[phantom, from=U, to=D, "\Downarrow\vec{\eta}_1"] & C^2 \arrow[r, bend left, "\mathrm{dom}"{name=U2}] \arrow[r, bend right, "E_2"'{name=D2}] \arrow[phantom, from=U2, to=D2, "\Downarrow\eta_2"] & C
\end{tikzcd}$$

$$\epsilon_{1\otimes 2} = \begin{tikzcd}
C^2 \arrow[r, "R_1"] & C^2 \arrow[r, bend left, "E_2"{name=U}] \arrow[r, bend right, "\mathrm{cod}"'{name=D}] \arrow[phantom, from=U, to=D, "\Downarrow\epsilon_2"] & C
\end{tikzcd}$$

  which also determines that $R_{1\otimes 2} = R_2 \circ R_1$.
- The horizontal unit I_C for $\otimes$ is $(\mathrm{dom}, \mathrm{id}, \kappa)$.
- The second horizontal composition $(E_1, \eta_1, \epsilon_1) \odot (E_2, \eta_2, \epsilon_2)$ is a horizontal 1-cell $(E_{1\odot 2}, \eta_{1\odot 2}, \epsilon_{1\odot 2})$, where

$$E_{1\odot 2} = \begin{tikzcd} C^2 \arrow[r, "L_1"] & C^2 \arrow[r, "E_2"] & C \end{tikzcd}$$

$$\eta_{1\odot 2} = \begin{tikzcd}
C^2 \arrow[r, "L_1"] & C^2 \arrow[r, bend left, "\mathrm{dom}"{name=U}] \arrow[r, bend right, "E_2"'{name=D}] \arrow[phantom, from=U, to=D, "\Downarrow\eta_2"] & C
\end{tikzcd}$$

$$\epsilon_{1\odot 2} = \begin{tikzcd}
C^2 \arrow[r, bend left, "L_1"{name=U}] \arrow[r, bend right, "\mathrm{id}"'{name=D}] \arrow[phantom, from=U, to=D, "\Downarrow\vec{\epsilon}_1"] & C^2 \arrow[r, bend left, "E_2"{name=U2}] \arrow[r, bend right, "\mathrm{dom}"'{name=D2}] \arrow[phantom, from=U2, to=D2, "\Downarrow\epsilon_2"] & C
\end{tikzcd}$$

  which also determines that $L_{1\odot 2} = L_2 \circ L_1$.
- The horizontal unit $\perp_C$ for $\odot$ is $(\mathrm{cod}, \kappa, \mathrm{id})$.

- 2-cells

$$\begin{tikzcd}
C \arrow[r, "{(E_1,\eta_1,\epsilon_1)}", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\theta"] & C \arrow[d, "F"] \\
D \arrow[r, "{(E_2,\eta_2,\epsilon_2)}"', "\shortmid"{marking}] & D
\end{tikzcd}$$

in $\mathbb{FF}(\mathbb{D})$ are given by 2-cells

$$\begin{tikzcd}
C^2 \arrow[r, "E_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "E_2"'] & D
\end{tikzcd}$$

in $\mathbb{D}$ such that

$$\begin{tikzcd}
C^2 \arrow[r, "E_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "E_2"] \arrow[r, bend right=50, "\mathrm{cod}"', ""{name=B}] & D
\end{tikzcd}
\overset{\Downarrow\epsilon_2}{} =
\begin{tikzcd}
C^2 \arrow[r, bend left=50, "E_1"] \arrow[r, "\mathrm{cod}"'] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_0"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "\mathrm{cod}"'] & D
\end{tikzcd} \tag{17}$$

and

$$\begin{tikzcd}
C^2 \arrow[r, "\mathrm{dom}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_1"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "\mathrm{dom}"] \arrow[r, bend right=50, "E_2"'] & D
\end{tikzcd}
\overset{\Downarrow\eta_2}{} =
\begin{tikzcd}
C^2 \arrow[r, bend left=50, "\mathrm{dom}"] \arrow[r, "E_1"'] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "E_2"'] & D
\end{tikzcd} \tag{18}$$

(with $\Downarrow\epsilon_1$ and $\Downarrow\eta_1$ between the curved and straight upper arrows on the right-hand sides.)

This also determines unique 2-cells

$$\begin{tikzcd}
C^2 \arrow[r, "R_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta^R"] & C^2 \arrow[d, "\hat{F}"] \\
D^2 \arrow[r, "R_2"'] & D^2
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
C^2 \arrow[r, "L_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta^L"] & C^2 \arrow[d, "\hat{F}"] \\
D^2 \arrow[r, "L_2"'] & D^2
\end{tikzcd}$$

such that composing horizontally with γ_0 or γ_1 gives γ_0, γ_1, or θ as appropriate. For instance:

$$\begin{tikzcd}
C^2 \arrow[r, "R_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta^R"] & C^2 \arrow[r, "\mathrm{dom}"] \arrow[d, "\hat{F}"] \arrow[dr, phantom, "\Downarrow\gamma_1"] & C \arrow[d, "F"] & = & C^2 \arrow[r, "E_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "R_2"'] & D^2 \arrow[r, "\mathrm{dom}"'] & D & & D^2 \arrow[r, "E_2"'] & D
\end{tikzcd}$$

