1 Functorial Factorizations

Now let $\mathbb{D}$ be a cyclic double category, and assume it has arrow objects in the sense of Section **??**. Let us spell out what this means for the single variable case:

- For every object C there is a diagram

$$\begin{tikzcd} C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\kappa"] & C. \end{tikzcd}$$

- Any 2-cell

$$\begin{tikzcd} A \arrow[r, bend left, "d_1"] \arrow[r, bend right, "d_0"'] \arrow[r, phantom, "\Downarrow\alpha"] & C \end{tikzcd}$$

  uniquely factors through κ, as

$$\begin{tikzcd} A \arrow[r, "\hat{\alpha}"] & C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\kappa"] & C. \end{tikzcd}$$

- For every vertical 1-cell $F\colon C \to D$ there is a vertical 1-cell $\hat{F}\colon C^2 \to D^2$ and 2-cells

$$\begin{tikzcd} C^2 \arrow[r, "\mathrm{dom}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_1"] & C \arrow[d, "F"] \\ D^2 \arrow[r, "\mathrm{dom}"'] & D \end{tikzcd} \qquad \begin{tikzcd} C^2 \arrow[r, "\mathrm{cod}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_0"] & C \arrow[d, "F"] \\ D^2 \arrow[r, "\mathrm{cod}"'] & D \end{tikzcd}$$

  such that

$$\begin{tikzcd} C^2 \arrow[r, "\mathrm{dom}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_1"] & C \arrow[d, "F"] \\ D^2 \arrow[r, "\mathrm{dom}"] \arrow[r, bend right=60, "\mathrm{cod}"'] \arrow[r, phantom, shift right=3, "\Downarrow\kappa"] & D \end{tikzcd} = \begin{tikzcd} C^2 \arrow[r, bend left=60, "\mathrm{dom}"] \arrow[r, "\mathrm{cod}"'] \arrow[r, phantom, shift left=3, "\Downarrow\kappa"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_0"] & C \arrow[d, "F"] \\ D^2 \arrow[r, "\mathrm{cod}"'] & D \end{tikzcd}$$

- Given any 2-cells

$$\begin{tikzcd} A \arrow[r, bend left, "d_1"] \arrow[r, bend right, "d_0"'] \arrow[r, phantom, "\Downarrow\alpha"] & C \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, bend left, "d_1'"] \arrow[r, bend right, "d_0'"'] \arrow[r, phantom, "\Downarrow\alpha'"] & D \end{tikzcd}$$

and

$$\begin{tikzcd}
A \arrow[r, "d_1"] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\lambda_1"] & C \arrow[d, "F"] \\
B \arrow[r, "d_1'"'] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "d_0"] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\lambda_0"] & C \arrow[d, "F"] \\
B \arrow[r, "d_0'"'] & D
\end{tikzcd}$$

such that

$$\begin{tikzcd}
A \arrow[r, "d_1"] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\lambda_1"] & C \arrow[d, "F"] \\
B \arrow[r, "d_1'"] \arrow[r, bend right, "d_0'"'] \arrow[r, phantom, shift right=2ex, "\Downarrow\alpha'"] & D
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, bend left, "d_1"] \arrow[r, phantom, shift left=2ex, "\Downarrow\alpha"] \arrow[r, "d_0"'] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\lambda_0"] & C \arrow[d, "F"] \\
B \arrow[r, "d_0'"'] & D
\end{tikzcd}$$

there is a unique 2-cell

$$\begin{tikzcd}
A \arrow[r, "\hat{\alpha}"] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\theta"] & C^2 \arrow[d, "\hat{F}"] \\
B \arrow[r, "\hat{\alpha}'"'] & D^2
\end{tikzcd}$$

such that the horizontal composition of θ with γ_0 and γ_1 is respectively equal to λ_0 and λ_1.

Remark 1.1 (TODO: Remark that this generalizes the 2-dimensional part of the usual universal property in 2-categories.).

