CS239: Lecture Notes on Performance Evaluation for Computer Networks

Kittipat Apicharttrisorn kapic001@ucr.edu
Computer Science and Engineering, University of California, Riverside

1 Probability Theory

We have an experiment like tossing a coin and observe the results which can be H or T. Let Ω be a sample set, and A be an event which can be a number or a range of numbers with $A \subset \Omega$. Properties of A are as follows.

1. $P(A) \geq 0$
2. $P(\Omega) = 1$
3. If A and B are disjoint (cannot occur at the same time) and together form Ω, $P(A)+P(B) = 1$
4. $P(A^c) = 1 - P(A)$
5. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
6. If $A \subseteq B$, then $P(A) \leq P(B)$

Next, we have conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

which suggest probabilty of A if we know for sure that B happens. Then, we have total proability from

$$B = \bigcup_{i=1}^{n} (B \cap A_i)$$

and then Bayes' Rule follows that

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_i P(A_i)P(B|A_i)}$$

. Next, look at Monty Hall (three door) problem.

...

Figure 1: Venn Diagram

1.1 Random Variables

Outcomes of an experiment that are mapped onto a real line. For example, we number $x = 0$ if we toss a head and $x = 1$ for a tail. Or, $P(x \leq 9.2)$ if x is the time to finish a race. We have two types of random variables.

1. Discrete RV $P(X = x_i)$ or $P(X \leq x_n)$
2. Continuous RV has continuous values, $P(X \leq x)$ where X is a random variable and x is a value. Continuous RV has two important and related functions called Cumulative Distribution Function $F_X(x)$ or $F(x)$ and Probability Density Function $f_X(x)$. The relationship is

$$f_X(x) = \frac{dF(x)}{dx}$$

$$F(x) = \int_{-\infty}^{\infty} f_X(x)dx$$

Next, we have important random variables here.

1. Exponential random variables

$$f_X(x) = \begin{cases} 0 & if \quad x \leq o \\ \lambda e^{-\lambda x} & if \quad x > 0 \end{cases}$$

$$F(x) = \int_0^x \lambda e^{-\lambda x} dx = 1 - e^{-\lambda x}$$

2. Binomial random variables is a discrete rv by which we do an experiment n times and succeed k times with p success probability. For example, we toss a coin 100 times and calculate probability that we get 40 heads.

$$P(X = k) = \binom{n}{k} \cdot p^k (1-p)^{n-k}$$

where $k = 0, 1, 2, \ldots, n$

3. Geometric distribution is the probability that represents the number of failures before seeing one success

$$P(X = k) = (1-p)^{k-1} \cdot p$$

4. Poisson distribution represents events where we have k successes within t time. For example, there is a question of probability of k packets arriving in a queue within t time interval.

$$P(X = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

Figure 2: Poisson Distribution

Expectations with expected values as follows

$$E[X] = \sum_{i=1}^{n} x_i P(X = x_i) \qquad \text{(discrete)}$$

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx \qquad \text{(continuous)}$$

Properties of expectations

1. $E[X_1 + X_2] = E[X_1] + E[X_2]$
2. If $y = g(X)$ then $E_Y[y] = E_X[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$
3. If X and Y are independent, then $E[XY] = E[X] \cdot E[Y]$. (Just like $P(X \cap Y) = P(X) \cdot P(Y)$)

Probability Generating Functions

1. Discrete RV (z-transforms)

$$E[z^X] = \sum_i z^{x_i} P(X = x_i)$$

where z is a variable. Let the values X takes be $x_i = 0, 1, 2, \ldots$

$$E[z^X] = \sum_{i=0}^{\infty} z^i P(X = i)$$

Then, we shall differentiate one time and replace z with 1.

$$\frac{d}{dz} E[z^X] = \sum_{i=0}^{\infty} i \cdot z^{i-1} P(X = i) \mid_{z=1} = \sum_{i=0}^{\infty} i \cdot P(X = i) = E[X]$$

Then, we differentiate one more time.

$$\frac{d^2}{dz^2} E[z^X] = \sum_{i=0}^{\infty} i \cdot (i-1) \cdot z^{i-2} P(X = i) \mid_{z=1} = \sum_{i=0}^{\infty} i^2 \cdot P(X = i) - \sum_{i=0}^{\infty} i \cdot P(X = i) = \underbrace{E[X^2]}_{\text{2nd moment}} - \underbrace{E[X]}_{\text{1st moment}}$$

2. Continuous RV (Laplace transforms)
   Let's define a variable s and X is a random variable.

$$L.T. = E[e^{-sX}] = \int_{-\infty}^{\infty} e^{-sx} \cdot f_X(x)dx$$

   Then differentiate one time

$$\frac{d}{ds}E[e^{-sX}] = \int_{-\infty}^{\infty} -x \cdot e^{-sx} \cdot f_X(x)dx \mid_{s=0} = -\int_{-\infty}^{\infty} xf(x)dx = -E[X] \tag{1}$$

Mean of z-transform of the sum = product of the mean

$$Y = X_1 + X_2$$

$$E[z^Y] = E[z^{X_1+X_2}] = E[z^{X_1}] \cdot E[z^{X_2}]$$

Mean of Laplace transform of the sum = product of the mean

$$E[e^{-s(X_1+X_2)}] = E[e^{-sX_1}] \cdot E[e^{-sX_2}]$$

Important Laplace transforms

- Exponential: Given

$$U(t) = \begin{cases} 0 & for \quad t < 0 \\ 1 & for \quad t \geq 0 \end{cases}$$

  Therefore,

$$\begin{aligned} \mathcal{L}(e^{-at}U(t)) &= \int_0^{\infty} e^{-at}e^{-st}dt \\ &= \frac{1}{s+a} \qquad \text{(use integration by parts)} \end{aligned}$$

- Poisson

$$\mathcal{L}\Big(\frac{t^n}{n!}\Big) = \frac{1}{s^{n+1}}$$

$$\mathcal{L}\Big(e^{-at}f(t)) = \underbrace{\mathcal{L}(f(s+a))}_{\text{shift "a" to the right}}$$

2 Birth Death Processes

Assuming there are packets arriving the queue of the router and we have k populations in the queue,

$$State: E_k \quad \rightarrow \quad \text{when there are k members in the population}$$

When we have a birth, we transition from

$$E_k \quad \rightarrow \quad E_{k+1}$$

and in case of death

$$E_k \quad \rightarrow \quad E_{k-1}$$

Let λ_k be the rate at which births occur when population is k and μ_k be the rate at which deaths occur when the population is k.

$$\begin{aligned}
B_1 &= P[\text{1 birth in the next } \Delta t \quad | \quad \text{population is } k \text{ at time } t] = \lambda_k \Delta t \\
D_1 &= P[\text{1 death in the next } \Delta t \quad | \quad \text{population is } k \text{ at time } t] = \mu_k \Delta t \\
B_2 &= P[\text{no birth in the next } \Delta t \quad | \quad \text{population is } k \text{ at time } t] = 1 - \lambda_k \Delta t \\
D_2 &= P[\text{no death in the next } \Delta t \quad | \quad \text{population is } k \text{ at time } t] = 1 - \mu_k \Delta t
\end{aligned}$$

If we consider marginal probability,

$$\begin{aligned}
P_k[t+\Delta t] &= \underbrace{P_k(t)\cdot(1-\lambda_k\Delta t)\cdot(1-\mu_k\Delta t)}_{\text{no birth or death at } k} + \underbrace{P_{k-1}(t)\cdot\lambda_{k-1}\Delta t}_{\text{there is one birth at } k-1} + \underbrace{P_{k+1}(t)\cdot\mu_{k+1}\Delta t}_{\text{there is one death at } k+1} \\
&= P_k(t)\cdot(1-\lambda_k\Delta t-\mu_k\Delta t)+\cdots \\
\frac{P_k[t+\Delta t]-P_k(t)}{\Delta t} &= -(\lambda_k+\mu_k)P_k(t)+P_{k-1}(t)\cdot\lambda_{k-1}+P_{k+1}(t)\cdot\mu_{k+1}
\end{aligned} \tag{2}$$

However, there cannot be death at $k = 0$ and there is no state such that $k < 0$, so we have an equation for $k = 0$ (initial state).

$$\frac{P_0(t+\Delta t)-P_0(t)}{\Delta t} = -\lambda_0 P_0(t) + \mu_1 P_1(t) \tag{3}$$

Then, we take the limits $\Delta t \rightarrow 0$ on both the equations (3) and (2) P_0 and P_k above.

$$\frac{dP_0(t)}{dt} = -\lambda_0 P_0(t) + \mu_1 P_1(t) \tag{4}$$

$$\frac{dP_k(t)}{dt} = -(\lambda_k+\mu_k)P_k(t)+P_{k-1}(t)\cdot\lambda_{k-1}+P_{k+1}(t)\cdot\mu_{k+1} \tag{5}$$

and do not forget the total probability $\sum_{k=0}^{\infty} P_k(t) = 1$. Look at the state diagram below.

Figure 3: State Transition Diagram

Let us now consider a pure birth process where there is no death. From differential difference equations (4) and (5), we have

$$\frac{dP_0(t)}{dt} = -\lambda_0 P_0(t) \tag{6}$$

$$\frac{dP_k(t)}{dt} = -\lambda_k P_k(t) + \lambda_{k-1} P_{k-1}(t) \tag{7}$$

Then, we take Laplace transform of Eq (7) and have the right hand side (RHS) of the transformed equation to be

$$-\lambda_k L_k(s) + \lambda_{k-1} L_{k-1}(s)$$

To find the L.T. of the LHS of Eq (7),

$$\int_0^\infty e^{-st} \frac{dP_k(t)}{dt} dt$$

Remember that

$$\int u dv = uv - \int v du$$

$$\int u \cdot \frac{dv}{dx} dx = uv - \int v \cdot \frac{du}{dx} dx$$

Therefore,

$$\begin{aligned}
\int_0^\infty e^{-st} \frac{dP_k(t)}{dt} dt &= e^{-st} P_k(t) \mid_0^\infty - \int_0^\infty P_k(t)(-se^{-st}) \quad dt \\
&= -P_k(0) + s \int_0^\infty e^{-st} P_k(t) \quad dt \\
&= -P_k(0) + sL_k(s)
\end{aligned}$$

Now we put together LHS and RHS of Eq (7).

$$-P_k(0) + sL_k(s) = -\lambda_k L_k(s) + \lambda_{k-1} L_{k-1}(s) \tag{8}$$

We can do the L.T. of the Eq 6 as well with the below result.

$$-P_0(0) + sL_0(s) = -\lambda_0 L_0(s) \tag{9}$$

Let's now assume that $P_0(0) = 1$ and $P_k(0) = 0, \forall k \neq 0$. (Pure Birth) Therefore,

$$L_0(s) = \frac{1}{s + \lambda_0}$$

$$L_k(s) = \frac{\lambda_{k-1} L_{k-1}(s)}{s + \lambda_k}$$

Let's assume that $\lambda_k = \lambda, \forall k$. Then,

$$L_0(s) = \frac{1}{s + \lambda}$$

$$\begin{aligned}
L_1(s) &= \frac{\lambda}{s + \lambda} \cdot L_0(s) \\
&= \frac{\lambda}{(s + \lambda)^2} \\
&\vdots
\end{aligned}$$

$$L_k(s) = \frac{\lambda^k}{(s+\lambda)^{k+1}}$$

Find inverse L.T. of $L(s)$ by looking up L.T. tables.

$$L_0(s) = \frac{1}{s+\lambda} \xrightarrow{\text{inv L.T.}} P_0(t) = e^{-\lambda t}$$

$$\frac{1}{s^{k+1}} \xrightarrow{\text{inv L.T.}} \frac{t^k}{k!}$$

$$\frac{1}{(s+\lambda)^{k+1}} \xrightarrow{\text{inv L.T.}} \frac{e^{-\lambda t} \cdot t^k}{k!}$$

Therefore,

$$P_k(t) = \frac{e^{-\lambda t} \cdot (\lambda t)^k}{k!} \tag{10}$$

In conclusion, from Eq. (10), the probability distribution for pure birth processes is Poisson.

Probability Generating Function (PGF) of a Poisson process

Let PGF be $G(z)$.

$$\begin{aligned} G(z) &= \sum_{k=0}^{\infty} \frac{z^k e^{-\lambda t} (\lambda t)^k}{k!} \\ &= e^{-\lambda t} \underbrace{\sum_{k=0}^{\infty} \frac{(z\lambda t)^k}{k!}}_{\text{Taylor Series}} \\ &= e^{-\lambda t} \cdot e^{\lambda z t} = e^{-\lambda t(1-z)} \end{aligned} \tag{11}$$

We take diff one time to get

$$\frac{d}{dz} e^{-\lambda t(1-z)} = e^{-\lambda t(1-z)} = e^{-\lambda t(1-z)} \cdot (\lambda t) \mid_{z=1}$$

Therefore, we have expectation of $E[k] = \lambda t$. And if we diff and take $z = 1$ one more time we get $E[k^2] = (\lambda t)^2$

Independent Increment Property of Poisson processes

Last time, we take a look at a random variable $X(0, t)$. Now, the same property holds for a Poisson process no matter when it starts.

$$X(\underbrace{h}_{\text{start time}}, \underbrace{h+t}_{\text{end time}}) = \frac{e^{-\lambda t} \cdot (\lambda t)^k}{k!} = P_k(t)$$

Inter-arrival time Probability Density Function

Figure 4: Interarrival Time PDF

$$\begin{aligned} P(\tau \le t) &= 1 - \overbrace{P(\tau > t)}^{\text{no arrival in time } t} \\ &= 1 - P_0(t) \\ &= 1 - e^{-\lambda t} \quad \text{(cdf of exponential distribution)} \end{aligned} \tag{12}$$

And density function of exponential distribution is $f(t) = \lambda e^{-\lambda t}$.

Memoryless Property of Exponential Distribution

Figure 5: Memoryless Property of Exponential Distribution

$$
\begin{aligned}
P(\tau \leq t_0 + t | \tau \geq t) &= \frac{P[t_0 < \tau \leq t_0 + t]}{P[\tau \geq t_0]} \\
&= \frac{\overbrace{P[\tau \leq t_0 + t]}^{\text{CDF of exponential dist}} \quad - \quad P[\tau \leq t_0]}{1 - P[\tau \leq t_0]} \\
&= \frac{1 - e^{-\lambda(t_0+t)} - (1 - e^{-\lambda t_0})}{1 - (1 - e^{-\lambda t_0})} \\
&= \frac{e^{-\lambda t_0} - e^{-\lambda(t_0+t)}}{e^{-\lambda t_0}} \\
&= \underbrace{1 - e^{-\lambda t} = P[\tau \leq t]}_{\text{Memoryless property}}
\end{aligned} \tag{13}
$$

Look at *Expected Value* by taking Laplace Transform of exponential PDF.

$$
\begin{aligned}
A(s) &= \int_0^\infty e^{-st} \cdot \underbrace{\lambda e^{-\lambda t}}_{\text{PDF}} dt \\
&= \frac{-\lambda}{s+\lambda} \cdot e^{-(\lambda+s)t} \mid_0^\infty \\
&= \frac{\lambda}{s+\lambda}
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
E[\tau] &= -\frac{d}{ds} A(s) \mid_{s=0} \\
&= -[\frac{-\lambda}{(s+\lambda)^2}]_{s=0} \\
&= \frac{1}{\lambda}
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
E[\tau^2] &= -\frac{d^2}{ds^2} A(s) = \frac{2\lambda}{(s+\lambda)^3}_{s=0} \\
&= \frac{2}{\lambda^2}
\end{aligned} \tag{16}
$$

Consequentially, we have *Variance of R.V.*, which is

$$
E[x^2] - E^2[x] = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2} \tag{17}
$$

Now, let's consider the time it takes for k arrivals.

