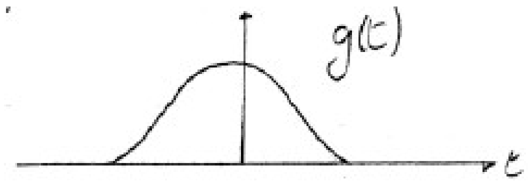
EE357 – Signals & Communications

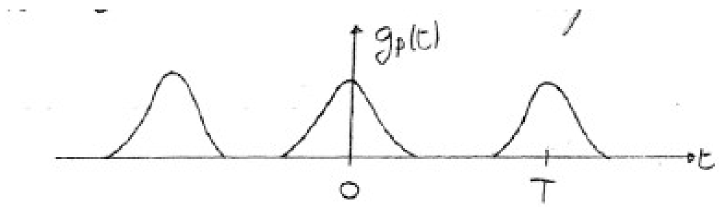
Section 2 – Fourier Transform

Fourier Series of aperiodic signals
Derivation of the Fourier Transform

- Fourier Series – eternal, periodic signals. But, suppose we have



- Construct $g_p(t)$ with period T, such that there is no overlap. Let $T \rightarrow \infty$ …



$$\lim_{T\to\infty} g_p(t) = g(t)$$

Fourier Series of aperiodic signals
Derivation of the Fourier Transform

$$g_p(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\omega_0 t}$$

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} g_p(t)e^{-j\omega_0 t}dt$$

As $T \to \infty, \omega_0 \to 0$, so replace ω_0 by $\Delta\omega$ => $T = \frac{2\pi}{\Delta\omega}$

Also, magnitudes of c_n become infinitesimally small

Derivation of the Fourier Transform

Define $c_n T = \int_{-\frac{T}{2}}^{\frac{T}{2}} g_p(t) e^{-jn\Delta\omega t} dt$

Denote $c_n T$ as $G(n\Delta\omega) \Rightarrow c_n = \frac{G(n\Delta\omega)}{T}$

$\Rightarrow g_p(t) = \sum_{n=-\infty}^{\infty} \frac{G(n\Delta\omega)}{T} e^{jn\Delta\omega t} = \sum_{n=-\infty}^{\infty} \frac{G(n\Delta\omega)\Delta\omega}{2\pi} e^{jn\Delta\omega t}$

i.e. a Fourier Series with components $\pm\Delta\omega$, $\pm 2\Delta\omega$, etc

$$g(t) = \lim_{T\to\infty} g_p(t) = \lim_{T\to\infty} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{G(n\Delta\omega)\Delta\omega}{2\pi} e^{jn\Delta\omega t} \Delta\omega$$

Derivation of the Fourier Transform

- As $T \to \infty$, we have a component at every possible frequency ($\Delta \to 0$, and $\Delta\omega \to \omega$)
- In the limit, we have the Fourier Transform synthesis equation:

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}d\omega$$

Fourier Transform Analysis equation:

$$G(n\Delta\omega) = \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} g_p(t)e^{-jn\Delta\omega t}dt$$

Which, in the limit, becomes

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt$$

Derivation of the Fourier Transform

Notation:

$$G(\omega) = \mathcal{F}[g(t)]$$
$$g(t) = \mathcal{F}^{-1}[G(\omega)]$$

$$G(\omega) = |G(\omega)|e^{+j\theta g(\omega)}$$

$$f(t) \rightleftharpoons F(\omega)$$

Derivation of the Fourier Transform

- Essentially, the Fourier Transform gives a representation of the signal in terms of an infinite sum of complex exponentials, each weighted by

$$G(\omega)\frac{d\omega}{2\pi} = G(\omega)df$$

- At any given frequency ω, the contribution to $g(t)$ is zero
- However, the contribution in a tine interval $d\omega$ is $G(\omega)e^{j\omega t}df$. Hence $G(\omega)$ can be regarded as a spectral density

Existence of the Fourier Transform

- The Dirichlet conditions for the existence of the Fourier Series also apply (in a slightly modified form) for the Fourier Transform
  - i) f(t) has a finite number of maxima and minima in any finite time interval
  - ii) f(t) has a finite number of discontinuities in a finite time interval
  - iii) f(t) is absolutely integrable i.e.

