Household Liquidity Policy
Supplemental Appendix*

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November 11, 2025

Contents

*The views expressed in this paper are solely those of the authors and do not represent the views of the Federal Reserve Board or the Federal Reserve System.

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A Defined contribution retirement accounts

The need to provide for retirement is universal and government intervention to meet some of this need is also common. The government's involvement is justified for a variety of reasons. These include a redistributive motive, to help avoid poverty in retirement, a protective motive, to short–circuit the moral hazard created by the redistributive motive (i.e. households neglect to save for retirement, anticipating the government will bail them out), and a paternalistic one, correcting for biases that reduce working–life saving (Feldstein, 1985; Beshears et al., 2015). The means with which countries address these needs vary a lot, but usually involve some mix of state and private provisions, with the latter becoming increasingly important as many governments grapple with the strain of unfunded state pension provisions coupled with ageing populations (OECD, 2018).

Our focus in this paper is on countries with private defined contribution (DC) schemes. In DC pension systems, working–age people make contributions into regulated investment accounts, and they have a claim on the balance and accumulated returns upon retirement. The exact design features of DC accounts differ across countries, but they can be broadly understood in terms of rules defining (a) liquidity during working life, (b) contributions, (c) tax treatment, and (d) the state pension they are combined with.

Liquidity during working life Regulations affecting access to DC accounts differ across countries. In some settings, like the USA and UK, participants are allowed to withdraw prior to retirement, but they pay a penalty (10% in the USA). In others, like Australia, withdrawals are not allowed except in dire personal circumstances (e.g. terminal illness or extreme financial hardship) effectively making the accounts completely illiquid. In either case, the illiquidity is created by regulation, rather than because the assets are difficult to transact in.

Contributions Regulations affecting contributions generally govern (a) whether any contributions are mandatory and (b) limits on voluntary contributions. Mandatory contributions, when they exist, are usually set as a proportion of pre–tax employment income, commonly in the range of 10–20% (see Table 1 for some examples from OECD countries). Voluntary contributions to these schemes are often allowed as well, but are usually limited to maximum nominal amounts per year because they attract tax concessions.

Tax treatment There are three potentially taxable flows in DC systems—contributions, investment returns, and withdrawals—and systems differ in whether each of these is taxed (potentially at concessional rates) or exempt, leading to a three-letter code describing them.

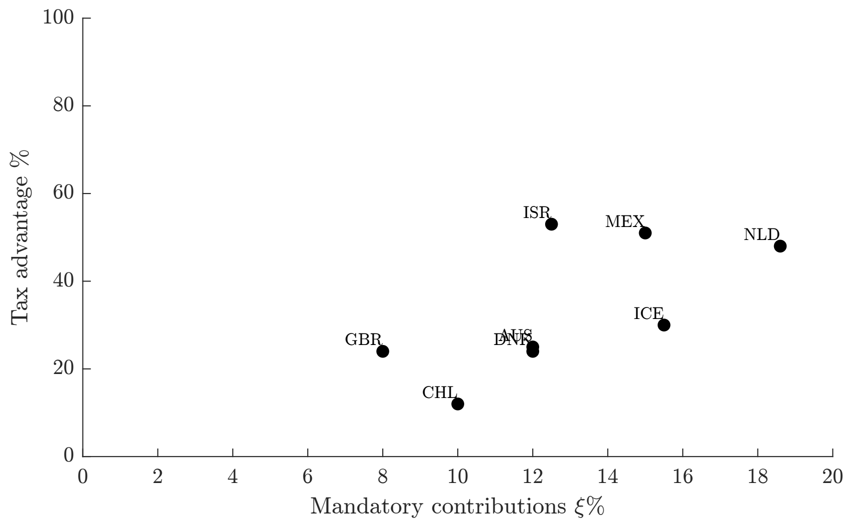
Table 1: Mandatory contribution rates in OECD DC systems

	AUS	CHL	DNK	ICE	ISR	MEX	NLD	UK*
ξ (p.p.)	12	10	12	15.5	13	15	18.6	8

Source: OECD (2023) Table 3.4, p. 141. For all OECD countries with privately funded DC schemes, and no other mandatory private system. *The UK's is a default, rather than mandatory.