Figure 6: Memoryless k-Arrivals

Let $X = \sum_{i=1}^{k} x_i$ where each x_i is exponentially distributed with mean $\frac{1}{\lambda}$. If we take Laplace transform, then we have

$$\begin{aligned} E[e^{-s(x_1+x_2+...+x_k)}] &= E[e^{-sx_1} \cdot e^{-sx_2} + ... + e^{-sx_k}] \\ &= \prod_{i=1}^{k} E[e^{-sx_i}] \\ &= \left[\frac{\lambda}{s+\lambda}\right]^k \xrightarrow{\text{inv L.T.}} \underbrace{f(x)}_{\text{PDF}} = \frac{\lambda e^{-\lambda x} \cdot (\lambda x)^{k-1}}{(k-1)!} \quad \text{for } x \geq 0 \end{aligned} \tag{18}$$

$$\text{k} = 1,\ f(x) = \lambda e^{-\lambda x}$$

$$\text{k=2},\ f(x) = \lambda e^{-\lambda x} \cdot \lambda x$$

$$\text{k=3},\ f(x) = \frac{\lambda e^{-\lambda x} \cdot (\lambda x)^2}{2}$$

$$\text{k=4},\ f(x) = \frac{\lambda e^{-\lambda x} \cdot (\lambda x)^3}{6}$$

These are called **Erlang Family of Densities**.

Stationarity

When we do experiments infinite times, the probability does not change. For a stationary system,

$$\text{as } t \to \infty, \frac{dP_k(t)}{dt} = 0$$

We say that the system is at equilibrium or in a steady state.

Figure 7: Stationary State Transition Diagram

At stationary, Output = Input.

$$\text{At state 0,} \quad \lambda_0 P_0 = \mu_1 P_1$$

$$\text{At state k,} \quad (\lambda_{k+1} + \mu_k) P_k = \mu_{k+1} P_{k+1} + \lambda_{k-1} P_{k-1}$$

Do not forget the total probability $\sum P_i = 1$.

$$P_1 = \frac{\lambda_0}{\mu_1} P_0 \tag{19}$$

$$(\lambda_1 + \mu_1) P_1 = \mu_2 P_2 + \lambda_0 P_0 \tag{20}$$

Plugging Eq. (19) into Eq. (20), we get

$$(\lambda_1 + \mu_1)(\frac{\lambda_0}{\mu_1})P_0 = \mu_2 P_2 + \lambda_0 P_0$$

$$P_2 = \frac{\lambda_0 \lambda_1 P_0}{\mu_1 \mu_2}$$

$$\text{By induction,} \quad P_k = \frac{\lambda_0 \cdots \lambda_{k-1}}{\mu_1 \cdots \mu_k} P_0$$

$$\text{Or,} \quad P_k = \prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}} P_0$$

$$\text{Total Probability,} \quad \sum_{k=0}^{\infty} \prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}} P_0 = 1$$

$$P_0 = \frac{1}{\sum_{k=0}^{\infty} \prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}}}$$

3 M/M/1 Queue

This queue is composed of

1. Poisson arrival processes with Inter-Arrival Time (IAT) as exponential distribution with PDF,

$$f(x) = \lambda e^{-\lambda x} \text{ , } x > 0$$

   which has arrival rate of λ or time $1/\lambda$

2. Exponentially distributed service time with rate μ or time $1/\mu$

$$f(t) = \mu e^{-\mu t} \text{ , } t > 0$$

3. Only 1 server

Figure 8: M/M/1 Queue

For M/M/1 Queue

$$\lambda_i = \lambda \quad ; \quad \forall i$$
$$\mu_k = \mu \quad ; \quad \forall k$$

Therefore,

$$P_1 = \frac{\lambda}{\mu} P_0$$
$$P_2 = \left(\frac{\lambda}{\mu}\right)^2 P_0$$
$$\vdots$$
$$P_k = \left(\frac{\lambda}{\mu}\right)^k P_0$$
$$\text{Total Probability,} \quad \sum_{k=0}^{\infty} P_k = 1$$
$$\sum_{k=0}^{\infty} P_0 \left(\frac{\lambda}{\mu}\right)^k = 1$$
$$P_0 = \frac{1}{\underbrace{\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k}_{\text{Geometric Series}}}$$

If $\frac{\lambda}{\mu} < 1$ then

$$\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k = \frac{1}{1 - \frac{\lambda}{\mu}}$$

Therefore,

$$P_0 = 1 - \frac{\lambda}{\mu}$$

Let $\frac{\lambda}{\mu} = \rho$ ("load" or "rho"), then

$$P_0 = 1 - \rho \tag{21}$$
$$P_k = \rho^k (1 - \rho) \tag{22}$$

$\rho < 1$; or else, $P_0 = 0$. We now want to find expected value $E[k]$ of k, the number of packets in queue. Therefore, we take z-transform of P_k and differentiate and place $z = 1$.

$$\begin{aligned} G(z) &= \sum_{k=0}^{\infty} z^k \cdot P_k \\ &= \sum_{k=0}^{\infty} z^k \cdot \rho^k (1 - \rho) \\ &= \frac{1 - \rho}{1 - \rho z} \end{aligned} \tag{23}$$

Differentiation,

$$\begin{aligned} E[k] &= \frac{-(1-\rho)}{(1-\rho z)^2} \cdot (-\rho) \mid_{z=1} \\ &= \frac{\rho(1-\rho)}{(1-\rho z)^2} \mid_{z=1} \\ &= \frac{\rho}{1-\rho} \end{aligned} \tag{24}$$

Figure 9: E[k] vs. ρ

$$\begin{aligned} P[\text{Number of packet waiting} = 0] &= \overbrace{P_0}^{\text{No person in Q}} + \overbrace{P_1}^{\text{1 person in service}} \\ &= (1-\rho) + \rho(1-\rho) \\ &= (1+\rho)\cdot(1-\rho) \\ &= 1-\rho^2 \end{aligned} \tag{25}$$

4 M/M/1/B Queue

M/M/1 Queue with Buffer

$$\lambda_k = \begin{cases} \lambda & \text{if} \quad k \le B \\ 0 & \text{if} \quad k > B \end{cases} \tag{26}$$

$$\mu_k = \mu \quad \text{for} \quad k = 1, 2, \ldots, B \tag{27}$$

That means if k population is greater than B buffer size, there can't be any arrival, and only packets in buffer depart.

Figure 10: M/M/1/B State Transition Diagram

$$P_k = P_0 \prod_{i=1}^{k-1} \frac{\lambda_i}{\mu_{i+1}}$$

$$P_k = P_0 \prod_{i=1}^{k-1} \frac{\lambda}{\mu} = P_0 \Big(\frac{\lambda}{\mu}\Big)^k \quad \text{for} \quad k \leq B$$

Summation of Geometric Series

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

$$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$$

$$\sum_{k=1}^{\infty} r^k = \frac{r}{1-r}$$

$$\sum_{k=1}^{n} r^k = \frac{r(1-r^n)}{1-r}$$

And again, we use the total probability to have

$$P_0 \cdot \sum_{k=0}^{B} \Big(\frac{\lambda}{\mu}\Big)^k = 1$$

$$P_0 \cdot \sum_{k=0}^{B} \rho^k = 1$$

$$P_0 \cdot \Big[\frac{1-\rho^{B+1}}{1-\rho}\Big] = 1$$

$$\text{Therefore,} \quad P_0 = \frac{1-\rho}{1-\rho^{B+1}}$$

Note that if $\rho > 1$, the buffer is filled up and packets are dropped.

$$P_k = \frac{(1-\rho)}{(1-\rho^{B+1})} \cdot \rho^k \tag{28}$$

$$P[\text{Queue is full}] = P_B = \frac{(1-\rho)}{(1-\rho^{B+1})} \cdot \rho^B \tag{29}$$

Then, we find expectation by using probability generating function as follows.

$$\begin{aligned}
P(z) &= E[z^k] \\
&= \sum_{k=0}^{B} z^k P_k \\
&= \sum_{k=0}^{B} z^k \frac{(1-\rho)}{(1-\rho^{B+1})} \cdot \rho^k \\
&= \frac{(1-\rho)}{(1-\rho^{B+1})} \cdot \frac{(1-(\rho z)^{B+1})}{1-\rho z}
\end{aligned} \tag{30}$$

Differentiation using Quotient Rules

$$\frac{d}{dx} \cdot \frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} \tag{31}$$

$$\begin{aligned}
E[k] &= \left(\frac{d}{dz} E[z^k]\right)_{z=1} \\
&= \frac{(1-\rho)}{(1-\rho^{B+1})} \cdot \frac{d}{dz}\left[\frac{(1-(\rho z)^{B+1})}{1-\rho z}\right]_{z=1} \quad \text{Solve this to find E[k]}
\end{aligned} \tag{32}$$

Problem

$$\sum_{k=\alpha_1}^{\alpha_2} (1-\rho) \cdot \rho^k = \rho^{\alpha_1} - \rho^{\alpha_2}$$

5 M/M/m Queue

= m-server queue

$$\lambda_k = \lambda, \quad \forall k$$

At Δt, for each packet, P[death] = $\mu \Delta t$ and P[no death] = $1 - \mu \Delta t$. Let's assume that j packets are currently in service ($j \leq m$)

$$\begin{aligned}
P[1 \text{ death in } \Delta t] &= \binom{j}{1} (\mu \Delta t)^1 (1-\mu \Delta t)^{j-1} \\
&\approx j\mu \Delta t
\end{aligned} \tag{33}$$

$$\begin{aligned}
P[2 \text{ deaths in } \Delta t] &= \binom{j}{2} (\mu \Delta t)^2 (1-\mu \Delta t)^{j-2} \\
&\approx 0
\end{aligned} \tag{34}$$

Then, we can conclude that the death rate becomes $j\mu$ when j is the number of occupied servers. And we have birth and death rates as follows.

$$\lambda_k = \lambda, \quad \forall k \tag{35}$$

$$\mu_k = \begin{cases} k\mu & \text{for} \quad k < m \\ m\mu & \text{for} \quad k \geq m \end{cases} \tag{36}$$

For $k < m$,

$$\begin{aligned} P_k &= P_0 \cdot \prod_{i=1}^{k-1} \frac{\lambda}{(i+1)\mu} \\ &= P_0 \cdot \Big(\frac{\lambda}{\mu}\Big)^k \cdot \prod_{i=1}^{k-1} (\frac{1}{i+1}) \\ &= P_0 \cdot \Big(\frac{\lambda}{\mu}\Big)^k \cdot \frac{1}{k!} \end{aligned} \tag{37}$$

state transition diagram M/M/m queue

For $k \geq m$,

$$\begin{aligned} P_k &= P_0 \cdot \prod_{i=1}^{k-1} \frac{\lambda}{\mu_{i+1}} \\ &= P_0 \cdot \prod_{i=1}^{m-1} \frac{\lambda}{(i+1)\mu} \prod_{i=m}^{k-1} \frac{\lambda}{m\mu} \\ &= P_0 \cdot \Big(\frac{\lambda}{\mu}\Big)^k \cdot \prod_{i=1}^{m-1} (\frac{1}{i+1}) \cdot \prod_{i=m}^{k-1} (\frac{1}{m}) \\ &= P_0 \cdot \Big(\frac{\lambda}{\mu}\Big)^k \cdot \frac{1}{m!} \cdot \frac{1}{m^{k-m}} \end{aligned} \tag{38}$$

under the condition that $\rho = \frac{\lambda}{m\mu} < 1$. And if we replace $\frac{\lambda}{\mu} = \rho m$, we have

$$P_k = \begin{cases} \dfrac{(m\rho)^k}{k!} \cdot P_0 & \text{if} \quad k < m \\ \dfrac{(m\rho)^k}{m! \cdot m^{k-m}} \cdot P_0 & \text{if} \quad k \geq m \end{cases} \tag{39}$$

In order to find P_0, we again use total probability $\sum_{k=0}^{\infty} P_k = 1$.

$$P_0 \cdot \Big[\underbrace{\sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!}}_{\text{finite sum}} + \underbrace{\sum_{k=m}^{\infty} \frac{(m\rho)^k}{m! \cdot m^{k-m}}}_{\text{infinite sum}} \Big] = 1 \tag{40}$$

Considering the $2^{nd} \sum$, we let $j = k - m$; hence, we have changed limits to become

$$\text{If} \quad k = m, \quad j = 0$$

$$\text{If} \quad k = \infty, \quad j = \infty$$

As a result, the $2^{nd} \sum$ becomes

$$\sum_{j=0}^{\infty} \frac{(m\rho)^{j+m}}{m! \cdot m^j} = \frac{(m\rho)^m}{m!} \sum_{j=0}^{\infty} \frac{(m\rho)^j}{m^j} = \frac{(m\rho)^m}{m! \cdot (1-\rho)} \tag{41}$$

Finally, we look at the probability that packets need to wait in an M/M/m queue.

$$P[Queuing] = \sum_{k=m}^{\infty} P_k = P_0 \cdot \frac{(m\rho)^m}{m! \cdot (1-\rho)} \tag{42}$$

We call this equation **Erland C Formula** used to predict voice calls.

6 M/M/m/m Queue

m-server loss queue

$$\lambda_k = \begin{cases} \lambda & \text{for} \quad k < m \\ 0 & \text{for} \quad k \geq m \end{cases} \tag{43}$$

$$\mu_k = k\mu \quad \text{for} \quad k \leq m \tag{44}$$

Therefore,

$$\begin{aligned} P_k &= P_0 \cdot \prod_{i=0}^{k-1} \frac{\lambda}{\mu_{i+1}} \\ &= P_0 \cdot \prod_{i=0}^{k-1} \frac{\lambda}{(i+1)\mu} \quad \text{for } k < m \\ &= P_0 \cdot \left(\frac{\lambda}{\mu}\right)^k \cdot \prod_{i=1}^{k-1} (\frac{1}{i+1}) \\ &= P_0 \cdot \left(\frac{\lambda}{\mu}\right)^k \cdot \frac{1}{k!} \end{aligned} \tag{45}$$

Then, we can find P_0 with

$$P_0 = \frac{1}{\sum_{k=0}^{m} \left(\frac{\lambda}{\mu}\right)^k \cdot \frac{1}{k!}}$$

Finally, we are interested in the probability that the queuing system is blocking the calls. This is a state where we have m packets in the M/M/m/m queue.

$$P[Blocking] = P_m = P_0 \cdot \frac{(m\rho)^m}{m!}$$

And this is **Erland B Formula** used in telephony.

7 Little's Law

It's the most important law in Queuing Theory. Now, we are interested in the delay **d** experienced by a packet.

$$\boldsymbol{d} = \overbrace{\boldsymbol{x}}^{\text{process time}} + \overbrace{\boldsymbol{\omega}}^{\text{waiting time}} \tag{46}$$

$$\text{Expectation:} \quad \bar{\boldsymbol{d}} = \bar{\boldsymbol{x}} + \bar{\boldsymbol{\omega}} \tag{47}$$

$$\text{Little's law:} \quad \boldsymbol{\lambda}\bar{\boldsymbol{d}} = \bar{\boldsymbol{k}} \quad (\bar{\boldsymbol{k}} = E[k]) \tag{48}$$

Here are the properties of Little's Law.

1. It is independent of the interarrival time (IAT) distribution.
2. It is independent of the service time distribution
3. It is independent of the number of servers
4. It is independent of the queuing discipline such as FIFO or Priority Queue

For the queue without server, we have

$$\bar{\boldsymbol{k_q}} = \boldsymbol{\lambda}\bar{\boldsymbol{\omega}} \tag{49}$$

where $\bar{\boldsymbol{k_q}}$ is the expected number of packets in the queue. Let's review what we have here.

- $\bar{\boldsymbol{d}}$ is the expected delay
- $\bar{\boldsymbol{k}}$ is the expected number of packets in the queue
- $\boldsymbol{\lambda}$ is the interarrival rate and $\bar{\boldsymbol{k}} = \boldsymbol{\lambda}\bar{\boldsymbol{d}}$

Delays in M/M/1 Queue

$$E[k] = \bar{\boldsymbol{k}} = \frac{\rho}{1-\rho}$$

$$\bar{\boldsymbol{d}} = \frac{\bar{\boldsymbol{k}}}{\boldsymbol{\lambda}} = \frac{\lambda/\mu}{1-\rho} \cdot \frac{1}{\lambda} = \frac{1}{\mu(1-\rho)}$$

And we have

$$\text{Poisson IAT:} \quad a(t) = \lambda e^{-\lambda t}, \quad t \geq 0 \tag{50}$$

$$\text{Service time:} \quad b(t) = \mu e^{-\mu t}, \quad t \geq 0 \tag{51}$$

And let $d(t)$ be the PDF of the delay which we take the Laplace transform to be.

$$D(s) = E[e^{-sd}]$$

...