$$\int_{-\infty}^{\infty} |f(t)| dt < \infty$$

or

$$\int_{-\infty}^{\infty} |f(t)|^2 dt < \infty$$

i.e. the integral has finite energy

Parseval's Theorem for energy signals

Recall $E = \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} f(t) f^*(t)\, dt$

In terms of frequency components:

$$E = \int_{-\infty}^{\infty} f(t) \left[1/2\pi \int_{-\infty}^{\infty} F^*(\omega) e^{-j\omega t} d\omega \right] dt$$

$$= \frac{1}{2\pi} \int F^*(\omega) \left[\int_{-\infty}^{\infty} f(t) e^{-j\omega}\ dt \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega)\, F(\omega) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

$$\therefore \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

- This is called *Parseval's Theorem* for energy signals
- The function $|F(\omega)|^2$ is often called the energy spectral density

Example i) causal exponential decay

$$g(t) = e^{-a}\ u(t)$$

$$G(\omega) = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} e^{(a+j\omega)t} dt$$

$$= \frac{1}{a + j\omega}$$

Magnitude:

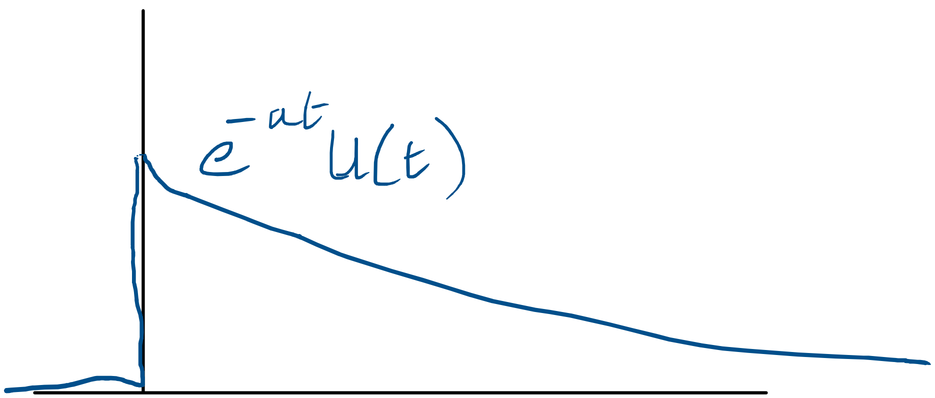
$$|G(\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}}$$

Phase:

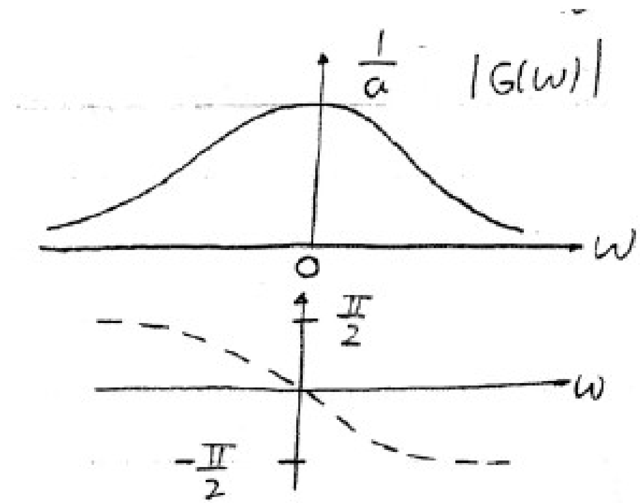
$$\theta_g(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

Note: complex FT because g(t) is asymmetric)

