Discrete Optimization

Mixed Integer Programming: Part V

Goals of the Lecture

- Branch and cut
  - Cover cuts
    - separation problem
  - TSP

Cover Cuts

- Consider constraints of the type

$$\sum_{j=1}^{n} a_j x_j \leq b$$

- Can we find facets for these constraints?
- Cover
  - a set $C \subseteq N = \{1, \ldots, n\}$ is a cover if

$$\sum_{j \in C} a_j > b$$

  - a cover is minimal if $C \setminus \{j\}$ is not a cover for any $j \in C$.

Cover Cuts

- Consider constraints of the type

$$\sum_{j=1}^{n} a_j x_j \leq b$$

- Can we find facets for these constraints?
- If $C \subseteq N = \{1, \ldots, n\}$ is a cover, then

$$\sum_{j \in C} x_j \leq |C| - 1$$

is a valid inequality.

Cover Cuts

- Consider

$$11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19$$

- Some minimal cover inequalities

$$x_1 + x_2 + x_3 \leq 2$$

$$x_3 + x_4 + x_5 + x_6 \leq 3$$

Stronger Cover Cuts

- If $C \subseteq N = \{1, \ldots, n\}$ is a cover, then

$$\sum_{j \in E(C)} x_j \leq |C| - 1$$

is a valid inequality

$$\sum_{j=1}^{n} a_j x_j \leq b$$

where

$$E(C) = C \cup \{j \mid \forall i \in C : a_j \geq a_i\}$$

Stronger Cover Cuts

- Consider

$$11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19$$

- And

$$x_3 + x_4 + x_5 + x_6 \leq 3$$

- A stronger cover inequality is

$$x_1 + \ldots + x_6 \leq 3$$

Branch and Cut

- Basic idea

1. formulate the application as a MIP;
2. solve the linear relaxation; if the linear relaxation is integral, terminate;
3. find a polyhedral cut which prunes the linear relaxation and is a facet if possible; if you can find such beautiful mathematical object, go back to step 2;
4. otherwise, settle for the poor man’s choice and branch

The Separation Problem

- Consider a solution x^* to the linear relaxation possibly enhanced by a number of cuts
- We wish to know whether there exists a cover cut that cut x^*

Separation for Cover Cuts

- The cover inequality

$$\sum_{j \in C} x_j \le |C| - 1$$

- can be rewritten into

$$\sum_{j \in C} (1 - x_j) \ge 1$$

- Does there exist $C \subseteq N$ that satisfies

$$\sum_{j \in C} (1 - x_j^*) < 1$$

$$\sum_{j \in C} a_j > b$$

Separation for Cover Cuts

- This is equivalent to a beautiful mathematical program

$$
\begin{aligned}
\min \quad & \sum_{j \in N} (1 - x_j^*) z_j \\
s.t. \quad & \\
& \sum_{j \in N} a_j z_j > b \\
& z_j \in \{0, 1\}
\end{aligned}
$$

- If the minimum value is lower than 1, then we have a cut! All the variables assigned to 1 are a cover.

Separation for Cover Cuts

- Consider the constraint

$$45x_1 + 46x_2 + 79x_3 + 54x_4 + 53x_5 + 125x_6 \leq 178$$

- And the fractional solution

$$x^* = (0, 0, \frac{3}{4}, \frac{1}{2}, 1, 0)$$

- The separation problem is

$$\begin{array}{lllllllll} min & z_1 & +z_2 & +\frac{1}{4}z_3 & +\frac{1}{2}z_4 & & +z_6 & \\ s.t & & & & & & & \\ & 45z_1 & +46z_2 & +79z_3 & +54z_4 & +53z_5 & +125z_6 & > 178 \end{array}$$