In a system like USA 401(k)s, for example, contributions are made from pre–tax income, and returns are exempt as well, but withdrawals are taxed at the personal marginal tax rate, so it is coded EET. This is the most common approach. By contrast, Australia's Superannuation contributions from pre–tax income are taxed at a concessional 15% rate, as are returns, and withdrawals are tax free, so it is coded TTE.[1]

Figure 1: DC system design in OECD countries



Source: mandatory contribution rates (OECD, 2023, Table 3.4) and tax advantage (OECD, 2018, Table 3.2)

Figure 1 shows the combinations of mandatory contribution rates (ξ) and tax advantages across the collection of OECD countries with mandatory DC systems.[2] The tax advantage variable comes from OECD (2018), and represents the tax savings from a given flow of contributions over a typical working life, relative to if they were saved in a regular investment account. Tax advantages are ubiquitous in these accounts, though they range from quite small (worth a discount of around 10% in Chile) to substantial (around 50% in Israel and Mexico).

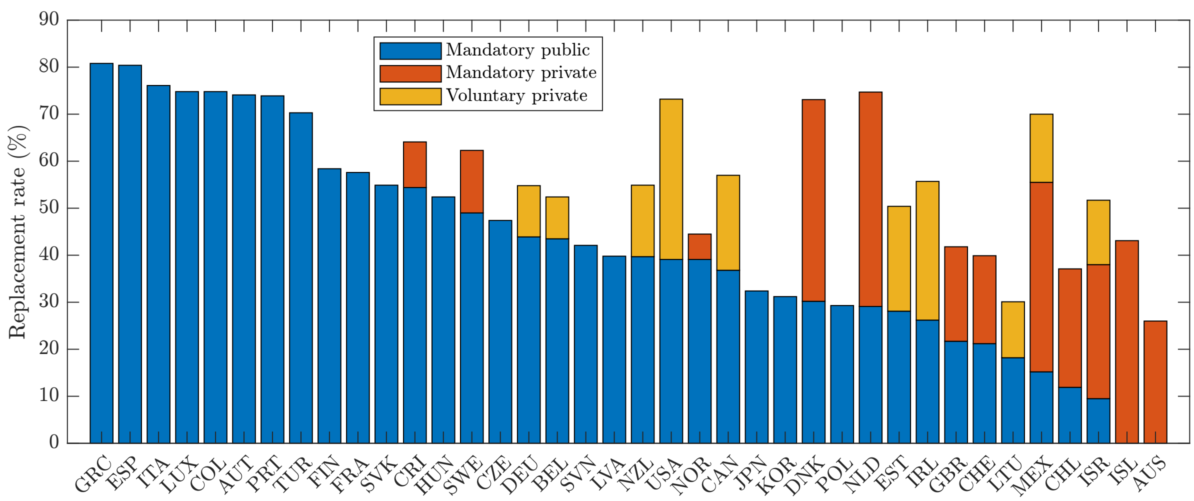
[1]In the model we use a TTE system so that we can control the tax concession granted for contributions and returns inside the account. EET systems, whilst more common, typically apply taxes at full marginal rates, and so offer fewer degrees of freedom.

[2]Note this excludes countries that mix these with other mandatory private options.

In the model introduced in the main paper, the retirement system features parameters encoding mandatory contributions and tax concessions, and we rationalise the features seen here as the optimal policy when society features a portion of the population that is present–biased, and the government needs to ensure workers to opt–in to the system at the start of their careers. The tax concessions, then, serve the *political* purpose of building buy–in into the system.

State pension DC systems are usually introduced to reduce the burden of retirement provision on current taxation by shifting the responsibility onto households themselves. In mandatory DC systems, government retirement provision is usually still present, but less generous. We can see this by looking at the state pension replacement rate for an average income earner across countries with and without mandatory private systems, plotted across OECD countries in Figure 2. This replacement rate for an average earner is 31% on average for OECD countries with mandatory systems, compared with 56% on average for OECD countries with only mandatory public systems, a substantial difference.[3]

Figure 2: Retirement system replacement rates for average earners, OECD countries



Source: OECD (2023) Table 4.2

[3]These are population weighted averages of the mandatory public gross replacement rates for an average earner across OECD countries with and without a mandatory private system in place (OECD, 2023, Table 4.2)

B Present–biased household

B.1 Instantaneous Gratification Preferences

In working life, unbiased households' recursive preferences are as follows[4]

$$v(x_t) = \lim_{\Delta\to 0} \max_{c,d} u(c)\Delta + e^{-\rho\Delta}\,\mathbb{E}\left[v\left(x_{t+\Delta}(c,d)\right)\right] \tag{1}$$

Where the maximisation is constrained by the state transition functions that define $x_{t+\Delta}(c,d)$. $v(\cdot)$ is the value the household places on states x in time t, which comes from a combination of the utility they gain from optimal consumption for the present moment Δ, and the expected, discounted value placed on the state variables they're left with in the next moment ($\mathbb{E}[\cdot]$ captures all idiosyncratic risk transitions).

