Quantum Algorithms
Homework 7 Solutions

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1 Book Problems

1. Exercise 8.7
   Prove Properties 8.10-8.13 of the operator norm.

Solution:

(i) $||XY|| \leq ||X|| \, ||Y||$

Proof. We know that $||X||$ is given by

$$||X|| = \sup_{|\xi\rangle \neq 0} \frac{||X\,|\xi\rangle\,||}{||\,|\xi\rangle\,||}$$

So it follows that $||X||\,||Y||$ can be given by

$$||X||\,||Y|| = \sup_{|\xi\rangle \neq 0} \frac{||X\,|\xi\rangle\,||}{||\,|\xi\rangle\,||} \sup_{|\xi\rangle \neq 0} \frac{||Y\,|\xi\rangle\,||}{||\,|\xi\rangle\,||}$$

but $||XY||$ is given by

$$||XY|| = \sup_{|\xi\rangle \neq 0} \frac{||XY\,|\xi\rangle\,||}{||\,|\xi\rangle\,||}$$

Take $|\xi\rangle$ to be the vector that maximizes the norm, i.e. $||\,|\xi\rangle\,|| = 1$. The following inequality can be established:

$$\begin{aligned} ||XY|| \leq ||XY\,|\xi\rangle\,|| &\leq ||X(Y\,|\xi\rangle)|| \\ &\leq \frac{||X(Y\,|\xi\rangle)||}{||Y\,|\xi\rangle\,||} \leq ||X|| \\ &\leq ||X||\,||Y\,|\xi\rangle\,|| \end{aligned}$$

At each step we utilized a constant factor for ξ, whose norm will be the same in each computation. So, we can establish the following chain of inequalities:

$$||XY|| \leq ||XY\,|\xi\rangle\,|| \leq ||X||\,||Y\,|\xi\rangle\,|| \leq ||X||\,||Y||$$

□

(ii) $||X^\dagger|| = ||X||$

Proof. It is known that the largest eigenvalue of $X^\dagger X = ||X||^2$ (page 72 in the text). Define $M = X^\dagger$. It then follows that

$$
\begin{aligned}
||M||^2 &= \max\{\lambda \,|\lambda \text{ is an eigenvalue of } M^\dagger M\} \\
\implies ||X^\dagger||^2 &= \max\{\lambda \,|\lambda \text{ is an eigenvalue of } X^{\dagger\dagger} X^\dagger\} \\
&= \max\{\lambda \,|\lambda \text{ is an eigenvalue of } X X^\dagger\}
\end{aligned}
$$

so it holds that $||X||^2 = ||X^\dagger||^2$ if $X^\dagger X = XX^\dagger$, or that $X^\dagger X, XX^\dagger$ share the same nonzero set of eigenvalues (since we picked λ to be the largest of the set of these eigenvalues)

Book problem 6.4 asserts that the eigenvalues of $X^\dagger X$ are the same as the eigenvalues of $XX^\dagger$, so the two sets must have the same maximum. The assertion of problem 6.4 has a proof that can be seen on page 204 of the text. Since this proves the assertion above, it proves item (ii). □

(iii) $||X \otimes Y|| = ||X||\,||Y||$

Proof. First, define operator Γ to be $X \otimes Y$.
Note that $X : A \to V$, $Y : B \to W$.
Take K to be in $A \otimes B$. Then:

