Risk Measures and their Applications in Quantitative Finance

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The Faculty of Business, Economics and Informatics of the University of Zurich hereby authorises the printing of this dissertation, without indicating an opinion of the views expressed in the work.

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To my beloved family

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Contents

Introduction

The understanding, measurement, and management of risk lay the ground for a safe, sustainable, and well-functioning modern world. The awareness of the concept of risk dates back to as far as Ancient Mesopotamia, according to a number of authors (see Hay-Gibson, 2008, and the references therein). But it was not until the emergence of early financial markets during the Renaissance in parts of Southern and Central Europe, that economic interest in understanding and managing risks sparked the rapid development of mathematical formalism (see Bernstein, 1996). Among the first contributors was Gerolamo Cardano, who compared empiricism to the theoretical probability of a certain outcome in a game of dice. Blaise Pascal and Pierre de Fermat are said to have laid the foundation for probability theory in their correspondence of 1654, where they solved the 'problem of points' (Lightner, 1991). Bernstein (1996) even argues that early probability theory and risk management, only possible after the adoption of Arabic numerals, constituted the key drivers that led to our modern society.

But this modern society is complicated. In the first half of the twentieth century, with sophisticated and complex financial markets in place, the number of academic publications around financial risk, portfolio diversification, and liquidity rose and found its first academic climax with the landmark papers of Markowitz (1952, 1959) and Roy (1952), marking the beginnings of what is now known as 'modern portfolio theory' (see also Markowitz, 1999). These publications sparked an abundance of research activity and led, among other things, to the 'capital asset pricing model' (see French, 2003, and the references therein). However, the increasing volatility and trading volume of derivatives in the 1980s, which ultimately led to the Black Monday stock market crash on October 19th, 1987, called for a revised concept of measuring risk. Early prototypes of Value-at-Risk (VaR) emerged as a tool to systematically separate extreme events, later termed 'Black Swans', from regular price movements. It was later adapted to be used to aggregate risk across trading desks of a single firm and gained public interest after its publication in 1994 by J.P. Morgan (see Jorion, 2006). The subsequent adoption of Value-at-Risk in the Basel Accords and later also in Solvency II, as well as the Expected Shortfall (ES) in the Swiss Solvency Test, increased the recognition of risk measures.

Finally, around the turn of the millennium, Artzner, Delbaen, Eber, and Heath (1999) formalised the concept of a risk measure. In their seminal contribution, they developed an axiomatic framework for measures of risk endowed with an explicit operational interpretation. Namely, raising a minimal amount of external capital and investing it in an asset S to meet the acceptability criteria specified as an acceptance set $\mathcal{A}$. This paradigm shift marked the beginning of the well-recognised theory of risk measures. It quickly became an integral part of the financial and academic world with applications ranging from capital adequacy, capital allocation and aggregation, pricing and hedging to portfolio selection and optimisation. The theory of risk measures has seen a plethora of debate, generalisations, and extensions. These range across a variety of topics including the introduction of convex risk measures (Föllmer and Schied, 2002; Frittelli and Rosazza Gianin, 2002), con-

ditional and dynamic risk measures (for a detailed overview see Acciaio and Penner, 2011), discussions about positive homogeneity, cash-(sub)additivity, (quasi-)convexity (Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio, 2011), the introduction of general eligible assets on quasi-ordered real topological vector spaces (Farkas, Koch-Medina, and Munari, 2014a,b; Munari, 2015) and many more. This increase in academic output led to an improved understanding of the concept of risk and the awareness of the pitfalls and shortcomings of the developed models.
While many of these risk measures find a variety of applications across the financial industry and beyond, the events during the Great Recession from 2007 to 2009 clearly demonstrated the insufficiency of these measures to capture systemic risk. As a result, the focus shifted towards a macroprudential perspective with the aim to regulate and mitigate the risk of the financial system as a whole. Many researchers and practitioners contributed to understanding and solving this problem, tackling it from many different angles. Brunnermeier (2009) describes and explains the economic mechanisms responsible for the events during the Great Recession. Adrian and Brunnermeier (2016) propose a conditional version of Value-at-Risk (CoVaR) which links the contribution of an institution to systemic risk with the increase of VaR of the entire financial system. Acharya, Pedersen, Philippon, and Richardson (2016) introduce the Systemic Expected Shortfall (SES) and the Marginal Expected Shortfall (MES) to measure the contribution of each financial institution to systemic risk. Acharya, Engle, and Richardson (2012) and Engle and Brownlees (2016) discuss the SRISK measure. For an extensive survey on systemic risk measures see (Silva, Kimura, and Sobreiro, 2017). A general approach with a broad range of applications is provided by the notion of set-valued risk measures (see for example Jouini, Meddeb, and Touzi, 2004; Hamel and Rudloff, 2008). Feinstein, Rudloff, and Weber (2017) introduce a methodology of measures of systemic risk in which many approaches such as for example (Chen, Iyengar, and Moallemi, 2013; Adrian and Brunnermeier, 2016; Brunnermeier and Cheridito, 2019) and also a special case of (Biagini, Fouque, Frittelli, and Meyer-Brandis, 2019) can be embedded directly or after a modification.
This very brief recapitulation of innovations dealing with risk and its concomitants attempts to show that we have come far in our understanding, but with ever more complex systems and technology to arise, so will new ways of handling the entailing risks.

Contribution This thesis contributes directly to the theory of risk measures by introducing a new concept of risk measure which is based around a management action that differs from the one suggested by (Artzner, Delbaen, Eber, and Heath, 1999) and adopts the notion of general eligible assets as introduced by Munari (2015). Our approach uses only internal capital and relies on restructuring a financial position instead of raising external capital. This approach is then extended and combined with the recently interest-gaining general approach of multivariate, set-valued measures of systemic risk. We show evidence that this treatment of systemic risk in networks of financial institutions has regulatory value, as it does not elevate the network with external capital, but rather shifts it to a safer eligible network structure.
The thesis contributes also indirectly to the theory of risk measures, demonstrating a powerful application of the Expected Shortfall and the Conditional Expected Shortfall in the context of capital allocation and aggregation in a network of insurance companies. We introduce a realistic risk-sharing mechanism between stand-alone insurance entities that allows them to optimise their profitability. Consulting principles of fairness from cooperative game theory, in particular Denault (1999), we are able to derive an optimal, fair, and unique risk-sharing scheme.

Structure The dissertation is divided into three main chapters. In the following, the main results are summarised.

In Chapter 1, *Intrinsic risk measures*, a version of which is published in (Farkas and Smirnow, 2018), we revisit the beginnings of modern risk measure theory and the formalism introduced in (Artzner, Delbaen, Eber, and Heath, 1999). Building on the framework of general eligible assets (Farkas, Koch-Medina, and Munari, 2014a,b; Munari, 2015), we suggest to restructure a non-acceptable financial position by selling parts of this position and investing the raised capital in the eligible asset. Mathematically, this approach results in a convex combination of two random variables, instead of a sum. The smallest convex coefficient such that the resulting convex combination lies in the acceptance set is the intrinsic risk of the position. We develop a framework for intrinsic risk measures and show that they satisfy important properties such as monotonicity and quasi-convexity. Furthermore, we derive a representation on conic acceptance sets and we provide a dual representation on convex acceptance sets.

In Chapter 2, *Intrinsic measures of systemic risk*, we combine the methodology developed in Chapter 1 with the notion of set-valued measures of systemic risk as treated in Feinstein, Rudloff, and Weber (2017). We first generalise the framework in Feinstein, Rudloff, and Weber (2017) to general eligible assets and then develop the framework of intrinsic measures for systemic risk. Since this approach requires an aggregation mechanism, we define the liability structure through the aggregation function and define our risk measures directly on the asset side of the financial institutions in the system. We show that intrinsic systemic risk measures satisfy set-valued equivalents of monotonicity and quasi-convexity. Furthermore, we derive a dual representation based on the work of Ararat and Rudloff (2020). Finally, we employ a complex aggregation mechanism based on networks as introduced in Eisenberg and Noe (2001) where assets and liabilities are treated separately. We incorporate a sink node ‘society’ as in Feinstein, Rudloff, and Weber (2017) to define acceptability criteria and discuss the ramifications of an unacceptable financial system in this regard. We see evidence that this approach has regulatory value to measure systemic risk and operational effectiveness to transform the system to a more stable system.

In Chapter 3, *Optimal risk-sharing across a network of insurance companies*, a version of which is published in (Ettlin, Farkas, Kull, and Smirnow, 2020), we turn to investigate networks of insurance companies. Insurance companies face strong regulations and must hold risk-based capital. We argue that common capital aggregation schemes cannot be applied to a network of insurance companies, since they face capital constraints on a stand-alone basis. Given this fact, we take an economic perspective to risk-based capital and introduce a realistic risk-sharing mechanism, incorporating the additional risk-based capital the companies have to hold in this reinsurance business setting. We show that risk-sharing can diversify the portfolios of the network participants and we derive risk transfer schemes that minimise the risk-based capital of the network, illustrating the effective use of the Expected Shortfall and the Conditional Expected Shortfall for capital allocations in this network setting. Finally, we apply principles from cooperative game theory developed in (Denault, 1999) to extract a unique optimal solution which is considered fair by the participants of the network.

Chapter 1

Intrinsic Risk Measures

A version of this chapter is published as: Walter Farkas and Alexander Smirnow. Intrinsic risk measures. In *Innovations in Insurance, Risk- and Asset Management*, Chapter 7, pp. 163-184, 2018. The notation has been adjusted to better match the notation in this thesis.

Monetary risk measures classify a financial position by the minimal amount of external capital that must be added to the position to make it acceptable.
We propose a new concept: intrinsic risk measures. The definition via external capital is avoided and only internal resources appear. An intrinsic risk measure is defined by the smallest percentage of the currently held financial position which has to be sold and reinvested in an eligible asset such that the resulting position becomes acceptable.
We show that this approach requires less nominal investment in the eligible asset to reach acceptability. It provides a more direct path from unacceptable positions towards the acceptance set and implements desired properties such as monotonicity and quasi-convexity solely through the structure of the acceptance set. We derive a representation on cones and a dual representation on convex acceptance sets and we detail the connections of intrinsic risk measures to their monetary counterparts.

1.1 Introduction

Risk measures associated with acceptance criteria as introduced by Artzner, Delbaen, Eber, and Heath (1999) are maps $\rho_{\mathcal{A},r}$ from a function space $\mathcal{X} \subseteq \mathbb{R}^{\Omega}$ to $\mathbb{R}$ of the form

$$\rho_{\mathcal{A},r}(X_T) = \inf \left\{ m \in \mathbb{R} \,|\, X_T + mr1_{\Omega} \in \mathcal{A} \right\}. \tag{1.1}$$

These maps are means to measure the 'risk' of a financial position $X_T \in \mathcal{X}$ with respect to certain acceptability criteria and a risk-free investment. The latter are specified as a subset $\mathcal{A} \subset \mathcal{X}$, the *acceptance set*, and the risk-free return rate $r > 0$, respectively. Geometrically[1], the risk of an unacceptable position $X_T \in \mathcal{X} \setminus \mathcal{A}$ in Equation (1.1) is defined as a scalar 'distance' to the acceptance set in direction $r1_{\Omega}$. Such risk measures are known as *cash-additive* risk measures. Evidently, the acceptance set forms the primary object, whereas the risk-free asset contributes only a constant factor. More recent research has revisited the original idea using eligible assets with random return rates $r : \Omega \to \mathbb{R}_{>0}$, as for example by Artzner, Delbaen, and Koch-Medina (2009) and Konstantinides and Kountzakis (2011). Farkas, Koch-Medina, and Munari (2014a,b) focus on general eligible assets $r : \Omega \to \mathbb{R}_{\geq 0}$, revealing significant shortcomings of the simplified constant approach.

[1]See Figure 1.1a for a visual example.

They point out that an appropriate interplay between eligible assets and acceptance sets is crucial for a consistent and successful risk measurement. They incorporate eligible assets as traded assets $S = (s_0, S_T)$ with initial unitary price $s_0 \in \mathbb{R}_{>0}$ and random payoff $S_T : \Omega \to \mathbb{R}_{\geq 0}$, and replace $r1_\Omega$ in Equation (1.1) by the random return S_T/s_0. This alteration yields the extended definition

$$\rho_{\mathcal{A},S}(X_T) = \inf \left\{ m \in \mathbb{R} \,\middle|\, X_T + \tfrac{m}{s_0} S_T \in \mathcal{A} \right\}. \tag{1.2}$$

Beside the geometric interpretation of $\frac{m}{s_0}S_T$ as a 'vector', it is economically interpreted as the payoff of $\frac{m}{s_0}$ units of asset S.
The more general definition in (1.2) can be consistently reduced to (1.1) if S_T is bounded away from zero, this means if $S_T \geq \varepsilon$, for some $\varepsilon > 0$.[2] This constitutes the basis for the simplified approach with constant return. However, payoffs of relevant financial instruments such as defaultable bonds and options do not satisfy this condition, and thus, the generalisation to *S-additive* risk measures in (1.2) is justified.

Referring to eligible assets, Artzner, Delbaen, Eber, and Heath (1999, Section 2.1, p. 205) suggest that

> '*The current cost of getting enough of this or these [commonly accepted] instrument(s) is a good candidate for a measure of risk of the initially unacceptable position.*'

Both cash-additive and S-additive risk measures are conceptually in line with this suggestion, and we broadly refer to them as *monetary risk measures*[3]. This is a suitable name as these risk measures are defined as actual money, which can be used to buy the eligible asset. Hence, they can be interpreted as more than just measurement tools. Referring to cash-additivity (or Axiom T), the authors claim in (Artzner, Delbaen, Eber, and Heath, 1999, Remark 2.7, p. 209) that

> '*By insisting on references to cash and to time, [...] our approach goes much further than the interpretation [...] that "the main function of a risk measure is to properly rank risks."*'

The application of this approach requires to raise the monetary amount $\rho_{\mathcal{A},S}(X_T)$ and carry it in the eligible asset S. However, the possible acquisition of additional capital is not completely accounted for by monetary risk measures. This raises the questions as to what effect this has on the risk measure and to which extent this method is applicable in reality.
Another approach is to restructure the portfolio and directly raise capital from the current position to invest it in the eligible asset, as was already mentioned in (Artzner, Delbaen, Eber, and Heath, 1999, Section 2.1, p. 205):

> '*For an unacceptable risk [...] one remedy may be to alter the position.*'

The aim of this chapter is to reflect on this thought and develop it towards a new class of risk measures, which we will call *intrinsic risk measures*. For great adaptability, we develop our approach based on acceptance sets $\mathcal{A} \subset \mathcal{X}$ as primary objects and the extended framework of general eligible assets $S = (s_0, S_T) \in \mathbb{R}_{>0} \times \mathcal{A}$.

In the 'future wealth' approach described in (Artzner, Delbaen, Eber, and Heath, 1999, p. 205), it is not possible to change the current financial position, representing the principle

[2] See (Farkas, Koch-Medina, and Munari, 2014a, Section 1, p. 146ff.) for a detailed discussion.
[3] A definition is given in Section 1.2.2.

of 'bygones are bygones'. The authors argue that the knowledge of the initial value of the position is not needed. So the risk measure is only used to determine the size of the buffer with respect to the eligible asset which sufficiently absorbs losses of this fixed position. However, we believe that a reconstruction of the financial position is possible and beneficial, since losses are not absorbed but essentially reduced as the eligible asset becomes part of the position. The intention to sell part of the current position requires the knowledge of the initial value. So while monetary risk measures are defined on $\mathcal{X}$, intrinsic risk measures take the initial value $x_0 \in \mathbb{R}_{>0}$ into account and are defined on $\mathbb{R}_{>0} \times \mathcal{X}$. For financial positions $X = (x_0, X_T)$ the intrinsic risk measure is given by

$$\rho^{\text{int}}_{\mathcal{A},S}(X) = \left\{\lambda \in [0,1] \mid (1-\lambda)X_T + \lambda \tfrac{x_0}{s_0} S_T \in \mathcal{A}\right\}. \tag{1.3}$$

In words, we search for the smallest $\lambda \in [0,1]$ such that selling the fraction λ of our initial position and investing the monetary amount λx_0 in the eligible asset S yields an acceptable position. Using the convex combination $(1-\lambda)X_T + \lambda \frac{x_0}{s_0} S_T$, $\lambda \in [0,1]$, instead of $X_T + \frac{m}{s_0} S_T$, $m \in \mathbb{R}$, changes the form of risk measures and suggests a new way to shift unacceptable positions towards the acceptance set.[4] Furthermore, standard properties such as monotonicity and, in contrast to monetary risk measures, also quasi-convexity are imposed solely through the structure of the underlying acceptance set.

We will introduce acceptance sets and traditional risk measures, give economic motivation, and review important properties in Section 1.2 to build a foundation for comparison. In Section 1.3, we define the new class of intrinsic risk measures and we derive basic properties. We derive an alternative representation on cones and show that intrinsic risk measures require less investment in the eligible asset to yield acceptable positions. Finally, we study a dual representation of intrinsic risk measures on convex acceptance sets.

1.2 Terminology and preliminaries

In this section, we establish the foundations on which we can build our framework. Common terminology such as acceptance sets and traditional risk measures are introduced and discussed.
Throughout this chapter we work on an atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For the sake of exposition we consider financial positions on the space of essentially bounded random variables $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ endowed with the $\mathbb{P}$-almost sure order and the $\mathbb{P}$-essential supremum norm. The majority of our results can be stated on arbitrary ordered real topological vector spaces.

1.2.1 Acceptance sets

In the financial world, it is a central task to hold positions that satisfy certain acceptability criteria, may they represent own preferences or be of regulatory nature. These criteria can be brought into a mathematical framework via what is known as acceptance sets.

Definition 1.2.1. *A subset $\mathcal{A} \subset \mathcal{X}$ is called an* acceptance set *if it satisfies*

1. Non-triviality*: $\mathcal{A} \neq \emptyset$ and $\mathcal{A} \subsetneq \mathcal{X}$, and*

2. Monotonicity*: $X_T \in \mathcal{A}$, $Y_T \in \mathcal{X}$, and $Y_T \geq X_T$ imply $Y_T \in \mathcal{A}$.*

[4] See Figure 1.1b.

An element $X_T \in \mathcal{A}$ is called $\mathcal{A}$-acceptable*, or just* acceptable *if the reference to $\mathcal{A}$ is clear. Similarly, we say $X_T \notin \mathcal{A}$ is* ($\mathcal{A}$-)unacceptable.

Non-triviality is mathematically important and also representative of real world requirements, as generally not every situation is acceptable and any event requires near-term reactions. Monotonicity implements the idea that any financial position dominating an acceptable position must be acceptable. These two axioms constitute the basis for acceptance sets and reflect the 'minimal' human rationale.
Depending on the context, it is often necessary to impose further structure and we recall three relevant properties.

Definition 1.2.2. *An acceptance set $\mathcal{A} \subset \mathcal{X}$ is called*

- *a* cone *or* conic *if $X_T \in \mathcal{A}$ implies for all $\lambda > 0 : \lambda X_T \in \mathcal{A}$,*
- convex *if $X_T, Y_T \in \mathcal{A}$ implies for all $\lambda \in [0,1] : \lambda X_T + (1-\lambda) Y_T \in \mathcal{A}$,*
- closed *if $\mathcal{A}$ is equal to its closure, $\mathcal{A} = \bar{\mathcal{A}}$.*

The cone property allows for arbitrary scaling of financial positions invariant of their acceptability status. Convexity represents the principle of diversification: given two acceptable positions, any convex combination of these will be acceptable. In Section 1.2.2, we will see how these two properties translate to monetary risk measures. Finally, closedness is of mathematical importance when considering limits of sequences of acceptable positions. Apart from this, it is economically motivated as it prohibits arbitrarily small perturbations to make unacceptable positions acceptable.

The next lemma summarises some useful properties of acceptance sets, which will be used in subsequent sections.

Lemma 1.2.3. *Let $\mathcal{A} \subset \mathcal{X}$ be an acceptance set. Then*

1. *$\mathcal{A}$ contains sufficiently large constants but no sufficiently small constants.*
2. *$S_T \in \operatorname{int}(\mathcal{A})$ if and only if there exists an $\varepsilon > 0$ such that $S_T - \varepsilon 1_\Omega \in \mathcal{A}$.*
3. *The interior $\operatorname{int}(\mathcal{A})$ and the closure $\bar{\mathcal{A}}$ are both acceptance sets, and $\operatorname{int}(\mathcal{A}) = \operatorname{int}(\bar{\mathcal{A}})$.*
4. *If $\mathcal{A}$ is a cone, then $\operatorname{int}(\mathcal{A})$ and $\bar{\mathcal{A}}$ are cones, and $0 \notin \operatorname{int}(\mathcal{A})$ and $0 \in \bar{\mathcal{A}}$.*

Proof. 1. Since $\mathcal{A}$ is a nonempty, proper subset of $\mathcal{X}$, the first assertion follows from monotonicity of $\mathcal{A}$.

2. The second assertion also follows directly from monotonicity of $\mathcal{A}$.

3. The proof of the third assertion goes along the lines of the proof of Lemma 2.3 in (Farkas, Koch-Medina, and Munari, 2014b, p. 60) and is omitted here.

4. Given $S_T \in \operatorname{int}(\mathcal{A})$, Assertion 2 together with the cone property imply $\lambda(S_T - \varepsilon 1_\Omega) \in \mathcal{A}$, for some $\varepsilon > 0$ and all $\lambda > 0$. The other direction of Assertion 2 implies $\lambda S_T \in \operatorname{int}(\mathcal{A})$. Given $S_T \in \bar{\mathcal{A}}$, take a sequence $\{S_T^n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ with limit S_T. Then conicity implies $\{\lambda S_T^n\}_{n\in\mathbb{N}} \subset \mathcal{A}$, for any $\lambda > 0$, and we conclude that λS_T belongs to $\bar{\mathcal{A}}$. The last two claims follow by similar arguments. □

We conclude this section with the well-known example of the *Value-at-Risk acceptance set*.

Example 1.2.4 (Value-at-Risk acceptance)**.** *For any probability level* $\alpha \in \left(0, \frac{1}{2}\right)$ *the set*

$$\mathcal{A}_\alpha = \{X_T \in \mathcal{X} \mid \mathbb{P}[X_T < 0] \le \alpha\}$$

defines a closed, conic acceptance set which, in general, is not convex.

Indeed, a few calculations show that $\mathcal{A}_\alpha$ *is a conic acceptance set. For closedness in* $L^\infty(\mathbb{P})$ *consider a sequence* $\{X_T^n\}_{n\in\mathbb{N}} \subset \mathcal{A}_\alpha$ *converging to some* X_T*. For any* $\delta > 0$ *and any* $n \in \mathbb{N}$ *the following inequality holds,*

$$\begin{aligned}\mathbb{P}[X_T < -\delta] &= \mathbb{P}[X_T < -\delta\,, X_T^n < -\tfrac{\delta}{2}] + \mathbb{P}[X_T < -\delta\,, X_T^n \ge -\tfrac{\delta}{2}]\\ &\le \alpha + \mathbb{P}[|X_T^n - X_T| > \tfrac{\delta}{2}]\,.\end{aligned}$$

Since norm convergence implies convergence in probability, letting $n \to +\infty$ *we get* $\mathbb{P}[X_T < -\delta] \le \alpha$*. It follows* $\mathbb{P}[X_T < 0] = \lim_{\delta\to 0}\mathbb{P}[X_T < -\delta] \le \alpha$*. To show that* $\mathcal{A}_\alpha$ *is not convex, we use its conicity to reduce the problem to finding* $X_T, Y_T \in \mathcal{A}_\alpha$ *such that* $X_T + Y_T \notin \mathcal{A}_\alpha$*. For two disjoint subsets* $A, B \in \mathcal{F}$ *with* $\mathbb{P}[A] = \mathbb{P}[B] = \alpha$ *the choices* $X_T = -1_A$ *and* $Y_T = -1_B$ *yield the desired inequality.*

1.2.2 Traditional risk measures

Traditional risk measures, commonly known as just risk measures, are instruments to measure risk in the financial world. Acceptance sets determine the meaning of 'good' and 'bad', acceptable or not. Traditional risk measures refine this differentiation and allow us to rank financial positions with respect to their distance to the acceptance set. To clearly distinguish between these risk measures and intrinsic risk measures, we define the broad class of traditional risk measures following Artzner, Delbaen, Eber, and Heath (1999, Definition 2.1, p. 207).

Definition 1.2.5. *A* traditional risk measure *is a map from* $\mathcal{X}$ *to* $\mathbb{R}$*.*

In Section 1.3, we will see that intrinsic risk measures are defined on $\mathbb{R}_{>0} \times \mathcal{X}$.
In what follows, we recall some well-known traditional risk measures. For the remainder of this section, let X_T, Y_T, Z_T and $r_T = r1_\Omega$ be elements of $\mathcal{X}$, and let ρ denote a traditional risk measure.

Coherent risk measures

Coherent risk measures form the historical foundation of modern risk measure theory. Artzner, Delbaen, Eber, and Heath (1999, Definition 2.4, p. 210) define them by the following set of axioms. A traditional risk measure is called *coherent* if it satisfies

- *Decreasing Monotonicity*: $X_T \ge Y_T$ implies $\rho(X_T) \le \rho(Y_T)$,
- *Cash-additivity*: for $m \in \mathbb{R}$ we have $\rho(X_T + mr_T) = \rho(X_T) - m$,
- *Positive Homogeneity*: for $\lambda \ge 0$ we have $\rho(\lambda X_T) = \lambda\rho(X_T)$, and
- *Subadditivity*: $\rho(X_T + Y_T) \le \rho(X_T) + \rho(Y_T)$.

Monotonicity allows us to rank financial positions according to their risk. It is cash-additivity that constitutes the basis for the interpretation of a risk measure as an additionally required amount of capital. Adding this capital to the financial position, its risk becomes 0, since by cash-additivity, $\rho(X_T + \rho(X_T)r_T) = 0$. These assumptions seem natural in the context of capital requirements and they are truly characterised by the term *monetary risk measures*, as coined by Föllmer and Schied (2004, Definition 4.1, p. 153).

Convex risk measures

Positive homogeneity, however, may not be satisfied, as risk can behave in non-linear ways. A possible variation is the following property around which Föllmer and Schied (2004) base their discussion of risk measures.

- *Convexity*: for all $\lambda \in [0,1]$ we have

$$\rho(\lambda X_T + (1-\lambda)Y_T) \leq \lambda\rho(X_T) + (1-\lambda)\rho(Y_T).$$

A short calculation reveals that under positive homogeneity, subadditivity and convexity are equivalent. Föllmer and Schied (2004, Definition 4.4, p. 154) decide to drop the homogeneity axiom and replace subadditivity by convexity, and call the result a *convex measure of risk* – a convex monetary risk measure.

The axioms we have seen so far form a canonical connection to our acceptance sets.

Proposition 1.2.6. *Any monetary risk measure $\rho : \mathcal{X} \to \mathbb{R}$ defines via*

$$\mathcal{A}_\rho = \{X_T \in \mathcal{X} \,|\, \rho(X_T) \leq 0\} \tag{1.4}$$

an acceptance set. Moreover, if ρ is positive homogeneous, then $\mathcal{A}_\rho$ is a cone, and if ρ is convex, then $\mathcal{A}_\rho$ is convex.

On the other hand, each acceptance set $\mathcal{A}$ defines a monetary risk measure

$$\rho_{\mathcal{A}}(X_T) = \inf\{m \in \mathbb{R} \,|\, X_T + m r_T \in \mathcal{A}\}. \tag{1.5}$$

Similarly, if $\mathcal{A}$ is a cone, then $\rho_{\mathcal{A}}$ is positive homogeneous, and if $\mathcal{A}$ is convex, then $\rho_{\mathcal{A}}$ is convex.
In particular, this means $\rho_{\mathcal{A}_\rho} = \rho$ and $\mathcal{A} \subseteq \mathcal{A}_{\rho_{\mathcal{A}}}$, with equality $\mathcal{A} = \mathcal{A}_{\rho_{\mathcal{A}}}$ if the acceptance set is closed.

Proof. The proof goes along the lines of the proofs of Proposition 4.6 and Proposition 4.7 in (Föllmer and Schied, 2004, p. 155f.) for bounded measurable functions on $(\Omega, \mathcal{F})$, and is omitted here. $\square$

Proposition 1.2.6 allows us to define acceptance sets via known risk measures and vice versa. Example 1.2.7 illustrates how properties can be inferred. A more general version of Proposition 1.2.6 is stated in Proposition 1.2.10.

Example 1.2.7 (Value-at-Risk acceptance)**.** *For a given probability level $\alpha \in \left(0, \frac{1}{2}\right)$ we define the risk measure* Value-at-Risk (VaR_α) *for all random variables on $(\Omega, \mathcal{F})$ by*

$$\mathrm{VaR}_\alpha(X_T) = \inf\{m \in \mathbb{R} \,|\, \mathbb{P}[X_T + m < 0] \leq \alpha\},$$

the negative of the α-quantile of X_T. Corresponding to Proposition 1.2.6, the VaR_α*-acceptance set is given by*

$$\mathcal{A}_{\mathrm{VaR}_\alpha} = \{X_T \in \mathcal{X} \,|\, \mathrm{VaR}_\alpha(X_T) \leq 0\}.$$

Recalling the closed, conic set $\mathcal{A}_\alpha = \{X_T \in \mathcal{X} \,|\, \mathbb{P}[X_T < 0] \leq \alpha\}$ from Example 1.2.4, we find that it defines the Value-at-Risk via Equation (1.5)*. So with Proposition 1.2.6 we conclude that $\mathcal{A}_\alpha = \mathcal{A}_{\mathrm{VaR}_\alpha}$ and that* VaR_α *is a positive homogeneous monetary risk measure which, in general, is not convex, and thus, not coherent.*

Convexity also allows for an alternative treatment of risk measures. The rich literature on convex functional analysis finds convenient application in the theory of risk measures. And risk measures are enriched with a dual representation and more possibilities of interpretation.
We recall two important results for completeness and for the comparison to the intrinsic dual representation in Section 1.3.4. The first one is given in Föllmer and Schied (2004, Theorem 4.31, p. 172).

Theorem 1.2.8. *Let $\mathcal{M}_\sigma(\mathbb{P}) = \mathcal{M}_\sigma(\Omega, \mathcal{F}, \mathbb{P})$ be the set of all σ-additive probability measures on $\mathcal{F}$ which are absolutely continuous with respect to $\mathbb{P}$. Let $\mathcal{A} \subset \mathcal{X}$ be a convex, $\sigma(L^\infty, L^1)$-closed (weak*-closed) acceptance set. Let $\rho_\mathcal{A}$ be defined as in Equation (1.5) with $r_T = 1_\Omega$. The risk measure has the representation*

$$\rho_\mathcal{A}(X_T) = \sup_{\mathbb{Q} \in \mathcal{M}_\sigma(\mathbb{P})} \left\{ \mathbb{E}^\mathbb{Q}[-X_T] - \alpha_{\min}(\mathbb{Q}, \mathcal{A}) \right\}, \tag{1.6}$$

with the minimal penalty function $\alpha_{\min}$ *defined for all $\mathbb{Q} \in \mathcal{M}_\sigma(\mathbb{P})$ by*

$$\alpha_{\min}(\mathbb{Q}, \mathcal{A}) = \sup_{X_T \in \mathcal{A}} \mathbb{E}^\mathbb{Q}[-X_T]\,. \tag{1.7}$$

Theorem 1.2.8 can now be directly applied to coherent risk measures, which of course are convex and positive homogeneous. But one can additionally show that with positive homogeneity we can restrict the supremum to a subset $\mathcal{M} \subset \mathcal{M}_\sigma(\mathbb{P})$ on which $\alpha_{\min}(\,\cdot\,, \mathcal{A}) = 0$. For further details see (Föllmer and Schied, 2004, Corollary 4.18 and Corollary 4.34, p. 165 and p. 175).