By contrast, the equivalent expression for present–biased households is[5]

$$v^\beta(x_t) = \lim_{\Delta\to 0} \max_{c,d} u(c)\Delta + \beta\cdot e^{-\rho\Delta}\,\mathbb{E}\left[v^E\left(x_{t+\Delta}(c,d)\right)\right] \tag{2}$$

The present–biased value in Equation (2) differs from (1) in two important ways. First, the continuation value is discounted by an extra $\beta \le 1$ on top of the exponential discount $e^{-\rho\Delta}$. This is the source of present–biased behaviour—the IG agent values the future less than their exponential counterpart. Second, the continuation value $v^E(x_{t+\Delta})$ represents the value the household *believes* they will place on expected states in the future, which may not be how they actually value them. A 'sophisticated' agent holds correct beliefs: they will have the same value function in future as in the present (i.e. $v^E(x) = v^\beta(x)$) whereas 'naivete' wedges them apart.

B.2 The present–biased household's problem

For a present–biased naif with bias parameter β, the HJB equation changes in two ways from the exponential equivalent—all terms except the flow utility are pre–pended by β, and

[4] This is derived by separating the present Δ from the future in the integral

$$v(x_t) = \max_{c_s,d_s} \mathbb{E}\int_t^\infty D(s-t)u(c_s)ds$$

where the discount function is $D(s-t) = e^{-\rho(s-t)}$

[5] As above, but where the discount is now the step function

$$D(s-t) = \begin{cases} 1 & \text{if } s-t=0 \\ \beta\cdot e^{-\rho(s-t)} & \text{if } s-t>0 \end{cases}$$

the value in the HJB is the rational value, not the household's actual valuation.

Working life

$$\beta\rho v(x) = \max_{\hat{c},\hat{d}} \Big\{ u(\hat{c}) + \beta\partial_b v(x)\cdot \dot{b}(x) + \beta\partial_a v(x)\cdot \dot{a}(x) \Big\} \\ + \beta \sum_{z'} \lambda^{z\to z'} \left[v(x') - v(x) \right] + \beta\delta_R \left[v_R(x) - v(x) \right] \tag{3}$$

This problem's FOC are

$$u'(\hat{c}(x)) = \beta(1+\tau)\partial_b v(x) = \beta u'(c(x)) \tag{4}$$

$$\partial_a v(x) = \partial_b v(x)(1 + \chi_d(\hat{d}(x), a)) + \kappa(x) \tag{5}$$

Where the latter is the same as the exponential agent's, and so $\hat{d}(x) = d(x)$

Retired Similarly, the retired present–biased naif's problem is to solve the following.

$$\beta(\rho+\delta)v_R(x) = \max_{\hat{c}_R,\hat{d}_R} \Big\{ u(\hat{c}_R) + \beta\partial_b v_R(x)\cdot \dot{b}(x) + \beta\partial_a v(x)\cdot \dot{a}(x) \Big\} \tag{6}$$

Where $\dot{b}(x)$ and $\dot{a}(x)$ are defined in the main paper. This problem's FOC are

$$u'(\hat{c}_R(x)) = \beta(1+\tau)\partial_b v_R(x) = \beta u'(c_R(x)) \tag{7}$$

$$\partial_a v_R(x) = \partial_b v_R(x)(1 + \chi_d(\hat{d}_R(x), a)) \tag{8}$$

B.3 The present–biased household's Euler equation

Present–biased naif, stationary Following Maxted (2025), Appendix A.4, but with an added consumption tax. The following solves the Euler for the working household. First, take the derivative of the working household's HJB, with respect to the liquid asset b.

$$\rho\partial_b v(x) = u'(c(x))\partial_b c(x) + \partial_{bb} v(x)\cdot \dot{b}(x) + \partial_b v(x)\partial_b \dot{b}(x) \\ + \sum_{z'} \lambda^{z\to z'} \left[\partial_b v(x') - \partial_b v(x) \right] + \delta_R \left[\partial_b v_R(x) - \partial_b v(x) \right]$$