We will now define a 2-fold double category $\mathbb{FF}(\mathbb{D})$ of functorial factorizations in $\mathbb{D}$, as follows:

- The objects and vertical 1-cells are the same as in $\mathbb{D}$.
- Horizontal 1-cells $C \relbar\joinrel\mapstochar\joinrel\rightarrow C$ in $\mathbb{FF}(\mathbb{D})$ are tuples (E, η, ϵ), where $E\colon C^2 \to C$ is a horizontal 1-cell in $\mathbb{D}$, and

$$\begin{tikzcd}
C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "E"'] \arrow[r, phantom, "\Downarrow\eta"] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
C^2 \arrow[r, bend left, "E"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\epsilon"] & C
\end{tikzcd}$$

are 2-cells in $\mathbb{D}$ such that

$$\begin{tikzcd}
C^2 \arrow[r, bend left=50, "\mathrm{dom}"] \arrow[r, phantom, shift left=2ex, "\Downarrow\eta"] \arrow[r, "E" description] \arrow[r, phantom, shift right=2ex, "\Downarrow\epsilon"] \arrow[r, bend right=50, "\mathrm{cod}"'] & C
\end{tikzcd}
=
\begin{tikzcd}
C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\kappa"] & C.
\end{tikzcd}$$

By the universal property of C^2, this also determines horizontal 1-cells $L, R\colon C^2 \to C^2$ such that $\mathrm{dom} \circ L = \mathrm{dom}$, $\mathrm{cod} \circ R = \mathrm{cod}$, $\mathrm{cod} \circ L = \mathrm{dom} \circ R = E$, $\kappa \circ L = \eta$, and $\kappa \circ R = \epsilon$, and 2-cells

$$\begin{tikzcd} C^2 \arrow[r, bend left, "L"{name=U}] \arrow[r, bend right, "\mathrm{id}"'{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow\vec{\epsilon}"] & C^2. \end{tikzcd} \qquad \begin{tikzcd} C^2 \arrow[r, bend left, "\mathrm{id}"{name=U}] \arrow[r, bend right, "R"'{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow\vec{\eta}"] & C^2. \end{tikzcd}$$

such that $\mathrm{dom} \circ \vec{\epsilon} = \mathrm{id}_{\mathrm{dom}}$, $\mathrm{cod} \circ \vec{\epsilon} = \epsilon$, $\mathrm{dom} \circ \vec{\eta} = \eta$, and $\mathrm{cod} \circ \vec{\eta} = \mathrm{id}_{\mathrm{cod}}$.

- The horizontal composition $(E_1, \eta_1, \epsilon_1) \otimes (E_2, \eta_2, \epsilon_2)$ of two horizontal 1-cells

  $$\begin{tikzcd} C \arrow[r, "{(E_1,\eta_1,\epsilon_1)}", "\shortmid"{marking}] & C \arrow[r, "{(E_2,\eta_2,\epsilon_2)}", "\shortmid"{marking}] & C \end{tikzcd}$$

  in $\mathbb{FF}(\mathbb{D})$ is a horizontal 1-cell $(E_{1\otimes 2}, \eta_{1\otimes 2}, \epsilon_{1\otimes 2})$, where

  $$E_{1\otimes 2} = \quad \begin{tikzcd} C^2 \arrow[r, "R_1"] & C^2 \arrow[r, "E_2"] & C \end{tikzcd}$$

  $$\eta_{1\otimes 2} = \quad \begin{tikzcd} C^2 \arrow[r, bend left, "\mathrm{id}"{name=U1}] \arrow[r, bend right, "R_1"'{name=D1}] \arrow[from=U1, to=D1, phantom, "\Downarrow\vec{\eta}_1"] & C^2 \arrow[r, bend left, "\mathrm{dom}"{name=U2}] \arrow[r, bend right, "E_2"'{name=D2}] \arrow[from=U2, to=D2, phantom, "\Downarrow\eta_2"] & C \end{tikzcd}$$

  $$\epsilon_{1\otimes 2} = \quad \begin{tikzcd} C^2 \arrow[r, "R_1"] & C^2 \arrow[r, bend left, "E_2"{name=U}] \arrow[r, bend right, "\mathrm{cod}"'{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow\epsilon_2"] & C \end{tikzcd}$$

  which also determines that $R_{1\otimes 2} = R_2 \circ R_1$.

- The horizontal unit I_C for $\otimes$ is $(\mathrm{dom}, \mathrm{id}, \kappa)$.