Corollary 1.2.9. *Let $\mathcal{A}$ be a conic, convex, $\sigma(L^\infty, L^1)$-closed acceptance set. Define the subset $\mathcal{M} = \{\mathbb{Q} \in \mathcal{M}_\sigma(\mathbb{P}) \,|\, \alpha_{\min}(\mathbb{Q}, \mathcal{A}) = 0\}$. Then the coherent risk measure $\rho_\mathcal{A} : \mathcal{X} \to \mathbb{R}$ can be written as*

$$\rho_\mathcal{A}(X_T) = \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^\mathbb{Q}[-X_T].$$

Cash-subadditivity and quasi-convexity of risk measures

El Karoui and Ravanelli (2009) point out that in presence of *stochastic interest rates* a financial position must be discounted before a cash-additive risk measure is applied. Consequently, the axiom of cash-additivity relies on the assumption that the discounting process does not carry additional risk. To relax this restriction they suggest the property of *cash-subadditivity*, where the equality in the cash-additivity condition is changed to the inequality '$\geq$'. However, Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011) explain that under cash-subadditivity, convexity is not a rigorous representative of the diversification principle, which translates into the following requirement for risk measures.

- *Diversification Principle*: if $\rho(X_T), \rho(Y_T) \leq \rho(Z_T)$ is satisfied, then

$$\text{for all } \lambda \in [0,1] : \rho(\lambda X_T + (1-\lambda) Y_T) \leq \rho(Z_T).$$

Substituting $\rho(Z_T)$ by $\max\{\rho(X_T), \rho(Y_T)\}$ yields the equivalent and recently importance gaining property of

- *Quasi-convexity*: for all $\lambda \in [0,1]$ we have

$$\rho(\lambda X_T + (1-\lambda) Y_T) \leq \max\{\rho(X_T), \rho(Y_T)\}\,.$$

Interestingly, quasi-convexity is equivalent to convexity under cash-additivity. Indeed, for any two positions with $\rho(X_T) \leq \rho(Y_T)$ we find an $m \in \mathbb{R}_{\geq 0}$ such that $\rho(X_T - mr_T) = \rho(Y_T)$ so that for any $\lambda \in [0,1]$ we get

$$\begin{aligned}\rho(\lambda X_T + (1-\lambda)Y_T) + \lambda m &\leq \max\{\rho(X_T - mr_T), \rho(Y_T)\} \\ &= \lambda\rho(X_T) + (1-\lambda)\rho(Y_T) + \lambda m\,.\end{aligned}$$

This equivalence does not hold under cash-subadditivity as shown in (Smirnow, 2016, Example 2.10, p. 12), resulting in the necessity to explicitly implement the diversification principle and thus, in the introduction of cash-subadditive, quasi-convex risk measures.

General monetary risk measures

Stochastic interest rates can also be directly addressed through risk measures of the form

$$\rho_{\mathcal{A},S}(X_T) = \inf\left\{m \in \mathbb{R} \mid X_T + \tfrac{m}{s_0} S_T \in \mathcal{A}\right\}, \tag{1.8}$$

as introduced in (Farkas, Koch-Medina, and Munari, 2014a,b). This approach avoids implicit discounting, since the stochastic eligible asset is now part of the risk measure. Munari (2015, Section 1.3, p. 26) provides a broad discussion of the discounting argument, revealing fundamental issues with discounting in the context of acceptance sets.

Equation (1.8) defines a generalised monetary risk measure which satisfies the following property for its defining eligible asset $S = (s_0, S_T)$,

- *S-additivity*: for $m \in \mathbb{R}$ we have $\rho_{\mathcal{A},S}(X_T + mS_T) = \rho_{\mathcal{A},S}(X_T) - ms_0$.

This general setup also yields the equivalence of quasi-convexity and convexity, and it exhibits a similar correspondence between acceptance sets and risk measures. The following result extends Proposition 1.2.6 to stochastic eligible assets.

Proposition 1.2.10. *Proposition 1.2.6 holds true if we replace $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ by any real ordered topological vector space, cash-additivity by S-additivity, and Equation* (1.5) *by Equation* (1.8)*, for any eligible asset $S = (s_0, S_T) \in \mathbb{R}_{>0} \times \mathcal{A}$.*

Proof. See the proofs of propositions 3.2.3, 3.2.4, 3.2.5, and 3.2.8 in (Munari, 2015, p. 87f.). The second claim in Proposition 1.2.6 follows from two short calculations. □

1.3 Intrinsic risk measures

The risk measures in the previous section all yield the same procedure to make an unacceptable position X_T acceptable – raise the required 'minimal' capital $\rho_{\mathcal{A},S}(X_T)$ and get the acceptable position $X_T^\rho := X_T + \frac{\rho_{\mathcal{A},S}(X_T)}{s_0} S_T$. A procedure to acquire the required capital-level and the risk of failing to obtain it are not addressed by these risk measures. But what if we do not use external capital?

1.3.1 Fundamental concepts

In this section, we explore a different procedure to obtain acceptable positions. We suggest to sell part of the risky position and invest the acquired capital in the acceptable eligible asset. Hereby, the distance to the acceptance set is directly reduced and therefore also the risk.

In order to sell our original position, we require the knowledge of the initial value $x_0 \in \mathbb{R}_{>0}$. Following the definition of general eligible assets $S = (s_0, S_T) \in \mathbb{R}_{>0} \times \mathcal{A}$ in Section 1.2.2, we consider financial positions $X = (x_0, X_T)$ on the product space $\mathbb{R}_{>0} \times \mathcal{X}$. The main object in this approach is the net worth of the convex combination of the risky position and a multiple of the eligible asset

$$X_T^{\lambda,S} := (1-\lambda)X_T + \lambda \tfrac{x_0}{s_0} S_T \in \mathcal{X}\,, \quad \lambda \in [0,1].$$

The notation $X_T^{\lambda,S}$ is convenient and we extend it to the whole position $X \in \mathbb{R}_{>0} \times \mathcal{X}$ as

$$X^{\lambda,S} := (x_0, X_T^{\lambda,S}) \in \mathbb{R}_{>0} \times \mathcal{X}.$$

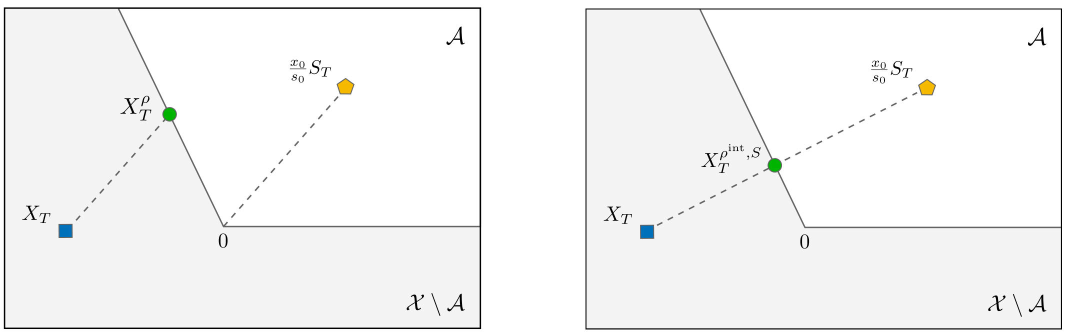
Hence, $X^{\lambda,S}$ describes a position with initial value x_0 which is split into $(1-\lambda)x_0$ and λx_0 and is then invested to get $(1-\lambda)X_T$ and $\lambda \frac{x_0}{s_0} S_T$, respectively. We aim to find the smallest λ such that $X_T^{\lambda,S}$ is acceptable, this defines the intrinsic risk measure.

Definition 1.3.1 (Intrinsic Risk Measure)**.** *For an acceptance set $\mathcal{A} \subset \mathcal{X}$ and an eligible asset $S \in \mathbb{R}_{>0} \times \mathcal{A}$ the intrinsic risk measure is a map $\rho^{\text{int}}_{\mathcal{A},S} \colon \mathbb{R}_{>0} \times \mathcal{X} \to [0,1]$ defined by*

$$\rho^{\text{int}}_{\mathcal{A},S}(X) = \inf \left\{ \lambda \in [0,1] \,\middle|\, X_T^{\lambda,S} \in \mathcal{A} \right\}. \tag{1.9}$$

For well-definedness two short considerations yield that the acceptance set must either be a cone or that 0 must be contained in it[5]. In both cases, $\lambda \frac{x_0}{s_0} S_T$ is acceptable for $\lambda \in (0,1]$, or $\lambda \in [0,1]$ if $\mathcal{A}$ is closed. This means selling all of the original position leaves us always with an acceptable net worth $\frac{x_0}{s_0} S_T$.

A brief comparison of the intrinsic approach and the traditional monetary approach is provided below. Consider the conceptual Figure 1.1 and imagine that $\mathcal{A}$ is an arbitrary closed acceptance set.



(a) Traditional approach (b) Intrinsic approach

Figure 1.1: The payoff of the eligible asset (yellow ⬠) is used to make the unacceptable position (blue □) acceptable (green ○).

While the monetary approach, illustrated in Figure 1.1a, yields the position $X_T^{\rho} := X_T + \frac{\rho_{\mathcal{A},S}(X_T)}{s_0} S_T$, the intrinsic approach, illustrated in Figure 1.1b, gives us

$$X_T^{\rho^{\text{int}}_{\mathcal{A},S}(X),S} := (1 - \rho^{\text{int}}_{\mathcal{A},S}(X))X_T + \rho^{\text{int}}_{\mathcal{A},S}(X) \tfrac{x_0}{s_0} S_T,$$

which we abbreviate with $X_T^{\rho^{\text{int}},S}$ if the reference to $\mathcal{A}, S$, and X is clear.

[5]The assumption $0 \in \mathcal{A}$ is widely used in the financial literature, as for example the equivalent Axiom 2.1 in (Artzner, Delbaen, Eber, and Heath, 1999, p. 206) or, if $\mathcal{A}$ is closed, the *normalisation property* $\rho(0) = 0$ in (Föllmer and Schied, 2004, above Remark 4.2, p. 154).

1. We notice that, since $\mathcal{A}$ is closed, both risk measures are strictly positive if and only if $X_T \notin \mathcal{A}$. In this case, and if $S_T \in \operatorname{int}(\mathcal{A})$, both altered positions X_T^{ρ} and $X_T^{\rho^{\text{int}},S}$ lie on the boundary of the acceptance set. Moreover, if $\mathcal{A}$ is either a cone or convex with $0 \in \mathcal{A}$, then the set $\{X_T^{\lambda,S} \mid \lambda \in [\rho^{\text{int}}_{\mathcal{A},S}(X), 1]\}$ belongs to $\mathcal{A}$. A similar results holds true for monetary risk measures.

2. If we assume a conic acceptance set as in Figure 1.1, we intuit that $X_T^{\rho^{\text{int}},S}$ must be a multiple of X_T^{ρ}. And indeed, in Corollary 1.3.7 we will derive the relation

$$X_T^{\rho^{\text{int}}_{\mathcal{A},S}(X),S} = (1 - \rho^{\text{int}}_{\mathcal{A},S}(X))X_T^{\rho}\,. \tag{1.10}$$

3. By Definition 1.3.1, it is apparent that intrinsic risk measures cannot attain infinite values as opposed to traditional risk measures. Farkas, Koch-Medina, and Munari (2014b, Theorem 3.3 and Corollary 3.4, p. 62) have shown that on closed, conic acceptance sets

$$\rho_{\mathcal{A},S} \text{ is finite if and only if } S_T \in \operatorname{int}(\mathcal{A}).$$

For a graphical illustration imagine that in Figure 1.1, $S_T \in \partial\mathcal{A}$. Then in Figure 1.1a, a possible X_T^{ρ} would move along a line 'parallel' to the boundary, thus it would never reach $\mathcal{A}$. Consequently, $\rho_{\mathcal{A},S}(X_T) = +\infty$ and X_T^{ρ} is actually not defined.

In contrast, one can show[6] that on closed, conic acceptance sets

$$\rho^{\text{int}}_{\mathcal{A},S} < 1 \text{ on } \mathbb{R}_{>0} \times \mathcal{X} \setminus \mathcal{A} \text{ if and only if } S_T \in \operatorname{int}(\mathcal{A})\,.$$

Hence, if $S_T \in \partial\mathcal{A}$ in Figure 1.1b, then $X_T^{\rho^{\text{int}},S}$ and $\frac{x_0}{s_0}S_T$ coincide on the boundary with $\rho^{\text{int}}_{\mathcal{A},S}(X) = 1$.

Having established a basic intuition for this approach, we will now take a deeper look at some of its properties. For this, we introduce the notions of monotonicity and convexity on $\mathbb{R}_{>0} \times \mathcal{X}$.

1. The monotonicity of $\mathcal{A}$ should be reflected by the corresponding intrinsic risk measure. So we need to extend the ordering on $\mathcal{X}$ to $\mathbb{R}_{>0} \times \mathcal{X}$. Two possible orderings are *element-wise* and *return-wise* defined respectively by

$$\begin{aligned} X &\geqslant_{\text{el}} Y \;\text{ if }\; x_0 \geq y_0 \text{ and } X_T \geq Y_T, \text{ and} \\ X &\geqslant_{\text{re}} Y \;\text{ if }\; \frac{X_T}{x_0} \geq \frac{Y_T}{y_0}. \end{aligned}$$

2. On $\mathbb{R}_{>0} \times \mathcal{X}$, we think of convex combinations element-wise as

$$\alpha X + (1-\alpha)Y := (\alpha x_0 + (1-\alpha)y_0\,,\, \alpha X_T + (1-\alpha)Y_T) \in \mathbb{R}_{>0} \times \mathcal{X}.$$

We can now show monotonicity and quasi-convexity of intrinsic risk measures with respect to these rules.

[6]For a direct proof one can use Lemma 1.2.3 and the fact that $X_T^{\rho^{\text{int}},S} \in \mathcal{A}$. For a proof via monetary risk measures consider Theorem 1.3.3 and Corollary 1.3.5 below.

Proposition 1.3.2 (Monotonicity, Quasi-convexity)**.** *Let $\mathcal{A}$ be an acceptance set containing* 0*, let $S \in \mathbb{R}_{>0} \times \mathcal{A}$ be an eligible asset and let $X, Y \in \mathbb{R}_{>0} \times \mathcal{X}$.*

1. *The order $X \geqslant_{el} Y$ implies $\rho^{\text{int}}_{\mathcal{A},S}(X) \leq \rho^{\text{int}}_{\mathcal{A},S}(Y)$. On conic acceptance sets, also the return-wise order $X \geqslant_{re} Y$ implies $\rho^{\text{int}}_{\mathcal{A},S}(X) \leq \rho^{\text{int}}_{\mathcal{A},S}(Y)$.*

2. *Let $\mathcal{A}$ be additionally convex. Then $\rho^{\text{int}}_{\mathcal{A},S}$ is quasi-convex, that means for all $\alpha \in [0,1]$, and any $X, Y \in \mathbb{R}_{>0} \times \mathcal{X}$*

$$\rho^{\text{int}}_{\mathcal{A},S}(\alpha X + (1-\alpha)Y) \leq \max\{\rho^{\text{int}}_{\mathcal{A},S}(X), \rho^{\text{int}}_{\mathcal{A},S}(Y)\}.$$

Proof. 1. If $X \geqslant_{\text{el}} Y$, then $X^{\lambda,S}_T \geq Y^{\lambda,S}_T$ for all $\lambda \in [0,1]$, and thus, by monotonicity of the acceptance set, $\rho^{\text{int}}_{\mathcal{A},S}(X) \leq \rho^{\text{int}}_{\mathcal{A},S}(Y)$. Similarly, $X \geqslant_{\text{re}} Y$ implies $X^{\lambda,S}_T \geq \frac{x_0}{y_0} Y^{\lambda,S}_T$. By conicity, we have $\frac{x_0}{y_0} Y^{\rho^{\text{int}}_{\mathcal{A},S}(Y),S}_T \in \mathcal{A}$ and again by monotonicity, we get $X^{\lambda,S}_T \in \mathcal{A}$.

2. Assume without loss of generality that $\rho^{\text{int}}_{\mathcal{A},S}(X) \leq \rho^{\text{int}}_{\mathcal{A},S}(Y)$. As mentioned above, since $\mathcal{A}$ is convex, $\{X^{\lambda,S}_T \,|\, \lambda \in [\rho^{\text{int}}_{\mathcal{A},S}(X), 1]\} \subset \mathcal{A}$. Hence, if $\lambda \in [\rho^{\text{int}}_{\mathcal{A},S}(Y), 1]$, then the convex combinations $Y^{\lambda,S}_T$ and $X^{\lambda,S}_T$ lie in $\mathcal{A}$. In particular, also their convex combinations $\alpha X^{\lambda,S}_T + (1-\alpha) Y^{\lambda,S}_T \in \mathcal{A}$, for all $\alpha \in [0,1]$. But these convex combinations commutate so that

$$\begin{aligned}\rho^{\text{int}}_{\mathcal{A},S}(\alpha X + (1-\alpha)Y) &= \inf\left\{\lambda \in [0,1] \,\middle|\, \alpha X^{\lambda,S}_T + (1-\alpha) Y^{\lambda,S}_T \in \mathcal{A}\right\} \\ &\leq \rho^{\text{int}}_{\mathcal{A},S}(Y) = \max\left\{\rho^{\text{int}}_{\mathcal{A},S}(X), \rho^{\text{int}}_{\mathcal{A},S}(Y)\right\},\end{aligned}$$

showing quasi-convexity of the intrinsic risk measure. □

So while monotonicity of $\mathcal{A}$ is passed on to the corresponding intrinsic risk measure, convexity of the acceptance set implies quasi-convexity and not convexity of the measure, as we have seen in Proposition 1.2.6 for monetary risk measures. A counter-example to convexity can be constructed with the transition property for unacceptable X and $\alpha \in [0, \rho^{\text{int}}_{\mathcal{A},S}(X)]$,

$$\rho^{\text{int}}_{\mathcal{A},S}(X^{\alpha,S}) = \frac{\rho^{\text{int}}_{\mathcal{A},S}(X) - \alpha}{1-\alpha},$$

which can be derived using the bijection $[0,1] \to [\alpha, 1]$ with $\lambda \mapsto (1-\lambda)\alpha + \lambda$, and the fact that $(1-\beta)X + \beta X^{\alpha,S} = X^{\alpha\beta,S}$. With help of Example 1.2.4 it can be shown that convexity of $\mathcal{A}$ is necessary for quasi-convexity of the intrinsic risk measure. Finally, a similar argument yields quasi-convexity with respect to eligible assets $S^1, S^2 \in \mathbb{R}_{>0} \times \mathcal{A}$ with same initial price $s^1_0 = s^2_0$,

$$R_{\mathcal{A},\alpha S^1 + (1-\alpha)S^2}(X) \leq \max\{R_{\mathcal{A},S^1}(X), R_{\mathcal{A},S^2}(X)\}.$$

1.3.2 Representation on conic acceptance sets

In this section, we will use cash- or S-additivity of monetary risk measures to derive an alternative representation of intrinsic risk measures on cones. This representation allows us to apply important results from monetary to intrinsic risk measures.

Theorem 1.3.3 (Representation on cones)**.** *Let $\rho_{\mathcal{A},S} : \mathcal{X} \to \mathbb{R}$ be a monetary risk measure defined by a closed, conic acceptance set $\mathcal{A}$ and let $S \in \mathbb{R}_{>0} \times \mathcal{A}$ be an eligible asset. Then the intrinsic risk measure with respect to $\mathcal{A}$ and S can be written as*

$$\rho^{\text{int}}_{\mathcal{A},S}(X) = \frac{(\rho_{\mathcal{A},S}(X_T))^+}{x_0 + \rho_{\mathcal{A},S}(X_T)}. \tag{1.11}$$

Proof. Since $\mathcal{A}$ is closed, we can use Proposition 1.2.10 to write

$$\rho^{\text{int}}_{\mathcal{A},S}(X) = \inf\{\lambda \in [0,1] \,|\, X_T^{\lambda,S} \in \mathcal{A}\} = \inf\{\lambda \in [0,1] \,|\, \rho_{\mathcal{A},S}(X_T^{\lambda,S}) \leq 0\}.$$

But $\rho_{\mathcal{A},S}$ is S-additive and positive homogeneous, so that we have

$$\rho^{\text{int}}_{\mathcal{A},S}(X) = \inf\big\{\lambda \in [0,1] \,|\, \rho_{\mathcal{A},S}(X_T) \leq \lambda\big(x_0 + \rho_{\mathcal{A},S}(X_T)\big)\big\}.$$

If $\rho_{\mathcal{A},S}(X_T) > 0$, then we can solve for λ to get the form in Equation (1.11). If $\rho_{\mathcal{A},S}(X_T) \leq 0$, then $X_T \in \mathcal{A}$ and therefore $\rho^{\text{int}}_{\mathcal{A},S}(X) = 0$. We abbreviate these two cases with $(\rho_{\mathcal{A},S}(X_T))^+$ in the numerator. □

Example 1.3.4. *For continuous X_T and constant eligible assets $S_T = rs_0 1_\Omega > 0$ we can directly derive the representation in Equation* (1.11) *on the conic Value-at-Risk acceptance set $\mathcal{A}_\alpha = \{X_T \in \mathcal{X} \,|\, \mathbb{P}[X_T < 0] \leq \alpha\}$ from Example 1.2.7. Let F_X be the continuous cumulative distribution function of X_T with inverse F_X^{-1}. For $X_T \notin \mathcal{A}_\alpha$, this means $F_X^{-1}(\alpha) < 0$, we get*

$$\begin{aligned} R_{\mathcal{A}_\alpha,S}(X) &= \inf\big\{\lambda \in (0,1) \,|\, \mathbb{P}[X_T^{\lambda,S} < 0] \leq \alpha\big\} \\ &= \inf\big\{\lambda \in (0,1) \,|\, F_X(-(1-\lambda)^{-1}\lambda r x_0) \leq \alpha\big\} \\ &= \frac{F_X^{-1}(\alpha)}{F_X^{-1}(\alpha) - rx_0} = \frac{\mathrm{VaR}_\alpha(X_T)}{rx_0 + \mathrm{VaR}_\alpha(X_T)}, \end{aligned}$$

an expression similar to Equation (1.11)*. Of course, when we use the constant eligible asset $S_T = rs_0 1_\Omega$, the Value-at-Risk is of the form $\rho_{\mathcal{A}_\alpha}(X) = \inf\{m \in \mathbb{R} \,|\, X_T + m1_\Omega \in \mathcal{A}_\alpha\}$ with $r = 1$.*

In our opinion, Theorem 1.3.3 is a very convenient result that allows us to draw connections to traditional risk measures. This is true for all conic acceptance sets, including the commonly used Value-at-Risk and Expected Shortfall acceptance sets. In particular, some important results from traditional risk measures can be directly applied to intrinsic risk measures.

Corollary 1.3.5. *Let $\mathcal{A}$ be a closed, conic acceptance set.*

1. $\rho^{\text{int}}_{\mathcal{A},S} < 1$ *on $\mathbb{R}_{>0} \times \mathcal{X} \setminus \mathcal{A}$ if and only if $S_T \in \operatorname{int}(\mathcal{A})$.*
2. *If $S_T \in \operatorname{int}(\mathcal{X}_+)$, then $\rho^{\text{int}}_{\mathcal{A},S}$ is continuous on $\mathbb{R}_{>0} \times \mathcal{X}$.*
3. *If $\mathcal{A}$ is additionally convex, then $S_T \in \operatorname{int}(\mathcal{A})$ implies continuity of $\rho^{\text{int}}_{\mathcal{A},S}$.*
4. $\rho^{\text{int}}_{\mathcal{A},S}$ *is scale-invariant, meaning $\rho^{\text{int}}_{\mathcal{A},S}(\alpha X) = \rho^{\text{int}}_{\mathcal{A},S}(X)$, for $\alpha > 0$.*

Proof. 1. With the representation in Theorem 1.3.3 and the finiteness result in (Farkas, Koch-Medina, and Munari, 2014b, Theorem 3.3, p. 62) the assertion follows directly.

2. As shown in (Farkas, Koch-Medina, and Munari, 2014a, Proposition 3.1, p. 154), if $S_T \in \mathrm{int}(\mathcal{X}_+)$, then $\rho_{\mathcal{A},S}$ is continuous. The map $f : (x_0, x) \mapsto \frac{x^+}{x_0+x}$ is jointly continuous on $\mathbb{R}_{>0} \times \mathbb{R}$. Therefore, as the composition of two continuous maps the intrinsic risk measures is continuous on $\mathbb{R}_{>0} \times \mathcal{X}$.

3. In this case, (Farkas, Koch-Medina, and Munari, 2014a, Theorem 3.16, p. 159) gives us continuity of $\rho_{\mathcal{A},S}$. The assertion follows as in the second part.

4. If $X_T \in \mathcal{A}$, then so is αX_T and thus, $\rho^{\mathrm{int}}_{\mathcal{A},S}(\alpha X) = \rho^{\mathrm{int}}_{\mathcal{A},S}(X) = 0$. If $X_T \notin \mathcal{A}$, then $\rho_{\mathcal{A},S}(X_T) > 0$ and the assertion follows from positive homogeneity of $\rho_{\mathcal{A},S}$ and Theorem 1.3.3. □

Another version of Theorem 1.3.3 is the representation of monetary risk measures on $\mathcal{X} \setminus \mathcal{A}$ in terms of intrinsic risk measures.

Corollary 1.3.6. *Let the acceptance set $\mathcal{A}$ be closed and conic, let $S \in \mathbb{R}_{>0} \times \mathrm{int}(\mathcal{A})$ and $X = (x_0, X_T) \in \mathbb{R}_{>0} \times \mathcal{X} \setminus \mathcal{A}$. Then*

$$\rho_{\mathcal{A},S}(X_T) = \frac{x_0\, \rho^{\mathrm{int}}_{\mathcal{A},S}(X)}{1 - \rho^{\mathrm{int}}_{\mathcal{A},S}(X)}. \tag{1.12}$$

Proof. We have $\rho_{\mathcal{A},S}(X_T) > 0$ on $\mathcal{X} \setminus \mathcal{A}$ and by Corollary 1.3.5, $S_T \in \mathrm{int}(\mathcal{A})$ implies $\rho^{\mathrm{int}}_{\mathcal{A},S} < 1$ on $\mathbb{R}_{>0} \times \mathcal{X} \setminus \mathcal{A}$. Setting $X = (x_0, X_T)$, for any $x_0 > 0$, and rearranging Equation (1.11) yields the assertion. □

With this representation we confirm our claim that $X^{\rho}_T = X_T + \frac{\rho_{\mathcal{A},S}(X_T)}{s_0} S_T$ is a multiple of $X^{\rho^{\mathrm{int}},S}_T$.

Corollary 1.3.7. *In the setting of Corollary 1.3.6, we have*

$$X^{\rho^{\mathrm{int}}_{\mathcal{A},S}(X),S}_T = (1 - \rho^{\mathrm{int}}_{\mathcal{A},S}(X)) X^{\rho}_T\,. \tag{1.13}$$

Proof. Dividing $X^{\rho^{\mathrm{int}},S}_T$ by $1 - \rho^{\mathrm{int}}_{\mathcal{A},S}(X)$ and using Equation (1.12) yields the desired relation. □

The representation in (1.11) does not hold for convex, non-conic acceptance sets. However, it does give us an upper bound.

Proposition 1.3.8. *Let $\mathcal{A}$ be a closed, convex acceptance set containing 0, which is not a cone. Then the following inequality holds,*

$$\rho^{\mathrm{int}}_{\mathcal{A},S}(X) \leq \frac{(\rho_{\mathcal{A},S}(X_T))^+}{x_0 + \rho_{\mathcal{A},S}(X_T)}. \tag{1.14}$$

Proof. Using Proposition 1.2.10, we establish with S-additivity, and then convexity and the fact that $\rho_{\mathcal{A},S}(0) \leq 0$ the inequality

$$\rho_{\mathcal{A},S}(X^{\lambda,S}_T) = \rho_{\mathcal{A},S}((1-\lambda)X_T) - \lambda x_0 \leq (1-\lambda)\rho_{\mathcal{A},S}(X_T) - \lambda x_0.$$

With this we arrive at the inclusion

$$\{\lambda \in [0,1] \,|\, (1-\lambda)\rho_{\mathcal{A},S}(X_T) - \lambda x_0 \leq 0\} \subseteq \{\lambda \in [0,1] \,|\, \rho_{\mathcal{A},S}(X^{\lambda,S}_T) \leq 0\},$$

which implies (1.14). □

1.3.3 Efficiency of the intrinsic approach

In the previous section, we have derived all necessary results to compare the intrinsic and the traditional approach on a monetary basis. We find that on conic or convex acceptance sets the intrinsic approach requires less investment in eligible assets. But on cones it yields positions with the same performance.

Corollary 1.3.9. *Let $\mathcal{A}$ be a closed acceptance set, either conic or convex. For an unacceptable position $X = (x_0, X_T)$ and an eligible asset S we have*

$$x_0 \rho^{\text{int}}_{\mathcal{A},S}(X) \leq \rho_{\mathcal{A},S}(X_T).$$

Proof. With Theorem 1.3.3 for conic acceptance sets, and Proposition 1.3.8 for the convex case we establish $x_0 \rho^{\text{int}}_{\mathcal{A},S}(X) \leq x_0 \frac{\rho_{\mathcal{A},S}(X_T)}{x_0 + \rho_{\mathcal{A},S}(X_T)}$. For unacceptable X_T the inequality $x_0 \frac{\rho_{\mathcal{A},S}(X_T)}{x_0 + \rho_{\mathcal{A},S}(X_T)} \leq \rho_{\mathcal{A},S}(X_T)$ holds true, proving the assertion. □

So while the magnitude of the initial value x_0 controls the required monetary amount, Corollary 1.3.9 shows that the amount $x_0 \rho^{\text{int}}_{\mathcal{A},S}(X)$ is always less than $\rho_{\mathcal{A},S}(X_T)$. This means using the intrinsic approach, less capital is transitioned to the eligible asset.

But since less money is invested in the eligible asset, one could think that the intrinsic approach yields worse acceptable positions compared to the traditional approach. However, comparing the resulting positions in terms of returns, for example with the (revised) Sharpe ratio, shows otherwise.
Given a financial position $X = (x_0, X_T)$, a monetary risk measure yields the acceptable position $X_T^{\rho} = X_T + \frac{\rho_{\mathcal{A},S}(X_T)}{s_0} S_T$. This means that at inception, the initial value must be $x_0^{\rho} := x_0 + \rho_{\mathcal{A},S}(X_T)$. On the other hand, an intrinsic risk measure does not change the initial value x_0 to get the acceptable position $X_T^{\rho^{\text{int}},S}$. Interestingly, the returns of these positions are equal on cones.

Corollary 1.3.10. *Let $\mathcal{A}$ be a closed, conic acceptance set, X an unacceptable position, and S an eligible asset. The returns of the positions $(x_0, X_T^{\rho^{\text{int}},S})$ and (x_0^{ρ}, X_T^{ρ}) are equal.*

Proof. Dividing both sides of Equation (1.13) by x_0 and using Equation (1.11) yield the assertion. □

1.3.4 Dual representations on convex acceptance sets

Referring to duality results of convex and coherent risk measures stated in Section 1.2.2, we derive a dual representation of intrinsic risk measures. The derivation is based on a representation of convex acceptance sets by $\mathcal{M}_\sigma(\mathbb{P})$, the set of σ-additive, absolutely continuous probability measures $\mathbb{Q} \ll \mathbb{P}$, similar to that of Drapeau and Kupper (2013, Lemma 2, p. 52).

Lemma 1.3.11. *Let $\mathcal{A}$ be a $\sigma(L^\infty, L^1)$-closed, convex acceptance set. Then $X_T \in \mathcal{A}$ if and only if for all probability measures $\mathbb{Q} \in \mathcal{M}_\sigma(\mathbb{P})$*

$$\inf_{Y_T \in \mathcal{A}} \mathbb{E}^{\mathbb{Q}}[Y_T] \leq \mathbb{E}^{\mathbb{Q}}[X_T].$$

Proof. The 'only if' implication is evidently true. We outline the proof of the 'if' direction. Using a version of the Hahn-Banach Separation Theorem, see for example (Dunford and Schwartz, 1958, Theorem V.2.10, p. 417), one shows that for any $X_T \in \mathcal{X} \setminus \mathcal{A}$ there is a linear functional ℓ in the topological dual space $\mathcal{X}^*$ such that $\inf_{y \in \mathcal{A}} \ell(y) > \ell(x)$. The structure of $\mathcal{A}$ implies that ℓ is positive on the positive cone $\{X_T \in \mathcal{X} \mid X_T \geq 0\}$. Under the weak*-topology $\sigma(L^\infty, L^1)$, using the Radon-Nikodým Theorem, as for example stated in (Dunford and Schwartz, 1958, Theorem III.10.2, p. 176), these linear functionals can be identified with expectations with respect to σ-additive, absolutely continuous probability measures $\mathbb{Q} \ll \mathbb{P}$ in $\mathcal{M}_\sigma(\mathbb{P})$. □

Using this result, we can now derive a dual representation for intrinsic risk measures.