Example i) causal exponential decay



Time Domain



Fourier Transform

Example i) causal exponential decay

For real $g(t)$

$$G(-\omega) = \int_{-\infty}^{\infty} g(t) e^{j\omega t} dt$$

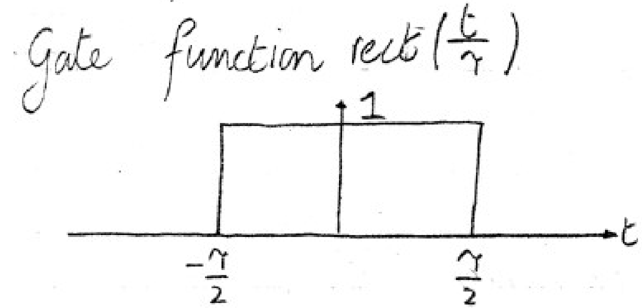
$$= G^*(\omega)$$

$$\Rightarrow G(-\omega) = |G(\omega)| e^{-j\theta_g(\omega)}$$

$$|G(-\omega)| = |G(\omega)| \quad (even)$$

$$\theta_g(-\omega) = -\theta_g(\omega) \quad (odd)$$

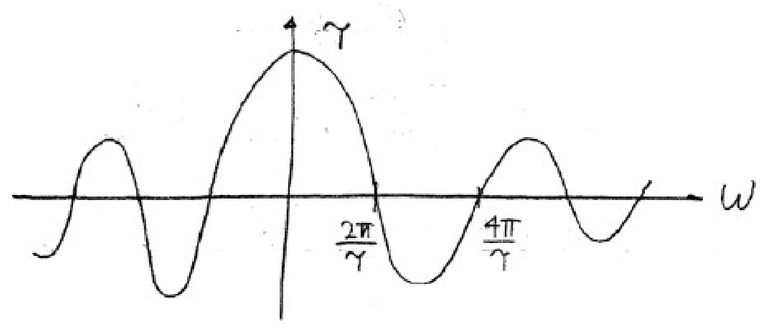
Example ii) gate function



$$\mathcal{F}\left[rect\left(\frac{\tau}{T}\right)\right] = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{-j\omega t}\, dt$$

$$= \frac{\tau \sin\left(\frac{\omega\tau}{2}\right)}{\frac{\omega\tau}{2}}$$

$$= \tau Sa\left(\frac{\omega\tau}{2}\right)$$



Note: $G(\omega)$ is

i) real-valued

ii) Symmetric

Because $rect(t/\tau)$ is real and symmetric

Zero crossings at $\frac{\omega\tau}{2} = n\pi$

Example iii) signum function (sgn(t))

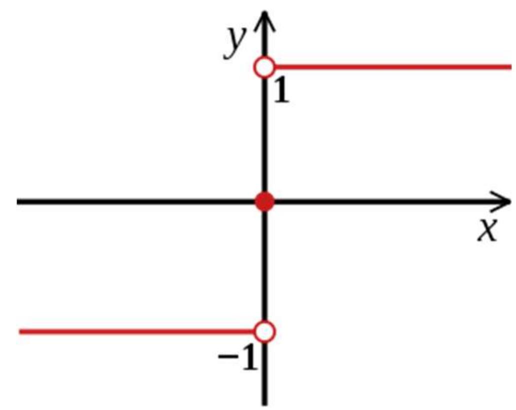
Consider the function as a limit (see figure below):

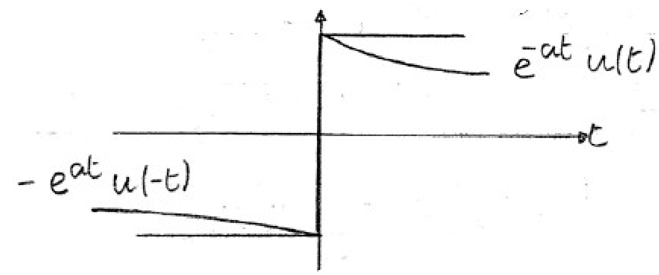
$$\mathcal{F}[\mathrm{sgn}(t)] = \lim_{a\to 0}\int_0^{\infty} e^{-at}e^{-j\omega t} - \int_{-\infty}^{0} e^{at}e^{-j} \quad dt$$

$$= \lim_{a\to 0}\left[\frac{1}{a+j\omega} - \frac{1}{a-j\omega}\right] = \lim_{a\to 0}\left[-\frac{j2\omega}{a^2+\omega^2}\right]$$