Figure 11: Delays in M/M/1

Let there be k packets in the queue when a tagged packet comes in.

$$\boldsymbol{d} = \underbrace{\boldsymbol{x}}_{\text{service time}} + \underbrace{\boldsymbol{\omega}}_{\text{waiting time}}$$

$$D(s) = X(s) \cdot W(s) = \frac{\mu}{\mu + s} \cdot W(s)$$

Let's first find $W(s|k)$, which is the waiting time given k packets in the queue.

$$\begin{aligned} W(s|k) &= E[e^{-s(x_1 + \ldots + x_k)}] \quad \text{(independent events)} \\ &= [X(s)]^k \quad \text{(service time until } k \text{ packets are serviced)} \\ &= \left(\frac{\mu}{\mu + s}\right)^k \end{aligned} \tag{52}$$

$$D(s|k) = X(s) \cdot W(s|k) = \left(\frac{\mu}{\mu + s}\right)^{k+1} \tag{53}$$

$$\begin{aligned} D(s) &= \sum_{k=0}^{\infty} D(s|k) \cdot P_k \\ &= \sum_{k=0}^{\infty} \left(\frac{\mu}{\mu + s}\right)^{k+1} \cdot \rho^k (1 - \rho) \\ &= \frac{\mu(1-\rho)}{\mu + s} \cdot \sum_{k=0}^{\infty} \left(\frac{\rho\mu}{\mu + s}\right)^k \\ &= \frac{\mu(1-\rho)}{\mu + s} \cdot \frac{1}{1 - \frac{\rho\mu}{\mu + s}} \\ &= \frac{\mu(1-\rho)}{s + \mu(1-\rho)} \end{aligned} \tag{54}$$

We now observer that $\mu(1-\rho)$ is a parameter of exponential distribution and that the delay itself has exponential distribution as follows.

$$d(t) = \mu(1-\rho) \cdot e^{-\mu(1-\rho)t} \quad \text{for} \quad t \geq 0 \tag{55}$$

$$\bar{\boldsymbol{d}} = \frac{1}{\mu(1-\rho)} \tag{56}$$

Proof of Little's Law in Specific Contexts

Let's take a look at when we do the same experiment for a number of times and find an average which we call *Ensemble Average*. In comparison, if we record a number of observations over the same experiment, we can have an average called *Time Average*. If a process has *Ensemble Average = Time average*, we call it **Ergodic Process**.

We are now proving Little's Law of a process with stationary and ergodic properties. (Proof is in the Data Networks book by Bertsekas and Gallager book)

In the end of Class 5's lecture notes

8 Burke's Theorem

We are considering inter-departure time (IDT) in M/M/1 queue. We have Possion arrival of rate λ and service time with rate μ. Now we want to know how the departure process looks like. Let's consider the two cases

Figure 12: IDT in M/M/1

1. the packet leaves the queue non-empty
2. the packet leaves the queue empty

Figure 13: Q-non-empty vs. Q-empty IDT

For case (1.), the next departure will be after the service time. Let PDF of IDT be $d^*(t)$ and its Laplace transform be $D^*(s)$. Therefore, we have

$$D^*(s \mid \text{pkt left Q non-empty}) = \frac{\mu}{\mu + s}$$

and

$$D^*(s \mid \text{pkt left Q empty}) = X(s) \cdot Y(s) = \frac{\mu}{\mu + s} \cdot \frac{\lambda}{\lambda + s}$$

Therefore, by the law of total probability,

$$\begin{aligned}
D^*(s) &= D^*(s \mid \text{pkt left Q non-empty}) \cdot \overbrace{P(\text{Q non-empty})}^{1-P_0=\rho} + D^*(s \mid \text{pkt left Q empty}) \cdot \overbrace{P(\text{Q empty})}^{P_0=1-\rho} \\
&= \frac{\mu}{\mu+s} \cdot \rho + \left(\frac{\mu}{\mu+s}\right) \cdot \left(\frac{\lambda}{\lambda+s}\right) \cdot (1-\rho) \\
&= \frac{\mu}{\mu+s} \cdot \left[\rho + \frac{\lambda}{\lambda+s} \cdot (1-\rho)\right] \\
&= \frac{\lambda}{\lambda+s}
\end{aligned} \tag{57}$$

In conclusion, the inter-departure time is exponentially distributed with the same rate λ as arrival process.

9 Aloha Systems

A.S. is used to analyze a shared medium or TDMA problems and understand what throughput such a system can achieve. We are considering the following scenario.

- A packet-switched system in which the channel is shared
- It is totally distributed
- When a station generates a packet, it immediately transmit regardless of what other stations do.
- If two or more stations transmit simultaneously, collisions happen.
- Let's ignore receivers' acknowledgement.

Moreover, we presume the following assumptions.

1. Stations generate fixed length packets of size m and it takes m seconds to transmit.
2. Many stations and each is lightly loaded, meaning each station has no more than 1 packet at anytime (no queuing)
3. Offered load: $\rho = \lambda m$, $(m = \frac{1}{\mu})$
4. Total load needs to consider both newly generated packets and retransmitted packets.

Figure 14: Collisions in Aloha Systems

Let the total arrival rate consisting of both new and retransmitted packets be Λ. Λ is a Poisson process and in order to avoid collision, no station should start transmitting in a $2m$ period, which is

$$P[k=0] = e^{-2\Lambda m}$$

Therefore,

$$P[\text{collision}] = 1 - e^{-2\Lambda m}$$

And the average rate of retransmission is

$$\Lambda \cdot [1 - e^{-2\Lambda m}]$$

We now have a total rate as

$$\text{Total rate:} \quad \Lambda = \lambda + \Lambda \cdot [1 - e^{-2\Lambda m}] \tag{58}$$

$$\text{Simplifying:} \quad \Lambda \cdot e^{-2\Lambda m} = \lambda \tag{59}$$

$$\text{Total load:} \quad R = \Lambda m \tag{60}$$

$$\text{Multiply both sides with } m\text{:} \quad R \cdot e^{-2R} = \rho \tag{61}$$

We can plot a graph representing relationship betwenn ρ and R and also $d\rho/dR = 0$ to find max ρ to be $1/2e$ or 0.18 at $R = 1/2$.

Figure 15: R vs. ρ (Aloha)

10 Slotted Aloha Systems

Slotted A.S. is an A.S. except

Figure 16: Slotted Aloha

- When a station generates a packet, it transmits the packet at the beginning of the next slot.
- We assume synchronization of slots.

Going through the same assumptions as A.S., we have

$$P[k = 0] = e^{-\Lambda m}$$

And the average rate of retransmission is

$$\Lambda \cdot [1 - e^{-\Lambda m}]$$

Finally, we have the following important relationship. $R \cdot e^{-R} = \rho$ and at $R = 1/2$, we have maximum ρ of $1/e$ or 0.36.

Figure 17: R vs. ρ (Slotted Aloha)

Stability Analysis of Slotted Aloha Systems

Considering Slotted A.S., we have the following variables.

- P[New message generated in a slot] $= 1 - e^{-\lambda m} = \sigma$
- P[Retransmission in a slot] $= \alpha$
- P[Retransmission after "i" slots] $= (1-\alpha)^{i-1} \cdot \alpha$
- The total number of stations is N.
- There are k backlogged stations attempting to retransmit.
- $(N-k)$ stations try to generate new packets.

Remember that each station has at most 1 packet at a time, either generating a new packet or retransmitting a previous one. Therefore,

$$P[\text{1 newly generated message succeeds in a given slot}] = \overbrace{\binom{N-k}{1}\sigma(1-\sigma)^{N-k-1}}^{\text{exactly 1 station gen new pkt}} \quad \overbrace{(1-\alpha)^k}^{\text{no stations try to retx}} \tag{62}$$

$$P[\text{1 retransmittion succeeds in a given slot}] = \overbrace{\binom{k}{1}\alpha(1-\alpha)^{k-1}}^{\text{exactly 1 station retx pkt}} \quad \overbrace{(1-\sigma)^{N-k}}^{\text{no new gen stations try to tx}} \tag{63}$$

And remember that $k \in [0, N]$. Now we can calculate the output rate of successfully transmitted packets.

$$\text{Output rate:} \quad T_k = \underbrace{(N-k)\sigma(1-\sigma)^{N-k-1}(1-\alpha)^k + k\alpha(1-\alpha)^{k-1}(1-\sigma)^{N-k}}_{\text{Two mutually exclusive events}} \tag{64}$$

Then, we have input rate of

$$\text{Input rate:} \quad S_k = (N-k)\sigma \tag{65}$$

$$\sigma = \frac{S_k}{N-k} \tag{66}$$

Taylor Series Approximation

$$\left(1 - \frac{S}{N}\right)^N \approx e^{-S} \approx \left(1 - \frac{S}{N-k}\right)^{N-k}$$

If N is large and σ is small, by approximation, we have

$$T_k = S_k e^{-S_k}(1-\alpha)^k + k\alpha(1-\alpha)^{k-1}e^{-S_k}$$

At equilibrium (input = output), we assume that $T_k = S_k$. Now, we can plot a relationship between S and k and observe stable and unstable points. Moreover, we can see the effect of choosing α and σ.

Figure 18: Stable Slotted Aloha

Figure 19: Unstable Slotted Aloha

Figure 20: Zero-throughput Stable Slotted Aloha

11 Bulk Arrivals in M/M/1 Queue

A case for burst arrivals is an example of split IP fragments, where a number of fracments may arrive at a queue at the same time.

Figure 21: Bulk Arrivals

We assume that we know the distribution of the bulk size.

$$P[\text{bulk size} = i] \;=\; g_i$$

$$\sum_{i=1}^{\infty} g_i = 1$$

Moreover, we assume IAT to be exponential and ST be exponential.

Figure 22: Bulk Arrival State Transition Diagram

At equilibrium, we have Output = Input to each state; therefore,

$$\text{For state 0:} \quad \lambda \cdot [\ \underbrace{g_1 + g_2 + \dots}_{\text{sum = 1 (total prob)}}\ ] \cdot P_0 = \mu P_1$$

$$\text{Therefore,} \quad \lambda P_0 = \mu P_1$$

$$\text{For state k:} \quad (\lambda\,[g_1 + g_2 + \dots] + \mu)\, P_k = \mu P_{k+1} \quad + \quad \underbrace{\lambda \sum_{i=0}^{k-1} P_i g_{k-i}}_{\text{transitions from preceding states}}$$

$$(\lambda + \mu) \cdot P_k = \mu P_{k+1} \quad + \lambda \sum_{i=0}^{k-1} P_i \cdot g_{k-i}$$

Given,

$$P(z) = \sum_{k=0}^{\infty} z^k P_k$$

Now we mulitply both sides of Eq. 11 with z^k and $\sum$ (Don't forget that $k > 0$).

$$\underbrace{(\lambda + \mu) \cdot \sum_{k=1}^{\infty} z^k P_k}_{\text{A}} = \underbrace{\mu \sum_{k=1}^{\infty} z^k P_{k+1}}_{\text{B}} + \underbrace{\lambda \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} P_i \cdot g_{k-i} \cdot z^k}_{\text{C}}$$

For **A**,

$$(\lambda + \mu)[P(z) - P_0]$$

For **B**,

$$\frac{\mu}{z}\sum_{k=1}^{\infty} P_{k+1} z^{k+1}$$

$$\text{Let} \quad j = k+1$$

$$\frac{\mu}{z}\sum_{j=2}^{\infty} P_j z^j = \frac{\mu}{z}[P(z) - P_0 - zP_1]$$

For **C**, we have nested sums that are difficult to solve, so let's switch the sums.

Figure 23: Nested Sum Switching

$$\lambda \sum_{i=0}^{\infty} \sum_{k=i+1}^{\infty} P_i \cdot g_{k-i} \cdot z^k$$

$$\lambda \sum_{i=0}^{\infty} P_i \sum_{k=i+1}^{\infty} g_{k-i} \cdot z^{k-i} z^i$$

$$\lambda \sum_{i=0}^{\infty} P_i z^i \sum_{k=i+1}^{\infty} g_{k-i} \cdot z^{k-i}$$

Let $j = k - i$, when $k = i + 1 \to j = 1$ and when $k = \infty \to j = \infty$.

$$\lambda \underbrace{\sum_{i=0}^{\infty} P_i z^i}_{\text{P(z)}} \quad \underbrace{\sum_{j=1}^{\infty} g_j z^j}_{\text{PGF of } g_i = G(z)} \quad = \lambda P(z) G(z)$$

Now, we put together **A**, **B** and **C**.

$$(\lambda + \mu)[P(z) - P_0] = \frac{\mu}{z}[P(z) - P_0 - zP_1] + \lambda P(z) G(z)$$

We know $P_1 = \frac{\lambda}{\mu} P_0$.

$$z \cdot (\lambda + \mu)[P(z) - P_0] = \mu \cdot [P(z) - P_0 - \frac{\lambda}{\mu} P_0 z] + z \cdot \lambda[P(z)G(z)]$$

$$P(z) = \frac{\mu P_0 (1 - z)}{\mu(1 - z) - \lambda z[1 - G(z)]}$$

Unfortunately, we still do not yet know P_0. To find P_0, we use the properties.

$$P(z) = \sum_{k=0}^{\infty} z^k P_k \qquad P(1) = \sum_{k=0}^{\infty} P_k = 1$$

To find P(1), we use L'Hospital rules. (diff numerator and diff denominator and z = 1)

$$P(1) = \frac{-\mu P_0}{-\mu - \lambda + \lambda + \lambda G'(1)} = 1$$
$$-\mu P_0 = -\mu + \lambda G'(1)$$
$$P_0 = 1 - \frac{\lambda}{\mu} \qquad \underbrace{G'(1)}_{\text{Expected value of } g_i}$$
$$P_0 = 1 - \frac{\lambda \bar{g}}{\mu}$$

Therefore, we have that $\frac{\lambda \bar{g}}{\mu} = \rho$ and $P_0 = 1 - \rho$ and for stability, $\rho < 1$. We also want to find expected value of the number of packets and delays.

12 Markov Chains

This section is covered by Chapter 2 of Kleinrock's book. Let's have a sequence of random variables $\{x_1, x_2, \ldots, x_n\}$

Figure 24: A Sequence of Random Variables

Let these random variables take a range of values which we call *state*.

$$x_i = j$$

If

$$P[x_n = j | x_1 = i_1, x_2 = i_2, \ldots, x_{n-1} = i_{n-1}] = \underbrace{P[x_n = j | x_{n-1} = i_{n-1}]}_{\text{only depends on the latest information}}$$

then $\{x_1, x_2, \ldots, x_n\}$ is a Markov chain (MC). Now we define the transition probability as.

$$\text{One-step Transition Probability,} \quad P_{ij} = P\big[x_n = j | x_{n-1} = i\big]$$

$$\text{m-Step Transition Probability,} \quad P_{ij}^{(m)} = P\big[x_{n+m} = j | x_n = i\big] = \sum_k P_{ik}^{(m-i)} P_{kj}$$

Here is a list of important definitions of MC.

Irreducible MC If all states of an MC are reachable from each other, the MC is irreducible; that is, $P_{ij}^{(m_0)} > 0$. This guarantees that we can go to other states.

A Closed Subset If there are a set of states from which we cannot reach other states, that set is call "a closed subset", and then MC is reducible.

An Absorbing State If a closed subset contains a single state then it is called "an absorbing state".

A Transient State Let

$$f_j^{(n)} = P[\text{the first return to state } j \text{ occurs after } n \text{ steps }]$$

and

$$P[\text{ever returning to state } j] = \sum_n f_j^{(n)}$$

If $f_j < 1$, then state j is called a transient state.

A Recurrent State If $f_j = 1$, then state j is called a recurrent state, such as a state in M/M/1 with $\rho < 1$

A Periodic State If state E_j is returned to every γ seconds (at 8, 18, 28 seconds), E_j is periodic with period γ.

An Aperiodic State If $\gamma = 1$, E_j is aperiodic.

A Recurrent Null State Let's define <u>Mean Recurrence Time</u> as.

$$M_j = \sum_{n=1}^{\infty} n \cdot f_j^{(n)}$$

If $M_j = \infty$, then the state is called a recurrent null state.

A Recurrent Non-Null State If $M_j < \infty$, then the state is called a recurrent non-null state.

