NatSci Maths Booklet

Patrick Nickols

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1 Mathematics

1.1 Algebra

1.1.1 Powers

$$
\begin{aligned}
&\frac{x^{-\frac{1}{5}} \times \left(x^{\frac{2}{3}}\right)^6}{x \times \sqrt[2]{x^5} \times \sqrt[5]{x^2}} \\
&= x^{-\frac{1}{5}} \times x^{\frac{12}{3}} \times x^{-1} \times x^{-\frac{5}{2}} \times x^{-\frac{2}{5}} \\
&= x^{\left(-\frac{1}{5}+4-1-\frac{5}{2}-\frac{2}{5}\right)} \\
&= x^{\left(\frac{-2+40-10-25-4}{10}\right)} \\
&= x^{-\frac{1}{10}}.
\end{aligned}
$$

1.1.2 Factorisation

1.
$$
\begin{aligned}
&x^2 - 1 \\
&= (x+1)(x-1).
\end{aligned}
$$

2.
$$
\begin{aligned}
&a^2 - 4ab + 4b^2 \\
&= (a-2b)^2.
\end{aligned}
$$

3.
$$
\begin{aligned}
&x^3 - 1 \\
&= (x-1)(x^2+x+1).
\end{aligned}
$$

1.1.3 Quadratic equations

1.

$$\begin{aligned} x^2 - 5x + 6 &= 0 \\ \iff (x-3)(x-2) &= 0 \\ \iff x &= 2, 3. \end{aligned}$$

2.

$$\begin{aligned} x^2 + 2x &= 0 \\ \iff (x)(x+2) &= 0 \\ \iff x &= -2, 0. \end{aligned}$$

3.

$$\begin{aligned} x^2 - x - 1 &= 0. \\ \iff x &= \frac{1 \pm \sqrt{5}}{2}. \end{aligned}$$

4.

$$x^4 - 3x^2 + 2 = 0.$$

We let $y = x^2$ so

$$\begin{aligned} y^2 - 3y + 2 &= 0 \\ \iff (y-2)(y-1) &= 0, \end{aligned}$$

and so

$$y = 1, 2$$

which means

$$x = \pm 1, \pm\sqrt{2}.$$

1.1.4 Completing the square

1.

$$\begin{aligned} &x^2 - 2x + 6 \\ &= (x-1)^2 + 5. \end{aligned}$$

Hence

$$x^2 - 2x + 6 \geq 5.$$

Because $(x-1)^2$ and therefore x^2-2x+6 is increasing in $[2,3]$ the minimum value it attains is attained in this interval when $x = 2$ and so $x^2-2x+6 = 6$.

2.
$$x^4 + 2x^2 + 2$$
$$= (x^2 + 1)^2 + 1.$$

We know that
$$x^2 \geq 0$$
and consequently
$$x^2 + 1 \geq 1$$
and so
$$(x^2 + 1)^2 \geq 1$$
and hence, finally,
$$x^4 + 2x^2 + 2 \geq 2.$$

1.1.5 Inequalities

1.
$$x^2 - 3x < 4$$
$$\iff x^2 - 3x - 4 < 0$$
$$\iff (x - 4)(x + 1) < 0$$
$$\iff x \in (-1, 4).$$

2.
$$y^3 > 2y^2 + 3y$$
$$\iff y^3 - 2y^2 - 3y < 0$$
$$\iff y(y - 3)(y + 1) < 0$$
$$\iff y \in (0, 3).$$

1.1.6 Factor theorem

1. Divide $x^3 + 5x^2 - 2x - 24$ by $(x + 4)$ and hence factorise it completely.
$$x^3 + 5x^2 - 2x - 24 = (x + 4)(x^2 + x - 6) = (x + 4)(x - 2)(x + 3).$$

2. Use the factor theorem to factorise $t^3 - 7t + 6$.
We note that $(1)^3 - 7(1) + 6 = 0$ and so by the factor theorem $(t - 1)$ is a factor. Then dividing reveals
$$t^3 - 7t + 6 = (t - 1)(t^2 + t - 6) = (t - 1)(t + 3)(t - 2).$$

3. Simplify $\dfrac{x^3+x^2-2x}{x^3+2x^2-x-2}$.

   We see that $(x-1)$ is a factor of both the numerator and denominator and proceed from there

$$\begin{aligned}\frac{x^3+x^2-2x}{x^3+2x^2-x-2} &= \frac{(x-1)\left(x^2+2x\right)}{(x-1)\left(x^2+3x+2\right)} \\ &= \frac{(x)(x+2)}{(x+2)(x+1)} \\ &= \frac{x}{x+1}.\end{aligned}$$

1.1.7 Partial fractions

1.
$$\frac{2}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}.$$

   So we see
$$2 = Ax - A + Bx + B.$$

   So
$$A + B = 0 \iff A = -B.$$
$$B - A = 2B = 2.$$

   So $A = -1$, $B = 1$. Thus
$$\frac{2}{(x+1)(x-1)} = \frac{1}{x-1} - \frac{1}{x+1}.$$

2.
$$\frac{x+13}{(x+1)(x-2)(x+3)} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x+3}.$$

   So
$$Ax^2 + Ax - 6A + Bx^2 + 4Bx + 3B + Cx^2 - Cx - 2C.$$
$$A + B + C = 0.$$
$$A + 4B - C = 1.$$
$$-6A + 3B - 2C = 13.$$

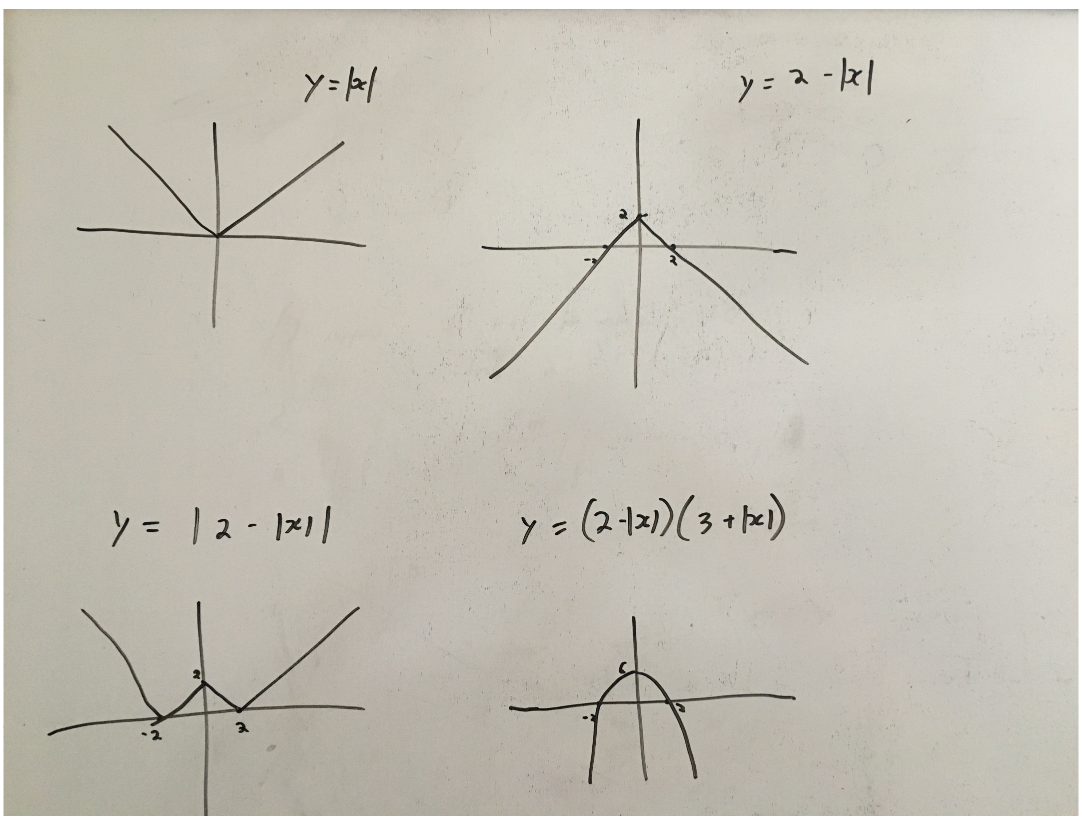
   Gaussian elimination yields
$$2A + 5B = 1.$$
$$8A + 5B = -11.$$

