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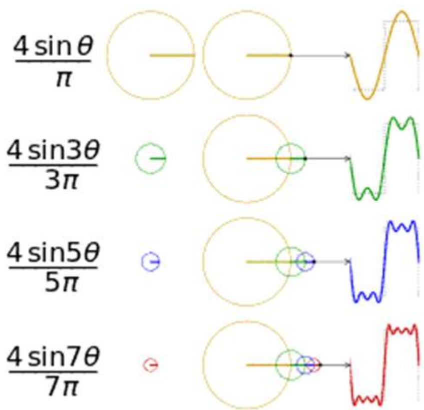
EE357 – Signals & Communications

Section 1 – Fourier Series

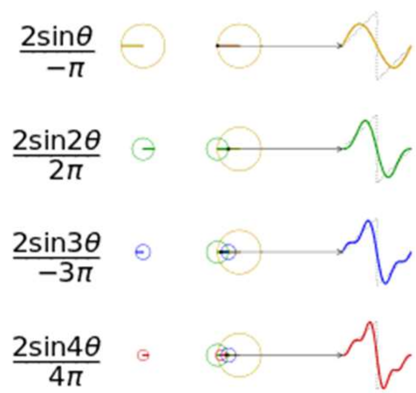
Introduction

- The Fourier series $s_N(x)$ represents a synthesis of a periodic function $s(x)$ by summing harmonically related sinusoids, whose coefficients are determined by harmonic analysis
- In simpler terms, almost any *periodic* function can be represented by summing harmonically related sinusoids with correctly chosen magnitude and phase components
- This section of the course explores how to use Fourier Series for signal/system analysis and definition

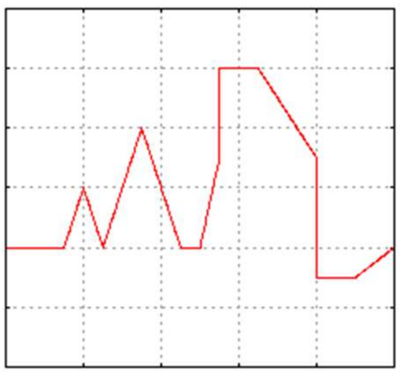
Visual examples
(https://en.wikipedia.org/wiki/Fourier_series)



Example 1: Fourier Series with 4 terms to approximate a square wave

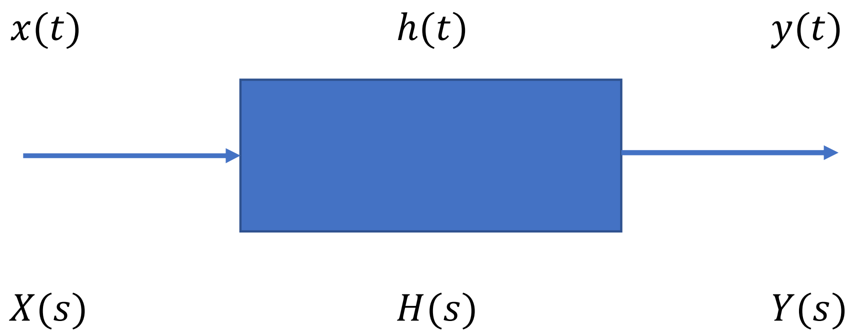


Example 2: Fourier Series with 4 terms to approximate a sawtooth wave



Example 3: Example of convergence of to a somewhat arbitrary function

Fourier Analysis of Periodic Signals
LTI Systems



$$y(t) = x(t) * h(t)$$

$$Y(s) = X(s).H(s)$$

Properties of LTI systems

- Superposition
  - If $x_1(t) \rightarrow y_1(t)$
    $x_2(t) \rightarrow y_2(t)$

  If a system is linear, then

$$Ax_1(t) + Bx_2(t) \rightarrow Ay_1(t) + By_2(t)$$

- Time-Invariance

$$x(t - T) \rightarrow y(t - T) \forall T$$

LTI system response

- We can analyse the response of an LTI system to any arbitrary waveform by examining the response to a particular set of “basis” functions, and then expressing our arbitrary waveform in terms of this set

i.e. if $x(t) = \sum_{n=-\infty}^{\infty} c_n \emptyset_n(t)$

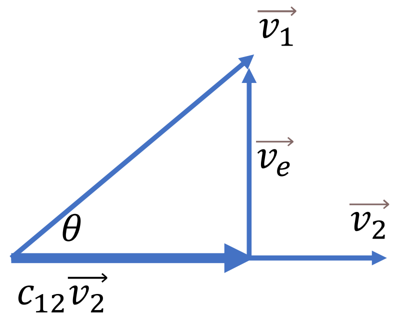
and if $\emptyset_n(t) \rightarrow \psi_n(t)$

then $y(t) = \sum_{n=-\infty}^{\infty} c_n \psi_n(t)$

$\emptyset$ = basis functions
ψ = system response

Characteristics of basis functions

- Vectors



Note: $c_{12} = \frac{|\overrightarrow{v_1}|}{|\overrightarrow{v_2}|}\cos\theta$

$$\overrightarrow{v_1} = c_{12}\overrightarrow{v_2} + \overrightarrow{v_e}$$

$$|C_{12}\vec{v}_2| = |\overrightarrow{v_1}|\cos\theta$$

- The component of $\overrightarrow{v_1}$ along $\overrightarrow{v_2}$ is given by $c_{12}\overrightarrow{v_2}$. c_{12} is a measure of the similarity between $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ and is calculated as follows in order to minimize $\overrightarrow{v_e}$

$$c_{12} = \frac{\overrightarrow{v_1} \cdot \overrightarrow{v_2}}{\overrightarrow{v_2} \cdot \overrightarrow{v_2}} = \frac{\overrightarrow{v_1} \cdot \overrightarrow{v_2}}{|\overrightarrow{v_2}|^2} \qquad (1)$$

