Quantum Algorithms
Homework 5 Solutions

Patrick Canny

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1 Book Problems

1. Exercise 6.5

   (i) Find $H[2]$.

   Solution: The matrix representation of $H[2]$ is as follows:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}$$

   This can be derived using the following steps:

$$\begin{aligned} H[2] &= \frac{1}{\sqrt{2}}\left(I_{\mathcal{B}^{\otimes 1}} \otimes H \otimes I_{\mathcal{B}^{\otimes 1}}\right) \\ &= \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \\ &= \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix} \end{aligned}$$

   (ii) Find $U[3,1]$.

   Solution: The matrix for $U[3,1]$ can be represented as follows:

$$\begin{pmatrix} u_{00,00} & u_{00,10} & 0 & 0 & u_{00,01} & u_{00,11} & 0 & 0 \\ u_{10,00} & u_{10,10} & 0 & 0 & u_{10,01} & u_{10,11} & 0 & 0 \\ 0 & 0 & u_{00,00} & u_{00,10} & 0 & 0 & u_{00,01} & u_{00,11} \\ 0 & 0 & u_{10,00} & u_{10,10} & 0 & 0 & u_{10,01} & u_{10,11} \\ u_{01,00} & u_{01,10} & 0 & 0 & u_{01,01} & u_{01,11} & 0 & 0 \\ u_{11,00} & u_{11,10} & 0 & 0 & u_{11,01} & u_{11,11} & 0 & 0 \\ 0 & 0 & u_{01,00} & u_{01,10} & 0 & 0 & u_{01,01} & u_{01,11} \\ 0 & 0 & u_{11,00} & u_{11,10} & 0 & 0 & u_{11,01} & u_{11,11} \end{pmatrix}$$

   This is derived using the following procedure:

First, we define U:

$$\begin{pmatrix} u_{00,00} & u_{00,01} & u_{00,10} & u_{00,11} \\ u_{01,00} & u_{01,01} & u_{01,10} & u_{01,11} \\ u_{10,00} & u_{10,01} & u_{10,10} & u_{10,11} \\ u_{11,00} & u_{11,01} & u_{11,10} & u_{11,11} \end{pmatrix} = \begin{pmatrix} v_{00} & v_{01} \\ v_{10} & v_{11} \end{pmatrix}$$

$$U[3,1] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}[3] * V_{00}[1] + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}[3] * V_{01}[1] + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}[3] * V_{10}[1] + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}[3] * V_{11}[1]$$

where V_{ij} is selected as an individual $2x2$ matrix selected from the $4x4$ matrix for U as defined. From here:

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}[3] &= \mathcal{I}_{\mathcal{B}^{\otimes 2}} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \end{pmatrix} \end{aligned}$$

Then, we compute $V_{00}[1]$.

$$\begin{aligned} V_{00}[1] &= \begin{pmatrix} u_{00,00} & u_{00,01} \\ u_{01,00} & u_{01,01} \end{pmatrix} \otimes \mathcal{I}_{\mathcal{B}^{\otimes 2}} \\ &= \begin{pmatrix} u_{00,00}&0&0&0&u_{00,01}&0&0&0 \\ 0&u_{00,00}&0&0&0&u_{00,01}&0&0 \\ 0&0&u_{00,00}&0&0&0&u_{00,01}&0 \\ 0&0&0&u_{00,00}&0&0&0&u_{00,01} \\ u_{01,00}&0&0&0&u_{01,01}&0&0&0 \\ 0&u_{01,00}&0&0&0&u_{01,01}&0&0 \\ 0&0&u_{01,00}&0&0&0&u_{01,01}&0 \\ 0&0&0&u_{01,00}&0&0&0&u_{01,01} \end{pmatrix} \end{aligned}$$

The two resulting matricies are then multiplied together, yielding

$$\begin{pmatrix} u_{00,00}&0&0&0&u_{00,01}&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&u_{00,00}&0&0&0&u_{00,01}&0 \\ 0&0&0&0&0&0&0&0 \\ u_{01,00}&0&0&0&u_{01,01}&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&u_{01,00}&0&0&0&u_{01,01}&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