- The second horizontal composition $(E_1, \eta_1, \epsilon_1) \odot (E_2, \eta_2, \epsilon_2)$ is a horizontal 1-cell $(E_{1\odot 2}, \eta_{1\odot 2}, \epsilon_{1\odot 2})$, where

  $$E_{1\odot 2} = \quad \begin{tikzcd} C^2 \arrow[r, "L_1"] & C^2 \arrow[r, "E_2"] & C \end{tikzcd}$$

  $$\eta_{1\odot 2} = \quad \begin{tikzcd} C^2 \arrow[r, "L_1"] & C^2 \arrow[r, bend left, "\mathrm{dom}"{name=U}] \arrow[r, bend right, "E_2"'{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow\eta_2"] & C \end{tikzcd}$$

  $$\epsilon_{1\odot 2} = \quad \begin{tikzcd} C^2 \arrow[r, bend left, "L_1"{name=U1}] \arrow[r, bend right, "\mathrm{id}"'{name=D1}] \arrow[from=U1, to=D1, phantom, "\Downarrow\vec{\epsilon}_1"] & C^2 \arrow[r, bend left, "E_2"{name=U2}] \arrow[r, bend right, "\mathrm{dom}"'{name=D2}] \arrow[from=U2, to=D2, phantom, "\Downarrow\epsilon_2"] & C \end{tikzcd}$$

  which also determines that $L_{1\odot 2} = L_2 \circ L_1$.

- The horizontal unit $\perp_C$ for $\odot$ is $(\mathrm{cod}, \kappa, \mathrm{id})$.

- 2-cells

$$\begin{tikzcd}
C \arrow[r, "{(E_1,\eta_1,\epsilon_1)}", "|" marking] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\theta"] & C \arrow[d, "F"] \\
D \arrow[r, "{(E_2,\eta_2,\epsilon_2)}"', "|" marking] & D
\end{tikzcd}$$

in $\mathbb{FF}(\mathbb{D})$ are given by 2-cells

$$\begin{tikzcd}
C^2 \arrow[r, "E_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "E_2"'] & D
\end{tikzcd}$$

in $\mathbb{D}$ such that

$$\begin{tikzcd}
C^2 \arrow[r, "E_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "E_2"] \arrow[r, bend right=50, "\mathrm{cod}"', "\Downarrow\epsilon_2"] & D
\end{tikzcd}
=
\begin{tikzcd}
C^2 \arrow[r, bend left=50, "E_1", "\Downarrow\epsilon_1"'] \arrow[r, "\mathrm{cod}"'] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_0"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "\mathrm{cod}"'] & D
\end{tikzcd}
\tag{1}$$

and

$$\begin{tikzcd}
C^2 \arrow[r, "\mathrm{dom}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_1"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "\mathrm{dom}"] \arrow[r, bend right=50, "E_2"', "\Downarrow\eta_2"] & D
\end{tikzcd}
=
\begin{tikzcd}
C^2 \arrow[r, bend left=50, "\mathrm{dom}", "\Downarrow\eta_1"'] \arrow[r, "E_1"'] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "E_2"'] & D
\end{tikzcd}
\tag{2}$$

This also determines unique 2-cells

$$\begin{tikzcd}
C^2 \arrow[r, "R_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta^R"] & C^2 \arrow[d, "\hat{F}"] \\
D^2 \arrow[r, "R_2"'] & D^2
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
C^2 \arrow[r, "L_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta^L"] & C^2 \arrow[d, "\hat{F}"] \\
D^2 \arrow[r, "L_2"'] & D^2
\end{tikzcd}$$

such that composing horizontally with γ_0 or γ_1 gives γ_0, γ_1, or θ as appropriate. For instance:

$$\begin{tikzcd}
C^2 \arrow[r, "R_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta^R"] & C^2 \arrow[r, "\mathrm{dom}"] \arrow[d, "\hat{F}"] \arrow[dr, phantom, "\Downarrow\gamma_1"] & C \arrow[d, "F"] \arrow[r, phantom, "="] & C^2 \arrow[r, "E_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "R_2"'] & D^2 \arrow[r, "\mathrm{dom}"'] & D & D^2 \arrow[r, "E_2"'] & D
\end{tikzcd}$$

- Given a pair of composable 2-cells in $\mathbb{FF}(\mathbb{D})$ as in

$$\begin{tikzcd}
C \arrow[r, "{(E_1,\eta_1,\epsilon_1)}", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\theta_1"] & C \arrow[r, "{(E_2,\eta_2,\epsilon_2)}", "\shortmid"{marking}] \arrow[d, "F"] \arrow[dr, phantom, "\Downarrow\theta_2"] & C \arrow[d, "F"] \\
D \arrow[r, "{(E_1',\eta_1',\epsilon_1')}"', "\shortmid"{marking}] & D \arrow[r, "{(E_2',\eta_2',\epsilon_2')}"', "\shortmid"{marking}] & D
\end{tikzcd}$$