- Given a pair of composable 2-cells in $\mathbb{FF}(\mathbb{D})$ as in

$$\begin{tikzcd}
C \arrow[r, "(E_1{,}\eta_1{,}\epsilon_1)", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\theta_1"] & C \arrow[r, "(E_2{,}\eta_2{,}\epsilon_2)", "\shortmid"{marking}] \arrow[d, "F"] \arrow[dr, phantom, "\Downarrow\theta_2"] & C \arrow[d, "F"] \\
D \arrow[r, "(E_1'{,}\eta_1'{,}\epsilon_1')"', "\shortmid"{marking}] & D \arrow[r, "(E_2'{,}\eta_2'{,}\epsilon_2')"', "\shortmid"{marking}] & D
\end{tikzcd}$$

the composite $\theta_1 \otimes \theta_2$ is given by

$$\begin{tikzcd}
C^2 \arrow[r, "R_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta_1^R"] & C^2 \arrow[r, "E_2"] \arrow[d, "\hat{F}"] \arrow[dr, phantom, "\Downarrow\theta_2"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "R_1'"'] & D^2 \arrow[r, "E_2'"'] & D
\end{tikzcd}$$

while the composite $\theta_1 \odot \theta_2$ is given by

$$\begin{tikzcd}
C^2 \arrow[r, "L_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta_1^L"] & C^2 \arrow[r, "E_2"] \arrow[d, "\hat{F}"] \arrow[dr, phantom, "\Downarrow\theta_2"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "L_1'"'] & D^2 \arrow[r, "E_2'"'] & D
\end{tikzcd}$$

It is a straightforward exercise to check that these definitions satisfy equations (17) and (18). To illustrate, we will demonstrate that $\theta_1 \otimes \theta_2$

satisfies (17):

$$\begin{tikzcd}
C^2 \arrow[r, "E_{1\otimes 2}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta_1\otimes\theta_2"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "E_{1'\otimes 2'}", "\Downarrow\epsilon_{1'\otimes 2'}"'] \arrow[r, bend right=50, "\mathrm{cod}"'] & D
\end{tikzcd}
=
\begin{tikzcd}
C^2 \arrow[r, "R_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta_1^R"] & C^2 \arrow[r, "E_2"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta_2"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "R_1'"'] & D^2 \arrow[r, "E_2'", "\Downarrow\epsilon_2'"'] \arrow[r, bend right=50, "\mathrm{cod}"'] & D
\end{tikzcd}$$

$$=
\begin{tikzcd}
C^2 \arrow[r, "R_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta_1^R"] & C^2 \arrow[r, bend left=50, "E_2"] \arrow[r, "\Downarrow\epsilon_2", "\mathrm{cod}"'] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_0"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "R_1'"'] & D^2 \arrow[r, "\mathrm{cod}"'] & D
\end{tikzcd}$$

$$=
\begin{tikzcd}
C^2 \arrow[r, bend left=50, "E_{1\otimes 2}"] \arrow[r, "\Downarrow\epsilon_{1\otimes 2}", "\mathrm{cod}"'] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_0"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "\mathrm{cod}"'] & D
\end{tikzcd}$$

It is straightforward to check that ⊗ and ⊙ are each associative and unital. It takes more work to provide the compatibility between ⊗ and ⊙, which is the content of the proof of the next proposition.

Proposition 6.2. $\mathbb{FF}(\mathbb{D})$ *has the structure of a 2-fold double category.*

Proof. The primary structure of $\mathbb{FF}(\mathbb{D})$ was given in the first part of this section. What is left is to provide the coherence data (10) and (11).

First, note that I_C is initial in the sense that, given any vertical morphism $F\colon C \to D$ and any functorial factorization (E, η, ϵ) on D, there is a unique 2-cell

$$\begin{tikzcd}
C \arrow[r, "I_C", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow"] & C \arrow[d, "F"] \\
D \arrow[r, "{(E,\eta,\epsilon)}"', "\shortmid"{marking}] & D
\end{tikzcd}$$

given by

$$\begin{tikzcd}
C^2 \arrow[r, "\mathrm{dom}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_1"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "\mathrm{dom}", "\Downarrow\eta"'] \arrow[r, bend right=50, "E"'] & D.
\end{tikzcd}$$

Similarly, $\bot_C$ is terminal. Thus there is only one possible way to define the 2-cells m, c, and j, and naturality and all other coherence equations follows immediately from this uniqueness.

We still need to construct the 2-cell z, which will take some work. We begin by defining 2-cells

$$\begin{tikzcd}
C \arrow[r, "E_1 \odot E_2", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow p_{E_1,E_2}"] & C \arrow[d, equal] \\
C \arrow[r, "E_1"', "\shortmid"{marking}] & C
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
C \arrow[r, "E_1", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow i_{E_1,E_2}"] & C \arrow[d, equal] \\
C \arrow[r, "E_1 \otimes E_2"', "\shortmid"{marking}] & C.
\end{tikzcd}$$

for any pair of functorial factorizations. The 2-cell p is given by the underlying 2-cell in $\mathbb{D}$

$$\begin{tikzcd}
C^2 \arrow[r, "L_1"] & C^2 \arrow[r, bend left, "E_2"{name=U}] \arrow[r, bend right, "\mathrm{cod}"'{name=D}] & C
\arrow[from=U, to=D, phantom, "\Downarrow \epsilon_2"]
\end{tikzcd}$$

and i is given by

$$\begin{tikzcd}
C^2 \arrow[r, "R_1"] & C^2 \arrow[r, bend left, "\mathrm{dom}"{name=U}] \arrow[r, bend right, "E_2"'{name=D}] & C.
\arrow[from=U, to=D, phantom, "\Downarrow \eta_2"]
\end{tikzcd}$$