From here, a similar process is repeated for the remaining matricies, which yield the following results:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}[3] * V_{01}[1] = \begin{pmatrix} 0&u_{00,10}&0&0&0&u_{00,11}&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&u_{00,10}&0&0&0&u_{00,11} \\ 0&0&0&0&0&0&0&0 \\ 0&u_{01,10}&0&0&0&u_{01,11}&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&u_{01,10}&0&0&0&u_{01,11} \\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}[3] * V_{10}[1] = \begin{pmatrix} 0&0&0&0&0&0&0&0 \\ u_{10,00}&0&0&0&u_{10,01}&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&u_{10,00}&0&0&0&u_{10,01}&0 \\ 0&0&0&0&0&0&0&0 \\ u_{11,00}&0&0&0&u_{11,01}&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&u_{11,00}&0&0&0&u_{11,01}&0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}[3] * V_{11}[1] = \begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&u_{10,10}&0&0&0&u_{10,11}&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&u_{10,10}&0&0&0&u_{10,11} \\ 0&0&0&0&0&0&0&0 \\ 0&u_{11,10}&0&0&0&u_{11,11}&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&u_{11,10}&0&0&0&u_{11,11} \end{pmatrix}$$

All of the resulting matricies are then summed, resulting in the final answer:

$$U[3,1] = \begin{pmatrix} u_{00,00}\ u_{00,10} & 0\ 0 & u_{00,01}\ u_{00,11} & 0\ 0 \\ u_{10,00}\ u_{10,10} & 0\ 0 & u_{10,01}\ u_{10,11} & 0\ 0 \\ 0\ 0 & u_{00,00}\ u_{00,10} & 0\ 0 & u_{00,01}\ u_{00,11} \\ 0\ 0 & u_{10,00}\ u_{10,10} & 0\ 0 & u_{10,01}\ u_{10,11} \\ u_{01,00}\ u_{01,10} & 0\ 0 & u_{01,01}\ u_{01,11} & 0\ 0 \\ u_{11,00}\ u_{11,10} & 0\ 0 & u_{11,01}\ u_{11,11} & 0\ 0 \\ 0\ 0 & u_{01,00}\ u_{01,10} & 0\ 0 & u_{01,01}\ u_{01,11} \\ 0\ 0 & u_{11,00}\ u_{11,10} & 0\ 0 & u_{11,01}\ u_{11,11} \end{pmatrix}$$

2 Additional Problems

1. Prove that the inner product and the tensor product commute:

$$\langle \alpha \otimes \beta \mid \gamma \otimes \delta \rangle = \langle \alpha \mid \gamma \rangle \langle \beta \mid \delta \rangle .$$

This is asserted on page 57 of the textbook.

Solution:

Claim 2.1. *The inner product and tensor product commute.*

Proof of claim.

$$\begin{aligned} \langle \alpha \otimes \beta \mid \gamma \otimes \delta \rangle &= \left\langle \sum_{j,k} \alpha_j \beta_k e_j \otimes f_k \mid \sum_{j,k} \gamma_j \delta_k e_j \otimes f_k \right\rangle \qquad e_i, f_i \in \mathcal{B}^{\otimes n} \\ &= \sum_{jk} \alpha_j \beta_k e_j \otimes f_k \sum_{j,k} \gamma_j \delta_k e_j \otimes f_k \\ &= \sum_{jk} \alpha_j e_j * \gamma_k e_k \otimes \beta_j f_j * \delta_k f_k \\ &= \langle \alpha \mid \gamma \rangle \otimes \langle \beta \mid \delta \rangle \\ &= \langle \alpha \mid \gamma \rangle \langle \beta \mid \delta \rangle \end{aligned}$$

The second simplification is possible due to the bilinear nature of the tensor peoduct. This allows us to pair the α and γ values with e_i, e_k, and likewise with β, δ, f_j, f_k.