the composite $\theta_1 \otimes \theta_2$ is given by

$$\begin{tikzcd}
C^2 \arrow[r, "R_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta_1^R"] & C^2 \arrow[r, "E_2"] \arrow[d, "\hat{F}"] \arrow[dr, phantom, "\Downarrow\theta_2"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "R_1'"'] & D^2 \arrow[r, "E_2'"'] & D
\end{tikzcd}$$

while the composite $\theta_1 \odot \theta_2$ is given by

$$\begin{tikzcd}
C^2 \arrow[r, "L_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta_1^L"] & C^2 \arrow[r, "E_2"] \arrow[d, "\hat{F}"] \arrow[dr, phantom, "\Downarrow\theta_2"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "L_1'"'] & D^2 \arrow[r, "E_2'"'] & D
\end{tikzcd}$$

It is a straightforward exercise to check that these definitions satisfy equations (1) and (2). To illustrate, we will demonstrate that $\theta_1 \otimes \theta_2$

satisfies (1):

$$\begin{tikzcd}
C^2 \arrow[r, "E_{1\otimes 2}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta_1\otimes\theta_2"] & C \arrow[d, "F"] & = & C^2 \arrow[r, "R_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta_1^R"] & C^2 \arrow[r, "E_2"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta_2"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "E_{1'\otimes 2'}", "\Downarrow\epsilon_{1'\otimes 2'}"'] \arrow[r, bend right=60, "\mathrm{cod}"'] & D & & D^2 \arrow[r, "R_1'"'] & D^2 \arrow[r, "E_2'", "\Downarrow\epsilon_2'"'] \arrow[r, bend right=60, "\mathrm{cod}"'] & D
\end{tikzcd}$$

$$\begin{tikzcd}
& C^2 \arrow[r, "R_1"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\theta_1^R"] & C^2 \arrow[r, bend left=60, "E_2"] \arrow[r, "\Downarrow\epsilon_2", "\mathrm{cod}"'] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_0"] & C \arrow[d, "F"] \\
= & D^2 \arrow[r, "R_1'"'] & D^2 \arrow[r, "\mathrm{cod}"'] & D
\end{tikzcd}$$

$$\begin{tikzcd}
& C^2 \arrow[r, bend left=60, "E_{1\otimes 2}"] \arrow[r, "\Downarrow\epsilon_{1\otimes 2}", "\mathrm{cod}"'] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_0"] & C \arrow[d, "F"] \\
= & D^2 \arrow[r, "\mathrm{cod}"'] & D
\end{tikzcd}$$

It is straightforward to check that $\otimes$ and $\odot$ are each associative and unital. It takes more work to provide the compatibility between $\otimes$ and $\odot$, which is the content of the proof of the next proposition.

Proposition 1.2. $\mathbb{FF}(\mathbb{D})$ *has the structure of a 2-fold double category.*

Proof. The primary structure of $\mathbb{FF}(\mathbb{D})$ was given in the first part of this section. What is left is to provide the coherence data (**??**) and (**??**).

First, note that I_C is initial in the sense that, given any vertical morphism $F\colon C \to D$ and any functorial factorization (E, η, ϵ) on D, there is a unique 2-cell

$$\begin{tikzcd}
C \arrow[r, "I_C", "|" marking] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow"] & C \arrow[d, "F"] \\
D \arrow[r, "{(E,\eta,\epsilon)}"', "|" marking] & D
\end{tikzcd}$$

given by

$$\begin{tikzcd}
C^2 \arrow[r, "\mathrm{dom}"] \arrow[d, "\hat{F}"'] \arrow[dr, phantom, "\Downarrow\gamma_1"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "\mathrm{dom}", "\Downarrow\eta"'] \arrow[r, bend right=60, "E"'] & D.
\end{tikzcd}$$

Similarly, $\perp_C$ is terminal. Thus there is only one possible way to define the 2-cells m, c, and j, and naturality and all other coherence equations follows immediately from this uniqueness.