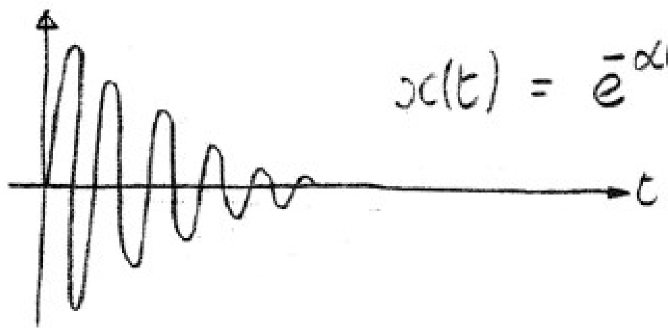
$$= \frac{2}{j\omega}$$

$$odd\ functionof\ time \Rightarrow purely\ imaginary\ FT$$





Example iv) Exponentially decaying sinusoid



$$x(t) = e^{-\alpha t} \sin \omega_1 t \; u(t)$$

$$\alpha > 0$$

Example iv) Exponentially decaying sinusoid

$$X(\omega) = \int_{-\infty}^{\infty} e^{-\alpha t} sin\omega_1 t u(t) e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} e^{-\alpha t} sin\omega_1 t e^{-j\omega t} dt$$

$$sin\omega_1 t = \frac{1}{2j}[e^{j\omega t} - e^{-j\omega t}]$$

$$X(\omega) = \frac{1}{2j}\int_{0}^{\infty} [e^{j\omega_1 t} - e^{-j\omega_1 t}] e^{-(\alpha + j\omega + j\omega_1)t} dt$$

$$= \frac{1}{2j}\int_{0}^{\infty} e^{-(\alpha + j\omega - j\omega_1)t} - \frac{1}{2j}\int_{0}^{\infty} e^{-(\alpha + j\omega + j\quad{}_1)t} dt$$

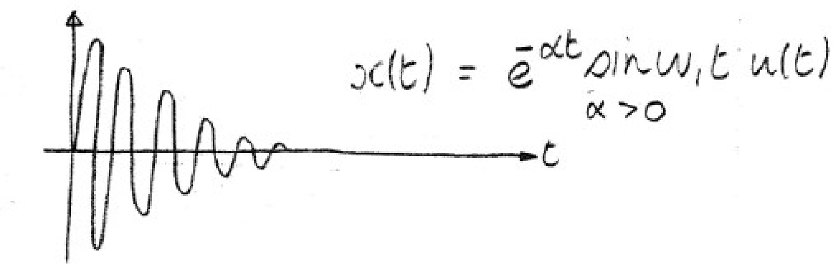
$$= \frac{1}{2j}\frac{-1}{\alpha + j\omega - j\omega_1}\left| e^{-(\alpha + j\omega - j\omega_1)t}\right|_0^{\infty}$$

$$+ \frac{1}{2j}\frac{1}{\alpha + j\omega - j\omega_1}\left| e^{-(\alpha + j\omega + j\omega_1)t}\right|_0^{\infty}$$

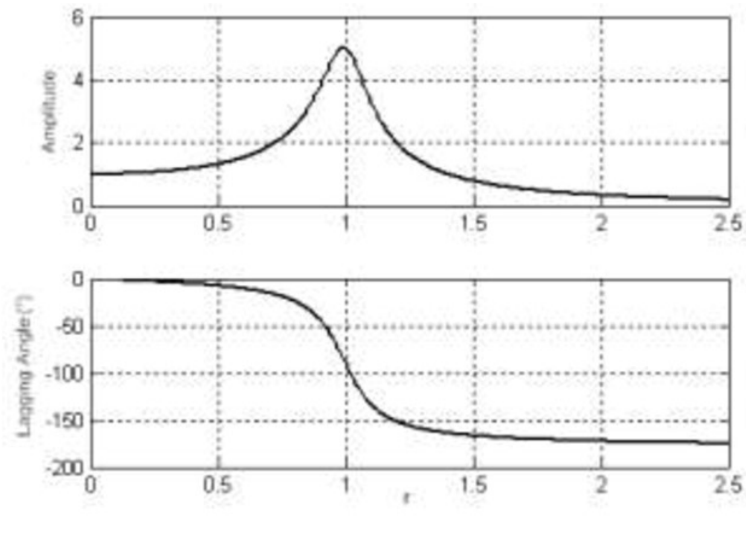
$$= \frac{1}{2}\left[\frac{1}{j\alpha - \omega + \omega_1} - \frac{1}{j\alpha - \omega - \omega_1}\right]$$