Where "." denotes scalar (or dot) product

Characteristics of basis functions

- If C_{12} is zero, either $\overrightarrow{v_1}$ or $\overrightarrow{v_2}$ has zero length, or the vectors are *orthogonal*
- Two vectors are orthogonal if

$$\vec{\emptyset}_n . \vec{\emptyset}_m = \begin{cases} L_n^2, & n = m \\ 0, & n \neq m \end{cases}$$

Where L_n is the length of either vector

Characteristics of basis functions

- In 3D, it is common to represent an arbitrary vector $\vec{v}$ in terms of a set of mutually orthogonal unit vectors $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$
  - i.e. $\vec{v} = v_1\vec{\imath} + v_2\vec{\jmath} + v_3\vec{k}$

  Where $v_1 = \frac{\vec{v}\cdot\vec{\imath}}{\vec{\imath}\cdot\vec{\imath}} = \vec{v}\cdot\vec{\imath}$

  ...etc
- Because the basis set is orthogonal, the calculation of any component v is independent of all of the other components

Back to Signals (1)

- Suppose we want to approximate $x_1(t)$ by $x_2(t)$ over an interval $[t_1, t_2]$. We can write

$$x_1(t) = c_{12}x_2(t) + x_e(t)$$

- We want to chose c_{12} in order to minimize the *mean squared value* of $x_e(t)$, which is given by

$$\overline{x_e^2(t)} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} x_e^2(t)\, dt$$

$$= \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} [x_1(t) - c_{12}x_2(t)]^2\, dt$$

Back to Signals (2)

- To minimize $x_e(t)$ w.r.t. c_{12},

$$\frac{d}{dC_{12}}[x_e^2(t)] = 0$$

- This will yield

$$c_{12} = \frac{\int_{t_1}^{t_2} x_1(t)x_2(t)dt}{\int_{t_1}^{t_2} x_2^2(t)\,dt} \qquad (2)$$

If $c_{12} = 0$, then $x_1(t)$ and $x_2(t)$ are orthogonal over $[t_1, t_2]$, i.e.

$$\int_{t_1}^{t_2} x_1(t)x_2(t)dt = 0$$

Back to Signals (3)

- In the most general case of complex-valued functions, two functions $\emptyset_1(t)$ and $\emptyset_2(t)$ are orthogonal if

$$\int_{t_1}^{t_2} \emptyset_1(t)\emptyset_2^*(t)\, dt = \int_{t_1}^{t_2} \emptyset_1^*(t)\emptyset^2(t)dt = 0$$

over the interval $[t_1, t_2]$

- The members of a set of complex-valued functions are mutually orthogonal if

$$\int_{t_1}^{t_2} \emptyset_n(t)\emptyset_m^*\,(t)dt = \begin{cases} k_n & n = m \\ 0 & n \neq m \end{cases}$$

Where $\int_{t_1}^{t_2} \emptyset_n(t)\emptyset_m^*\,(t)dt$ is called the inner product of $\emptyset_n(t)$ and $\emptyset_m^*(t)$, and $k_n = \int_{t_1}^{t_2}|\emptyset_n(t)|^2\, dt$

Back to Signals (4)

- The square root of k_n is called the *norm* of $\emptyset_n(t)$.
- Note the similarity between the inner product and the dot product, and the norm and vector length
  - Review Equations (1) and (2)

Choice of basis functions

- Recall that LTI systems are described by linear, constant coefficient differential equations.
- If we apply an input $\emptyset(t)$ to such a system such that the output is given by $y(t) = 6 \cdot \emptyset(t)$, i.e. the output is a scaled version of the inputs, then $\emptyset(t)$ is called an **eigenfunction** of the system

Choice of basis functions

- It so happens that the set of complex exponentials

$$\emptyset(t) = e^{st} \qquad\qquad s = \sigma + j\omega$$

are eigenfunctions of LTI systems, thus such functions would seem to be a natural choice for a set of basis functions with which to represent an arbitrary input to an LTI system

- For convergence reasons, $\sigma = 0$, thus the basis functions are of the form $e^{Ij\omega t}$

Fourier Series

- Suppose we are dealing with periodic signals i.e. $x(t) = x(t + T)$, where T is the period
- Suppose also that we are dealing with a set of complex exponentials, which are harmonically related i.e $\emptyset_n(t) = e^{jn\omega_0 t}$

  where ω_0 is the fundamental frequency, equal to $\frac{2\pi}{T}$
- It can be shown (Stremer, section 2.7) that $\emptyset_n(t)$ are orthogonal over a single period of $x(t)$

Fourier Series

- IT can also be shown that

$$x(t) = \sum_{n-\infty}^{\infty} c_n e^{jn\omega_0 t} \text{ where } \omega_0 = \frac{2\pi}{T}$$

- i.e. an arbitrary periodic signal $x(t)$ can be represented by an infinite series of complex exponentials which are harmonically related
- This is known as the Fourier Series representation of $x(t)$

Fourier Series

- Also note that if

$$e(t) = x(t) - \sum_{n=-N}^{N} c_n e^{jn\omega_0 t}$$

Then $\lim_{N \to \infty} e(t) = 0$

- i.e. the error can be made as small as desired by using a large enough number of terms in the Fourier Series. The advantage of using orthogonal basis functions lies in the fact that the calculation of any c_n is independent of the calculation of any other, so the coefficient of a given harmonic function is unaffected by adding other harmonics

Calculation of c_n

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jm\omega_0 t}$$

- Multiply both sides by $e^{-jm\omega_0 t}$, and integrate over one period

$$\int_0^T x(t)e^{-j^m\omega_0 t}\,dt = \int_0^T \sum_{n=-\infty}^{\infty} C_n e^{j^n\omega_0 t}e^{-jm\omega_0 t}\,dt$$

- Interchange the integral and the summation of the RHS to obtain

$$\int_0^{-T} x(t)e^{-jm\omega_0 t}\,dt = \sum_{n=-\infty}^{\infty} C_n \int_0^T e^{jn\quad {}_0 t}e^{-jm\omega_0 t}\,dt$$