We still need to construct the 2-cell z, which will take some work. We begin by defining 2-cells

$$\begin{tikzcd}
C \arrow[r, "E_1 \odot E_2", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow p_{E_1,E_2}"] & C \arrow[d, equal] \\
C \arrow[r, "E_1"', "\shortmid"{marking}] & C
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
C \arrow[r, "E_1", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow i_{E_1,E_2}"] & C \arrow[d, equal] \\
C \arrow[r, "E_1 \otimes E_2"', "\shortmid"{marking}] & C.
\end{tikzcd}$$

for any pair of functorial factorizations. The 2-cell p is given by the underlying 2-cell in $\mathbb{D}$

$$\begin{tikzcd}
C^2 \arrow[r, "L_1"] & C^2 \arrow[r, bend left, "E_2"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow \epsilon_2"] & C
\end{tikzcd}$$

and i is given by

$$\begin{tikzcd}
C^2 \arrow[r, "R_1"] & C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "E_2"'] \arrow[r, phantom, "\Downarrow \eta_2"] & C.
\end{tikzcd}$$

To illustrate the verification that these give well-defined 2-cells in $\mathbb{FF}(\mathbb{D})$, we will show that i satisfies (1) (keep in mind that when F is an identity, γ_0 and γ_1 are also identities):

$$\begin{tikzcd}
C^2 \arrow[r, "L_1"] & C^2 \arrow[r, bend left=60, "\mathrm{dom}"] \arrow[r, dashed, "E_2" description] \arrow[r, bend right=60, "\mathrm{cod}"'] \arrow[r, phantom, bend left=30, "\Downarrow \eta_2"] \arrow[r, phantom, bend right=30, "\Downarrow \epsilon_2"] & C
\end{tikzcd}
=
\begin{tikzcd}
C^2 \arrow[r, "L_1"] & C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow \kappa"] & C
\end{tikzcd}$$

$$=
\begin{tikzcd}
C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "E_1"'] \arrow[r, phantom, "\Downarrow \eta_1"] & C.
\end{tikzcd}$$

Moreover, it is straightforward to check that i and p are natural families of 2-cells. Specifically, for any pair of 2-cells θ_1 and θ_2

$$\begin{tikzcd} C \arrow[r, "E_1\odot E_2", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow p_{E_1,E_2}"] & C \arrow[d, equal] \\ C \arrow[r, "E_1", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow \theta_1"] & C \arrow[d, "F"] \\ D \arrow[r, "E_1'"', "\shortmid"{marking}] & D \end{tikzcd} = \begin{tikzcd} C \arrow[r, "E_1\odot E_2", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow \theta_1\odot\theta_2"] & C \arrow[d, "F"] \\ D \arrow[r, "E_1'\odot E_2'", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow p_{E_1',E_2'}"] & D \arrow[d, equal] \\ D \arrow[r, "E_1'"', "\shortmid"{marking}] & D \end{tikzcd}$$

$$\begin{tikzcd} C \arrow[r, "E_1", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow i_{E_1,E_2}"] & C \arrow[d, equal] \\ C \arrow[r, "E_1\otimes E_2", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow \theta_1\otimes\theta_2"] & C \arrow[d, "F"] \\ D \arrow[r, "E_1'\otimes E_2'"', "\shortmid"{marking}] & D \end{tikzcd} = \begin{tikzcd} C \arrow[r, "E_1", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow \theta_1"] & C \arrow[d, "F"] \\ D \arrow[r, "E_1'", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow i_{E_1',E_2'}"] & D \arrow[d, equal] \\ D \arrow[r, "E_1'\otimes E_2'"', "\shortmid"{marking}] & D \end{tikzcd}$$

As with any 2-cell in $\mathbb{FF}(\mathbb{D})$, p and i induce 2-cells in $\mathbb{D}$

$$\begin{tikzcd} C^2 \arrow[r, bend left, "R_{1\odot 2}"] \arrow[r, bend right, "R_1"'] \arrow[r, phantom, "\Downarrow p^R"] & C^2 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C^2 \arrow[r, bend left, "L_1"] \arrow[r, bend right, "L_{1\otimes 2}"'] \arrow[r, phantom, "\Downarrow i^L"] & C^2. \end{tikzcd}$$

such that

$$\begin{tikzcd} C^2 \arrow[r, bend left, "R_{1\odot 2}"] \arrow[r, bend right, "R_1"'] \arrow[r, phantom, "\Downarrow p^R"] & C^2 \arrow[r, "\mathrm{dom}"] & C \end{tikzcd} = \begin{tikzcd} C^2 \arrow[r, "L_1"] & C^2 \arrow[r, bend left, "E_2"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow \epsilon_2"] & C \end{tikzcd} \tag{3}$$

$$\begin{tikzcd} C^2 \arrow[r, bend left, "L_1"] \arrow[r, bend right, "L_{1\otimes 2}"'] \arrow[r, phantom, "\Downarrow i^L"] & C^2 \arrow[r, "\mathrm{cod}"] & C \end{tikzcd} = \begin{tikzcd} C^2 \arrow[r, "R_1"] & C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "E_2"'] \arrow[r, phantom, "\Downarrow \eta_2"] & C \end{tikzcd} \tag{4}$$