$$= \frac{\omega_1}{\alpha^2 + \omega_1^2 - \omega^2 + j2\alpha\omega}$$

Example iv) Exponentially decaying sinusoid



Time domain



Fourier Spectrum

Unit Impulse functions

$$\mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t} dt$$
$$= e^{j0}\ (sifting\ property)$$
$$= 1\ for\ all\ \omega$$

- Recall sifting property of δ function

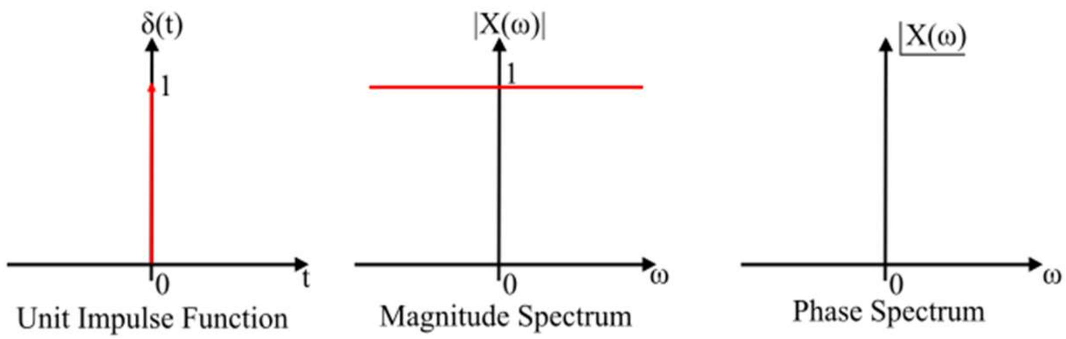
$$\int_{-\infty}^{\infty} \delta(t) f(t) dt = f(0)$$

Delayed impulse response:

$$\mathcal{F}[\delta(t - t_0)] = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega t} dt$$
$$= e^{-j\omega_0 t}$$

- Phase spectrum of shifted unit impulse is linear with slope t_0, i.e.

$$|\Delta(\omega)| = 1$$
$$\theta(\omega) = -\omega t_0$$



Complex exponentials

- What is $\mathcal{F}\left[e^{j\omega_0 t}\right]$?
- We would expect that the frequency components of $e^{j\omega_0 t}$ would be concentrated at ω_0 i.e. $\delta(\omega - \omega_0)$

$$\mathcal{F}\{\delta(\omega - \omega_0)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} e^{j\omega_0 t}$$

$$\Rightarrow \mathcal{F}\left\{e^{j\omega_0 t}\right\} = 2\pi\delta(\omega - \omega_0)$$

Eternal sinusoids

$$\mathcal{F}\{sin\omega_0 t\} = \mathcal{F}\left\{\frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right]\right\}$$

$$= \frac{1}{2j}2\pi\delta(\omega - \omega_0) - \frac{1}{2j}2\pi\delta(\omega - \omega_0)$$

$$= \frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega - \omega_0)]$$

- Note that finding the Fourier Transform of signals of infinite energy involves the use of delta functions, i.e. it is not as straightforward as for signals of finite energy

Fourier Transform of a constant (DC)

- What is $\mathcal{F}^{-1}\{\delta(\omega)\}$?

$$\mathcal{F}^{-1}\{\delta(\omega\} = 1/2\pi \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi}$$

This leads to the Fourier Transform pair

$$2\pi\delta(\omega) \rightleftharpoons 1$$

Fourier Transform of a step u(t)