Calculation of c_n

- Since the complex exponentials are orthogonal over $[0, T]$, the integral will be zero, except where $m = n$, where it equals T. Hence

$$\int_0^T x(t)e^{-jn\omega_0 t}\,dt = c_n T$$

Therefore

$$c_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}\,dt$$

Note that for $n = 0$,

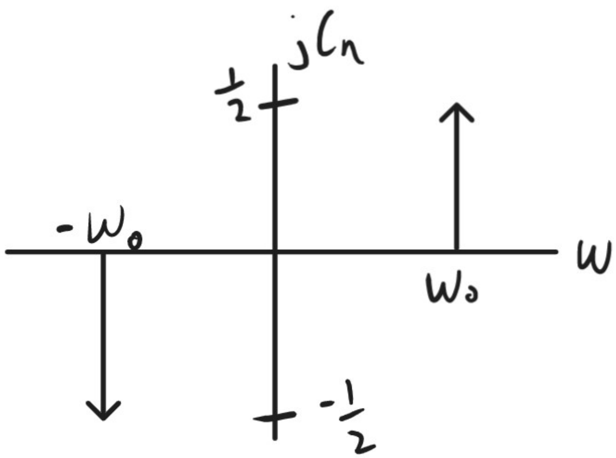
$$C_n = \frac{1}{T}\int_0^T x(t)\,dt$$

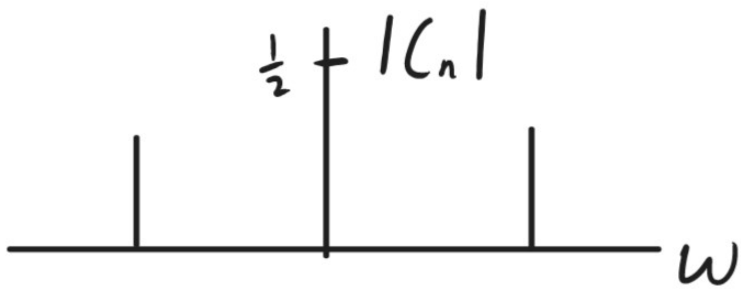
i.e. the **DC level**

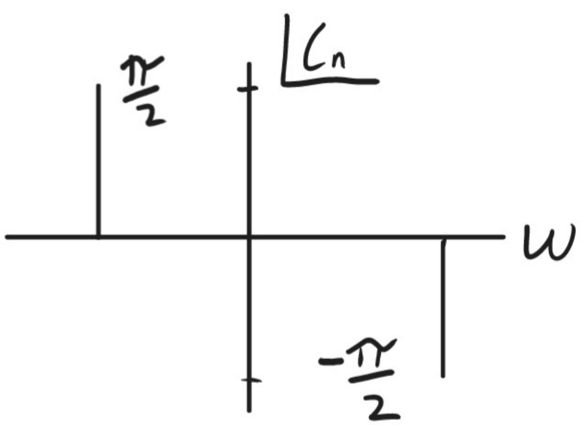
- This is the Fourier Series Analysis equation

Spectrum – discrete or line spectrum…

- E.g. $\sin \omega_0 t$







When does the Fourier Series exist?

- It can be shown that, if a function $f(t)$ satisfies a number of conditions, then its $F(s)$ can be found
- These conditions are as follows

i) $\int_{t_0}^{t_0+T} |f(t)|^2\, dt < \infty$ (i.e. finite energy in a single period)

ii) $f(t)$ has a finite number of maxima and minima over $[t_0, t_0 + T]$

iii) $f(t)$ has a finite number of discontinuities over $[t_0, t_0 + T]$

- These conditions are called the Dirichlet conditions, and are sufficient, but not necessary, for the existence of the Fourier Series of $f(t)$

Fourier Spectrum

- The Fourier Series can be represented graphically by means of the spectrum which shows the variation of c_n vs n or ω
- Since the Fourier Series is non-zero only for discrete frequencies, the spectrum is a discrete spectrum
- Generally, c_n is complex, so two diagrams are used
  - One for the magnitude of c_n
  - One for the angle of c_n
- In some cases, there is a simple expression for c_n, and a single diagram will suffice

Example

$$f(t) = sin\omega_0 t$$

- By Euler's identities:

$$cos\omega_0 t = \left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$$

$$sin\omega_0 t = \left[e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t}\right]$$

$$f(t) = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t}$$

n=1 n=-1

Example

- Note that to represent a real function requires that its Fourier Series consists of pairs of complex conjugate components
- Each c_n represents the magnitude and starting angle of the corresponding complex exponential
- To represent a real valued function, the magnitudes of the two conjugate phases must be equal and their instantaneous phase angles must be equal by opposite

Example

- If

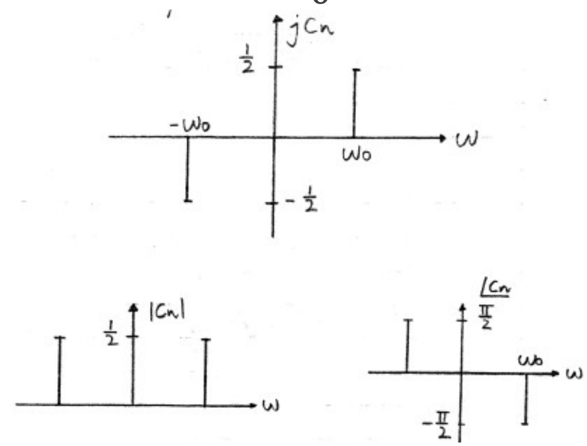
$$c_n = a_n + jb_n$$

$$c_n = e^{j\theta n}$$

$$|c_n| = \sqrt{a_n^2 + b_n^2}$$

$$\theta_n = tan^{-1}\frac{b_n}{a_n}$$

- Spectrum of $sin\omega_0 t$



- Note that for real $f(t)$, $|c_n|$ is an even function of ω or n, and the angle of c_n is an odd function of ω or n

When does the Fourier Series exist?