To illustrate the verification that these give well-defined 2-cells in $\mathbb{FF}(\mathbb{D})$, we will show that i satisfies (17) (keep in mind that when F is an identity, γ_0 and γ_1 are also identities):

$$\begin{tikzcd}
C^2 \arrow[r, "L_1"] & C^2 \arrow[r, bend left=60, "\mathrm{dom}"{name=U}] \arrow[r, "E_2"{name=M}, dashed] \arrow[r, bend right=60, "\mathrm{cod}"'{name=D}] & C
\arrow[from=U, to=M, phantom, "\Downarrow \eta_2"]
\arrow[from=M, to=D, phantom, "\Downarrow \epsilon_2"]
\end{tikzcd}
=
\begin{tikzcd}
C^2 \arrow[r, "L_1"] & C^2 \arrow[r, bend left, "\mathrm{dom}"{name=U}] \arrow[r, bend right, "\mathrm{cod}"'{name=D}] & C
\arrow[from=U, to=D, phantom, "\Downarrow \kappa"]
\end{tikzcd}$$

$$=
\begin{tikzcd}
C^2 \arrow[r, bend left, "\mathrm{dom}"{name=U}] \arrow[r, bend right, "E_1"'{name=D}] & C.
\arrow[from=U, to=D, phantom, "\Downarrow \eta_1"]
\end{tikzcd}$$

Moreover, it is straightforward to check that i and p are natural families of 2-cells. Specifically, for any pair of 2-cells θ_1 and θ_2

$$\begin{tikzcd}
C \arrow[r, "E_1\odot E_2", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow p_{E_1,E_2}"] & C \arrow[d, equal] & & C \arrow[r, "E_1\odot E_2", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow \theta_1\odot\theta_2"] & C \arrow[d, "F"] \\
C \arrow[r, "E_1", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow \theta_1"] & C \arrow[d, "F"] & = & D \arrow[r, "E_1'\odot E_2'", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow p_{E_1',E_2'}"] & D \arrow[d, equal] \\
D \arrow[r, "E_1'"', "\shortmid"{marking}] & D & & D \arrow[r, "E_1'"', "\shortmid"{marking}] & D
\end{tikzcd}$$

$$\begin{tikzcd}
C \arrow[r, "E_1", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow i_{E_1,E_2}"] & C \arrow[d, equal] & & C \arrow[r, "E_1", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow \theta_1"] & C \arrow[d, "F"] \\
C \arrow[r, "E_1\otimes E_2", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow \theta_1\otimes\theta_2"] & C \arrow[d, "F"] & = & D \arrow[r, "E_1'", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow i_{E_1',E_2'}"] & D \arrow[d, equal] \\
D \arrow[r, "E_1'\otimes E_2'"', "\shortmid"{marking}] & D & & D \arrow[r, "E_1'\otimes E_2'"', "\shortmid"{marking}] & D
\end{tikzcd}$$

As with any 2-cell in $\mathbb{FF}(\mathbb{D})$, p and i induce 2-cells in $\mathbb{D}$

$$\begin{tikzcd}
C^2 \arrow[r, bend left, "R_{1\odot 2}"] \arrow[r, bend right, "R_1"'] \arrow[r, phantom, "\Downarrow p^R"] & C^2 & \text{and} & C^2 \arrow[r, bend left, "L_1"] \arrow[r, bend right, "L_{1\otimes 2}"'] \arrow[r, phantom, "\Downarrow i^L"] & C^2.
\end{tikzcd}$$

such that

$$\begin{tikzcd}
C^2 \arrow[r, bend left, "R_{1\odot 2}"] \arrow[r, bend right, "R_1"'] \arrow[r, phantom, "\Downarrow p^R"] & C^2 \arrow[r, "\mathrm{dom}"] & C & = & C^2 \arrow[r, "L_1"] & C^2 \arrow[r, bend left, "E_2"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow \epsilon_2"] & C
\end{tikzcd} \tag{19}$$

$$\begin{tikzcd}
C^2 \arrow[r, bend left, "L_1"] \arrow[r, bend right, "L_{1\otimes 2}"'] \arrow[r, phantom, "\Downarrow i^L"] & C^2 \arrow[r, "\mathrm{cod}"] & C & = & C^2 \arrow[r, "R_1"] & C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "E_2"'] \arrow[r, phantom, "\Downarrow \eta_2"] & C
\end{tikzcd} \tag{20}$$

Now suppose given three functorial factorizations E_1, E_2, E_3 on an object C. We define a 2-cell in $\mathbb{D}$

$$\begin{tikzcd}
& C^2 \arrow[dr, "L_3"] & \\
C^2 \arrow[ur, "R_{1\otimes 2}"] \arrow[dr, "L_{1\otimes 3}"'] & \Downarrow w & C^2 \\
& C^2 \arrow[ur, "R_2"'] &
\end{tikzcd}$$

such that

$$\begin{tikzcd}
& C^2 \arrow[dr, bend left, "L_3"] & & & & & & \\
C^2 \arrow[ur, bend left, "R_{1\otimes 2}"] \arrow[dr, bend right, "L_{1\otimes 3}"'] & \Downarrow w & C^2 \arrow[r, "\mathrm{dom}"] & C & = & C^2 \arrow[r, bend left, "L_1"] \arrow[r, bend right, "L_{1\otimes 3}"'] \arrow[r, phantom, "\Downarrow i^L"] & C^2 \arrow[r, "E_2"] & C \\
& C^2 \arrow[ur, bend right, "R_2"'] & & & & & &
\end{tikzcd} \tag{21}$$

$$\begin{tikzcd}
& C^2 \arrow[dr, bend left, "L_3"] & & & & & & \\
C^2 \arrow[ur, bend left, "R_{1\otimes 2}"] \arrow[dr, bend right, "L_{1\otimes 3}"'] & \Downarrow w & C^2 \arrow[r, "\mathrm{cod}"] & C & = & C^2 \arrow[r, bend left, "R_{1\odot 2}"] \arrow[r, bend right, "R_1"'] \arrow[r, phantom, "\Downarrow p^R"] & C^2 \arrow[r, "E_3"] & C. \\
& C^2 \arrow[ur, bend right, "R_2"'] & & & & & &
\end{tikzcd} \tag{22}$$