Theorem 1.3.12 (Dual representation)**.** *Let $\mathcal{A}$ be a $\sigma(L^\infty, L^1)$-closed, convex acceptance set containing* 0 *and let S be an eligible asset. For $\mathbb{Q} \in \mathcal{M}_\sigma(\mathbb{P})$ define the* penalty function[7] $\alpha(\mathbb{Q}, \mathcal{A}) = \inf_{X_T \in \mathcal{A}} \mathbb{E}^{\mathbb{Q}}[X_T]$. *The intrinsic risk measure can be written as*

$$\rho^{\text{int}}_{\mathcal{A},S}(X) = \sup_{\mathbb{Q} \in \mathcal{M}_\sigma(\mathbb{P})} \frac{(\alpha(\mathbb{Q}, \mathcal{A}) - \mathbb{E}^{\mathbb{Q}}[X_T])^+}{\frac{x_0}{s_0}\mathbb{E}^{\mathbb{Q}}[S_T] - \mathbb{E}^{\mathbb{Q}}[X_T]}. \tag{1.15}$$

Proof. As shown in Lemma 1.3.11, we have the equivalence $X_T^{\lambda,S} \in \mathcal{A}$ if and only if for all $\mathbb{Q} \in \mathcal{M}_\sigma(\mathbb{P}) :\ \mathbb{E}^{\mathbb{Q}}\big[X_T^{\lambda,S}\big] \geq \alpha(\mathbb{Q}, \mathcal{A})$, or rewritten,

$$\lambda \mathbb{E}^{\mathbb{Q}}\big[\tfrac{x_0}{s_0} S_T - X_T\big] \geq \alpha(\mathbb{Q}, \mathcal{A}) - \mathbb{E}^{\mathbb{Q}}[X_T].$$

For $X_T \in \mathcal{A}$, Lemma 1.3.11 directly implies that the infimum over λ is equal to 0, for all $\mathbb{Q} \in \mathcal{M}_\sigma(\mathbb{P})$. For $X_T \notin \mathcal{A}$, Lemma 1.3.11 gives the inequality $\mathbb{E}^{\mathbb{Q}}[\frac{x_0}{s_0} S_T] - \mathbb{E}^{\mathbb{Q}}[X_T] \geq \alpha(\mathbb{Q}, \mathcal{A}) - \mathbb{E}^{\mathbb{Q}}[X_T] > 0$ so that we can solve for λ and get

$$\begin{aligned}\rho^{\text{int}}_{\mathcal{A},S}(X) &= \inf \left\{ \lambda \in [0,1] \,\middle|\, \forall \mathbb{Q} \in \mathcal{M}_\sigma(\mathbb{P}) :\ \lambda \geq \frac{\alpha(\mathbb{Q}, \mathcal{A}) - \mathbb{E}^{\mathbb{Q}}[X_T]}{\frac{x_0}{s_0}\mathbb{E}^{\mathbb{Q}}[S_T] - \mathbb{E}^{\mathbb{Q}}[X_T]} \right\} \\ &= \sup_{\mathbb{Q} \in \mathcal{M}_\sigma(\mathbb{P})} \frac{\alpha(\mathbb{Q}, \mathcal{A}) - \mathbb{E}^{\mathbb{Q}}[X_T]}{\frac{x_0}{s_0}\mathbb{E}^{\mathbb{Q}}[S_T] - \mathbb{E}^{\mathbb{Q}}[X_T]}.\end{aligned}$$

From here the representation in (1.15) follows. □

It is interesting to find the same terms in the numerator in Equation (1.15) and the expression in Equation (1.6). But here, the numerator is normalised by an expected distance between financial position and eligible asset before the supremum over $\mathcal{M}_\sigma(\mathbb{P})$ is taken.
In case of a conic acceptance set and a constant eligible asset, we can link Theorem 1.3.12 via the dual representation of coherent risk measures in Corollary 1.2.9 to Theorem 1.3.3.

Corollary 1.3.13. *Let $\mathcal{A}$ be a $\sigma(L^\infty, L^1)$-closed, convex cone and $S_T = s_0 1_\Omega$. Then we recover the representation in Equation* (1.11)*.*

Proof. A short calculation confirms that on cones, $\alpha(\mathbb{Q}, \mathcal{A}) = \lambda\alpha(\mathbb{Q}, \mathcal{A})$ is satisfied for all $\lambda > 0$, and thus, $\alpha(\mathbb{Q}, \mathcal{A}) \in \{0, \pm\infty\}$. Using Theorem 1.3.12, but taking the supremum over $\mathcal{M} = \{\mathbb{Q} \in \mathcal{M}_\sigma(\mathbb{P}) \mid \alpha(\mathbb{Q}, \mathcal{A}) = 0\}$, yields

$$\rho^{\text{int}}_{\mathcal{A},S}(X) = \sup_{\mathbb{Q} \in \mathcal{M}} \frac{(\mathbb{E}^{\mathbb{Q}}[-X_T])^+}{x_0 + \mathbb{E}^{\mathbb{Q}}[-X_T]}.$$

But for any constant $c > 0$ the map $x \mapsto \frac{x}{c+x}$ is increasing on $\mathbb{R}_{\geq 0}$ and therefore, we can split the supremum and then use the dual representation of coherent risk measures from Corollary 1.2.9 to get

$$\rho^{\text{int}}_{\mathcal{A},S}(X) = \frac{\sup_{\mathbb{Q} \in \mathcal{M}}(\mathbb{E}^{\mathbb{Q}}[-X_T])^+}{x_0 + \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[-X_T]} = \frac{(\rho_{\mathcal{A},S}(X_T))^+}{x_0 + \rho_{\mathcal{A},S}(X_T)},$$

the representation of intrinsic risk measures on cones from Theorem 1.3.3. □

[7] The negative of the minimal penalty function $\alpha_{\min}$ in Equation (1.7).

1.4 Conclusion

In this chapter, we have extended the methodology of risk measurement with a new type of risk measure: the intrinsic risk measure. We argued that since traditional risk measures are defined via hypothetical external capital, it is natural to consider risk measures that only allow the usage of internal capital contained in the financial position.
We discussed basic properties of intrinsic risk measures and provided some examples. We derived an alternative representation on conic acceptance sets, such as the ones associated with Value-at-Risk and Expected Shortfall. With this we showed that the intrinsic approach requires less investment in the eligible asset, and at the same time yields acceptable positions with the same performance. As the representation on cones does not hold on convex acceptance sets, we established a dual representation in terms of σ-additive probability measures.

Finally, we mention two ideas for further studies. First of all, the extension to general ordered topological vector spaces is necessary to provide greater adaptivity.
Furthermore, in the setting we presented, the random variable X_T represents the net worth of an institution and can assume positive and negative values. It is of interest to restrict X_T to $L^\infty_{\geq 0}$ so that it can be interpreted as the asset side of the balance sheet. This is the framework we will study in Chapter 2 in the context of systemic risk in networks, where liabilities are incorporated through an aggregation function. However, in the univariate setting of this chapter, liabilities would need to be incorporated through the acceptance set, which would entail the loss of conicity. It is valuable to investigate which properties of the intrinsic risk measure remain and how intrinsic risk measures can be used to optimise the portfolio structure of an institution.

Chapter 2

Intrinsic Measures of Systemic Risk

This chapter is a joint work with Jana Hlavinová and Birgit Rudloff. Section 2.2 has been shortened compared to the planned publication for better embedding into this dissertation.

In recent years, it has become apparent that an isolated microprudential approach to capital adequacy requirements of individual institutions is insufficient. It can increase the homogeneity of the financial system and ultimately the cost to society. For this reason, the focus of the financial and mathematical literature has shifted towards the macroprudential regulation of financial networks as a whole. In particular, systemic risk measures have been discussed as a risk measurement and mitigation tool. In this spirit, we adopt a general approach of multivariate, set-valued risk measures and combine it with the notion of intrinsic risk measures. We translate our methodology of using only internal capital from Chapter 1 into the systemic framework and show that intrinsic systemic risk measures have desirable properties such as the set-valued equivalents of monotonicity and quasi-convexity. Furthermore, for convex acceptance sets we derive a dual representation of the intrinsic systemic risk measure. We apply our methodology to a modified Eisenberg-Noe network of banks and discuss the appeal of this approach from a regulatory perspective, as it does not elevate the financial system with external capital. We show evidence that this approach allows to mitigate systemic risk by moving the network towards more stable assets.

2.1 Introduction

A key task of risk management is the quantification and assessment of the riskiness of a certain financial position, a portfolio, or even a financial system. As described in Chapter 1, Artzner, Delbaen, Eber, and Heath (1999) were the first to axiomatically define risk measures with a specific management action in mind, namely, raising external capital and holding it in a reference asset. We call these measures monetary risk measures, a term coined by Föllmer and Schied (2002), and interpret their value as the minimal monetary amount to be added either directly or through a reference asset to the existing financial position to make it acceptable. Farkas, Koch-Medina, and Munari (2014a,b) formalised this approach for general eligible assets, including the case of defaultable investment vehicles. Monetary risk measures have a straightforward operational interpretation and, seemingly, they can be directly applied as a risk mitigation tool in the real world. However, the question of how and at which cost it is possible to raise the necessary capital remains unanswered. External capital is often not readily available and therefore we explored in Chapter 1 the methodology of using internal resources only. We suggested a different management action to alter the existing position, namely, selling a fraction of the current unacceptable position and investing the acquired funds into a general eligible asset as defined in (Farkas, Koch-Medina, and Munari, 2014b,a). This way, in addition to gaining the benefit of holding capital in a safe eligible asset, the existing unacceptable position is reduced. Choosing a suitable eligible asset, for example one with negative correlation to the existing position, can further reduce the risk of the new altered position.

The question of capital adequacy is of even higher importance in the setting of financial systems. During the last two decades and especially considering the events during the Great Recession from 2007 to 2009, it has become apparent that applying a scalar risk measure to each participant in a system individually, and thereby ignoring dependencies within the system, is not an appropriate approach to measure systemic risk. Since then, many contributions have improved the understanding of systemic risk and discussed necessary countermeasures. A discussion of the events during the recession and the economic mechanisms behind them is given in (Brunnermeier, 2009). To better quantify systemic risk, Adrian and Brunnermeier (2016) propose a conditional version of Value-at-Risk (CoVaR) and Acharya, Pedersen, Philippon, and Richardson (2016) introduce the Systemic Expected Shortfall (SES), both to measure the contribution of each financial entity to the overall risk in the system. An extensive survey on systemic risk measures including publications before 2016 is provided in (Silva, Kimura, and Sobreiro, 2017). Chen, Iyengar, and Moallemi (2013) go on to employ an axiomatic approach to define measures of systemic risk. Their approach results in risk measures of the form $\rho(\Lambda(\boldsymbol{X}_T))$, where ρ is a scalar risk measure, $\Lambda\colon \mathbb{R}^d \to \mathbb{R}$ is a non-decreasing aggregation function and $\boldsymbol{X}_T$ is a d-dimensional random vector representing wealths or net worths of each player in a financial system. Feinstein, Rudloff, and Weber (2017) then argue that applying a scalar risk measure to the aggregated outcome of the system leads to identifying the bailout costs. These, however, are the costs of saving the system after it has been disrupted, rather than capital requirements that would prevent the system from experiencing severe distress. Moreover, using just a single number to quantify the systemic risk of a system with $d \geq 2$ participants, important information can get lost, for instance the way in which different participants contribute to the overall risk.
Motivated by this realisation, Feinstein, Rudloff, and Weber (2017) introduce set-valued measures of systemic risk. In their framework, the risk measure of a financial system is a collection of all vectors of capital allocations that, added to the individual participants' positions, yield an acceptable system. The system is deemed acceptable if the random variable describing the aggregated outcome of the system is an element of the acceptance set. Choosing an appropriate vector from the set-valued risk measure, capital requirements can be posed by the regulator. The approach of Feinstein, Rudloff, and Weber (2017) is general in the sense that many risk measures such as the ones in (Chen, Iyengar, and Moallemi, 2013) and (Adrian and Brunnermeier, 2016) can be embedded into their framework.

In this chapter, we combine the management action of intrinsic risk measures introduced in Chapter 1 and the set-valued approach to measure systemic risk introduced in (Feinstein, Rudloff, and Weber, 2017). For a financial system with $d \geq 2$ participants we define the set-valued intrinsic measure of systemic risk as the collection of all vectors of fractions $\boldsymbol{\lambda} \in [0,1]^d$ such that if the participants sell the respective fraction of their assets and invest this raised capital in an eligible asset, the aggregated system will be acceptable. In the framework of a simulated network, we show evidence that these acceptable aggregated systems are less volatile, have milder worst case outcomes, and are more likely to repay more of their liabilities to society compared to the monetary approach. Intrinsic systemic risk measures therefore provide not only an alternative to measuring risk as capital injections, but also provide alternative risk-reducing management actions that are practical in cases where external monetary injections are unfavourable.

The rest of this chapter is structured as follows. In Section 2.2, we introduce the terminology and lay down the mathematical framework. We briefly repeat the underlying notion of acceptability and recapitulate scalar risk measures and set-valued measures of systemic risk. In Section 2.3, we introduce our novel set-valued intrinsic risk measures. We derive their properties and juxtapose them to scalar intrinsic risk measures and set-valued measures of systemic risk. Furthermore, we present two algorithms to approximate intrinsic systemic risk measurements numerically. We derive a dual representation of intrinsic systemic risk measures in Section 2.4. Finally, we apply set-valued intrinsic risk measures to an Eisenberg-Noe network including a sink node and highlight how they can provide asset allocations which improve an unacceptable financial system in Section 2.5. In Section 2.6, we conclude our findings and discuss possible extensions and further research avenues.

2.2 Monetary, intrinsic, and set-valued risk measures

In this section, we define acceptance sets axiomatically and use them to define monetary and intrinsic risk measures. Then we proceed to introduce set-valued measures of systemic risk with general eligible assets.

We briefly introduce our notation. Throughout this chapter, we work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We employ a one-period model from time $t = 0$ to $t = T$. Financial positions or assets have a known initial value at time $t = 0$ and a random value at a fixed point in time $t = T$. These scalar future outcomes are represented by random variables in $L^p = L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$, $p \in [1, \infty]$, the space of equivalence classes of p-integrable random variables endowed with the L^p-norm or, in the case $p = \infty$, of essentially bounded random variables endowed with the essential supremum norm. This is an extension to the case studied in Chapter 1. Future outcomes of $d \in \mathbb{N}$, $d \geq 2$ parties in a financial network are represented by multivariate random variables in $L^p_d = L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ endowed with the canonical norm induced by the L^p-norm and the p-norm on $\mathbb{R}^d$.
As in Chapter 1, we indicate scalar future outcomes by capital letters with the subscript T, for example X_T, and their known initial prices by lower-case letters with the subscript 0, for example x_0. In the multivariate case, we use a bold font to simplify the differentiation, for example $\boldsymbol{X}_T$ and $\boldsymbol{x}_0$.
We use the componentwise ordering $\leq$ on $\mathbb{R}^d$, that means for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$ we write $\boldsymbol{x} \leq \boldsymbol{y}$ if and only if $x_k \leq y_k$ for all $k \in \{1, \dots, d\}$, and we write $\boldsymbol{x} < \boldsymbol{y}$ if all inequalities are strict. Furthermore, we use the notation $\mathbb{R}^d_+ = \{\boldsymbol{x} \in \mathbb{R}^d \mid \boldsymbol{x} \geq 0\}$ and $\mathbb{R}^d_{++} = \{\boldsymbol{x} \in \mathbb{R}^d \mid \boldsymbol{x} > 0\}$ to denote the non-negative and positive orthant of $\mathbb{R}^d$, respectively. For $d = 1$ we suppress the superscript. For any set $A \subseteq \mathbb{R}^d$ we denote its power set by $\mathcal{P}(A)$.
For random vectors $\boldsymbol{X}_T, \boldsymbol{Y}_T \in L^p_d$ we write $\boldsymbol{X}_T \leq \boldsymbol{Y}_T$ $\mathbb{P}$-a.s. if and only if $X^k_T \leq Y^k_T$ $\mathbb{P}$-a.s. for all $k \in \{1, \dots, d\}$, and $\boldsymbol{X}_T < \boldsymbol{Y}_T$ if all inequalities are strict. We define the cones $(L^p_d)_+ = \{\boldsymbol{X}_T \in L^p_d \mid \boldsymbol{X}_T \geq 0 \text{ } \mathbb{P}\text{-a.s.}\}$ and $(L^p_d)_{++} = \{\boldsymbol{X}_T \in L^p_d \mid \boldsymbol{X}_T > 0 \text{ } \mathbb{P}\text{-a.s.}\}$.
Finally, we denote the Hadamard product and the Hadamard division of $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$ by

$$\boldsymbol{x} \odot \boldsymbol{y} = (x_1 y_1, \dots, x_d y_d)^\intercal \in \mathbb{R}^d \quad \text{and} \quad \boldsymbol{x} \oslash \boldsymbol{y} = \left(\frac{x_1}{y_1}, \dots, \frac{x_d}{y_d}\right)^\intercal \in \mathbb{R}^d,$$

respectively.

2.2.1 Monetary and intrinsic risk measures

In risk measure theory, we can differentiate between acceptable and non-acceptable outcomes with the help of acceptance sets. For completeness we include here the definition of acceptance sets in L^p.

Definition 2.2.1. *A set $\mathcal{A} \subset L^p$ is called an* acceptance set *if it satisfies*

(A1) Non-triviality*: $\mathcal{A}$ is neither empty nor the whole L^p space, and*

(A2) Monotonicity*: if $X_T \in \mathcal{A}$ and $Y_T \in L^p$ with $X_T \leq Y_T$ $\mathbb{P}$-a.s., then $Y_T \in \mathcal{A}$.*

We call the future outcome X_T of a financial position acceptable *if and only if $X_T \in \mathcal{A}$. We will often assume that $\mathcal{A}$ is a* closed *set, that is, $\mathcal{A}$ is equal to its closure, $\mathcal{A} = \bar{\mathcal{A}}$. Furthermore, acceptance sets can be* conic *and* convex*, see Definition 1.2.2.*

More details and a discussion of these properties have already been given in Section 1.2.1.

To quantify the risk of a financial position, we can use this binary structure of acceptability and non-acceptability imposed on the underlying space to construct risk measures. Similarly to Section 1.2.2, we introduce an *eligible asset* as a tuple $S = (s_0, S_T) \in \mathbb{R}_{++} \times (L^p)_+$. Such an eligible asset is a traded asset with initial unitary price s_0 and random payoff S_T at time T and serves as an investment vehicle. For more information on this form of eligible assets see (Farkas, Koch-Medina, and Munari, 2014a,b).

As discussed in Section 1.2.2, monetary risk measures quantify the risk of a random variable by its distance to the boundary of the acceptance set. The distance is measured by the additional monetary amount that needs to be added through the eligible asset to the current financial position to make it acceptable. For completeness we extend the notation of monetary risk measures with general eligible assets to L^p.

Definition 2.2.2. *Let $S \in \mathbb{R}_{++} \times (L^p)_+$ be an eligible asset and let $\mathcal{A} \subset L^p$ be an acceptance set. A* monetary risk measure *$\rho_{\mathcal{A},S} \colon L^p \to \mathbb{R} \cup \{+\infty, -\infty\}$ is defined by*

$$\rho_{\mathcal{A},S}(X_T) = \inf \left\{ m \in \mathbb{R} \mid X_T + \tfrac{m}{s_0} S_T \in \mathcal{A} \right\}.$$

By definition and by the structure of the acceptance set, monetary risk measures are non-constant, S-additive, and decreasing. Furthermore, monetary risk measures can be positively homogeneous, subadditive, and convex. A detailed discussion and an interpretation of these properties were already provided in Section 1.2.2.

We have seen in Proposition 1.2.6 and Proposition 1.2.10 that acceptance sets and monetary risk measures exhibit a canonical correspondence which allows us to define one from the other. As in Chapter 1, we interpret acceptance sets as given regulatory restrictions and use them to define risk measures, instead of the axiomatic definition of risk measures.

The approach with general eligible assets is versatile and, in particular, allows the use of defaultable investment vehicles. Intrinsic risk measures were also defined via eligible assets and acceptance sets in Definition 1.3.1. For completeness we extend Definition 1.3.1 to L^p.

Definition 2.2.3. *Let $\mathcal{A}$ be an acceptance set and let $S = (s_0, S_T) \in \mathbb{R}_{++} \times \mathcal{A}$ be an eligible asset. The* intrinsic risk measure *is a functional $\rho^{\text{int}}_{\mathcal{A},S} \colon \mathbb{R}_{++} \times L^p \to [0,1]$ defined as*

$$\rho^{\text{int}}_{\mathcal{A},S}(X) = \inf \left\{ \lambda \in [0,1] \mid (1-\lambda) X_T + \lambda \tfrac{x_0}{s_0} S_T \in \mathcal{A} \right\}. \tag{2.1}$$

As in Section 1.3, we adopt the notation

$$X_T^{\lambda,S} := (1-\lambda)X_T + \lambda\tfrac{x_0}{s_0}S_T\,. \tag{2.2}$$

We have seen in Proposition 1.3.2 that, similarly to monetary risk measures, intrinsic risk measures are decreasing when we choose a suitable order on $\mathbb{R}_{++} \times L^p$. Moreover, we have seen that, in contrast monetary risk measures, convexity of the acceptance set corresponds to quasi-convexity of the intrinsic risk measure. We show in Proposition 2.3.4 and Proposition 2.3.5, that these properties translate to the intrinsic measure of systemic risk.

2.2.2 Measures of systemic risk

While monetary and intrinsic risk measures can quantify the risk of a single isolated financial institution or position, they are not suitable to measure the systemic risk of a financial system. In this section, we turn to financial systems with $d \geq 2$ participants. As mentioned in the introduction, risk measures of the form $\rho \circ \Lambda : L_d^p \to \mathbb{R}$ might discard crucial information, as capital is added after aggregation to the whole system so that identifying and understanding the source of the risk becomes difficult. In order to be most useful for regulators to recognise and mitigate systemic risk and prevent cascades of risk, one can add risk capital to each institution separately before aggregation. This way, the risk capital of single participants or groups of participants can be adjusted while observing the effects on the whole system. For this reason we adopt the set-valued approach of Feinstein, Rudloff, and Weber (2017), where the authors search for all capital allocations $\boldsymbol{k} \in \mathbb{R}^d$ such that $\Lambda(\boldsymbol{X}_T + \boldsymbol{k})$ belongs to the acceptance set $\mathcal{A}$. For the sake of consistency, we generalise Definition 2.2 in (Feinstein, Rudloff, and Weber, 2017) to $\boldsymbol{S}$-additive systemic risk measures.

Consider a network of $d \geq 2$ financial institutions enumerated by $\{1, \ldots, d\}$. Let the random vector $\boldsymbol{X}_T = (X_T^1, \ldots, X_T^d)^\intercal \in L_d^p$ denote their future wealths. In order to use univariate acceptance sets, we need the concept of an aggregation function as a mechanism to map random vectors to univariate random variables.

Definition 2.2.4. *An* aggregation function *is a non-constant, non-decreasing function $\Lambda\colon \mathbb{R}^d \to \mathbb{R}$. This means that Λ has at least two distinct values and that if $\boldsymbol{x} \leq \boldsymbol{y}$, then $\Lambda(\boldsymbol{x}) \leq \Lambda(\boldsymbol{y})$.*

Furthermore, an aggregation function can be concave or positively homogeneous. If these properties are required, we list them explicitly. For an overview and a discussion of specific examples of aggregation functions see (Feinstein, Rudloff, and Weber, 2017, Example 2.1) and the references given therein.

Definition 2.2.5. *Let $\mathcal{A}$ be an acceptance set, let Λ be an aggregation function, and let $\boldsymbol{S} = (\boldsymbol{s}_0, \boldsymbol{S}_T) \in \mathbb{R}_{++}^d \times (L_d^p)_+$ be a vector of eligible assets. A* set-valued measure of systemic risk *is a functional $R_{\boldsymbol{S}}\colon L_d^p \to \mathcal{P}(\mathbb{R}^d)$ defined by*

$$R_{\boldsymbol{S}}(\boldsymbol{X}_T) = \left\{\boldsymbol{k} \in \mathbb{R}^d \mid \Lambda(\boldsymbol{X}_T + \boldsymbol{k} \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0) \in \mathcal{A}\right\}. \tag{2.3}$$

We will refer to these measures also as *monetary* measures of systemic risk, as an additional monetary amount $\boldsymbol{k}$ is added to the system. This will make it easier to differentiate these measures from the intrinsic type defined in Definition 2.3.1.

In the following, we collect some important properties of $R_{\boldsymbol{S}}$. The proofs can be found in Section 2.7.1.

Proposition 2.2.6. *Let $\mathcal{A}$, Λ, and $\boldsymbol{S}$ be as in Definition 2.2.5. Then $R_{\boldsymbol{S}}\colon L^p_d \to \mathcal{P}(\mathbb{R}^d)$ as defined in* (2.3) *satisfies the following properties:*

(i) Values of $R_{\boldsymbol{S}}$ are upper sets: *for all $\boldsymbol{X}_T \in L^p_d$: $R_{\boldsymbol{S}}(\boldsymbol{X}_T) = R_{\boldsymbol{S}}(\boldsymbol{X}_T) + \mathbb{R}^d_+$.*

(ii) $\boldsymbol{S}$-additivity: *for all $\boldsymbol{X}_T \in L^p_d$ and $\boldsymbol{\ell} \in \mathbb{R}^d$: $R_{\boldsymbol{S}}(\boldsymbol{X}_T + \boldsymbol{\ell} \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0) = R_{\boldsymbol{S}}(\boldsymbol{X}_T) - \boldsymbol{\ell}$.*

(iii) Monotonicity: *for any $\boldsymbol{X}_T, \boldsymbol{Y}_T \in L^p_d$ with $\boldsymbol{X}_T \leq \boldsymbol{Y}_T$ $\mathbb{P}$-a.s.: $R_{\boldsymbol{S}}(\boldsymbol{X}_T) \subseteq R_{\boldsymbol{S}}(\boldsymbol{Y}_T)$.*

Moreover, if $\mathcal{A}$ is a cone and Λ is positively homogeneous, then $R_{\boldsymbol{S}}$ also satisfies

(iv) Positive Homogeneity: *for all $\boldsymbol{X}_T \in L^p_d$ and $c > 0$: $R_{\boldsymbol{S}}(c\boldsymbol{X}_T) = cR_{\boldsymbol{S}}(\boldsymbol{X}_T)$.*

Finally, if $\mathcal{A}$ is convex and Λ concave, the following properties hold:

(v) Convexity: *for all $\boldsymbol{X}_T, \boldsymbol{Y}_T \in L^p_d$ and $\alpha \in [0,1]$ we have*

$$R_{\boldsymbol{S}}(\alpha\boldsymbol{X}_T + (1-\alpha)\boldsymbol{Y}_T) \supseteq \alpha R_{\boldsymbol{S}}(\boldsymbol{X}_T) + (1-\alpha)R_{\boldsymbol{S}}(\boldsymbol{Y}_T)\,,$$

(vi) $R_{\boldsymbol{S}}$ has convex values: *for all $\boldsymbol{X}_T \in L^p_d$ the set $R_{\boldsymbol{S}}(\boldsymbol{X}_T)$ is a convex set in $\mathbb{R}^d$.*

The properties in Proposition 2.2.6 show that measures as in Equation (2.3) are a natural generalisation of univariate monetary risk measures. Properties (ii)-(v) can be interpreted in analogy to the corresponding properties of monetary risk measures. Properties (i) and (vi) are particularly useful for the numerical approximation of the values. Furthermore, the upper set property allows the efficient communication of risk measurements $R_{\boldsymbol{S}}(\boldsymbol{X}_T)$ through efficient cash invariant allocation rules (EARs). For more details see (Feinstein, Rudloff, and Weber, 2017) and the discussions in Remark 2.3.8 and Remark 2.3.13 in this chapter. In addition to these properties, the set $R_{\boldsymbol{S}}(\boldsymbol{X}_T)$ is closed whenever Λ is continuous and $\mathcal{A}$ is closed, see also (Feinstein, Rudloff, and Weber, 2017, Lemma 2.4 (iii)). Some of these properties will be further discussed in Section 2.3.

In analogy to univariate monetary risk measures, the defining management action dictates the institutions in the financial system to adjust their capital holdings such that the system can be deemed acceptable. In the following section, we will explore a different management action without the use of external capital.

2.3 Intrinsic measures of systemic risk

In this section, we introduce the novel intrinsic measures of systemic risk. We continue working in the framework of an interconnected network of $d \geq 2$ participants. However, in contrast to monetary measures of systemic risk as defined in Definition 2.2.5, we will not rely on external capital. Instead, each participant of the network needs to improve their own position by shifting it towards a specified eligible position. A regulatory authority can specify these eligible positions or restrict their choice to certain classes of assets for each participant individually. A network consisting of only eligible positions should be acceptable when aggregated, as described in Proposition 2.3.3.
In the following section, we define the intrinsic systemic risk measures and derive and discuss their most important properties.

2.3.1 Intrinsic measures and their properties

As in Section 2.2.2, we collect eligible assets and their initial values in a tuple of vectors $\boldsymbol{S} = (\boldsymbol{s}_0, \boldsymbol{S}_T) \in \mathbb{R}^d_{++} \times (L^p_d)_+$. In addition to the random vector $\boldsymbol{X}_T$, we also need the initial values of each of the participants' future values, $\boldsymbol{x}_0$, so we extend financial positions to tuples $\boldsymbol{X} = (\boldsymbol{x}_0, \boldsymbol{X}_T) \in \mathbb{R}^d_{++} \times (L^p_d)_+$.
The shift from the current position towards an eligible position can mathematically be expressed as a convex combination of two random variables. Since each participant's position can be altered individually, we extend the notation introduced in Equation (2.2) element-wise to a multivariate random variable. For a financial network $\boldsymbol{X} = (\boldsymbol{x}_0, \boldsymbol{X}_T) \in \mathbb{R}^d_{++} \times (L^p_d)_+$, a collection of eligible assets $\boldsymbol{S} = (\boldsymbol{s}_0, \boldsymbol{S}_T) \in \mathbb{R}^d_{++} \times (L^p_d)_+$, and a vector of coefficients $\boldsymbol{\lambda} \in [0,1]^d$ we define the random vector

$$\boldsymbol{X}_T^{\boldsymbol{\lambda},\boldsymbol{S}} = (1-\boldsymbol{\lambda}) \odot \boldsymbol{X}_T + \boldsymbol{\lambda} \odot \boldsymbol{x}_0 \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0 = \Big((X_T^1)^{\lambda^1, S^1} \,,\, \ldots \,,\, (X_T^d)^{\lambda^d, S^d} \Big)^\intercal \in (L^p_d)_+ \,.$$

This is the element-wise convex combination of $\boldsymbol{X}_T$ and $\boldsymbol{x}_0 \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0$ given by the coefficients collected in $\boldsymbol{\lambda}$. In this framework, each participant's financial position $X_T^1, \ldots, X_T^d$ can be altered by a different fraction $\lambda^1, \ldots, \lambda^d$ and a different eligible asset $S_T^1, \ldots, S_T^d$, respectively. Assuming a choice for all d eligible assets has been made, we aim to find all vectors $\boldsymbol{\lambda} \in [0,1]^d$ such that the aggregated position $\Lambda(\boldsymbol{X}_T^{\boldsymbol{\lambda},\boldsymbol{S}})$ belongs to the acceptance set $\mathcal{A}$.

Definition 2.3.1. *Let Λ be an aggregation function and let $\mathcal{A}$ be an acceptance set. Let $\boldsymbol{S} \in \mathbb{R}^d_{++} \times (L^p_d)_+$ be a vector-valued eligible asset. An* intrinsic measure of systemic risk *is a map $R^{\text{int}}_{\boldsymbol{S}} \colon \mathbb{R}^d_{++} \times (L^p_d)_+ \to \mathcal{P}([0,1]^d)$ defined as*

$$R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X}) = \big\{ \boldsymbol{\lambda} \in [0,1]^d \mid \Lambda\big(\boldsymbol{X}_T^{\boldsymbol{\lambda},\boldsymbol{S}}\big) \in \mathcal{A} \big\} \,. \tag{2.4}$$

It is important to note that we define intrinsic systemic risk measures on $\mathbb{R}^d_{++} \times (L^p_d)_+$. This means X_T^k represents the future value of the asset side of the balance sheet of institution $k \in \{1, \ldots, d\}$. In particular, $\boldsymbol{X}_T$ has non-negative values.