- It can be shown that, if a function f(t) satisfies a number of conditions, then its FS can be found. These conditions are as follows:
  - $\int_{t_0}^{t_0+T} |f(t)|^2 \, dt < 0$ (i.e. finite energy in a single period)
  - f(t) has a finite number of maxima and minimal over $[t_0, t_0 + T]$
  - f(t) has a finite number of discontinuities over $[t_0, t_0 + T]$
- These conditions are called the Duvichet conditions, and are sufficient but not necessary for the existence of the FS of f(t)

The Fourier Spectrum

- The FS can be represented graphically by means of the spectrum which show the variation of c_n vs n or ω
- Since the FS is non-zero only for discrete frequencies, the spectrum is a discrete spectrum
- Generally, c_n is complex, so two diagrams are used, one for the magnitude of c_n, and one for the phase angle
- In some cases, there is a simple expression for c_n, and a single diagram may suffice. This is unusual, however

Example

- $f(t) = \sin\omega_0 t$
- By Euler's identities:

$$\cos\omega_0 t = \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$$

$$sin\,\omega_0 t = \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right]$$

Substituting…

$$f(t) = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t}$$

Representing real functions

If

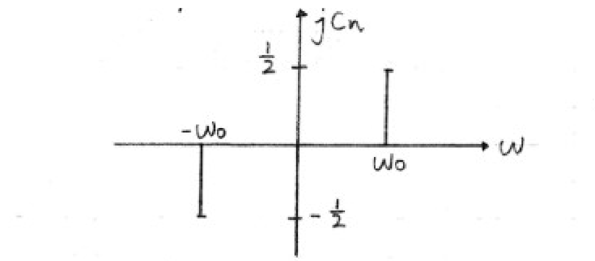
$$c_n = a_n + b_n$$

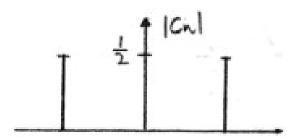
$$c_n = |c_n|e^{j\theta_n}$$

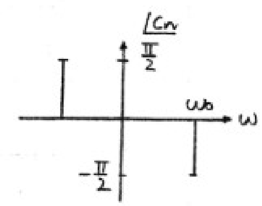
$$|c_n| = \sqrt{a_n^2 + b_n^2}$$

$$\theta_n = \tan^{-1}\frac{b_n}{a_n}$$

- Spectrum of $\sin\omega_0 t$







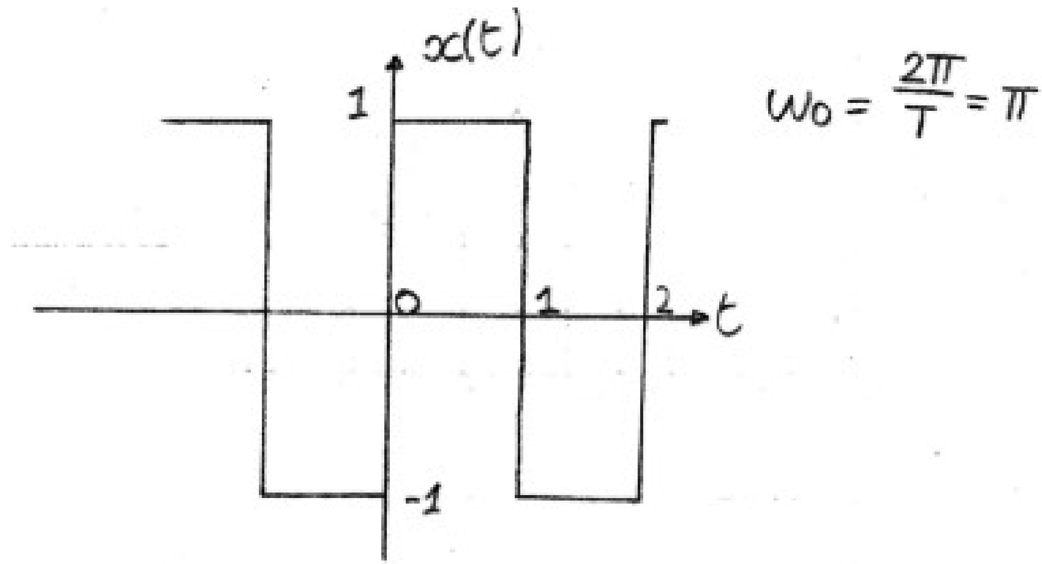
- Note that for real (f(t), $|c_n|$ is an even function of ω (or n), and the phase angle of c_n is an odd function of ω (or n)

Representing a real function

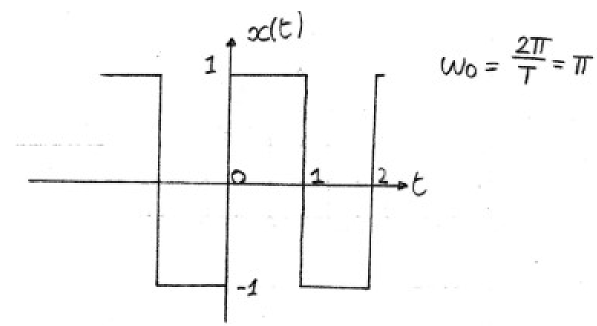
- To represent a real function requires that its FS consists of pairs of complex conjugate components
- Each c_n represents the magnitude and starting angle of the corresponding complex exponential
- To represent a real valued function, the magnitudes of two conjugate phases must be equal and their instantaneous phase angles must be equal but opposite

Examples...