Figure 25: A Closed Subset and An Absorbing State

Theorem 1. *All states of an irreducible MC are either transient, recurrent null, or recurrent non-null, or periodic with the same γ.*

Considering marginal probability

$$\Pi_j^{(n)} = P(X_n = j)$$

For an irreducible, aperiodic MC, equilibrium probabilities exist and are independent of the initial state $\Pi^{(0)}$.

$$\lim_{n\to\infty} \Pi_j^{(n)} = \Pi_j$$

We can find $\Pi_j^{(n)}$ by

$$\Pi_j^{(n)} = \sum_k \Pi_k^{(n-1)} \cdot P_{kj}$$
$$\lim_{n\to\infty} \Pi_j^{(n)} = \lim_{n\to\infty} \sum_k \Pi_k^{(n-1)} \cdot P_{kj}$$
$$\Pi_j = \sum_k \Pi_k P_{kj}$$

Definition 1. *<u>Ergodic States:</u> If E_j is aperiodic and recurrent non-null, it is ergodic and if all states of an MC are ergodic, the MC is ergodic.*

Now that we have steady state probability Π_j and transition probability P_{ij}, let's take a look at an MC.

Figure 26: A Three-State Markov Chain

Given $P = \left[P_{ij}\right]$ and

$$\lim_{n\to\infty} \Pi^{(n)} = \lim_{n\to\infty} \Pi^{(n-1)} \cdot P$$
$$\Pi = \Pi \cdot P$$

For example,

$$P = \begin{bmatrix} 0 & 3/4 & 1/4 \\ 1/4 & 0 & 3/4 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

Observe that summation on each row is 1. Therefore,

$$\begin{bmatrix} \Pi_1 & \Pi_2 & \Pi_3 \end{bmatrix} = \begin{bmatrix} \Pi_1 & \Pi_2 & \Pi_3 \end{bmatrix} \cdot \begin{bmatrix} 0 & 3/4 & 1/4 \\ 1/4 & 0 & 3/4 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

As a result, we have the following equations to solve.

$$\Pi_1 = \frac{1}{4}\Pi_2 + \frac{1}{4}\Pi_3$$
$$\Pi_2 = \frac{3}{4}\Pi_1 + \frac{1}{4}\Pi_3$$
$$\Pi_3 = \frac{1}{4}\Pi_1 + \frac{3}{4}\Pi_2 + \frac{1}{2}\Pi_3$$

Also, do not forget that $\sum \Pi_j = 1$. Solving these, we have the solutions

$$\Pi_1 = \frac{1}{5}, \quad \Pi_2 = \frac{7}{25}, \quad \Pi_3 = \frac{13}{25}$$

These solutions are at steady states. How about solutions at transient states?

$$\Pi^{(n)} = \Pi^{(n-1)}P \tag{67}$$

Let's define

$$\Pi(z) = \sum_{n=0}^{\infty} \Pi^{(n)} z^n$$

Now we multiply both sides of Eq. (67) by "z" and sum from 1 to ∞.

$$\underbrace{\sum_{n=1}^{\infty} \Pi^{(n)} z^n}_{=\Pi(z)-\Pi^{(0)}} = \underbrace{\sum_{n=1}^{\infty} \Pi^{(n-1)} \cdot P \cdot z^n}_{m=n-1;\ \sum_{m=0}^{\infty} \Pi^{(m)}Pz^{m+1}=z\Pi(z)P}$$
$$\Pi(z) - z\Pi(z)\cdot P = \Pi^{(0)}$$
$$\Pi(z)\big[I - zP\big] = \Pi^{(0)}$$
$$\Pi(z) = \Pi^{(0)}\big[I - zP\big]^{-1}$$

See Sec. (20.6) for matrix inversion. Now, let's go back to our example.

$$\big[\ I - zP\big] = \begin{bmatrix} 1 & -\frac{3}{4}z & -\frac{1}{4}z \\ -\frac{1}{4}z & 1 & -\frac{3}{4}z \\ -\frac{1}{4}z & -\frac{1}{4}z & 1-\frac{1}{2}z \end{bmatrix} \tag{68}$$

We have to inverse this matrix by

$$Inv(A) = \frac{\text{Transpose of Matrix of Cofactors}}{\text{Determinant}}$$

From (68), we have

$$\begin{aligned} det[I - zP] &= \Big[\big[(1-\frac{1}{2}z) + (-\frac{3}{4}z)(-\frac{3}{4}z)(-\frac{1}{4}z) + (-\frac{1}{4}z)(-\frac{1}{4}z)(-\frac{1}{4}z)\big] \\ &\quad - \big[(-\frac{1}{4}z)(-\frac{1}{4}z) + (-\frac{3}{4}z)(-\frac{1}{4}z) + (-\frac{3}{4}z)(-\frac{1}{4}z)(1-\frac{1}{2}z)\big]\Big] \\ &= 1 - \frac{1}{2}z - \frac{7}{16}z^2 - \frac{1}{16}z^3 \\ &= (1-z)(1+\frac{1}{4}z)^2 \end{aligned}$$

$$\left[\, I - zP \right]^{-1} = \frac{1}{(1-z)(1+\frac{1}{4}z)^2} \begin{bmatrix} 1-\frac{1}{2}z-\frac{3}{16}z^2 & \frac{3}{4}z-\frac{5}{16}z^2 & \frac{1}{4}z+\frac{9}{16}z^2 \\ \frac{1}{4}z+\frac{1}{16}z^2 & 1-\frac{1}{2}z-\frac{1}{16}z^2 & \frac{3}{4}z+\frac{1}{16}z^2 \\ \frac{1}{4}z+\frac{1}{16}z^2 & \frac{1}{4}z+\frac{3}{16}z^2 & 1-\frac{3}{16}z^2 \end{bmatrix}$$

Taking a partial fraction expansion, we have

$$\left[\, I - zP \right]^{-1} = \frac{1/25}{1-z} \begin{bmatrix} 5 & 7 & 13 \\ 5 & 7 & 13 \\ 5 & 7 & 13 \end{bmatrix} + \frac{1/5}{(1+z/4)^2} \begin{bmatrix} 0 & -8 & 8 \\ 0 & 2 & -2 \\ 0 & 2 & -2 \end{bmatrix} + \frac{1/25}{1+z/4} \begin{bmatrix} 20 & 33 & -53 \\ -5 & 8 & -3 \\ -5 & -17 & 22 \end{bmatrix}$$

Finally, we take inverse-z transform on the matrices to have

$$\boldsymbol{P^n} = \frac{1}{25} \begin{bmatrix} 5 & 7 & 13 \\ 5 & 7 & 13 \\ 5 & 7 & 13 \end{bmatrix} + \frac{1}{5}(n+1)(-\frac{1}{4})^n \begin{bmatrix} 0 & -8 & 8 \\ 0 & 2 & -2 \\ 0 & 2 & -2 \end{bmatrix} + \frac{1}{25}(-\frac{1}{4})^n \begin{bmatrix} 20 & 33 & -53 \\ -5 & 8 & -3 \\ -5 & -17 & 22 \end{bmatrix}$$

13 Bulk Service Systems

Figure 26: Bulk Services

Figure 27: Bulk Service State Diagram

At equilibrium, (out = in)

$$\text{For state } k\text{:} \quad (\lambda+\mu)\cdot P_k = \lambda P_{k-1} + \mu P_{k+r} \tag{69}$$

$$\text{For state 0:} \quad \lambda P_0 = [\mu P_1 + \mu P_2 + \ldots + \mu P_r] = \mu \sum_{j=1}^{r} P_j \tag{70}$$

Now we take z-transform of Eq. 69 and have

$$(\lambda+\mu)\sum_{k=1}^{\infty} z^k P_k = \overbrace{\lambda \sum_{k=1}^{\infty} z^k P_{k-1}}^{\text{let } j=k-1} + \overbrace{\mu \sum_{k=1}^{\infty} z^k P_{k+r}}^{\text{let } i=k+r}$$

$$(\lambda+\mu)\sum_{k=1}^{\infty} z^k P_k = \lambda z \sum_{j=0}^{\infty} z^j P_j + \frac{\mu}{z^r} \sum_{i=1+r}^{\infty} z^i P_i$$

Let $P(z) = \sum_{k=0}^{\infty} z^k P_k$ and then

$$(\lambda+\mu)[P(z) - P(0)] = \lambda z P(z) + \frac{\mu}{z^r}\left[P(z) - \sum_{j=0}^{r} z^j P_j\right]$$

Simplifying, we have

$$P(z) = \frac{\mu \sum_{j=0}^{r} z^j P_j - (\lambda+\mu) P_0 \cdot z^r}{\lambda z^{r+1} - (\lambda+\mu) z^r + \mu}$$

From state 0 Eq. 70, we have that

$$(\lambda+\mu) P_0 = \mu \sum_{j=0}^{r} P_j$$

Therefore,

$$\begin{aligned} P(z) &= \frac{\mu \sum_{j=0}^{r} P_j z^j - \mu \sum_{j=0}^{r} P_j z^r}{\lambda z^{r+1} - (\lambda+\mu) z^r + \mu} \\ &= \frac{\mu \cdot \left[\sum_{j=0}^{r} P_j z^j - \sum_{j=0}^{r} P_j z^r\right]}{\lambda z^{r+1} - (\lambda+\mu) z^r + \mu} \end{aligned}$$

Because $\rho = \dfrac{\lambda}{r\mu}$ and μ can be pulled to the denominator, we have

$$P(z) = \frac{\sum_{j=0}^{r} P_j z^j - \sum_{j=0}^{r} P_j z^r}{r\rho z^{r+1} - (1+r\rho) z^r + 1}$$

Figure 28: A unit circle in a complex plane

Within the unit circle, we have this relationship

$$|P(z)| = \sum_{k=1}^{\infty} P_k |z|^k = 1 \quad \text{if } z = 1$$

Again, bulk service implies one-by-one arrival and r bulk to be serviced. Also, we must have

$$\rho = \frac{\lambda}{r\mu} < 1$$

Now, we rewrite the $P(z)$ equation.

$$P(z) = \frac{\sum_{j=0}^{r} P_j (z^j - \overbrace{z^r}^{\text{polynomial with degree } r})}{r\rho \underbrace{z^{r+1}}_{\text{polynomial with degree } r+1} - (1+r\rho)z^r + 1}$$

Therefore, $P(z)$ is bounded with $|z| < 1$.

Definition 2. *Analytic Function: Fourier transform from one domain to another. A function of a complex variable, z, is analytic at a point z_0 if the derivative exists at z_0 and at each point in the neighborhood of z_0.*

Theorem 2. *Rouche's Theorem: If $f(z)$ and $g(z)$ are analytic functions within and on a closed contour C and if $|f(z)| > |g(z)|$ on C, then $f(z)$ and $f(z) + g(z)$ have the same number of zeros within C.*

$$P(z) = \frac{\boxed{\text{Numerator}}}{\underbrace{\boxed{\text{Denominator}}}_{\text{one root should coincide with numerator's to be cancelled out}}}$$

Now, the question is that how many roots in denominator appear in numerator? Let

$$f(z) = -(1 + r\rho)z^r$$

$$g(z) = r\rho z^{r+1} + 1$$

Let the closed contour be a circle of radius $1+\delta$. Now let's take a look at the Taylor series expansion of a function $h(z)$.

$$h(z) = \sum_{m=0}^{\infty} \frac{h^{(m)}(a)(z-a)^m}{m!} \tag{71}$$

where $h^{(m)}$ is m^{th} derivative and $(z - a)$ is a point around which we are finding the Taylor series. Now, we want to find $|f(z)|$ and $|g(z)|$.

$$|f(z)| = (1 + r\rho)z^r$$
$$f(1) = 1 + r\rho$$
$$f'(1) = rz^{r-1}((1 + r\rho) \,|_{z=1} = r(1 + r\rho)$$

However, we are interested in the contour $1+\delta$, so we can replace $z = 1+\delta$ in the expansion (Eq. 71).

$$|f(z)| = |f(1)| + |f'(1)|\delta = (1+r\rho) + r(1+r\rho)\delta$$

For $g(z)$,

$$\begin{aligned}|g(1)| &= 1+r\rho \\ |g'(1)| &= r\rho(r+1)z^r \mid_{z=1} = r\rho(r+1) \\ |g(z)| &= (1+r\rho) + r\rho(r+1)\delta\end{aligned}$$

Let's check if $|f(z)| > |g(z)|$.

$$\begin{gathered}\cancel{(1+r\rho)} + \cancel{r}(1+r\rho)\cancel{\delta} > \cancel{(1+r\rho)} + \cancel{r}\rho(r+1)\cancel{\delta} \\ 1 + \cancel{r\rho} > \cancel{r\rho} + \rho \\ \underbrace{1 > \rho}_{\text{which is true}}\end{gathered}$$

This means $f(z)$ and $f(z)+g(z)$ have the same number of zeroes within $|z| = 1+\delta$. Since $f(z)$ has r roots at the origin, the denominator of $P(z)$ has r roots within the unit circle. Therefore, we are having one root outside the unit circle.

$$P(z) = \frac{\cancel{(z-z_1)(z-z_2)\cdots(z-z_r)}}{k\cdot\cancel{(z-z_1)(z-z_2)\cdots(z-z_r)}\cdot(z-z^*)}$$

$z = 1$ is obvious?

$$\begin{gathered}P(z) = \frac{1}{k\cdot(z-z^*)} \\ P(1) = 1 \rightarrow \frac{1}{k\cdot(1-z^*)} = 1 \rightarrow k = \frac{1}{1-z^*} \\ P(z) = \frac{1-z^*}{z-z^*} = \frac{z^*-1}{z^*-z} = \frac{\left(1-\dfrac{1}{z^*}\right)}{\left(1-\dfrac{z}{z^*}\right)}\end{gathered}$$

Finally, we inverse-z-transform $P(z)$ to have

$$P_k = \big(1-\frac{1}{z^*}\big)\big(\frac{1}{z^*}\big)^k \tag{72}$$

14 Networks of Queues

Figure 29: Global vs. Local Balance

From Fig. 29, we have the global balance equation at state k as follows

$$\lambda_k P_k + \mu_k P_k = \mu_{k+1} P_{k+1} + \lambda_{k-1} P_{k-1} \tag{73}$$

Also at boundary, we have the local balance equations.

$$\lambda_{k-1} P_{k-1} = \mu_k P_k \quad \text{and} \quad \lambda_k P_k = \mu_{k+1} P_{k+1} \tag{74}$$

If Eq. 73 holds, then Eq. 74 holds and if Eq. 74 holds, then Eq. 73 holds. Both have uniqueness.

There are two types of queueing networks - closed and open.

14.1 Closed Queueing Networks

In this type, packets cannot leave the network.

Figure 30: Closed Queueing Networks

$$P(002) = P[2 \text{ packets in } Q_3]$$

Look at state **110**.

$$\mu_1 P(110) + \mu_2 P(110) = \mu_3 P(101) + \mu_2 P(020)$$

At **002**,

$$\mu_1 P(101) = \mu_3 P(002)$$

And do not forget the total probability,

$$\sum P(i,j,k) = 1$$

If there are 3 packets in the network, these equeation will be more complex. Therefore, we want to have local balance equations because they are easier to be solved. For local balance equation, we track arrival to a queue and departure from that queue.

For state **011**,

$$\mu_3 P(002) = \mu_2 P(011)$$

$$\mu_1 P(110) = \mu_3 P(011)$$

For state **110**,

$$\mu_2 P(020) = \mu_1 P(110)$$

$$\mu_3 P(101) = \mu_2 P(110)$$

For state **101**,

$$\mu_2 P(011) = \mu_1 P(101)$$

$$\mu_1 P(200) = \mu_3 P(101)$$

For state **020**,

$$\mu_3 P(011) = \mu_2 P(020)$$

For state **002**,

$$\mu_1 P(101) = \mu_3 P(002)$$

For state **200**,

$$\mu_2 P(110) = \mu_1 P(200)$$

14.2 Open Queueing Networks

Figure 31: A Sequence of Queues

$$P(k_1, k_2) = P(k_1)P(k_2) = (1-\rho_1)\rho_1^{k_1} \cdot (1-\rho_2)\rho_2^{k_2}$$

where k_1, k_2 are the number of packets in Q_1, Q_2 respectively.

Figure 32: Open Queueing Networks

where γ_i is an external arrival rate.