Now suppose given three functorial factorizations E_1, E_2, E_3 on an object C. We define a 2-cell in $\mathbb{D}$

$$\begin{tikzcd} & C^2 \arrow[dr, "L_3"] & \\ C^2 \arrow[ur, "R_{1\otimes 2}"] \arrow[dr, "L_{1\otimes 3}"'] & \Downarrow w & C^2 \\ & C^2 \arrow[ur, "R_2"'] & \end{tikzcd}$$

such that

$$\begin{tikzcd}
& C^2 \arrow[dr, bend left, "L_3"] \arrow[dd, phantom, "\Downarrow w"] & & & & & & \\
C^2 \arrow[ur, bend left, "R_{1\otimes 2}"] \arrow[dr, bend right, "L_{1\otimes 3}"'] & & C^2 \arrow[r, "\mathrm{dom}"] & C \arrow[r, phantom, "="] & C^2 \arrow[r, bend left, "L_1"] \arrow[r, bend right, "L_{1\otimes 3}"'] \arrow[r, phantom, "\Downarrow i^L"] & C^2 \arrow[r, "E_2"] & C \\
& C^2 \arrow[ur, bend right, "R_2"'] & & & & & &
\end{tikzcd} \tag{5}$$

$$\begin{tikzcd}
& C^2 \arrow[dr, bend left, "L_3"] \arrow[dd, phantom, "\Downarrow w"] & & & & & & \\
C^2 \arrow[ur, bend left, "R_{1\otimes 2}"] \arrow[dr, bend right, "L_{1\otimes 3}"'] & & C^2 \arrow[r, "\mathrm{cod}"] & C \arrow[r, phantom, "="] & C^2 \arrow[r, bend left, "R_{1\odot 2}"] \arrow[r, bend right, "R_1"'] \arrow[r, phantom, "\Downarrow p^R"] & C^2 \arrow[r, "E_3"] & C. \\
& C^2 \arrow[ur, bend right, "R_2"'] & & & & & &
\end{tikzcd} \tag{6}$$

Using the universal property for C^2, it suffices to check that

$$\begin{tikzcd}
C^2 \arrow[r, bend left=60, "L_1"] \arrow[r, "L_{1\otimes 3}"'] \arrow[r, phantom, bend left=30, "\Downarrow i^L"] & C^2 \arrow[r, "E_2"] \arrow[r, bend right=60, "\mathrm{cod}"'] \arrow[r, phantom, bend right=30, "\Downarrow\epsilon_2"] & C \arrow[r, phantom, "="] & C^2 \arrow[r, "R_{1\odot 2}"] \arrow[r, bend right=60, "R_1"'] \arrow[r, phantom, bend right=30, "\Downarrow p^R"] & C^2 \arrow[r, bend left=60, "\mathrm{dom}"] \arrow[r, "E_3"'] \arrow[r, phantom, bend left=30, "\Downarrow\eta_3"] & C
\end{tikzcd}$$

and a quick check using equations (3) and (4) shows that both are equal to

$$\begin{tikzcd}
& C^2 \arrow[dr, "\mathrm{cod}"'] \arrow[dr, bend left=60, "E_2"] \arrow[dr, phantom, bend left=30, "\Downarrow\epsilon_2"] & \\
C^2 \arrow[ur, "L_1"] \arrow[dr, "R_1"'] & & C \\
& C^2 \arrow[ur, "\mathrm{dom}"] \arrow[ur, bend right=60, "E_3"'] \arrow[ur, phantom, bend right=30, "\Downarrow\eta_3"] &
\end{tikzcd}$$

where the inner diamond is the equality $\operatorname{cod} L_1 = \operatorname{dom} R_1 = E_1$.