Using the universal property for C^2, it suffices to check that

$$\begin{tikzcd}
C^2 \arrow[r, bend left=60, "L_1"] \arrow[r, "L_{1\otimes 3}"'] \arrow[r, phantom, bend left=30, "\Downarrow i^L"] & C^2 \arrow[r, "E_2"] \arrow[r, bend right=60, "\mathrm{cod}"'] \arrow[r, phantom, bend right=30, "\Downarrow \epsilon_2"] & C & = & C^2 \arrow[r, "R_{1\odot 2}"] \arrow[r, bend right=60, "R_1"'] \arrow[r, phantom, bend right=30, "\Downarrow p^R"] & C^2 \arrow[r, bend left=60, "\mathrm{dom}"] \arrow[r, "E_3"'] \arrow[r, phantom, bend left=30, "\Downarrow \eta_3"] & C
\end{tikzcd}$$

and a quick check using equations (19) and (20) shows that both are equal to

$$\begin{tikzcd}
& C^2 \arrow[dr, "\mathrm{cod}"] \arrow[dr, bend left=60, "E_2"] \arrow[dr, phantom, bend left=30, "\Downarrow \epsilon_2"] & \\
C^2 \arrow[ur, "L_1"] \arrow[dr, "R_1"'] & & C \\
& C^2 \arrow[ur, "\mathrm{dom}"] \arrow[ur, bend right=60, "E_3"'] \arrow[ur, phantom, bend right=30, "\Downarrow \eta_3"] &
\end{tikzcd}$$

where the inner diamond is the equality $\mathrm{cod}\, L_1 = \mathrm{dom}\, R_1 = E_1$.

We also check that w is natural with respect to 2-cells in $\mathbb{FF}(\mathbb{D})$ in the following sense: given three 2-cells θ_1, θ_2, and θ_3, there is an equality

$$\begin{tikzcd}
& C^2 \arrow[dr, bend left, "L_3"] & & & & C^2 \arrow[dr, bend left, "L_3"] \arrow[dd, "\hat{F}"] & \\
C^2 \arrow[ur, bend left, "R_{1\odot 2}"] \arrow[dr, bend right, "L_{1\otimes 3}"] \arrow[dd, "\hat{F}"'] & \Downarrow w & C^2 \arrow[dd, "\hat{F}"] & & C^2 \arrow[ur, bend left, "R_{1\odot 2}"] \arrow[dd, "\hat{F}"'] & & C^2 \arrow[dd, "\hat{F}"] \\
\Downarrow(\theta_1 \otimes \theta_3)^L & C^2 \arrow[ur, bend right, "R_2"] \arrow[dd, "\hat{F}"] & \Downarrow \theta_2^R & = & \Downarrow(\theta_1 \odot \theta_2)^R & D^2 \arrow[dr, bend left, "L'_3"] & \Downarrow \theta_3^L \\
D^2 \arrow[dr, bend right, "L'_{1\otimes 3}"'] & & D^2 & & D^2 \arrow[ur, bend left, "R'_{1\odot 2}"'] \arrow[dr, bend right, "L'_{1\otimes 3}"'] & \Downarrow w & D^2. \\
& D^2 \arrow[ur, bend right, "R'_2"'] & & & & D^2 \arrow[ur, bend right, "R'_2"'] &
\end{tikzcd}$$

To verify this equation, it suffices to check equality upon right composition with γ_0 and γ_1. We will illustrate the γ_1 case, making use of the naturality of i:

$$\begin{tikzcd}
& C^2 & & & \\
C^2 & & C^2 & C \\
& C^2 & & \\
D^2 & & D^2 & D \\
& D^2 & &
\arrow[from=2-1, to=1-2, bend left, "R_{1\odot2}"]
\arrow[from=1-2, to=2-3, bend left, "L_3"]
\arrow[from=2-1, to=3-2, bend right, "L_{1\otimes3}"]
\arrow[from=3-2, to=2-3, bend right, "R_2"]
\arrow[from=2-3, to=2-4, "\mathrm{dom}"]
\arrow[from=2-1, to=4-1, "\hat{F}"']
\arrow[from=3-2, to=5-2, "\hat{F}"]
\arrow[from=2-3, to=4-3, "\hat{F}"]
\arrow[from=2-4, to=4-4, "F"]
\arrow[from=4-1, to=5-2, bend right, "L'_{1\otimes3}"']
\arrow[from=5-2, to=4-3, bend right, "R'_2"']
\arrow[from=4-3, to=4-4, "\mathrm{dom}"']
\arrow[from=1-2, to=3-2, phantom, "\Downarrow w"]
\arrow[from=3-1, to=4-2, phantom, "\Downarrow(\theta_1\otimes\theta_3)^L"]
\arrow[from=3-2, to=4-3, phantom, "\Downarrow\theta_2^R"]
\arrow[from=2-3, to=4-4, phantom, "\Downarrow\gamma_1"]
\end{tikzcd}
=
\begin{tikzcd}
C^2 & C^2 & C \\
D^2 & D^2 & D
\arrow[from=1-1, to=1-2, bend left=50, "L_1"]
\arrow[from=1-1, to=1-2, "L_{1\otimes3}"']
\arrow[from=1-1, to=1-2, phantom, shift left=3ex, "\Downarrow i^L"]
\arrow[from=1-2, to=1-3, "E_2"]
\arrow[from=1-1, to=2-1, "\hat{F}"']
\arrow[from=1-2, to=2-2, "\hat{F}"]
\arrow[from=1-3, to=2-3, "F"]
\arrow[from=2-1, to=2-2, "L_{1'\otimes3'}"']
\arrow[from=2-2, to=2-3, "E'_2"']
\arrow[from=1-1, to=2-2, phantom, "\Downarrow(\theta_1\otimes\theta_3)^L"]
\arrow[from=1-2, to=2-3, phantom, "\Downarrow\theta_2"]
\end{tikzcd}$$