   So $6A = -12$, so $A = -2$. Hence $B = 1$ and $C = 1$ so
$$\frac{x+13}{(x+1)(x-2)(x+3)} = \frac{1}{x-2} + \frac{1}{x+3} - \frac{2}{x+1}.$$

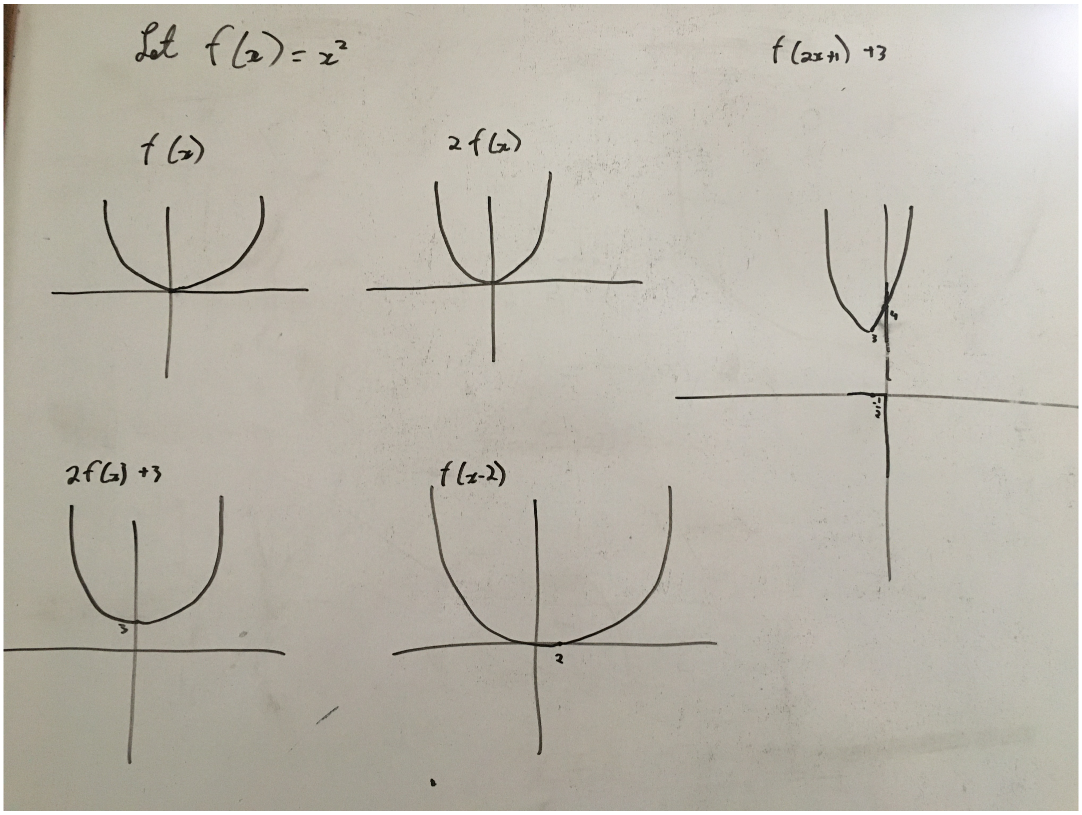
3.
$$\frac{4x+1}{(x+1)^2(x-2)} = \frac{Ax+B}{(x+1)^2} + \frac{C}{x-2}.$$
So
$$Ax^2 + (B-2A)x - 2B + Cx^2 + 2Cx + C = 4x + 1.$$
$$A + C = 0.$$
$$-2A + B + 2C = 4.$$
$$-2B + C = 1.$$
Subbing in $A = -C$,
$$B + 4C = 4.$$
$$-2B + C = 1.$$
So $9C = 9$, so $C = 1$ so $B = 0$ and $A = -1$. Hence
$$\frac{4x+1}{(x+1)^2(x-2)} = \frac{1}{x-2} - \frac{x}{(x+1)^2}.$$

