Quantum Algorithms
Homework 9 Solutions

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1. Carefully work through the two paragraphs after problem 9.3 on page 87 of the text, dealing with Grover's algorithm.
   (i) Why are U and V reflections through hyperplanes? What are the two hyperplanes? [*Hint: compare the angle between* $|x\rangle$ *and* $|\xi\rangle$ *and between* $V|x\rangle$ *and* $|\xi\rangle$. *Likewise for* U.]
   (ii) Prove that VU acting on $\mathbb{C}$-span$(|y_0\rangle, |\xi\rangle)$ is rotation by twice the angle between $|y_0\rangle$ and $|\xi\rangle$. [*Hint: use the previous part and a little bit of geometry.*]
   (iii) There is an error in the text — $\langle \xi \mid y_0\rangle = 1/\sqrt{N} = \cos(\varphi/2)$, where $\varphi/2$ is the angle between $|\xi\rangle$ and $|y_0\rangle$. Explain why this is correct.
   (iv) Using the correction above, explain why $\varphi \approx 2/\sqrt{N}$ for large N.

Solution:

(i) U reflects through the hyperplane orthogonal to $|y_0\rangle$, while V reflects through the hyperplane orthogonal to $|\xi\rangle$.

Since the goal of Grover's algorithm is to "rotate" the input vector $|x\rangle$ to $|y_0\rangle$ after s many applications of VU, it is important that both U and V represent reflections through the hyperplanes made up by taking all vectors in $\mathcal{L} = \mathbb{C}\{|\xi\rangle, |y_0\rangle\}$ that are orthogonal to $|y_0\rangle$ and $|\xi\rangle$, respectively. This is because geometrically two reflections always result in a rotation, regardless of the dimension of the space where these refletions are taking place.

The application of U to $|x\rangle$ will be a reflection through the plane orthogonal to y_0. Call this new vector $|\varphi\rangle$. The angle between this new vector and the hyperplane orthogonal to $|y_0\rangle$ will be $\pi - \theta$.

The application $V|\varphi\rangle$ results in a reflection across the plane orthoogonal to $|\xi\rangle$. This will bring $|\varphi\rangle$ up to the original plane with angle 2θ greater than the original θ between $|y_0\rangle$ and $|\xi\rangle$.

Since a unitary operator preserves the inner product by definition, and both U and V are unitary, the angle between the hyperplane being reflected across and the input vector will be preserved in both cases.

(ii) *Proof.* Take $|x\rangle$ to be an input vector.
Take θ to be the angle between $|\xi\rangle$ and $|y_0\rangle$.
Define φ to be the angle between $|\xi\rangle$ and the plane orthogonal to y_0

Then, consider the effect of U on $|x\rangle$:

$$\begin{aligned}
U|x\rangle &= U(a|\xi\rangle + b|y_0\rangle) \\
&= (I - 2|y_0\rangle\langle y_0|)(a|\xi\rangle + b|y_0\rangle) \\
&= a|\xi\rangle - b|y_0\rangle \\
&= -|x\rangle
\end{aligned}$$

Which corresponds to a reflection about the plane orthogonal to $|y_0\rangle$, resulting in angle $\pi - \theta$ from $|y_0\rangle$

Consider now the application of V to generic vector $|\psi\rangle$ (this vector is different from $|x\rangle$ above):

$$\begin{aligned}
V|\psi\rangle &= (I - 2|\xi\rangle\langle\xi|)|\psi\rangle \\
&= |\psi\rangle - 2\langle\xi \mid \psi\rangle|\xi\rangle
\end{aligned}$$

Geometrically, this is equivilent to a reflection about the hyperplane orthogonal to $|\xi\rangle$. This can be seen because $2\langle\xi \mid \psi\rangle$ will be a scalar, so $2\langle\xi \mid \psi\rangle|\xi\rangle$ yields a multiple of $|\xi\rangle$. Subtracting this from $|\psi\rangle$ from this takes into account the direction of $|\psi\rangle$. When considering the tip-to-tail geometry of this action, we note that the angle between $V|\psi\rangle$ and $|\xi\rangle$ is the same as the angle between $|\xi\rangle$ and $|\psi\rangle$.

Considering our original system, the angle θ is between $|y_0\rangle$ and $|\xi\rangle$. $U|x\rangle$ has angle 2θ between itself and $|\xi\rangle$ due to the fact that it has angle $\pi - \theta$ with $|y_0\rangle$.

This angle 2θ is then reflected across the plane orthogonal to $|\xi\rangle$ when V is applied to $U|x\rangle$. The total rotation that occurs in $VU|\xi\rangle$ is 2θ because V preserves it. $\square$

(iii) Take $\varphi/2$ to be the angle between $|y_0\rangle$ and $|\xi\rangle$. Call it θ.

$$\begin{aligned}
\cos(\theta) &= \frac{\langle\xi \mid y_0\rangle}{||\xi|| \quad ||y_0||} \\
&= \langle\xi \mid y_0\rangle \\
&= \frac{1}{\sqrt{N}}
\end{aligned}$$

If we are considering $\varphi/2$ to be the complementary angle to θ where θ is the angle between $|y_0\rangle$ and $|\xi\rangle$, i.e.:

$$\varphi/2 = \pi/2 - \theta \implies \theta = \pi/2 - \varphi/2$$

Then

$$\cos(\theta) = \cos(\pi/2 - \varphi/2) = 1/\sqrt{N} = \sin(\varphi/2)$$

(iv)

$$\cos(\varphi/2) = \frac{1}{\sqrt{N}} \implies \varphi = 2\arccos(\frac{1}{\sqrt{N}})$$

So, we need to take

$$\lim_{N\to\infty} 2\arccos(1/\sqrt{N})$$

However, this yields π, because

$$\lim_{N\to\infty} \arccos(1/N) = \pi/2$$

So, a different approach must be used to approximate the form of φ as the size of N becomes large. We can use linear approximation to generate this new function. The forumla for this is as follows:

$$f(x) = f(a) + f'(a)(x-a) + R_2 \implies f(x) \approx f(a) + f'(a)(x-a)$$

In this case, $f(x)$ is given by $\arccos(x)$. Take $a = 0$ So:

$$\begin{aligned} f(x) &= \arccos(x) \\ f'(x) &= -\frac{1}{\sqrt{1-x^2}} \\ f(a) &= \pi/2 \\ f'(a) &= -1 \end{aligned}$$

This gives approximation $f(x) \approx \pi/2 - x$. From here we need to use our actual values for x:

$$\begin{aligned} f(1/\sqrt{N}) &= \pi/2 - 1/\sqrt{N} \\ 2f(1/\sqrt{N}) &= \pi - 2/sqrtN \end{aligned}$$

So φ is actually approximated by $\pi - 2/\sqrt{N}$.