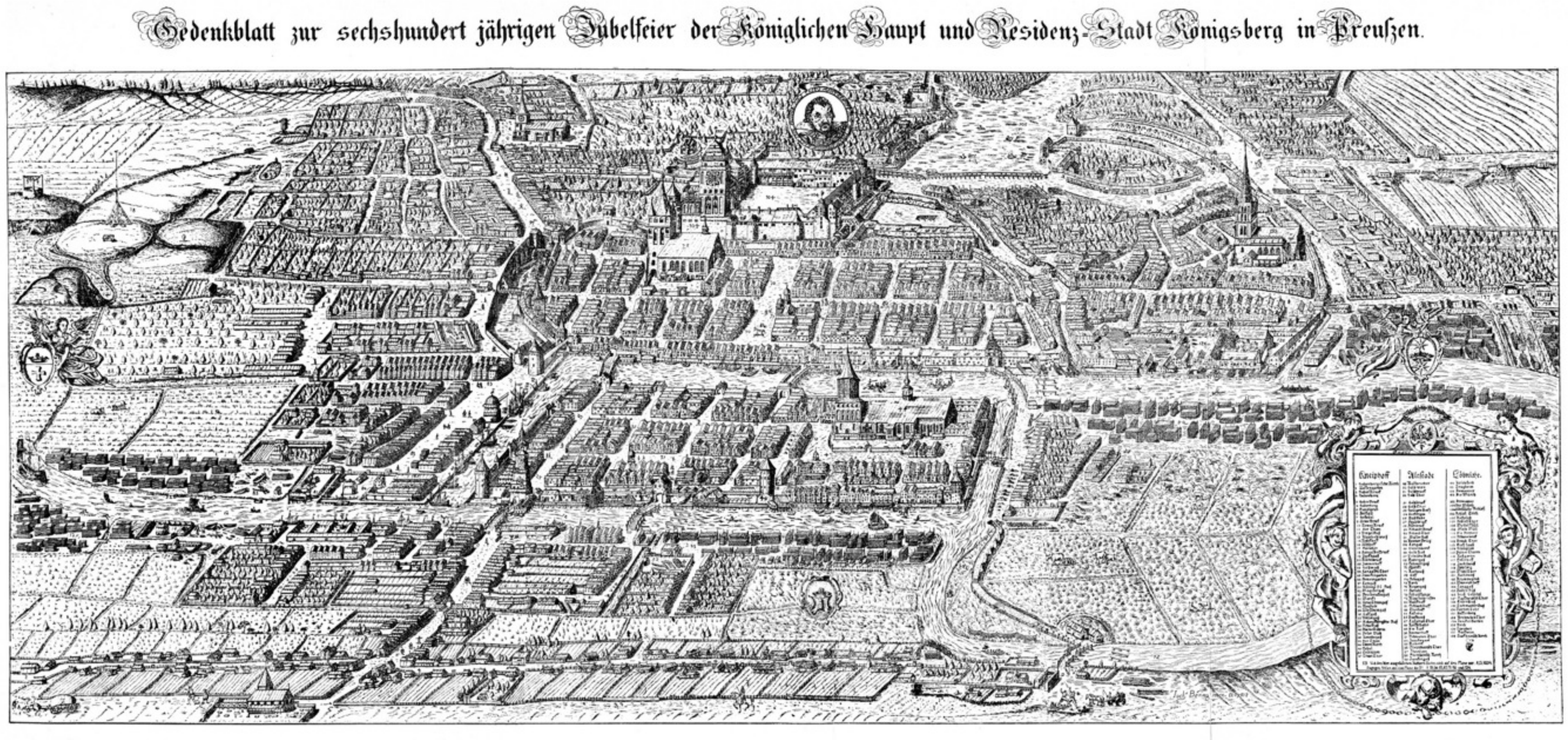
Separation for Cover Cuts

- This is equivalent to a beautiful mathematical program

$$
\begin{aligned}
\min \quad & \sum_{j \in N} (1 - x_j^*) z_j \\
s.t. \quad & \\
& \sum_{j \in N} a_j z_j > b \\
& z_j \in \{0, 1\}
\end{aligned}
$$