1.2 Functions and Curve Sketching

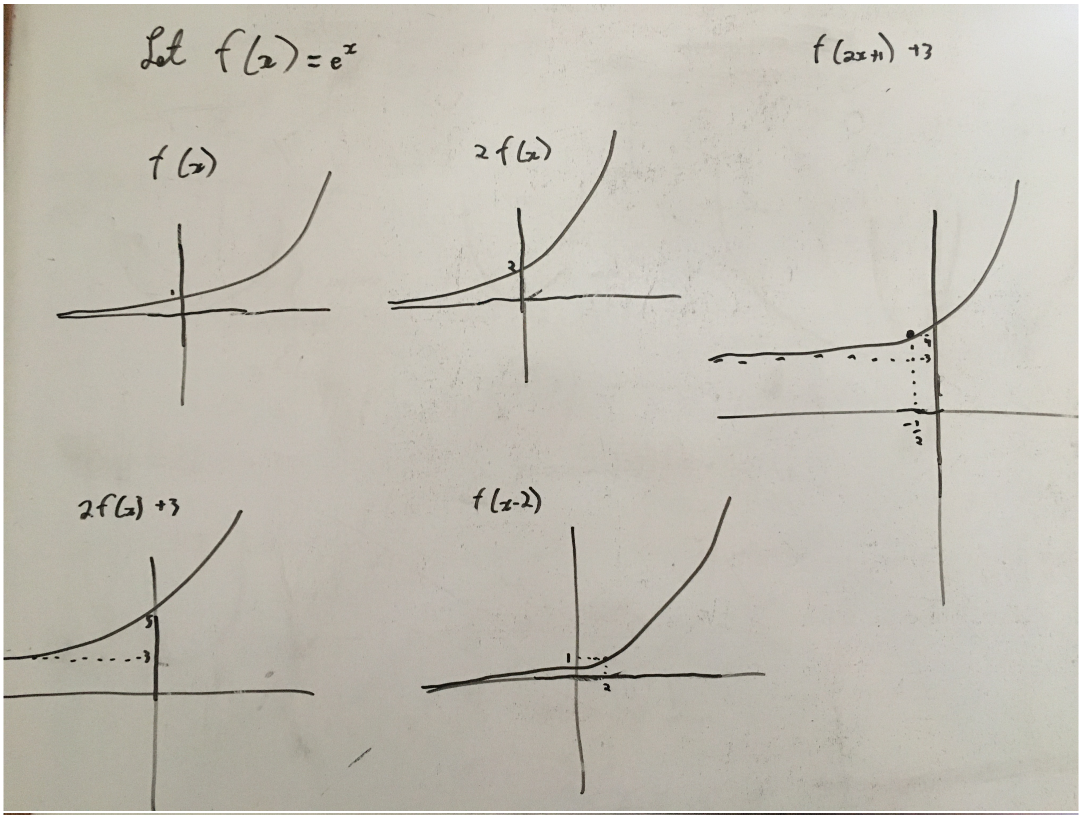
1.2.1 Modulus function

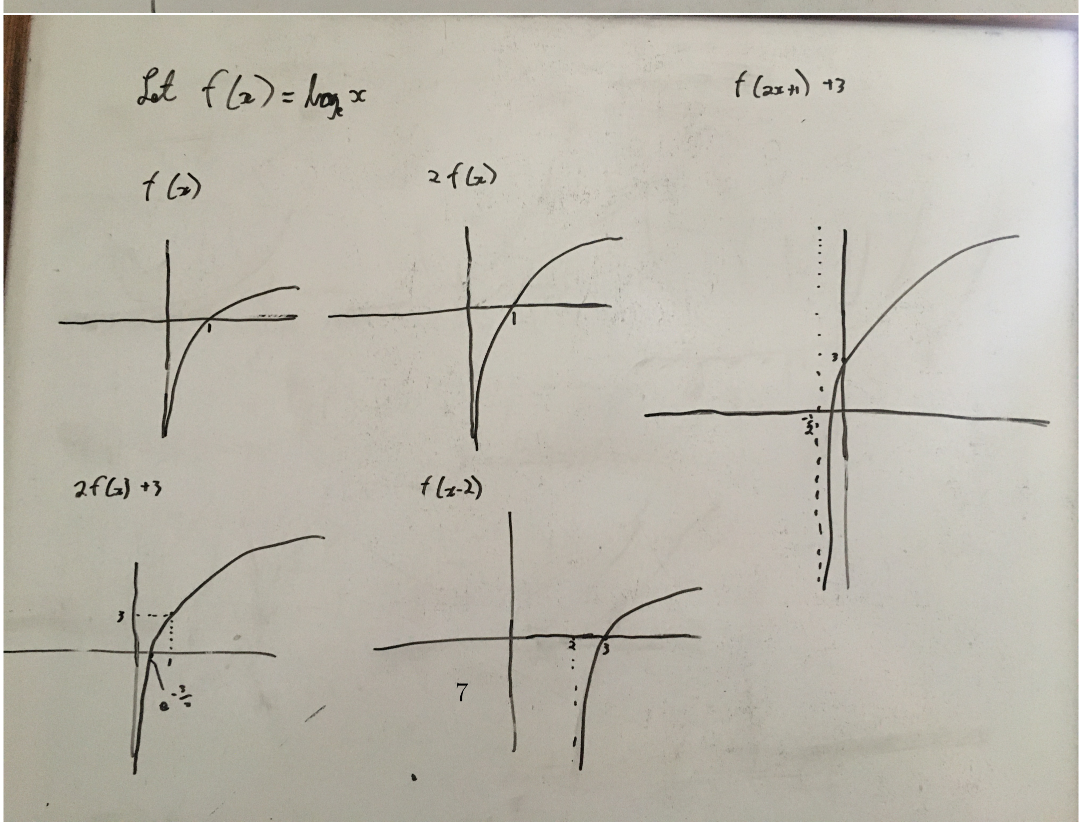


1.2.2 Transformations of functions

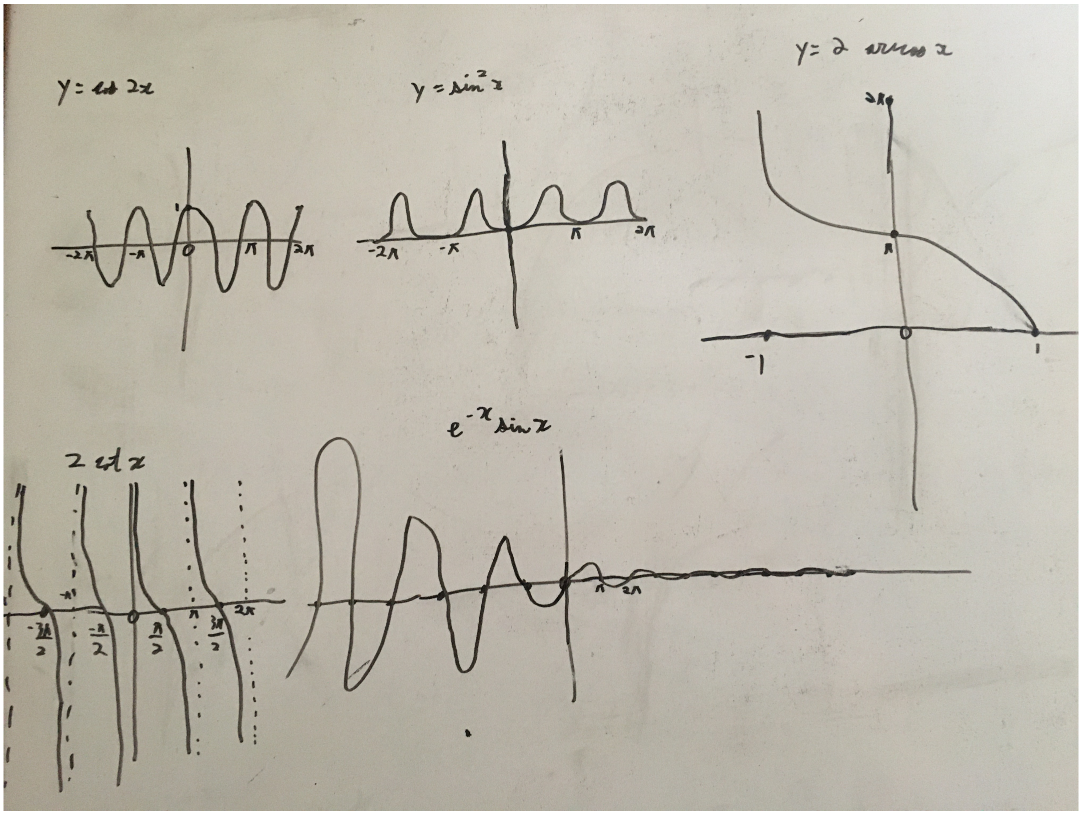


1.2.3 Transformations of functions





1.2.4 Trig and inverse trig functions



1.2.5 Logarithms

1. If $3 = 9^{-x}$, find x.

$$\begin{aligned} 3 &= 9^{-x} \\ \log_9 3 &= -x \\ \frac{1}{2} &= -x \\ x &= \frac{1}{2}. \end{aligned}$$