Remark 2.3.2. *The choice of non-negative $\boldsymbol{X}_T$ allows for the most useful operational interpretation of the intrinsic systemic risk measure. In the context of the element-wise convex combination $\boldsymbol{X}_T^{\boldsymbol{\lambda},\boldsymbol{S}}$, the term $(1-\boldsymbol{\lambda}) \odot \boldsymbol{X}_T$ should be interpreted as the future value of a system $\boldsymbol{X}_T$ after fractions $\boldsymbol{\lambda} \in [0,1]^d$ have been sold. In general, it would not have the intended operational interpretation if $\boldsymbol{X}_T$ denoted the net worth of the institutions, that is, assets minus liabilities, as liabilities would also be scaled. Only in specific situations, for example with operational costs that are reduced when assets are sold, this interpretation may be accurate. For this reason, we consider assets and liabilities separately. Since in the multivariate framework we make use of aggregation functions, we can incorporate liabilities through them, or if not possible, through the acceptance set. Network models as studied in Section 2.5 go hand in hand with our framework. This stands in contrast to univariate intrinsic risk measures, where X_T can assume negative values. There, if one wants to split assets and liabilities, liabilities need to be incorporated via the acceptance set.*

As a technical remark, we note that $R^{\text{int}}_{\boldsymbol{S}}$ is always well-defined, since $\emptyset \in \mathcal{P}([0,1]^d)$ is vacuously true. This is different for the univariate intrinsic risk measure, where $\rho^{\text{int}}_{\mathcal{A},S}$ is only well defined if $\frac{x_0}{s_0} S_T \in \mathcal{A}$, see the discussion below Definition 1.3.1. However, an empty risk measure has a similar meaning as the value $+\infty$ for univariate monetary risk measures, namely that the choices of $\mathcal{A}$, Λ, and $\boldsymbol{S}$ cannot yield an acceptable system.

Since the aggregation function and the acceptance set can be thought of as restrictions on the system imposed by a regulatory authority, we assume that these objects are given and have certain properties. In this case, we must choose suitable eligible assets to ensure that $R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$ is not an empty set.

Proposition 2.3.3. *Let Λ be an aggregation function and let $\mathcal{A}$ be an acceptance set. Let $\boldsymbol{X}$ be an unacceptable system in the sense that $\Lambda(\boldsymbol{X}_T) \notin \mathcal{A}$. If the eligible asset $\boldsymbol{S}$ satisfies $\Lambda(\boldsymbol{x}_0 \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0) \in \mathcal{A}$, then $R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X}) \neq \emptyset$.*

Proof. Notice that $\Lambda(\boldsymbol{x}_0 \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0) = \Lambda(\boldsymbol{X}^{\boldsymbol{1},\boldsymbol{S}}_T)$ and hence, $\boldsymbol{1} \in R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$. □

This requirement makes sense intuitively. A system in which each agent is fully invested in their eligible asset needs to be acceptable. This system corresponds to the coefficient vector $\boldsymbol{1}$ and marks the end point of the path from $\boldsymbol{X}_T$ to $\boldsymbol{x}_0 \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0$. However, the condition $\Lambda(\boldsymbol{x}_0 \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0) \in \mathcal{A}$ is not necessary for $R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X}) \neq \emptyset$, as can be seen in Figure 2.6.

In the following, we collect basic properties of intrinsic measures of systemic risk. To this end let Λ be an aggregation function and let $\mathcal{A}$ be an acceptance set.

Proposition 2.3.4 (Monotonicity)**.** *$R^{\text{int}}_{\boldsymbol{S}}$ is monotonic in the sense that if $\boldsymbol{x}_0 \leq \boldsymbol{y}_0$ and $\boldsymbol{X}_T \leq \boldsymbol{Y}_T$ $\mathbb{P}$-a.s., then $R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X}) \subseteq R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{Y})$.*

Proof. Let $\boldsymbol{x}_0 \leq \boldsymbol{y}_0$ and $\boldsymbol{X}_T \leq \boldsymbol{Y}_T$, and take $\boldsymbol{\lambda} \in R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$. Notice that

$$(1 - \boldsymbol{\lambda}) \odot \boldsymbol{X}_T + \boldsymbol{\lambda} \odot \boldsymbol{x}_0 \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0 \leq (1 - \boldsymbol{\lambda}) \odot \boldsymbol{Y}_T + \boldsymbol{\lambda} \odot \boldsymbol{y}_0 \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0 \,.$$

Since Λ is non-decreasing, the assertion follows by the monotonicity of $\mathcal{A}$. □

Proposition 2.3.5 (Quasi-convexity)**.** *If $\mathcal{A}$ is convex and Λ is concave, then $R^{\text{int}}_{\boldsymbol{S}}$ is quasi-convex, that is, for all $\alpha \in [0, 1]$ and for all $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{R}^d_{++} \times (L^p_d)_+$ we have*

$$R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X}) \cap R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{Y}) \subseteq R^{\text{int}}_{\boldsymbol{S}}(\alpha \boldsymbol{X} + (1 - \alpha)\boldsymbol{Y}) \,. \tag{2.5}$$

Proof. If $R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X}) \cap R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{Y}) = \emptyset$, there is nothing to prove. So assume $R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X}) \cap R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{Y})$ is not empty. Take any $\boldsymbol{\lambda} \in R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X}) \cap R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{Y})$, then $\Lambda\big(\boldsymbol{X}^{\boldsymbol{\lambda},\boldsymbol{S}}_T\big)$ and $\Lambda\big(\boldsymbol{Y}^{\boldsymbol{\lambda},\boldsymbol{S}}_T\big)$ are contained in $\mathcal{A}$. Notice that

$$\begin{aligned}(1 - \boldsymbol{\lambda}) &\odot (\alpha \boldsymbol{X}_T + (1 - \alpha)\boldsymbol{Y}_T) + \boldsymbol{\lambda} \odot (\alpha \boldsymbol{x}_0 + (1 - \alpha)\boldsymbol{y}_0) \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0 \\ &= \alpha \boldsymbol{X}^{\boldsymbol{\lambda},\boldsymbol{S}}_T + (1 - \alpha)\boldsymbol{Y}^{\boldsymbol{\lambda},\boldsymbol{S}}_T \,.\end{aligned}$$

By convexity of $\mathcal{A}$, the convex combination $\alpha\Lambda\big(\boldsymbol{X}^{\boldsymbol{\lambda},\boldsymbol{S}}_T\big) + (1 - \alpha)\Lambda\big(\boldsymbol{Y}^{\boldsymbol{\lambda},\boldsymbol{S}}_T\big)$ is contained in $\mathcal{A}$ and by monotonicity of $\mathcal{A}$ and concavity of Λ, also $\Lambda\big(\alpha \boldsymbol{X}^{\boldsymbol{\lambda},\boldsymbol{S}}_T + (1 - \alpha)\boldsymbol{Y}^{\boldsymbol{\lambda},\boldsymbol{S}}_T\big) \in \mathcal{A}$. Hence, $\boldsymbol{\lambda} \in R^{\text{int}}_{\boldsymbol{S}}(\alpha \boldsymbol{X} + (1 - \alpha)\boldsymbol{Y})$. □

Property (2.5) is a set-valued version of quasi-convexity. The intersection is the set-valued counterpart of a maximum and the subset relation corresponds to the ordering relation $\geq$. Monotonicity and quasi-convexity are the most important properties a risk measure should satisfy. Monotonicity implies that all the management actions that make a system of assets $\boldsymbol{X}$ acceptable will also make a system $\boldsymbol{Y}$ with larger asset values acceptable. This is also important from a modelling perspective, as the management actions resulting from overestimating risk will not have an adverse effect on the systemic risk. Quasi-convexity implements the notion of the diversification principle. In our set-valued framework, this means that management actions which make both systems $\boldsymbol{X}$ and $\boldsymbol{Y}$ acceptable will also make any convex combination of them acceptable.

We turn to show two regularity properties which are of particular value for the numerical approximation of intrinsic systemic risk measures.

Proposition 2.3.6 (Closed values)**.** *Let Λ be continuous and let $\mathcal{A}$ be closed. Then $R^{\text{int}}_{\boldsymbol{S}}$ has closed values, that is, for all $\boldsymbol{X} \in \mathbb{R}^d_{++} \times (L^p_d)_+$ the set $R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$ is a closed subset of $[0,1]^d$.*

Proof. Let $R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X}) \neq \emptyset$ and let $(\boldsymbol{\lambda}_n)_{n\in\mathbb{N}} \subset R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$ be a sequence that converges to some $\boldsymbol{\lambda} \in [0,1]^d$. Notice that $\boldsymbol{X}^{\boldsymbol{\lambda}_n,\boldsymbol{S}}_T - \boldsymbol{X}^{\boldsymbol{\lambda},\boldsymbol{S}}_T = (\boldsymbol{\lambda} - \boldsymbol{\lambda}_n) \odot (\boldsymbol{X}_T - \boldsymbol{x}_0 \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0)$. This gives us

$$\|\boldsymbol{X}^{\boldsymbol{\lambda}_n,\boldsymbol{S}}_T - \boldsymbol{X}^{\boldsymbol{\lambda},\boldsymbol{S}}_T\|_{L^p_d} \leq \max_{k\in\{1,\dots,d\}} \left\| X^k_T - \frac{x^k_0 S^k_T}{s^k_0} \right\|_{L^p} \cdot |\boldsymbol{\lambda} - \boldsymbol{\lambda}_n|_p \,.$$

Since Λ is continuous, we get a sequence $\Lambda(\boldsymbol{X}^{\boldsymbol{\lambda}_n,\boldsymbol{S}}_T)_{n\in\mathbb{N}} \subset \mathcal{A}$ that converges to $\Lambda(\boldsymbol{X}^{\boldsymbol{\lambda},\boldsymbol{S}}_T)$. Since $\mathcal{A}$ is closed, the limit $\Lambda(\boldsymbol{X}^{\boldsymbol{\lambda},\boldsymbol{S}}_T)$ is also contained in $\mathcal{A}$. Hence $\boldsymbol{\lambda} \in R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$. □

Many important examples of acceptance sets are closed, such as the ones associated with Value-at-Risk and Expected Shortfall, see also Example 1.2.4. While in the financial literature closedness is often required to simplify mathematical technicalities, it has also a financial relevance. Closedness of the acceptance set prevents unacceptable positions from becoming acceptable through arbitrarily small perturbations, see also (Munari, 2015, Section 2.2.3). With continuity of Λ this interpretation translates to the set-valued framework.

Proposition 2.3.7 (Convex values)**.** *If Λ is concave and $\mathcal{A}$ is convex, then for any $\boldsymbol{X} \in \mathbb{R}^d_{++} \times L^p_d$ the set $R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$ is convex.*

Proof. Assume that $R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$ is not empty, or otherwise there is nothing to show. Let $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \in R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$ and for $\alpha \in [0,1]$ define $\boldsymbol{\lambda}_\alpha = \alpha\boldsymbol{\lambda}_1 + (1-\alpha)\boldsymbol{\lambda}_2 \in [0,1]^d$. Notice that for all $\alpha \in [0,1]$,

$$\begin{aligned}\boldsymbol{X}^{\boldsymbol{\lambda}_\alpha,S}_T &= (1 - \alpha\boldsymbol{\lambda}_1 - (1-\alpha)\boldsymbol{\lambda}_2) \odot \boldsymbol{X}_T + (\alpha\boldsymbol{\lambda}_1 + (1-\alpha)\boldsymbol{\lambda}_2) \odot \boldsymbol{x}_0 \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0 \\ &= \alpha \boldsymbol{X}^{\boldsymbol{\lambda}_1,\boldsymbol{S}}_T + (1-\alpha)\boldsymbol{X}^{\boldsymbol{\lambda}_2,\boldsymbol{S}}_T \,.\end{aligned}$$

Hence, by concavity of Λ we have

$$\Lambda(\boldsymbol{X}^{\boldsymbol{\lambda}_\alpha,\boldsymbol{S}}_T) \geq \alpha\Lambda(\boldsymbol{X}^{\boldsymbol{\lambda}_1,\boldsymbol{S}}_T) + (1-\alpha)\Lambda(\boldsymbol{X}^{\boldsymbol{\lambda}_2,\boldsymbol{S}}_T) \,.$$

Since both $\Lambda(\boldsymbol{X}^{\boldsymbol{\lambda}_1,\boldsymbol{S}}_T)$ and $\Lambda(\boldsymbol{X}^{\boldsymbol{\lambda}_2,\boldsymbol{S}}_T)$ are contained in $\mathcal{A}$, we know by convexity and monotonicity of $\mathcal{A}$ that $\Lambda(\boldsymbol{X}^{\boldsymbol{\lambda}_\alpha,\boldsymbol{S}}_T) \in \mathcal{A}$ and thus, $\boldsymbol{\lambda}_\alpha \in R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$ for all $\alpha \in [0,1]$. □

Convexity of the acceptance set corresponds to the diversification principle. However, since we work with random vectors, we must aggregate them accordingly, which is achieved by requiring that Λ is concave. Indeed, concavity is in line with the diversification principle, as the aggregation of a convex combination of two systems should not reduce the value when compared to the convex combination of two aggregated systems. This way, convexity of $\mathcal{A}$ is passed on to $R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$. Furthermore, there is a wide range of numerical methods for the approximation and representation of convex sets. In Section 2.3.2, we will see that convexity allows for a modification of the algorithm for the computation of $R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$ that makes the approximation either faster or more accurate.

Remark 2.3.8. *An important and useful property of monetary systemic risk measures is the upper set property stated in Proposition 2.2.6 (i). It motivates the definition of EARs in (Feinstein, Rudloff, and Weber, 2017, Definition 3.3), which can be interpreted as minimal capital requirements for the participants of a financial system, and the communication of which is easier compared to the whole systemic risk measure. Furthermore, the upper set property lays the basis for the algorithm presented in (Feinstein, Rudloff, and Weber, 2017, Section 4). In contrast, intrinsic systemic risk measures do not, in general, exhibit this property, as one can see for example in Figure 2.4. We will attempt to give an intuitive explanation for this.*
The upper set property is essentially a consequence of the monotonicity of Λ and $\mathcal{A}$, as shown in Section 2.7.1. This means, any participant of an acceptable system is free to add an arbitrary positive amount of capital through their eligible asset without influencing the acceptability of the system. In particular, intra- and inter-dependencies of $\boldsymbol{X}_T$ and $\boldsymbol{S}_T$ are irrelevant.
In the intrinsic approach, however, no external capital is injected into the system. Instead, the system is translated element-wise to a system of eligible assets and, in general, the partial order on L_d^p is not enough to compare the resulting positions. In particular, the set $R_{\boldsymbol{S}}^{\text{int}}(\boldsymbol{X})$ depends on the interplay of $\boldsymbol{X}$ and $\boldsymbol{S}$. For example, assume the elements of $\boldsymbol{X}_T = (X_T^1, X_T^2)^\intercal$ are negatively correlated and the eligible vector $\boldsymbol{S}_T \oslash \boldsymbol{s}_0 = (r,r)^\intercal$ is constant. If one institution increases its holding in the eligible asset, thereby decreasing its holding in its original position, the correlation between the institutions increases. This in turn can result in an unacceptable aggregate system. So since the management action in the intrinsic approach does not rely on external capital injections, it is more sensitive to the overall dependency structure of the system and the eligible assets.
The definition of a concept similar to EARs is still possible, as will be discussed in Section 2.3.2. However, on a stand-alone basis, institutions are not allowed to increase their holdings in the eligible assets in general.
For convex $R_{\boldsymbol{S}}^{\text{int}}(\boldsymbol{X})$ the lack of the upper set property is not a drawback for the computational approximation. For conic acceptance sets and concave aggregation functions, an additional assumption is sufficient to apply the algorithm described in Section 2.3.2, see Proposition 2.3.9.

The following result can be interpreted as the set-valued counterpart of

$$\left\{X_T^{\lambda,S} \mid \lambda \in [\rho_{\mathcal{A},S}^{\text{int}}(X),1]\right\} \subseteq \mathcal{A}$$

for univariate intrinsic risk measures on conic acceptance sets mentioned below Definition 1.3.1.

Proposition 2.3.9. *Let Λ be concave and let $\mathcal{A}$ be a cone. Assume that the eligible asset satisfies $\Lambda(\boldsymbol{x}_0 \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0) \geq 0$. If $\boldsymbol{\lambda} \in R_{\boldsymbol{S}}^{\text{int}}(\boldsymbol{X})$, then for $\alpha \in [0,1]$ we have $(1-\alpha)\boldsymbol{\lambda} + \alpha\mathbf{1} \in R_{\boldsymbol{S}}^{\text{int}}(\boldsymbol{X})$.*

Proof. Let $\alpha \in [0,1]$. Since $\boldsymbol{\lambda} \in R_{\boldsymbol{S}}^{\text{int}}(\boldsymbol{X})$, the aggregated position $\Lambda(X_T^{\boldsymbol{\lambda},\boldsymbol{S}})$ is contained in $\mathcal{A}$, and since $\mathcal{A}$ is a cone, also $(1-\alpha)\Lambda(X_T^{\boldsymbol{\lambda},\boldsymbol{S}})$ is contained in $\mathcal{A}$. By concavity of Λ, we arrive at

$$\begin{aligned}\Lambda\left((1-\alpha)\boldsymbol{X}_T^{\boldsymbol{\lambda},\boldsymbol{S}} + \alpha(\boldsymbol{x}_0 \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0)\right) &\geq (1-\alpha)\Lambda(\boldsymbol{X}_T^{\boldsymbol{\lambda},\boldsymbol{S}}) + \alpha\Lambda(\boldsymbol{x}_0 \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0) \\ &\geq (1-\alpha)\Lambda(\boldsymbol{X}_T^{\boldsymbol{\lambda},\boldsymbol{S}}) \in \mathcal{A}\,,\end{aligned}$$

and thus, by monotonicity of $\mathcal{A}$, the position $\Lambda\left((1-\alpha)\boldsymbol{X}_T^{\boldsymbol{\lambda},\boldsymbol{S}}+\alpha\boldsymbol{x}_0\odot\boldsymbol{S}_T\oslash\boldsymbol{s}_0\right)$ is acceptable. A short calculation shows that

$$(1-\alpha)\boldsymbol{X}_T^{\boldsymbol{\lambda},\boldsymbol{S}}+\alpha(\boldsymbol{x}_0\odot\boldsymbol{S}_T\oslash\boldsymbol{s}_0)=\boldsymbol{X}_T^{(1-\alpha)\boldsymbol{\lambda}+\alpha\mathbf{1},\boldsymbol{S}},$$

proving the assertion. □

Remark 2.3.10. *Notice that a statement similar to the one in Proposition 2.3.9 is true if Λ is concave, $\mathcal{A}$ is convex and we demand $\Lambda(\boldsymbol{x}_0\odot\boldsymbol{S}_T\oslash\boldsymbol{s}_0)\in\mathcal{A}$. In this case, we know by Proposition 2.3.7 that $R_{\boldsymbol{S}}^{\text{int}}(\boldsymbol{X})$ is convex and hence, $(1-\alpha)\boldsymbol{\lambda}+\alpha\mathbf{1}\in R_{\boldsymbol{S}}^{\text{int}}(\boldsymbol{X})$ for all $\boldsymbol{\lambda}\in R_{\boldsymbol{S}}^{\text{int}}(\boldsymbol{X})$ and $\alpha\in[0,1]$.*

This observation and Proposition 2.3.9 are used in the algorithms described in Section 2.3.2 to approximate values of intrinsic systemic risk measures.

2.3.2 Computation of intrinsic measures of systemic risk

As a set-valued measure, the intrinsic systemic risk measure is more difficult to calculate compared to scalar risk measures. Except for simple toy examples, intrinsic systemic risk measures have no explicit or closed representations. So one has to rely on numerical methods to approximate these sets. A natural approximation would consist of a collection of points $\boldsymbol{\lambda}\in R_{\boldsymbol{S}}^{\text{int}}(\boldsymbol{X})$ which lie close to the boundary $\partial R_{\boldsymbol{S}}^{\text{int}}(\boldsymbol{X})$. Assuming a concave aggregation function, we know by Proposition 2.3.9 for conic acceptance sets and Remark 2.3.10 for convex acceptance sets that all line segments which connect points $\boldsymbol{\lambda}\in R_{\boldsymbol{S}}^{\text{int}}(\boldsymbol{X})$ with $\mathbf{1}$ are contained in $R_{\boldsymbol{S}}^{\text{int}}(\boldsymbol{X})$.

In the following, we will use these results to construct a simple bisection method which approximates the boundary of the intrinsic risk measure with a prespecified accuracy. In this section, we assume that either the assumptions in Proposition 2.3.9 or Remark 2.3.10 hold. The algorithm is illustrated for a network of two and three participants in Figure 2.1. Since the intrinsic measure maps into the power set of $[0,1]^d$, we can a priori restrict our search to $[0,1]^d$.

We consider the d faces of the cube $[0,1]^d$ which contain the point $\mathbf{0}\in\mathbb{R}^d$ and construct a grid on each of these faces. The resolution of this grid influences the spacing of points that approximate the boundary of $R_{\boldsymbol{S}}^{\text{int}}(\boldsymbol{X})$.

1. Let $\boldsymbol{\lambda}_0$ be a point on this grid. If $\boldsymbol{\lambda}_0\in R_{\boldsymbol{S}}^{\text{int}}(\boldsymbol{X})$, add it to the collection of approximation points and proceed with a new grid point $\boldsymbol{\lambda}_0$, otherwise continue with Step 2.

2. Do a bisection search along the line connecting $\boldsymbol{\lambda}_0$ and $\mathbf{1}$. To this end, define $\boldsymbol{\lambda}_{a_0}=\boldsymbol{\lambda}_0\notin R_{\boldsymbol{S}}^{\text{int}}(\boldsymbol{X})$ and $\boldsymbol{\lambda}_{b_0}=\mathbf{1}\in R_{\boldsymbol{S}}^{\text{int}}(\boldsymbol{X})$ as the initial end points.

3. Iterate over $k\geq 1$ and construct a sequence that tends to a point $\boldsymbol{\lambda}$ on the boundary of $R_{\boldsymbol{S}}^{\text{int}}(\boldsymbol{X})$. In each iteration, define $\boldsymbol{\lambda}_k=\frac{1}{2}(\boldsymbol{\lambda}_{a_{k-1}}+\boldsymbol{\lambda}_{b_{k-1}})$ and check whether the aggregated position corresponding to $\boldsymbol{\lambda}_k$ is acceptable.

4. If $\Lambda(\boldsymbol{X}_T^{\boldsymbol{\lambda}_k,\boldsymbol{S}})\in\mathcal{A}$, set $\boldsymbol{\lambda}_{a_k}=\boldsymbol{\lambda}_{a_{k-1}}$ and $\boldsymbol{\lambda}_{b_k}=\boldsymbol{\lambda}_k$, otherwise $\boldsymbol{\lambda}_{a_k}=\boldsymbol{\lambda}_k$ and $\boldsymbol{\lambda}_{b_k}=\boldsymbol{\lambda}_{b_{k-1}}$.

5. Stop this procedure at $n\in\mathbb{N}$ at which the distance $\|\boldsymbol{\lambda}_{b_n}-\boldsymbol{\lambda}_{a_n}\|$ is smaller than some desired threshold $\epsilon>0$ and take $\hat{\boldsymbol{\lambda}}=\boldsymbol{\lambda}_{b_n}$ as the approximation of $\boldsymbol{\lambda}$. This also covers the rare case that for some k, $\boldsymbol{\lambda}_k=\boldsymbol{\lambda}$.

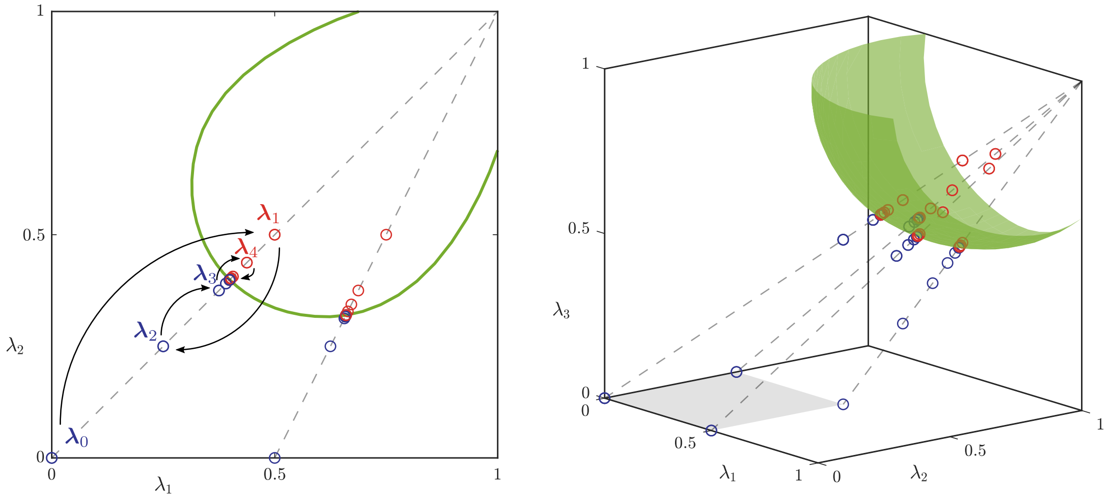


Figure 2.1: Illustration of grid search algorithm on $[0,1]^2$ and $[0,1]^3$.

By definition, $\hat{\boldsymbol{\lambda}} = \boldsymbol{\lambda}_{b_n}$ lies in $R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$ and is arbitrarily close to the boundary, as $\|\hat{\boldsymbol{\lambda}} - \boldsymbol{\lambda}\| \leq \|\boldsymbol{\lambda}_{b_n} - \boldsymbol{\lambda}_{a_n}\| = 2^{-n}\|\mathbf{1} - \boldsymbol{\lambda}_0\| \leq 2^{-n}\sqrt{d}$. Since for any point $\boldsymbol{\lambda}_0$ on the grid we have $\mathbf{1} - \boldsymbol{\lambda}_0 \geq 0$, we also know that $\boldsymbol{\lambda}_{a_n} \leq \boldsymbol{\lambda} \leq \boldsymbol{\lambda}_{b_n}$.
We repeat this procedure for all points $\boldsymbol{\lambda}_0$ in the grid on the faces of the cube that contain $\mathbf{0}$. This establishes an approximation of the boundary of $R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$ as a collection of points $\{\hat{\boldsymbol{\lambda}}_k\}_{k=1}^N \subset R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$, where N is the number of grid points. Notice, this algorithm approximates only the boundary marked in green in Figure 2.1. The rest of the boundary is approximated by all lines connecting the algorithmically found points on the faces of $[0,1]^d$ with $\mathbf{1}$.

Remark 2.3.11. *This collection of points constitutes an inner approximation of the set $R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$. Notice that an outer approximation of the set as part of a $\boldsymbol{v}$-approximation as described in (Feinstein, Rudloff, and Weber, 2017, Definition 4.1) is in general not possible, since the values of $R^{\text{int}}_{\boldsymbol{S}}$ are in general not upper sets and therefore $R^{\text{int}}_{\boldsymbol{S}} \not\subset R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X}) - \boldsymbol{v}$, for $\boldsymbol{v} \in \mathbb{R}^d_{++}$. However, this is not a drawback, since we are only interested in elements of $R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$.*

We can easily communicate this approximation of the set $R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$ for small dimensions. In particular, since all line segments connecting points of the approximation and $\mathbf{1}$ are contained in $R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$, we get a 'discrete cover' of $R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$ by lines. For a fine grid, this may be precise enough for practical purposes. However, interpolation between these lines is, in general, not possible if the acceptance set is not convex.
If, however, the acceptance set is convex, then we know by Proposition 2.3.7 that $R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$ is convex. In this case, we can define a stronger approximation as the convex hull of $\{\hat{\boldsymbol{\lambda}}_k\}_{k=1}^N$ and the vector $\mathbf{1}$,

$$\hat{R}^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X}) := \operatorname{conv}\left\{\{\hat{\boldsymbol{\lambda}}_k\}_{k=1}^N \cup \{\mathbf{1}\}\right\} \subset R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})\,.$$

In particular, any interpolation between the line segments is also contained in $R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$. This allows for the tradeoff between accuracy at the boundary of $R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$ and substantially faster computation by coarsening the grid on the faces, while still covering the majority of $R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$.

Remark 2.3.12. *This algorithm can be applied to high-dimensional systems at the expense of considerably longer runtime and memory usage. Feinstein, Rudloff, and Weber (2017, Remark 4.3) suggest to reduce the dimension of the problem by dividing the set of institutions into groups with equal capital requirements for the computation of their measure of systemic risk. This is also possible in the intrinsic framework. In analogy to (Feinstein, Rudloff, and Weber, 2017, Example 2.1 (iv)), for $k < d$ groups we can restrict the risk measurement to vectors of the form $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_2, \ldots, \lambda_k, \ldots, \lambda_k)^\intercal \in [0,1]^d$. However, since the values of intrinsic systemic risk measures are not upper sets, players cannot, in general, deviate from this position. This means a system is not guaranteed to remain acceptable if an institution in group $j \in \{1, \ldots, k\}$ increases its position in the eligible asset, that is, chooses to sell a greater fraction of its position than λ_j.*

Remark 2.3.13. *In analogy to (Feinstein, Rudloff, and Weber, 2017, Definition 3.3), we can define a notion similar to EARs for intrinsic systemic risk measures. For a convex, closed risk measurement $R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X}) \notin \{\emptyset, [0,1]^d\}$ we define the set of minimal points as*

$$\operatorname{Min} R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X}) = \{\boldsymbol{\lambda} \in [0,1]^d \mid (\boldsymbol{\lambda} - [0,1]^d) \cap R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X}) = \{\boldsymbol{\lambda}\}\}\,.$$

However, since $R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$ is not an upper set in general, it is not true that for $\boldsymbol{\lambda}^* \in \operatorname{Min} R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$ the set $(\boldsymbol{\lambda}^* + [0,1]^d) \cap [0,1]^d$ is a maximal subset of $R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$, nor is it a subset at all. In particular, there are points $\boldsymbol{\lambda}^* \in \operatorname{Min} R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$ such that $\boldsymbol{\lambda}^* + \epsilon \boldsymbol{e}_k \notin R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$ for any $\epsilon > 0$ and some standard unit vector $\boldsymbol{e}_k \in \mathbb{R}^d$. This means, agents cannot deviate from allocations in $\operatorname{Min} R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$. To tackle this problem and allow small perturbations without losing acceptability, we could further restrict the set of minimal points for $\epsilon > 0$ to

$$\operatorname{Min}_\epsilon R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X}) = \{\boldsymbol{\lambda} \in \operatorname{Min} R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X}) \mid \boldsymbol{\lambda} + \epsilon[0,1]^d \in R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})\}\,.$$

However, a priori this set is not guaranteed to be non-empty. Take for example a pointed, closed, convex cone, C, which does not include $[0,1]^d$. Then for any $\boldsymbol{x} \in (0,1)^d$ the set $(C + \boldsymbol{x}) \cap [0,1]^d$ has only one minimal point, $\boldsymbol{x}$, and $\boldsymbol{x} + \epsilon \boldsymbol{e}_k \notin (C + \boldsymbol{x}) \cap [0,1]^d$, for any $\epsilon > 0$ and standard unit vector $\boldsymbol{e}_k$. However, it remains to be investigated whether convex intrinsic systemic risk measurements can take this form.