$$
\begin{aligned}
||X \otimes Y|| &= \sup_{|\xi\rangle \neq 0} \frac{||\Gamma\,|K\rangle\,||}{||K||} \\
&= \frac{\sqrt{\langle \xi \mid \xi \rangle}}{\sqrt{\langle K \mid K \rangle}} \qquad \text{where } \xi \text{ is the result of } \Gamma\,|K\rangle \text{ that is sup} \\
&= \frac{\sqrt{\left\langle \sum_{j,k} \lambda^* \mu^* \langle e_j \otimes f_k| \mid \sum_{l,m} \lambda\mu\,|e_l \otimes f_m\rangle \right\rangle}}{\sqrt{\langle K \mid K \rangle}} \qquad e_{j,l} \in \mathcal{B}_V, f_{k,m} \in \mathcal{B}_W \\
&= \frac{\sqrt{\langle X\alpha \mid X\alpha \rangle \langle Y\beta \mid Y\beta \rangle}}{\sqrt{\langle \alpha \mid \alpha \rangle \langle \beta \mid \beta \rangle}} \qquad \alpha \in A, \beta \in B \\
&= \frac{\sqrt{\langle X\alpha \mid X\alpha \rangle}\sqrt{\langle Y\beta \mid Y\beta \rangle}}{\sqrt{\langle \alpha \mid \alpha \rangle}\sqrt{\langle \beta \mid \beta \rangle}} \\
&= \frac{||X\,|\alpha\rangle\,||\;||Y\,|b\rangle\,||}{||\alpha||\;||\beta||} \\
&= ||X||\;||Y||
\end{aligned}
$$

So, this proves item (iii). □

(iv) $||U|| = 1$

(over)

Proof. First, introduce vector $|\xi\rangle$ as the vector that maximizes the norm:

$$
\begin{aligned}
||U|| &= \frac{||U\,|\xi\rangle\,||}{|\xi\rangle} \\
&= \frac{\sqrt{\langle U\xi \mid U\xi\rangle}}{\sqrt{\langle \xi \mid \xi\rangle}} \\
&= \frac{\sqrt{\langle \xi|\, U^\dagger U\,|\xi\rangle}}{\sqrt{\langle \xi \mid \xi\rangle}} \\
&= \frac{\sqrt{\langle \xi \mid \xi\rangle}}{\sqrt{\langle \xi \mid \xi\rangle}} \\
&= 1
\end{aligned}
$$

Note that the property $U^\dagger U = I$ is utilized in this proof, this fact has been previously utilized and proven (specifically Quiz 5). This proves item (iv). □

2. Exercise 8.8
Prove the two basic properties of approximation i with ancillas:
   (i) If $\tilde{U}$ approximates U with precision δ, then $\tilde{U}^{-1}$ approximates U^{-1} with the same precision δ
   (ii) If unitary operators $\tilde{U_k}$ approximate unitary operators U_k $(1 \leq k \leq L)$ with precision δ_k, then $\tilde{U_L}\ldots\tilde{U_1}$ approximates $U_L \ldots U_1$ with precision $\sum_k \delta_k$

Solution:

(i) *Proof.* We can use the fact that

$$||\tilde{U}V - UV|| \leq \delta$$

In order to show that this can hold for $\tilde{U^{-1}}$, the above expression must be recreated using U and $\tilde{U^{-1}}$:

$$||U^{-1}V - V\tilde{U}^{-1}|| \leq \delta$$

so multiply the original expression on either side by the norms of the two inverse unitary matrices.

$$
\begin{aligned}
||\tilde{U}V - UV|| &= ||\tilde{U}^{-1}||\;||\tilde{U}V - VU||\;||U^{-1}|| \\
&= ||\tilde{U}^{-1}V\tilde{U}U^{-1} - \tilde{U}^{-1}UU^{-1}|| \\
&= ||IU^{-1}V - V\tilde{U}^{-1}I|| \\
&= ||U^{-1}V - V\tilde{U}^{-1}|| \leq \delta
\end{aligned}
$$

□

(ii) *Proof.* To show that the errors propagate linearly, we can consider a concrete example of the propagation since it is a fact that the product of unitary matrices is also unitary. Take $L = 2$:

$$
\begin{aligned}
||\tilde{U}_2\tilde{U}_1 - U_2U_1|| &= ||\tilde{U}_2(\tilde{U}_1V - VU_1) + (\tilde{U}_2V - VU_2)U_1|| \\
&\leq ||\tilde{U}_2(\tilde{U}_1V - VU_1)|| + ||(\tilde{U}_2V - VU_2)U_1|| \\
&\leq ||\tilde{U}_2||\;||(\tilde{U}_1V - U_1)|| + ||(\tilde{U}_2V - VU_2)||\;||U_1|| \\
&= ||(\tilde{U}_1V - VU_1)|| + ||(\tilde{U}_2V - VU_2)|| \leq \delta_1 + \delta_2
\end{aligned}
$$

□