2. If $\log_a b = c$, show that $c = \dfrac{\log_\alpha b}{\log_\alpha a}$ for any base α.

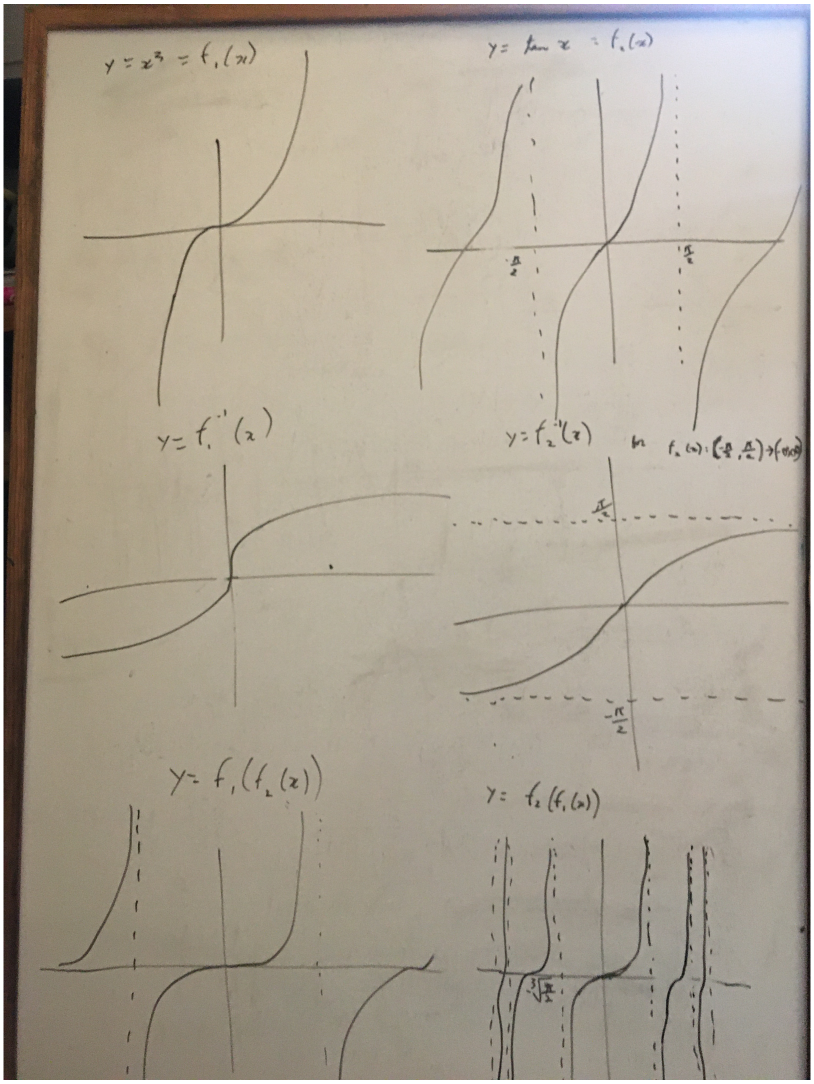
$$\begin{aligned} \log_a b &= c \\ b &= a^c \end{aligned}$$

$$\begin{aligned}
\log_\alpha b &= \log_\alpha(a^c) \\
\log_\alpha b &= c \log_\alpha a \\
c &= \frac{\log_\alpha b}{\log_\alpha a}.
\end{aligned}$$

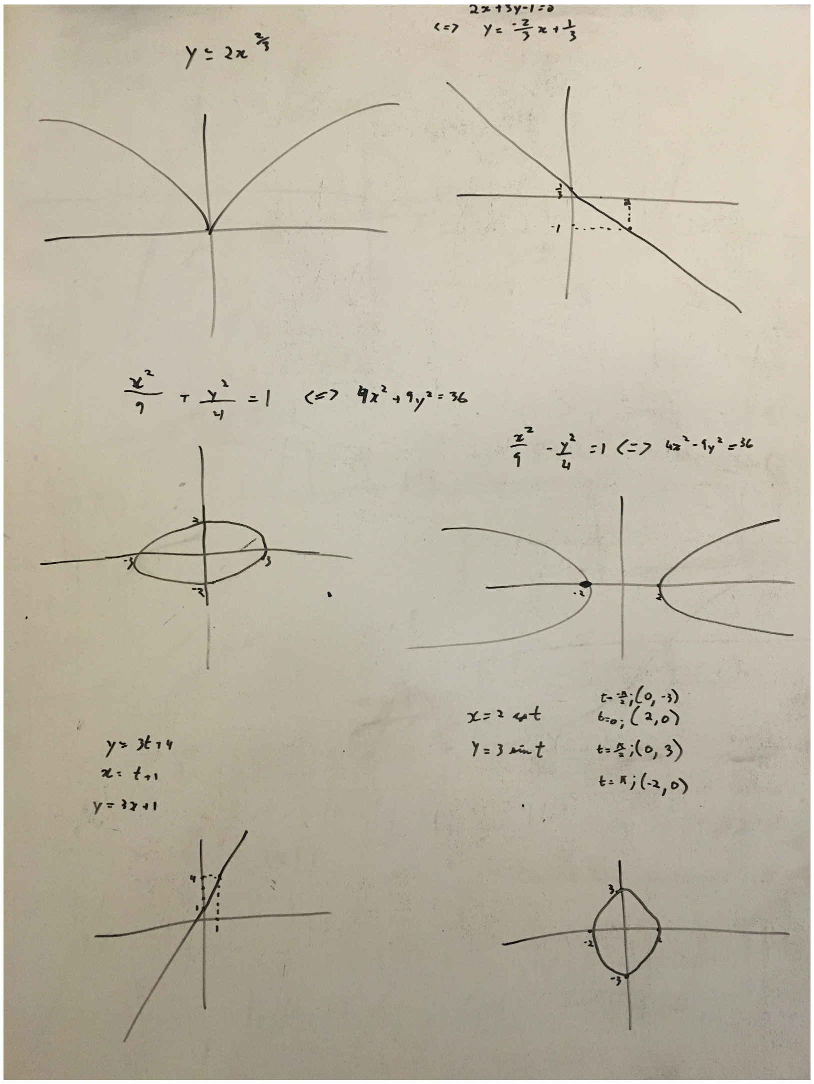
3. Find x if $16 \log_x 3 = \log_3 x$.

$$\begin{aligned}
16 \log_x 3 &= \log_3 x \\
16 \frac{\ln 3}{\ln x} &= \frac{\ln x}{\ln 3} \\
16(\ln 3)^2 &= (\ln x)^2. \\
\ln x &= \pm 4 \ln 3 \\
x &= e^{\pm 4 \ln 3} \\
&= \frac{1}{81}, 81.
\end{aligned}$$

1.2.6 Composition of functions



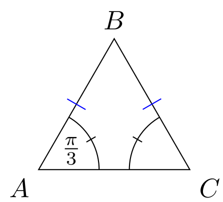
1.2.7 Curve sketching



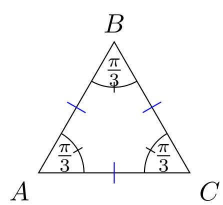
1.3 Geometry

1.3.1 Triangles

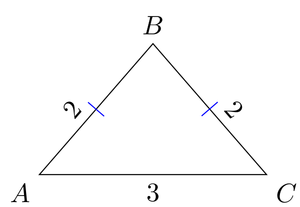
1. In triangle ABC, $AB = 1$, $BC = 1$, and $\angle A = \dfrac{\pi}{3}$ radians. Find CA and $\angle B$.