Apply the realised FOC i.e. that $u'(\hat{c}(x)) = \beta(1+\tau)\partial_b v(x)$ so $\partial_b v(x) = \frac{u'(\hat{c}(x))}{\beta(1+\tau)}$

$$
\begin{aligned}
(\rho - \partial_b r(b)b - r(b))\, u'(\hat{c}(x)) &= (1+\tau)\left(\beta u'(c(x)) - u'(\hat{c}(x))\right)\partial_b c(x) + \partial_b \hat{c}(x) u''(\hat{c}(x)) \cdot \dot{b}(x) \\
&\quad + \sum_{z'} \lambda^{z \to z'} \left[u'(\hat{c}(x')) - u'(\hat{c}(x))\right] + \delta_R \left[u'(\hat{c}_R(x)) - u'(\hat{c}(x))\right]
\end{aligned}
$$

Optimisation implies the relationship $u'(\hat{c}(x)) = \beta u'(c(x))$, which we can use to eliminate the first term on the RHS so the equation simplifies to

$$
\begin{aligned}
(\rho - \partial_b r(b)b - r(b))\, u'(\hat{c}(x)) &= \partial_b \hat{c}(x) u''(\hat{c}(x)) \cdot (r(b)b - (1+\tau_c)c(x) + other) \\
&\quad + \sum_{z'} \lambda^{z \to z'} \left[u'(\hat{c}(x')) - u'(\hat{c}(x))\right] + \delta_R \left[u'(\hat{c}_R(x)) - u'(\hat{c}(x))\right]
\end{aligned}
$$

Note that we can collect many of these terms into the time–derivative of expected marginal utility ($\mathbb{E}[du'(c(x))]/dt$), by Ito's Lemma, after we add and subtract $u''(\hat{c}(x))\partial_b\hat{c}(x)\hat{c}(x)(1+\tau)$

$$
(\rho - \partial_b r(b)b - r(b))\, u'(\hat{c}(x)) = (1+\tau)u''(\hat{c}(x))\partial_b \hat{c}(x)(\hat{c}(x) - c(x)) + \mathbb{E}[du'(c(x))]/dt
$$

And using CRRA utility we know $c(x) = \beta^{1/\sigma}\hat{c}(x)$

$$
\begin{aligned}
(\rho - \partial_b r(b)b - r(b)) &= (1+\tau)\frac{u''(\hat{c}(x))\hat{c}(x)}{u'(\hat{c}(x))}\partial_b \hat{c}(x)(1-\beta^{1/\sigma}) + \frac{\mathbb{E}[du'(c(x))]/dt}{u'(c(x))} \\
\mathbb{E}\left[\frac{\dot{c}}{c}\right] &= \frac{1}{\sigma}\left[\partial_b r(b)b - r(b) - \rho - \underbrace{\sigma(1+\tau)(1-\beta^{1/\sigma})\partial_b \hat{c}(x)}_{\text{Naive bias distortion}}\right]
\end{aligned}
\tag{9}
$$

Hence the naif's Euler equation is a distorted version of the standard one, where the distortion scales with the bias, marginal propensity to consumer, and consumption tax.

Present–biased naif, dynamic Using the same steps as above, but with an added term $\partial_t v_t(x)$ on the HJB equation, which eventually introduces an influence from drift in the tax rate.

$$
\mathbb{E}\left[\frac{\dot{c}}{c}\right] = \frac{1}{\sigma}\left[(r_b(b)b + r(b)) - \rho - \sigma(1+\tau_t)(1-\beta^{1/\sigma})\partial_b \hat{c}_t(x) - \frac{\dot{\tau}}{1+\tau_t}\right] \tag{10}
$$

The dynamic tax adds two distortions, relative to the stationary Euler. First, its level alters the bias distortion we found in the previous section, relative to its stationary level. Second, expected drift in the tax rate introduces inter–temporal smoothing distortions common to all agents. The analysis in the main text focuses on the latter.

C Calibration

Life–cycle As well as the standard life–cycle transitions discussed in the main text, we impose two extra rules on the transition from working life to retirement that households do not anticipate. The first is a 'forced–retirement' level in working households' illiquid account. This limit is necessary to ensure the state–space is compact and that retirement balances don't get out of hand, but is set to $a_{max} = 15$, sufficiently high that it only affects a small measure of workers and keeps the retired population at realistic levels. And second, households that retire with negative total assets (liquid plus illiquid) are bankrupted. In the model, this means their gross positions in both accounts are returned to zero (the same as if they were born into retirement). Without bankruptcy, a small measure of households retire with the maximum debt, and they are stuck there because it is an absorbing state for present–biased households. This is a disastrous position to be in for these households—their consumption is close to zero and so their value is orders of magnitude lower than at other points in the state space. If the risk of destitution is not addressed, then the government's retirement policy is primarily focused on managing it, rather than the more prosaic concern of general retirement adequacy. The bankruptcy rule avoids the issue.

