EE357 – Signals & Communications

Section 1 – Fourier Series

Part 2

Trigonometric Fourier Series

- Given a real function f(t), it should be possible to express f(t) in terms of a set of *real* basis functions
- Given two complex functions, f_1 and f_2, we can write

$$f_1 = f_{1r} + jf_{1i}, f_2 = f_{2r} + jf_{2i}$$

$$Re\{f_1 f_2\} = Re\{f_1\}Re\{f_2\} - Im\{f_1\}Im\{f_2\}$$

Trigonometric Fourier Series

- Given

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

We can write

$$f(t) = f_r(t) = \sum_{n=-\infty}^{\infty} Re\{c_n\}Re\left\{e^{jn\;{}_0 t}\right\} - \sum_{n=-\infty}^{\infty} Im\{c_n\}Im\{e^{jn\omega_0 t}\}$$

$$= \sum_{n=-\infty}^{\infty} Re(c_n)cosn\omega_0 t - \sum_{n=-\infty}^{\infty} Im(c_n)sinn\omega_0 t$$

Recall

$$c_n = \frac{1}{T}\int_0^T f(t)e^{-jn\omega_0 t}$$

Since f(t) is real, $c_n^* = c_n$

Trigonometric Fourier Series

- Since
  - $f_r = \frac{1}{2}(f + f^*), f_i = \frac{1}{2j}(f - f^*)$
- We can write
  - $Re\{c_n\} = \frac{1}{2}[c_n + c_{-n}]$
  - $Im\{c_n\} = \frac{1}{2j}[c_n - c_{-n}]$
- Let us define new variables as follows
  - $a_0 \equiv c_0$
  - $a_n \equiv [c_n + c_{-n}] = 2Re\{c_n\}\ for\ n \neq 0$
  - $b_n \equiv [c_n - c_{-n}] = -2Im\{c_n\}\ for\ n \neq 0$
  - $c_n = \frac{1}{2}(a_n - jb_n)\ for\ n \neq 0$

Trigonometric Fourier Series

- Since $a_n = 2Re\{c_n\}$, and since $Re\{c_n\} = Re\{c_n\}$ $(c_n^* = c_{-n})$, we can see that $a_n$is an even function of n
- Likewise, since $b_n = -2Im\{c_n\}$ and sine $Im\{c_n\} = -Im\{c_n\}$, b_n is an odd function of n
- Thus, we can double the coefficient in $\sum_{n=-\infty}^{\infty} Re(c_n) cosn\omega_0 t - \sum_{n=-\infty}^{\infty} Im(c_n) sinn\omega_0 t$
- And sum from 1 to ∞ …

Trigonometric Fourier Series

- This yields

$$f(t) = c_0 + \sum_{n=1}^{\infty} 2Re(c_n)cosn\omega_0 t - \sum_{n=1}^{\infty} 2Im(c_n)sinn\omega_0 t$$

Substituting for a_0, a_n, b_n, we get

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n cosn\omega_0 t - \sum_{n=1}^{\infty} b_n sinn\omega_0 t$$

- This is the Trigonometric Fourier Series for a real valued function f(t)

Trigonometric Fourier Series

- To calculate $a_0, a_n, and\ b_n$, we multiply both sides of the equation on the previous slide by $cosm\omega_0 t$ and $sinm\omega_0 t$ and integrate over one period. We will obtain

$$a_n = \frac{\int_{t_0}^{t_0+T} f(t)cosn\quad_0 tdt}{\int_{t_0}^{t_0+T} \cos^2 n\omega_0 tdt} = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)cosn\omega_0 tdt$$

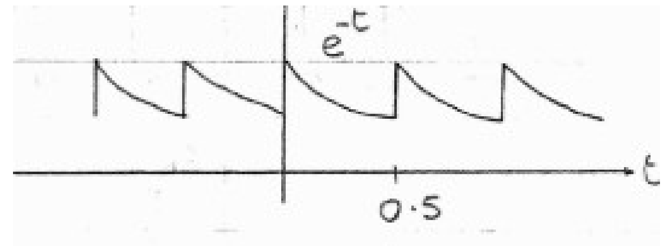
$$b_n = \frac{\int_{t_0}^{t_0+T} f(t)sinn\omega_0 tdt}{\int_{t_0}^{t_0+T} si\quad^2 n\omega_0 tdt} = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)sinn\omega_0 tdt$$

- To obtain a_0, we integrate both sides of the equation in the previous slide to obtain

$$a_0 = \frac{1}{T}\int f(t)dt$$

- These equations are the Fourier Series analysis equations

Example



- Find the trigonometric FS of the periodic exponential decaying waveform

$$T = {}^{1}/_{2} = \omega_0 = 4\pi$$

$$g(t) = a_0 + \sum_{n=-\infty}^{\infty} a_n cos4\pi nt + \sum_{n=-\infty}^{\infty} b_n sin4\pi nt$$

$$a_0 = 2\int_0^{1/2} \mathrm{e}^{-t}\,\mathrm{d}t = 0.79$$

$$a_n = 4\int_0^{\frac{1}{2}} e^{-t} cos4\pi ntdt \ \ (*)$$

$$\Rightarrow a_n = 0.79\left(\frac{2}{1+16\pi^2 n^2}\right)$$

$$b_n = 4\int_0^{\frac{1}{2}} e^{-t} sin4\pi ntdt = 0.79\left(\frac{8\pi n}{2+16\pi^2 n^2}\right)$$

$$(*)\ note\!: \int \mathrm{e}^{ax}\cos bx\,\mathrm{d}x = \frac{\mathrm{e}^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$$and\!: \int \mathrm{e}^{ax}\sin bx\,\mathrm{d}x = \frac{\mathrm{e}^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