Since we know the triangle is isosceles since $BC = AB$, we see that $\angle C = \frac{\pi}{3}$ radians. But that means that $\angle B = \frac{\pi}{3}$ radians as well (since the three internal angles must sum to π) and so the triangle is equilateral. Consequently every side has a length of 1, and so $AC = 1$. In fact the picture should look like the following.



2. In triangle ABC, $AB = 2$, $BC = 2$ and $AC = 3$. Find the angles of the triangle.



By the law of cosines we have

$$9 = 8 - 8\cos(\angle B),$$

or equivalently

$$\cos(\angle B) = -\frac{1}{8}.$$

$$\angle B \approx 1.696 \text{ radians}.$$

Then since $\angle A = \angle C$ and the three angles sum to π, we have

$$\angle A = \angle C \approx 0.723 \text{ radians}.$$

1.3.2 Circles

Find, for a sector of angle $\frac{\pi}{3}$ radians of a disc of radius 3:

1. the length of the permiter; and
2. the area.

1. Since the sector has as an angle of $\frac{\pi}{3}$ it is a sixth of the circle. Consequently its arc length is
$$\frac{1}{6}2\pi \times 3 = \pi.$$
In addition, it has two sides that are radii and so have lengths of 3 each. Hence the perimter is $6 + \pi$.
2. Again, the area is a sixth and so is equal to
$$\frac{1}{6} \times 9\pi = \frac{3}{2}\pi \text{ units squared.}$$

1.4 Sequences And Series

1.4.1 Arithmetic progressions

An arithmetic progression has third term α and ninth term β. Find the sum to thirty terms.
If the first term is u and the common difference is d, then we have

$$\alpha = u + 2d$$

$$\beta = u + 8d$$

and the sum of the overall series is

$$15(2u + 29d) = 30u + 435d.$$

We thus wish to write
$$a\alpha + b\beta = 30u + 435d.$$
Expanding gives
$$(a + b)u + (2a + 8b)d = 30u + 435d.$$
We thus have a two variable system of equations:

$$a + b = 30.$$

$$2a + 8b = 435.$$

So
$$b = \frac{125}{2}$$
and
$$a = \frac{-65}{2}.$$
So we have that $\Sigma = -32.5\alpha + 62.5\beta$.

1.4.2 Binomial expansions

Expand the following expressions, using the binomial expansion, as far as the fourth term:

1. $(1+x)^3$
2. $(2+x)^4$
3. $\left(2+\frac{3}{x}\right)^5$.

1. $$(1+x)^3 = 1 + 3x + 3x^2 + x^3.$$
2. $$(2+x)^4 = 16 + 32x + 24x^2 + 8x^3 + (x^4).$$
3. $$\left(2+\frac{3}{x}\right)^5 = 32 + \frac{240}{x} + \frac{720}{x^2} + \frac{1080}{x^3} + \frac{810}{x^4} + \frac{243}{x^5}.$$

1.4.3 Arithmetic and geometric progressions

Prove that $\sum_1^N n = \frac{1}{2}N(N+1)$.
Evaluate:

1. the sum of the odd integers from 11 to 99 inclusive.
2. $\sum_{n=1}^{5}(3n+2)$
3. $\sum_{n=0}^{N}(an+b)$ (a and b are constants)
4. $\sum_{r=0}^{10} 2^r$
5. $\sum_{n=0}^{N} ar^{2n}$ (a and r are constants).

Proof. We have

$$\sum_{n=1}^{N} n = 1 + 2 + 3 + \ldots + N.$$

This is a sum of N terms. The mean of these terms, since they are consecutive numbers, is the middle term. That is half the $(N+1)^{\text{th}}$ term. So we have a sum of N terms, which have a mean of $\frac{1}{2}(N+1)$. Hence their sum is the product of the mean and the number of terms:

$$\frac{1}{2}N(N+1).$$

□

1.
$$\sum_{n=0}^{44} 11 + 2n = 45(55) = 2475.$$

2.
$$\sum_{n=1}^{5}(3n+2) = 5 + 8 + 11 + 14 + 17 + 20 = 75.$$

3.
$$\sum_{n=0}^{N}(an+b) = \frac{1}{2}(aN+2b)(N).$$

4.
$$\sum_{r=0}^{10} 2^r = 2^{11} - 1 = 2047.$$

5.
$$\sum_{n=0}^{N} ar^{2n} = \frac{a(r^{2N}-1)}{r^2-1}.$$

1.4.4 Iterative sequences

The sequence u_n satisfies $u_{n+1} = ku_n$ where k is a fixed number, and $u_0 = 1$. Express u_n in terms of k. Describe the behaviour of u_n for large n in the different cases that arise according to the value of k.

$$u_{n+1} = ku_n.$$
$$u_0 = 1.$$
$$u_n = k^n.$$

- If $k > 1$, then $u_n \to \infty$.
- If $k = 1$, then $u_n = 1$ for all n.
- If $0 < k < 1$ then $u_n \to 0$.
- If $k = 0$ then $u_n = 0$, $n \neq 0$.
- If $-1 < k < 0$ then $u_n \to 0$.
- If $k = -1$ then $u_{2n} = 1$ and $u_{2n+1} = -1$.
- If $k < -1$ then $u_{2n} \to \infty$ and $u_{2n+1} \to -\infty$.

1.4.5 Binomial expansion for rational powers

Find the first four terms in the expansions in ascending powers of x of the following expressions, stating for what values of x the expansion is valid in each case:

1. $(1+x)^{\frac{1}{2}}$
2. $(2+x)^{\frac{2}{5}}$
3. $\dfrac{(1+2x)^{\frac{1}{2}}}{(2+x)^{\frac{1}{3}}}$

1.

$$\begin{aligned}(1+x)^{\frac{1}{2}} &\approx 1+\frac{1}{2}x+\frac{1}{2!}\frac{1}{2}\cdot\frac{-1}{2}x^2+\frac{1}{3!}\cdot\frac{1}{2}\cdot\frac{-1}{2}\cdot\frac{-3}{2}x^3\\ &= 1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3, |x|<1.\end{aligned}$$

2.

$$(2+x)^{\frac{2}{5}} = 2^{\frac{2}{5}}(1+\frac{x}{2})^{\frac{2}{5}} \approx 2^{\frac{2}{5}}\left(1+\frac{2}{5}\cdot\frac{x}{2}+\frac{1}{2!}\cdot\frac{2}{5}\cdot\frac{-3}{5}\cdot\frac{x^2}{4}+\frac{1}{3!}\cdot\frac{2}{5}\cdot\frac{-3}{5}\cdot\frac{-8}{5}\cdot\frac{x^3}{8}\right), |x|<2.$$

$$= 2^{\frac{2}{5}}\left(1+\frac{1}{5}x-\frac{3}{100}x^2+\frac{1}{125}x^3\right)$$

3.