$$=
\begin{tikzcd}
C^2 & C^2 & C \\
D^2 & D^2 & D
\arrow[from=1-1, to=1-2, "L_1"]
\arrow[from=1-2, to=1-3, "E_2"]
\arrow[from=1-1, to=2-1, "\hat{F}"']
\arrow[from=1-2, to=2-2, "\hat{F}"]
\arrow[from=1-3, to=2-3, "F"]
\arrow[from=2-1, to=2-2, "L'_1"]
\arrow[from=2-1, to=2-2, bend right=50, "L'_{1\otimes3}"']
\arrow[from=2-1, to=2-2, phantom, shift right=3ex, "\Downarrow i^L"]
\arrow[from=2-2, to=2-3, "E'_2"']
\arrow[from=1-1, to=2-2, phantom, "\Downarrow\theta_1^L"]
\arrow[from=1-2, to=2-3, phantom, "\Downarrow\theta_2"]
\end{tikzcd}
=
\begin{tikzcd}
& C^2 & & \\
C^2 & & C^2 & C \\
& D^2 & & \\
D^2 & & D^2 & D. \\
& D^2 & &
\arrow[from=2-1, to=1-2, bend left, "R_{1\odot2}"]
\arrow[from=1-2, to=2-3, bend left, "L_3"]
\arrow[from=2-3, to=2-4, "\mathrm{dom}"]
\arrow[from=2-1, to=4-1, "\hat{F}"']
\arrow[from=1-2, to=3-2, "\hat{F}"]
\arrow[from=2-3, to=4-3, "\hat{F}"']
\arrow[from=2-4, to=4-4, "F"]
\arrow[from=4-1, to=3-2, bend left, "R'_{1\odot2}"']
\arrow[from=3-2, to=4-3, bend left, "L'_3"']
\arrow[from=4-1, to=5-2, bend right, "L'_{1\otimes3}"']
\arrow[from=5-2, to=4-3, bend right, "R'_2"']
\arrow[from=4-3, to=4-4, "\mathrm{dom}"']
\arrow[from=2-1, to=3-2, phantom, "\Downarrow(\theta_1\odot\theta_2)^R"]
\arrow[from=1-2, to=3-3, phantom, "\Downarrow\theta_3^L"]
\arrow[from=2-3, to=4-4, phantom, "\Downarrow\gamma_1"]
\arrow[from=3-2, to=5-2, phantom, "\Downarrow w"]
\end{tikzcd}$$

Finally, given four functorial factorizations E_1, E_2, E_3, E_4 on an object C, we define the 2-cell

$$\begin{tikzcd}
C & C \\
C & C
\arrow[from=1-1, to=1-2, "(1\odot2)\otimes(3\odot4)", "\shortmid" marking]
\arrow[from=1-1, to=2-1, equal]
\arrow[from=1-2, to=2-2, equal]
\arrow[from=2-1, to=2-2, "(1\otimes3)\odot(2\otimes4)"', "\shortmid" marking]
\arrow[from=1-1, to=2-2, phantom, "\Downarrow z_{1,2,3,4}"]
\end{tikzcd}$$

in $\mathbb{FF}(\mathbb{D})$, where $(1 \odot 2)$ is shorthand for $(E_1, \eta_1, \epsilon_1) \odot (E_2, \eta_2, \epsilon_2)$, to have the underlying 2-cell in $\mathbb{D}$

$$\begin{tikzcd}
& C^2 & & \\
C^2 & & C^2 & C. \\
& C^2 & &
\arrow[from=2-1, to=1-2, bend left, "R_{1\otimes2}"]
\arrow[from=1-2, to=2-3, bend left, "L_3"]
\arrow[from=2-1, to=3-2, bend right, "L_{1\otimes3}"']
\arrow[from=3-2, to=2-3, bend right, "R_2"']
\arrow[from=2-3, to=2-4, "E_4"]
\arrow[from=1-2, to=3-2, phantom, "\Downarrow w"]
\end{tikzcd}$$