- Example – square wave



Square wave



Synthesis equation: $x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\pi t}$ $\quad (note: set\ \omega_0 = \frac{2\pi}{T} = \pi)$

Analysis equation: $c_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jn\pi}\ dt$

Integrate over one period…

$$c_n = \frac{1}{2}\int_0^1 1.e^{-jn\pi}\ dt - \frac{1}{2}\int_1^2 1.e^{-jn\pi}\ dt$$

$$= \frac{1}{2jn\pi}\left[\left|-e^{-jn\pi t}\right|_0^1 + \left|e^{-jn\pi t}\right|_0^1\right]$$

$$= \frac{1}{j2n\pi}\left[-e^{-jn\pi} + e^0 + e^{-j2n\pi} - e^{-jn\pi}\right]$$

$$= \frac{1}{j2n\pi}\left[2 - 2e^{-jn\pi}\right]$$

$$= \frac{1}{jn\pi}\left[1 - e^{-jn\pi}\right]$$

$$n\ odd, e^{-jn\pi} = -1 \Rightarrow c_n = \frac{2}{jn\pi}$$

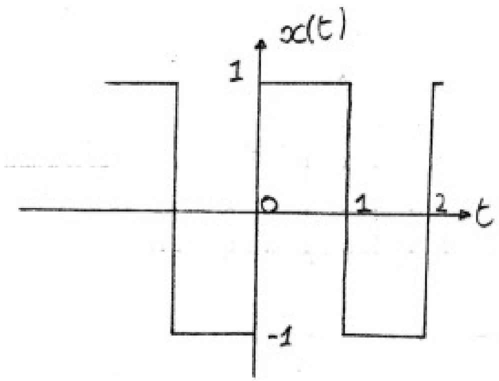
$$n\ even, e^{-jn\pi} = 1 \Rightarrow c_n = 0$$

$$\Rightarrow x(t) = \sum_{\substack{n=-\infty \\ n\ odd}}^{\infty} \frac{2}{jn\pi} e^{jn\omega_0 t}$$

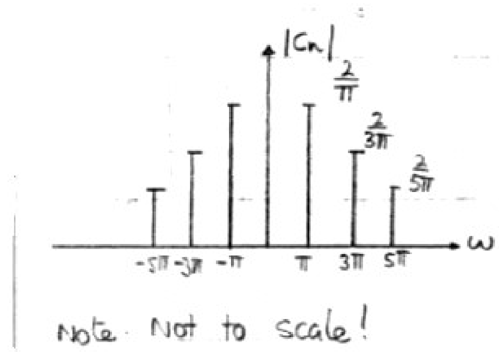
Square wave - spectrum

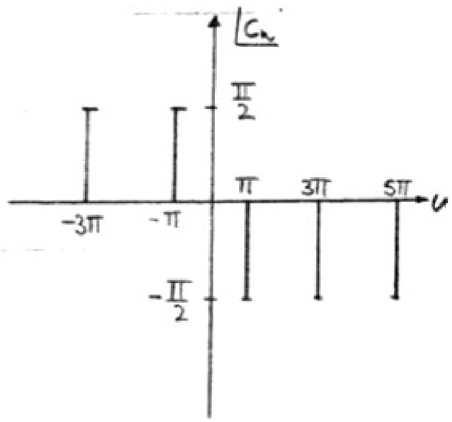
$$|c_n| = \frac{2}{n\pi}$$

$$\phi = \frac{1}{jn} = -jn$$



$\omega_0 = \frac{2\pi}{T} = \pi$



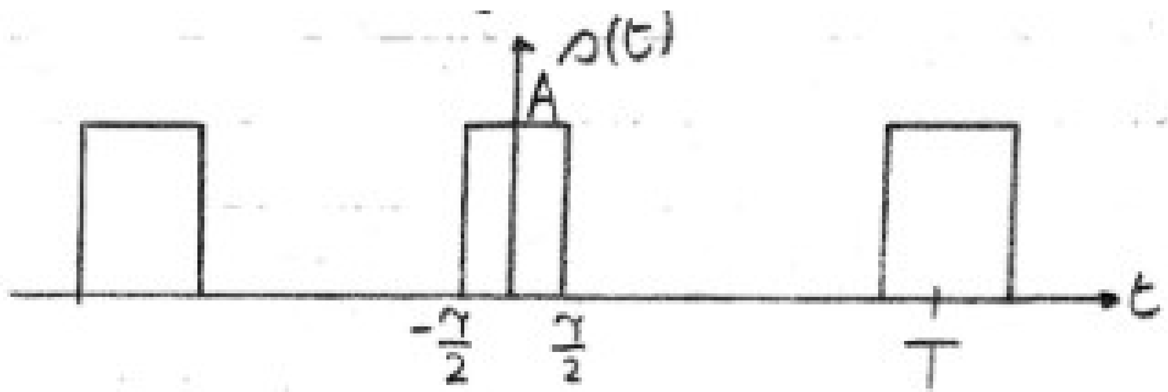


Time domain

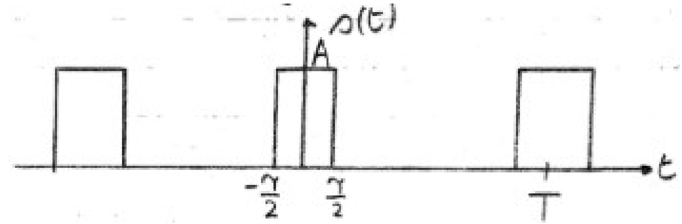
Fourier Series Spectrum

Examples…

- Example – rectangular pulse train



Example – rectangular pulse train



$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} Ae^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{T}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} Ae^{-jn\omega_0 t}\,dt$$

$$= \frac{-A}{jn\omega_0 T}\cdot e^{-j\omega_0 t}\Big|_{-\frac{\tau}{2}}^{\frac{\tau}{2}}$$

$$= -\frac{A}{jn2\pi}[e^{-jn\omega_0\left(\frac{\tau}{2}\right)} - e^{jn\quad_0\left(\frac{\tau}{2}\right)}]$$

$$= -\frac{A}{jn2\pi}[e^{jn\quad_0\left(\frac{\tau}{2}\right)} - e^{-jn\omega_0\left(\frac{\tau}{2}\right)}]$$

Note: $\omega_0 = 2\pi/T$

$$= \frac{A}{n\pi}\sin(n\pi\left(\frac{\tau}{T}\right))$$

Mult by $\frac{\frac{n\pi\tau}{T}}{\frac{n\pi\tau}{T}}$...

