SICP: Ex. 1.19, p. 47

Exercise

Given:

$$T_{pq}(a,b) = (bq + aq + ap, bp + aq) \tag{1}$$
$$T_{p'q'}(a,b) = T_{pq}(T_{pq}(a,b)) \tag{2}$$

Solution

$$a' = bq + aq + ap \tag{3}$$
$$b' = bq + aq \tag{4}$$
$$a'' = b'q + a'q + a'p \tag{5}$$
$$b'' = b'p + a'q \tag{6}$$

Insert a' and b':

$$a'' = (bp + aq)q + (bq + aq + ap)q + (bq + aq + ap)p \tag{7}$$
$$b'' = (bp + aq)p + (bq + aq + ap)q \tag{8}$$

Expanded:

$$a'' = bpq + aq^2 + bq^2 + aq^2 + apq + bpq + apq + ap^2 \tag{9}$$
$$b'' = bp^2 + apq + bq^2 + aq^2 + apq \tag{10}$$

Added up and sorted:

$$a'' = ap^2 + 2apq + 2aq^2 + 2bpq + bq^2 \tag{11}$$
$$b'' = 2apq + aq^2 + bp^2 + bq^2 \tag{12}$$

Factorize b'':

$$b'' = a(2pq + q^2) + b(p^2 + q^2)$$

From:

$$b' = bp + aq \tag{13}$$
$$b'' = bp' + aq' \tag{14}$$

Follows:

$$p' = 2pq + q^2 \tag{15}$$
$$q' = p^2 + q^2 \tag{16}$$

Check using a'':

$$
\begin{aligned}
a'' &= bq' + aq' + ap' &\quad (17)\\
&= b(2pq + q^2) + a(2pq + q^2) + a(p^2 + q^2) &\quad (18)\\
&= 2bpq + bq^2 + 2apq + aq^2 + ap^2 + aq^2 &\quad (19)\\
&= ap^2 + 2apq + 2aq^2 + 2bpq + bq^2 \qquad \text{q.e.d.} &\quad (20)
\end{aligned}
$$

Scheme implementation:

```scheme
(define (fib n)
  (fib-iter 1 0 0 1 n))

(define (fib-iter a b p q count)
  (cond ((= count 0)
         b)
        ((even? count)
         (fib-iter a
                   b
                   (+ (* p p) (* q q))
                   (+ (* 2 p q) (* q q))
                   (/ count 2)))
        (else
         (fib-iter (+ (* b q)
                      (* a q)
                      (* a p))
                   (+ (* b p)
                      (* a q))
                   p
                   q
                   (- count 1)))))

(define (even? x)
  (= (remainder x 2) 0))
```