The naturality of z follows immediately from that of w, but we still need to check that this satisfies equations (17) and (18). We will leave the details to the reader, but note that (18) comes down to the verification of the equality

$$\begin{tikzcd}
& C^2 & & \\
C^2 & & C^2 & C \\
& C^2 & &
\arrow[from=2-1, to=1-2, bend left=70, "\mathrm{id}"]
\arrow[from=2-1, to=1-2, bend left, "R_{1\otimes2}"']
\arrow[from=2-1, to=1-2, phantom, bend left=40, "\Downarrow\vec{\eta}_{1\odot2}"]
\arrow[from=1-2, to=2-3, bend left, "L_3"]
\arrow[from=2-1, to=3-2, bend right, "L_{1\otimes3}"']
\arrow[from=3-2, to=2-3, bend right, "R_2"']
\arrow[from=1-2, to=3-2, phantom, "\Downarrow w"]
\arrow[from=2-3, to=2-4, bend left=50, "\mathrm{dom}"]
\arrow[from=2-3, to=2-4, "E_4"']
\arrow[from=2-3, to=2-4, phantom, shift left=3ex, "\Downarrow\eta_4"]
\end{tikzcd}
=
\begin{tikzcd}
C^2 & C^2 & C^2 & C,
\arrow[from=1-1, to=1-2, "L_{1\otimes3}"]
\arrow[from=1-2, to=1-3, bend left, "\mathrm{id}"]
\arrow[from=1-2, to=1-3, bend right, "R_2"']
\arrow[from=1-2, to=1-3, phantom, "\Downarrow\vec{\eta}_2"]
\arrow[from=1-3, to=1-4, bend left, "\mathrm{dom}"]
\arrow[from=1-3, to=1-4, bend right, "E_4"']
\arrow[from=1-3, to=1-4, phantom, "\Downarrow\eta_4"]
\end{tikzcd}$$

which follows from equation (21) and the fact that $\mathrm{dom} \circ i^L = \mathrm{id}_{\mathrm{dom}}$. □

Up to this point, we have demonstrated that given any double category $\mathbb{D}$ having arrow objects, there is a 2-fold double category $\mathbb{FF}(\mathbb{D})$ of functorial factorizations in $\mathbb{D}$. The last thing we want to say about this construction is that a cyclic action on $\mathbb{D}$ lifts to one on $\mathbb{FF}(\mathbb{D})$, and hence also to one on $\mathrm{Bimon}(\mathbb{FF}(\mathbb{D}))$.

The cyclic action on objects and vertical morphisms is given directly by that on $\mathbb{D}$. Given a horizontal 1-cell (E, η, ϵ) on an object C, we define the 1-cell $(E, \eta, \epsilon)^\bullet$ on $C^\bullet$ to be $(E^\bullet, \epsilon^\bullet, \eta^\bullet)$. This also implies that the cyclic action swaps L and R for any given functorial factorization.

A quick look at the definitions of the two horizontal compositions is now enough to see that for any two functorial factorizations E_1 and E_2, we have

$$(E_1 \otimes E_2)^\bullet = E_1^\bullet \odot E_2^\bullet \qquad \text{and} \qquad (E_1 \odot E_2)^\bullet = E_1^\bullet \otimes E_2^\bullet$$

Similarly, the cyclic action on 2-cells in $\mathbb{FF}(\mathbb{D})$ is given by the cyclic action in $\mathbb{D}$ on the underlying 2-cell. This gives a valid 2-cell in $\mathbb{FF}(\mathbb{D})$ since the cyclic action simply swaps the equations (17) and (18).

7 Algebraic Weak Factorization Systems

For this section, let $\mathcal{D}$ be the 2-category of (small) categories $\mathcal{C}\mathrm{at}$, and let $\mathbb{D}$ be the canonical double category of squares in $\mathcal{D}$, whose horizontal and vertical morphisms are simply the 1-cells of $\mathcal{D}$, and whose 2-cells are the 2-cells of $\mathcal{D}$ of the appropriate shape.

In this section we will show that bimonoids in $\mathbb{FF}(\mathbb{D})$ are precisely algebraic weak factorization systems, and more generally that the morphisms in $\mathrm{Bimon}(\mathbb{D})$ are given by (co)lax morphisms of algebraic weak factorization systems.

Suppose that $E = (E, \eta, \epsilon)$ is a functorial factorization on a category $\mathcal{C}$, and consider a monoid structure on E. As I_C is initial, the unit of the monoid is forced, and is simply η. The multiplication is given by a natural transformation $\mu\colon ER \Rightarrow E$ satisfying equations (17) and (18), which now take the form $\epsilon \circ \mu = \epsilon R$ and $\mu \circ (\eta \cdot \vec{\eta}) = \eta$.

The unit axioms for the monoid give the equations $\mu \circ E\vec{\eta} = \mathrm{id}_E = \mu \circ \eta R$, which together imply the equation $\mu \circ (\eta \cdot \vec{\eta}) = \eta$ above. And finally, writing $\vec{\mu} = \mu^R\colon R^2 \to R$ for the natural transformation induced by the 2-cell μ, the associativity axiom gives the equation $\mu \circ E\vec{\mu} = \mu \circ \mu R$.

Proposition 7.1. *A monoid structure on an object* (E, η, ϵ) *in* $\mathbb{FF}(\mathbb{D})$ *is given by a natural transformation* $\mu\colon ER \Rightarrow E$*, satisfying equations*

$$\epsilon \circ \mu = \epsilon R \qquad \mu \circ E\vec{\eta} = \mathrm{id}_E = \mu \circ \eta R \qquad \mu \circ E\vec{\mu} = \mu \circ \mu R. \tag{23}$$

This determines a monad $\mathbb{R} = (R, \vec{\eta}, \vec{\mu})$*, such that* $\mathrm{dom}\,\vec{\mu} = \mu$ *and* $\mathrm{cod}\,\vec{\mu} = \mathrm{id}_{\mathrm{cod}}$.