- Does this remind you of something?
  - replace z_j by $(1 - y_j)$

Seven Bridges of Königsberg

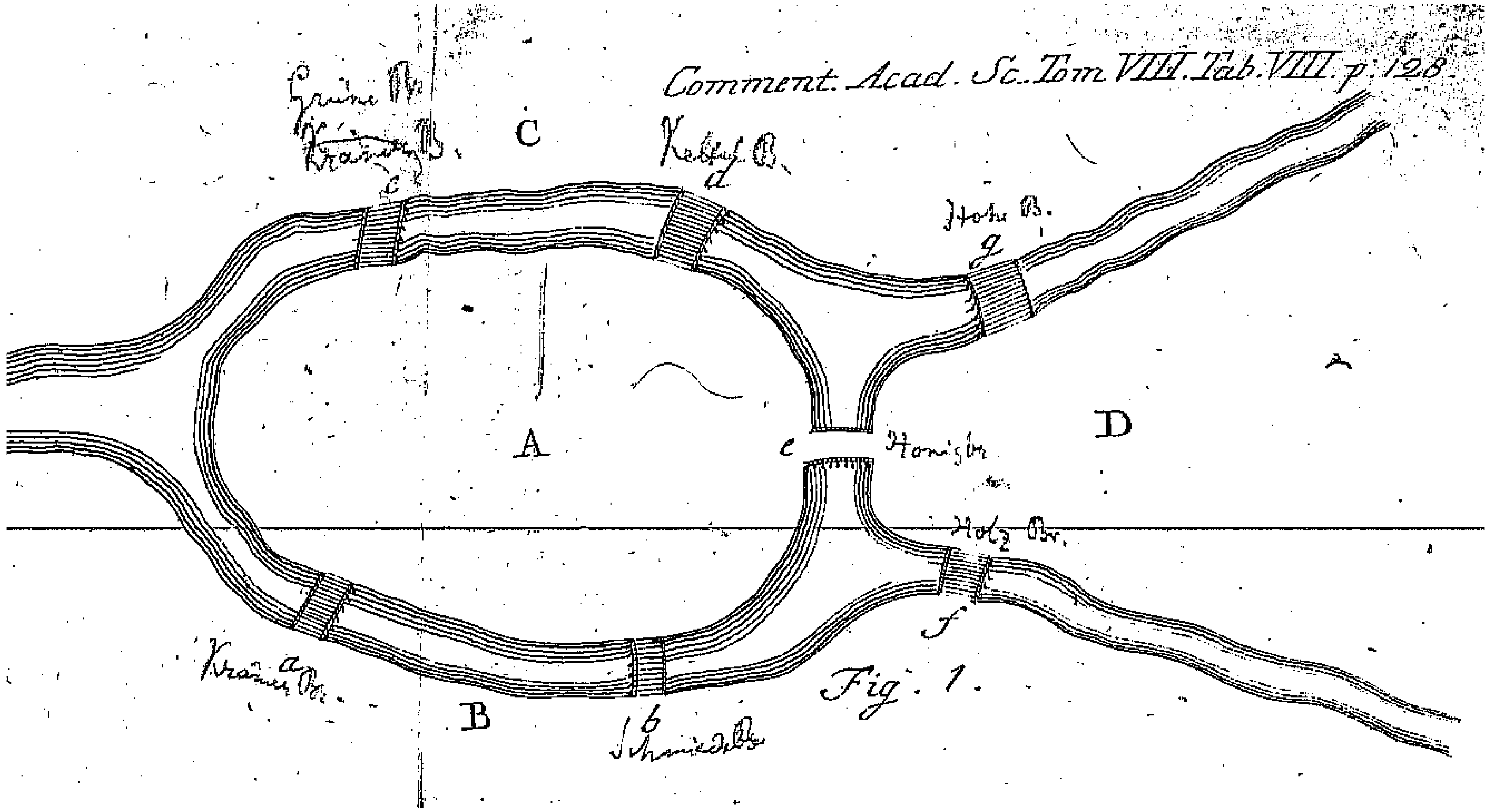


Seven Bridges of Konigsberg

Seven Bridges of Königsberg

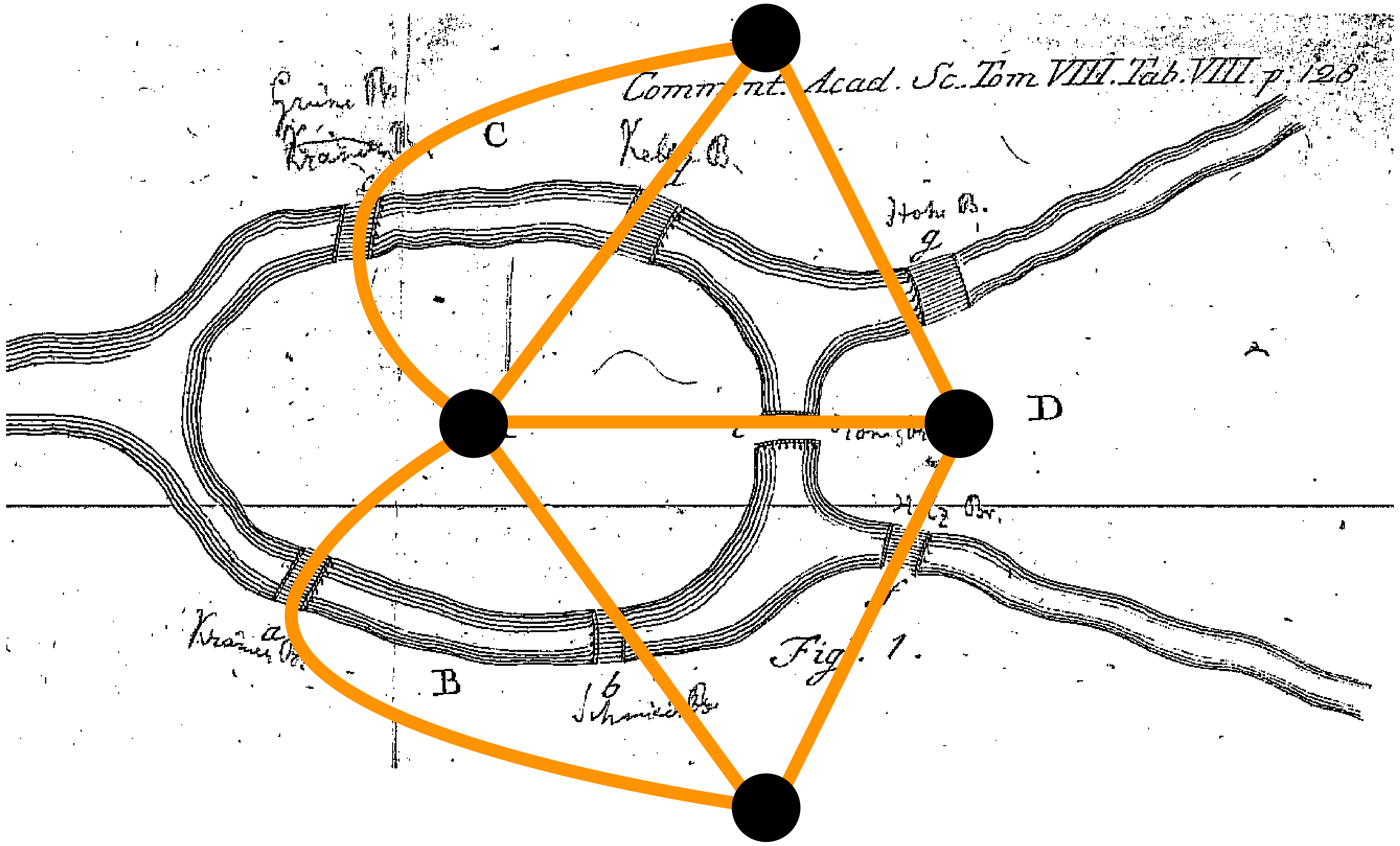
Leonhard Euler

By Jakob Emanuel Handmann

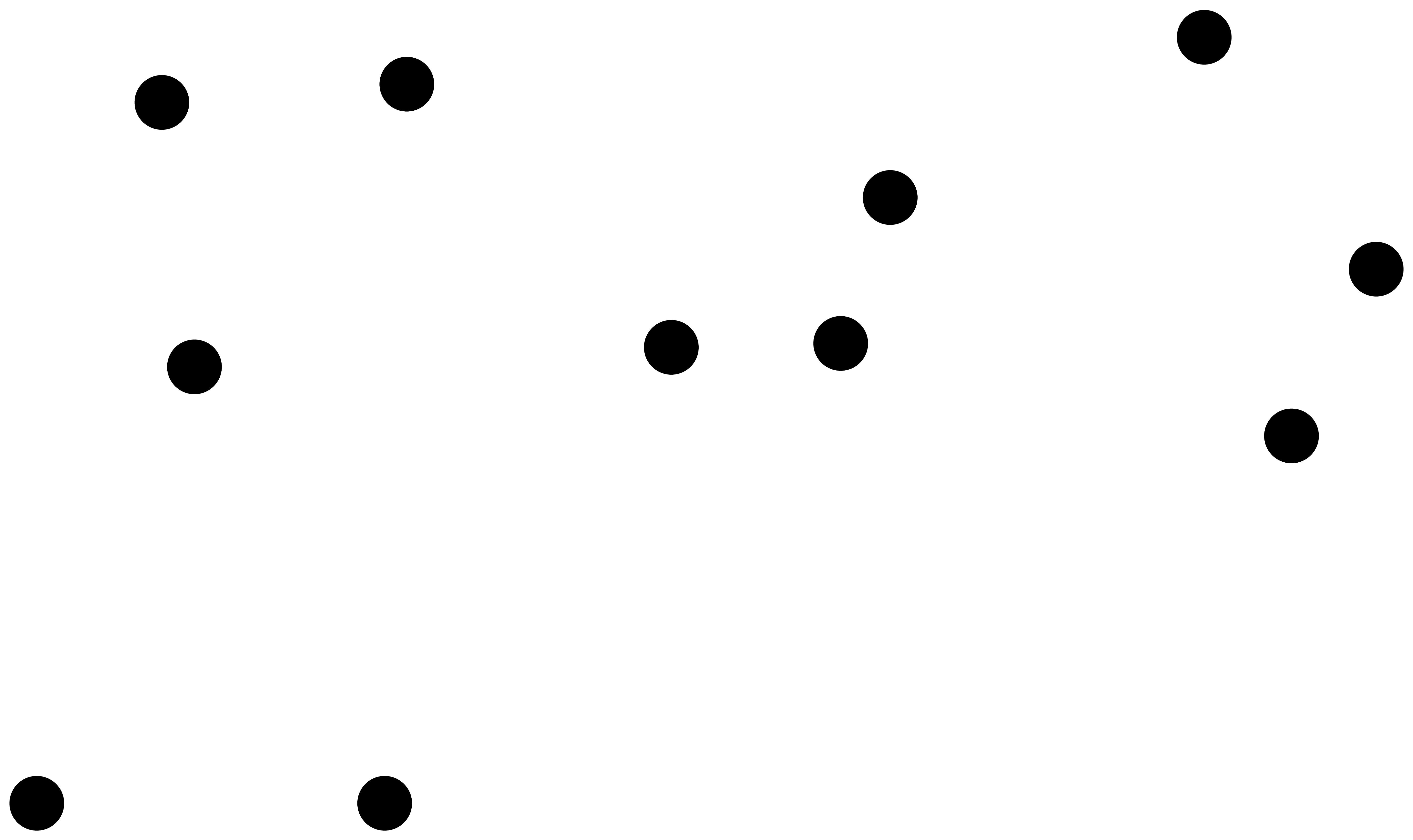


Bridges of Königsberg, 1741

Seven Bridges of Königsberg



Traveling Salesman Problem



MIP for TSP

- How to express the TSP as a MIP?
  - decision variables, constraints, objectives
  - several models obviously

MIP for TSP

- How to express the TSP as a MIP?
  - decision variables, constraints, objectives
  - several models obviously
- Decision variables
  - decide whether an edge is part of the tour