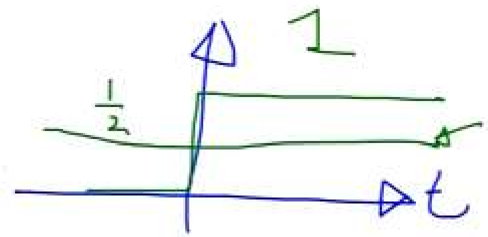
- Recall

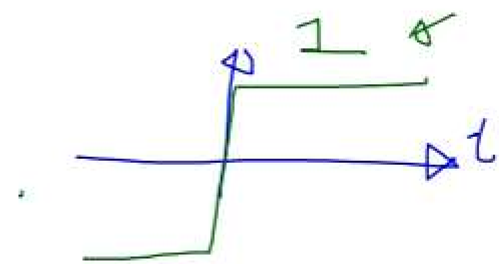
$$\mathrm{sgn}(y) \rightleftharpoons \frac{2}{j\omega}$$

$$u(t) = \frac{1}{2} + \frac{1}{2}\mathrm{sgn}(t)$$

$$\mathcal{F}\{u(t)\} = \mathcal{F}\left\{\frac{1}{2} + \frac{1}{2}\mathrm{sgn}(t)\right\}$$

$$= \pi\delta(\omega) + \frac{1}{j\omega}$$





Note that treating the step as $\lim_{\alpha \to 0} e^{-at}$ will not give the correct result

Periodic functions

$$f_T(t) - periodic, with\ period\ T$$

$$f_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\omega_0 t}, \omega_0 = \frac{2\pi}{T}$$

$$\mathcal{F}\{f_T(t)\} = \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} e^{jn\quad {}_0 t}\right\}$$

$$= \sum_{n=-\infty}^{\infty} c_n \mathcal{F}\{e^{jn\omega_0 t}\}$$

$$= 2\pi \sum_{n=-\infty}^{\infty} c_n \delta(\omega - n\omega_0)$$

Thus the spectrum of a periodic signal consists of a set of weighted impulses at the frequencies of the harmonics

Fourier Transform of a train of unit impulses

Recall

$$\delta_t(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}$$

Thus

$$\delta_t(t) \rightleftharpoons \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$$

Properties of the Fourier Transform

linearity	$af(t) + bg(t)$	$aF(\omega) + bG(\omega)$
time scaling	$f(at)$	$\frac{1}{\lvert a \rvert} F(\frac{\omega}{a})$
time shift	$f(t - T)$	$e^{-j\omega T} F(\omega)$
differentiation	$\frac{df(t)}{dt}$	$j\omega F(\omega)$
	$\frac{d^k f(t)}{dt^k}$	$(j\omega)^k F(\omega)$
integration	$\int_{-\infty}^{t} f(\tau) d\tau$	$\frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$
multiplication with t	$t^k f(t)$	$j^k \frac{d^k F(\omega)}{d\omega^k}$
convolution	$\int_{-\infty}^{\infty} f(\tau) g(t - \tau)\, d\tau$	$F(\omega)G(\omega)$
multiplication	$f(t)g(t)$	$\frac{1}{2\pi} \int_{-\infty}^{\infty} F(\widetilde{\omega}) G(\omega - \widetilde{\omega})\, d\widetilde{\omega}$

Properties of the Fourier Transform (Stremler 3.6)

- Linearity
$$\mathcal{F}\{af_1(t) + bf_2(t)\} = aF_1(\omega) + bF_2(\omega)$$

- Complex conjugate

$$For\ f(t)\ complex,$$

$$\mathcal{F}\{f^*(t)\} = F^*(-\omega)$$

$$f(t) = \mathcal{F}^{-1}\{F(\omega)\}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

$$f^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega)e^{-j\omega t}d\omega$$

Replace ω by $-\omega$ to get

$$f^*(t) = -\frac{1}{2\pi}\int_{\infty}^{-\infty} F^*(-\omega)e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(-\omega)e^{j\omega t}d\omega$$

i.e. $f^*(t)$ is the inverse Fourier Transform of $F^*(-\omega)$

Properties of the Fourier Transform (Stremler 3.6)