Similarly, a comonoid structure on (E, η, ϵ) *is given by a natural transformation* $\delta\colon L \Rightarrow EL$*, satisfying equations*

$$\delta \circ \eta = \eta L \qquad E\vec{\epsilon} \circ \delta = \mathrm{id}_E = \epsilon L \circ \delta \qquad E\vec{\delta} \circ \delta = \delta L \circ \delta, \tag{24}$$

which determines a comonad $\mathbb{L} = (L, \vec{\epsilon}, \vec{\delta})$*, such that* $\mathrm{dom}\,\vec{\delta} = \mathrm{id}_{\mathrm{dom}}$ *and* $\mathrm{cod}\,\vec{\delta} = \delta$.

Hence a functorial factorization which simultaineously has a monoid structure and a comonoid structure in $\mathbb{FF}(\mathbb{D})$ is precisely a weak factorization system, missing only the second bullet of Definition 1.9, the distributive law condition. This is not surprising, as it is the only condition requiring a compatability between the monad and comonad structures. We will see that a bialgebra in $\mathbb{FF}(\mathbb{D})$ adds precisely this compatibility.

Proposition 7.2. *A bimonoid structure on a horizontal morphism* $(E, \eta, \epsilon)\colon C \to C$ *in* $\mathbb{FF}(\mathbb{D})$ *is precisely a weak factorization system on* C *with underlying functorial factorization system* (E, η, ϵ).

Proof. We have already shown how the monoid an comonoid structures give rise to the monod and comonad of the awfs. All that remains is to show that the equations (16) amount to just the distributive law, i.e. the equation

$$\begin{tikzcd}
& C^2 \arrow[dr, "L"] \arrow[drr, bend left, "E"] \arrow[r, phantom, "\Downarrow\delta"] & & & & & C^2 \arrow[dr, "E"] \arrow[d, phantom, "\Downarrow\mu"] & \\
C^2 \arrow[ur, "R"] \arrow[dr, "L"'] & \Downarrow\Delta & C^2 \arrow[r, "E"] & C & = & C^2 \arrow[ur, "R"] \arrow[rr, "E"] \arrow[dr, "L"'] & & C. \\
& C^2 \arrow[ur, "R"'] \arrow[urr, bend right, "E"'] \arrow[r, phantom, "\Downarrow\mu"] & & & & & C^2 \arrow[ur, "E"'] \arrow[u, phantom, "\Downarrow\delta"] &
\end{tikzcd} \tag{25}$$

First of all, notice that the first three equations of (16) follow trivially from the initiality of I_C and the terminality of $\bot_C$ in $\mathbb{FF}(\mathbb{D})$, hence they do not impose any further conditions.

The fourth equation here takes the form

$$\begin{tikzcd}
C^2 \arrow[r, "R"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\delta^R"] & C^2 \arrow[rr, "E"] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\delta"] & & C \arrow[d, equal] \\
C^2 \arrow[r, "R_{E\odot E}"'] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow w"] & C^2 \arrow[r, "L"] & C^2 \arrow[r, "E"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_E"] & C \arrow[d, equal] \\
C^2 \arrow[r, "L_{E\otimes E}"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mu^L"] & C^2 \arrow[r, "R"'] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\mu"] & C^2 \arrow[r, "E"'] & C \arrow[d, equal] \\
C^2 \arrow[r, "L"'] & C^2 \arrow[rr, "E"'] & & C
\end{tikzcd}
=
\begin{tikzcd}
C^2 \arrow[r, "R"] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\mu"] & C^2 \arrow[r, "E"] & C \arrow[d, equal] \\
C^2 \arrow[rr, "E"] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\delta"] & & C \arrow[d, equal] \\
C^2 \arrow[r, "L"'] & C^2 \arrow[r, "E"'] & C,
\end{tikzcd}$$

and so to prove (25), it suffices to show that

$$\begin{tikzcd}
& C^2 \arrow[dr, "L"] & & & & C^2 \arrow[dr, "L"] & \\
C^2 \arrow[ur, bend left=60, "R"] \arrow[ur, phantom, bend left=30, "\Downarrow\delta^R"] \arrow[ur, "R_{E\odot E}"'] \arrow[dr, "L_{E\otimes E}"] \arrow[dr, phantom, bend right=30, "\Downarrow\mu^L"] \arrow[dr, bend right=60, "L"'] & \Downarrow w & C^2 & = & C^2 \arrow[ur, "R"] \arrow[dr, "L"'] & \Downarrow\Delta & C^2. \\
& C^2 \arrow[ur, "R"'] & & & & C^2 \arrow[ur, "R"'] &
\end{tikzcd}$$

We can check this using the universal property of C^2 by composing with dom and cod. First, use (21) and (22) to check that