- Constraints
  - degree constraints:
    - if a edge is selected, the nodes of the edge must be present in another edge (a.k.a. each node has exactly two edges)

MIP for TSP

- Decision variables
  - x_e is 1 if edge e is in the solution
- Notations
  - V is the set of vertices
  - E is the set of edges
  - δ(v): edges adjacent to vertex v
  - δ(S): edges with exactly one vertex in S ⊆V
  - γ(S): edges with both vertices in S ⊆V
  - $x_{\{e_1,\ldots,e_n\}} = x_{e_1} + \ldots + x_{e_n}$

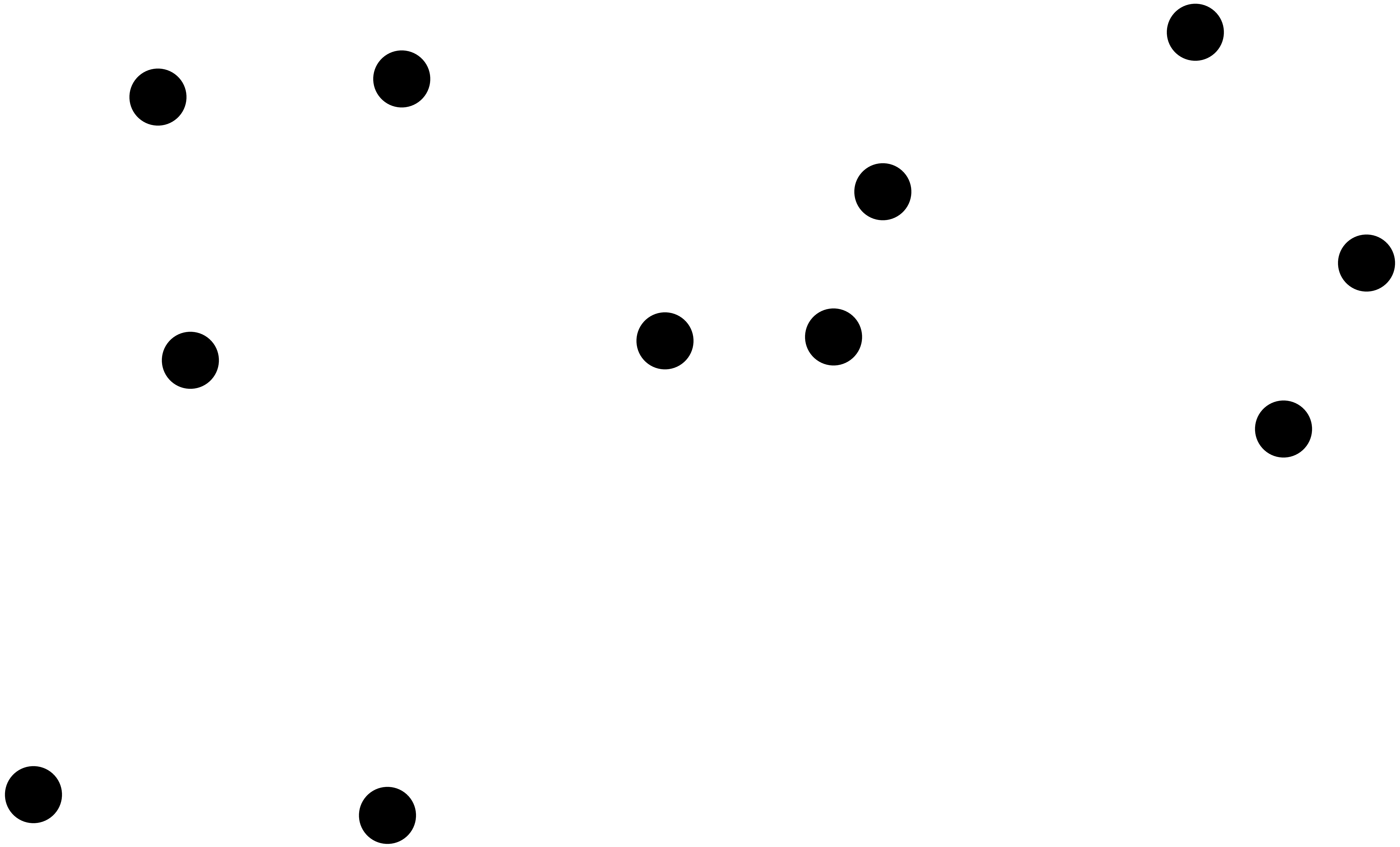
MIP for TSP

$$
\begin{aligned}
\min \quad & \sum_{e \in E} c_e x_e \\
\text{subject to} \quad & \\
& x_{\delta(v)} = 2 \qquad v \in V \\
& x_e \in \{0,1\} \quad e \in E
\end{aligned}
$$