Compact trigonometric representation

- The trigonometric Fourier Series can also be represented in a slightly more compact form as follows*:

$$f(t) = \sum_{n=0}^{\infty} d_n \cos(n\omega_0 t + \phi n)$$

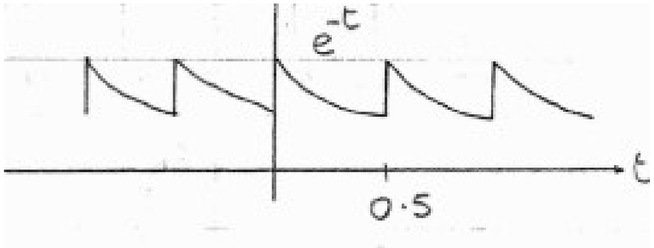
Where

$$d_n = \sqrt{a_n^2 + b_n^2}$$

$$\phi_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$$

*Note: $\mathrm{a}\cos(x) + bsin(x) = \sqrt{a^2 + b^2}\cos(x - \tan^{-1}\frac{b}{a})$

Example



- Comparing this with the expressions for a_n and b_n in terms of c_n, we get

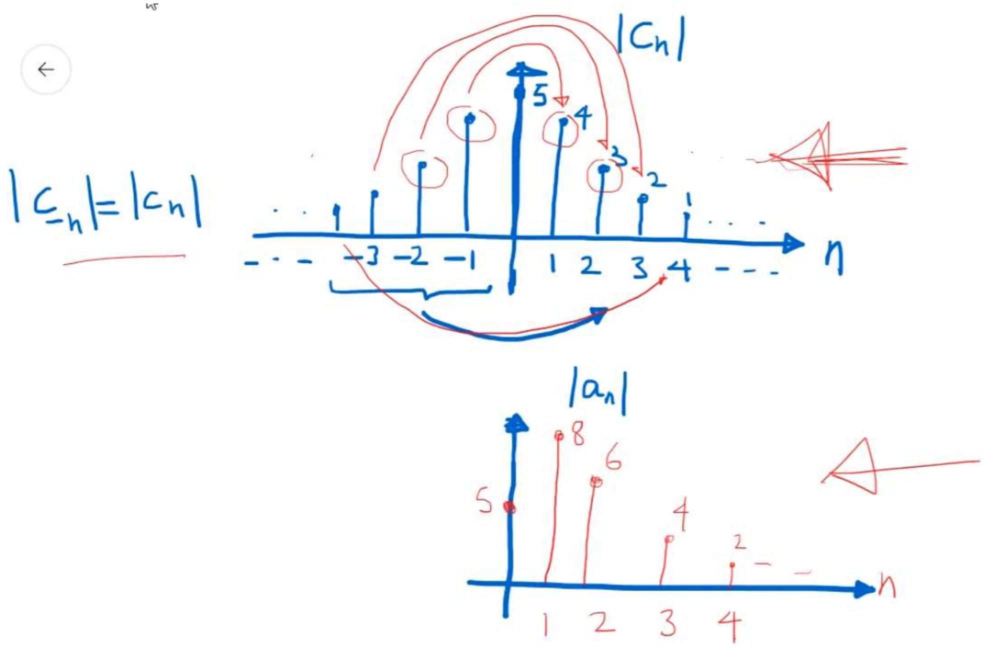
$d_n = 2|c_n|$ (i.e. $d_n = \sqrt{4Re^2\{c_n\} = 4Im^2\{c_n\}}$)

$$\phi_n = \tan^{-1}\frac{Im\{c_n\}}{Re\{c_n\}}$$

$$d_0 = c_0$$

- Recall that the exponential Fourier Series involves both $+ve$ and $-ve$ frequencies, and these components combine in conjugate pairs to form a real function of time
- Hence, a signal of frequency $n\omega_0$ (in the "ordinary" sense) is expressed as a sum of two signals of frequency $n\omega_0$ and $-n\omega_0$ (in the "exponential" sense)
- Thus, it would appear that the trigonometric Fourier Series spectrum is formed by folding the exponential Fourier Series Spectrum about the vertical axis and adding the superimposed components at $n\omega_0$ and $-n\omega_0$

Relationship between exponential and trigonometric spectra



Odd and Even Functions: half wave symmetry

- Recall, even symmetry: *f(-t) = f(t)*
- Such a function can only be built up using functions which are themselves even, i.e. the trigonometric Fourier Series for an even function will be

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n cos n\omega_0 t$$

i.e. $b_n = 0$ for all n

Odd and Even Functions: half wave symmetry

- Likewise, an odd function: *f(-t) = -f(t)*
- Has a trigonometric Fourier Series

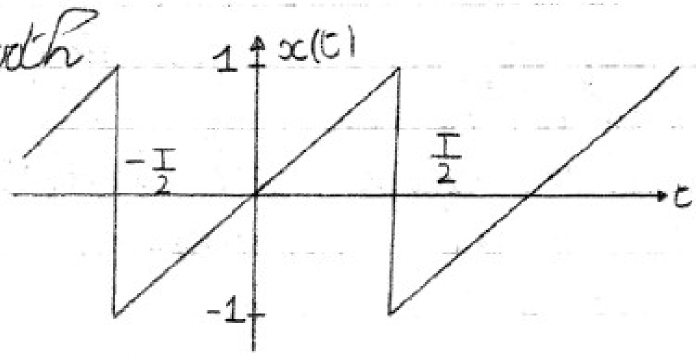
$$f(t) = a_0 + \sum_{n=1}^{\infty} b_n sin\omega_0 t$$

i.e. $a_n = 0$ for all *n > 0*

- Note that these simplifications are possible because $cosn\omega_0 t$ and $sinn\omega_0 t$ are even and odd functions
- These simplifications cannot be applied in the case of the exponential Fourier Series, since $e^{j\omega_0 t}$ is neither even nor odd