Let r_{ij} be the probability of a packet going from node i to j. Now we can create a routing matrix.

$$\begin{bmatrix} 0 & 1/4 & 0 & 1/2 \\ 1/3 & 0 & 1/3 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \end{bmatrix}$$

$$P[\text{packet leave systems at node } i] = 1 - \sum_{j=1}^{N} r_{ij}$$

Therefore, we have aggregate arrival at node i as

$$\lambda_i = \gamma_i + \sum_{j=1}^{N} \lambda_j r_{ij}$$

Or, we can express in the matrix form.

$$\boldsymbol{\lambda} = \boldsymbol{\gamma} + \boldsymbol{\lambda R}$$

$$P(k_1 k_2 \cdots k_N) = P(k_1)P(k_2)\cdots P(k_i)\cdots P(k_N)$$

where $P(k_i) = (1-\rho_i)\rho_i^{k_i}$ and $\rho_i = \dfrac{\lambda_i}{\mu_i}$. Please note that μ_i is known and λ_i is to be computed.

Now, let's go back to a closed network where k packets circulating in the network and no external input or output.

$$\lambda_i = \sum_{j=1}^{N} \lambda_j r_{ij}$$

$$\boldsymbol{\lambda} = \boldsymbol{\lambda R}$$

These equations are linearly dependent, so they are unable to be solved.

Let $\rho_i = \dfrac{\lambda_i}{\mu_i}$.

$$P(k_1 k_2 \dots k_N) = \frac{1}{G(k)} \prod_{i=1}^{N} \rho_i^{k_i}$$

$$\sum_{k_1} \sum_{k_2} \cdots \sum_{k_N} P(k_1 \dots k_N) = 1$$

15 BCMP Theorem

1. This allows for L classes of users. Each class is identified by a different routing matrix, elements of which are $r_{ij}(l)$.

2. This allows for a stage type server.

Figure 33: Stage Type Servers

If $A(s)$ is the service time distribution,

$$A(s|i=0) = 1 \quad \text{where } i \text{ is the number of stages}$$

$$A(s|i) = \prod_{j=1}^{i} \frac{\mu_j}{s+\mu_j}$$

$$A(s) = 1 \times \gamma_1 + \sum_{i=1}^{r} \prod_{j=1}^{i} \frac{\mu_j}{s+\mu_j} \beta_1 \dots \beta_i \gamma_{i+1}$$

If the service time has a Laplace transform which is a rational function in S_1, it can work with the BCMP theorem. Now we can have four types of nodes in the network.

Let $\Gamma_i(l)$ be external arrival rate of the L^{th} class to node i and $e_i(l)$ be total arrival rate of the L^{th} class to node i.

$$e_i(l) = \Gamma_i(l) + \sum_{j=1}^{N} e_j(l) r_{ji}(l)$$

	Server	Service Time	Service Discipline
Type 1	/M/1 single server	-	FCFS
Type 2	Single server	Stage type	Round robin
Type 3	Infinite number of servers	Stage type	-
Type 4	Single server	Stage type	Pre-emptive LCFS

Table 1: Four Types of Nodes in the Network

Let α_i be the state of queue I, which is not only the number of packets, but also what class, what service stage, etc.

$$P(\alpha_1\alpha_2\ldots\alpha_N) = c\cdot g(\alpha_1)g(\alpha_2)\cdots g(\alpha_N)$$

We can remove information regarding the stage of services the packets are in.

$$P(y_1y_2\ldots y_N) = c'\cdot f(y_1)f(y_2)\cdots f(y_N)$$

where

$$y_i = [n_{i1}, n_{i2}, \ldots, n_{il}]$$

If we remove class information, then

$$P(n_1n_2\ldots n_N) = c''\cdot h(n_1)h(n_2)\cdots h(n_N)$$

where $n_i = \sum_l n_{il}$, $c'' = \prod(1-\rho_i)$, and $h(n_i) = \rho_i^{n_i}$.

If a node is of **Type 1**,

$$f(y_i) = n_i!\prod_{l=1}^{L}\frac{1}{(n_{il})!}\cdot[e_i(l)]^{n_{il}}\cdot(\frac{1}{\mu_i})^{n_i}$$

Type 2 or 4

$$f(y_i) = n_i!\prod_{l=1}^{L}(\frac{1}{n_{il}})\cdot[\frac{e_i(l)}{\mu_{il}}]^{n_{il}}$$

Type 3

$$f(y_i) = \prod_{l=1}^{L}\frac{1}{(n_{il})!}\cdot[\frac{e_i(l)}{\mu_{il}}]^{n_{il}}$$

For the case of aggregated states,

$$P(k_1k_2\ldots k_N) = c''\cdot h(k_1)h(k_2)\cdots h(k_N)$$

For **Type 1**,

$$h(k_i) = \Big(\sum_l \frac{e_i(l)}{\mu_i}\Big)^{k_i}$$

For **Type 2 or 4**

$$h(k_i) = \Big(\sum_l \frac{e_i(l)}{\mu_{il}}\Big)^{k_i}$$

For **Type 3**

$$h(k_i) = \frac{1}{k_i!}\Big(\sum_l \frac{e_i(l)}{\mu_{il}}\Big)^{k_i}$$

Basically, we want find c' and c''. For open networks, we can find them.

$$P(k_1 k_2 \dots k_N) = P(k_1)P(k_2)\cdots P(k_N)$$

$$\text{where,} \quad P(k_i) = \begin{cases} (1-\rho_i)\,\rho_i^{k_i} & \text{for Type 1, 2, and 4} \\ \dfrac{\rho_i^{k_i}\,e^{-\rho_i}}{k!} & \text{for Type 3} \end{cases}$$

$$\text{where,} \quad \rho_i = \begin{cases} \sum_l \dfrac{e_i(l)}{\mu_i} & \text{for Type 1} \\ \sum_l \dfrac{e_i(l)}{\mu_{il}} & \text{for Type 2, 3, and 4} \end{cases}$$

15.1 Verifying BCMP for Open Networks

We assume that there are only one class of packets and Type 1 nodes. Let service rate at node i be μ_i and number of nodes be N.

$$P(\text{packets go from node } i \text{ to } j) = q_{ij}$$

$$P(\text{packets leave systems at node } i) = q_{i(N+1)}$$

$$\sum_{j=1}^{N+1} q_{ij} = 1$$

Let λ_i be external arrival rate to node i and Λ_i be total arrival rate at node i.

$$\Lambda_i = \lambda_i + \sum_{j=1}^{N} \Lambda_j\, q_{ji} \tag{75}$$

Flow out of state $(k_1 k_2 \dots k_N)$ is

$$P(k_1 k_2 \dots k_N)\Big[\underbrace{\sum_{i=1}^{N} \lambda_i}_{\text{external arrival}} + \underbrace{\sum_{i=1}^{N+1} \mu_i\,(1-q_{ii})}_{\text{departure from node to another}}\Big]$$

Flow into state $(k_1 k_2 \dots k_N)$ is

$$\textbf{(1)} \sum_{i=1}^{N} P(k_1 \dots (k_i-1) \dots k_N)\,\lambda_i$$

$$\textbf{(2)} + \sum_{i=1}^{N} P(k_1 \dots (k_i+1) \dots k_N)\,\mu_i\, q_{i,N+1}$$

$$\textbf{(3)} + \sum_{i=1}^{N}\sum_{j=1}^{N} P(k_1 \dots (k_i-1) \dots (k_j+1) \dots k_N)\,\mu_i\, q_{ij} \tag{76}$$

(1) external arrival

(2) departure from the systems

(3) packet from one node went to another

Eq. 76 is a global balance equation.

Local Balance Equations:
We consider arrival and departure at only one node.

$$P[\text{departure from node } i] = P[\text{arrival to node } i]$$

$$P(k_1 k_2 \dots k_N)\, \mu_i\, (1 - q_{ii}) = P(k_1 \dots (k_i - 1) \dots k_N)\, \lambda_i + \sum_{j=1,\, j\neq i}^{N} P(k_1 \dots (k_i - 1) \dots (k_j + 1) \dots k_N)\, \mu_i\, q_{ij} \quad (77)$$

If BCMP theorem holds,

$$P(k_1 \dots k_N) = \prod_{i=1}^{N} (1 - R_i)\, R_i^{k_i} \quad \text{where} \quad R_i = \frac{\Lambda_i}{\mu_i}$$

$$P(k_1 \dots (k_i - 1) \dots k_N) = P(k_1 k_2 \dots k_N) \cdot \frac{1}{R_i}$$

$$P(k_1 \dots (k_i - 1) \dots (k_j + 1) \dots k_N) = P(k_1 k_2 \dots k_N) \cdot \frac{R_j}{R_i}$$

We substitute these into Eq. 77 to have

$$\cancel{P(k_1 \dots k_N)}\, \mu_i\, (1 - q_{ii}) = \cancel{P(k_1 \dots k_N)}\, \frac{\lambda_i}{R_i} + \sum_{j=1,\, j\neq i}^{N} \cancel{P(k_1 \dots k_N)}\, \mu_i\, q_{ij} \cdot \frac{R_j}{R_i}$$

$$\mu_i (1 - q_{ii}) = R_i^{-1} \lambda_i + \sum_{j=1,\, j\neq i}^{N} q_{ji}\, \mu_j\, R_j\, R_i^{-1}$$

$$\cancel{\mu_i} (1 - q_{ii}) = \frac{\cancel{\mu_i}}{\Lambda_i} \lambda_i + \sum_{j=1,\, j\neq i}^{N} q_{ji}\, \cancel{\mu_j}\, \frac{\Lambda_j}{\cancel{\mu_j}}\, \frac{\cancel{\mu_i}}{\Lambda_i}$$

$$\Lambda_i (1 - q_{ii}) = \lambda_i + \sum_{j=1,\, j\neq i}^{N} \Lambda_j\, q_{ji}$$

$$\Lambda_i = \lambda_i + \sum_{j=1}^{N} \Lambda_j\, q_{ji} \quad \leftarrow \textbf{Identity}$$

It is an important note that the last equation is the identity equation of open networks (Eq. 75); therefore, we succesfully verify BCMP theorem for open networks of queues.

15.2 Verifying BCMP for Closed Networks

$$P(k_1 \ldots k_N) = \frac{1}{G(k)} \prod_{i=1}^{N} \rho_i^{k_i}$$

$$\text{Total arrival rate to node } i: \quad e_i = \sum_{j=1}^{N} e_j \, q_{ji} \tag{78}$$

Note that these equations cannot be solved because of linear dependency. Now, let's take a look at local balance equations.

$$P[\text{departure from node } i] = P[\text{arrival to node } i]$$

$$P(k_1 \ldots k_N) \, \mu_i \, (1 - q_{ii}) = \sum_{\substack{j=1 \\ j \neq i}}^{N} P(k_1 \ldots (k_i - 1) \ldots (k_j + 1) \ldots k_N) \, \mu_j \, q_{ji} \tag{79}$$

If BCMP theorem holds,

$$\rho_i = \frac{e_i}{\mu_i}$$

$$\begin{aligned} P(k_1 k_2 ... (k_i - 1) ... (k_j + 1) ... k_N) &= \frac{1}{G(k)} \cdot \rho_1^{k_1} \cdot \rho_2^{k_2} \cdots \rho_i^{k_i - 1} \cdots \rho_j^{k_j + 1} \cdots \rho_N^{k_N} \\ &= \frac{\rho_j}{\rho_i} \, P(k_1 \ldots k_N) \end{aligned}$$

We substitute this into Eq 79.

$$\cancel{P(k_1 \ldots k_N)} \, \mu_i \, (1 - q_{ii}) = \sum_{\substack{j=1 \\ j \neq i}}^{N} \cancel{P(k_1 \ldots k_N)} \, \frac{\rho_j}{\rho_i} \, \mu_j \, q_{ji}$$

$$\cancel{\mu_i} \, (1 - q_{ii}) = \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{e_j}{\cancel{\mu_j}} \, \frac{\cancel{\mu_i}}{e_i} \, \cancel{\mu_j} \, q_{ji}$$

$$e_i \, (1 - q_{ii}) = \sum_{\substack{j=1 \\ j \neq i}}^{N} e_j \, q_{ji}$$

$$e_i = \sum_{j=1}^{N} e_j \, q_{ji} \quad \leftarrow \textbf{Identity}$$

Again, we arrive at the identity equation (Eq. 78) of closed queueing networks; therefore, BCMP passes the verification.

15.2.1 Finding G(k)

$$\sum_{n_1}\sum_{n_2}\cdots\sum_{n_N} P(n_1 \dots n_N) = 1$$

$$\sum_{n_1}\sum_{n_2}\cdots\sum_{n_N} \frac{1}{G(k)} \cdot \prod_{i=1}^{N} \rho_i^{n_i} = 1$$

$$G(k) = \sum_{n_1}\sum_{n_2}\cdots\sum_{n_N} \prod_{i=1}^{N} \rho_i^{n_i}$$

Let's define

$$g_i(z) = \sum_{n_i=0}^{\infty} \rho_i^{n_i}\, z^{n_i}$$

Note that ρ_i is not probability, so $g(z)$ is not PGF. Also, we have

$$\begin{aligned} g(z) &= \prod_{i=1}^{N} g_i(z) \\ &= \prod_{i=1}^{N} \sum_{n_i=0}^{\infty} \rho_i^{n_i}\, z^{n_i} \end{aligned}$$

Therefore,

$$g(z) = \underbrace{\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_N=0}^{\infty}}_{=\text{large}\sum} \rho_1^{n_1}\rho_2^{n_2}\cdots\rho_N^{n_N}\; z^{(n_1+n_2+\cdots+n_N)}$$

Note that $n_1, \dots, n_N$ are independent. As a result, we can represent $g(z)$ as

$$g(z) = \sum_{n_1+\ldots+n_N=0} \prod_{i=1}^{N} \rho_i^{n_i}\, z^0 + \sum_{n_1+\ldots+n_N=1} \prod_{i=1}^{N} \rho_i^{n_i}\, z^1 + \cdots + \underbrace{\sum_{n_1+\ldots+n_N=k} \prod_{i=1}^{N} \rho_i^{n_i}}_{G(k)}\, z^k + \cdots\infty$$

We may notice that the coefficient of z^k in $g(z)$ is $G(k)$ and recall that

$$P(n_1 \dots n_N) = \frac{1}{G(k)} \cdot \prod_{i=1}^{N} \rho_i^{n_i}$$

Now let us define

$$\gamma_i(z) = \prod_{j=1}^{i} g_j(z)$$

so that

$$g(z) = \gamma_N(z)$$

and

$$\gamma_i(z) = \gamma_{i-1}(z) \cdot g_i(z)$$

Also, recall that

$$g_i(z) = \sum_{n_i=0}^{\infty} (\rho_i z)^{n_i} = \frac{1}{1-\rho_i z}$$

Therefore,

$$\begin{aligned} \gamma_i(z) &= \gamma_{i-1}(z) \cdot \frac{1}{1-\rho_i z} \\ \gamma_i(z) - \rho_i \gamma_i(z) \cdot z &= \gamma_{i-1}(z) \\ \underbrace{\gamma_i(z)}_{\text{polynomial}} &= \underbrace{\rho_i\, z\, \gamma_i(z) + \gamma_{i-1}(z)}_{\text{polynomial}} \end{aligned}$$

From the equation above, we have equal polynomials, so the coefficients should be equal as well. Let $\mathcal{G}_i(j)$ be the coefficient of z^j in $\gamma_i(z)$ and by equating the coefficients of z^j on both sides of the equation, we have

$$\mathcal{G}_i(j) = \rho_i \mathcal{G}_i(j-1) + \mathcal{G}_{i-1}(j)$$

Observe that a coefficient depends on previous ones $\mathcal{G}_i(j-1), \mathcal{G}_{i-1}(j)$. Finally, we can have $\mathcal{G}_N(k)$, which is $G(k)$, the coefficient of z^k in $\gamma_N(z)$.