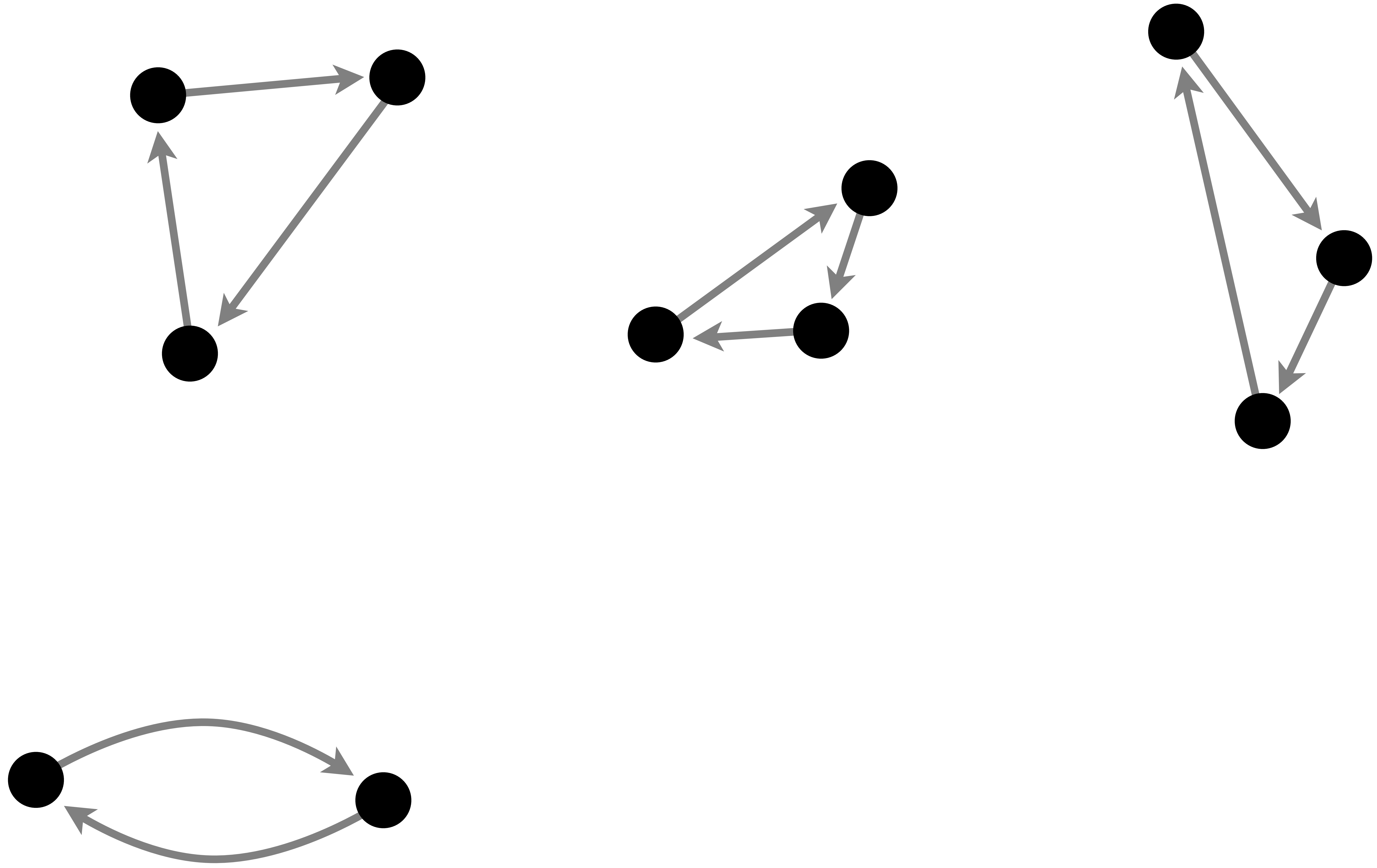
flow conservation

minimize cost

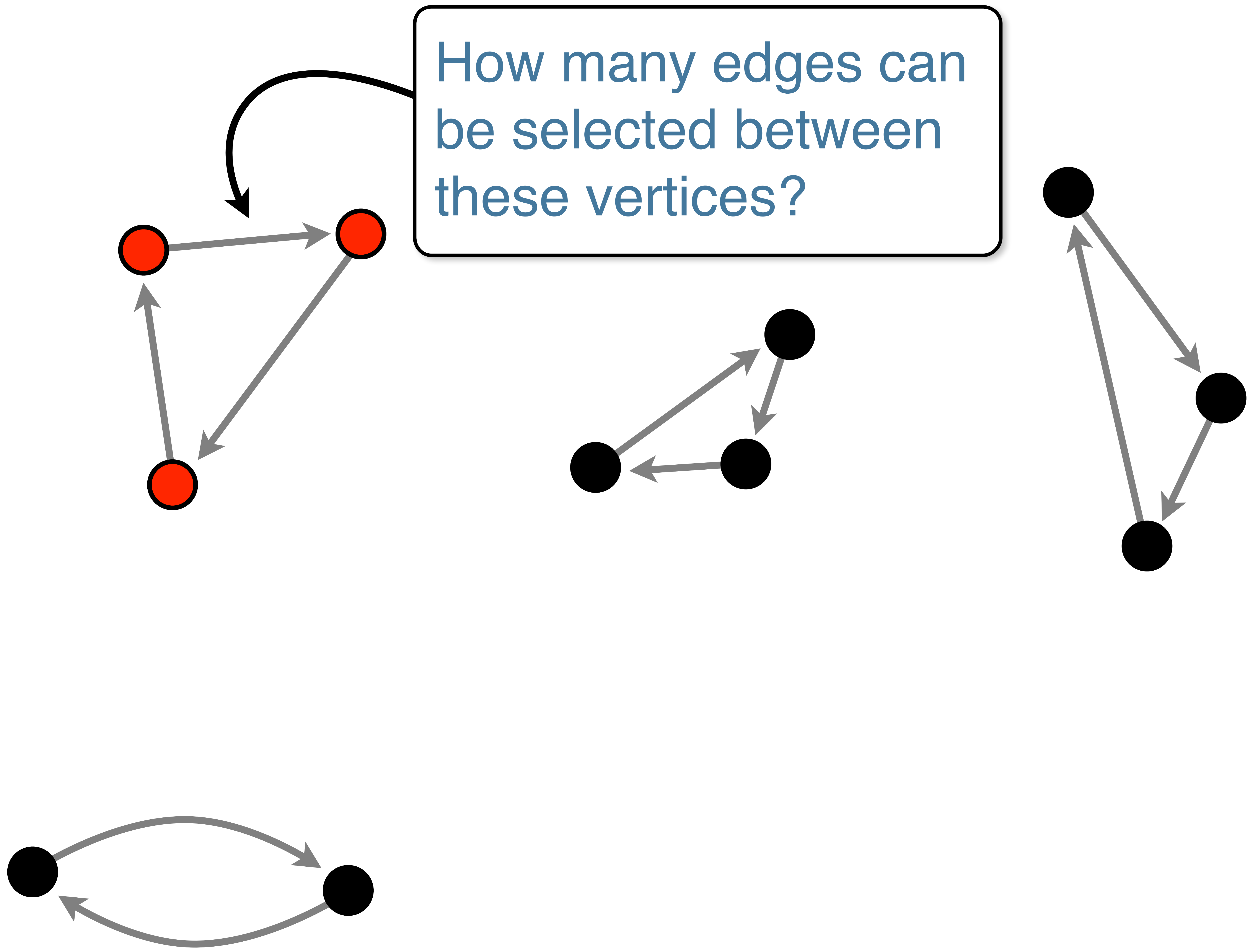
Traveling Salesman Problem



The Degree Constraint Formulation



The Degree Constraint Formulation



The Subtour Elimination Problem

$$
\begin{array}{lll}
\min & \sum_{e \in E} c_e x_e & \\
\text{subject to} & & \\
& x_{\delta(v)} = 2 & v \in V \\
& \boxed{x_{\gamma(S)} \leq |S| - 1 \quad S \subset V} & \\
& x_e \in \{0, 1\} & e \in E
\end{array}
$$

The Subtour Elimination Problem