Working–life idiosyncratic risk Working households face idiosyncratic risk from jumps into, and out of, unemployment, and diffusion in their employed labour productivity. The jump transitions are governed by finding and separation rates of 0.0587 and 1.2 respectively; the former matches the quarterly separation rate in Shimer (2005), and the latter matches the mean 2.5 months spent in unemployment from the US Bureau of Labor Statistics.[6] The labour productivity process is calibrated to match the estimated AR(1) process in log–income residuals after individual characteristics effects are stripped out (Floden and Lindé, 2001).[7]

$$\ln z_t = 0.9136 \ln z_{t-1} + u_t,\ u_t \sim N(0, 0.0426)$$

We cast this in continuous time, following Achdou et al. (2022), so it becomes a quarterly Ornstein–Uhlenbeck process, which we discretise over $k = 3$ points with reflecting barri-

[6] Table A.12 'Unemployed persons by duration of unemployment: Monthly, Seasonally Adjusted'; recent average outside of recessions.

[7] This calibration is used in Maxted et al. (2024) and Guerrieri and Lorenzoni (2017). The estimates are from PSID data covering 1988–1991. More recent estimates of the same AR(1) process all get numbers around this i.e. with auto–regressive parameter at least 0.9, and standard–deviation at least 0.2 (e.g. Kaplan et al., 2020; Chang et al., 2013; Guvenen et al., 2023).

ers at one standard–deviation $\ln z \in [-\sqrt{0.0426}, \sqrt{0.0426}]$, and normalise so the stationary distribution of z, in levels, has a mean of one.[8]

Preferences Households have CRRA preferences over consumption, $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$, with a standard coefficient of relative risk aversion equal to $\sigma = 2$. Households discount the future at $\rho = 0.0025$, which corresponds to an annual discount rate of 0.99. Recall that households also face the risk of retirement and death. This boosts the effective annual discount rate to 0.97 and 0.94 for employed and retired households, within the range of standard estimates (Carroll et al., 2017).

Adjustment costs The adjustment cost function serves no purpose other than to deliver analytical solutions for the voluntary contribution policy d.[9] The baseline parameters that set common adjustment costs between working and retired households are set so the costs are trivial $(\chi_0, \chi_1) = (0, 0.001)$.

Prices & fiscal The base rate of return is $r = 0.0051$ per quarter (2% p.a.) and the borrowing penalty is $\omega = 0.4024$ (500% p.a.), set to be extortionate to impose a soft–borrowing constraint. The annual wage for a unit of effective labour is the numeraire (quarterly wage $w = 0.25$) so all other monetary values are relative to this. The government pays an unemployed benefit $w_u = 0.1$ to match the standard replacement rate of 0.4 from Shimer (2005). The baseline income tax is set to $\tau = 25\%$, the OECD average personal income tax rate for a single person with no children on the average wage. In both the stationary solution and the dynamic exercises later, the consumption tax τ_c is set internally to meet the fiscal rule.

Steady state government spending is $G = 0.0238$, targeting 15% of GDP, and steady state debt levels are $\bar{B} = 0.1589$, or 25% annual GDP. In both cases, GDP here is taken to be the aggregate income of the steady state employed population multiplied by 3/2 to adjust for the capital share.

D Computational solution

We solve the model using a finite–difference scheme (Achdou et al., 2022), with discrete, non–linear grids for the two endogenous assets, and three states for the idiosyncratic productivity

[8] This yields a stationary distribution defined by

$$\{z_L, z_M, z_H\} = \{0.681, 0.9516, 1.4652\} \text{ and } \{\pi_L, \pi_M, \pi_H\} = \{0.2686, 0.4628, 0.2686\}$$

Where π_i represent the stationary probability of being in state i.

[9] It is simple to use this function to impose an early withdrawal penalty during working life, as in the US.

process. The value–function updates are computed using the semi–implicit method (Achdou et al., 2022), and the policies are derived using the nested—drift algorithm introduced by Sabet and Schneider (2024).

D.1 The nested–drift algorithm

To find the optimal policy functions for a given value, this paper uses a version of the nested–drift algorithm introduced in Sabet and Schneider (2024). Combined with implicit updates of the value, described in Supplemental Appendix D.2, this is an approach to solving discretised systems of partial differential equations that is consistent and monotone, and so admits a solution that is stable to parameter choices. The below gives a sketch of the context for the algorithm, and its steps; see the original paper for more depth.