We also check that w is natural with respect to 2-cells in $\mathbb{FF}(\mathbb{D})$ in the following sense: given three 2-cells θ_1, θ_2, and θ_3, there is an equality

$$\begin{tikzcd}
& C^2 \arrow[dr, bend left, "L_3"] \arrow[d, phantom, "\Downarrow w"] & & & & C^2 \arrow[dr, bend left, "L_3"] \arrow[dd, "\hat{F}"] & \\
C^2 \arrow[ur, bend left, "R_{1\odot 2}"] \arrow[dr, bend right, "L_{1\otimes 3}"] \arrow[dd, "\hat{F}"'] & {} & C^2 \arrow[dd, "\hat{F}"] & & C^2 \arrow[ur, bend left, "R_{1\odot 2}"] \arrow[dd, "\hat{F}"'] \arrow[r, phantom, "\Downarrow(\theta_1\odot\theta_2)^R"] & {} \arrow[r, phantom, "\Downarrow\theta_3^L"] & C^2 \arrow[dd, "\hat{F}"] \\
{} \arrow[r, phantom, "\Downarrow(\theta_1\otimes\theta_3)^L"] & C^2 \arrow[ur, bend right, "R_2"] \arrow[dd, "\hat{F}"] \arrow[r, phantom, "\Downarrow\theta_2^R"] & {} & = & & D^2 \arrow[dr, bend left, "L'_3"] \arrow[d, phantom, "\Downarrow w"] & \\
D^2 \arrow[dr, bend right, "L'_{1\otimes 3}"'] & & D^2 & & D^2 \arrow[ur, bend left, "R'_{1\odot 2}"'] \arrow[dr, bend right, "L'_{1\otimes 3}"'] & {} & D^2. \\
& D^2 \arrow[ur, bend right, "R'_2"'] & & & & D^2 \arrow[ur, bend right, "R'_2"'] &
\end{tikzcd}$$

To verify this equation, it suffices to check equality upon right composition with γ_0 and γ_1. We will illustrate the γ_1 case, making use of the naturality of i:

$$\begin{tikzcd}
& C^2 \arrow[dr, bend left, "L_3"] & & & \\
C^2 \arrow[ur, bend left, "R_{1\odot 2}"] \arrow[dr, bend right, "L_{1\otimes 3}"] \arrow[ddd, "\hat{F}"'] & \Downarrow w & C^2 \arrow[r, "\mathrm{dom}"] \arrow[ddd, "\hat{F}"] & C \arrow[ddd, "F"] \\
& C^2 \arrow[ur, bend right, "R_2"'] \arrow[ddd, "\hat{F}"] & & & \\
\Downarrow(\theta_1\otimes\theta_3)^L & & \Downarrow\theta_2^R & \Downarrow\gamma_1 \\
& & & & \\
D^2 \arrow[dr, bend right, "L'_{1\otimes 3}"'] & & D^2 \arrow[r, "\mathrm{dom}"'] & D \\
& D^2 \arrow[ur, bend right, "R'_2"'] & & &
\end{tikzcd}
=
\begin{tikzcd}
C^2 \arrow[r, bend left=60, "L_1"] \arrow[r, "L_{1\otimes 3}"'] \arrow[r, phantom, bend left=30, "\Downarrow i^L"] \arrow[d, "\hat{F}"'] & C^2 \arrow[r, "E_2"] \arrow[d, "\hat{F}"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "L_{1'\otimes 3'}"'] \arrow[ur, phantom, "\Downarrow(\theta_1\otimes\theta_3)^L"] & D^2 \arrow[r, "E'_2"'] \arrow[ur, phantom, "\Downarrow\theta_2"] & D
\end{tikzcd}$$

$$=
\begin{tikzcd}
C^2 \arrow[r, "L_1"] \arrow[d, "\hat{F}"'] & C^2 \arrow[r, "E_2"] \arrow[d, "\hat{F}"] & C \arrow[d, "F"] \\
D^2 \arrow[r, "L'_1"] \arrow[r, bend right=60, "L'_{1\otimes 3}"'] \arrow[r, phantom, bend right=30, "\Downarrow i^L"] \arrow[ur, phantom, "\Downarrow\theta_1^L"] & D^2 \arrow[r, "E'_2"'] \arrow[ur, phantom, "\Downarrow\theta_2"] & D
\end{tikzcd}
=
\begin{tikzcd}
& C^2 \arrow[dr, bend left, "L_3"] \arrow[dd, "\hat{F}"] & & & \\
C^2 \arrow[ur, bend left, "R_{1\odot 2}"] \arrow[ddd, "\hat{F}"'] & \Downarrow(\theta_1\odot\theta_2)^R & C^2 \arrow[r, "\mathrm{dom}"] \arrow[ddd, "\hat{F}"] & C \arrow[ddd, "F"] \\
& & \Downarrow\theta_3^L & \Downarrow\gamma_1 \\
& D^2 \arrow[dr, bend left, "L'_3"] & & & \\
D^2 \arrow[ur, bend left, "R'_{1\odot 2}"'] \arrow[dr, bend right, "L'_{1\otimes 3}"'] & \Downarrow w & D^2 \arrow[r, "\mathrm{dom}"'] & D. \\
& D^2 \arrow[ur, bend right, "R'_2"'] & & &
\end{tikzcd}$$