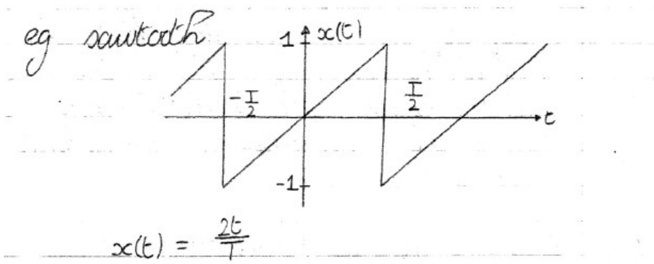
Example – sawtooth wave

eg sawtooth



$$x(t) = \frac{2t}{T}$$

Example – sawtooth wave



- Clearly, the average value = 0, $\Rightarrow a_n = 0 \,\forall\, n > 0$

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{2t}{T}\sin(n\omega_0 t)\,dt$$

$$= \left(\frac{2}{T}\right)^2 \int_{-\frac{T}{2}}^{\frac{T}{2}} t\sin(n\omega_0 t)\,dt$$

$$= \left(\frac{2}{T}\right)^2 \frac{1}{(n\omega_0)^2}\,[\sin(n\omega_0 t) - n\omega_0 t cos(n\omega_0 t]_{-\frac{T}{2}}^{\frac{T}{2}}$$

$$= \frac{4}{(n\omega_0 T)^2}\left[\sin\left(\frac{n\omega_0 T}{2}\right) - \frac{n\omega_0 T}{2}\cos\left(\frac{n\omega_0 T}{2}\right) - \sin\left(-\frac{n\omega_0 T}{2}\right) - \frac{n\omega_0 T}{2}\cos\left(\frac{n\omega_0 T}{2}\right)\right]$$

$$\omega_0 T = 2\pi, \sin(n\pi) = 0\forall n$$

$$b_n = \frac{4}{4n^2\pi^2}[\sin(n\pi) - n\pi\cos(n\pi) + \sin(n\pi) - n\pi\cos(n\pi)]$$

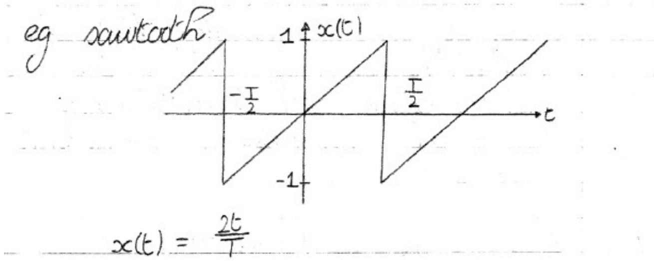
$$b_n = \frac{\cancel{4}}{\cancel{4}n^2\pi^2}[\cancel{\sin(n\pi)} - n\pi\cos(n\pi) + \cancel{\sin(n\pi)} - n\pi\cos(n\pi)]$$

$$= \frac{1}{n^2\pi^2}[-2n\pi\cos(n\pi)$$

$$n\ odd \Rightarrow cosn\pi = -1 \Rightarrow b_n = \frac{2}{n\pi}$$

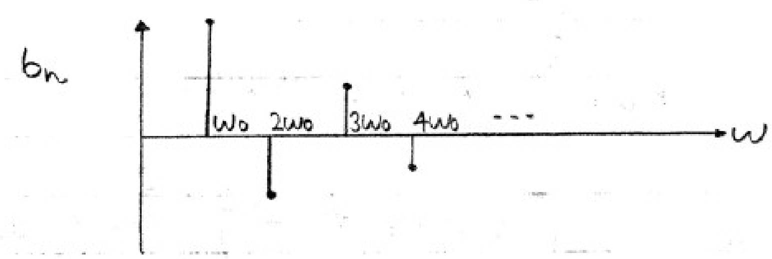
$$n\ even \Rightarrow cosn\pi = 1 \Rightarrow b_n = -\frac{2}{n\pi}$$

Example – sawtooth wave



$$\Rightarrow x(t) = \frac{2}{\pi}\sin(\omega_0 t) - \frac{1}{\pi}\sin(2\omega_0 t) + \frac{2}{3\pi}\sin(3\omega_0 t) - \cdots$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} sinn\omega_0 t$$

