Machine Learning

Personal Formula Collection

Patrick Bucher

December 13, 2020

Contents

1 Linear Regression

1.1 One Variable

Prediction:

$$y = h(x) = \theta_0 + \theta_1 x = \theta^T x$$

Cost Function (Squared Error):

$$J(\theta) = \frac{1}{2m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})^2$$

Gradient Descent:

$$\theta = \theta - \alpha\frac{1}{m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})x^{(i)}$$

1.2 Multiple Variables

Prediction:

$$y = h(x) = \theta_0 + \theta_1 x_1 = + \ldots + \theta_n x_n = \theta^T x$$

Cost Function:

$$J(\theta_j) = \frac{1}{2m}\sum_{i=1}^{m}(h_\theta(x_j^{(i)}) - y_j^{(i)})^2$$

Gradient Descent:

$$\theta_j = \theta_j - \alpha\frac{1}{m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})x_j^{(i)}, j := 0 \ldots n$$

An additional feature $x_0 = 1$ is introduced, so that the vector x becomes $n+1$ dimensional, which simplifies the matrix calculations.

Normal Equation:

$$\theta = (X^T X)^{-1} X^T y$$

Octave (Complexity with n features: $O(n^3)$):

```
theta = pinv(X'*X)*X'*y
```

1.3 Normalization

$$x_i = \frac{x_i - \mu_i}{s_i}$$

Octave:

```
X = (X - mean(X)) ./ std(X)
```

2 Classification

Binary Classification: $y \in \{0, 1\}$, where 0 signifies negative or absent, and 1 signifies positive or present.

2.1 Logistic Regression

$$0 \leq h_{\theta(x)} \leq 1$$

Sigmoid Activation Function g (with asymptotes at y 0 and 1, to be interpreted as probabilities):

$$h_\theta = g(\theta^T x), g(z) = \frac{1}{1+e^{-z}}$$

$$h_\theta(x) = \frac{1}{1+e^{-\theta^T x}}$$

Cost Function: $-log(h_\theta(x))$ for $y = 1$ and $-log(1 - h_\theta(x))$ for $y = 0$, combined:

$$C(h_\theta(x), y) = -y \cdot log(h_\theta(x)) - (1-y) \cdot log(1 - h_\theta(x))$$

$$J(\theta) = \frac{1}{m} \sum_{i=1}^{m} C(h_\theta(x^{(i)}), y^{(i)})$$

With maximum likelihood estimation:

$$J(\theta) = \frac{1}{m} \left[\sum_{i=1}^{m} y^{(i)} \cdot log(h_\theta(x^{(i)})) + (1 - y^{(i)}) \cdot log(1 - h_\theta(x^{(i)})) \right]$$

Prediction:

$$h_\theta(x) = \frac{1}{1+e^{-\theta^T x}}$$

Gradient Descent (for each j in θ):

$$\theta_j := \theta_j - \frac{\alpha}{m} \sum_{i=1}^{m} (h_\theta(x^{(i)}) - y^{(i)}) x_j^{(i)}$$

Vectorized:

$$\theta := \theta - \frac{\alpha}{m} \sum_{i=1}^{m} \left[(h_\theta(x^{(i)}) - y^{(i)}) x^{(i)} \right]$$

$$\theta := \theta - \frac{\alpha}{m} X^T (g(X\theta) - \vec{y})$$

2.1.1 Regularization (Gradient Descent)

Regularization mitigates the problem of overfitting for higher-order polynomials.
Regularization term (only regularize θ_j for $j \geq 1$, but not θ_0):

$$\lambda \sum_{j=1}^{m} \theta_j^2$$

Regularized Cost Function:

$$J(\theta) = \frac{1}{m}\left[\sum_{i=1}^{m} y^{(i)} \cdot log(h_\theta(x^{(i)})) + (1-y^{(i)}) \cdot log(1-h_\theta(x^{(i)}))\right] + \frac{\lambda}{2m}\sum_{j=1}^{n}\theta_j^2$$

Regularized Gradient Descent (for θ_j with $j \geq 1$):

$$\theta_0 := \theta_0 - \alpha\left[\frac{1}{m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})x_j^{(i)}\right]$$
$$\theta_j := \theta_j - \alpha\left[\frac{1}{m}\sum_{i=1}^{m}\Big((h_\theta(x^{(i)}) - y^{(i)})x_j^{(i)}\Big) + \frac{\lambda}{m}\theta_j\right]$$

2.1.2 Regularization (Normal Equation)

To regularize using the normal equation, $(n+1)(n+1)$ matrix L with i rows and j columns and the values 1 (for $i = j \wedge i \geq 1 \wedge j \geq 1$) and 0 (all the other indices), respectively, has to be created. (This is an identity matrix of size $n+1$ with the value at $(0, 0)$ set to 0.)

$$\theta = (X^T X + \lambda L)^{-1} X^T y$$

With regularization, the matrix is always inversible.