$$= \frac{A\tau}{T}\frac{\sin\left(n\pi\left(\frac{\tau}{T}\right)\right)}{n\pi(\frac{\tau}{T})}$$

$$= \frac{A\tau}{T}sinc(\frac{n\tau}{T})$$

Where $sinc(x) = \frac{\sin(\pi x)}{\pi x}$

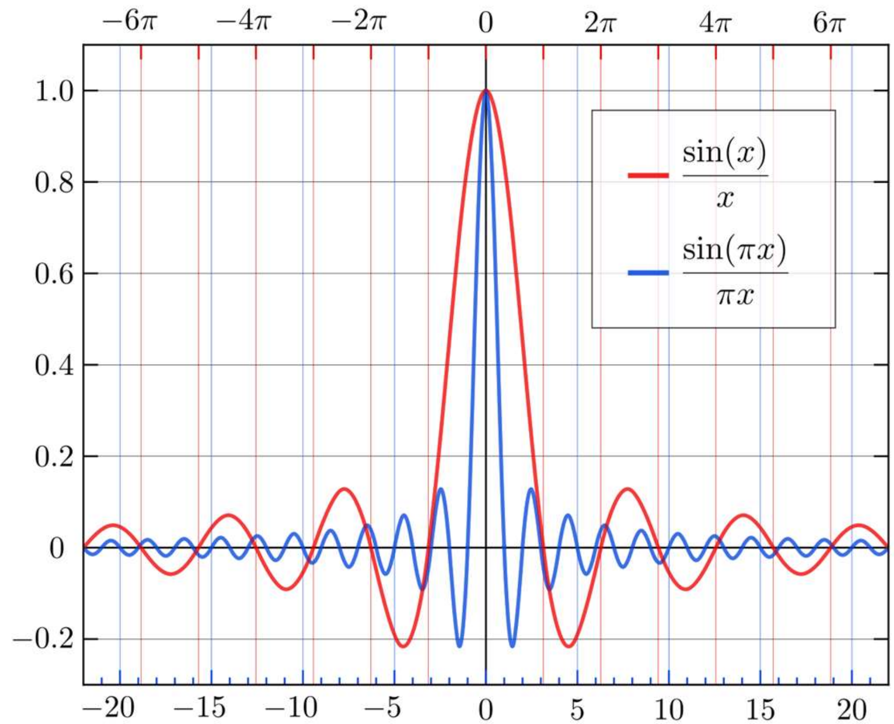
The sinc function

- Sinc(x) and Sa(x) appear frequently in signals and systems theory:

$$sinc(x) = \frac{\sin(\pi x)}{\pi x}$$

$$Sa(x) = \frac{\sin(x)}{x}$$

Note: $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$



Rectangular pulse train

- Returning to the rectangular pulse train,

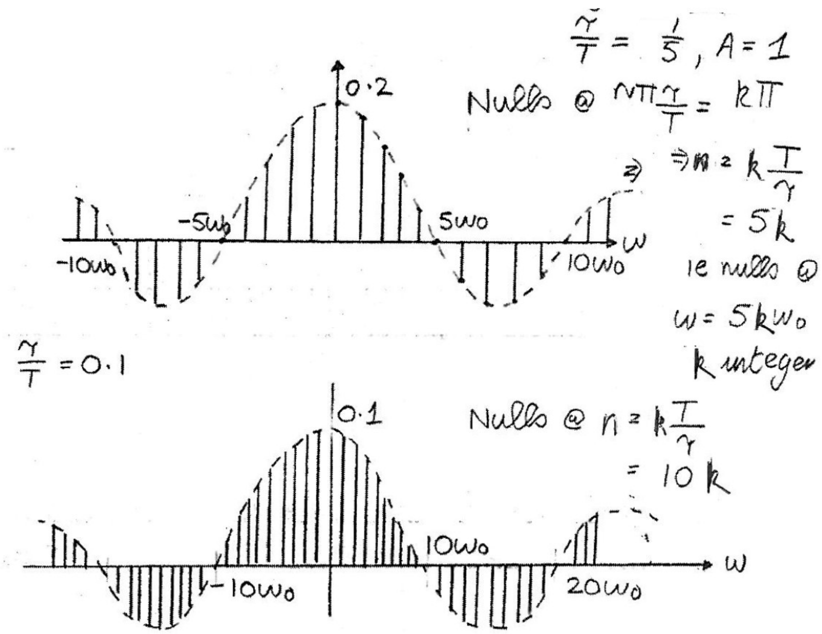
$$c_n = \frac{A\tau}{T} Sa(\frac{n\pi\tau}{T})$$

$$\therefore s(t) = \frac{A\tau}{T} \sum_{n=-\infty}^{\infty} Sa\left(\frac{n\pi\tau}{T}\right) e^{jn\ _0 t}$$