Finally, given four functorial factorizations E_1, E_2, E_3, E_4 on an object C, we define the 2-cell

$$\begin{tikzcd}
C \arrow[r, "(1\odot 2)\otimes(3\odot 4)", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow z_{1,2,3,4}"] & C \arrow[d, equal] \\
C \arrow[r, "(1\otimes 3)\odot(2\otimes 4)"', "\shortmid" marking] & C
\end{tikzcd}$$

in $\mathbb{FF}(\mathbb{D})$, where $(1 \odot 2)$ is shorthand for $(E_1, \eta_1, \epsilon_1) \odot (E_2, \eta_2, \epsilon_2)$, to have the underlying 2-cell in $\mathbb{D}$

$$\begin{tikzcd}
& C^2 \arrow[dr, bend left, "L_3"] & & \\
C^2 \arrow[ur, bend left, "R_{1\otimes 2}"] \arrow[dr, bend right, "L_{1\otimes 3}"'] & \Downarrow w & C^2 \arrow[r, "E_4"] & C. \\
& C^2 \arrow[ur, bend right, "R_2"'] & &
\end{tikzcd}$$

The naturality of z follows immediately from that of w, but we still need to check that this satisfies equations (1) and (2). We will leave the details to the reader, but note that (2) comes down to the verification of the equality

$$\begin{tikzcd}
& C^2 \arrow[dr, bend left, "L_3"] & & \\
C^2 \arrow[ur, bend left=70, "\mathrm{id}"] \arrow[ur, bend left, "R_{1\otimes 2}"'] \arrow[ur, phantom, bend left=40, "\Downarrow\vec{\eta}_{1\odot 2}"] \arrow[dr, bend right, "L_{1\otimes 3}"'] & \Downarrow w & C^2 \arrow[r, bend left=60, "\mathrm{dom}"] \arrow[r, "E_4"'] \arrow[r, phantom, bend left=30, "\Downarrow\eta_4"] & C \\
& C^2 \arrow[ur, bend right, "R_2"'] & &
\end{tikzcd}
=
\begin{tikzcd}
C^2 \arrow[r, "L_{1\otimes 3}"] & C^2 \arrow[r, bend left, "\mathrm{id}"] \arrow[r, bend right, "R_2"'] \arrow[r, phantom, "\Downarrow\vec{\eta}_2"] & C^2 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "E_4"'] \arrow[r, phantom, "\Downarrow\eta_4"] & C,
\end{tikzcd}$$

which follows from equation (5) and the fact that $\mathrm{dom} \circ i^L = \mathrm{id}_{\mathrm{dom}}$. □

Up to this point, we have demonstrated that given any double category $\mathbb{D}$ having arrow objects, there is a 2-fold double category $\mathbb{FF}(\mathbb{D})$ of functorial factorizations in $\mathbb{D}$. The last thing we want to say about this construction is that a cyclic action on $\mathbb{D}$ lifts to one on $\mathbb{FF}(\mathbb{D})$, and hence also to one on Bimon($\mathbb{FF}(\mathbb{D})$).

The cyclic action on objects and vertical morphisms is given directly by that on $\mathbb{D}$. Given a horizontal 1-cell (E, η, ϵ) on an object C, we define the 1-cell $(E, \eta, \epsilon)^\bullet$ on $C^\bullet$ to be $(E^\bullet, \epsilon^\bullet, \eta^\bullet)$. This also implies that the cyclic action swaps L and R for any given functorial factorization.

A quick look at the definitions of the two horizontal compositions is now enough to see that for any two functorial factorizations E_1 and E_2, we have

$$(E_1 \otimes E_2)^\bullet = E_1^\bullet \odot E_2^\bullet \qquad \text{and} \qquad (E_1 \odot E_2)^\bullet = E_1^\bullet \otimes E_2^\bullet$$

Similarly, the cyclic action on 2-cells in $\mathbb{FF}(\mathbb{D})$ is given by the cyclic action in $\mathbb{D}$ on the underlying 2-cell. This gives a valid 2-cell in $\mathbb{FF}(\mathbb{D})$ since the cyclic action simply swaps the equations (1) and (2).