SICP: Ex. 1.13, p. 42

Exercise

Given:

$$\phi = \frac{1+\sqrt{5}}{2}, \psi = \frac{1-\sqrt{5}}{2},$$

And:

$$Fib(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ \mathrm{Fib}(n-1) + \mathrm{Fib}(n-2) & n > 1 \end{cases}$$

Proof that:

$$\mathrm{Fib}(n) = \frac{\phi^n - \psi^n}{\sqrt{5}}$$

Proof

$$\mathrm{Fib}(2) = \mathrm{Fib}(1) + \mathrm{Fib}(0) \tag{1}$$
$$\frac{\phi^2 - \psi^2}{\sqrt{5}} = \frac{\phi - \psi}{\sqrt{5}} + \frac{\phi^0 - \psi^0}{\sqrt{5}} \tag{2}$$
$$\frac{\phi^2 - \psi^2}{\sqrt{5}} = \frac{\phi - \psi}{\sqrt{5}} \tag{3}$$
$$\phi^2 - \psi^2 = \phi - \psi \tag{4}$$
$$(\phi - \psi)(\phi + \psi) = \phi - \psi \tag{5}$$

With the definitions of ψ and ϕ:

$$\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} \tag{6}$$
$$\frac{(1+\sqrt{5})^2 - (1-\sqrt{5})^2}{4} = \frac{1+\sqrt{5}-1+\sqrt{5}}{2} \tag{7}$$
$$\frac{1+2\sqrt{5}+5-(1-2\sqrt{5}+5)}{4} = \frac{2\sqrt{5}}{2} \tag{8}$$
$$\frac{6+2\sqrt{5}-6+2\sqrt{5}}{4} = \sqrt{5} \tag{9}$$
$$\frac{4\sqrt{5}}{4} = \sqrt{5} \tag{10}$$
$$\sqrt{5} = \sqrt{5} \qquad \text{q.e.d.} \tag{11}$$