- Because c_n are all real, we only need one diagram to display the spectrum

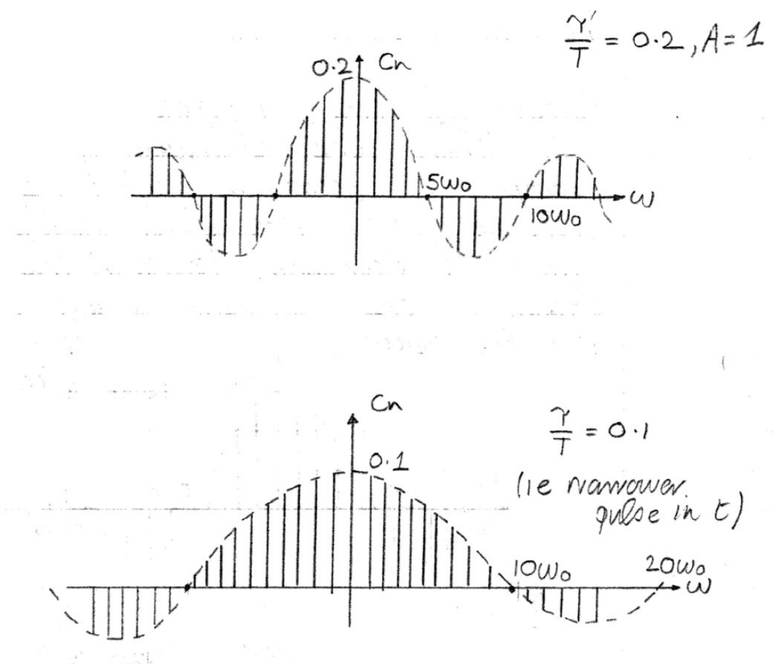
Rectangular pulse train

- Suppose we keep τ fixed, and increase T
- Then, the amplitude of each component decreases $\propto \frac{1}{T}$, and the spectral lines move closer together (since $\omega_0 = \frac{2\pi}{T}$ decreases)
- However, the overall shape of the spectrum remains constant



Rectangular pulse train

- If we keep T contrast and increase τ (i.e. make the pulse width wider), we find
  - 1) the amplitude increase ($A \propto \tau$)
  - 2) the frequency content of the signal is compressed within a narrow range of frequencies
- This illustrates the inverse relationship between duration in time and duration in frequency
  - The narrower the pulse in the time domain, the wider the spectrum in the frequency domain, and vice versa

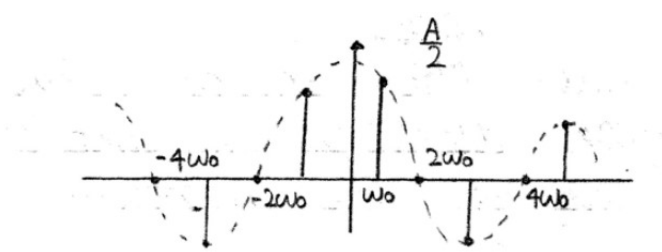


Note that ω_0 remains the same in both cases, since T is constant

Example – limiting cases of pulse train

- i) square pulse train, $\frac{\tau}{T} = 0.5$

$$x(t) = \frac{A}{2} \sum_{n=-\infty}^{\infty} Sa(\frac{n\pi}{2})\, e^{jn\omega_0 t}$$



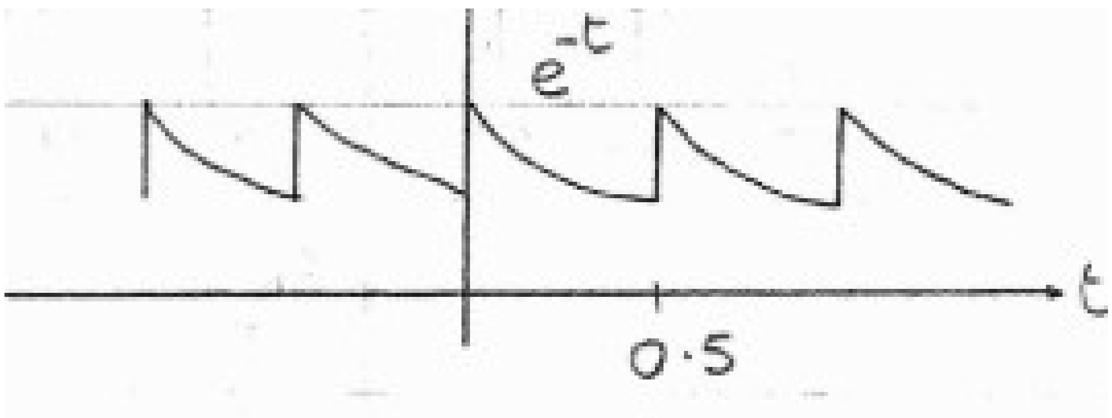
- ii) impulse train
- As $\tau \to 0$, let $A \to \infty$, so that $A\tau \to 1$ (recall $\int_{-\infty}^{\infty} \delta(t)dt = 1$)

$$c_n = \lim_{\tau \to 0} \frac{A\tau}{T} Sa\left(\frac{n\pi\tau}{T}\right) = \frac{1}{T}$$

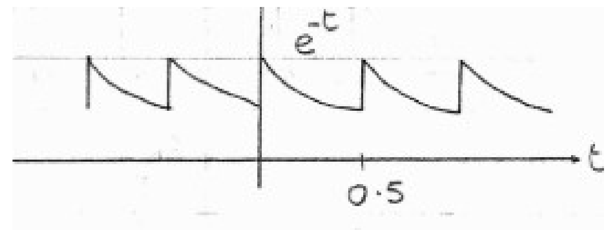
- i.e. each component in the Fourier Series has amplitude $\frac{1}{T}$

Examples

- periodic decaying exponential



Example - periodic decaying exponential



$$T = 0.5 \Rightarrow \omega_0 = 4\pi \; (i.e. \, \omega_0 = \frac{2\pi}{T})$$

$$c_n = 2\int_0^{0.5} e^{-t}\, e^{-j4n\pi}\; dt$$

$$= 2\int_0^{0.5} e^{-(1+j4n\ \ )}\; dt$$

$$= \frac{2}{1+j4n\pi} e^{-(1+j4n\ \ )}\Big|_{0.5}^{0}$$

$$= \frac{2}{1+j4n\pi} e^{-t}\Big|_{0.5}^{0}$$

$$(since\ e^{-t} e^{-(1+j4n\ \ )} = 1)$$

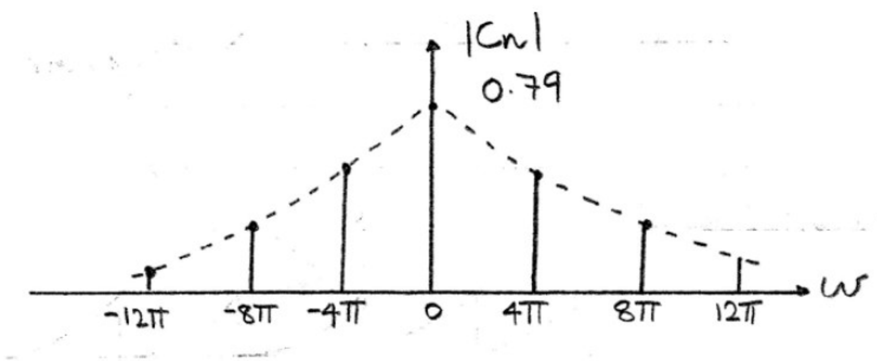
$$\Rightarrow c_n = \frac{2}{1+j4n\pi}[e^{0} - e^{-\frac{1}{2}}]$$