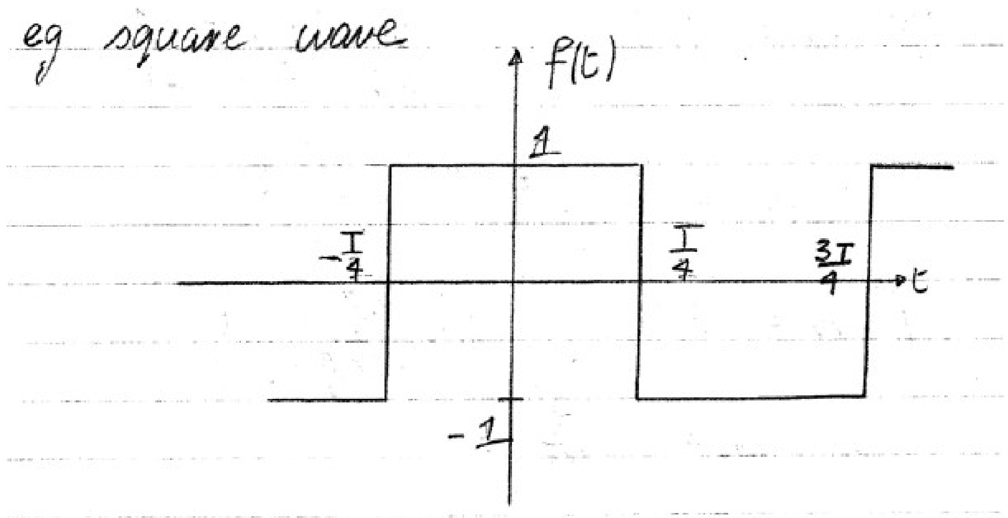


Half-wave symmetry

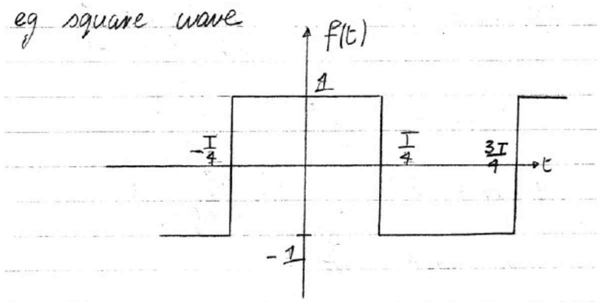
$$x(t) = -x(t + \frac{T}{2})$$

- i.e. in any two points separated by $\frac{T}{2}$ are equal in magnitude by opposite in sign
- In general, odd-order harmonic functions exhibit this property, so the Fourier Series of a periodic signal which displays such symmetry will not contain even-order components

Half-wave symmetry - Square wave



Half-wave symmetry
Square wave



$$Even\ function \Rightarrow b_n = 0$$

$$zero\ average\ value \Rightarrow a_0 = 0$$

$$a_n = \frac{2}{T}\int_{-\frac{T}{4}}^{\frac{3T}{4}} f(t) cos n\omega_0 t dt$$

$$a_n = \frac{2}{T}\int_{-\frac{T}{4}}^{\frac{T}{4}} cos n\omega_0 t dt - \frac{2}{T}\int_{\frac{T}{4}}^{\frac{3T}{4}} cos n\omega_0 t dt$$

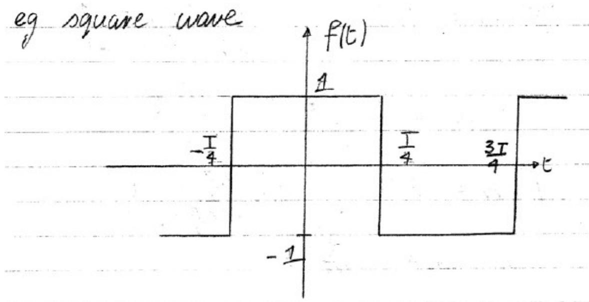
$$= \frac{2}{n\omega_0 T}\left[sin n\omega_0 t \Big|_{-\frac{T}{4}}^{\frac{T}{4}} - sin n\omega_0 t \Big|_{\frac{T}{4}}^{\frac{3T}{4}} \right]$$

$$= \frac{2}{n\omega_0 T}\left[sin n\omega_0\left(\frac{T}{4}\right) - sin n\omega_0\left(-\frac{T}{4}\right) - sin n\omega_0\left(\frac{3T}{4}\right) + sin n\omega_0\left(\frac{T}{4}\right)\right]$$

$$= \frac{1}{n\pi}\left[\sin\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{3n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{1}{n\pi}\left[3\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{3n\pi}{2}\right)\right]$$

Half-wave symmetry
Square wave



$$if\ n\ is\ even{:}\ \sin\left(\frac{n\pi}{2}\right) = 0,$$
$$\sin\left(\frac{3n\pi}{2}\right) = 0$$

$$if\ n\ is\ 1,5,9,\ldots\ \sin\left(\frac{n\pi}{2}\right) = 1,$$
$$\sin\left(\frac{3n\pi}{2}\right) = -1$$
$$\Rightarrow a_n = \frac{1}{n\pi[3-(-1)]} = \frac{4}{n\pi}$$

$$if\ n = 3,7,11,\ldots \sin\left(\frac{n\pi}{2}\right) = -1\,,$$
$$\sin\left(\frac{3n\pi}{2}\right) = 1$$
$$\Rightarrow a_n = -\frac{4}{n\pi}$$

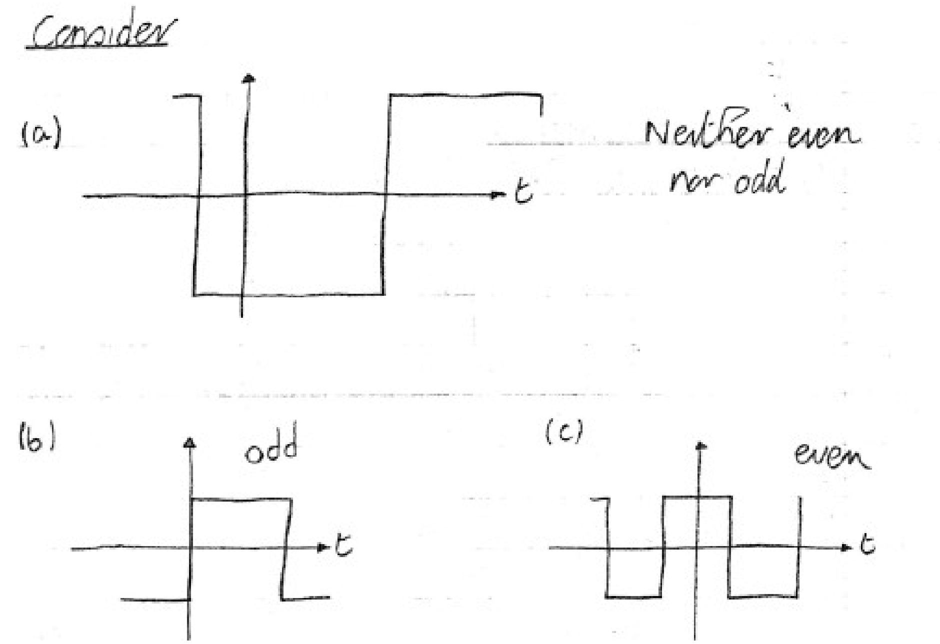
$$\therefore f(t) = \frac{4}{pi}\left[cos\omega_0 t - \frac{1}{3}cos3\omega_0 t + \frac{1}{5}cos5\omega_0 t - \cdots\right]$$