$$\begin{aligned}&\frac{(1+2x)^{\frac{1}{2}}}{(2+x)^{\frac{1}{3}}}\\ &= (1+2x)^{\frac{1}{2}}(2+x)^{-\frac{1}{3}}\\ &= 2^{-\frac{1}{3}}(1+2x)^{\frac{1}{2}}\left(1+\frac{x}{2}\right)^{-\frac{1}{3}}\\ &\approx 2^{-\frac{1}{3}}\left(1+\frac{1}{2}2x+\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{-1}{2}4x^2+\frac{1}{3!}\cdot\frac{1}{2}\cdot\frac{-1}{2}\cdot\frac{-3}{2}8x^3\right)\times\dots\\ &\dots\left(1-\frac{1}{3}\cdot\frac{x}{2}+\frac{1}{2!}\cdot\frac{-1}{3}\cdot\frac{-4}{3}\cdot\frac{x^2}{4}+\frac{1}{3!}\cdot\frac{-1}{3}\cdot\frac{-4}{3}\cdot\frac{-7}{3}\frac{x^3}{8}\right)\\ &= 2^{-1\frac{1}{3}}\left(1+x-\frac{1}{2}x^2+\frac{1}{2}x^3\right)\left(1-\frac{x}{6}+\frac{x^2}{18}-\frac{7x^3}{324}\right)\\ &2^{-\frac{1}{3}}\left(1+\frac{5}{6}x-\frac{11}{18}x^2+\frac{50}{81}x^3\right), |x|<\frac{1}{2}.\end{aligned}$$

1.4.6 Composition of approximations

Given that, for small $\theta, \sin\theta \approx \theta - \frac{1}{6}\theta^3$ and $\cos\theta \approx 1 - \frac{1}{2}\theta^2$, find an approximation, ignoring powers of θ greater than 3, for $\sin(\frac{1}{2}\theta)\cos\theta + \sec 2\theta$.
Substituing the approximations, we obtain:

$$\begin{aligned}
&\left(\frac{\theta}{2} - \frac{\theta^3}{48}\right)\left(1 - \frac{\theta^2}{2}\right) + (1 - 2\theta^2)^{-1} \\
&\approx \frac{\theta}{2} - \frac{13\theta^3}{48} + (1 - 2\theta^2)^{-1} \\
&\approx \frac{\theta}{2} - \frac{13\theta^3}{48} + (1 + 2\theta^2) \\
&= 1 + \frac{\theta}{2} + 2\theta^2 - \frac{13\theta^2}{48}.
\end{aligned}$$

1.5 Trigonometry

1.5.1 Solving trig equations

Find the four values of θ in the range 0 to 2π that satisfy the equation $2\sin^2\theta = 1$.

$$\begin{aligned}
2\sin^2\theta &= 1 \\
\sin^2\theta &= \frac{1}{2} \\
\sin\theta &= \pm\frac{\sqrt{2}}{2} \\
\theta &= \pi/4, 3\pi/4, 5\pi/4, 7\pi/4.
\end{aligned}$$

1.5.2 Trig identities

Prove that $\dfrac{\cot^2 x + \sin^2 x}{\cos x + \text{cosec } x} = \text{cosec } x - \cos x$.
We start by multiplying the right hand side by a carefully chosen form of 1 and proceed from there:

$$\begin{aligned}
&(\text{cosec } x - \cos x)\frac{\text{cosec } x + \cos x}{\text{cosec } x + \cos x} \\
&= \frac{\frac{1}{\sin^2 x} - \cos^2 x}{\text{cosec } x + \cos x} \\
&= \frac{\frac{1}{\sin^2 x} - (1 - \sin^2 x)}{\text{cosec } x + \cos x} \\
&= \frac{\frac{1 - \sin^2 x}{\sin^2 x} + \sin^2 x}{\text{cosec } x + \cos x} \\
&= \frac{\cot^2 x + \sin^2 x}{\text{cosec } x + \cos x}.
\end{aligned}$$

1.5.3 Trig identities

By writing $\dfrac{\pi}{12} = \dfrac{\pi}{3} - \dfrac{\pi}{4}$, use trigonometric identities to evaluate:

1. $\cos\dfrac{\pi}{12}$
2. $\sin\dfrac{\pi}{12}$
3. $\cot\dfrac{\pi}{12}$

1.

$$\begin{aligned}\cos\frac{\pi}{12} &= \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)\\ &= \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}\\ &= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}\\ &= \frac{\sqrt{6}+\sqrt{2}}{4}.\end{aligned}$$

2.

$$\begin{aligned}\sin\frac{\pi}{12} &= \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)\\ &= \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\cos\frac{\pi}{3}\\ &= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}.\end{aligned}$$

3.

$$\begin{aligned}\cot\frac{\pi}{12} &= \frac{\cos\frac{\pi}{12}}{\sin\frac{\pi}{12}}\\ &= \frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}}\\ &= \frac{8+4\sqrt{3}}{4}\\ &= 2+\sqrt{3}.\end{aligned}$$

1.5.4 Trig identities

If $t = \tan \frac{1}{2}\theta$, express the following in terms of t:

1. $\cos\theta$
2. $\sin\theta$
3. $\tan\theta$

1. We first note that

$$t^2 + 1 = \frac{\sin^2 \frac{\theta}{2}}{\cos^2 \frac{\theta}{2}} + 1 = \frac{\sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2}}{\cos^2 \frac{\theta}{2}} = \frac{1}{\cos^2 \frac{\theta}{2}}.$$

   Hence

$$\cos^2 \frac{\theta}{2} = \frac{1}{1+t^2}.$$

   Then

$$\begin{aligned} \cos\theta &= 2\cos^2 \frac{\theta}{2} - 1 \\ &= \frac{2}{1+t^2} - 1 \\ &= \frac{1-t^2}{1+t^2}. \end{aligned}$$

2. Similarly,

$$\begin{aligned} \sin\theta &= 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ &= 2t\cos^2\frac{\theta}{2} \\ &= \frac{2t}{1+t^2}. \end{aligned}$$

3. Finally, from the tan double angle identity,

$$\tan\theta = \frac{2t}{1-t^2}.$$

1.5.5 Trig identities

Simplify $\tan(\arctan \frac{1}{3} + \arctan \frac{1}{4})$.
We proceed via the tan sum-angle identity and obtain:

$$\begin{aligned} \tan\left(\arctan\frac{1}{3} + \arctan\frac{1}{4}\right) &= \frac{\frac{1}{3}+\frac{1}{4}}{1-\frac{1}{3}\cdot\frac{1}{4}} \\ &= \frac{7}{11}. \end{aligned}$$

1.5.6 Trig identities

If A, B and C are the angles of a triangle, prove that

$$\cos\left(\frac{B-C}{2}\right) - \sin\left(\frac{A}{2}\right) = 2\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right).$$

We first apply a compound angle identity backwards to to the right hand side to obtain

$$2\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) = \cos\left(\frac{B-C}{2}\right) - \cos\left(\frac{B+C}{2}\right).$$

Since A, B and C are three internal angles in a triangle we know that $A+B+C = \pi$ and consequently $B+C = \pi - A$, which, when substituted gives us

$$2\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) = \cos\left(\frac{B-C}{2}\right) - \cos\left(\frac{\pi-A}{2}\right).$$

But, $\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$ and so $\cos\left(\frac{\pi-A}{2}\right) = \sin\left(\frac{A}{2}\right)$, which when substituted gives us, as desired,

$$2\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) = \cos\left(\frac{B-C}{2}\right) - \sin\left(\frac{A}{2}\right).$$