To find initial values, remember that $\mathcal{G}_1(j)$ is the coefficient of z^j in $\gamma_1(z)$. Since

$$\gamma_1(z) = g_1(z) = \sum_{n_1=0}^{\infty} \rho_1^{n_1} z^{n_1}$$

,

$$\mathcal{G}_1(j) = \rho_1^j$$

And because $\mathcal{G}_i(0)$ is the coefficient of z^0 in $\gamma_i(z)$ and

$$g_i(z) = \sum_{n_i=0}^{\infty} \rho_i^{n_i}\, z^{n_i}, \quad \gamma_i(z) = \prod_{j=1}^{i} g_j(z)$$

$$\mathcal{G}_i(0) = \rho_1^0 \cdot \rho_2^0 \cdots \rho_i^0 = 1$$

we have that

$$\mathcal{G}_1(0) = 1, \quad \mathcal{G}_2(0) = 1, \ldots, \mathcal{G}_i(0) = 1,\ \forall i$$

$$\mathcal{G}_1(1) = \rho_1, \quad \mathcal{G}_1(2) = \rho_1^2, \ldots, \mathcal{G}_1(j) = \rho_1^j,\ \forall j$$

$$\mathcal{G}_2(1) = \rho_2 \mathcal{G}_2(0) + \mathcal{G}_1(1) = \rho_2 + \rho_1$$

$$\mathcal{G}_2(2) = \rho_2 \mathcal{G}_2(1) + \mathcal{G}_1(2) = \rho_2(\rho_2 + \rho_1) + \rho_1^2$$

$$\mathcal{G}_2(3) = \rho_2 \mathcal{G}_2(2) + \mathcal{G}_1(3) = \rho_2(\rho_2(\rho_2 + \rho_1) + \rho_1^2) + \rho_1^3$$

$$\mathcal{G}_3(1) = \rho_3 \mathcal{G}_3(0) + \mathcal{G}_2(1) = \rho_3 + (\rho_2 + \rho_1)$$

$$\mathcal{G}_3(2) = \rho_3 \mathcal{G}_3(1) + \mathcal{G}_2(2) = \rho_3(\rho_3 + \rho_2 + \rho_1) + \rho_2(\rho_2 + \rho_1) + \rho_1^2$$

$$\mathcal{G}_3(3) = \rho_3\mathcal{G}_3(2) + \mathcal{G}_2(3) = \rho_3(\rho_3(\rho_3 + \rho_2 + \rho_1) + \rho_2(\rho_2 + \rho_1) + \rho_1^2) + \rho_2(\rho_2(\rho_2 + \rho_1) + \rho_1^2) + \rho_1^3$$

we can repeatedly calculate $\mathcal{G}_i(j)$ until we find $\mathcal{G}_N(k) = G(k)$.

Now, let us take a look at the marginal probability

$$\begin{aligned} P(n_i \geq j) &= \sum_{n_1}\sum_{n_2}\cdots\sum_{n_N} P(n_1 \ldots n_N) \\ &\quad {\scriptstyle n_i \geq j,\; n_1+n_2+\ldots+n_N=k} \\ &= \frac{1}{G(k)} \cdot \underbrace{\sum_{n_1}\sum_{n_2}\cdots\sum_{n_N}}_{\substack{n_i \geq j \\ n_1+n_2+\ldots+n_N=k}} \prod_{i=1}^{N} \rho_i^{n_i} \end{aligned} \tag{80}$$

Note that without the condition, the summations are equal to $G(k)$. To compute an n-fold summation of the equation above, consider the following.

$$\tilde{g}(z) = \frac{g(z)}{g_i(z)}\left[g_i(z) - \sum_{n_i=1}^{j-1} \rho_i^{n_i}\, z^{n_i}\right]$$

This polynimial is essentially

$$\sum_{\substack{\forall\, n_1+n_2+\ldots+n_N=j \\ n_i \geq j}} \prod_{i=1}^{N} \rho_i^{n_i}\, z^j + \sum_{\substack{\forall\, n_1+n_2+\ldots+n_N=j+1 \\ n_i \geq j}} \prod_{i=1}^{N} \rho_i^{n_i}\, z^{j+1} + \cdots$$

which means that queue in the network is at least j.

$$\tilde{g}(z) = \frac{g(z)\left[\frac{1}{\cancel{1-\rho_i z}} - \frac{1-(\rho_i z)^j}{\cancel{1-\rho_i z}}\right]}{\frac{1}{\cancel{1-\rho_i z}}}$$

$$\tilde{g}(z) = g(z)\,\rho_i^j\, z^j$$

The coefficient of z^k is $\mathcal{G}_N(k-j)\cdot\rho_i^j$; thus,

$$P(n_i \geq j) = \frac{\mathcal{G}_N(k-j)\cdot\rho_i^j}{\mathcal{G}_N(k)}$$

So,

$$\underbrace{P(n_i \geq 1)}_{\text{queue } i \text{ is not empty}} = \frac{\mathcal{G}_N(k-1)\cdot\rho_i}{\mathcal{G}_N(k)} = u_i$$

where u_i is absolute utilization at queue i. Recall that for M/M/1, $\rho = 1 - P_0$ is the probability that Q is non-empty.

$$\begin{aligned} \text{Absolute utilization at queue } i &= u_i \\ \text{Absolute arrival rate to queue } i &= u_i\mu_i = \lambda_i \\ \text{Relative utilization at queue } i &= \rho_i \\ \text{Relative arrival rate to queue } i &= e_i \end{aligned}$$

Recall that

$$e_i = \sum_{j=1}^{N} e_j \, r_{ji} \quad , \quad e_1 = 1 \quad , \quad \rho_i = \frac{e_i}{\mu_i}$$

Steps to find the solution are as follows.

1. Find e_i by fixing one e_i and find other e_i
2. Find ρ_i
3. Find $\mathcal{G}_N(k)$ by using recursion
4. Find u_i
5. Find λ_i

$$\begin{aligned}
P(n_i \geq 1) &= P(n_i = 1) + P(n_i = 2) + \cdots + P(n_i = k) \\
&+ \\
P(n_i \geq 2) &= P(n_i = 2) + P(n_i = 3) + \cdots + P(n_i = k) \\
&+ \\
P(n_i \geq 3) &= P(n_i = 3) + P(n_i = 4) + \cdots + P(n_i = k) \\
&+ \\
&\vdots \\
P(n_i \geq k) &= P(n_i = k)
\end{aligned}$$

Therefore,

$$\sum_{j=1}^{k} P(n_i \geq j) = \sum_{j=1}^{N} j \, P(n_i = j) = E[n_i]$$

which is the expected number of queues in each node. To find the expected delay, we use Little's Law.

$$E[D_i] = \frac{E[n_i]}{\lambda_i}$$

16 M/G/1 Queue

M is for Markovian memoryless arrival process and G is for general distribution. Now we lost assumption about memoryless exponential distribution of service process.

Imbedded Markov Chain

Now we only consider departure instances.

Figure 34: Imbedded Markov Chain

Let n_i be the number of packets in the queue at the i^{th} packet departure instance.

$$n_{i+1} = n_i - 1 + a_{i+1} \quad \text{if} \quad n_i > 0 \tag{81}$$

where a_{i+1} is the number of arrivals in the $(i+1)^{th}$ service time. Note that this equation is true if $n_i > 0$ or left queue non-empty.

$$n_{i+1} = a_{i+1} \quad \text{if} \quad n_i = 0 \tag{82}$$

Let

$$U(n_i) = \begin{cases} 0 & \text{if} \quad n_i \leq 0 \\ 1 & \text{otherwise} \end{cases}$$

We now can combine Eq. 81 and 82 to get.

$$n_{i+1} = n_i - U(n_i) + a_{i+1} \tag{83}$$

At equilibrium, we have

$$E[n_{i+1}] = E[n_i]$$

By taking expectation on Eq. 83

$$\begin{aligned} \cancel{E[n_{i+1}]} &= \cancel{E[n_i]} - E[U(n_i)] + E[a_{i+1}] \\ E[U(n_i)] &= E[a_{i+1}] \end{aligned}$$

Let's see $E[U(n_i)]$

$$\begin{aligned} E[U(n_i)] &= 0 \cdot P[n_i = 0] + 1 \cdot P[n_i > 0] \\ &= P[n_i > 0] \\ &= 1 - P[n_i = 0] \\ &= 1 - P_0 \end{aligned}$$

Let's take a look at $E[a_{i+1}]$ by introducing $m(t)$. Let $m(t)$ be the pdf of the service time or message transmission time and $\mathcal{M}(s)$ be the Laplace transform of $m(t)$.

$$\mathcal{M}(s) = \int_0^\infty e^{-st}\, m(t)\, dt$$

$$\begin{aligned}
E[a_{i+1}] &= \int_0^\infty E[a_{i+1}|\text{S.T. } = t] \cdot m(t)\, dt \\
&= \int_0^\infty \lambda\, t \cdot m(t)\, dt \\
&= \lambda \int_0^\infty t\, m(t)\, dt \quad \rightarrow \text{expected msg length or service time} \\
&= \lambda\, \bar{m} \\
1 - P_0 &= \lambda\, \bar{m} = \rho
\end{aligned}$$

Therefore,

$$P_0 = 1 - \rho$$

Now, we look at how to find expected number of packets in the queue.

$$P_i(z) = E[z^{n_i}]$$

At equilibrium, $P_i(z) = P_{i+1}(z)$.

$$\begin{aligned}
P_{i+1}(z) &= E[z^{n_{i+1}}] \\
&= E[z^{n_i - U(n_i) + a_{i+1}}] \\
&= E[z^{n_i - U(n_i)} \cdot z^{a_{i+1}}] \\
&= E[z^{n_i - U(n_i)}] \cdot E[z^{a_{i+1}}]
\end{aligned}$$

Let's take a look at the first term.

$$E[z^{n_i - U(n_i)}] = P_0 + \sum_{k=1}^{\infty} z^{k-1} \cdot P_k$$

$$\text{where} \quad P_k = P(n_i = k)$$

$$\begin{aligned}
&= P_0 + \frac{1}{z} \sum_{k=1}^{\infty} z^k \cdot P_k \\
&= P_0 + \frac{1}{z}[P(z) - P_0]
\end{aligned}$$

Next we let $E[z^{a_i+1}] = A(z)$; therefore,

$$\begin{aligned}
P(z) &= \Big[P_0 + \frac{1}{z}[P(z) - P_0]\Big] \cdot A(z) \\
z\,P(z) &= P_0\,z\,A(z) + P(z)\,A(z) - P_0\,A(z) \\
P_0\Big[A(z)\,[1-z]\Big] &= P(z)[A(z) - z] \\
P(z) &= \frac{P_0\Big[A(z)\,[1-z]\Big]}{[A(z)-z]} \\
&= \frac{(1-\rho)\Big[A(z)\,[1-z]\Big]}{[A(z)-z]}
\end{aligned}$$

Subscript of $P(z)$ disappears because of stationarity. Now, we want to find $P'(1)$, so let us go back to the equation.

$$P(z)[A(z) - z] = (1-\rho)\Big[A(z)\,[1-z]\Big]$$

Differentiate both sides with respect to z.

$$P'(z)[A(z) - z] + P(z)[A'(z) - 1] = (1-\rho)[A'(z)(1-z)] + (1-\rho)[A(z) - 1]$$

Differentiate again

$$\begin{aligned}
P''(z)[A(z) - z] + P'(z)[A'(z) - 1] + P'(z)[A'(z) - 1] + P(z)[A''(z)] \\
= (1-\rho)\Big[[A''(z)(1-z)] + A'(z)(-1) - A'(z)\Big]
\end{aligned}$$

Now, we put $z = 1$ and simplify to have (note that $A(1) = 1$)

$$P'(1) = \bar{n} = \frac{(1-\rho)A'(1)}{1 - A'(1)} + \frac{A''(1)}{z[1 - A'(1)]} \tag{84}$$

Now, we want to know what $A(z)$ is?

$$\begin{aligned}
A(z) = E[z^a] &= \int_0^\infty \overbrace{E[z^a|\text{msg tx time} = t]}^{\text{PGF of Poisson process}}\; m(t)\,dt \\
&= \int_0^\infty e^{-\lambda t(1-z)}\; m(t)\,dt \\
&= \mathcal{M}\big(\lambda(1-z)\big)
\end{aligned} \tag{85}$$

where $\mathcal{M}\big(\lambda(1-z)\big)$ is the Laplace transform of message transmission time $m(t)$ computed at $s = \lambda(1-z)$. Now, we need to differentiate Eq. 85 with respect to z.

$$\begin{aligned}
A'(z)\,|_{z=1} &= \mathcal{M}'(\lambda(1-z))(-\lambda)\,|_{z=1} \\
&= -\bar{m} \cdot (-\lambda) \\
&= \lambda \cdot \bar{m} = \rho \\
A''(z)\,|_{z=1} &= \mathcal{M}''(\lambda(1-z))(\lambda^2)\,|_{z=1} \\
&= \bar{m^2} \cdot \lambda^2
\end{aligned}$$

Note that $\mathcal{M}'(0) = -\bar{m}$ and $\mathcal{M}''(0) = \bar{m^2}$, where $\bar{m^2}$ is the second moment of message transmission time. We put all these in Eq. 84.

$$\bar{n} = \rho + \frac{\lambda^2 \bar{m^2}}{2(1-\rho)}$$

Therefore, we have expected delay as.

$$\bar{d} = \frac{\bar{n}}{\lambda} = \bar{m} + \frac{\lambda \bar{m^2}}{2(1-\rho)} \tag{86}$$

where the first term is message transmission delay and the second is queueing delay. This equation is called **Pollackzek Khinchin Formula for M/G/1** or abbreviated **PK Formula**. Now, we can check the correctness of this M/G/1 formular by applying it to M/M/1 (memoryless exponential). For M/M/1 queue,

$$\bar{m} = \frac{1}{\mu} \qquad \bar{m^2} = \frac{2}{\mu^2}$$

Therefore,

$$\begin{aligned}
\bar{n} &= \frac{\lambda}{\mu} + \frac{\lambda^2 \cdot 2}{\mu^2 \cdot 2 \cdot (1-\rho)} \\
&= \rho + \frac{\rho^2}{1-\rho} \\
&= \frac{\rho}{1-\rho}
\end{aligned}$$

16.1 Delay in M/G/1 Queue

Figure 35: Delay in M/G/1 queue

$$\begin{aligned}
P(z) = E[z^n] &= \int_0^\infty E[z^n|\text{delay} = t] \cdot d(t)\, dt \\
&= \int_0^\infty e^{-\lambda t(1-z)}\, d(t)\, dt \\
&= \mathcal{D}(\lambda(1-z))
\end{aligned}$$

where $d(t)$ is pdf of the delay and $\mathcal{D}$ is its Laplace transform. Note that the term $e^{-\lambda t(1-z)}$ comes because when i^{th} message looks back to the queue, it is the total packets left behind, which is

Poisson.

$$
\begin{aligned}
P(z) = \mathcal{D}(\lambda(1-z)) &= \frac{(1-\rho)(1-z)A(z)}{A(z)-z} \\
&= \frac{(1-\rho)(1-z)\cdot\mathcal{M}(\lambda(1-z))}{\mathcal{M}(\lambda(1-z))-z}
\end{aligned}
$$

Let $s = \lambda(1-z)$; therefore, $(1-z) = \dfrac{s}{\lambda}$ and $z = 1 - \dfrac{s}{\lambda}$. Consequentially, we have

$$\mathcal{D}(s) = \frac{s\,(1-\rho)\,\mathcal{M}(s)}{s-\lambda+\lambda\,\mathcal{M}(s)}$$

As we know that

$$D = M + Q$$

$$\mathcal{D}(s) = \mathcal{M}(s)\cdot\mathcal{Q}(s)$$

where $\mathcal{Q}(s)$ is the Laplace transform of the queueing delay. It is easy to see that

$$\mathcal{Q}(s) = \frac{s\,(1-\rho)}{s-\lambda+\lambda\,\mathcal{M}(s)}$$

Exercise If we differentiate $\mathcal{D}(s)$ twice and put $s = 0$, we will get the PK formula.

17 Examples of M/G/1 Queue

In this section we have a couple of examples to which we can apply M/G/1 queue to solve problems.