- Duality

$$if\ g(t) \rightleftharpoons G(\omega), then$$

$$G(t) \rightleftharpoons 2\pi g(-\omega)$$

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}d\omega$$

$$\Rightarrow g(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{-j\omega t}d\omega$$

Interchanging t and ω, we get

$$g(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(t)e^{-jt\omega}dt$$

$$\Rightarrow 2\pi g(-\omega) = \int_{-\infty}^{\infty} G(t)e^{-j\omega t}dt$$

$$i.e. \mathcal{F}\{G(t)\} = 2\pi g(-\omega)$$

For example,

$$if\ rect(t) \rightleftharpoons Sa\left(\frac{\omega}{2}\right), what\ is\ \mathcal{F}\left\{Sa\left(\frac{t}{2}\right)\right\}?$$

$$ans: 2\pi rect(\omega)$$

Properties of the Fourier Transform (Stremler 3.6)

- Time scaling

$$if\ f(t) \rightleftharpoons F(\omega), then$$

$$f(\alpha t) \rightleftharpoons F\left(\frac{\omega}{\alpha}\right).\frac{1}{|\alpha|}$$

$$\mathcal{F}\{f(\alpha t)\} = \int_{-\infty}^{\infty} f(\alpha t)e^{-j\omega}\ dt$$

$\alpha > 0$:

$$x = \alpha t \Rightarrow dx = \alpha dt \rightarrow dt = \frac{dx}{\alpha}$$

$$\Rightarrow \mathcal{F}\{f(\alpha t)\}$$

$$= \frac{1}{\alpha}\int_{-\infty}^{\infty} f(x)e^{-j\omega\frac{x}{\alpha}}dx$$

$$= \frac{1}{\alpha}F\left(\frac{\omega}{\alpha}\right)$$

Properties of the Fourier Transform (Stremler 3.6)

$$\alpha < 0:$$

$$\mathcal{F}\{f(\alpha t)\} = \frac{1}{|\alpha|} F\left(\frac{\omega}{\alpha}\right)$$

(note: with $\alpha - ve$, the limits of integration are reversed with the variable of integration is changed)

$$= -\frac{1}{\alpha}\int_{-\infty}^{\infty} f(x) e^{-j\omega\frac{x}{\alpha}} dx$$

$$= \frac{1}{|\alpha|} F\left(\frac{\omega}{\alpha}\right)$$

Combining the two cases, we have

$$\mathcal{F}\{f(\alpha t)\} = \frac{1}{|\alpha|} F\left(\frac{\omega}{\alpha}\right)$$

- i.e. if a time function is expanded by a factor α, its spectral density is compressed
- The scaling factor $\frac{1}{|\alpha|}$ is necessary to maintain an energy balance between the time and frequency domains

Properties of the Fourier Transform (Stremler 3.6)

- Time shifting

$$\mathcal{F}\{f(t-t_0) = F(\omega)e^{j\omega t_0}$$

$$\mathcal{F}\{f(t-t_0)\} = \int_{-\infty}^{\infty} f(t-t_0)e^{j\omega t}dt$$

$$Let\ x = t - t_0 \Rightarrow dx = dt, \mathrm{t} = \mathrm{x} - \mathrm{t}_0$$

$$\Rightarrow \mathcal{F}\{f(t-t_0)\} = \int_{-\infty}^{\infty} f(x)e^{-j\omega(x+t_0)}dx$$

$$= e^{-j\omega t_0}\int_{-\infty}^{\infty} f(x)e^{-j\omega x}dx$$

$$= e^{-j\omega t_0}F(\omega)$$

- Thus, if a signal is shifted in time, its magnitude is unchanged, but its phase is modified by an additional $-\omega \mathrm{t}_0$

Properties of the Fourier Transform (Stremler 3.6)