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\end{aligned}
$$

- What is the issue with the subtour constraints?
  - there are exponentially many of them
- Branch and cut
  - generate them on demand: separation

The Subtour Elimination Problem

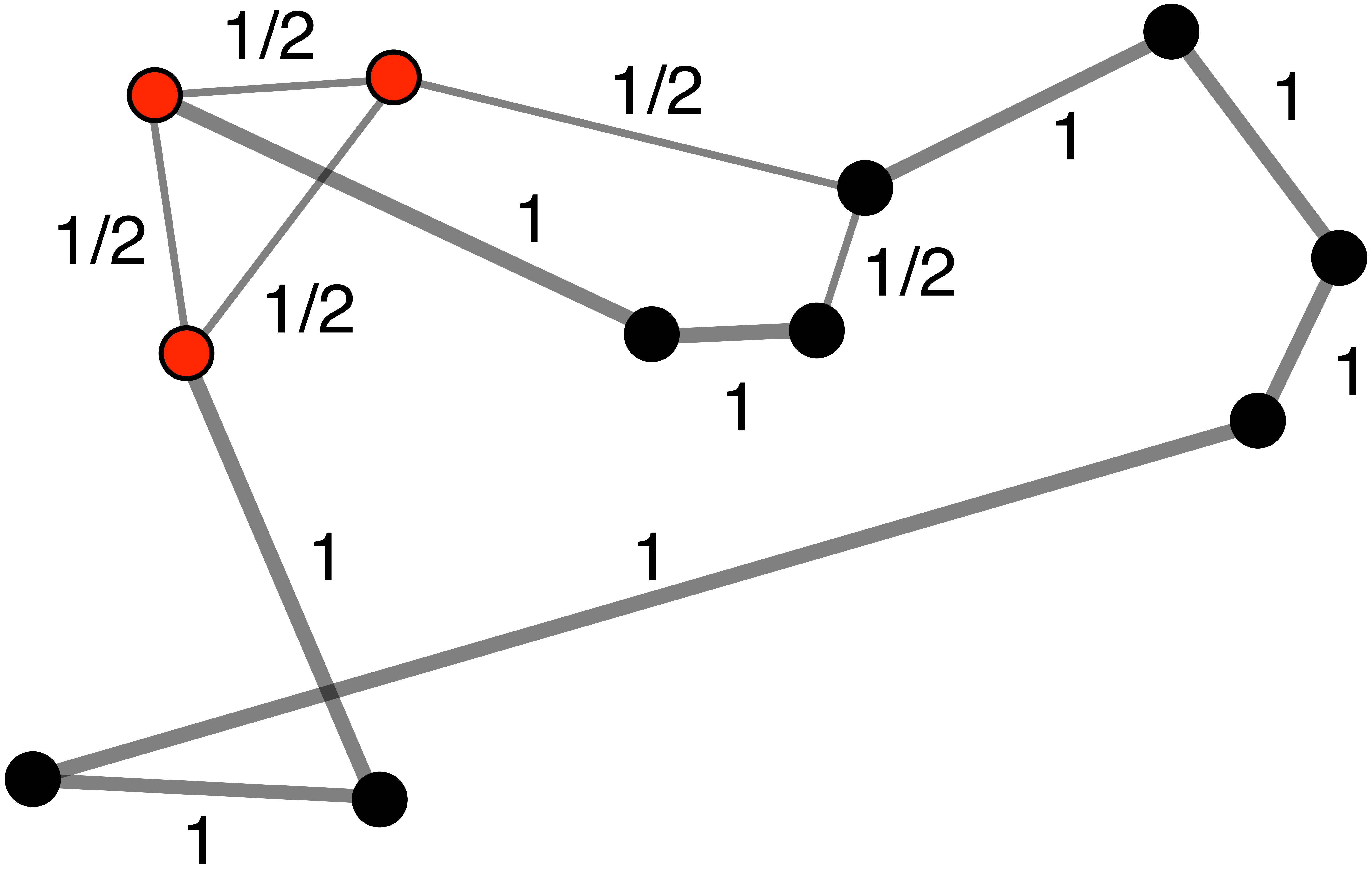
$$
\begin{aligned}
\min \quad & \sum_{e \in E} c_e x_e \\
\text{subject to} \quad & \\
& x_{\delta(v)} = 2 && v \in V \\
& \boxed{x_{\delta(S)} \geq 2 \qquad S \subset V} \\
& x_e \in \{0,1\} && e \in E
\end{aligned}
$$

- How to separate subtour constraints?