17.1 Stop and Wait Transmission

Figure 36: Stop and Wait Transmission

Let $B(t)$ be the distribution of the acknowledgement (ack) delay.

$$B(t) = P(\text{delay } \leq t) \qquad b(t) = \frac{d}{dt}\,B(t)$$

Let Q be the probability that packets or acks are not lost.

$$P(\text{receiving an ack before timeout}) = Q\cdot P(\text{delay } \leq t) = Q\cdot B(t) = G$$

$$P(n \text{ transmissions before success}) = (1-G)^{n-1} \cdot G$$

If there are n retransmissions,

$$\underbrace{n \cdot m}_{\text{L.T. } = e^{-smn}} + \underbrace{(n-1) \cdot T}_{\text{L.T. } = e^{-s(n-1)T}} + \underbrace{R}_{\text{time it took for ack}}$$

Note for a fixed length message, L.T. is $\int_0^\infty e^{-smn} \delta(m) dt = e^{-smn}$.

$$P[\text{delay } \leq t | \overbrace{\text{delay } \leq T}^{\text{finally, an ack comes}}] = \frac{P[\text{delay } \leq t \cup \text{ delay } \leq T]}{P[\text{delay } \leq T]} = \frac{B(t)}{B(T)}$$

Thus,

$$R(s) = \frac{1}{B(t)} \int_0^T e^{-st}\, b(t)\, dt$$

$$\mathcal{M}(s \mid n \text{ retransmissions}) = e^{-snm} \cdot e^{-s(n-1)T} \cdot R(s)$$

$$\mathcal{M}(s) = \sum_{n=1}^{\infty} e^{-snm} \cdot e^{-s(n-1)T} \cdot R(s) \cdot (1-G)^{n-1} \cdot G$$

If we assume that ack takes exactly T seconds,

$$b(t) = \delta(t-T) \qquad \text{then,} \qquad R(s) = e^{-sT}$$

$$\begin{aligned}
\mathcal{M}(s) &= \sum_{n=1}^{\infty} e^{-sn(m+T)} \cdot (1-G)^{n-1} \cdot G \\
&= G\, e^{-s(m+T)} \sum_{n=1}^{\infty} e^{-s(m+1)(n-1)} (1-G)^{(n-1)} \\
&= \frac{G\, e^{-s(m+T)}}{1 - \big((1-G)\, e^{-s(m+T)}\big)}
\end{aligned}$$

17.2 Initiator of a Busy Period

Again, we can combine Eq. 81 and 82 to have

$$n_{i+1} = n_i - U(n_i) + U(n_i) \cdot a_{i+1} + \big(1 - U(n_i)\big) \cdot \tilde{a}_{i+1}$$

where a is the number of arrivals when Q is non-empty and $\tilde{a}$ is the number of arrivals when Q is empty.

$$n_{i+1} = n_i - U(n_i) + \tilde{a}_{i+1} + U(n_i)[a_{i+1} - \tilde{a}_{i+1}]$$

Now, let us take expectation on both sides of the equation and because of stationarity, $E[n_{i+1}] = E[n_i]$ and $E[U(n_i)] = 1 - P_0$. Therefore,

$$1 - P_0 = \frac{E[\tilde{a}]}{1 - E[a] + E[\tilde{a}]}$$

where $E[a] = \lambda\bar{m}$ and $E[\tilde{a}] = \lambda\bar{\tilde{m}}$. Note that $m(t)$ is service time or message transmission time for Q non-empty and $\tilde{m}(t)$ is when Q is empty.

$$\begin{aligned}
P(z) = E[z^{n_{i+1}}] &= E[z^{n-U(n)+a(U(n))+\tilde{a}(1-U(n))}] \\
&= P_0 E[z^{\tilde{a}}] + \sum_{i=1}^{\infty} P(n=i) z^{i-1} \cdot E[z^a] \\
&= P_0 \tilde{A}(z) + z^{-1}\big[[P(z) - P_0]A(z)\big] \\
&= \frac{P_0[A(z) - \tilde{A}(z)]}{A(z) - z}
\end{aligned}$$

18 Busy Periods of M/G/1 Queue

Figure 37: Busy Periods of M/G/1 Queue

We finally want to find expected busy periods but now let us introduce the assumptions.

1. FIFO services
2. 1^{st} generation message $\rightarrow$ Initiator of a busy period (A_1)
   2^{nd} generation message $\rightarrow$ Arrive during service time of 1^{st} gen (A_2)
   3^{rd} generation message $\rightarrow$ Arrive during service time of 2^{nd} gen (A_3)
   and so on...

Let $\bar{N}_i$ be E[number of msg $\in i^{th}$ generation]

$$\begin{aligned}
\bar{N}_1 &= 1 \\
\bar{N}_2 &= \int_0^\infty E[a \mid t]\, m(t)\, dt = \int_0^\infty \lambda t\, m(t)\, dt = \lambda \bar{m} = \rho \\
\bar{N}_3 &= \rho^2 = \bar{N}_2 \cdot \rho \\
&\vdots \\
\bar{N}_i &= \rho^{i-1}
\end{aligned}$$

Therefore,

$$E[\text{num of msg in BP}] = \sum_{i=1}^{\infty} \bar{N}_i = \sum_{i=1}^{\infty} \rho^{i-1} = \frac{1}{1-\rho}$$

and

$$E[\bar{BP}] = E[\text{msg length}] \cdot E[\text{num of msg}] = \frac{\bar{m}}{1-\rho}$$

18.1 Distribution of Busy Periods

Let pdf or density of the BP be $b(t)$ where each message starts its own BP.

Figure 38: Distribution of Busy Periods

$$\begin{aligned}
b(t)dt &= P[t \leq BP \leq t + dt] \\
&= \sum_{n=0}^{\infty} P[t \leq BP \leq t + dt, n \text{ packets in 2nd gen}] \\
&= \int_0^\infty \sum_{n=0}^{\infty} P[u \leq m \leq u + du,\ n \text{ packets in 2nd gen},\ t \leq BP \leq t + dt]
\end{aligned}$$

Now, let us break up the $P[\cdots]$ into three parts.

$$
\begin{aligned}
&P[u \le m \le u+du,\ n \text{ packets in 2nd gen},\ t \le BP \le t+dt] \\
&\qquad = P[t \le BP \le t+dt \mid n \text{ packets in 2nd gen},\ u \le m \le u+du] \\
&\qquad\qquad \cdot \underbrace{P[n \text{ packets in 2nd gen} \mid u \le m \le u+du]}_{\text{Poisson}=\frac{e^{-\lambda u}(\lambda u)^n}{n!}} \cdot \underbrace{P[u \le m \le u+du]}_{m(u)du}
\end{aligned}
$$

Figure 39: $P[t \le BP \le t+dt \mid n$ packets in 2nd gen, $u \le m \le u+du]$

Therefore,

$$b(t)dt = \sum_{n=0}^{\infty} \int_0^t m(u)du \cdot \frac{e^{-\lambda u}(\lambda u)^n}{n!} \cdot b^{(n)}(t-u)dt$$

Now, take Laplace transform to both sides of the equation.

$$
\begin{aligned}
B(s) = \int_0^\infty e^{-st} b(t)dt &= \int_0^\infty e^{-st} \int_0^t m(u)du \sum_{n=0}^{\infty} \frac{e^{-\lambda u}(\lambda u)^n}{n!} \cdot b^{(n)}(t-u)dt \\
&= \int_0^\infty \int_u^\infty e^{-st} m(u)du \sum_{n=0}^{\infty} \frac{e^{-\lambda u}(\lambda u)^n}{n!} \cdot b^{(n)}(t-u)dt \\
&= \int_0^\infty m(u)du \sum_{n=0}^{\infty} \frac{e^{-\lambda u}(\lambda u)^n}{n!} \cdot \int_u^\infty e^{-st} b^{(n)}(t-u)dt
\end{aligned}
$$

Let $t-u=x$ if $t=u$, $x=0$ and if $t=\infty$, $x=\infty$.

$$
\begin{aligned}
B(s) &= \int_0^\infty m(u)du \sum_{n=0}^{\infty} \frac{e^{-\lambda u}(\lambda u)^n}{n!} \cdot \int_0^\infty e^{-s(u+x)} b^{(n)}(x)dx \\
&= \int_0^\infty e^{-su} m(u)du \sum_{n=0}^{\infty} \frac{e^{-\lambda u}(\lambda u)^n}{n!} \cdot \underbrace{\int_0^\infty e^{-sx} b^{(n)}(x)dx}_{B^n(s)} \\
&= \int_0^\infty e^{-u(s+\lambda)} m(u)du \underbrace{\sum_{n=0}^{\infty} \frac{(\lambda u)^n}{n!} \cdot B^n(s)}_{e^{\lambda u B(s)}} \\
&= \int_0^\infty e^{-u(s+\lambda-\lambda B(s))} m(u)du
\end{aligned}
$$

Therefore,

$$\begin{aligned}
B(s) &= \mathcal{M}(\lambda + s - \lambda B(s)) \\
B'(s) \mid_{s=0} &= \mathcal{M}'(s)(1 - \lambda B'(s)) \mid_{s=0} \\
B'(0) &= \mathcal{M}'(0)(1 - \lambda B'(0)) \\
\bar{BP} &= \bar{m}(1 - \lambda \bar{BP}) \\
\bar{BP} &= \frac{\bar{m}}{1 - \rho}
\end{aligned}$$

19 Priority Queue: Preemptive Resume

There are three types of priority queue

(1) Preemptive Priority: stop serving lower-priority messages when higher-priority messages come in

(1a) Preemptive Resume: preemptive priority that resumes serving lower-priority messages after all higher-priority messages left

(1b) Preemptive Non-Resume

(2) Non-Preemptive Priority

Note that (1b) incurs lost of work while (1a) and (2) do not. In this course, we focus only on pre-emptive resume priority queues, in which there are two classes of messages Class 1 and 2 with arrival rate λ_1 and λ_2 respectively. Class 1 has higher priority; therefore, Class 1 messages behave the same way as in M/G/1 queues with λ_1.

Figure 40: Pre-emptive Resume PQ

Therefore, we now focus on how to handle Type 2 messages, which have two cases.

- Arrival to a non-empty queue: as soon as previous packets leave, this packet is ushered into service

- Arrival to an empty queue: server may not be available, so there can be a waiting period with service time = waiting time + msg tx time

...

Figure 41: Class 2 Messages in PRPQ

$$n_{i+1} = n_i - 1 + a_{i+1} \quad \text{if } n_i > 0$$
$$n_{i+1} = \tilde{a}_{i+1} \quad \text{if } n_i = 0$$
$$P(z) = \frac{P_0[A(z) - zA(z)]}{A(z) - z}$$
$$P_0 = \frac{1 - A'(1)}{1 - A'(1) - \tilde{A}(1)}$$

19.1 Class 2 Msg Tx Time in PRPQ

...

Figure 42: Class 2 Msg Tx Time in PRPQ

Note that Type 1 messages have arrival rate λ_1 and message length m_1 and Type 2 has λ_2 and m_2. Therefore,

$$\text{msg tx time } = T = \sum_{i=1}^{n} F_i + m_2$$

Now, what is the distribution of n (interruptions by Class 1 arrivals)? The answer is a Poisson distribution with rate λ_1. Hence,

$$E[T] = \bar{F} \cdot \bar{n} + \bar{m_2} \quad \text{and} \quad \bar{n} = \lambda_1 \, \bar{m_2}$$

Therefore,

$$E[T] = \bar{F} \, \lambda_1 \, \bar{m_2} + \bar{m_2}$$

19.2 Distribution of Class 2 Msg Tx Time in PRPQ

$$P(t \le T \le t + dt) = T(t) \, dt$$

Now, we want to find $P(t \le T \le t + dt \mid n \text{ interruptions}, u \le m_2 \le u + du)$ and we should see $F_1 + F_2 + \ldots + F_n$ consume $t - u$ seconds. Therefore,

$$F_1 + F_2 + \ldots + F_n = f^{(n)}(t-u) \, dt \quad \text{(n-fold convolutions)}$$

Then, we can remove the condition on n interruptions by summation for $\forall n$.

$$P(t \le T \le t + dt \mid u \le m_2 \le u + du) = \sum_{n=0}^{\infty} \frac{e^{-\lambda_1 u}(\lambda_1 u)^n}{n!} \; f^{(n)}(t-u) \, dt$$

Next, we remove the other condition on msg length m_2.

$$T(t) \, dt = \int_0^t \sum_{n=0}^{\infty} \frac{e^{-\lambda_1 u}(\lambda_1 u)^n}{n!} \; f^{(n)}(t-u) \, m_2(u) \, du \, dt$$

By taking Laplace transform on both sides of the equation, we have

$$\begin{aligned}
\mathcal{T}(s) = \int_0^{\infty} e^{-st} \, T(t) \, dt &= \int_0^{\infty} e^{-st} \int_0^t \sum_{n=0}^{\infty} \frac{e^{-\lambda_1 u}(\lambda_1 u)^n}{n!} \; f^{(n)}(t-u) \, m_2(u) \, du \, dt \\
&= \int_0^{\infty} m_2(u) \, du \sum_{n=0}^{\infty} \frac{e^{-\lambda_1 u}(\lambda_1 u)^n}{n!} \int_u^{\infty} e^{-st} f^{(n)}(t-u) \, dt
\end{aligned}$$

Let $t - u = x$, so $t = u, x = 0$ and $t = \infty, x = \infty$.

$$\begin{aligned}
\mathcal{T}(s) &= \int_0^{\infty} m_2(u) \, du \sum_{n=0}^{\infty} \frac{e^{-\lambda_1 u}(\lambda_1 u)^n}{n!} \int_0^{\infty} e^{-s(u+x)} f^{(n)}(x) \, dx \\
&= \int_0^{\infty} e^{-su} \, m_2(u) \, du \sum_{n=0}^{\infty} \frac{e^{-\lambda_1 u}(\lambda_1 u)^n}{n!} \underbrace{\int_0^{\infty} e^{-sx} f^{(n)}(x) \, dx}_{\substack{\text{Laplace transform of} \\ F_1 + \cdots + F_n \\ = \mathcal{F}^n(s)}} \\
&= \int_0^{\infty} e^{-u(s+\lambda_1)} \, m_2(u) \, du \underbrace{\sum_{n=0}^{\infty} \frac{(\lambda_1 u)^n \; \mathcal{F}^n(s)}{n!}}_{e^{\lambda_1 u \mathcal{F}(s)}} \\
&= \int_0^{\infty} e^{-u\left(s + \lambda_1 - \lambda_1 \mathcal{F}(s)\right)} \, m_2(u) \, du
\end{aligned}$$

As a result,

$$\mathcal{T}(s) = \mathcal{M}_2\big(s + \lambda_1 - \lambda_1 \mathcal{F}(s)\big)$$

And we know that

$$A(z) = \mathcal{T}\big(\lambda_2(1-z)\big)$$

The next step is to find $\tilde{A}(z)$, which is divided into two cases.

1. Type 2 messages come when Type 1 queue is empty
2. Type 2 messages come when Type 1 queue is non-empty

For case 1,

...