- Frequency shifting

$$\mathcal{F}\left\{f(t)e^{j\omega_0 t}\right\} = F(\omega - \omega_0)$$

Proof

$$\mathcal{F}\left\{f(t)e^{j\omega_0 t}\right\}$$

$$= \int_{-\infty}^{\infty} f(t)e^{j\omega_0 t}e^{-j\omega_0 t}dt$$

$$= \int_{-\infty}^{\infty} f(t)e^{-j(\omega-\omega_0)t}dt$$

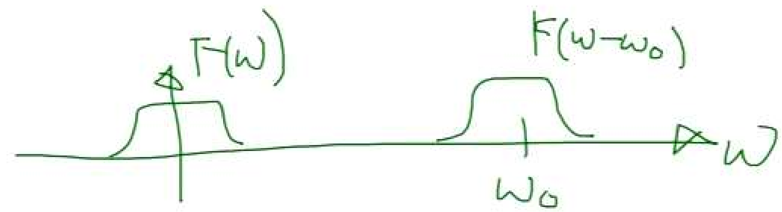
$$= F(\omega - \omega_0)$$

$$e.g.\, f(t)\cos(\omega_0)\,t$$

$$= f(t)\frac{1}{2}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$$

$$\mathcal{F}\{f(t)cos\omega_0 t\}$$

$$= \frac{1}{2}[F(\omega - \omega_0) + f(\omega + \omega_0)]$$



Properties of the Fourier Transform (Stremler 3.6)

- Time convolution

$$\mathcal{F}\{f_1(t) * f_2(t)\} = F_1(\omega)F_2(\omega)$$

$$f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)d\tau$$

$$\mathcal{F}\{f_1(t) * f_2(t)\} = \int_{-\infty}^{\infty} [f_1(\tau)f_2(t-\tau)d\tau]e^{-j\omega t}dt$$

Changing the order of integration, we get

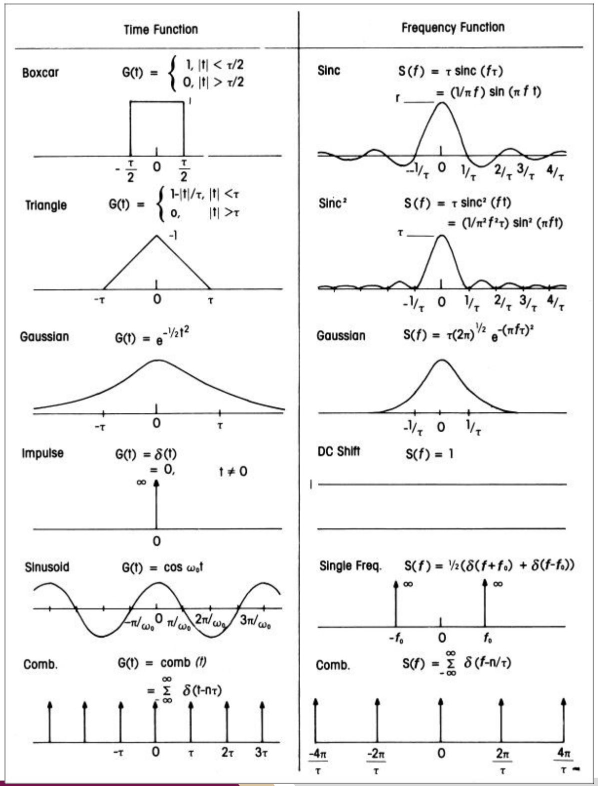
$$\mathcal{F}\{f_1(t) * f_2(t)\} = \int_{-\infty}^{\infty} f_1(\tau)\left[\int_{-\infty}^{\infty} f_{2(t-\tau)}e^{-j\omega t}\,dt\right]d\tau$$

$$\mathcal{F}\{f_2(t-\tau)\} = F_2(\omega)e^{-j\omega}$$

$$\Rightarrow \mathcal{F}\{f_1(t) * f_2(t)\} = F_2(\omega)\int_{-\infty}^{\infty} f_1(\tau)e(^{-j\omega\tau})d\tau$$

$$= F_1(\omega)F_2(\omega)$$

Some useful Fourier Transform pairs



https://wiki.seg.org/wiki/Dictionary:Fourier_transform