The last simplification is possible because the $\alpha_j e_j * \gamma_k e_k$ and the corresponding γ, β will all be 1-D matricies, meaning that their tensor product will just be scalar multiplication, which yields the desired form. •

2. Let $T : \mathbb{A} \to \mathbb{B}$ be a linear transformation between vector spaces with ordered bases

$$\mathcal{B}_{\mathbb{A}} = \{ |1\rangle, |2\rangle, |3\rangle \} \qquad\qquad \mathcal{B}_{\mathbb{B}} = \{ |1\rangle, |2\rangle \}.$$

Suppose that T has matrix with respect to these bases

$$T = \begin{pmatrix} 9 & 6 & -3 \\ -4 & -8 & 8 \end{pmatrix}.$$

(i) Show that the matrix for T can be written

$$T = \sum_{\substack{|j\rangle \in \mathcal{B}_{\mathbb{A}} \\ |i\rangle \in \mathcal{B}_{\mathbb{B}}}} a_{ij} |i\rangle \langle j|$$

(note that $|1\rangle \in \mathcal{B}_{\mathbb{A}}$ is a 3-dimensional vector, while $|1\rangle \in \mathcal{B}_{\mathbb{B}}$ is a 2-dimensional vector).

Solution: From the matrix above, we can construct the following:

$$a_{11} |1\rangle \langle 1| + \ldots + a_{23} |2\rangle \langle 3|$$

which can be expanded into:

$$a_{11} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \ldots + a_{23} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

From here, the a_{ij} can be chosen from the corresponding ij position in the matrix T, since this will isolate the element at the given ij position in the final matrix.

(ii) Show that for fixed $|i\rangle \in \mathcal{B}_{\mathbb{A}}$ and $|j\rangle \in \mathcal{B}_{\mathbb{B}}$

$$\big(|j\rangle \langle i| \big) |v\rangle = \langle i \mid v\rangle |j\rangle$$

for all $\langle v| \in \mathbb{A}$. From this, prove that $|j\rangle \langle i|$ defines a linear transformation from $\mathbb{A} \to \mathbb{B}$.

Solution:

$$\begin{aligned}
\big(|j\rangle \langle i| \big) |v\rangle &= \big(|a\rangle \langle b| \big) |v\rangle \text{ (relabel for ease of use)} \\
&= \big(\begin{pmatrix} a_i \\ a_j \end{pmatrix} \begin{pmatrix} b_i & b_j & b_k \end{pmatrix} \big) \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix} \\
&= \begin{pmatrix} a_i * b_i & a_i * b_j & a_i * b_k \\ a_j * b_i & a_j * b_j & a_j * b_k \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix} \\
&= \begin{pmatrix} \sum_x a_i b_x v_x \\ \sum_x a_j b_x v_x \end{pmatrix} \\
&= \begin{pmatrix} a_i (\sum_x b_x v_x) \\ a_j (\sum_x b_x v_x) \end{pmatrix} \\
&= \begin{pmatrix} a_i \langle b \mid v\rangle \\ a_j \langle b \mid v\rangle \end{pmatrix} \\
&= \begin{pmatrix} a_i \\ a_j \end{pmatrix} \langle b \mid v\rangle \\
&= |a\rangle \langle b \mid v\rangle \\
&= \langle i \mid v\rangle |j\rangle \text{ (relabel)}
\end{aligned}$$

This implies that $|j\rangle \langle i|$ defines a linear transformation from $\mathbb{A} \to \mathbb{B}$, because $\langle i \mid v\rangle$ is a scalar, which is then multiplied by $|j\rangle$, a vector in $\mathbb{B}$. Since the vector that this operator was applied to was originally in $\mathbb{A}$, this is enough to show that this is a linear transformation from $\mathbb{A} \to \mathbb{B}$

(iii) Suppose that

$$R = \sum_{\substack{|j\rangle \in \mathcal{B}_{\mathbb{A}} \\ |i\rangle \in \mathcal{B}_{\mathbb{B}}}} b_{ij} |i\rangle \langle j|$$

for $b_{ij} \in \mathbb{C}$. Use the previous part to prove that R is a linear transformation from $\mathbb{A} \to \mathbb{B}$.

Solution: R represents a matrix that is $\dim(\mathbb{A}) \times \dim(\mathbb{B})$. By the previous part, if any element from $\mathbb{A}$ is applied to R, the resulting matrix will be filled with elements from $\mathbb{B}$ because the linear operator $|j\rangle \langle i|$ is present at each index in R. This allows us to create a one-to-one mapping of elements in $\mathbb{A}$ to elements in $\mathbb{B}$.