From a practical perspective, the objective of the regulator might be to make a network acceptable with as little alteration to existing positions as possible. What exactly this means depends on the choice of the objective function. The most straightforward choice would be to minimise the overall percentage change of all positions in the network. In that case, we are interested in all $\boldsymbol{\lambda} \in R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$ with minimal sum over their components, or equivalently on $[0,1]^d$, with minimal 1-norm,

$$\operatorname*{arg\,min}_{\boldsymbol{\lambda} \in R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})} \boldsymbol{\lambda}^\intercal \mathbf{1} = \operatorname*{arg\,min}_{\boldsymbol{\lambda} \in R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})} |\boldsymbol{\lambda}|_1\,. \tag{2.6}$$

Alternatively, one could minimise the total nominal change in the sense of the value $\boldsymbol{x}_0 \odot \boldsymbol{\lambda}$. The optimisation for this, and in fact any weighted sum with non-negative weights $\boldsymbol{w}$, can be written in the form of Equation (2.6) by minimising over the set $\boldsymbol{w} \odot R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X}) = \{\boldsymbol{w} \odot \boldsymbol{\lambda} \mid \boldsymbol{\lambda} \in R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})\}$. So in the following, we will concentrate on the case in (2.6).
To simplify the problem, we define for $k \geq 0$ the plane $E_k = \{\ell \in [0,1]^d \mid \ell^\intercal \mathbf{1} = k\}$, on which each element has the same 1-norm. For increasing $k \in [0,d]$ these planes

contain elements with increasing 1-norms. In particular, the first non-empty intersection $E_k \cap R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$ for increasing k contains all points with minimal 1-norm in $R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$,

$$\underset{\boldsymbol{\lambda} \in R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})}{\arg\min} \ \boldsymbol{\lambda}^\intercal \mathbf{1} = E_{k_{\min}} \cap R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X}),$$

where $k_{\min} = \min\{k \in [0,d] \mid E_k \cap R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X}) \neq \emptyset\}$. So if we are interested in these minimal points, we do not need to approximate the whole boundary of the risk measurement. Instead, we can take advantage of this observation and adapt the algorithm to find only the minimal points. This also reduces the computational load.

To implement this procedure, we denote by $\mathbf{1}^\perp = \{\ell \in \mathbb{R}^d \mid \ell^\intercal \mathbf{1} = 0\}$ the orthogonal complement of $\mathbf{1} \in \mathbb{R}^d$, and we write $E_k = (\frac{k}{d}\mathbf{1} + \mathbf{1}^\perp) \cap [0,1]^d$. The idea is to do a modified bisection search, where in each iteration we check whether the intersection of E_k and $R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$ is empty or not. The algorithm is described below and its modification using the method described in Remark 2.3.14 is illustrated in Figure 2.2.

1. Generate a grid on the plane $\mathbf{1}^\perp$, such that it covers the whole cube $[0,1]^d$ when translated along the vector $\mathbf{1}$.

2. Define $k_{a_0} = 0$ and $k_{b_0} = d$ as the initial end points of the search. Check that $\mathbf{0} \notin R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$, or otherwise $\arg\min_{\boldsymbol{\lambda} \in R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})} \boldsymbol{\lambda}^\intercal \mathbf{1} = \{\mathbf{0}\}$.

3. In each iteration, define $k_\ell = \frac{1}{2}(k_{a_{\ell-1}} + k_{b_{\ell-1}})$ and translate the grid from $\mathbf{1}^\perp$ to E_{k_ℓ}. Calculate which grid points lie in $E_{k_\ell} \cap R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$. If the intersection contains no grid points, set $k_{a_\ell} = k_\ell$ and $k_{b_\ell} = k_{b_{\ell-1}}$, otherwise set $k_{a_\ell} = k_{a_{\ell-1}}$ and $k_{b_\ell} = k_\ell$.

4. Repeat Step 3 until $k_{b_\ell} - k_{a_\ell} < \delta$, for some prespecified threshold $\delta > 0$.

This leaves us with a sequence of planes that converge to $E_{k_{\min}}$ and we can easily check which grid points lie in $E_{k_{\min}} \cap R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$.

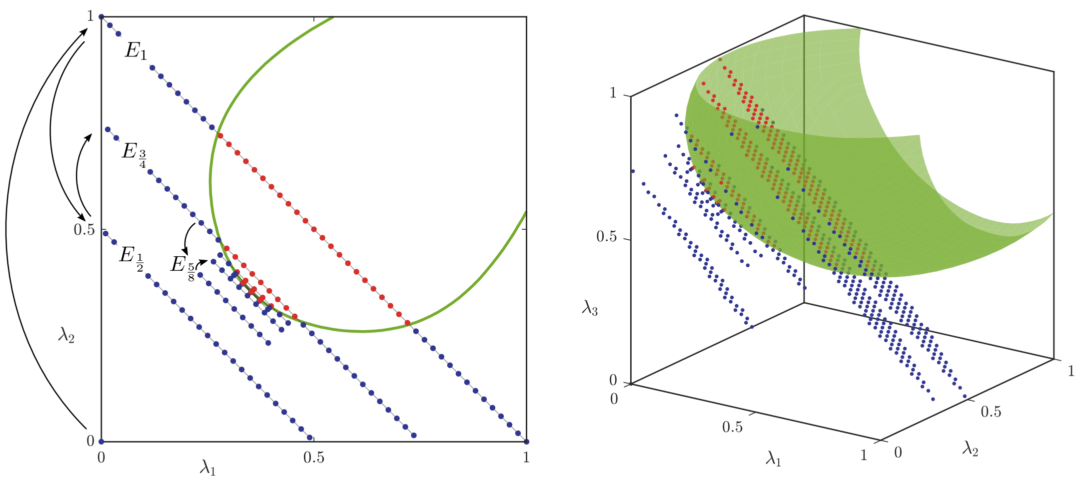


Figure 2.2: Visualisation of grid search algorithm to find minimal points.

Remark 2.3.14. *If the acceptance set is convex, we can reduce the computational time even further with the help of Lemma 2.3.15. This method is briefly outlined below. In addition to the steps described above, we also keep track of the current acceptable grid points in $E_{k_\ell} \cap R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$. Then, whenever we consider grid points of a non-empty intersection $E_{k_\ell - \epsilon} \cap R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$ for some $\epsilon > 0$, we compare them to the current acceptable grid* *points. If all the grid points in $E_{k_\ell-\epsilon} \cap R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$ are contained in the set of grid points in $(E_{k_\ell} \cap R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})) - \frac{\epsilon}{d}\mathbf{1}$, we can, by Lemma 2.3.15 , restrict our search to only the former grid points.*

Alternatively, we can increase the accuracy of the algorithm by refining the grid. Whenever we use Lemma 2.3.15 and restrict our search from a grid on some E_k to a smaller grid on $E_{k-\epsilon}$, we can decrease the step size between grid points such that the number of grid points per iteration stays the same.

Lemma 2.3.15. *Let $A \subset \mathbb{R}^d$ be a closed, convex set. Let $A_k = \{\boldsymbol{x} \in A \mid \boldsymbol{x}^\intercal \mathbf{1} = k\}$. If there exists $\epsilon > 0$ such that $A_{k-\epsilon} \subset A_k - \frac{\epsilon}{d}\mathbf{1}$, then for all $\delta > \epsilon$ also $A_{k-\delta} \subset A_{k-\epsilon} - \frac{\delta-\epsilon}{d}\mathbf{1}$.*

The proof of Lemma 2.3.15 is given in Section 2.7.1.

2.4 Dual representation

The dual representation for univariate monetary risk measures is a classical result, which can be found for example in (Föllmer and Schied, 2004, Section 4.2) and was already stated in Theorem 1.2.8. For completeness, we state the definition of the minimal penalty function again,

$$\alpha(\mathbb{Q}) = \sup_{X \in \mathcal{A}} \mathbb{E}^{\mathbb{Q}}[-X]. \tag{2.7}$$

The dual representation can be seen as considering all possible probabilistic models, with their plausibility and closeness to $\mathbb{P}$ being conveyed in the penalty function α. The value of the risk measure then corresponds to the worst case expectation over all possible models, penalised by α.
In Theorem 1.3.12, we derived a dual representation for scalar intrinsic risk measures, whereas Ararat and Rudloff (2020) provide the dual representations for monetary systemic risk measures with constant eligible assets under appropriate assumptions on the underlying acceptance set and aggregation function.

In this section, we build on these results and derive the dual representation for the intrinsic systemic risk measure. We denote by $\rho_{\mathcal{A}}$ the scalar risk measure associated with the acceptance set $\mathcal{A}$ and a constant eligible assets $S = (1, 1) \in \mathbb{R}_+ \times L^\infty_+$. In the following, we consider only the space $(L^\infty_d)_+$ and we assume that the aggregation function Λ is concave. Moreover, we assume that $\mathcal{A}$ is convex and closed. Note that if $\mathcal{A}$ is convex, we know by Proposition 1.2.10 that $\rho_{\mathcal{A}}$ is convex. Furthermore, closedness of $\mathcal{A}$ implies that $\rho_{\mathcal{A}}$ is lower semi-continuous, see for example (Munari, 2015, Proposition 3.3.7).
Let $g : \mathbb{R}^d \to \mathbb{R}$ be the Legendre-Fenchel conjugate of the convex function $f : \mathbb{R}^d \to \mathbb{R}$ defined by $f(\boldsymbol{x}) = -\Lambda(-\boldsymbol{x})$, that is,

$$g(\boldsymbol{z}) = \sup_{\boldsymbol{x} \in \mathbb{R}^d} \left(\Lambda(\boldsymbol{x}) - \boldsymbol{z}^\intercal \boldsymbol{x}\right).$$

Let $\mathcal{M}_d(\mathbb{P})$ be the set of all vector probability measures $\mathbb{Q} = (\mathbb{Q}_1, \dots, \mathbb{Q}_d)^\intercal$ whose components $\mathbb{Q}_k$ are in $\mathcal{M}(\mathbb{P})$, $k \in \{1, \dots, d\}$, and let α be the penalty function defined in Equation (2.7). We recall the definition of a systemic penalty function introduced in (Ararat and Rudloff, 2020, Definition 3.1).

Definition 2.4.1. *The function $\alpha^{\text{sys}} : \mathcal{M}_d(\mathbb{P}) \times (\mathbb{R}^d_+ \setminus \{0\}) \to \mathbb{R} \cup \{+\infty\}$ defined by*

$$\alpha^{\text{sys}}(\mathbb{Q}, \boldsymbol{w}) = \inf_{\substack{\mathbb{S} \in \mathcal{M}(\mathbb{P}): \\ \forall k:\, w_k \mathbb{Q}_k \ll \mathbb{S}}} \left\{ \mathbb{E}^{\mathbb{S}}\left[g\left(\boldsymbol{w} \odot \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{S}} \right) \right] + \alpha(\mathbb{S}) \right\}$$

is called the systemic penalty function.

Now we can formulate the dual representation of intrinsic systemic risk measures.

Proposition 2.4.2. *Let $\boldsymbol{S} \in \mathbb{R}^d_{++} \times (L^\infty_d)_+$ be an eligible asset, let Λ be a concave aggregation function and let $\mathcal{A}$ be a closed, convex acceptance set. Moreover, assume that $\rho_{\mathcal{A}}(0) \in \Lambda(\mathbb{R}^d)$. Then the intrinsic measure of systemic risk $R^{\text{int}}_{\boldsymbol{S}} \colon \mathbb{R}^d_{++} \times (L^\infty_d)_+ \to \mathcal{P}([0,1]^d)$ defined in Equation (2.4) admits the following dual representation,*

$$R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X}) = \bigcap_{\substack{\mathbb{Q}\in\mathcal{M}_d(\mathbb{P}),\\ \boldsymbol{w}\in\mathbb{R}^d_+\setminus\{0\}}} \left\{ \boldsymbol{\lambda} \in [0,1]^d \;\middle|\; \begin{array}{r} \boldsymbol{\lambda}^\intercal\big(\boldsymbol{w}\odot\mathbb{E}^{\mathbb{Q}}\left[\boldsymbol{x}_0\odot\boldsymbol{S}_T\oslash\boldsymbol{s}_0-\boldsymbol{X}_T\right]\big) \geq \ldots \\ \ldots\,\boldsymbol{w}^\intercal\mathbb{E}^{\mathbb{Q}}\left[-\boldsymbol{X}_T\right]-\alpha^{\text{sys}}(\mathbb{Q},\boldsymbol{w}) \end{array} \right\}.$$

Proof. From (Ararat and Rudloff, 2020, Proposition 3.4) we have

$$\rho_{\mathcal{A}}(\Lambda(\boldsymbol{X_T})) = \sup_{\substack{\mathbb{Q}\in\mathcal{M}_d(\mathbb{P}),\\ \boldsymbol{w}\in\mathbb{R}^d_+\setminus\{0\}}} \Big(\boldsymbol{w}^\intercal\mathbb{E}^{\mathbb{Q}}\left[-\boldsymbol{X}_T\right]-\alpha^{\text{sys}}(\mathbb{Q},\boldsymbol{w})\Big)\,. \tag{2.8}$$

By Proposition 1.2.10, we can write

$$R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X}) = \Big\{\boldsymbol{\lambda}\in[0,1] \mid \Lambda\big(\boldsymbol{X}^{\boldsymbol{\lambda},\boldsymbol{S}}_T\big)\in\mathcal{A}\Big\} = \Big\{\boldsymbol{\lambda}\in[0,1] \mid \rho_{\mathcal{A}}\big(\Lambda\big(\boldsymbol{X}^{\boldsymbol{\lambda},\boldsymbol{S}}_T\big)\big)\leq 0\Big\}\,.$$

Together with Equation (2.8) it follows that $\boldsymbol{\lambda}\in[0,1]^d$ lies in $R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$ if and only if

$$\forall\mathbb{Q}\in\mathcal{M}_d(\mathbb{P}), \boldsymbol{w}\in\mathbb{R}^d_+\setminus\{0\} : \boldsymbol{w}^\intercal\mathbb{E}^{\mathbb{Q}}\left[-\boldsymbol{X}^{\boldsymbol{\lambda},\boldsymbol{S}}_T\right]-\alpha^{\text{sys}}(\mathbb{Q},\boldsymbol{w})\leq 0\,,$$

or, rewritten,

$$\begin{aligned} &\forall\mathbb{Q}\in\mathcal{M}_d(\mathbb{P}), \boldsymbol{w}\in\mathbb{R}^d_+\setminus\{0\}\colon \\ &\boldsymbol{\lambda}^\intercal\big(\boldsymbol{w}\odot\mathbb{E}^{\mathbb{Q}}\left[\boldsymbol{X}_T-\boldsymbol{x}_0\odot\boldsymbol{S}_T\oslash\boldsymbol{s}_0\right]\big)\leq\alpha^{\text{sys}}(\mathbb{Q},\boldsymbol{w})+\boldsymbol{w}^\intercal\mathbb{E}^{\mathbb{Q}}\left[\boldsymbol{X}_T\right]\,. \end{aligned}$$

From here, the claim follows. □

Remark 2.4.3. *Note that Ararat and Rudloff (2020) assume that $\rho_{\mathcal{A}}$ satisfies the Fatou property. Satisfying the Fatou property means satisfying lower semi-continuity with respect to the topology $\sigma(L^\infty, L^1)$, see (Delbaen, 2002, Theorem 3.2), which is weaker than lower semi-continuity implied by the closedness of $\mathcal{A}$. Therefore, the assumptions used in Proposition 2.4.2 can be weakened.*

The dual representation of $R^{\text{int}}_{\boldsymbol{S}}$ given in Proposition 2.4.2 can be interpreted in a similar way as in Ararat and Rudloff (2020, p. 147f). Consider a network consisting of d institutions, represented by the elements of $\boldsymbol{x}_0$ and $\boldsymbol{X}_T$, as well as society. The dual representation collects the possible restructuring actions in the presence of model uncertainty and weight ambiguity.
Society is assigned a probability measure $\mathbb{S}$ and each institution is assigned its own probability measure $\mathbb{Q}_k$ along with a weight w_k with respect to society, $k\in\{1,\ldots,d\}$. The penalty function α^{sys} combines two penalty terms. One is the distance of the network to society, captured as the multivariate g-divergence of $\mathbb{Q}$ with respect to $\mathbb{S}$, $\mathbb{E}^{\mathbb{S}}\left[g\big(\boldsymbol{w}\odot\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{S}}\big)\right]$. The other is the penalty $\alpha(\mathbb{S})$ incurred for the choice of $\mathbb{S}$. The penalty function α^{sys} is then given as the infimum of the sum of these penalties over all choices of $\mathbb{S}$.
Finally, a vector of fractions $\boldsymbol{\lambda}\in[0,1]^d$ is deemed feasible for a specific choice of $\mathbb{Q}$ and $\boldsymbol{w}$, if the weighted sum of expected return of the eligible asset held in the restructured

portfolios of institutions exceeds the weighted expected negative return of the original positions $\mathbf{X}$ held in the restructured portfolios of the institutions penalised by $\alpha^{\text{sys}}(\mathbb{Q}, \boldsymbol{w})$,

$$\boldsymbol{w}^\intercal \mathbb{E}^{\mathbb{Q}} \left[\boldsymbol{\lambda} \odot \boldsymbol{x}_0 \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0\right] \geq \boldsymbol{w}^\intercal \mathbb{E}^{\mathbb{Q}} \left[-(1-\boldsymbol{\lambda}) \odot \boldsymbol{X}_T\right] - \alpha^{\text{sys}}(\mathbb{Q}, \boldsymbol{w})\,.$$

To be considered a feasible action to compensate the systemic risk in the system, the vector of fractions $\boldsymbol{\lambda}$ has to be deemed feasible for all possible choices of probability measures $\mathbb{Q}$ and weights $\boldsymbol{w}$.

2.5 The network approach

In this section, we will investigate the effects of the management actions underlying intrinsic systemic risk measures on networks as originally proposed by Eisenberg and Noe (2001). We will complement their model with an additional sink node called *society* and define the aggregation function as the net equity of society after receiving the clearing payments as described in (Ararat and Rudloff, 2020, Section 4.4), see also (Feinstein, Rudloff, and Weber, 2017, Section 5.2). This enables us to derive statements about the repercussions of an under-capitalised financial system and to monitor the impact of intrinsic management actions on the wider economy. Furthermore, this study provides insights into current regulatory policies and challenges the current approach to capital regulation.

2.5.1 Network model

In the following, we recall the network model. An illustration of the network structure can be seen in Figure 2.3. A financial system consists of $d+1$, $d \geq 2$, nodes. Nodes $\{1, \dots, d\}$ represent financial institutions participating in the network and node 0 represents society. The network is interconnected via liabilities towards each other. Throughout this section, we assume for simplicity that liabilities are deterministic, whereas future endowments of participants of the network are represented by a random vector $\boldsymbol{X}_T$ with an initial value $\boldsymbol{x}_0$. For $i, j \in \{0, \dots, d\}$ let $L_{ij} \geq 0$ denote the nominal liability of node i towards node j. Self-liabilities are disregarded, that is, $L_{ii} = 0$ for all $i \in \{0, \dots, d\}$. Node 0 takes the role of a sink node, as was already described in Eisenberg and Noe (2001, Section 2.2), and we assume that it has no liabilities towards the other d nodes in the system, that is, $L_{0i} = 0$ for $i \in \{1, \dots, d\}$. In line with (Feinstein, Rudloff, and Weber, 2017), we interpret society as part of the wider economy. That means, node 0 represents all outside factors which are not explicitly part of the financial system. In particular, we assume that all institutions have liabilities towards society, that is, for $i \in \{1, \dots, d\}$ we have $L_{i0} > 0$. Furthermore, we define relative liabilities for $i, j \in \{0, \dots, d\}$, $i \neq 0$ as

$$\Pi_{ij} = \frac{L_{ij}}{\hat{L}_i} \quad \text{with} \quad \hat{L}_i = \sum_{j=0}^{d} L_{ij} > 0\,,$$

where $\hat{L}_i$ is the aggregate nominal liability of i towards all other nodes in the network.

At time point T, liabilities are cleared. This means that all participants in the network repay all or, if not possible, part of their liabilities. For a realised state $\boldsymbol{x}_T := \boldsymbol{X}_T(\omega) \in \mathbb{R}^d_+$ at time T we collect these payments in a vector $p(\boldsymbol{x}_T) = (p_1(\boldsymbol{x}_T), \dots, p_d(\boldsymbol{x}_T))^\intercal \in \mathbb{R}^d_+$, so that node i pays node j the amount $\Pi_{ij} p_i(\boldsymbol{x}_T)$. We call $p(\boldsymbol{x}_T)$ a *clearing payment vector* if it solves the fixed point problem

$$p_i(\boldsymbol{x}_T) = \min\left\{\hat{L}_i\ ,\ x_T^i + \sum_{j=1}^{d} \Pi_{ji} p_j(\boldsymbol{x}_T)\right\}, \quad i \in \{1, \ldots, d\}\,.$$

In the case that institution i stays in business and no default occurs, it pays all of its liabilities to the rest of the network, $p_i(\boldsymbol{x}_T) = \hat{L}_i$. In the case of default, the payment is equal to the realised wealth, x_T^i, plus the income from the other participants of the network, $\sum_{j=1}^{d} \Pi_{ji} p_j(\boldsymbol{x}_T)$. The clearing payment vector can be calculated with the 'fictitious default algorithm' introduced by Eisenberg and Noe (2001, p. 243, see also Lemma 3 and Lemma 4).

We choose the aggregation function $\Lambda : \mathbb{R}^d \to \mathbb{R}$ for the intrinsic systemic risk measure to represent the impact of the financial system on society. For $\beta \in (0, 1)$ we define

$$\Lambda(\boldsymbol{x}) = \sum_{i=1}^{d} \Pi_{i0} p_i(\boldsymbol{x}) - \beta \sum_{i=1}^{d} L_{i0}\,. \tag{2.9}$$

Note that the first term in Equation (2.9) lies in the interval $[0, \sum_{i=1}^{d} L_{i0}]$. If we were to use only this term, then every system would be acceptable with regard to

$$\mathcal{A}_{\mathrm{VaR}_\alpha} = \{X_T \in L^p \mid \mathrm{VaR}_\alpha(X_T) \leq 0\} \ \text{ or } \ \mathcal{A}_{\mathrm{ES}_\alpha} = \{X_T \in L^p \mid \mathrm{ES}_\alpha(X_T) \leq 0\}\,.$$

Therefore, we subtract the second term $\beta \sum_{i=1}^{d} L_{i0}$. The interpretation is in line with (Feinstein, Rudloff, and Weber, 2017). We deem a system acceptable if the Value-at-Risk, respectively the Expected Shortfall, of the aggregated payments to society minus a percentage β of the total liabilities to society does not exceed 0. For the Value-at-Risk, this means that a system is acceptable if with at least the probability $(1 - \alpha)$ at least the percentage β of the liabilities towards society can be repaid.

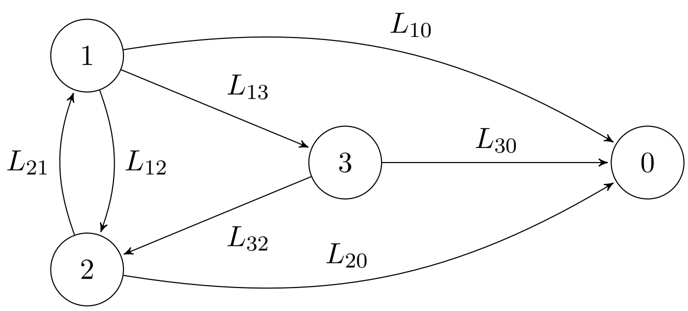


Figure 2.3: Network of financial institutions including society as node 0.

Remark 2.5.1. *It is possible to extend this network model in a variety of ways. A natural extension would incorporate random liabilities. Furthermore, one can incorporate the illiquidity of assets during fire sales, see for example (Cifuentes, Ferrucci, and Shin, 2005; Feinstein, 2017; Hurd, 2016), bankruptcy costs as in (Rogers and Veraart, 2013), cross holdings as in (Elsinger, 2009), or a combination of the above as in (Weber and Weske, 2017).*

2.5.2 Numerical case studies

In order to get a better understanding of the behaviour of intrinsic systemic risk measures under different model parameters, it is helpful to consider a simple network of $d = 2$ institutions. The following model parameters define the base model. Throughout this section, we will adjust them separately to see their effects. The marginal distributions of the agents' wealths at time T are assumed to be beta distributions, $X_T^k \sim \text{Beta}(a_k, b_k)$, with $a_k = 2, b_k = 5$, $k \in \{1, 2\}$. We choose the initial value $\boldsymbol{x}_0$ such that the expected return of each institution is 15%, that is, $\mathbb{E}[\boldsymbol{X}_T] = 1.15\,\boldsymbol{x}_0$. The eligible assets have a log-normal marginal distribution, $S_T^k \sim \log\mathcal{N}(\mu_k, \sigma_k^2)$, $k \in \{1, 2\}$. We specify μ_k and σ_k such that the expectations of the eligible assets are equal to the expectations of the agent's positions, $\mathbb{E}[\boldsymbol{X}_T] = \mathbb{E}[\boldsymbol{S}_T]$, and such that the variances are a fifth of the variances of $\boldsymbol{X}_T$, $\mathbb{V}[\boldsymbol{S}_T] = 0.2\,\mathbb{V}[\boldsymbol{X}_T]$. We set the initial cost of the eligible assets such that the expected return is 10%, $\mathbb{E}[\boldsymbol{S}_T] = 1.1\,\boldsymbol{s}_0$. In this model, the eligible assets are more secure in the sense that their distributions are a lot narrower. But, in turn, this security comes at an additional cost, since $\boldsymbol{x}_0 \leq \boldsymbol{s}_0$.
Dependence is incorporated via a Gaussian copula with a correlation $\rho \in [-1, 1]$ between X_T^1 and X_T^2. The eligible assets are uncorrelated to each other and to X_T^k, $k \in \{1, 2\}$.
Furthermore, we assume that the agents have symmetric liabilities to each other, $L_{12} = L_{21} = 0.6$, and to society, $L_{10} = L_{20} = 0.2$. We use the aggregation function specified in Equation (2.9). We deem a system acceptable if the Expected Shortfall at probability level $\alpha = 5\%$ of the aggregated network is less or equal 0, that is, $\boldsymbol{\lambda} \in R_{\boldsymbol{S}}^{\text{int}}(\boldsymbol{X})$ if and only if $\text{ES}_\alpha(\Lambda(\boldsymbol{X}_T^{\boldsymbol{\lambda},\boldsymbol{S}})) \leq 0$.

The following numerical studies consist of 10^5 simulated samples of the described multivariate distribution. The step size of the grid on the axes is set to 0.05 and the size of the interval at which we stop the bisection search is set to 10^{-6}, see also Section 2.3.2.
In the following, for $\boldsymbol{\lambda} \in R_{\boldsymbol{S}}^{\text{int}}(\boldsymbol{X})$ we will refer to $\boldsymbol{X}_T^{\boldsymbol{\lambda},\boldsymbol{S}}$ as an intrinsic system and to $\Lambda(\boldsymbol{X}_T^{\boldsymbol{\lambda},\boldsymbol{S}})$ as an aggregate intrinsic system. We use the analogous nomenclature for the monetary case.

Influence of dependency structure In the following figures, we depict the boundaries of $R_{\boldsymbol{S}}^{\text{int}}(\boldsymbol{X})$ on the left-hand side and the boundaries of $R_{\boldsymbol{S}}(\boldsymbol{X}_T)$ on the right-hand side.

In Figure 2.4, we illustrate the risk measures for different correlations between the elements of $\boldsymbol{X}_T$. We observe that as the correlation between the elements of $\boldsymbol{X}_T$ increases, the risk measures become smaller in the sense that $R_{\boldsymbol{S}}^{\text{int}}(\boldsymbol{X}_\rho) \subset R_{\boldsymbol{S}}^{\text{int}}(\boldsymbol{X}_{\hat{\rho}})$ for correlations $\rho > \hat{\rho}$. This is expected, since higher correlation between the participants of the network results in higher probability of cascades of defaults and hence, the inability to repay society. Furthermore, allocations which are acceptable for highly correlated agents are also acceptable if the correlation decreases while other dependencies stay unaltered.
We also observe that the lines representing the boundaries of all the sets meet in two points on the boundary of $[0, 1]^2$. This comes from the fact that $\boldsymbol{X}$ and $\boldsymbol{S}$ are uncorrelated, so if one agent translates fully to the eligible asset, the correlation between X_T^1 and X_T^2 becomes irrelevant. A similar statement can be made for the monetary risk measurements, where the whole system is deemed acceptable when enough capital is added to either of the two agents.
It is also noticeable that in this symmetric case, the cheapest way to acceptance, in the sense that $\lambda_1 + \lambda_2$ or $k_1 + k_2$ is minimal, is when both agents adjust their position equally, that is, $\lambda_1 = \lambda_2$ or $k_1 = k_2$.

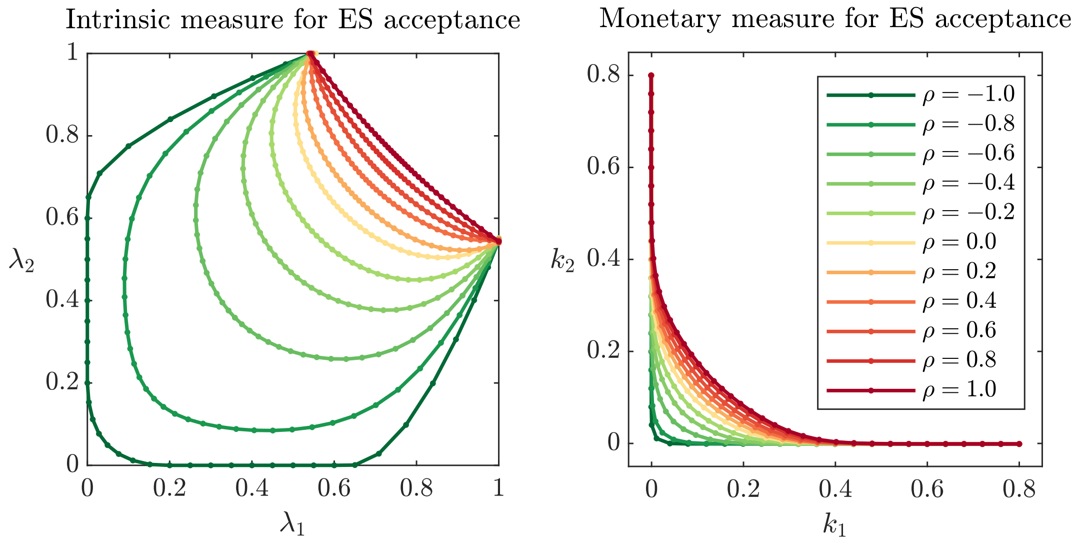


Figure 2.4: Visualisation of the influence of different correlations between the agents' positions.

In the next scenario, we adjust the previous setup and assume different correlations between institution 1 and their eligible asset. In Figure 2.5, we sample distributions such that X_T^1 and S_T^1 are correlated with parameter ρ, while the rest of the system remains uncorrelated. The yellow lines correspond to the yellow lines in Figure 2.4, as in both cases, X_T^1, X_T^2, S_T^1, and S_T^2 are uncorrelated. We observe an accumulation point on the set $\{\lambda_1 = 1\}$, since X_T^2, S_T^2, and S_T^1 are uncorrelated, and no accumulation point on $\{\lambda_2 = 1\}$, since X_T^1 and S_T^1 are correlated. Furthermore, close to $\{\lambda_1 = 1\}$, the sets are almost identical to the ones depicted in Figure 2.4. This means that when institution 1 is almost fully invested in their eligible asset, the correlation between X_T^1 and S_T^1 has little effect on the managements actions of institution 2. However, close to $\{\lambda_2 = 1\}$, we observe that a negative correlation results in less strict management actions for institution 1, whereas a positive correlation results in stricter management actions compared to Figure 2.4.