Half wave symmetry

- In the case of an even or off function, it is sometimes sufficient to integrate over one-half of the period, and multiply the result by 2
- Furthermore, if the function also displays half-wave symmetry, it is sufficient to integrate over ¼ of a period, and multiply by 4
  - Note: this applies to the trigonometric Fourier Series only
- Thus, the amount of work involved in calculating a Trigonometric Fourier series is reduced in the case of signals with such symmetries
- This can often be arranged by a judicious choice of time origin…

Example

- Consider waveform opposite
- Between (b) and (c), a shift in time origin converts a sine series into a cosine series, but the amplitude of a given component is unaltered
- Part (a) contains both cosine and sine terms in the Fourier Series



Parseval's Theorem for Trigonometric Fourier Series

- Power in any one component:

$$P = \frac{1}{2}(a_n^2 + b_n^2)$$

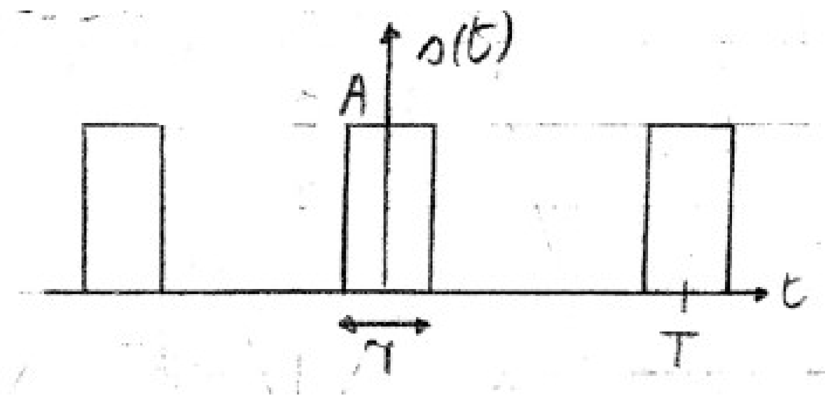
DC power is a_0^2

$$P_T = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

According to Parseval's theorem, we have

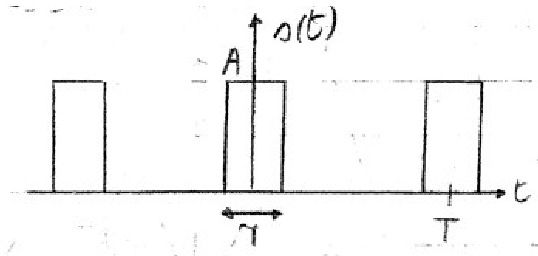
$$\frac{1}{T}\int_{t_0}^{t_0+T} |x(t)|^2 dt = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

More examples – Rectangular Pulse train



even function
$\Rightarrow b_n = 0$

Rectangular Pulse train



$$s(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$$

$$a_0 = \frac{A}{T}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} dt = \frac{A\tau}{T}$$

$$a_n = \frac{2}{T}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A cos n\omega_0 t dt$$

$$= \frac{2A}{T}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} cos n\omega_0 t dt$$

$$= \frac{2A}{n\omega_0 T} sin n\omega_0 t\Big|_{-\frac{\tau}{2}}^{\frac{\tau}{2}}$$

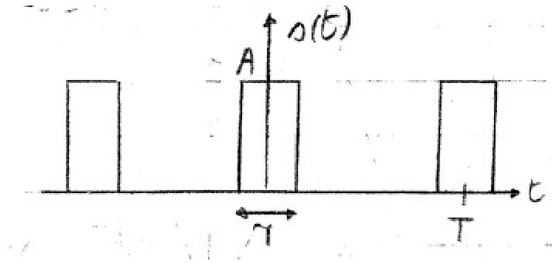
$$= \frac{A}{n\pi}\left[sin n\omega_0\left(\frac{\tau}{2}\right) - sin n\omega_0\left(-\frac{\tau}{2}\right)\right]$$

$$= \frac{2A}{n\pi}\sin(n\pi\left(\frac{\tau}{T}\right)$$

$$= \frac{\frac{2A\tau}{T}\sin\left(n\pi\left(\frac{\tau}{T}\right)\right)}{n\pi\left(\frac{\tau}{T}\right)}$$

$$= \frac{2A\tau}{T} Sa\left(\frac{n\pi\tau}{T}\right)$$

Rectangular Pulse train



- For trigonometric Fourier Series:

$$a_n \frac{2A\tau}{T} Sa\left(\frac{n\pi\tau}{T}\right)$$

- Recall that the value of the exponential Fourier Series coefficient is

$$c_n = \frac{A\tau}{T} Sa\left(\frac{n\pi\tau}{T}\right)$$

- This, the trigonometric Fourier Series coefficient is twice this value
  - i.e. the exponential spectrum has been "folded" about the vertical axis