Figure 43: Type 2 Msg Arrive When Type 1 Queue Empty

The probability that waiting time = 0 is

$$P[w=0] = \sum_{n=0}^{\infty} \underbrace{P\Big[\sum_{i=0}^{n} A_i + F_i \le \tau \le \sum_{i=0}^{n} A_i + F_i + A_{n+1}\Big]}_{\boxed{1}}$$

Note that

$$P[\tau \le a+b] = 1 - e^{-\lambda_2(a+b)}$$

Therefore,

$$P[\tau > a+b] = e^{-\lambda_2(a+b)} = e^{-\lambda_2 a} \cdot e^{-\lambda_2 b} = P[\tau > a] \cdot P[\tau > b]$$

Now we want to find

$$\boxed{1} = \underbrace{P\Big[\tau \le \sum_{i=0}^{n} A_i + F_i + A_{n+1} \;\Big|\; \tau > \sum_{i=0}^{n} A_i + F_i\Big]}_{\substack{=P[\tau \le A_{n+1}] \\ \text{because of memoryless}}} \cdot \underbrace{P\Big[\tau > \sum_{i=0}^{n} A_i + F_i\Big]}_{\prod_{i=1}^{n} P(\tau > A_i) \cdot P(\tau > F_i)}$$

$$= \prod_{i=1}^{n} \underbrace{P(\tau > A_i)}_{\boxed{A}} \cdot \underbrace{P(\tau > F_i)}_{\boxed{B}} \cdot \underbrace{P[\tau \le A_{n+1}]}_{\boxed{C}}$$

$$\begin{aligned}
\boxed{A} = P[\tau > A_i] &= \int_0^\infty P[\tau > x]\, \lambda_1 e^{-\lambda_1 x} dx \\
&= \int_0^\infty \int_x^\infty \lambda_2 e^{-\lambda_2 t}\, \lambda_1 e^{-\lambda_1 x} dx\, dt \\
&= \int_0^\infty \lambda_1 \lambda_2 e^{-\lambda_1 x} dx \int_x^\infty e^{-\lambda_2 t}\, dt \\
&= \frac{\lambda_1}{\lambda_1 + \lambda_2}
\end{aligned}$$

$$\begin{aligned}
\boxed{C} = P[\tau \le A_{n+1}] &= 1 - \frac{\lambda_1}{\lambda_1 + \lambda_2} \\
&= \frac{\lambda_2}{\lambda_1 + \lambda_2}
\end{aligned}$$

$$\begin{aligned}
\boxed{B} = P[\tau > F_i] &= \int_0^\infty P[\tau > x] \cdot f(x) dx \\
&= \int_0^\infty \int_x^\infty \lambda_2 e^{-\lambda_2 t}\, dt \cdot f(x) dx \\
&= \int_0^\infty \cancel{\lambda_2} f(x) dx \cdot \Big[\frac{e^{-\lambda_2 t}}{-\cancel{\lambda_2}}\Big]_x^\infty \\
&= \int_0^\infty e^{-\lambda_2 x} f(x) dx \\
&= \mathcal{F}(\lambda_2)
\end{aligned}$$

As a result,

$$\boxed{1} = \Big[\prod_{i=1}^{n} \big(\frac{\lambda_1}{\lambda_1 + \lambda_2}\big) \cdot \mathcal{F}(\lambda_2)\Big] \cdot \frac{\lambda_2}{\lambda_1 + \lambda_2} = \Big(\frac{\lambda_1 \cdot \mathcal{F}(\lambda_2)}{\lambda_1 + \lambda_2}\Big)^n \cdot \frac{\lambda_2}{\lambda_1 + \lambda_2}$$

Finally, we have

$$\sum_{n=0}^{\infty} \boxed{1} = \sum_{n=0}^{\infty} \Big(\frac{\lambda_1 \cdot \mathcal{F}(\lambda_2)}{\lambda_1 + \lambda_2}\Big)^n \cdot \frac{\lambda_2}{\lambda_1 + \lambda_2} = \frac{\lambda_2}{\lambda_1 + \lambda_2 - \lambda_1 \cdot \mathcal{F}(\lambda_2)} = P(w = 0)$$

For Case 2, where $w > 0$, or Type 2 messages have to wait for Type 1 to finish transmitting.

...

Figure 44: Type 2 Msg Arrive When Type 1 Queue Non- Empty

$$
\begin{aligned}
P[t \le w \le t+dt] &= P[t \le \sum_{i=o}^{n+1}(A_i+F_i) - \tau \le t+dt] \\
&= \underbrace{P\big[t \le \sum_{i=o}^{n+1}(A_i+F_i) - \tau \le t+dt \,\big|\, \tau > (\sum_{i=1}^{n} A_i + F_i) + A_{n+1}\big]}_{\boxed{A}} \\
&\quad \cdot \underbrace{P\big[\tau > (\sum_{i=1}^{n} A_i + F_i) + A_{n+1}\big]}_{\boxed{B}}
\end{aligned}
$$

So, for $\boxed{B}$

$$
\begin{aligned}
\boxed{B} &= P(\tau > A_{n+1}) \cdot \Big\{ \prod_{i=1}^{n} P[\tau > A_i] \cdot P[\tau > F_i] \Big\} \\
&= \Big(\frac{\lambda_1}{\lambda_1+\lambda_2}\Big) \cdot \Big(\prod_{i=1}^{n} \frac{\lambda_1}{\lambda_1+\lambda_2} \mathcal{F}(\lambda_2) \Big) \\
&= \Big(\frac{\lambda_1}{\lambda_1+\lambda_2}\Big) \cdot \Big(\frac{\lambda_1\, \mathcal{F}(\lambda_2)}{\lambda_1+\lambda_2} \Big)^n
\end{aligned}
$$

And for $\boxed{A}$,

$$
\begin{aligned}
\boxed{A} &= P[t \le F_{n+1} - \tau \le t+dt] \\
&= P[t+\tau \le F_{n+1} \le t+\tau+dt] \\
&= f(t+\tau)dt \\
&= \int_0^\infty f(t+r)dt\ \lambda_2 e^{-\lambda_2 r} dr
\end{aligned}
$$

Combining these things together, we have

$$
\begin{aligned}
w(t)dt &= \sum_{n=0}^{\infty} P[t \le \sum_{i=o}^{n+1}(A_i+F_i) - \tau \le t+dt] \\
&= \sum_{n=0}^{\infty} \Big(\frac{\lambda_1}{\lambda_1+\lambda_2}\Big) \cdot \Big(\frac{\lambda_1\, \mathcal{F}(\lambda_2)}{\lambda_1+\lambda_2} \Big)^n \cdot \int_0^\infty f(t+r)dt\ \lambda_2 e^{-\lambda_2 r} dr \\
&= \underbrace{\frac{\lambda_1}{\lambda_1+\lambda_2-\lambda_1 \cdot \mathcal{F}(\lambda_2)}}_{\boldsymbol{C}} \cdot \int_0^\infty f(t+r)dt\ \lambda_2 e^{-\lambda_2 r} dr \\
&= \boldsymbol{C} \cdot \int_0^\infty \lambda_2 e^{-\lambda_2 r} f(t+r)\ dr\ dt
\end{aligned}
$$

Let $x = r + t, r = x - t$, so when $r = 0$, $x = t$ and when $r = \infty$, $x = \infty$.

$$w(t)dt = \boldsymbol{C} \cdot \int_t^\infty \lambda_2 e^{-\lambda_2(x-t)} f(x)\; dx\; dt$$

Now, let us take the Laplace transform on both sides of the equation.

$$\begin{aligned}
\mathcal{W}(s) &= \boldsymbol{C} \cdot \int_0^\infty e^{-st} \int_t^\infty \lambda_2 e^{-\lambda_2(x-t)} f(x)\; dx\; dt \\
&= \boldsymbol{C} \cdot \int_0^\infty \lambda_2 e^{-(s-\lambda_2)t} \int_t^\infty e^{-\lambda_2 x} f(x)\; dx\; dt \\
&= \boldsymbol{C} \cdot \int_0^\infty e^{-\lambda_2 x} f(x)\; dx\; \int_0^x \lambda_2 e^{-(s-\lambda_2)t}\; dt \\
&= \boldsymbol{C} \cdot \int_0^\infty e^{-\lambda_2 x} f(x)\; dx\; \Big[\frac{\lambda_2 e^{-(s-\lambda_2)t}}{-(s-\lambda_2)}\Big]_0^x \\
&= \frac{\lambda_2 \cdot \boldsymbol{C}}{\lambda_2 - s} \cdot \int_0^\infty e^{-\lambda_2 x} f(x)\; dx\; \Big[e^{-(s-\lambda_2)x} - 1\Big]
\end{aligned}$$

Let $\lambda_2 \boldsymbol{C} = \boldsymbol{C'}$

$$\begin{aligned}
\mathcal{W}(s) &= \frac{\boldsymbol{C'}}{\lambda_2 - s} \cdot \Big[\int_0^\infty e^{-sx} f(x)\; dx - \int_0^\infty e^{-\lambda_2 x} f(x)\; dx\Big] \\
&= \frac{\boldsymbol{C'}}{\lambda_2 - s} \cdot \Big[\mathcal{F}(s) - \mathcal{F}(\lambda_2)\Big]
\end{aligned}$$

Putting the two cases together to have

$$\mathcal{W}(s) = \frac{\lambda_2}{\lambda_1 + \lambda_2 - \lambda_1 \mathcal{F}(\lambda_2)} + \frac{\lambda_1 \lambda_2 \big[\mathcal{F}(s) - \mathcal{F}(\lambda_2)\big]}{[\lambda_1 + \lambda_2 - \lambda_1 \mathcal{F}(\lambda_2)] \cdot (\lambda_2 - s)}$$

Finally, we have

$$\tilde{\mathcal{T}}(s) = \mathcal{W}(s) \cdot \mathcal{T}(s)$$

since service time in this case is waiting time + tx time. And recall that

$$A(z) = \mathcal{T}\big(\lambda_2(1-z)\big)$$

$$\tilde{A}(z) = \tilde{\mathcal{T}}\big(\lambda_2(1-z)\big)$$

And these two terms can be used to calculate $P(z)$.

20 Mathematical Backgrounds

This section is devoted to helping readers recollect mathematical backgrounds necessary for use with the context of the course.

20.1 Differentiation

1. $\frac{dc}{dx} = 0$
2. $\frac{d(cu)}{dx} = c\frac{du}{dx}$
3. $\frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx}$
4. $\frac{d}{dx}(\frac{u}{v}) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
5. $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
6. $\frac{d}{dx}x^n = nx^{n-1}$
7. $\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$
8. $\frac{d}{dx}a^x = (\ln a) \cdot a^x$
9. $\frac{d}{dx}a^u = (\ln a) \cdot a^u \frac{du}{dx}$
10. $\frac{d}{dx}e^x = e^x$
11. $\frac{d}{dx}e^u = e^u \frac{du}{dx}$
12. $\frac{d}{dx}\log_a x = \frac{1}{(\ln a)x}$
13. $\frac{d}{dx}\ln x = \frac{1}{x}$
14. $\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$

20.2 Integration

1. $\int x^n dx = \frac{1}{n+1}x^{n+1}$
2. $\int \frac{1}{x}dx = \ln x$
3. $\int udv = uv - \int vdu$
4. $\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx$
5. $\int \frac{1}{ax+b}dx = \frac{1}{a}\ln(ax+b)$
6. $\int \frac{1}{(x+a)^2}dx = \frac{-1}{x+a}$
7. $\int e^{ax}dx = \frac{1}{a}e^{ax}$
8. $\int xe^x dx = (x-1)e^x$
9. $\int u(x)v(x)dx = u(x)\int v(x)dx$
   $- \int (u'(x) \cdot \int v(x)dx)dx$
10. $\int xe^{ax}dx = \left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax}$
11. $\int x^2 e^x dx = e^x(x^2 - 2x + 2)$

20.3 z-Transform

1. $F(z) = \sum_{n=0}^{\infty} f_n z^n$
2. $F(1) = \sum_{n=0}^{\infty} f_n$
3. $af_n + bg_n = aF(z) + bG(z)$
4. $u = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} \iff 1$
5. u_{n-k} (or k-shifted) $\iff z^k$
6. $\delta_n = 1$, n=0,1,2,... $\iff \frac{1}{1-z}$
7. $\delta_{n-k} \iff \frac{z^k}{1-z}$

20.4 Laplace Transform

$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt$

1. $f(t)\ ; t \geq 0 \iff F(s) = \int_0^\infty f(t)e^{-st}dt$
2. $af(t) + bg(t) \iff aF(s) + bG(s)$
3. $f(\frac{t}{a})\ ; a \geq 0 \iff aF(as)$
4. Shifted function: $f(t-a) \iff e^{-as}F(s)$
5. $e^{-at}f(t) \iff F(s+a)$
6. $t \cdot f(t) \iff -\frac{d}{ds}F(s)$
7. $t^n \cdot f(t) \iff (-1)^n \frac{d^n}{ds^n}F(s)$
8. Convolution: $f(t) * g(t) \iff F(s) \cdot G(s)$
9. Differentiation: $\frac{d}{dt}f(t) \iff sF(s)$
10. Differentiation: $\frac{d^n}{dt^n}f(t) \iff s^nF(s)$
11. s=0: $F(0) = \int_0^\infty f(t)dt$
12. Unit Impulse: $u_0(t) \iff 1$
13. Shifted Unit Impulse: $u_0(t-a) \iff e^{-as}$
14. Unit Step: $\delta(t) \iff \frac{1}{s}$
15. Shifted Unit Step: $\delta(t-a) \iff \frac{e^{-as}}{s}$
16. $A \cdot e^{-at}\delta(t) \iff \frac{A}{s+a}$
17. $t \cdot e^{-at}\delta(t) \iff \frac{1}{(s+a)^2}$
18. $\frac{t^n}{n!} \cdot e^{-at}\delta(t) \iff \frac{1}{(s+a)^{n+1}}$

20.5 Summation of Series

1. Sum of First Few Values

$$\sum_{k=1}^{m} k = \frac{m(m+1)}{2}$$
$$\sum_{k=1}^{m} k^2 = \frac{m(m+1)(2m+1)}{6}$$
$$\sum_{k=1}^{m} k^3 = \Big[\frac{m(m+1)}{2}\Big]^2$$

2. Taylor Series Approximation:

$$\Big(1 - \frac{S}{N}\Big)^N \approx e^{-S} \approx \Big(1 - \frac{S}{N-k}\Big)^{N-k}$$

3. Summation of Geometric Series:

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$
$$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$$
$$\sum_{k=1}^{\infty} r^k = \frac{r}{1-r}$$
$$\sum_{k=1}^{n} r^k = \frac{r(1-r^n)}{1-r}$$

4. Exponential Functions:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$
$$x\, e^x = \sum_{n=0}^{\infty} n\frac{x^n}{n!} \quad \text{(mean of Poisson dist, P)}$$
$$(x + x^2)\, e^x = \sum_{n=0}^{\infty} n^2 \cdot \frac{x^n}{n!} \quad \text{(second moment P)}$$

20.6 Matrix Inversion

20.6.1 A 2x2 Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

20.6.2 A 3x3 Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned} det[A] &= a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} \\ &- a_{11}a_{32}a_{23} - a_{31}a_{22}a_{13} - a_{21}a_{12}a_{33} \end{aligned}$$

$$A^{-1} = \frac{1}{det[A]} \cdot \begin{bmatrix} a_{22}a_{33} - a_{23}a_{32} & a_{13}a_{32} - a_{12}a_{33} & a_{12}a_{23} - a_{13}a_{22} \\ a_{23}a_{31} - a_{21}a_{33} & a_{11}a_{33} - a_{13}a_{31} & a_{13}a_{21} - a_{11}a_{23} \\ a_{21}a_{32} - a_{22}a_{31} & a_{12}a_{31} - a_{11}a_{32} & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix}$$

20.7 Partial Fraction Expansion

Given an example of functions

$$F(s) = \frac{s+3}{s^3 + 7s^2 + 10s}$$

which can be represented as

$$\begin{aligned} F(s) &= \frac{s+3}{s(s+2)(s+5)} \\ &= \frac{A_1}{s} + \frac{A_2}{s+2} + \frac{A_3}{s+5} \end{aligned}$$

To find A_1, we multiply $F(s)$ by s and set $s = 0$.

$$\begin{aligned} sF(s) &= s\frac{A_1}{s} + s\frac{A_2}{s+2} + s\frac{A_3}{s+5} \\ = \frac{s+3}{(s+2)(s+5)} &= A_1 + s\frac{A_2}{s+2} + s\frac{A_3}{s+5} \\ \left[\frac{s+3}{(s+2)(s+5)}\right]_{s=0} &= \left[A_1 + s\frac{A_2}{s+2} + s\frac{A_3}{s+5}\right]_{s=0} \\ \frac{3}{10} &= A_1 \end{aligned}$$

Next, we find A_2 by multiplying $F(s)$ with $s+2$ and set $s = -2$.

$$\begin{aligned} &(s+2)F(s) = \\ &(s+2)\frac{A_1}{s} + (s+2)\frac{A_2}{s+2} + (s+2)\frac{A_3}{s+5} \\ &\left[\frac{s+3}{s(s+5)}\right]_{s=-2} \\ &= \left[(s+2)\frac{A_1}{s} + A_2 + (s+2)\frac{A_3}{s+5}\right]_{s=-2} \\ &-\frac{1}{6} = A_2 \end{aligned}$$

Finally, we find A_3 by multiplying $F(s)$ with $s+5$ and set $s = -5$.

$$\begin{aligned} &(s+5)F(s) = \\ &(s+5)\frac{A_1}{s} + (s+5)\frac{A_2}{s+2} + (s+5)\frac{A_3}{s+5} \\ &\left[\frac{s+3}{s(s+2)}\right]_{s=-5} \\ &= \left[(s+2)\frac{A_1}{s} + (s+5)\frac{A_2}{s+2} + A_3\right]_{s=-5} \\ &-\frac{2}{15} = A_3 \end{aligned}$$

20.8 Miscellaneous Math

1. Permutation of N objects

$$\frac{N!}{(N-k)!}$$

2. Combination of N objects

$$\binom{N}{k} = \frac{N!}{k!(N-k)!}$$