The problem Continuous time problems with multiple endogenous state variables, such as the one in this paper, typically have a solution defined by an HJB equation and first order conditions.

For the sake of communicating the algorithm, consider a simplified problem where households choose consumption and deposits between two accounts, where the latter are subject to adjustment costs. Their income is a random process y which switches between discrete states according to some transition intensities. The solution is thus defined by the HJB (and state transition equations) and first order conditions.

$$\begin{aligned}
\rho v(x) &= u(c) + \partial_b v(x) \cdot (r_b b + y - d - \chi(d,a) - c) + \partial_a v(x) \cdot (r_a a + d) \\
&\quad + \sum_{y' \in \mathcal{Y}} \lambda^{y \to y'} \left[v(x') - v(x) \right] \\
u'(c) &= \partial_b v(x) \\
\partial_a v(x) &= \partial_b v(x)(1 + \chi_d(d,a))
\end{aligned}$$

The above equations define the solution, but to implement it on a computer in a finite–difference scheme, we discretise the state–space x. Doing so introduces some imprecision that can be managed by using non–linear grids, placing a higher density of grid–points at areas where the value and policy functions are more likely to have curvature or kinks.

One form of imprecision is that the partial derivatives cannot be known exactly—we can find them with the forward and backward derivatives of the value at any point in the discretised state–space, or anywhere in between. So, which to choose?

Upwinding The standard approach to choosing between these options is to use 'upwinding'. That is, selecting the derivative that, when used, leads to drift in the state that goes in the direction assumed. In a single asset problem (e.g. if we fix $d = 0$ in the above), this consists of adding an additional condition to the solution. For a point in the state–space i the derivative is defined by

$$\partial_b v(x_i) = \begin{cases} \frac{v_{i+1}-v_i}{b_{i+1}-b_i} & \text{if } \dot{b}(x_i, c_i) > 0 \\ \frac{v_i-v_{i-1}}{b_i-b_{i-1}} & \text{if } \dot{b}(x_i, c_i) < 0 \\ u'(r_b b_i + y_i) & \text{if } \dot{b}(x_i, c_i) = 0 \end{cases}$$

Where the consumption policy is found from the FOC in the first two cases. In the third case, where there is no drift in the asset, the policy is identified by the drift equation and the derivative can be recovered from the first order condition.

In a two asset problem, things are more complex because one must consider the permutations of directions that each asset could be going in—both forward, both back, one still and the other drifting up, and so on. And also because multiple policies enter the one of the drift equations, and so the policies are jointly determined and can't necessarily be identified from the drift equations. In this case, they can be recovered using a root–finding algorithm based on the FOCs.

The nested–drift algorithm The algorithm we use here, originally from Sabet and Schneider (2024), solves this problem by exploring all possible cases (9 in a two–asset problem) of drift directions, using a root–finding algorithm to solve for policies in cases where necessary. The algorithm is monotone and consistent, and so satisfies the conditions necessary for local convergence (Barles and Souganidis, 1991). And it is also efficient because it places the most expensive root–finding steps last in the process, so they will only be reached if other alternatives have already failed.

The logic is as follows for each point in the state space x_{ijk}[10]:

1. Calculate key objects

    – The deposit policy that causes zero illiquid drift $\tilde{d} = -r_a a$

[10] Practically many of these operations are done simultaneously for all points in the state–space, but it's simpler to show one point at a time.

- The directional derivatives of the value

$$V_b^F = \frac{v_{i+1,j,k} - v_{i,j,k}}{b_{i+1} - b_i} \quad \text{and} \quad V_b^B = \frac{v_{i,j,k} - v_{i-1,j,k}}{b_i - b_{i-1}}$$
$$V_a^F = \frac{v_{i,j+1,k} - v_{i,j,k}}{a_{j+1} - a_j} \quad \text{and} \quad V_a^B = \frac{v_{i,j,k} - v_{i,j-1,k}}{a_j - a_{j-1}}$$

2. Assume forward liquid drift, setting $\partial_b v = V_b^F$.

   - Calculate the consumption policy from the FOC

   $$c^F = u^{-1}\left(V_b^F\right)$$

   - Calculate the deposit policy from the FOC under each assumed illiquid direction, using the FOC for forward and backward drifts, and the state transition equation for zero drift. Check the resulting deposit policies against the drift in the illiquid asset they're based on

   $$d^F = \begin{cases} d^{FF} & \text{if } d^{FF} > \tilde{d} \\ d^{FB} & \text{if } d^{FB} < \tilde{d} \\ \tilde{d} & \text{otherwise} \end{cases}$$

     This step nests the upwinding for the illiquid asset within the process for finding the liquid asset's drift.