Special cases

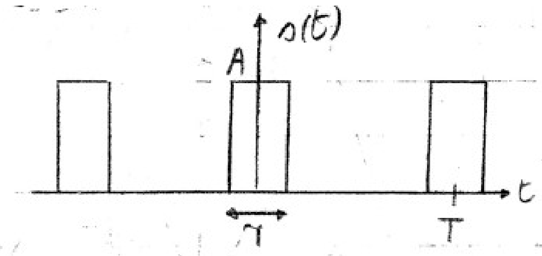
- Square pulse train, $\frac{\tau}{T} = \frac{1}{2}, a_0 = \frac{A}{2}$

$$a_n = \frac{2A}{n\pi} \sin\left(\frac{n\pi}{2}\right)$$

$$n\ even, \sin\left(\frac{n\pi}{2}\right) = 0 \Rightarrow a_n = 0$$

$$n\ odd, \sin\left(\frac{n\pi}{2}\right) = \frac{-1(n-1)}{2}$$

$$sq(t) = \frac{A}{2} + \frac{2A}{\pi} \sum_{\substack{n=1 \\ n\ odd}}^{\infty} \frac{(-1)^{(n-1)/2}}{n} cosn\omega_0 t$$



even function
⇒ $b_n = 0$

$$= \frac{A}{2}\left[1 + \frac{4}{pi}\left(cos\omega_o t - \frac{1}{3} cos3\omega_0 t + \frac{1}{5} cos5\omega_0 t - \cdots\right)\right]$$

- We see that this is very similar to the even square wave we had earlier, except for the scaling factor A, and the addition of the DC term

Special cases

- Limit as $\tau \to 0$ and $A \to \infty$

$$a_0 = \lim_{\substack{\tau \to 0 \\ A \to \infty}} \frac{A\tau}{T} = \frac{1}{T}$$

$$a_n = \lim_{\substack{\tau \to 0 \\ A \to \infty}} \frac{2A\tau}{T} Sa\left(\frac{n\pi\tau}{T}\right) = \frac{2}{T} \; \forall n \neq 0$$

Compare this with the exponential Fourier Series (i.e. $a_n = \frac{1}{T}$)

Differentiation of the Fourier Series

- Differentiation of exponential Fourier Series:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$\frac{df(t)}{dt} = \sum_{n=-\infty}^{\infty} c_n jn\omega_0 e^{jn\omega_0 t}$$

- Factor of j introduces a 90 degree phase shift
- $n\omega_0$ magnifies higher frequencies

Differentiation of the Fourier Series

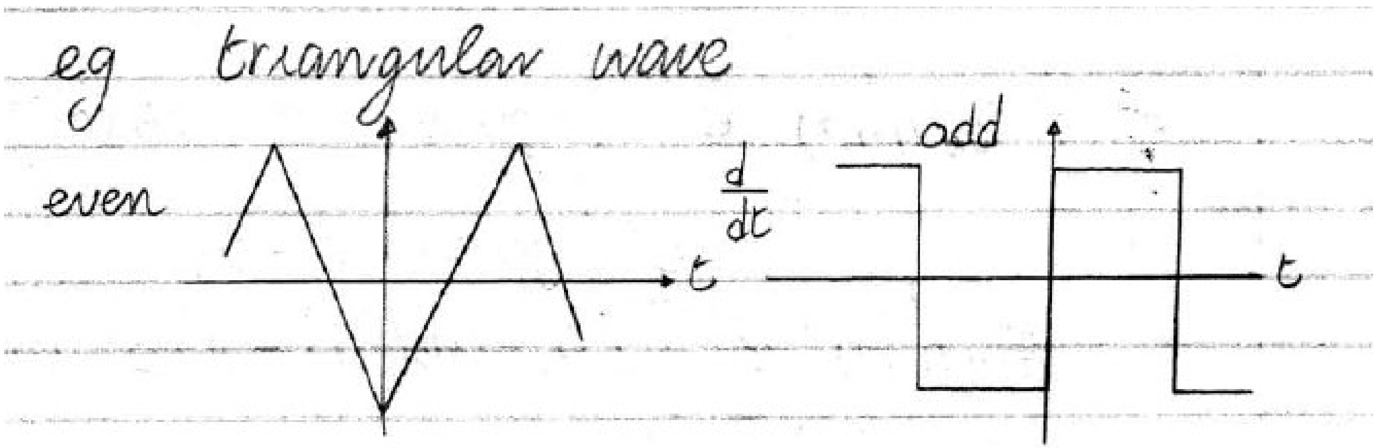
- Differentiation of Trigonometric Fourier series:

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n cosn\omega_0 t + \sum_{n=1}^{\infty} b_n sinn\omega_0 t$$

$$\frac{df(t)}{dt} = \sum_{n=1}^{\infty} -n\omega_0 a_n sinn\omega_0 t + \sum_{n=1}^{\infty} n\omega_0 a_n cosn\omega_0 t$$

- The sine and cosine terms have “swapped” places, which means an odd function becomes an even function, and vice versa