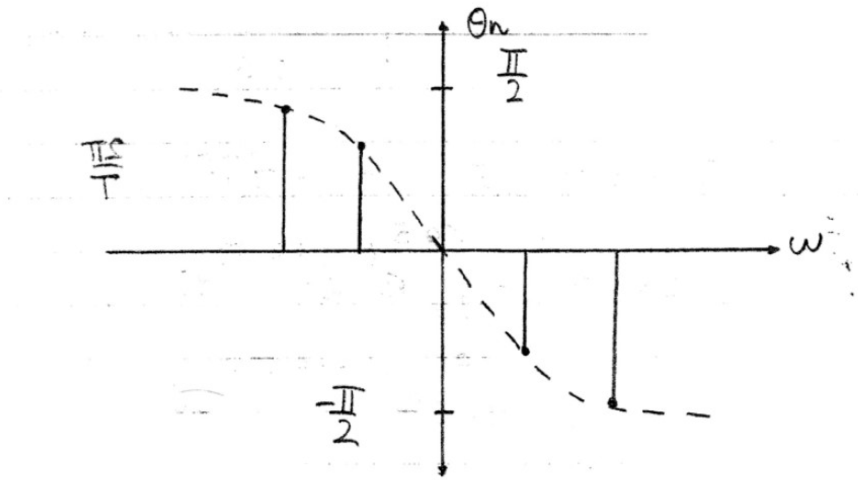
$$= \frac{0.79}{1+j4n\pi}$$

$$|c_n| = \frac{0.79}{1+16n^2\pi^2}$$

$$\theta_n = -\tan^{-1} 4n\pi$$

Example - periodic decaying exponential





Note that, since $x(t)$ is a real function, $|c_n|$ is an even function, and θ_n is an odd function of ω, as expected

Example - periodic decaying exponential

- In general, for real-valued $x(t)$,

$$c_{-n} = c_n^*$$

Where * is the complex conjugate

- To show that this is true for real x(t), recall

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

- Take the complex conjugate to obtain

$$x^*(t) = \sum_{n=-\infty}^{\infty} c_n e^{-jn\omega_0 t}$$

- Replacing –n by n, and noting that $x^*(t) = x(t)$ for real-valued $x(t)$, we have

$$x(t) = \sum_{n=-\infty}^{\infty} c_{-n}^* e^{jn\omega_0 t}$$

$$\Rightarrow c_n = c_{-n}^*$$

$$\Rightarrow c_n^* = c_{-n}$$

Parseval's Theorem for Periodic Functions

- For the general case of a complex-valued function f(t) = f_r + f_i, we have:

$$f^* = f_r - jf_i$$

$$f_r = \frac{1}{2}(f + f^*)$$

$$f_i = \frac{1}{2j}(f - f^*)$$

$$ff^* = |f^2| = |f^r|^2 + \left|f^i\right|^2$$

Parseval's Theorem for Periodic Functions

- The average power in such a signal is

$$P = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) f^*(t) dt = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 dt$$

- Substituting for f(t) and f*(t), we have

$$P = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{m=-\infty}^{\infty} c_m \, e^{jm\omega_0 t} \sum_{n=-\infty}^{\infty} c_n^* \, e^{jn\omega_0 t}$$

$$= \sum_{m=-\infty}^{\infty} c_m \sum_{n=-\infty}^{\infty} c_n^* \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{j(m-n)\omega_0 t} dt$$

Parseval's Theorem for Periodic Functions

- But, the complex exponentials are orthogonal over one period, hence the integral on the RHS is zero except for m=n. Hence, the double summation reduces to a single summation, and

$$P = \sum_{n=-\infty}^{\infty} c_n c_n^* = \sum_{n=-\infty}^{\infty} |c_n|^2$$

- Comparing this with the expression for P in the time domain, we get

$$P = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Parseval's Theorem for Periodic Functions

- This is called Parseval's Theorem for periodic signals. It means that the average power in a signal can be calculated in either the time domain or from the Fourier Spectrum coefficients
- E.g. f(t)=Asinω_0t

$$P = \frac{A^2}{2} \qquad (i.e.\ \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |Asin\omega_0 t|^2 dt)$$

- The FS is given by

$$f(t) = \frac{A}{2j}e^{j\omega_0 t} - \frac{A}{2j}e^{-j\omega_0 t}$$

$$c_1 = \frac{A}{2j}; c_{-1} = -\frac{A}{2j}$$

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = |c_1|^2 + |C_{-1}|^2 = \frac{A^2}{4} + \frac{A^2}{4} = \frac{A^2}{2}$$

Summary

- Almost any *periodic* function can be represented by summing harmonically related sinusoids with correctly chosen magnitude and phase components (i.e. Fourier Series)
- Complex exponentials are a common choice for Fourier Series basis functions
- Given enough terms, the error in a Fourier Series representation converges to zero

Summary

- A Fourier Series can be represented graphically by means of a Fourier spectrum plot, which typically is composed of a magnitude plot and a phase plot
- There is an inverse relationship between the extent of a signal in the time domain and the extent of the signal in the frequency domain
  - i.e. a narrow extent in time gives a wider spectral response, and vice versa
- From Parseval's theorem, the average power in a signal can be calculated either from the time domain or from the Fourier Spectrum coefficients