   - Find the liquid drift implied by (c^F, d^F)

   $$\dot{b}^F = r_b b + y - c^F - d^F - \chi(d^F, a)$$

     and if it is positive move to the next point in the state–space. Otherwise continue.

3. Assume backward liquid drift, setting $\partial_b v = V_b^B$, and follow the equivalent steps to above to find the backward policies. This time, check whether the resulting liquid drift is backward

   $$\dot{b}^B = r_b b + y - c^B - d^B - \chi(d^B, a) < 0$$

   If it is, move to the next point in the state–space. Otherwise continue.

4. Assume zero liquid drift, setting $\partial_b v = u'(c^0)$. Given this assumption, we know the consumption policy from the state transition equation

   $$c^0 = r_b b + y - d^0 - \chi(d^0, a)$$

and the deposit policy will be defined implicitly by the FOC

$$\partial_a v(x) = u'\left(r_b b + y - d^0 - \chi(d^0, a)\right)(1 + \chi_d(d^0, a))$$

And we can solve for d^0 using this equation, and searching through different regions.

(a) Check if $d^0 > \tilde{d}$ using $\partial_a v = V_a^F$. If not, continue

(b) Check if $d^0 < \tilde{d}$ using $\partial_a v = V_a^B$. If not, continue

(c) Set $d^0 = \tilde{d}$

This algorithm results in the household's optimal policy functions $(\mathbf{c}, \mathbf{d})$ for a given value $\mathbf{V}$. It nests the up–winding of the illiquid asset within that of the liquid one. This is the efficient order to use because the illiquid asset's state transitions depend on only one policy function, and so the costly route–finding in the final steps is only necessary if all alternatives have been exhausted.

D.2 HJB and KFE

Suppose we know the household's optimal policy functions. The stationary solution to the household's problem can be expressed as a linear system, discretised over the state space, as follows

$$\rho\mathbf{V} = u(\mathbf{c}) + \mathbf{A}\mathbf{V}$$

Where $\mathbf{A}$ captures the finite–difference transition rates between all the states, from the perspective of the households (i.e. they anticipate retirement and death but not rebirth), taking into account their optimal policies, and $\mathbf{V} = \begin{bmatrix}\mathbf{v}' & \mathbf{v}_R'\end{bmatrix}'$ stacks the discretised working and retired values together. The solution to this linear system is

$$\mathbf{V} = [\rho\mathbf{I} - \mathbf{A}]^{-1} u(\mathbf{c})$$

Where updates are implemented using the semi–implicit scheme with update control step–size Δ

$$\mathbf{V}^{n+1} = [(1/\Delta + \rho)\mathbf{I} - \mathbf{A}^n]^{-1} [u(\mathbf{c}^n) + \mathbf{V}^n/\Delta]$$

We use the nested–drift algorithm in Sabet and Schneider (2024) to find the policy functions that define $\mathbf{A}^n$ and $\mathbf{c}^n$ for a given value guess $\mathbf{V}^n$, described in Appendix D.1. In practice, we solve the retired value first, and then use this as an input into the working–life value solution; doing so reduces the computational burden of inverting the matrix in the semi–implicit update step. Following Achdou et al. (2022), the stationary measure[11] discretised over the same state grids ($\mathbf{g}$) is the solution to the linear system

$$0 = \tilde{\mathbf{A}}'\mathbf{g}$$

Where $\tilde{\mathbf{A}}$ adjusts $\mathbf{A}$ for the state transitions households do not anticipate, which are (1) rebirth as a zero–asset worker after dying in retirement, (2) forced retirement if illiquid assets reach the threshold, and (3) bankruptcy upon retirement if total assets are negative.

Present–biased agents For known policy functions, the solution process and formulae are the same, with the exception that their state transition matrix $\mathbf{A}^\beta$ solve for their *long–run* value function, not their perceived value.