Differentiation of the Fourier Series Example – Triangular wave

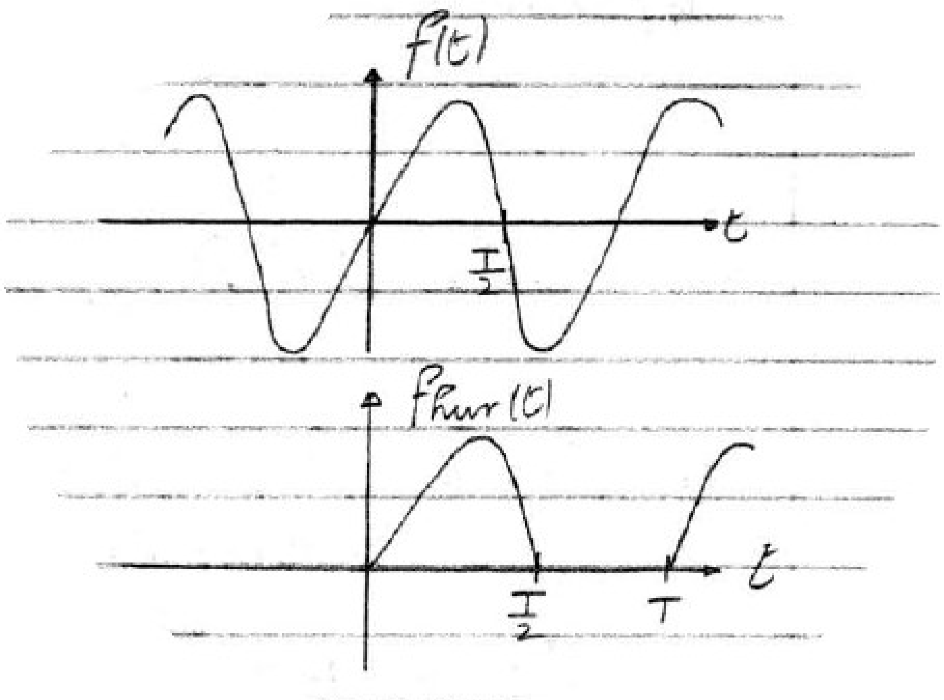


$$c_n = -\frac{4}{n^2\pi^2}$$

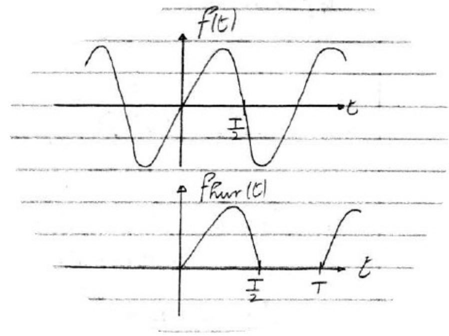
$$c_n' = jn\omega_0 c_n$$
$$= -\frac{jn4\omega_0}{n^2\pi^2}$$

i.e. changed from purely real to purely imaginary

Half wave rectified sinusoid



Half wave rectified sinusoid



$$f_{hwr}(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$c_n = \frac{1}{T}\int_0^{\frac{T}{2}} sin\omega_0 t e^{-jn\omega_0 t} dt$$

$$= \frac{\frac{1}{T}e^{-jn\omega_0 t}}{\omega_0^2 - n^2\omega_0^2}\left[-jn\omega_0 sin\omega_0 t - \omega_0 \cos\omega_0 t\right]_0^{\frac{T}{2}}$$

$$= \left(\frac{1}{T}\right)\frac{1}{\omega_0^2(1-n^2)}\{e^{-jn\pi}(-jn\omega_0 sin\pi$$
$$- \omega_0 cos\pi - e^0(-jn\omega_0 sin0 - \omega_0 cos0)\}$$

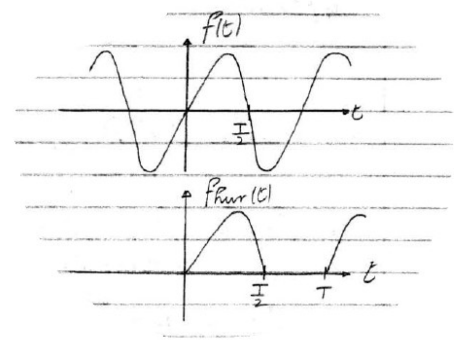
$$= \frac{1}{2\pi\omega_0(1-n^2)}\left[\omega_0 e^{-jn\pi} + \omega_0\right]$$

$$= \frac{1}{2\pi(1-n^2)}\left[e^{-jn\pi} + 1\right]$$

$$n\ odd, e^{-jn\pi} = -1, \Rightarrow c_n = 0$$

$$n\ even, e^{-jn\pi} = 1, \Rightarrow c_n = \frac{1}{\pi(1-n^2)}\ n \neq \pm 1$$

Half wave rectified sinusoid



$$for\ n = 1\ (i.e. n\omega_0\ for\ n = 1)$$

$$c_1 = \frac{1}{T}\int_0^{\frac{T}{2}} sin\omega_0 t e_0^{-j\omega t} dt$$

$$= \frac{1}{T}\int_0^{\frac{T}{2}} sin\omega_0 t[cos\omega_0 t - jsin\omega_0 t]dt$$

$$= \frac{1}{T}\int_0^{\frac{T}{2}} sin\omega_0 tcos\omega_0 tdt - j\frac{1}{T}\int_0^{\frac{T}{2}} \sin^2 \omega_0 t\, dt$$

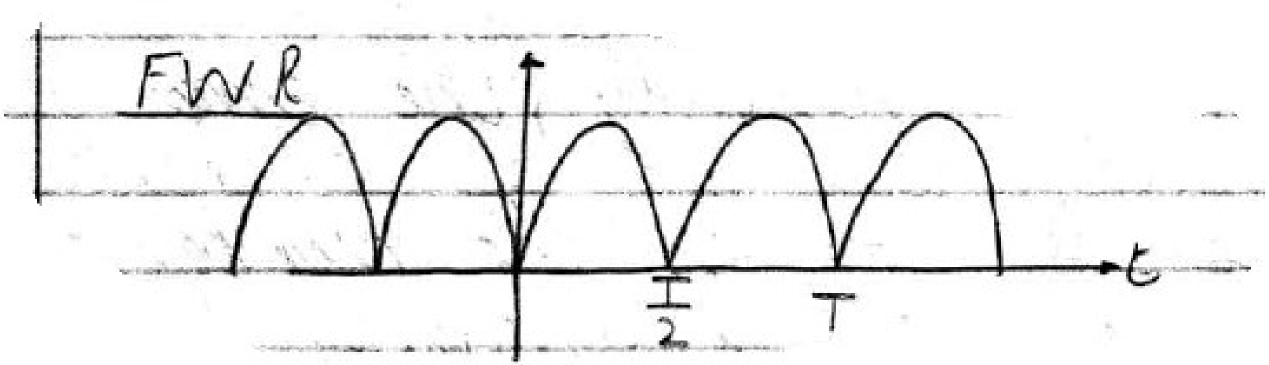
$$= -\frac{j}{2T}\int_0^{\frac{T}{2}} (1 - cos2\omega_0 t)dt$$

$$= -\frac{j}{2T}\int_0^{\frac{T}{2}} dt + \frac{j}{2T}\int_0^{\frac{T}{2}} cos2\omega_0 tdt$$