$$\begin{tikzcd}
 & \mathcal{C}^2 \arrow[dr, bend left, "L"] & & & & & & \\
\mathcal{C}^2 \arrow[ur, bend left=60, "R"] \arrow[ur, phantom, bend left=30, "\Downarrow\delta^R"] \arrow[ur, "R_{E\odot E}"'] \arrow[dr, "L_{E\otimes E}"] \arrow[dr, phantom, bend right=30, "\Downarrow\mu^L"] \arrow[dr, bend right=60, "L"'] & \Downarrow w & \mathcal{C}^2 \arrow[r, "\mathrm{dom}"] & \mathcal{C} & = & \mathcal{C}^2 \arrow[r, bend left, "L"] \arrow[r, phantom, "\Downarrow i^L"] \arrow[r, bend right, "L_{E\otimes E}"'] \arrow[r, phantom, bend right=60, "\Downarrow\mu^L"] \arrow[r, bend right=90, "L"'] \arrow[rr, bend left=60, "E"] \arrow[rr, phantom, bend left=30, "\Downarrow\delta"] & \mathcal{C}^2 \arrow[r, "E"'] & \mathcal{C} \\
 & \mathcal{C}^2 \arrow[ur, bend right, "R"'] & & & & & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathcal{C}^2 \arrow[dr, bend left, "L"] & & & & & & \\
\mathcal{C}^2 \arrow[ur, bend left=60, "R"] \arrow[ur, phantom, bend left=30, "\Downarrow\delta^R"] \arrow[ur, "R_{E\odot E}"'] \arrow[dr, "L_{E\otimes E}"] \arrow[dr, phantom, bend right=30, "\Downarrow\mu^L"] \arrow[dr, bend right=60, "L"'] & \Downarrow w & \mathcal{C}^2 \arrow[r, "\mathrm{cod}"] & \mathcal{C} & = & \mathcal{C}^2 \arrow[r, bend left=90, "R"] \arrow[r, phantom, bend left=60, "\Downarrow\delta^R"] \arrow[r, bend left, "R_{E\odot E}"] \arrow[r, phantom, "\Downarrow p^R"] \arrow[r, bend right, "R"'] \arrow[rr, bend right=60, "E"'] \arrow[rr, phantom, bend right=30, "\Downarrow\mu"] & \mathcal{C}^2 \arrow[r, "E"] & \mathcal{C}. \\
 & \mathcal{C}^2 \arrow[ur, bend right, "R"'] & & & & & &
\end{tikzcd}$$

Then use the definitions of i and p to check that $\mu \circ i = \mu \circ \eta R = \mathrm{id}_E$ and $p \circ \delta = \epsilon L \circ \delta = \mathrm{id}_E$, so that the first row above just equals δ, and the second row equals μ. Since Δ also (by definition) satisfies $\mathrm{dom}\,\Delta = \delta$ and $\mathrm{cod}\,\Delta = \mu$, we are done. ☐

The appropriate notion of morphism between awfs, analagous to left/right Quillen functors and Quillen adjunctions, is (to our knowledge) first given in [Rie11].

Definition 7.3. Suppose that $(E_1, \eta_1, \mu_1, \epsilon_1, \delta_1)$ and $(E_2, \eta_2, \mu_2, \epsilon_2, \delta_2)$ are awfs on $\mathcal{C}$ and $\mathcal{D}$ respectively.

- A *lax morphism of awfs* $(G, \rho): E_1 \to E_2$ consists of a functor $G: \mathcal{C} \to \mathcal{D}$ and a natural transformation $\rho: E_2\hat{G} \Rightarrow GE_1$, such that $(1, \rho): L_2\hat{G} \Rightarrow GL_1$ is a lax morphism of comonads and $(\rho, 1): R_2\hat{G} \Rightarrow GR_1$ is a lax morphism of monads.
- A *colax morphism of awfs* $(F, \lambda): E_1 \to E_2$ consists of a functor $F: \mathcal{C} \to \mathcal{D}$ and a natural transformation $\lambda: FE_1 \Rightarrow E_2\hat{F}$, such that $(1, \lambda): FL_1 \Rightarrow L_2\hat{F}$ is a colax morphism of comonads and $(\lambda, 1): FR_1 \Rightarrow R_2\hat{F}$ is a colax morphism of monads.

Notice that a lax morphism of awfs induces a lift of the functor $\hat{G}$ to a functor $\mathbb{R}_1\mathrm{Alg} \to \mathbb{R}_2\mathrm{Alg}$. In that sense, G "preserves the right class," so is analagous to a right Quillen functor. Similarly, a colax morphism of awfs induces a lift of $\hat{F}$ to $\mathbb{L}_1\mathrm{Coalg} \to \mathbb{L}_2\mathrm{Coalg}$, so is analagous to a left Quillen functor.

Proposition 7.4. *Morphisms in* $\mathrm{Bimon}(\mathbb{FF}(\mathbb{D}))$ *are precisely the colax morphisms of awfs.*

Proof. As above, let $(E_1, \eta_1, \mu_1, \epsilon_1, \delta_1)$ and $(E_2, \eta_2, \mu_2, \epsilon_2, \delta_2)$ be awfs on $\mathcal{C}$ and $\mathcal{D}$ respectively. A morphism of bimonoids is given by a 2-cell

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "{(E_1,\eta_1,\epsilon_1)}", "|" marking] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\lambda"] & \mathcal{C} \arrow[d, "F"] \\
\mathcal{D} \arrow[r, "{(E_2,\eta_2,\epsilon_2)}"', "|" marking] & \mathcal{D}
\end{tikzcd}$$

which commutes with the monoid and comonoid structures. It is straightforward to check that this implies the natural transformations

$$\begin{tikzcd}
\mathcal{C}^2 \arrow[r, "L_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\lambda^L"] & \mathcal{C}^2 \arrow[d, "\hat{F}"] \\
\mathcal{D}^2 \arrow[r, "L_2"'] & \mathcal{D}^2
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}^2 \arrow[r, "R_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\lambda^R"] & \mathcal{C}^2 \arrow[d, "\hat{F}"] \\
\mathcal{D}^2 \arrow[r, "R_2"'] & \mathcal{D}^2
\end{tikzcd}$$

are colax morphisms of comonads and monads respectively. □

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