1.5.7 Solving trig equations

Write $\sqrt{3}\sin\theta+\cos\theta$ in the form $A\sin(\theta+\alpha)$, where A and α are to be determine. Working backwards, we see

$$A\sin(\theta+\alpha) = A(\sin\theta\cos\alpha + \sin\alpha\cos\theta).$$

From here we equate coefficients and see that

$$A\cos\alpha = \sqrt{3}$$

$$A\sin\alpha = 1.$$

Dividing these two equations by each other yields

$$\tan\alpha = \frac{1}{\sqrt{3}}$$

and so one solution is

$$\alpha = \frac{\pi}{6}.$$

From there we can substitute this into either of the equations to obtain

$$A = 2.$$

So

$$\sqrt{3}\sin\theta + \cos\theta = 2\sin\left(\theta + \frac{\pi}{6}\right).$$

1.5.8 Solving trig equations

Find the values of θ in the range 0 to 2π which satisfy the equation

$$\cos\theta + \cos 3\theta = \sin\theta + \sin 3\theta.$$

We apply the compound angle identities repeatedly and obtain

$$\cos\theta + \cos\theta\cos(2\theta) - \sin\theta\sin(2\theta) = \sin\theta + \sin\theta\cos(2\theta) + \sin(2\theta)\cos\theta$$

$$\iff \cos\theta + \cos\theta(\cos^2\theta - \sin^2\theta) - \sin\theta(2\sin\theta\cos\theta) = \sin\theta + \sin\theta(\cos^2\theta - \sin^2\theta) + \cos\theta(2\sin\theta\cos\theta)$$

$$\iff \cos\theta + \cos^3\theta - \cos\theta\sin^2\theta - 2\sin^2\theta\cos\theta = \sin\theta + \cos^2\theta\sin\theta - \sin^3\theta + 2\cos^2\theta\sin\theta.$$

Now we try and factor cleverly and look for where we can replace $1 - \sin^2\theta$ with $\cos^2\theta$.

$$\cos\theta(1 + \cos^2\theta - \sin^2\theta - 2\sin^2\theta) = \sin\theta + \cos^2\theta\sin\theta - \sin^3\theta + 2\cos^2\theta\sin\theta$$

$$\cos\theta(2\cos^2\theta - 2\sin^2\theta) = \sin\theta(1 + \cos^2\theta - \sin^2\theta + 2\cos^2\theta)$$
$$\cos\theta(2\cos^2\theta - 2\sin^2\theta) = \sin\theta(4\cos^2\theta)$$

Before we divide on both sides by $\cos\theta$, we must check if $\cos\theta$ could be equal to zero. If it were then $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. If $\theta = \frac{\pi}{2}$, this would mean that $\cos\theta + \cos 3\theta = 0$ while $\sin\theta + \sin 3\theta = 0$. Hence $\theta = \frac{\pi}{2}$ is a valid solution. Similarly, checking reveals that so too is $\frac{3\pi}{2}$. Now we can divide:

$$2\cos^2\theta - 2\sin^2\theta = 4\sin\theta\cos\theta$$
$$2\cos(2\theta) = 2\sin(2\theta)$$
$$\cos(2\theta) = \sin(2\theta)$$
$$\tan(2\theta) = 1.$$

Hence:

$$\theta = \frac{\pi}{8}, \frac{\pi}{2}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{3\pi}{2}, \frac{13\pi}{8}.$$

1.6 Vectors

1.6.1 Vectors in 3D

Consider the four vectors

$$\mathbf{A} = \begin{pmatrix} 16 \\ -6 \\ 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 4 \\ 14 \\ -9 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} -15 \\ 7 \\ 4 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 12 \\ 12 \\ 1 \end{pmatrix}.$$

1. Order the vectors by magnitude.
2. Calculate the distance between the points with position vectors **A** and **B**.

1.

$$
\begin{aligned}
|\mathbf{A}| &= \left| \begin{pmatrix} 16 \\ -6 \\ 1 \end{pmatrix} \right| = \sqrt{16^2 + 6^2 + 1^2} \\
&= \sqrt{293}. \\
|\mathbf{B}| &= \left| \begin{pmatrix} 4 \\ 14 \\ -9 \end{pmatrix} \right| = \sqrt{4^2 + 14^2 + 9^2} \\
&= \sqrt{293}. \\
|\mathbf{C}| &= \left| \begin{pmatrix} -15 \\ 7 \\ 4 \end{pmatrix} \right| = \sqrt{15^2 + 7^2 + 4^2} \\
&= \sqrt{290}. \\
|\mathbf{D}| &= \left| \begin{pmatrix} 12 \\ 12 \\ 1 \end{pmatrix} \right| = \sqrt{12^2 + 12^2 + 1^2} \\
&= \sqrt{288}.
\end{aligned}
$$

Consequently

$$|\mathbf{A}| = |\mathbf{B}| > |\mathbf{C}| > |\mathbf{D}|.$$

2.

$$
\begin{aligned}
|\mathbf{A} - \mathbf{B}| &= \left| \begin{pmatrix} 12 \\ 20 \\ 10 \end{pmatrix} \right| = \sqrt{12^2 + 20^2 + 10^2} \\
&= \sqrt{644}.
\end{aligned}
$$

1.7 Differentiation

1.7.1 Stationary points

Find the stationary points of the following functions, stating whether they are local maxima, minima or points of inflexion:

1. $y = x^2 + 2$
2. $y = x^3 - 3x + 3$
3. $y = x^3 - 3x^2 + 3x$
4. $y = x^3 + 3x + 3$.

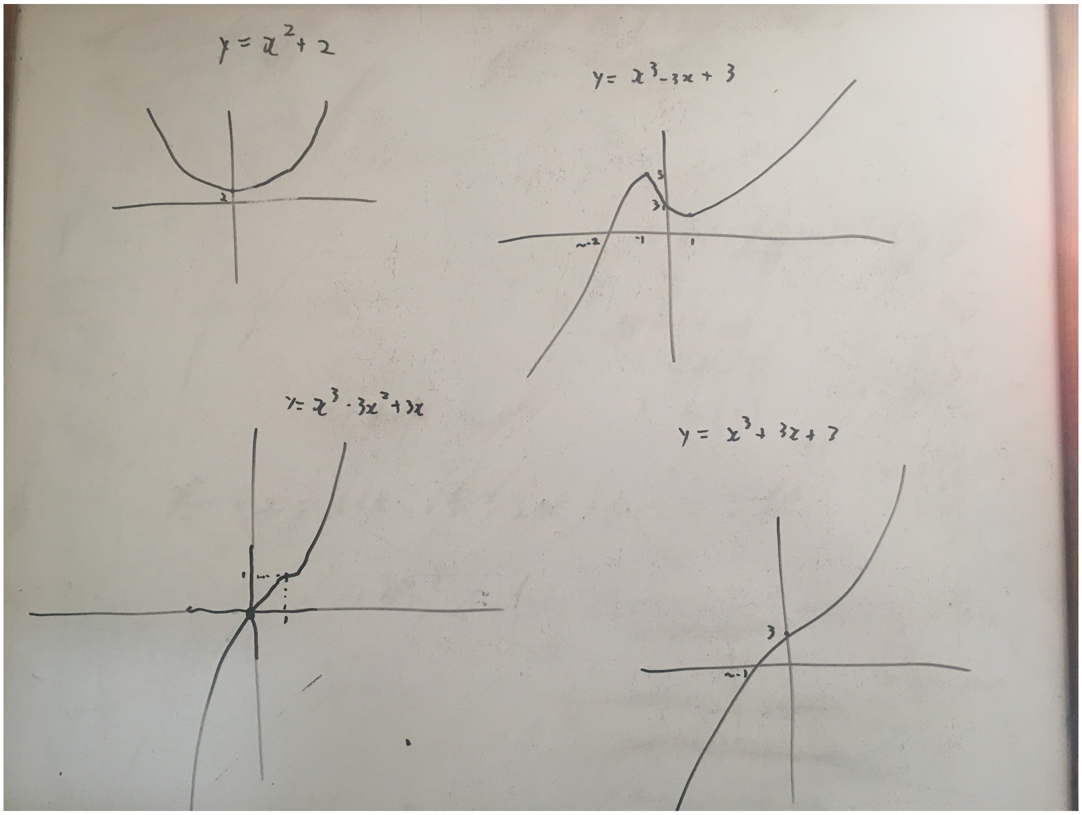
Sketch the graphs of the functions.