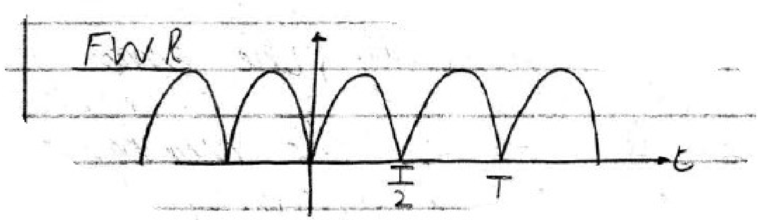
$$= -\frac{j}{2T}.\frac{T}{2} = -\frac{j}{4}$$

$$c_{-1} = c_1^* = \frac{j}{4}$$

Full wave rectified sinusoid



Full wave rectified sinusoid



$$c_n = \frac{1}{T}\int_0^T |sin\omega_0 t| e^{-jn\omega_0 t} dt$$

(care is required because the integrand is the product of $|\sin\omega_0 t|$ and $e^{-jn\omega_0 t}$)

$$c_n = \frac{1}{T}\int_0^{\frac{T}{2}} sin\omega_0 t e^{-jn\omega_0 t} dt - \frac{1}{T}\int_{\frac{T}{2}}^{T} sin\omega_0 t e^{-jn\omega_0 t} dt$$

$$= \frac{e^{-jn\omega_0 t}}{T\omega_0^2(1-n^2)}\left\{[-jnsin\omega_0 t - \omega_0 cos\omega_0 t]_0^{\frac{T}{2}} - [-jn\omega_0 sin\omega_0 t - cos\omega_0 t]_{-\frac{T}{2}}^{\frac{T}{2}}\right\}$$

$$c_n = \frac{2}{\pi(1-n^2)}\left[1 + e^{-jn\pi}\right]$$

$$n\ odd, e^{-jn\pi} = -1 \Rightarrow c_n = 0$$

$$n\ even, e^{-jn\pi} = \pm 1 \Rightarrow c_n = \frac{2}{\pi(1-n^2)}$$

Question: what is the value of c_n for $n \pm 1$?

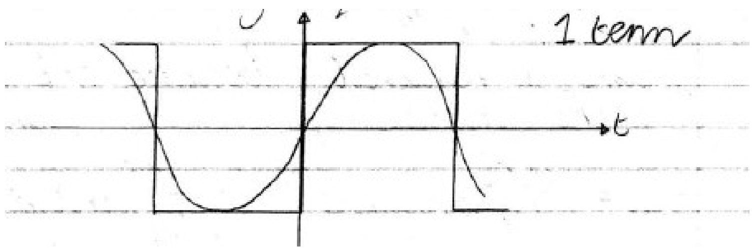
Ans: $c_n = 0$

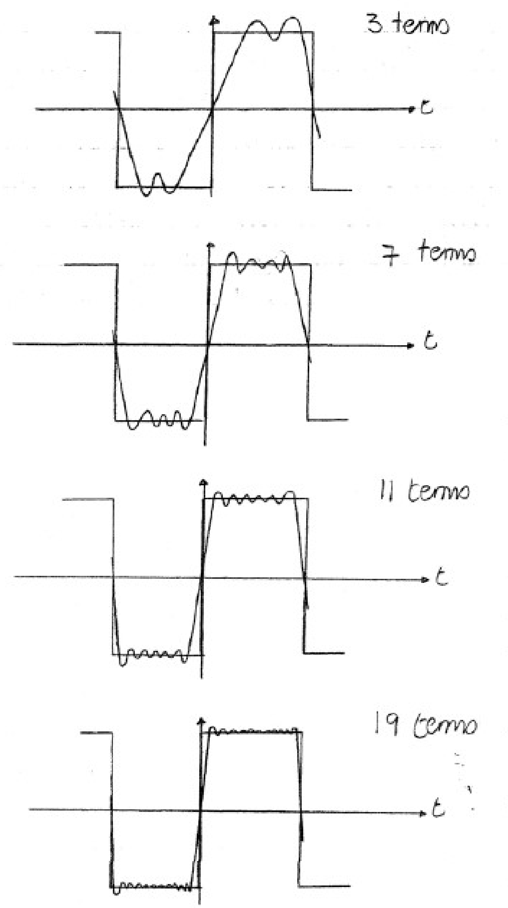
This means there is no 50Hz component (i.e.$1\omega_0$)

Gibbs phenomenon – convergence of FS

- As mentioned previously, the error between a signal x(t) and its Fourier Series representation decreases to zero as the number of terms in the Fourier Series goes to infinity
  - i.e. the Fourier Series converges to x(t)
- However, this convergence is not perfect in the case of signals with discontinuities e.g. a square wave…

Gibb's Phenomenon





Gibb's phenomenon

- At points of discontinuity, the Fourier Series converges to the average of the points on either side of the discontinuity.
- As the number of terms is increased, the Fourier Series exhibits overshoot near the points of discontinuity, even though the mean-square error goes to zero
- The overshoot reaches a value of about 9% of the height of the discontinuity
- As the number of terms increases, the amplitude of the overshoot moves closer to the discontinuity
- This behaviour is known as Gibb's phenomenon
  - We will return to this again in our study of filter responses