E Conditional expectations & MPC in continuous–time

The goal in this section is to find the expected path of a variable (a policy or state variable), given a starting point in the state space (x_0), from the perspective of an agent at that point in the state–space.[12]

$$y^e(x_0, t_n) = \mathbb{E}[y(x, t_n)|x_0 = x]$$

Suppose we have solved our problem, with time discretised over a sequence of N segments of a span from $t \in [0, T]$, where t_n denotes the end–point of the n–the segment such that $t_1 > 0$ and $t_N = T$. The solution is a sequence of policy vectors, transition matrices, and distributions (the measure, not the density) stacked over the state–space

$$\{\mathbf{y}(t_n), \mathcal{A}(t_n), \mathbf{g}(t_n)\}_{n=1}^N$$

Where $\mathbf{y}(t_n)$ is the policy that applies during the span $\Delta(n) = t_n - t_{n-1}$ and $\mathbf{g}(t_n)$ is the state–distribution at the point in time t_n.

[12] Important if (a) expectations are not fully rational, or (b) agents don't internalise some transitions like reincarnation.

Finding the distribution updates The sequence of distributions is found using the implicit, discretised Kolmogorov Forward equation, following Achdou et al. (2022)

$$\frac{\mathbf{g}(t_n) - \mathbf{g}(t_{n-1})}{\Delta(n)} = \mathcal{A}(t_n)'\mathbf{g}(t_n)$$

Where the starting distribution is the stationary solution $\mathbf{g}(t_0) = \mathbf{g}_{ss}$ The discretised KF equation rearranges into the implicit updating equation

$$\mathbf{g}(t_n) = \mathcal{L}(t_n)\mathbf{g}(t_{n-1}) = \prod_{j=1}^{n} \mathcal{L}(t_j)\mathbf{g}_{ss}$$

With the implicit transition matrix defined by $\mathcal{L}(t_n) = [\mathcal{I} - \Delta(n)\mathcal{A}(t_n)']^{-1}$.

Aggregation To build a sequence of aggregate policies, we use

$$Y(t_n) = \mathbf{y}(t_n)'\mathbf{g}(t_{n-1})$$

So the expected aggregate at the n–th point in the time grid is the policy from that point, integrated over the distribution at that point. If we substitute the definition of the distribution into this expression we have

$$Y(t_n) = \begin{cases} \mathbf{y}(t_1)'\mathbf{g}_{ss} & \text{if } n = 1 \\ \mathbf{y}(t_n)' \prod_{j=1}^{n} \mathcal{L}(t_j)\mathbf{g}_{ss} & \text{if } n > 1 \end{cases}$$

Focusing on the cases where $n > 1$, we can see that the aggregate is made up of three terms—the future policy: $\mathbf{y}(t_n)$, the starting distribution: $\mathbf{g}_{ss}$, and a matrix that affects the expected movement of measure around the state–space between the starting point and t_n: $\prod_{j=1}^{n} \mathcal{L}(t_n)$.

We can collect these terms in two different ways to get the same aggregate. In the standard aggregation, the matrix is applied to the starting distribution, and so the expected aggregate in t_n is the policy in that period integrated over the distribution at the beginning of that period. If we instead apply the matrix to the future policy, then it produces the expected policy in t_n from the point of view of a point in the state space at t_0: $\mathbf{y}^e(t_n)$. Aggregation is then achieved by

$$Y(t_n) = \mathbf{y}^e(t_n)\mathbf{g}_{ss} \text{ where } \mathbf{y}^e(t_n) = \mathbf{y}(t_n)' \prod_{j=1}^{n} \mathcal{L}(t_j)$$

The two approaches measure the same aggregate, but they create very different sub–aggregate objects—the first a future distribution, and the second an expected policy. The latter $\mathbf{y}^e(t_n)$ is exactly the conditional expectation we are seeking—it tells us the *expected* value of the policy y at some future point in time t for each *starting–point* in the state–space.[13]

With the conditionally expected policy in hand, we can use it for other purposes than aggregation. Say we wish to describe expected policies by some quantile of the distribution over states. This is very easy with above object, with only two steps required:

1. Calculate the relevant quantile for each point in the starting state space

2. Find the average of the conditional policy, conditional on the quantile x_0 is in

$$y^e(t_n, q) = \mathbb{E}[y^e(t_n, x_0)|x_0 \in q] = \frac{\int_{x \in q} y^e(t_n, x) dG(x)}{\int_{x \in q} dG(x)}$$

Marginal propensity to consume Following Achdou et al. (2022) we calculate marginal propensity to consume over a discrete time–interval τ (always one–quarter in this paper) using the following formula

$$MPC_\tau(x_0) = \partial_b C_\tau(x_0) \text{ where } C_\tau(x_0) = \int_0^\tau c^e(x_0, t) dt$$

Where the conditional expected consumption policy $c^e(t, x)$ is defined in Section E.

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[13] This is a discretised implementation of the Feynman–Kac formula.

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