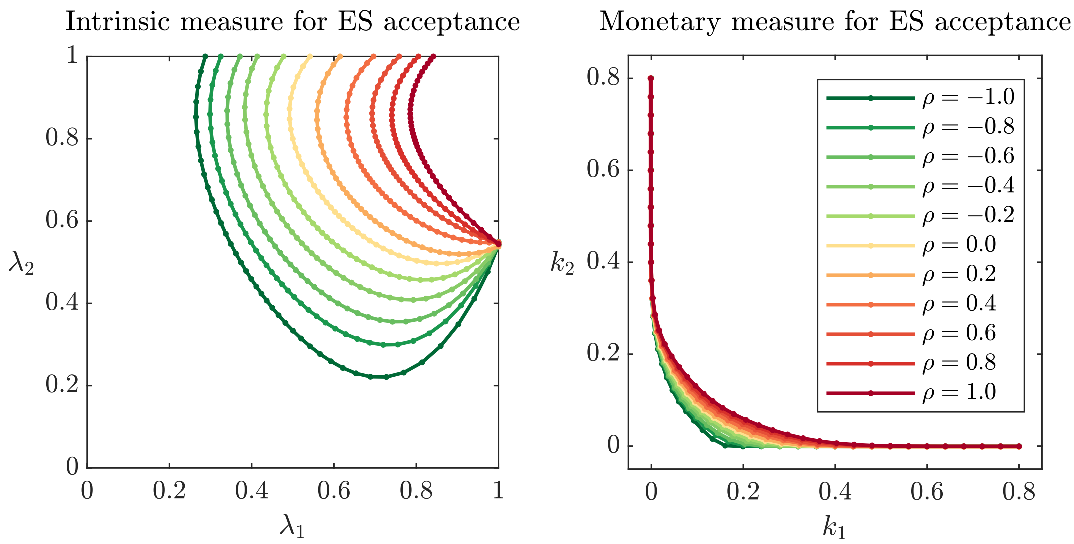


Figure 2.5: Visualisation of the influence of different correlations between the position of one agent and their corresponding eligible asset.

Now we move on to discuss the effect of correlated eligible assets. For this we adapt the base setup by setting the correlation between S_T^1 and S_T^2 to ρ and all other correlations to 0. The resulting risk measures are depicted in Figure 2.6. Compared to the monetary measure of systemic risk, the intrinsic measure is more sensitive to the choice of eligible assets. In this example, the aggregate position of the system consisting of only the eligible assets is acceptable up to approximately a correlation of 0.24. Correlations higher than this result in an intrinsic risk measure which does not include $\mathbf{1}$. However, this does not necessarily mean that the set $R_{\boldsymbol{S}}^{\text{int}}(\boldsymbol{X})$ is empty. For completeness we have included the risk measurement for $\rho = 0.4$, which demonstrates that the condition $\Lambda(\boldsymbol{x}_0 \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0) \in \mathcal{A}$ is conservative in the sense that it is sufficient but not necessary for a non-empty risk measurement. This means that it can be possible to construct an acceptable system even if $\Lambda(\boldsymbol{x}_0 \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0) \notin \mathcal{A}$. However, this boundary cannot be calculated with the algorithm described in Section 2.3.2 and we used a brute force approach on the full grid on $[0,1]^2$ instead.

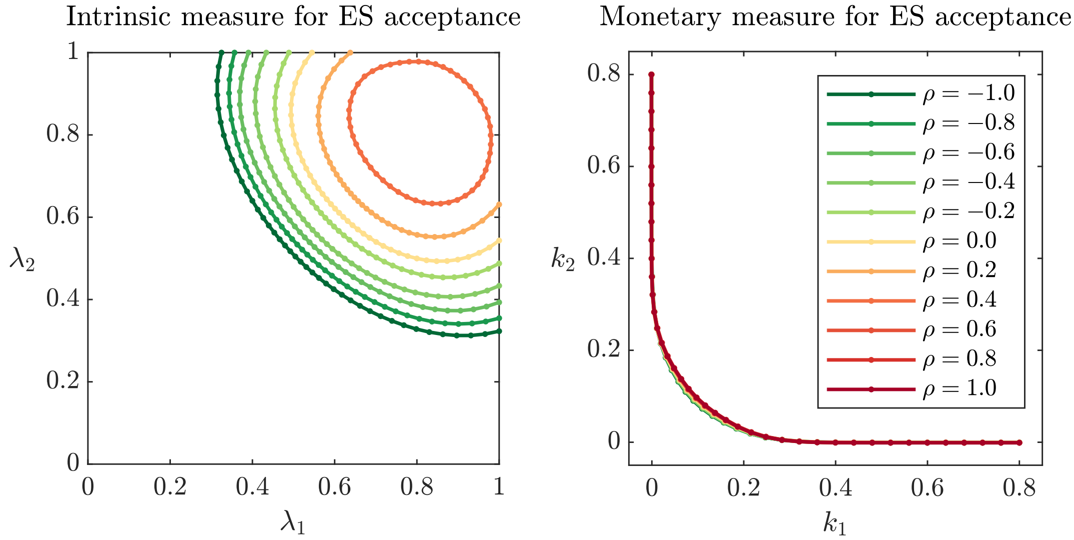


Figure 2.6: Visualisation of the influence of different correlations between the eligible assets.

Whereas from a regulatory point of view, a first thought might be to adapt the agents' positions in an unacceptable system with only one single eligible asset or asset class, this result underlines the importance of a diversified network. In particular, it is beneficial to choose eligible assets which are negatively correlated or uncorrelated to each other. This effect is less apparent for monetary measures, since adding eligible assets inherently increases the overall capital level and the dependence between eligible assets has a smaller impact by comparison. Nevertheless, adding positively correlated eligible assets increases the correlations between the agents.

Influence of volatility We briefly discuss how the variance of an agent's position influences the risk measurements. In Figure 2.7, we start with the base case with uncorrelated random variables. We then decrease the variance of the beta distribution of agent 1 (from green to red) while keeping the expectation at $\frac{a_1}{a_1+b_1} = \frac{2}{2+5}$. As expected, we observe that both risk measurements increase with decreasing variance. In particular, agent 1 needs to adjust their position less in comparison with agent 2.

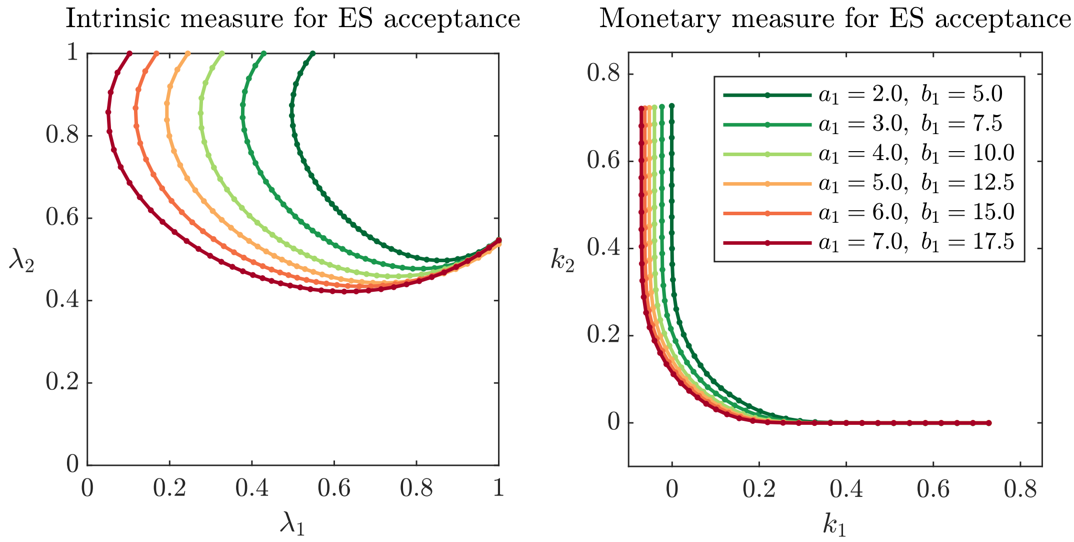


Figure 2.7: Visualisation of the influence of decreasing variance of agent 1.

Influence of liability structure In this paragraph, we investigate the impact of the liability structure on the systemic intrinsic risk measure. We use the parameters of the base case with uncorrelated X_T^k, S_T^k, $k \in \{1, 2\}$.

In Figure 2.8, we leave the bilateral liabilities of the institutions at $L_{12} = L_{21} = 0.6$ and we gradually increase both their liabilities towards society from $L_{10} = L_{20} = 0.1$ to $L_{10} = L_{20} = 0.2$. We observe that the risk measurements are decreasing with increasing liabilities towards society. This is expected, as the term $\beta \sum_{i=1}^{d} L_{i0}$ in Equation (2.9) has a linear influence on the aggregation. It is noticeable that for low liabilities the monetary systemic risk measurements have considerable parts intersecting with $(\mathbb{R}_+^2)^\complement$. This means that if one institution raises enough external capital, then the other one can extract capital from the system while the system remains acceptable. Compared to Figure 2.4, where changes in correlation changed the shape of the sets and made them 'pointier', changes in liabilities to society rather translate the whole set. It is apparent that the amount of liabilities towards society strongly influences the size of the risk measurement.

In the next example, we keep liabilities towards society constant at $L_{10} = L_{20} = 0.2$ and vary bilateral liabilities between the agents. At first, the result in Figure 2.9 might seem counter-intuitive, as both intrinsic and monetary risk measurements increase with increasing liabilities. However, increasing liabilities between the institutions in this network essentially means adding capital to the system. In particular, if one institution is doing poorly and goes bankrupt while the other is doing well, it will still receive the full payment from the other institution. The higher this payment, the higher is the payment from the defaulting institution towards society. It is an interesting observation that the intrinsic risk measurements appear to converge to a 'maximal set'. This set is very close to the one represented by the red line in Figure 2.9. In the case of the monetary measure, it is not clear from this preliminary investigation whether the sets approach a half-space which is supported at a point with $k_1 = k_2$. See also Figure 2.11 in Section 2.7.2. However, the higher the bilateral liabilities and the more external capital one of the agents holds, the more capital the other agent can extract from their position. This could be a dangerous feature of the monetary approach if kept unmonitored.

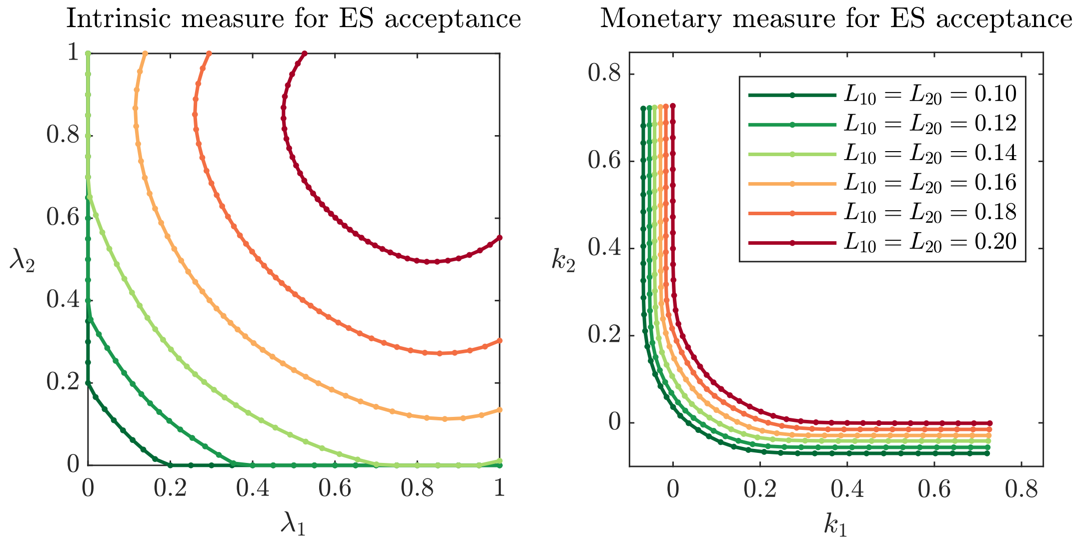


Figure 2.8: Visualisation of the influence of increasing symmetric liabilities towards society.

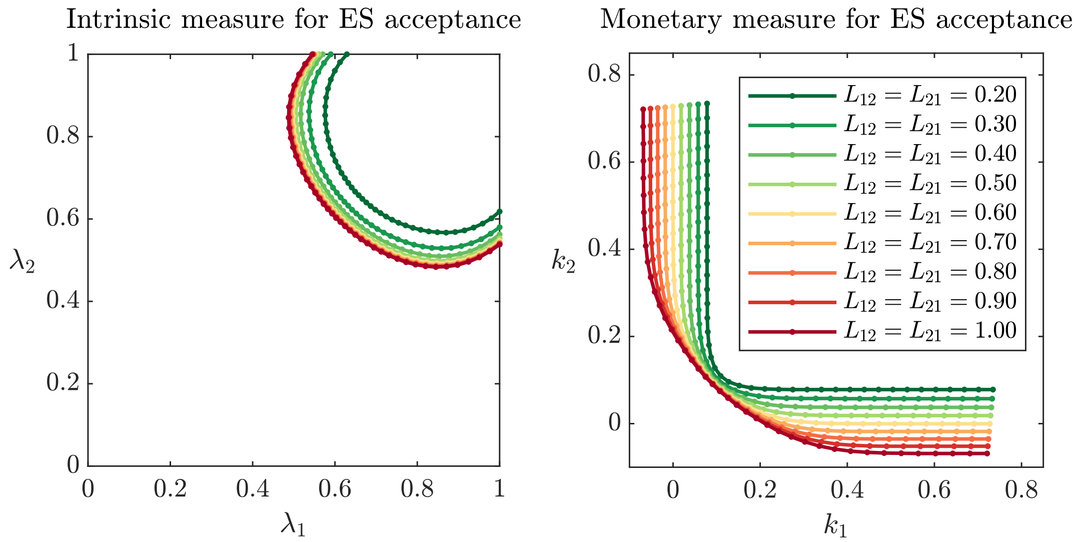
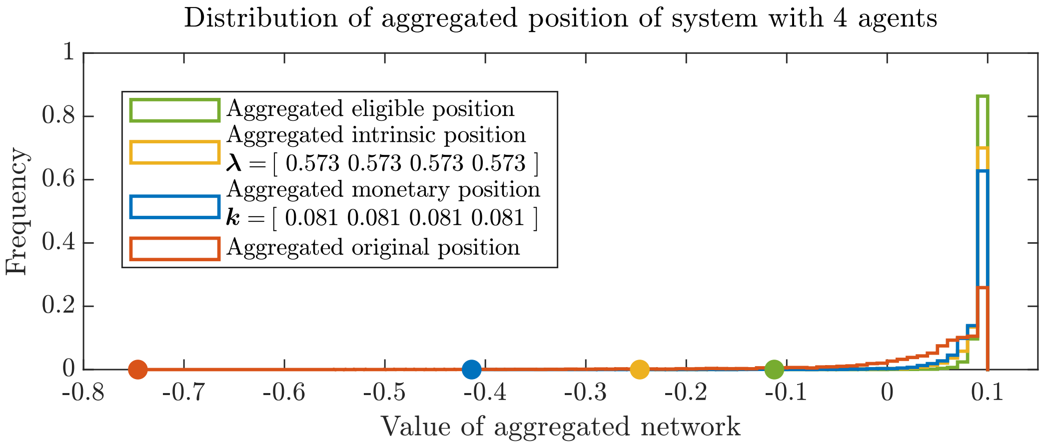


Figure 2.9: Visualisation of the influence of increasing symmetric bilateral liabilities between the agents.

Intrinsic management actions and aggregate network outcomes We conclude this section with a discussion about the aggregated outcomes resulting from management actions of $\boldsymbol{\lambda}$ and $\boldsymbol{z}$ on the boundaries of $R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$ and $R_{\boldsymbol{S}}(\boldsymbol{X}_T)$, respectively. In the following, we assume that all X^k_T, S^k_T, $k \in \{1, \dots, d\}$ are uncorrelated. Furthermore, since there are more players, we adjust the liability structure. For $d = 4$ we set $L_{ij} = 0.6$ and $L_{i0} = 0.23$ and for $d = 20$ we set $L_{ij} = 0.2$ and $L_{i0} = 0.25$, for $i, j \in \{1, \dots, d\}$. The rest of the parameters remain unchanged.

In Figure 2.10, the histograms of the aggregated outcomes of systems with four and 20 agents are depicted. The aggregate eligible systems $\Lambda(\boldsymbol{x}_0 \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0)$ are presented in green, the intrinsic systems $\Lambda(\boldsymbol{X}^{\boldsymbol{\lambda},\boldsymbol{S}}_T)$ in yellow, the monetary systems $\Lambda(\boldsymbol{X}_T + \boldsymbol{k})$ in blue, and the original unacceptable systems $\Lambda(\boldsymbol{X}_T)$ in red. The vectors $\boldsymbol{\lambda}$ and $\boldsymbol{k}$ lie in $R^{\text{int}}_{\boldsymbol{S}}(\boldsymbol{X})$ and $R_{\boldsymbol{S}}(\boldsymbol{X}_T)$, respectively, and they are multiples of $\mathbf{1}$, that is, $\boldsymbol{\lambda} = \lambda\mathbf{1}$ and $\boldsymbol{k} = k\mathbf{1}$. In particular, the Expected Shortfall of $\Lambda(\boldsymbol{x}_0 \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0)$ is negative and the

Expected Shortfall of $\Lambda(\boldsymbol{X}_T^{\boldsymbol{\lambda},\boldsymbol{S}})$ and $\Lambda(\boldsymbol{X}_T+\boldsymbol{k})$ is approximately equal to 0. The dots of corresponding colour indicate the minimum of the support of the histogram.
First we notice that the minimum value of the aggregated intrinsic position (yellow dot) is greater than the one of the monetary position (blue dot). Since the expected shortfall of both positions is approximately 0, the mass in the tail of the distribution of the aggregate monetary position is more spread out. In this sense, the worst cases of the intrinsic positions are milder compared to the monetary positions. Furthermore, we observe that the distribution of the intrinsic system is more right-skewed. This means that the intrinsic system is more likely to repay more of its liabilities to society. From a regulatory perspective this is a valuable insight, as it demonstrates that changing the structure of a financial system can be more beneficial to society than elevating it by external capital.



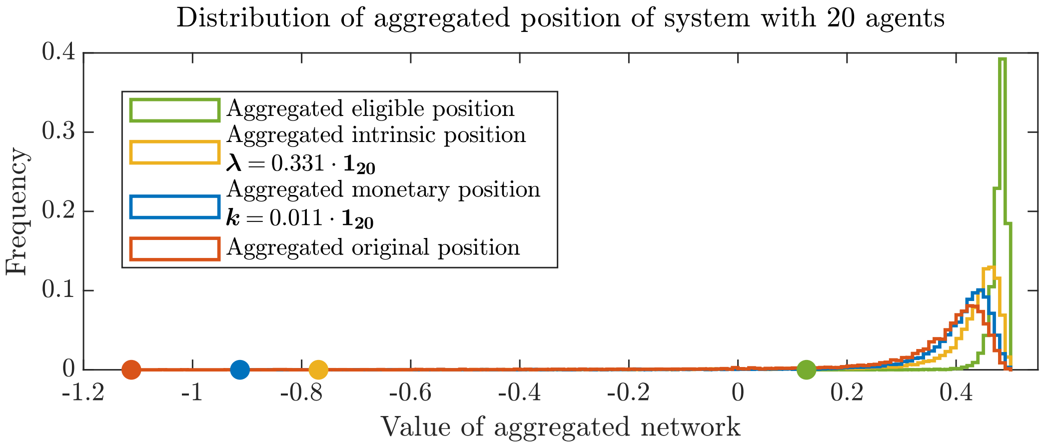


Figure 2.10: Value distributions of the aggregated systems.

From a preliminary statistical analysis, we observe that the variance of the aggregate intrinsic system is in general slightly smaller compared to the variance of the aggregate monetary system, whereas the expected value is slightly bigger. However, this needs to be verified in a more elaborate study. In particular, as the number of agents in the system increases, the symmetry of the risk measurement appears to become less pronounced. Therefore, points on the boundary of $R_{\boldsymbol{S}}^{\text{int}}(\boldsymbol{X})$, different from scales of $\mathbf{1}$, may be more appropriate to use.

In order to understand the effect of additionally incurred costs due to the implementation of the management actions, we have also studied the effects of a crude implementation of transaction costs and cost of debt. For the intrinsic measure we implemented transaction costs of 50 basis points of $\boldsymbol{\lambda} \odot \boldsymbol{x}_0$ once for selling the original position and once for buying the eligible asset. In the monetary case, we implemented transaction costs for buying the eligible asset and the cost of debt of 2.64% for holding the necessary external capital $\boldsymbol{k}$. In both cases, the resulting aggregate positions do not change considerably.

2.6 Conclusion and outlook

We proposed a novel approach to measuring systemic risk. We challenged the paradigm of using external capital injections to the financial system and suggested realistic management actions that fundamentally change the structure of the system such that it can become less volatile and less correlated. We developed two algorithms, one to approximate the boundary of intrinsic systemic risk measurements and the other to find specific minimal points without calculating the whole boundary. Furthermore, we derived a dual representation of intrinsic systemic risk measures. Finally, on the basis of numerical case studies, we demonstrated that intrinsic systemic risk measures are a useful tool to analyse and mitigate systemic risk.

We mention here possible extensions and further research avenues.
The notion of EARs associated with monetary systemic risk measures needs to be adapted to intrinsic systemic risk measures in a meaningful way. In particular, the absence of the upper set property calls for the introduction of further conditions to allow agents in the network to deviate from the suggested management actions in a controlled way without losing acceptability. Furthermore, it remains to be shown that this would allow to group together similar institutions to reduce the complexity of the model and computational time.
The Eisenberg-Noe model has seen numerous extensions which can also be applied to our framework. These include in particular extensions to random liabilities and the incorporation of illiquidity during fire sales. This is of interest, as intrinsic measures rely on selling parts of risky portfolios and buying safer assets. While the primary objective of intrinsic systemic risk measures is to mitigate risk to prevent crises, it is valuable to know which restrictions it faces during a crisis.
Moreover, it is necessary to study more asymmetrically structured networks and develop 'fair' allocation rules, as it may for example be more adequate if players with higher liability to society are required to change their position more.
Furthermore, it would be valuable to conduct an empirical case study including systemically relevant banks and their actual liabilities towards each other and towards participants of the greater economy.

2.7 Appendix

2.7.1 Proofs

In the following, we prove the assertions made in Proposition 2.2.6.

Proof of Proposition 2.2.6. Let $\boldsymbol{X}_T \in L_d^p$ and $\boldsymbol{\ell} \in \mathbb{R}^d$. $\boldsymbol{S}$-additivity follows directly from

$$\begin{aligned} R_{\boldsymbol{S}}(\boldsymbol{X}_T + \ell \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0) &= \Big\{ \boldsymbol{k} \in \mathbb{R}^d \mid \Lambda(\boldsymbol{X}_T + (\boldsymbol{\ell} + \boldsymbol{k}) \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0) \in \mathcal{A} \Big\} \\ &= \Big\{ \hat{\boldsymbol{k}} \in \mathbb{R}^d \mid \Lambda(\boldsymbol{X}_T + \hat{\boldsymbol{k}} \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0) \in \mathcal{A} \Big\} - \boldsymbol{\ell} = R_{\boldsymbol{S}}(\boldsymbol{X}_T) - \boldsymbol{\ell}. \end{aligned}$$

Monotonicity follows from monotonicity of $\mathcal{A}$ and Λ. For $\boldsymbol{X}_T, \boldsymbol{Y}_T \in L_d^p$ with $\boldsymbol{X}_T \leq \boldsymbol{Y}_T$ $\mathbb{P}$-a.s. and $\boldsymbol{k} \in R_{\boldsymbol{S}}(\boldsymbol{X}_T)$ we have $\Lambda(\boldsymbol{Y}_T + \boldsymbol{k} \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0) \geq \Lambda(\boldsymbol{X}_T + \boldsymbol{k} \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0) \in \mathcal{A}$. This implies $\boldsymbol{k} \in R_{\boldsymbol{S}}(\boldsymbol{Y}_T)$.
$\boldsymbol{S}$-additivity and monotonicity together imply that the values of $R_{\boldsymbol{S}}$ are upper sets. Let $\boldsymbol{X}_T \in L_d^p$ and $\boldsymbol{y} \in \mathbb{R}_+^d$. Then $\boldsymbol{X}_T - \boldsymbol{y} \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0 \leq \boldsymbol{X}_T$ and we have

$$R_{\boldsymbol{S}}(\boldsymbol{X}_T) + \boldsymbol{y} = R_{\boldsymbol{S}}(\boldsymbol{X}_T - \boldsymbol{y} \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0) \subseteq R_{\boldsymbol{S}}(\boldsymbol{X}_T).$$

Since the above holds for any $\boldsymbol{y} \in \mathbb{R}_+^d$, the claim follows.
For positive homogeneity, assume that $\mathcal{A}$ is a cone, Λ is positively homogeneous and let $X \in L_d^p$ and $c > 0$. Notice that in this case $\Lambda(c\boldsymbol{X}_T + \boldsymbol{k} \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0) \in \mathcal{A}$ is equivalent to $\Lambda(\boldsymbol{X}_T + \frac{1}{c}\boldsymbol{k} \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0) \in \mathcal{A}$. Therefore,

$$\begin{aligned} R_{\boldsymbol{S}}(c\boldsymbol{X}_T) &= \Big\{ \boldsymbol{k} \in \mathbb{R}^d \mid \Lambda(\boldsymbol{X}_T + \frac{1}{c}\boldsymbol{k} \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0) \in \mathcal{A} \Big\} \\ &= c \Big\{ \hat{\boldsymbol{k}} \in \mathbb{R}^d \mid \Lambda(\boldsymbol{X}_T + \hat{\boldsymbol{k}} \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0) \in \mathcal{A} \Big\} = cR_{\boldsymbol{S}}(\boldsymbol{X}_T). \end{aligned}$$

Finally, for properties (v) and (vi) we assume that $\mathcal{A}$ is convex and Λ is concave and let $\boldsymbol{X}_T, \boldsymbol{Y}_T \in L_d^p$, $\alpha \in [0,1]$. To show convexity, let $\boldsymbol{x} \in R_{\boldsymbol{S}}(\boldsymbol{X}_T)$, $\boldsymbol{y} \in R_{\boldsymbol{S}}(\boldsymbol{Y}_T)$. We get

$$\begin{aligned} &\Lambda\big(\alpha\boldsymbol{X}_T + (1-\alpha)\boldsymbol{Y}_T + (\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0\big) \\ &\quad = \Lambda\big(\alpha(\boldsymbol{X}_T + \boldsymbol{x} \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0) + (1-\alpha)(\boldsymbol{Y}_T + \boldsymbol{y} \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0)\big) \\ &\quad \geq \alpha\Lambda(\boldsymbol{X}_T + \boldsymbol{x} \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0) + (1-\alpha)\Lambda(\boldsymbol{Y}_T + \boldsymbol{y} \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0) \in \mathcal{A}, \end{aligned}$$

where the element inclusion is implied by the convexity of $\mathcal{A}$. By monotonicity of $\mathcal{A}$ the assertion follows.
To show that $R_{\boldsymbol{S}}(\boldsymbol{X}_T)$ has convex values, let $\boldsymbol{k}, \boldsymbol{\ell} \in R_{\boldsymbol{S}}(\boldsymbol{X}_T)$. Notice that

$$\boldsymbol{X}_T + (\alpha\boldsymbol{k} + (1-\alpha)\boldsymbol{\ell}) \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0 = \alpha(\boldsymbol{X}_T + \boldsymbol{k} \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0) + (1-\alpha)(\boldsymbol{X}_T + \boldsymbol{\ell} \odot \boldsymbol{S}_T \oslash \boldsymbol{s}_0).$$

The assertion follows as in the proof of (v). □

In the following, we prove Lemma 2.3.15.

Proof of Lemma 2.3.15. Assume by contradiction that for some $\epsilon > 0$ with $A_{k-\epsilon} \subset A_k - \frac{\epsilon}{d}\boldsymbol{1}$ there exists a $\delta > \epsilon$ and an $\boldsymbol{x}_\delta \in A_{k-\delta}$ such that $\boldsymbol{x}_\delta \notin A_{k-\epsilon} - \frac{\delta-\epsilon}{d}\boldsymbol{1}$. Since A is closed, $\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x} \in A_{k-\epsilon}} \|\boldsymbol{x} - \boldsymbol{x}_\delta\|$ exists and is contained in $A_{k-\epsilon}$. Notice that by assumption, $\hat{\boldsymbol{x}} \neq \boldsymbol{x}_\delta + \frac{\delta-\epsilon}{d}\boldsymbol{1}$. Therefore there exists $\boldsymbol{y} \in \mathbb{R}^d \setminus \{\boldsymbol{0}\}$ with $\boldsymbol{y}^\intercal\boldsymbol{1} = 0$ such that $\boldsymbol{x}_\delta = \hat{\boldsymbol{x}} - \frac{\delta-\epsilon}{d}\boldsymbol{1} + \boldsymbol{y}$. Furthermore, notice that $\|\boldsymbol{x}_\delta - (\hat{\boldsymbol{x}} + \beta\boldsymbol{y})\| = \|(1-\beta)\boldsymbol{y} - \frac{\delta-\epsilon}{d}\boldsymbol{1}\|$ is decreasing in $\beta \in [0,1]$. In particular, for $\beta \in (0,1]$, $\hat{\boldsymbol{x}} + \beta\boldsymbol{y}$ cannot lie in $A_{k-\epsilon}$, and since $(\hat{\boldsymbol{x}} + \beta\boldsymbol{y})^\intercal\boldsymbol{1} = k - \epsilon$, $\hat{\boldsymbol{x}} + \beta\boldsymbol{y} \notin A$.
Now let $\boldsymbol{x}_\epsilon = \frac{\epsilon}{\delta}\boldsymbol{x}_\delta + (1 - \frac{\epsilon}{\delta})(\hat{\boldsymbol{x}} + \frac{\epsilon}{d}\boldsymbol{1}) = \hat{\boldsymbol{x}} + \frac{\epsilon}{\delta}\boldsymbol{y}$. From the previous observation, we see that $\boldsymbol{x}_\epsilon \notin A$. However by assumption, $\hat{\boldsymbol{x}} + \frac{\epsilon}{d}\boldsymbol{1} \in A_k$. So by convexity of A, $\boldsymbol{x}_\epsilon$ must lie in A. This is a contradiction and therefore, such an $\boldsymbol{x}_\delta$ cannot exist. □

2.7.2 Note on increasing bilateral liabilities

This section complements the discussion around Figure 2.9. As bilateral liabilities between two agents increase, both systemic risk measures increase. The following figure illustrates the 'limit set' of the intrinsic systemic risk measure. For monetary systemic risk measurements it is not clear from our simulations whether they converge to a half-space or not.

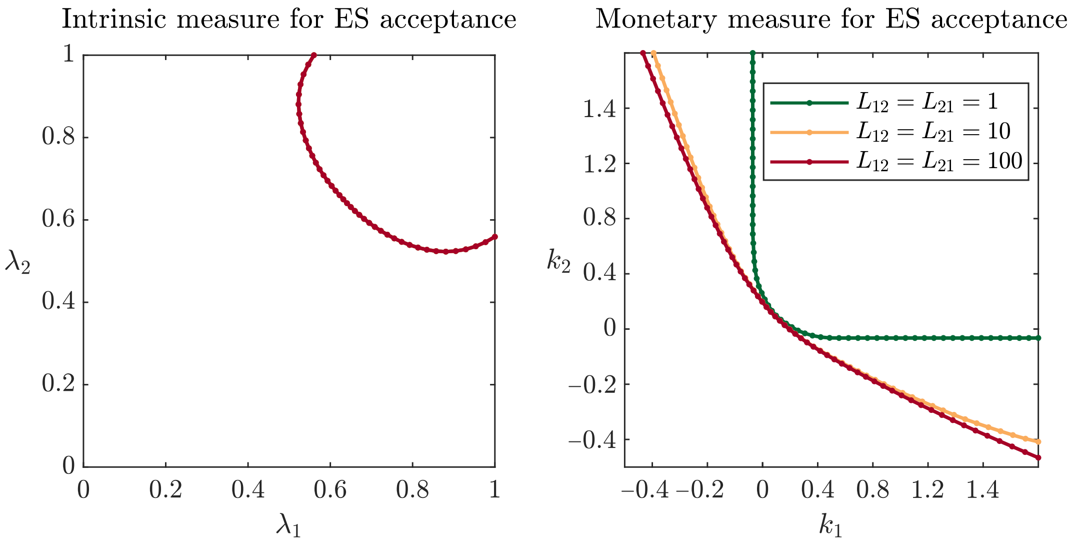


Figure 2.11: Visualisation of the limit set for increasing bilateral liabilities.

Acknowledgements The authors are thankful to Gabriela Kováčová for assisting in the proof of Lemma 2.3.15.