3 Neural Networks

Definitions:

- x_0: bias unit
- $a_i^{(j)}$: activation unit i of layer j
- $\Theta^{(j)}$: weight matrix between layer j and $j+1$

Given layer j with s_j units, and layer $j+1$ with s_{j+1} units, the matrix $\Theta^{(j)}$ has the dimensions $s_{j+1} \times (s_j + 1)$.

3.1 Activation

Neural network with three units in the one hidden layer:

$$a_1^{(2)} = g(\Theta_{10}^{(1)} x_0 + \Theta_{11}^{(1)} x_1 + \ldots)$$
$$a_2^{(2)} = g(\Theta_{20}^{(2)} x_0 + \Theta_{21}^{(2)} x_1 + \ldots)$$
$$a_3^{(2)} = g(\Theta_{30}^{(3)} x_0 + \Theta_{31}^{(3)} x_1 + \ldots)$$
$$h_\Theta(x) = a_1^3 = g(\Theta_{10}^{(2)} a_0^{(2)} + \Theta_{11}^{(2)} a_1^{(2)} + \ldots)$$

3.1.1 Vectorization

With (forward propagation):

$$a_1^{(2)} = g(\Theta_{10}^{(1)}x_0 + \Theta_{11}^{(1)}x_1 + \Theta_{12}^{(1)}x_2 + \Theta_{13}^{(1)}x_3)$$

And:

$$z_1^{(2)} = \Theta_{10}^{(1)}x_0 + \Theta_{11}^{(1)}x_1 + \Theta_{12}^{(1)}x_2 + \Theta_{13}^{(1)}x_3$$

Follows:

$$a_1^{(2)} = g(z_1^{(2)})$$

So that:

$$z^{(2)} = \Theta^{(1)}x = \Theta^{(1)}a^{(1)}$$

Output layer:

$$h_\Theta = a^{(3)} = g(z^{(3)})$$

3.2 Cost Function

$$J(\Theta) = -\frac{1}{m}\left[\sum_{i=1}^{m}\sum_{k=1}^{K} y_k^{(i)} log(h_\Theta(x^{(i)}))_k + (1-y_k^{(i)}) log(1-(h_\Theta(x^{(i)}))_k\right] + \frac{\lambda}{2m}\sum_{l=1}^{L-1}\sum_{i=1}^{s_l}\sum_{j=1}^{s_{l+1}}(\Theta_{ji}^{(l)})^2$$

With $(h_\Theta(x))_i$ being the i^{th} output. Note that regularization is *not* added to the bias unit, i.e. only for $j \geq 1$.

3.3 Forward Propagation

With a single training example (x, y). The first activation is the input (a bias unit $a_0^{(1)} = 1$ must be added before):

$$a^{(1)} = x$$

The second activation is computed using Θ and the sigmoid function $g(z)$:

$$z^{(2)} = \Theta^{(2)}a^{(2)}$$
$$a^{(2)} = g(z^{(2)})$$

The bias unit $a_0^{(2)} = 1$ must be added again, then the further activations (l) are computed:

$$z^{(l)} = \Theta^{(l)}a^{(l)}$$
$$a^{(l)} = g(z^{(l)})$$

Finally, the output (layer L) is computed:

$$z^{(L)} = \Theta^{(L)} a^{(L)}$$
$$a^{(L)} = g(z^{(L)}) = h_\Theta(x)$$

3.4 Backpropagation

The δ for the rightmost layer L is computed as:

$$\delta^L = a^{(L)} - y$$

The further δ values are computed from right to left, down to $l = 2$ (no δ for the input layer):

$$\delta^{(l)} = \delta^{(l+1)} \Theta^{(l)} g'(z^{(l)})$$

With (bias unit included in $a^{(l)}$):

$$g'(z^{(l)}) = a^{(l)}(1 - a^{(l)})$$

The Δ values are computed as ($a^{(l)}$ *without* bias unit):

$$\Delta^{(l)} = (\delta^{(l+1)})^T a^{(l)}$$

Finally, the gradients D for $j \geq 1$ are computed as follows:

$$D_{ij}^{(l)} = \frac{1}{m}(\Delta_{ij}^{(l)} + \lambda \Theta_{ij}^{(l)})$$

And without regularization for $j = 0$, respectively:

$$D_{ij}^{(l)} = \frac{1}{m}(\Delta_{ij}^{(l)}$$

Which is the partial derivative of the cost function:

$$\frac{\partial}{\partial \Theta_{ij}^{(l)}} J(\Theta) = D_{ij}^{(l)}$$

3.5 Gradient Checking

Estimate the derivative of $J(\Theta)$ with $\varepsilon \approx 10^{-4}$ (two-sided difference):

$$\frac{d}{d\Theta} J(\Theta) \approx \frac{J(\Theta + \varepsilon) - J(\Theta - \varepsilon)}{2\varepsilon}$$

The result should only deviate from the D values by a rounding margin.

3.6 Random initialization

When working with neural networks, Θ must be initialized to a random value symmetrically around 0. A (10×11) matrix is initialized as follows (Octave):

```
init_epsilon = 0.1;
Theta = rand(10,11) * (2 * init_epsilon) - init_epsilon;
```