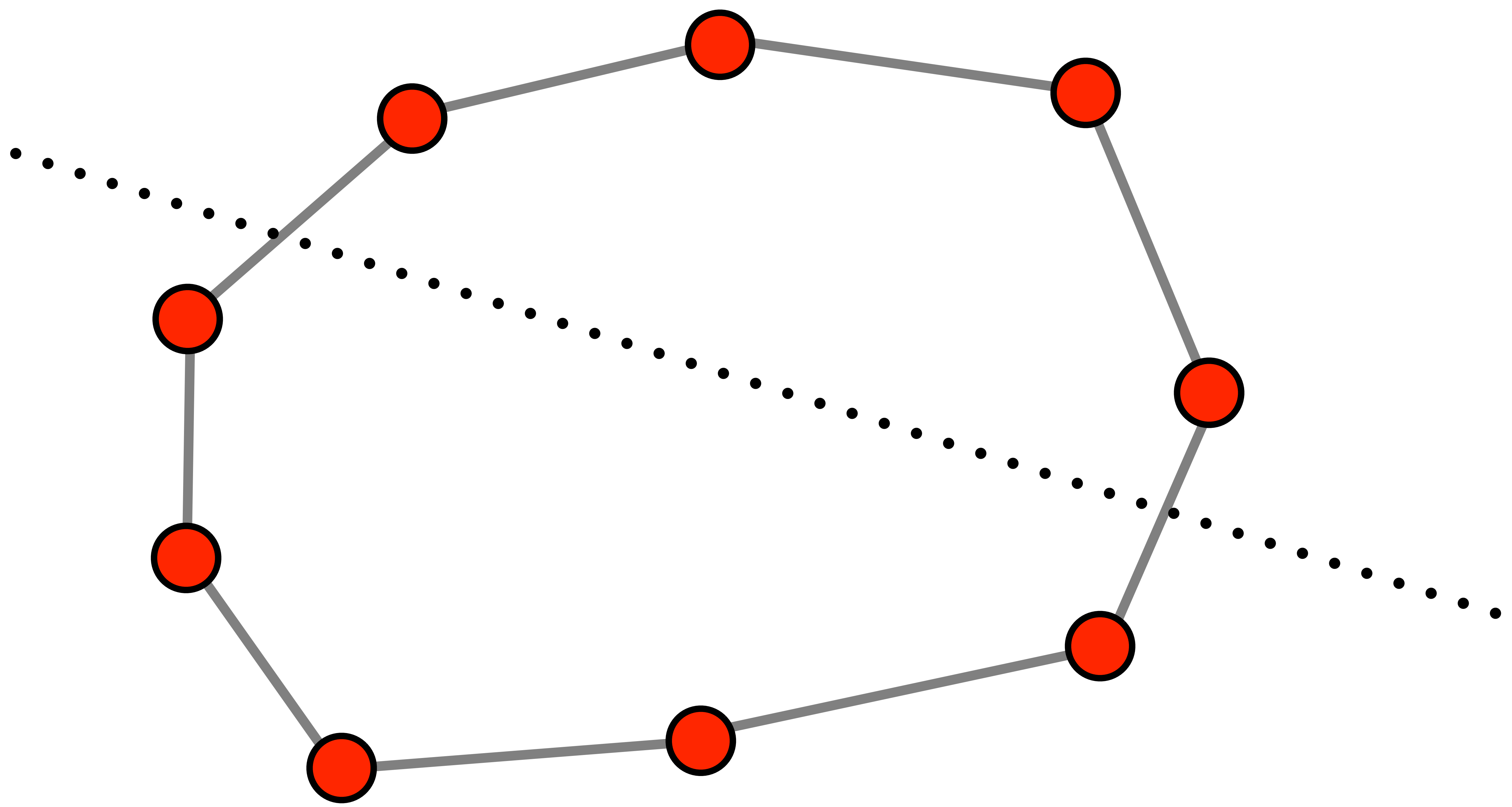
Separation of Subtour Constraints

- Build a graph G* = (V,E) where
  - the weight of edge e is $w(e) = x_e^*$
- Finding a separation consists of finding
  - a minimum cut in G*
  - if the weight of the cut is smaller than 2, then we have isolated a subtour constraint violated by the linear relaxation
  - finding such a cut takes polynomial time