Chapter 3

Optimal risk-sharing across a network of insurance companies

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Risk transfer is a key risk and capital management tool for insurance companies. Transferring risk between insurers is used to mitigate risk and manage capital requirements. We investigate risk transfer in the context of a network environment of insurers and consider capital costs and capital constraints at the level of individual insurance companies. We demonstrate that the optimisation of profitability across the network can be achieved through risk transfer. Considering only individual insurance companies, there is no unique optimal solution and, a priori, it is not clear which solutions are fair. However, from a network perspective, we derive a unique fair solution in the sense of cooperative game theory. Implications for systemic risk are briefly discussed.

3.1 Introduction

Insurance companies can transfer their risk to other entities, for example reinsurance companies, to mitigate loss potential and reduce the cost of capital. For a single insurance company optimising risk transfer is straightforward as it means to compare the price for risk transfer to its benefits in terms of mitigated losses and reduced cost of risk capital. For a network of insurers, however, the optimisation can take different perspectives and therefore the meaning of optimising risk transfer in a network is not clear. In this chapter, we investigate risk transfer in the context of a network of insurance companies. We use a general framework with passive and active reinsurance, that means insurers can transfer their risk to reinsurance companies and also between themselves.
We take an economic point of view and consider capital costs and capital constraints at the level of individual insurers. A key result is that while there is no unique solution optimising risk transfer for all insurance companies at the individual level, there exists a unique, fair and optimal solution in the sense of cooperative game theory if the network perspective is adopted.

Optimal risk-sharing between insurance and reinsurance companies has been considered by various authors. Research either takes the perspective of individual entities (see for example de Finetti, 1940; Kaluszka, 2001; Centeno, 2002) or, more recently, insurance groups (see for example Kupper and Filipović, 2007; Keller, 2007; Mayer, Kull, Keller, and Portmann, 2009; Haier, Molchanov, and Schmutz, 2016; Asimit, Badescu, Haberman, and Kim, 2016). Because of the ownership structure and related transparency of risk exposures, an optimisation of risk transfer for insurance groups can start from the comprehensive group standpoint. Related research results are not only relevant for the economics of insurance groups but also for insurance supervision and regulation. Indeed, recent findings point towards a significant impact of regulatory frameworks on optimising risk transfer and systemic risk as for example discussed by Asimit, Badescu, and Tsanakas (2013). Our starting point in this chapter differs in the sense that we do not assume a group structure but consider risk transfer across a network of insurers. Optimal risk transfer thus is not considered from an overarching insurance group perspective but from the perspective of a network of individual insurers and game theory.

Risk transfer across a network of insurance companies has been considered by other authors as well. For example, Hamm, Knispel, and Weber (2019) show that risk transfer can significantly decrease overall capital requirements when using Value-at-Risk based risk measures. Other approaches aim to minimise the aggregated risk in a system by redistributing its components across participating agents, see for example the work of Liebrich and Svindland (2019) for an acceptance set based approach. A utility based framework has been suggested by Biagini, Doldi, Fouque, Frittelli, and Meyer-Brandis (2021).
The chapter builds in part on the work of Ettlin (2018) and generalises results on optimisation of risk transfer for two insurers as discussed by Kull (2009) to multiple insurers. Furthermore, it draws a connection to cooperative game theory and related concepts of capital allocation as developed by Denault (1999).

The rest of this chapter is organised as follows. In Section 3.2, we define monetary risk measures and the Conditional Expected Shortfall. Furthermore, we explain the notion of risk capital, capital costs, capital allocation, risk aggregation, and related economics. In Section 3.3, we draw links between these concepts and risk transfer by introducing a network of insurance companies. We define the optimisation of risk transfer for individual insurance companies and the network in Section 3.4. We discuss that there is no unique optimal solution without further constraints. However, using principles from cooperative game theory, we can identify a unique and in that sense fair solution to the optimisation of risk transfer. We conclude with a discussion of implications for insurance groups and the systemic risk debate in Section 3.6.

3.2 Economics of risk transfer

The inherent uncertainty of the future as it manifests itself, for example in uncertain loss experience, is usually referred to as risk. Mathematically, the impact of the uncertainty of the future is described by random variables. To this end, let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space. Let $\mathcal{X}$ be some real topological vector space equipped with an appropriate preorder $\leq$. For us $\mathcal{X}$ is a subset of $L^0(\Omega, \mathcal{F}, \mathbb{P})$, the space of all equivalence classes with respect to $\mathbb{P}$-a.s. equality of measurable maps from $(\Omega, \mathcal{F})$ to $\mathbb{R}$ with its Borel σ-algebra. Since we will work with Expected Shortfall, we will usually choose $\mathcal{X} = L^p(\Omega, \mathcal{F}, \mathbb{P})$, for $p \in [1, \infty]$. Furthermore, we will work with continuous random variables.

3.2.1 Measuring risk

As already discussed in Chapter 1, Artzner, Delbaen, Eber, and Heath (1999) introduce four axioms a practicable risk measure should fulfil and define what is known as coherent risk measures on the space of bounded random variables. In this chapter, we will adapt the framework of McNeil, Frey, and Embrechts (2005) and define risk measures on random variables which represent losses. This means that for our risk measures losses are positive. Notice that therefore the four coherency axioms differ slightly from the ones presented in Section 1.2.2. In particular, instead of decreasing monotonicity we have increasing monotonicity, and for $L \in \mathcal{X}$, $\ell \in \mathbb{R}$ the translation property reads $\rho(L+\ell) = \rho(L) + \ell$.

In the following, we will use the (right-tail) Expected Shortfall or Tail Value-at-Risk as the coherent risk measure.[1] For a loss L with $\mathbb{E}[|L|] < \infty$ the Expected Shortfall at a confidence level $\alpha \in (0,1)$ is defined as

$$\mathrm{ES}_\alpha(L) = \frac{1}{1-\alpha} \int_\alpha^1 \mathrm{VaR}_\beta(L)\, \mathrm{d}\beta\,,$$

where $\mathrm{VaR}_\beta \colon \mathcal{X} \to \mathbb{R}$ denotes the Value-at-Risk at a probability level β. The Value-at-Risk of a loss random variable L with continuous and strictly increasing cumulative distribution function $F_L : \mathbb{R} \to [0,1]$ is given by its inverse, $\mathrm{VaR}_\alpha(L) = F_L^{-1}(\alpha)$.[2] It can be shown, see for example (McNeil, Frey, and Embrechts, 2005, Lemma 2.16), that in this case, the Expected Shortfall coincides with the Conditional Value-at-Risk, which is defined as the expectation of L conditioned on that L exceeds its Value-at-Risk at level α,

$$\mathrm{ES}_\alpha(L) = \mathrm{CVaR}_\alpha(L) = \mathbb{E}[L \mid L \geq \mathrm{VaR}_\alpha(L)]\,.$$

Notice that in general, this equation does not hold for discontinuous loss distributions as described in (McNeil, Frey, and Embrechts, 2005, Remark 2.17) and proved by Acerbi and Tasche (2002, Proposition 3.2). We will use this equality to draw a connection to the related conditional risk measure termed Conditional Expected Shortfall $\mathrm{CES}_{\alpha,Y}$. It is defined as the expectation of L conditioned on another loss random variable Y at a probability level α,

$$\mathrm{CES}_{\alpha,Y}(L) = \mathbb{E}[L \mid Y \geq \mathrm{VaR}_\alpha(Y)]\,.$$

This conditional risk measure is favourable due to its properties, in particular, its additivity which allows for an additive and consistent allocation of risk and capital. A wider context is given by Venter (2004).

3.2.2 Risk capital and capital costs

For insurance companies capital provides protection against insolvency or ruin due to large unexpected losses. Insurers hold certain amounts of capital, termed risk-based capital, in order to ensure the ability to sustain significant, unexpected losses. Determining the risk-based capital for a specific risk is usually done by applying risk measures. For instance, the supervisory framework of Solvency II prescribes the usage of Value-at-Risk and the Swiss Solvency Test of Expected Shortfall as risk measures for calculating regulatory capital requirements.

The exact definition of risk-based capital and the different ways to calculate it is a widely discussed topic. Definitions depend on scopes ranging from regulatory aspects

[1] For a proof that the Expected Shortfall is indeed coherent see for example (McNeil, Frey, and Embrechts, 2005, Proposition 6.9).

[2] For more details see for example (McNeil, Frey, and Embrechts, 2005, Definition 2.10 and Definition 2.12).

to company-internal capital management and performance measurement. Considering a portfolio with underwriting profits reflected by the random variable $X \in \mathcal{X}$ and a coherent risk measure ρ, the risk-based capital corresponding to the portfolio X is given by

$$\mathrm{RBC}(X) = \rho(-X)\,.$$

Notice that since X represents returns, we use $-X$ inside the risk measure. Holding capital comes at a cost proportional to the amount of risk-based capital,

$$C(X) = \eta\mathrm{RBC}(X)\,,$$

where the cost of capital rate $\eta \in (0, 1)$ depends on various factors. These factors include the capital structure of the company, the type and riskiness of the business, and other specificities. The cost of capital rate η for an insurer typically lies in the range of 4% to 15%, see for example the cost of capital study by KPMG (2019). Any meaningful economic analysis of the economics of insurance and risk transfer should take this cost of capital into account.

We consider now the case of a simplified non-life insurer that underwrites short-tail business over a single time period. Neglecting the cost of capital, the net underwriting profit is given by

$$U = P - L\,,$$

where P stands for the premium income and L for the accumulated gross loss amount. To keep things simple, consider the premium $P \geq 0$ to be a constant. The underwriting profit Z, after including the cost of capital, is defined as

$$Z = U - C(Z) = U - \eta\mathrm{RBC}(Z) = U - \eta\rho(-Z)\,. \tag{3.1}$$

Here, $\mathrm{RBC}(Z)$ stands for the amount of capital allocated to support the risk related to Z.

In general, insurance companies hold risk-based capital only up to a probability corresponding to a company-specific risk appetite level α or regulatory requirements and not to the full range of potential adverse results. For example, the Swiss Solvency Test prescribes $\alpha = 0.99$. Furthermore, the determination of the risk-based capital $\mathrm{RBC}(Z)$ also depends on the situation relative to which it is defined.

Lemma 3.2.1. *The recursive formula for Z in Equation* (3.1) *can be explicitly stated as*

$$Z = U - \frac{\eta}{1-\eta}\rho(-U)\,.$$

Proof. Inserting Z itself in Equation (3.1) and using the translation property of ρ, we get the identity $Z = U - \eta\rho(-U) - \eta^2\rho(-Z)$. Writing $U = \eta U + (1-\eta)U$ and then using Equation (3.1) again, we arrive at

$$Z = (1-\eta)U - \eta\rho(-U) + \eta(U - \eta\rho(-Z)) = (1-\eta)U - \eta\rho(-U) + \eta Z\,.$$

Solving for Z concludes the proof. $\square$

Hence, to calculate the risk-based capital of Z we apply the risk measure to $-Z$ and use the translation property to get

$$\rho(-Z) = \rho(-U)\Big(1 + \frac{\eta}{1-\eta}\Big) = \frac{\rho(-U)}{1-\eta}\,. \tag{3.2}$$

As mentioned before, we choose the Expected Shortfall as our risk measure so that

$$\mathrm{RBC}(Z) = \frac{\mathrm{ES}_\alpha(-U)}{1-\eta}. \tag{3.3}$$

We will assume capital costs η and risk appetite levels α to be fixed. Both of these assumptions are significant simplifications which, however, can be motivated by similar regulatory constraints.

Remark 3.2.2. *Notice that we can use the recursive formula in Equation* (3.1) *directly to calculate the risk-based capital of Z as*

$$\rho(-Z) = \rho(-U + \eta\rho(-Z)) = \rho(-U) + \eta\rho(-Z)\,.$$

Solving this with respect to $\rho(-Z)$, we arrive at Equation (3.2).

3.2.3 Capital allocation and capital aggregation

For various purposes ranging from management structure to risk and performance measurement, insurance companies group their business into sub-portfolios consisting of business of similar risk characteristics, known as "lines of business". For company-level risk and performance management the quantification of the contribution of risk-based capital at the sub-portfolio level to the overall risk capital, or vice versa, the allocation of overall risk-based capital to a sub-portfolio, is fundamental. For a review of capital allocation principles see for example (Venter, 2004) and the references therein.

Consider the case of a simplified non-life insurer. We want to split its business with underwriting profit Z given in (3.1) into $n \in \mathbb{N}$ business lines. For $i \in \{1, \dots, n\}$ their underwriting profit should be given by

$$Z_i = U_i - \eta \mathrm{RBC}_Z(Z_i)\,, \tag{3.4}$$

where $U = \sum_{i=1}^n U_i$ and $\mathrm{RBC}_Z(Z_i)$ stands for the allocated risk-based capital of Z_i, but with respect to the total economic profit Z. Here, we explicitly choose the Conditional Expected Shortfall,

$$\mathrm{RBC}_Z(Z_i) = \mathrm{CES}_{\alpha,-Z}(-Z_i)\,. \tag{3.5}$$

So the risk-based capital of Z_i is allocated dependent on its contribution to the overall risk. However, to ensure that we have indeed $Z = \sum_{i=1}^n Z_i$, we must in particular have $\mathrm{RBC}(Z) = \sum_{i=1}^n \mathrm{RBC}_Z(Z_i)$. The two lemmas in Section 3.7.1 show that the Conditional Expected Shortfall is indeed consistent with our requirements.

In a strict sense, allocating capital is only meaningful for risks that are pooled in a single portfolio, or on the balance sheet of a single insurer. Only in this case, diversification effects materialise economically through reducing capital costs. If individual insurance companies are considered which operate on a stand-alone basis, then diversification may be present in the risk they underwrite. However, from an economic perspective this diversification may not be realised as capital costs are incurred at the level of individual insurers. Thus, rather than considering an individual insurer we consider a network of insurers each determining its risk-based capital $\mathrm{RBC}(Z_i)$ on a stand-alone basis. Assuming the coherence of the underlying risk measure, subadditivity implies

$$\mathrm{RBC}_{\mathrm{Market}} \coloneqq \mathrm{RBC}\left(\sum_{i=1}^n Z_i\right) \le \sum_{i=1}^n \mathrm{RBC}(Z_i) \eqqcolon \mathrm{RBC}_{\mathrm{Network}}\,, \tag{3.6}$$

where $\mathrm{RBC}_{\mathrm{Market}}$ denotes the risk-based capital for a situation in which all risks are pooled, the market portfolio, and $\mathrm{RBC}_{\mathrm{Network}}$ stands for the sum of stand-alone risk-based capital. The inequality in (3.6) implies that the aggregated stand-alone capital cannot be less than the risk-based capital $\mathrm{RBC}_{\mathrm{Market}}$ for fully pooled risks. The difference $\mathrm{RBC}_{\mathrm{Network}} - \mathrm{RBC}_{\mathrm{Market}}$ is non-negative and it is related to the diversification achieved by the network of insurance companies relative to the overall market portfolio. The higher the diversification achieved by the insurers in the network, the lower this difference. Full diversification relative to the market portfolio is reached if $\mathrm{RBC}_{\mathrm{Network}} - \mathrm{RBC}_{\mathrm{Market}} = 0$.

Next, we consider the profitability distributions at the market level,

$$Z_{\mathrm{Market}} = \sum_{i=1}^{n} (P_i - L_i) - \eta \mathrm{RBC}\left(\sum_{i=1}^{n} Z_i\right) = \sum_{i=1}^{n} (P_i - L_i) - \eta \mathrm{RBC}_{\mathrm{Market}}\,,$$

and at the level of aggregated stand-alone profitability,

$$Z_{\mathrm{Network}} = \sum_{i=1}^{n} P_i - L_i - \eta \mathrm{RBC}(Z_i) = \sum_{i=1}^{n} (P_i - L_i) - \eta \mathrm{RBC}_{\mathrm{Network}}\,.$$

In terms of expected profitability, Equation (3.6) implies that

$$\mathbb{E}[Z_{\mathrm{Market}}] = \sum_{i=1}^{n} \mathbb{E}\left[P_i - L_i\right] - \eta \mathrm{RBC}_{\mathrm{Market}} \geq \sum_{i=1}^{n} \mathbb{E}\left[P_i - L_i - \eta \mathrm{RBC}(Z_i)\right] = \mathbb{E}[Z_{\mathrm{Network}}]\,.$$

The difference between $\mathbb{E}[Z_{\mathrm{Market}}]$ and $\mathbb{E}[Z_{\mathrm{Network}}]$ is driven by the level of diversification achieved by the insurance companies which are part of the network and is equal to

$$\mathbb{E}[Z_{\mathrm{Market}}] - \mathbb{E}[Z_{\mathrm{Network}}] = \eta\big(\mathrm{RBC}_{\mathrm{Network}} - \mathrm{RBC}_{\mathrm{Market}}\big)\,.$$

This means that $\mathbb{E}[Z_{\mathrm{Market}}] - \mathbb{E}[Z_{\mathrm{Network}}]$ is the difference in the cost of capital for the fully diversified market portfolio and the sum of the stand-alone cost of capital at the level of individual insurance companies. The question then arises whether risk transfer between insurers can improve profitability at the level of individual insurers and the network as a whole. To answer this question, the economics of risk transfer needs to be clarified.

3.3 Risk transfer across a network of insurers

Pooling and sharing risk generally increases diversification and thus may reduce risk-based capital and related capital costs. Beyond risk management purposes and meeting capital constraints, transferring risk may also be driven by purely economic considerations as for an insurer the cost of transferring risk may be lower than the cost of keeping the risk on its balance sheet. Assuming an adequate premium for risk transfer is paid, a "win-win situation", whereby both parties increase their profitability, can occur. In essence, risk transfer will enable insurers to access and share the diversification inherent to the network.

3.3.1 Risk transfer network

Risk transfer usually takes place between an insurer and a reinsurer. We adopt the more general view that insurance companies can transfer risk between themselves. In practice, this situation occurs if insurance companies participate in passive reinsurance, that is, transferring risk to a reinsurer, active reinsurance, that is, accepting risk from another insurer, or in insurance groups which optimise capital requirements by internal risk transfer.

Including risk transfer into the model of the simplified non-life insurer introduced in Section 3.2.2, the underwriting profit after risk transfer is given by

$$Z^\star = U - \eta \mathrm{RBC}(Z^\star) = P - L + R - P^\star - \eta \mathrm{RBC}(Z^\star)\,,$$

where R represents the reinsurance recoveries due to risk transfer and $P^\star$ is the reinsurance premium paid for the risk transfer.

To structure the problem, we represent our network of insurers by a graph. The vertices of the graph represent the insurers and the edges represent their risk transfers. For each risk transfer related quantity we use two indices, i and j, to indicate the direction of risk transfer. The first index denotes the risk assuming party and the second index the risk ceding party. For example, $P^\star_{ij}$ denotes the premium that insurer j pays insurer i, whereas R_{ij} denotes the recovery that insurer i pays insurer j. See Figure 3.1 for an illustration. The arrows indicate the direction of the risk transfer.

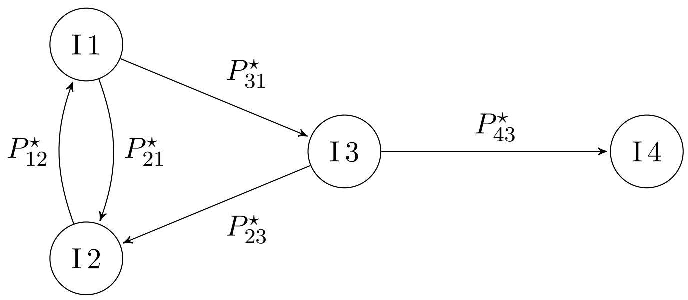

Figure 3.1: System of insurers and network of risk transfer.

3.3.2 The price of transferring risk

Principles defining the cost of risk transfer are generally referred to as premium principles. Most premium principles split the premium into the expected loss originating from risk transfer and the loading. Common loadings principles are, for example, the Expected Value principle or the Variance principle. For a review see for example (Daykin, Pentikainen, and Pesonen, 1993) and for the broader context see (Dacorogna, 2018). In line with economic principles laid out in Section 3.2.2, we define the loading as the additional cost of capital incurred due to the risk transfer by the risk accepting party. Notice that by this definition, it is the risk accepting party which defines the price for risk transfer. This is a simplification as in reality, pricing of risk transfer is often negotiated by the risk transferring and risk accepting parties.

Following concepts discussed in Section 3.2.2 and Section 3.2.3, the net underwriting profit of the risk accepting insurer i after risk transfer from insurer j is

$$Z^\star_i = Z_i + P^\star_{ij} - R_{ij} - \eta \mathrm{RBC}_{Z^\star_i}(P^\star_{ij} - R_{ij})\,. \tag{3.7}$$

The risk-based capital $\mathrm{RBC}_{Z^\star_i}$ allocated to the risk transfer from insurer j to i relative to the profit $Z^\star_i$ is defined as

$$\mathrm{RBC}_{Z^\star_i}(P^\star_{ij} - R_{ij}) = \mathrm{CES}_{\alpha,-Z^\star_i}(R_{ij} - P^\star_{ij})\,. \tag{3.8}$$

Based on the discussion above and Equation (3.7), we define the premium for risk transfer as

$$P_{ij}^{\star} = \mathbb{E}[R_{ij}] + \eta \mathrm{RBC}_{Z_i^{\star}}(P_{ij}^{\star} - R_{ij})\,.$$

This premium principle states that the premium $P_{ij}^{\star}$ for the risk transfer from insurer j to insurer i is equal to the expected loss due to paid recoveries $\mathbb{E}[R_{ij}]$ plus the loading $\eta \mathrm{RBC}_{Z_i^{\star}}(P_{ij}^{\star} - R_{ij})$. The latter reflects the incurred capital costs for the risk accepting insurer i. Together with Equation (3.8) and the linearity of the Conditional Expected Shortfall this leads to

$$P_{ij}^{\star} = \frac{1}{1+\eta}\Big(\mathbb{E}[R_{ij}] + \eta\, \mathrm{CES}_{\alpha,-Z_i^{\star}}(R_{ij})\Big)\,. \tag{3.9}$$

3.3.3 Proportional risk transfer

While risk transfer can take proportional and non-proportional forms, all relations in Section 3.3.2 are of general form. For the sake of simplicity and analytical tractability, we will consider the case of proportional risk transfer here. If non-proportionality is assumed, analytical expressions (Equation (3.10) to Equation (3.13)) for non-proportional risk transfer are not readily available. However, because of the universality of our approach, a numerical evaluation is feasible and it is left for later work. Here, parties transfer proportions of their risk exposures for an adequate premium. Proportional risk transfer can be characterised by one parameter, the proportion of risk that is ceded from one to another party or the related retention of risk that is kept.
To increase readability, we will omit the boundaries of the running variable when summing over insurers. So for some fixed $i \in \{1,\ldots,n\} \subset \mathbb{N}$, we define

$$\sum_{j\neq i} a_j := \sum_{j\in J_i} a_j\,,$$

where $J_i = \{j \in \mathbb{N} \mid 1 \leq j \leq n,\, j \neq i\}$.
For a network of $n \in \mathbb{N}$ insurers mutually transferring risk, the proportional risk transfer can be represented by an $(n \times n)$ - matrix $C \in [0,1]^{n\times n}$. For $i, j \in \{1,\ldots,n\}$, $i \neq j$, let C_{ij} denote the portion of the risk L_j insurer j is transferring to insurer i, or equivalently, the proportion of risk L_j the insurer i is accepting from insurer j. Furthermore, set $C_{ii} = 1 - \sum_{j\neq i} C_{ji}$. Then the matrix becomes

$$C = \begin{pmatrix} 1-\sum_{j\neq 1} C_{j1} & C_{12} & \ldots & C_{1n} \\ C_{21} & 1-\sum_{j\neq 2} C_{j2} & & C_{2n} \\ \vdots & & \ddots & \vdots \\ C_{n1} & \ldots & & 1-\sum_{j\neq n} C_{jn} \end{pmatrix}. \tag{3.10}$$

Notice that the matrix C satisfies the following properties.

- $R_{ij}(C) = C_{ij}L_j$ (for $i \neq j$) corresponds to losses incurred by insurer i due to risk transfer from insurer j to insurer i.
- $R_{ji}(C) = C_{ji}L_i$ (for $i \neq j$) corresponds to recoveries received by insurer i due to risk transfer from insurer i to insurer j.
- Elements $C_{ii} = 1 - \sum_{j\neq i} C_{ji}$ represent the retention levels for risk L_i of insurer i. Assuming that no shorting of risk positions is possible, we have $\sum_{j\neq i} C_{ji} \leq 1$ and $C_{ii} \geq 0$.

- For each insurer i we have $\sum_{j=1}^{n} C_{ji} = 1$.

Notice that the matrix C can be decomposed into the adjacency matrix characterising risk transfer where the diagonal elements are equal to zero, and the diagonal matrix representing risk retention.

The net underwriting profit of insurer $i \in \{1, \ldots, n\}$ taking into account all risk transfer is given by

$$Z_i^\star(C) = P_i - L_i^\star(C) - \sum_{j\neq i}(P_{ji}^\star(C) - P_{ij}^\star(C)) - \eta \mathrm{RBC}(Z_i^\star(C)) \,, \tag{3.11}$$

where the total net loss $L_i^\star(C)$ incurred by insurer i is the operational loss adjusted by the received and paid recoveries,

$$\begin{aligned} L_i^\star(C) &\coloneqq L_i - \sum_{j\neq i}(R_{ji}(C) - R_{ij}(C)) = L_i\Big(1 - \sum_{j\neq i} C_{ji}\Big) + \sum_{j\neq i} C_{ij} L_j \\ &= L_i C_{ii} + \sum_{j\neq i} C_{ij} L_j = \sum_{j=1}^{n} C_{ij} L_j \,. \end{aligned} \tag{3.12}$$

$P_{ij}^\star(C)$ and $P_{ji}^\star(C)$ are defined as in Equation (3.9). In analogy to Remark 3.2.2, we derive the risk-based capital as

$$\mathrm{RBC}(Z_i^\star(C)) = \frac{1}{1-\eta} \mathrm{ES}_\alpha \Big(L_i^\star(C) - P_i + \sum_{j\neq i}(P_{ji}^\star(C) - P_{ij}^\star(C)) \Big). \tag{3.13}$$

Notice that we use the Expected Shortfall as we are considering the situation of a single insurer.

Expressions (3.11) and (3.13) are fundamental. They describe the economics of proportional risk transfer and related capital requirements at the level of individual insurer i as well as for the network itself. In line with (3.6), the overall amount of capital held by all insurance companies part of the network to cover their capital requirements is

$$\mathrm{RBC}_{\mathrm{Network}}^\star(C) = \sum_{i=1}^{n} \mathrm{RBC}(Z_i^\star(C)) \,.$$

Furthermore,

$$\mathbb{E}[Z_{\mathrm{Network}}^\star(C)] = \sum_{i=1}^{n} \mathbb{E}[Z_i^\star(C)] \tag{3.14}$$

is the total expected net underwriting profit generated by all insurers part of the network.

3.4 Optimising risk transfer

Risk transfer affects the profitability of risk exchanging parties as demonstrated in (3.7) and it also impacts the overall profitability of the network in (3.14). This gives rise to a series of questions. Can risk transfer be optimised such that it benefits all risk exchanging parties and the network overall? Will all parties be able to maximise their profitability through risk transfer? Is there a unique “optimal” or “fair” solution?

3.4.1 The network perspective

There are in principle two different approaches to investigate the questions from above. The first approach relies on agent-based models where risk pricing plays the role of a utility function. The second approach depends on network-based models that describe dependencies and interactions between market participants from a more global point of view. We take the latter perspective and start our analysis from the expressions (3.11) and (3.13). Consider the maximisation problem of the profitability of the network $\mathbb{E}[Z^\star_{\text{Network}}(C)]$ with respect to $C \in [0,1]^{n\times n}$,

$$\max_{C\in[0,1]^{n\times n}} \mathbb{E}[Z^\star_{\text{Network}}(C)]\,. \tag{3.15}$$

This maximisation problem can be simplified by observing that the overall amount of premiums paid for risk transfer amongst the insurers disappears. The reason for this is that the network is closed, that is, any premium paid by an insurer will be received by another. Furthermore, we notice that the aggregate total loss of all insurers after risk transfer is equal to the overall gross loss.

Lemma 3.4.1. *The total net underwriting profit generated by all insurers in the network is given by*

$$\sum_{i=1}^n Z_i^\star(C) = \sum_{i=1}^n P_i - L_i - \eta \mathrm{RBC}(Z_i^\star(C))\,.$$

Furthermore, the sum over the risk-based capital is given by

$$\sum_{i=1}^n \mathrm{RBC}(Z_i^\star(C)) = \frac{1}{1-\eta}\sum_{i=1}^n \mathrm{ES}_\alpha(L_i^\star(C)) - P_i\,.$$

Proof. We start with the sum over $Z_i^\star(C)$ as given in Equation (3.11). For the sake of argument define $P^\star_{ii} = 0$ and add $\sum_{i=1}^n P^\star_{ii} - P^\star_{ii} = 0$ to the sum. We immediately see that

$$\sum_{i=1}^n \sum_{j\neq i} (P^\star_{ji}(C) - P^\star_{ij}(C)) = \sum_{i=1}^n \sum_{j=1}^n (P^\star_{ji}(C) - P^\star_{ij}(C)) = 0\,. \tag{3.16}$$

For the loss we start with Equation (3.12), exchange the sums, and use the fact that for all $j \in \{1,\dots,n\}$ we have $\sum_{i=1}^n C_{ij} = 1$. We get

$$\sum_{i=1}^n L_i^\star(C) = \sum_{i=1}^n \sum_{j=1}^n C_{ij} L_j = \sum_{j=1}^n L_j \sum_{i=1}^n C_{ij} = \sum_{j=1}^n L_j\,.$$

The last assertion follows from Equation (3.13) and the translation property of the Expected Shortfall together with Equation (3.16). □

With help of Lemma 3.4.1 we can now restate the maximisation problem of $\mathbb{E}[Z^\star_{\text{Network}}(C)]$ as a minimisation problem of the overall risk-based capital $\mathrm{RBC}^\star_{\text{Network}}(C)$. Indeed, we get

$$\begin{aligned}\max_{C\in[0,1]^{n\times n}} \mathbb{E}[Z^\star_{\text{Network}}(C)] &= \max_{C\in[0,1]^{n\times n}} \left\{ \sum_{i=1}^n P_i - \mathbb{E}[L_i] - \eta \mathrm{RBC}(Z_i^\star(C)) \right\} \\ &= \sum_{i=1}^n P_i - \mathbb{E}[L_i] - \eta \min_{C\in[0,1]^{n\times n}} \mathrm{RBC}^\star_{\text{Network}}(C)\,.\end{aligned}$$

Hence, we can concentrate on the minimisation problem

$$\min_{C \in [0,1]^{n\times n}} \mathrm{RBC}^{\star}_{\mathrm{Network}}(C)\,. \tag{3.17}$$

In analogy to Equation (3.6), $\mathrm{RBC}_{\mathrm{Market}}$ is a lower bound to our minimisation problem. Using Lemma 3.4.1 together with subadditivity of the Expected Shortfall, we get

$$\begin{aligned} \mathrm{RBC}^{\star}_{\mathrm{Network}}(C) &= \frac{1}{1-\eta} \sum_{i=1}^{n} \mathrm{ES}_{\alpha}(L_i^{\star}(C)) - P_i \\ &\geq \frac{1}{1-\eta} \left(\mathrm{ES}_{\alpha}\left(\sum_{i=1}^{n} L_i \right) - \sum_{i=1}^{n} P_i \right) = \mathrm{RBC}_{\mathrm{Market}}\,. \end{aligned} \tag{3.18}$$

In terms of the optimisation in (3.17), the question can be restated. How should the risk be shared amongst the insurers such that $\mathrm{RBC}^{\star}_{\mathrm{Network}}(C) = \mathrm{RBC}_{\mathrm{Market}}$? To answer this question, notice that the inequality in (3.18) can equivalently be stated as

$$\sum_{i=1}^{n} \mathrm{ES}_{\alpha}\left(L_i^{\star}(C)\right) - \mathrm{ES}_{\alpha}\left(\sum_{i=1}^{n} L_i \right) \geq 0.$$

Therefore, any solution C to the equation

$$\sum_{i=1}^{n} \mathrm{ES}_{\alpha}\left(L_i^{\star}(C)\right) - \mathrm{ES}_{\alpha}\left(\sum_{i=1}^{n} L_i \right) = 0 \tag{3.19}$$

is also a minimiser of $\mathrm{RBC}^{\star}_{\mathrm{Network}}(C)$. The set of solutions to Equation (3.19) is given by

$$\left\{ C \in [0,1]^{n\times n} \;\middle|\; \forall i,j \in \{1,\dots,n\} : C_{ij} = C_{ii} \text{ and } \sum_{i=1}^{n} C_{ij} = 1 \right\}. \tag{3.20}$$

This can be seen from the following short calculation, where we use positive homogeneity of the Expected Shortfall and the fact that $\sum_{i=1}^{n} C_{ii} = \sum_{i=1}^{n} C_{ij} = 1$, for any $j \in \{1,\dots,n\}$,

$$\begin{aligned} \sum_{i=1}^{n} \mathrm{ES}_{\alpha}(L_i^{\star}(C)) &= \sum_{i=1}^{n} \mathrm{ES}_{\alpha}\left(\sum_{j=1}^{n} C_{ij} L_j \right) = \sum_{i=1}^{n} \mathrm{ES}_{\alpha}\left(\sum_{j=1}^{n} C_{ii} L_j \right) \\ &= \mathrm{ES}_{\alpha}\left(\sum_{j=1}^{n} L_j \right) \sum_{i=1}^{n} C_{ii} = \mathrm{ES}_{\alpha}\left(\sum_{j=1}^{n} L_j \right). \end{aligned}$$

The set of solutions in (3.20) represents the first main result of this chapter. It has a straightforward interpretation: the profitability at network level is optimised if after risk transfer all insurance companies hold a share of the market portfolio. In this situation, the sum of stand-alone risk-based capital corresponds to the overall risk-based capital of the market portfolio. This is a plausible and intuitive result. However, it should be kept in mind that it has been derived on the back of a consistent model of risk transfer realistically reflecting economics of capital costs.

Rewriting the set of optimal solutions in matrix form as in (3.10) yields

$$C^{\mathrm{opt}} = \begin{pmatrix} 1 - \sum_{j\neq 1}^{n} c_j & c_1 & \dots & c_1 \\ c_2 & 1 - \sum_{j\neq 2}^{n} c_j & & c_2 \\ \vdots & & \ddots & \vdots \\ c_n & \dots & & 1 - \sum_{j\neq n}^{n} c_j \end{pmatrix},$$

where $c_1, \dots, c_n \in [0,1]$ are constants such that $\sum_{i=1}^{n} c_i = 1$. This is an equation with n unknowns and consequently, there is no unique solution. An additional $n-1$ constraints will determine risk transfer unambiguously. From a practical perspective, the natural choice for the constraints is the capital available to support risk-based capital at the level of individual insurers. Such an approach reflects economic reality with limited capital available to support risk taking. Notice that following this approach, for one insurer the risk-based capital is determined implicitly. In economic terms, this insurer would need to adjust its capital base to ensure the overall network after risk transfer is adequately capitalised. This function of providing capital support to a network of insurance companies through risk transfer is typically performed by reinsurers. In Figure 3.1, this is illustrated by "I 4".

3.4.2 Unique, fair and optimal solution

As demonstrated in Section 3.4.1, there is no unique solution for the maximisation problem of the network without additional constraints, but rather a set of solutions. This raises the question whether one of these solutions can be regarded as "better" or "fair" towards all the insurers? In the following, we answer this question by applying principles of cooperative game theory to identify a unique fair solution for risk transfer. To this end, we follow the concept of fairness developed by Denault (1999).

In more detail, Denault develops an approach to allocate risk-based capital between subportfolios of one single company. He considers coalitional games with fractional players for general coherent risk measures and identifies a unique fair allocation. We show that this fair allocation concept defines a unique, fair, and optimal risk transfer scheme C^{f} for a network of different insurance companies. In what follows, we will briefly explain the framework and demonstrate that the risk transfer scheme C^{f} results in an allocation of stand-alone risk-based capital that equals the Aumann-Shapley allocation. In the sense of Denault, this is considered fair and optimal.

Consider a coalitional game with fractional players (N, Λ, R_ρ) consisting of

- a finite set $N = \{1, \dots, n\} \subset \mathbb{N}$ of players with $|N| = n$, where in our case each player corresponds to a portfolio of an insurer,
- a positive vector $\Lambda \in [0,1]^n$, where each element Λ_i represents the maximal possible involvement of player $i \in N$, and
- a risk measure function $R_\rho : [0,1]^n \to \mathbb{R}$ with $\lambda \mapsto R_\rho(\lambda)$ such that $R_\rho(0) = 0$.

So each player represents the portfolio of an insurer i which is characterised by Z_i, the profit distribution. The vector $\Lambda \in [0,1]^n$ represents the maximal portion of each player's portfolio included in the coalition,

$$Z = \sum_{i=1}^{n} \Lambda_i Z_i \,.$$

We introduce another vector $\lambda \in [0,1]^n$, $\lambda \leq \Lambda$, which defines the level of actual involvement of each player in the coalition. So the ratio $\frac{\lambda_i}{\Lambda_i}$ gives the proportion of the actual to the maximal involvement of player $i \in N$.

For fixed random variables $\{Z_i\}_{i\in\{1,..,n\}}$, we define the risk measure function associated with a coherent risk measure ρ as $R_\rho : [0,1]^n \to \mathbb{R}$ with

$$R_\rho(\lambda) = \rho\left(-\sum_{i=1}^{n} \lambda_i Z_i\right) .$$

Notice that $R_\rho(\Lambda) = \rho(-Z)$. The notion of coherency can be extended to these risk measure functions, see (Denault, 1999, Definition 11). However, we will only need positive homogeneity, that is, for all $\gamma > 0$ and $\lambda \in [0,1]^n$ we have $R_\rho(\gamma\lambda) = \gamma R_\rho(\lambda)$. This follows directly from the positive homogeneity of the coherent risk measure ρ.

Following Denault, we introduce the notion of fuzzy values. These are maps φ which assign to each coalitional game with fractional players (N, Λ, R_ρ) a unique per unit allocation vector,

$$\varphi : (N, \Lambda, R_\rho) \mapsto \big(\varphi_1(N, \Lambda, R_\rho), \ldots, \varphi_n(N, \Lambda, R_\rho)\big)^\intercal \in \mathbb{R}^n ,$$

such that

$$\Lambda^\intercal \varphi(N, \Lambda, R_\rho) = R_\rho(\Lambda) .$$

The concept of coherency is then further extended for fuzzy values and the implications for the fairness of these allocations are discussed. For the exact definition and the justification see (Denault, 1999, Definition 13) and the subsequent discussion.

A well known coherent fuzzy value is the Aumann-Shapley value.[3] Given the existence of the partial derivative of R_ρ in direction λ_i, $\partial_{\lambda_i} R_\rho$, it is for $i \in \{1, \ldots, n\}$ defined as

$$\varphi_i^{\mathrm{AS}}(N, \Lambda, R_\rho) = \int_0^1 \partial_{\lambda_i} R_\rho(\gamma\Lambda)\ \mathrm{d}\gamma .$$

So the per unit allocation is the mean of the marginal risk of the i-th insurer as the involvement of all insurers increases uniformly from 0 to 1. Using the fact that for a differentiable, k–homogeneous function $f : \mathbb{R}^n \to \mathbb{R}$, that means, for all $\gamma > 0$, $x \in \mathbb{R}^n$ we have $f(\gamma x) = \gamma^k f(x)$, any of its partial derivatives are $(k-1)$–homogeneous, we can simplify the integral from above. Since the risk measure function R_ρ is positively homogeneous, that means 1–homogeneous, we get

$$\varphi_i^{\mathrm{AS}}(N, \Lambda, R_\rho) = \partial_{\lambda_i} R_\rho(\Lambda) \int_0^1 1\, \mathrm{d}\gamma = \partial_{\lambda_i} R_\rho(\Lambda) .$$

This corresponds to the Euler allocation principle. The resulting per portfolio allocation is the gradient of R_ρ evaluated at the maximal presence level Λ,

$$\varphi^{\mathrm{AS}}(N, \Lambda, R_\rho) = \nabla R_\rho(\Lambda) .$$

This gradient is the Aumann-Shapley risk allocation. It refers to the amount of risk allocated to each portfolio when $\Lambda = [1, \ldots, 1]^\intercal \in \mathbb{R}^n$.
Hence, choosing $\Lambda = [1, \ldots, 1]^\intercal \in \mathbb{R}^n$ and the Expected Shortfall, $\rho = \mathrm{ES}_\alpha$, we get

$$\mathrm{RBC}(Z) = \mathrm{ES}_\alpha(-Z) = R_{\mathrm{ES}_\alpha}(\Lambda).$$

[3]For a proof of the coherency of the Aumann-Shapley value see (Denault, 1999, Corollary 1).

Therefore, the resulting risk capital allocation is

$$\mathrm{RBC}_Z(Z_i) = \varphi_i^{\mathrm{AS}}(N, \Lambda, R_{\mathrm{ES}_\alpha}) = \partial_{\lambda_i} R_{\mathrm{ES}_\alpha}(\Lambda) = \partial_{\lambda_i} \mathrm{ES}_\alpha\bigg(- \sum_{i=1}^{n} \lambda_i Z_i \bigg)\bigg|_{\lambda=\Lambda} . \tag{3.21}$$

This expression can be simplified. Indeed, the Aumann-Shapley allocation with the Expected Shortfall as the risk measure corresponds to the allocation scheme based on the Conditional Expected Shortfall.

Proposition 3.4.2. *For $i \in \{1, \dots, n\}$ we have*

$$\partial_{\lambda_i} \mathrm{ES}_\alpha\bigg(- \sum_{i=1}^{n} \lambda_i Z_i \bigg) = \mathrm{CES}_{\alpha, -\sum_{i=1}^{n} \lambda_i Z_i}(-Z_i) .$$

Proof. A proof is given in (Scaillet, 2004, Proposition 2.1, Lemma A.1). Compare also (Tasche, 2000). □

Using Proposition 3.4.2 and Equation (3.21) with $\Lambda = (1, \dots, 1)^\intercal \in \mathbb{R}^n$ yields

$$\mathrm{RBC}_Z(Z_i) = \mathrm{CES}_{\alpha, -Z}(-Z_i).$$

Following Section 3.4.1, the maximisers of the optimisation in (3.15) are the solutions of Equation (3.19) and only depend on losses before risk transfer, L_i, and after risk transfer, $L_i^\star$. Thus, let the n insurers be represented by their gross loss amounts L_i and the gross loss amount of the grand coalition be represented by the network itself. Using the Expected Shortfall as our risk measure, the overall risk of the network is quantified by

$$\mathrm{ES}_\alpha(L_{\mathrm{Network}}) = \mathrm{ES}_\alpha\bigg(\sum_{i=1} L_i \bigg) .$$

Assuming again $\Lambda = (1, \dots, 1)^\intercal \in \mathbb{R}^n$, the resulting Aumann-Shapley allocation,

$$\varphi_i^{\mathrm{AS}}(N, \Lambda, R_{\mathrm{ES}_\alpha}) = \mathrm{CES}_{\alpha, L_{\mathrm{Network}}}(L_i) , \tag{3.22}$$

is the fair risk allocation to insurer i. From Section 3.4.1 we know that the optimal risk transfer C^{f} belongs to the set

$$\bigg\{ C \in [0,1]^{n \times n} \;\bigg|\; \forall i, j \in \{1, \dots, n\} : C_{ij} = C_{ii} \text{ and } \sum_{i=1}^{n} C_{ij} = 1 \bigg\} ,$$

and it is fair if the retained risk $L_i^\star$ after risk transfer equals the fair risk allocation in (3.22). Therefore, the optimal and fair risk transfer C^{f} is determined by the solution of the equation

$$C_{ii}^{\mathrm{f}} \; \mathrm{ES}_\alpha(L_{\mathrm{Network}}) = \mathrm{CES}_{\alpha, L_{\mathrm{Network}}}(L_i) ,$$

for all $i \in \{1, \dots, n\}$. Hence, we have

$$C_{ii}^{\mathrm{f}} = \frac{\mathrm{CES}_{\alpha, L_{\mathrm{Network}}}(L_i)}{\mathrm{ES}_\alpha(L_{\mathrm{Network}})} , \tag{3.23}$$

where both $\mathrm{CES}_{\alpha, L_{\mathrm{Network}}}(L_i)$ and $\mathrm{ES}_\alpha(L_{\mathrm{Network}})$ are assumed to be positive. Equation (3.23) is the second main result of this chapter. It defines a unique, fair and optimal solution for the risk transfer C^{f} across the network. This solution can be understood as defining risk transfer such that the diversification present at network level is effectively and efficiently allocated in a fair, optimal and unique way to individual insurers. As such the solution ensures that the positive impact of diversification on economic profitability is realised at the level of individual insurers in a fair and optimal way.

3.5 Impact of risk transfer on systemic risk

The network approach to risk transfer developed in this chapter lends itself naturally to an analysis of systemic risk. Our framework can shed light on a range of aspects of systemic risk, including interconnectedness, complexity, and resilience at the level of individual insurance companies and the network itself. While a subsequent study will analyse systemic risk in more detail, here we just highlight three critical aspects directly linked to risk transfer, sharing risk and diversification.

- Firstly, we notice that sharing risk and diversification through risk transfer will distribute risk across the network and thereby improve resilience at the level of individual insurers.
- On the other hand, we observe that risk transfer is increasing interconnectedness and may thus contribute to systemic risk itself.
- Lastly, we note that the optimisation of risk transfer through increasing diversification at the level of individual insurers may lead to a reduction of overall capital requirements.

The impact of optimisation of risk transfer on capital requirements at network level as compared to the capital requirement at market level is illustrated best by Equation (3.6). The ratio

$$\frac{\mathrm{RBC}_{\mathrm{Network}} - \mathrm{RBC}_{\mathrm{Market}}}{\mathrm{RBC}_{\mathrm{Market}}} \tag{3.24}$$

describes how close the network is to optimal risk transfer. At the same time it also describes capital redundancy of the network relative to the market. See Figure 3.2 for a visual illustration. The graph illustrates Equation (3.24) as a function of different risk retention levels. For this we simulated the losses of two insurers. We assume that the losses have the same log-normal distribution and that they are independent. For each retention level pair we evaluate Equation (3.24). One can clearly see that the optimal risk retention pairs lie on the diagonal line connecting the points $(0, 1)$ and $(1, 0)$ on the xy-plane, as predicted in Section 3.4. In contrast, if both insurers either do not participate in risk transfer, point $(1, 1)$, or decide to just swap their losses, point $(0, 0)$, the network is furthest away from the optimum.

Optimising risk transfer reduces the ratio which means overall capital remaining at network level is reduced. From the point of systemic risk this may indicate that through optimising risk transfer, systemic risk is increased as the overall capital available at network level to cope with a systemic stress event is reduced.

Optimisation of risk transfer on the other hand leads to each insurance company of the network holding a fraction of the total market portfolio. Risk, including systemic risk, is shared evenly across all insurance companies preventing concentration of systemic risk at level of a single insurance company thereby increasing resilience. However, risk transfer is increasing interconnectedness. As a result, a systemic event through optimised risk transfer can impact the whole network thereby increasing systemic risk.

Further analysis is needed to understand in more detail these competing effects of optimised risk transfer on systemic risk exposure and resilience at network level. Such an analysis will have to distinguish between idiosyncratic risk and systemic risk factors and their role in the optimisation of risk transfer through optimally sharing diversification.

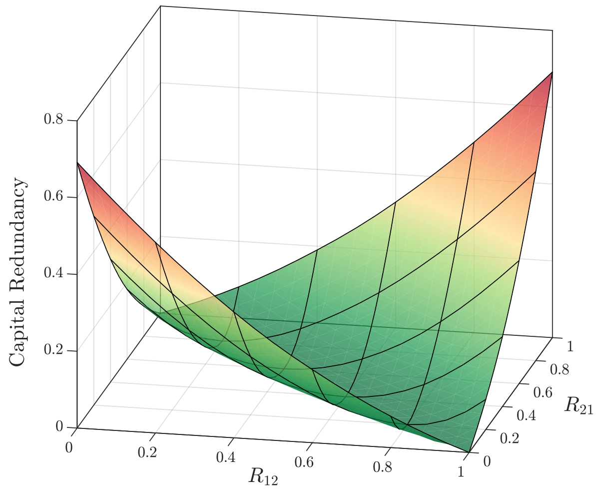

Figure 3.2: The graph visualises capital redundancy as defined by Equation (3.24). The R_{12} and R_{21} axes represent the level of risk retention of the first and second insurer, respectively.

3.6 Conclusions and outlook

Risk transfer is a risk and capital management tool which enables insurers to meet capital and risk appetite constraints. At the same time, risk transfer is also a tool to enhance performance as it can balance portfolios and thereby improve risk-return characteristics. For a single insurer, optimising risk transfer essentially means trading-off the cost of risk transfer to mitigated loss potential and reduced capital costs.

For a network of insurers, optimising risk transfer is less evident as optimisation can take different perspectives. To analyse this, we considered risk transfer in the context of a network of insurance companies. Prompted by supervisory frameworks as, for example, Solvency II or SST requiring insurers to quantify risk-based capital using prescribed modelling principles and risk measures, a common, realistic, and economical premium principle is applied. Assuming an equal risk appetite and capital cost for all insurers, there is, in general, no solution which maximises the profitability of all insurance companies in the network. However, there is a set of solutions which maximise the overall profitability at the network level. These solutions are characterised by risk transfer such that all insurers hold a proportion of the market portfolio. The analysis also demonstrates that additional constraints will determine risk transfer unambiguously.
Absent of additional constraints cooperative game theory defines a unique, fair and optimal solution related to coherent risk allocation principles based on the Aumann-Shapley value developed in (Denault, 1999). This solution defines risk transfer such that the diversification present at network level is effectively and efficiently allocated in a fair, optimal, and unique way to individual insurers.

In practice, the fair and optimal solution defined by the Aumann-Shapley value can be realised closest for insurance groups. Reason for this is that for insurance groups there exists an overarching rationale to optimise profitability not only at the level of individual insurance companies, but at the level of the insurance group itself. Thus, insurers which are part of the group are directly incentivised to form coalitions in terms of risk transfer that optimises profitability at the group level. The solution defined by the Aumann-Shapley value not only optimises profitability, but also is fair in the sense that it recognises the contribution to group level diversification and through risk transfer efficiently allocates it back to individual insurers. It should be noted that the realisation of group level diversification through risk transfer instruments is the only way to recognise diversification under supervisory frameworks building on risk-based capital quantification. In this context, the interplay of optimal ownership structure and optimal risk transfer remains a topic for further analysis.

A related topic for further research is the impact of risk transfer on systemic risk. The network approach to risk transfer developed in this chapter sheds light on a wide range of aspects of systemic risk, including in particular interconnectedness, complexity, and resilience at the level of individual insurance companies and the network itself. There are two critical facets to this. Firstly, sharing risk and diversification through risk transfer will distribute risk across the network and thereby improve resilience at the level of individual insurers. Risk transfer, on the other hand, is increasing interconnectedness and may thus contribute to systemic risk itself. The optimisation of risk transfer discussed in this chapter may lead to a reduction of overall capital supporting risk and thus increase systemic risk exposure accordingly. Connecting concepts developed in this chapter with related research like (Acharya, Pedersen, Philippon, and Richardson, 2016) and (Engle and Brownlees, 2016) is a possible starting point to further investigate these and related questions.

3.7 Appendix

3.7.1 Proofs of Section 3.2

The following lemmas show that with the choices in (3.4) and (3.5) our requirements described in Section 3.2.3 are satisfied. In analogy to Lemma 3.2.1, we get the following result.

Lemma 3.7.1. *With the choices in* (3.4) *and* (3.5) *we get*

$$\mathrm{RBC}_Z(Z_i) = \frac{\mathrm{CES}_{\alpha,-Z}(-U_i)}{1-\eta}\,.$$

Proof. Using Equation (3.4), Equation (3.5), and the linearity of the conditional expectation, we get

$$\mathrm{RBC}_Z(Z_i) = \mathrm{CES}_{\alpha,-Z}(-U_i + \eta\mathrm{RBC}_Z(Z_i)) = \mathrm{CES}_{\alpha,-Z}(-U_i) + \eta\mathrm{RBC}_Z(Z_i)\,.$$

Solving for $\mathrm{RBC}_Z(Z_i)$ yields the result. □

Furthermore, the Conditional Expected Shortfall defines an additive risk-based capital.

Lemma 3.7.2. *With the choices in* (3.4) *and* (3.5) *we get*

$$\sum_{i=1}^{n} \mathrm{RBC}_Z(Z_i) = \mathrm{RBC}(Z)\,.$$

Proof. Applying Lemma 3.7.1 in the first equality and using the linearity of the conditional expectation in the second equality, we get

$$\sum_{i=1}^{n} \mathrm{RBC}_Z(Z_i) = \frac{1}{1-\eta}\sum_{i=1}^{n} \mathrm{CES}_{\alpha,-Z}(-U_i) = \frac{1}{1-\eta}\,\mathrm{CES}_{\alpha,-Z}(-U)\,.$$

Notice that by cash-additivity of the Value-at-Risk as well as Lemma 3.2.1 the condition $-Z \geq \mathrm{VaR}_\alpha(-Z)$ is equivalent to $-U \geq \mathrm{VaR}_\alpha(-U)$. Hence,

$$\mathrm{CES}_{\alpha,-Z}(-U) = \mathrm{CES}_{\alpha,-U}(-U) = \mathrm{ES}_\alpha(-U)\,.$$

Applying Lemma 3.7.1 again yields the result. □

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Bibliography

Acciaio, B. and Penner, I., 2011. Dynamic risk measures. In: G. Di Nunno and B. Øksendal, editors, *Advanced Mathematical Methods for Finance*, pages 1–34. Springer Berlin Heidelberg, Berlin, Heidelberg.

Acerbi, C. and Tasche, D., 2002. On the coherence of expected shortfall. *Journal of Banking & Finance*, 26(7), 1487–1503.

Acharya, V.V., Engle, R., and Richardson, M., 2012. Capital shortfall: A new approach to ranking and regulating systemic risks. *The American Economic Review*, 102(3), 59–64.

Acharya, V.V., Pedersen, L.H., Philippon, T., and Richardson, M., 2016. Measuring Systemic Risk. *The Review of Financial Studies*, 30(1), 2–47.

Adrian, T. and Brunnermeier, M.K., 2016. CoVaR. *The American Economic Review*, 106(7), 1705–1741.

Ararat, Ç. and Rudloff, B., 2020. Dual representations for systemic risk measures. *Mathematics and Financial Economics*, 14, 139–174.

Artzner, P., Delbaen, F., Eber, J.M., and Heath, D., 1999. Coherent measures of risk. *Mathematical Finance*, 9(3), 203–228.

Artzner, P., Delbaen, F., and Koch-Medina, P., 2009. Risk measures and efficient use of capital. *Astin Bulletin*, 39(1), 101–116.

Asimit, A.V., Badescu, A.M., Haberman, S., and Kim, E.S., 2016. Efficient risk allocation within a non-life insurance group under Solvency II regime. *Insurance: Mathematics and Economics*, 66, 69–76.

Asimit, A.V., Badescu, A.M., and Tsanakas, A., 2013. Optimal risk transfers in insurance groups. *European Actuarial Journal*, 3, 159–190.

Bernstein, P.L., 1996. *Against the gods: The remarkable story of risk.* New York: John Wiley & Sons.

Biagini, F., Doldi, A., Fouque, J.P., Frittelli, M., and Meyer-Brandis, T., 2021. Systemic optimal risk transfer equilibrium. *Mathematics and Financial Economics*, 15, 233–274.

Biagini, F., Fouque, J.P., Frittelli, M., and Meyer-Brandis, T., 2019. A unified approach to systemic risk measures via acceptance sets. *Mathematical Finance*, 29(1), 329–367.

Brunnermeier, M.K., 2009. Deciphering the liquidity and credit crunch 2007-2008. *Journal of Economic Perspectives*, 23(1), 77–100.

Brunnermeier, M.K. and Cheridito, P., 2019. Measuring and allocating systemic risk. *Risks*, 7(2).

Centeno, M.d.L., 2002. Measuring the effects of reinsurance by the adjustment coefficient in the sparre anderson model. *Insurance: Mathematics and Economics*, 30, 37–49.

Cerreia-Vioglio, S., Maccheroni, F., Marinacci, M., and Montrucchio, L., 2011. Risk measures: Rationality and diversification. *Mathematical Finance*, 21(4), 743–774.

Chen, C., Iyengar, G., and Moallemi, C.C., 2013. An Axiomatic Approach to Systemic Risk. *Manag. Sci.*, 59(6), 1373–1388.

Cifuentes, R., Ferrucci, G., and Shin, H.S., 2005. Liquidity risk and contagion. *Journal of the European Economic Association*, 3(2/3), 556–566.

Dacorogna, M., 2018. A change of paradigm for the insurance industry. *Annals of Actuarial Science*, 12, 211–232.

Daykin, C., Pentikainen, T., and Pesonen, M., 1993. *Practical Risk Theory for Actuaries.* New York: Chapman and Hall.

de Finetti, B., 1940. *Il problema dei pieni*, volume 11, pages 1–88. Roma.

Delbaen, F., 2002. Coherent risk measures on general probability spaces. In: *Advances in Finance and Stochastics: Essays in Honour of Dieter Sondermann*, pages 1–37. Springer Berlin Heidelberg, Berlin, Heidelberg.

Denault, M., 1999. Coherent allocation of risk capital. *Journal of Risk*, 4, 1–34.

Drapeau, S. and Kupper, M., 2013. Risk preferences and their robust representation. *Mathematics of Operations Research*, 38(1), 28–62.

Dunford, N. and Schwartz, J.T., 1958. *Linear Operators. Part I: General Theory.* Interscience Publishers Inc., New York.

Eisenberg, L. and Noe, T.H., 2001. Systemic risk in financial systems. *Management Science*, 47(2), 236–249.

El Karoui, N. and Ravanelli, C., 2009. Cash subadditive risk measures and interest rate ambiguity. *Mathematical Finance*, 19(4), 561–590.

Elsinger, H., 2009. *Financial Networks, Cross Holdings, and Limited Liability.* Working papers. Österreichische Nationalbank.

Engle, R.F. and Brownlees, C., 2016. SRISK: A conditional capital shortfall measure of systemic risk. *The Review of Financial Studies*, 30(1), 48–79.

Ettlin, N., 2018. Optimal risk sharing: An analysis of risk-based capital allocation and risk transfer in the insurance industry. Master's Thesis, University of Basel.

Ettlin, N., Farkas, W., Kull, A., and Smirnow, A., 2020. Optimal risk-sharing across a network of insurance companies. *Insurance: Mathematics and Economics*, 95, 39–47.

Farkas, W., Koch-Medina, P., and Munari, C., 2014a. Beyond cash-additive risk measures: when changing the numéraire fails. *Finance and Stochastics*, 18(1), 145–173.

Farkas, W., Koch-Medina, P., and Munari, C., 2014b. Capital requirements with defaultable securities. *Insurance: Mathematics and Economics*, 55, 58–67.

Farkas, W. and Smirnow, A., 2018. Intrinsic risk measures. In: *Innovations in Insurance, Risk- and Asset Management*, pages 163–184.

Feinstein, Z., 2017. Financial contagion and asset liquidation strategies. *Operations Research Letters*, 45(2), 109–114.

Feinstein, Z., Rudloff, B., and Weber, S., 2017. Measures of systemic risk. *SIAM Journal on Financial Mathematics*, 8(1), 672–708.

Föllmer, H. and Schied, A., 2002. Convex measures of risk and trading constraints. *Finance Stoch.*, 6(4), 429–447.

Föllmer, H. and Schied, A., 2004. *Stochastic Finance. An Introduction in Discrete Time.* Walter de Gruyter, Berlin, Boston.

French, C.W., 2003. The treynor capital asset pricing model. *Journal of Investment Management*, 1(2), 60–72.

Frittelli, M. and Rosazza Gianin, E., 2002. Putting order in risk measures. *Journal of Banking & Finance*, 26(7), 1473–1486.

Haier, A., Molchanov, I., and Schmutz, M., 2016. Intragroup transfers, intragroup diversification and their risk assessment. *Annals of Finance*, 12, 363–392.

Hamel, A. and Rudloff, B., 2008. Continuity and finite-valuedness of set-valued risk measures. In: C. Tammer and F. Heyde, editors, *Festschrift in Celebration of Prof. Dr. Wilfried Grecksch's 60th Birthday*, page 46–64. Shaker Verlag.

Hamm, A.M., Knispel, T., and Weber, S., 2019. Optimal risk sharing in insurance networks - an application to asset-liability management. https://www.stochastik.uni-hannover.de/fileadmin/institut/pdf/ORS_20190323.pdf. Accessed July 15, 2019.

Hay-Gibson, N.V., 2008. A river of risk : a diagram of the history and historiography of risk management.

Hurd, T.R., 2016. *Contagion! Systemic Risk in Financial Networks.* Springer Cham.

Jorion, P., 2006. *Value at Risk, 3rd Ed.: The New Benchmark for Managing Financial Risk.* McGraw-Hill Education.

Jouini, E., Meddeb, M., and Touzi, N., 2004. Vector-valued coherent risk measures. *Finance and Stochastics*, 8, 531–552.

Kaluszka, M., 2001. Optimal reinsurance under mean-variance premium principles. *Insurance: Mathematics and Economics*, 28, 61–67.

Keller, P., 2007. Group diversification. *The Geneva Papers on Risk and Insurance - Issues and Practice*, 32, 382–392.

Konstantinides, D.G. and Kountzakis, C.E., 2011. Risk measures in ordered normed linear spaces with non-empty cone-interior. *Insurance: Mathematics and Economics*, 48(1), 111–122.

KPMG, 2019. Cost of capital study 2019. https://assets.kpmg/content/dam/kpmg/ch/pdf/cost-of-capital-study-2019.pdf. Accessed April 15, 2020.

Kull, A., 2009. Sharing risk - an economic perspective. *ASTIN Bulletin*, 39(2), 591–613.

Kupper, M. and Filipović, D., 2007. On the group level swiss solvency test. 2007(1), 97–115.

Liebrich, F.B. and Svindland, G., 2019. Risk sharing for capital requirements with multidimensional security markets. *Finance and Stochastics*, 23(4), 925–973.

Lightner, J.E., 1991. A brief look at the history of probability and statistics. *The Mathematics Teacher*, 84(8), 623–630.

Markowitz, H.M., 1952. Portfolio selection. *The Journal of Finance*, 7(1), 77–91.

Markowitz, H.M., 1959. *Portfolio Selection: Efficient Diversification of Investments*. Yale University Press.

Markowitz, H.M., 1999. The early history of portfolio theory: 1600-1960. *Financial Analysts Journal*, 55(4), 5–16.

Mayer, T., Kull, A., Keller, P., and Portmann, H., 2009. *Risk Sharing in Insurance Groups*. Risk Books.

McNeil, A.J., Frey, R., and Embrechts, P., 2005. *Quantitative Risk Management: Concepts, Techniques, and Tools*. Princeton Series in Finance. Princeton University Press.

Munari, C., 2015. *Measuring risk beyond the cash-additive paradigm*. Ph.D. thesis, Diss. ETH no. 22541, ETH Zürich.

Rogers, L.C.G. and Veraart, L.A.M., 2013. Failure and Rescue in an Interbank Network. *Manag.Sci*, 59(4), 882–898.

Roy, A.D., 1952. Safety first and the holding of assets. *Econometrica*, 20(3), 431–449.

Scaillet, O., 2004. Nonparametric estimation and sensitivity analysis of expected shortfall. *Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics*, 14(1), 115–129.

Silva, W., Kimura, H., and Sobreiro, V.A., 2017. An analysis of the literature on systemic financial risk: A survey. *Journal of Financial Stability*, 28, 91–114.

Smirnow, A., 2016. *Risk measures: recent developments and new ideas*. Master's thesis, Universität Zürich, ETH Zürich.

Tasche, D., 2000. Risk contributions and performance measurement. https://pdfs.semanticscholar.org/2659/60513755b26ada0b4fb688460e8334a409dd.pdf. Accessed March 30, 2018.

Venter, G.G., 2004. Capital allocation survey with commentary. *North American Actuarial Journal*, 8(1), 96–107.

Weber, S. and Weske, K., 2017. The joint impact of bankruptcy costs, fire sales and cross-holdings on systemic risk in financial networks. *Probability, Uncertainty and Quantitative Risk*, 2(9).