The Subtour Elimination Problem

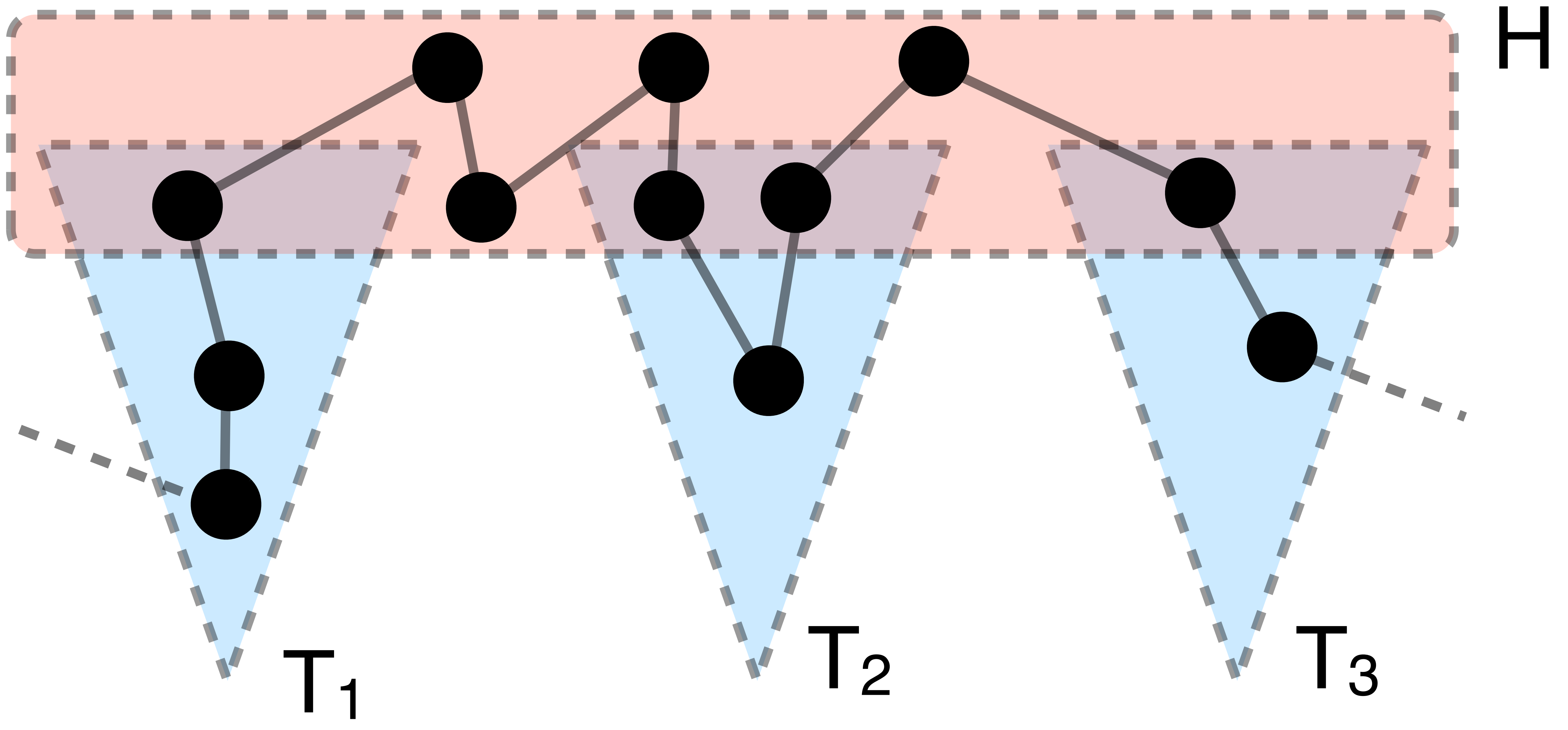


Comb Constraints

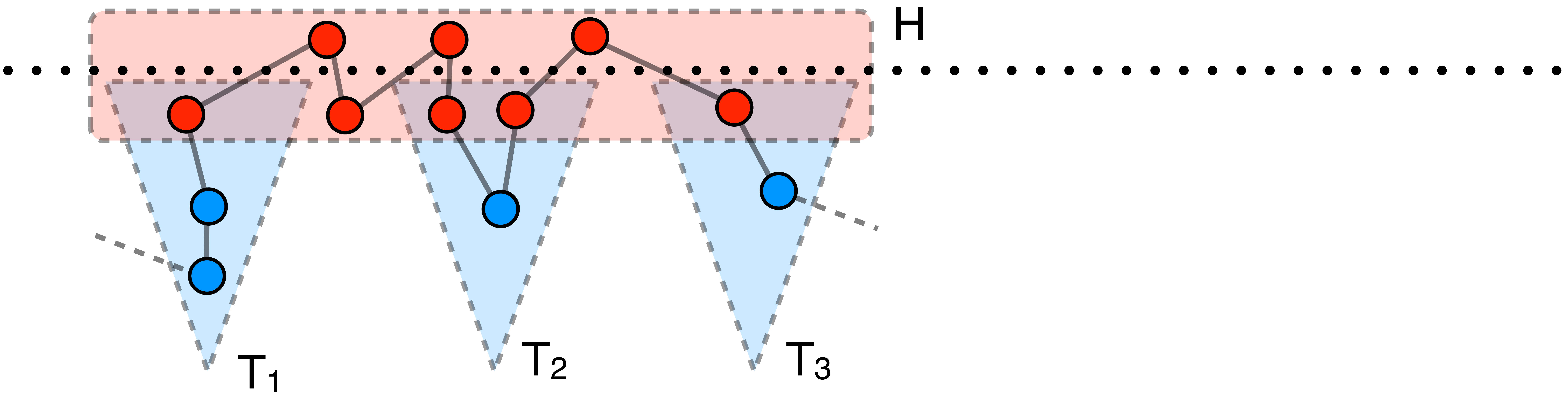


- How many edges do you cross?

Comb Constraints



Comb Constraints



- comb inequalities

$$x_{\gamma(H)} + \sum_{i=1}^{t} x_{\gamma(T_i)} \leq |H| + \sum_{i=1}^{k} |T_i| - \lceil \frac{3k}{2} \rceil$$

Branch and Cut on the TSP

- On benchmarks from the TSPLIB
  - subtour elimination: 2% of optimality gap
  - subtour + comb cuts: 0.5% of optimality gap

- Other cuts are needed on very large instances

Until Next Time

Citations

Map of Koenigsberg (https://commons.wikimedia.org/wiki/File:Koenigsberg,_Map_by_Bering_1613.jpg#globalusage) by Joachim Bering [Public Domain], via Wikimedia Commons.   Leonhard Euler (http://commons.wikimedia.org/wiki/File:Leonhard_Euler_2.jpg) by Jakob Emanuel Handmann [Public domain], via Wikimedia Commons (Euler, Leonhard) Fig 1, Fig 2, Fig 3 [aka Bridges of Konigsberg] from Solutio problematis ad geometriam situs pertinentis in Commentarii academiae scientiarum Petropolitanae 8, 1741, pp. 128-140