1.
$$\frac{dy}{dx} = 2x.$$
Consequently, this graph has a minimum at $(0, 2)$.

2.
$$\frac{dy}{dx} = 3x^2 - 3.$$
Consequently this graph has turning points at
$$x = \pm 1.$$
So there is a local maximum at $(-1, 5)$ and a local minimum at $(1, 1)$.

3.
$$\frac{dy}{dx} = 3x^2 - 6x + 3.$$
Consequently this graph has turning points at
$$x = 1 \pm \sqrt{36 - 36}.$$
Hence this graph has its only turning point at $(1, 1)$. The second derivative
$$y''(1) = 6(1) - 6 = 0$$
and so this is an inflexion point (which was expected since its the only turning point in a cubic anyway).

4.
$$\frac{dy}{dx} = 3x^2 + 3.$$
Since the derivative is strictly positive (from the trivial inequality), there are no turning points in this graph.



1.7.2 Differentiation from first principles

Calculate the derivative of $y = x^2 + 1$ from first principles (i.e. by considering the derivative of a function as the limit of the gradient of a chord).
We do as requested and note that the gradient would be given by the change in y over the change in x where we let the change in x tend very small.

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{[(x+h)^2 + 1] - (x^2 + 1)}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} 2x + h$$

$$= 2x.$$

1.7.3 Chain rule and product rule

Using the chain and product rules etc., find the derivatives of:

1. $y = \sin(x^2)$

2. $y = a^x$(hint: take logs)

3. $y = \ln(x^a + x^{-a})$

4. $y = x^x$

5. $y = \sin^{-1} x$

where a is a positive constant.

1.
$$\frac{dy}{dx} = 2x \cos x^2$$

2.
$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a} = a^x \ln a.$$

3.
$$\frac{dy}{dx} = \frac{ax^{a-1} - ax^{-a-1}}{x^a + x^{-a}}$$

4.
$$\begin{aligned} y &= x^x. \\ \ln y &= x \ln x \\ \frac{1}{y}\frac{dy}{dx} &= \ln x + 1 \\ \frac{dy}{dx} &= \ln(x)x^x + x^x. \end{aligned}$$

5.
$$\begin{aligned} \sin y &= x \\ \cos y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\cos y} \\ &= \frac{1}{\sqrt{1 - x^2}}. \end{aligned}$$

1.7.4 Implicit differentiation

If $y + e^y = x + x^3 + 1$, find $\frac{dy}{dx}$ in terms of y and x.

$$\begin{aligned} y + e^y &= x + x^3 + 1 \\ \frac{dy}{dx} + e^y \frac{dy}{dx} &= 3x^2 + 1. \\ \frac{dy}{dx} &= \frac{3x^2 + 1}{1 + e^y}. \end{aligned}$$

1.7.5 Implicit differentiation

If $y = \dfrac{t+1}{t-2}$ and $x = \dfrac{2t+1}{t-3}$, find $\dfrac{dy}{dx}$ when $t = 1$.

$$y = \frac{t+1}{t-2}$$
$$x = \frac{2t+1}{t-3}$$
$$\frac{dy}{dt} = \frac{-3}{(t-2)^2}$$
$$\frac{dx}{dt} = \frac{-7}{(t-3)^2}.$$

Consequently, when $t = 1$, at $(-1/2, -2)$,

$$\frac{dy}{dx} = \frac{-3(1-3)^2}{-7(1-2)^2} = \frac{12}{7}.$$

1.8 Integration

1.8.1 Integration techniques

Find the following indefinite integrals (stating the values of x for which the integrand is a real function):

1. $\int \frac{1}{2+x^2}\ dx$ (set $x = \sqrt{2}\tan\theta$)
2. $\int \frac{1}{\sqrt{3+2x-x^2}}\ dx$ (set $x - 1 = 2\sin\theta$)
3. $\int \frac{1}{x\sqrt{1-x}}\ dx$
4. $\int \ln x\ dx$.

1.
$$I = \int \frac{1}{2+x^2} dx.$$

Let $x = \sqrt{2}\tan\theta$ so $dx = \sqrt{2}\sec^2\theta d\theta$. Then

$$I = \int \frac{\sqrt{2}\sec^2\theta}{2+2\tan^2\theta} d\theta$$
$$= \int \frac{\sqrt{2}}{2\cos^2\theta + 2\sin^2\theta} d\theta$$
$$= \int \frac{\sqrt{2}}{2} d\theta$$

$$= \frac{\sqrt{2}}{2}\theta$$

$$= \frac{\sqrt{2}}{2} \arctan \frac{x}{\sqrt{2}} + C.$$

2.
$$I = \int \frac{1}{\sqrt{3+2x-x^2}} dx$$

Let $x - 1 = 2\sin\theta$ so $dx = 2\cos\theta d\theta$.

$$I = \int \frac{1}{\sqrt{-(x-1)^2+4}}$$

$$I = \int \frac{2\cos\theta}{\sqrt{-4\sin^2\theta+4}} d\theta$$

$$= \int \frac{\cos\theta}{\sqrt{1-\sin^2\theta}} d\theta$$

$$= \int d\theta$$

$$= \theta$$

$$= \arcsin \frac{x-1}{2}.$$

3.
$$I = \int \frac{1}{x\sqrt{1-x}} dx.$$

Let $u = \sqrt{1-x}$ so $x = 1 - u^2$ and $du = \frac{-1}{2u}$. Then

$$I = \int \frac{2u}{u(1-u^2)} du$$

$$= 2 \int \frac{1}{1-u^2} du.$$

Now we try to use partial fractions and write

$$\frac{1}{1-u^2} = \frac{A}{1+u} + \frac{B}{1-u}.$$

Hence

$$1 = A - uA + B + uB.$$

So $A - B = 0$ and $A + B = 1$. Hence

$$\frac{1}{1-u^2} = \frac{1}{2(1+u)} + \frac{1}{2(1-u)}.$$

Hence
$$\begin{aligned}I &= \int \frac{1}{1+u} + \frac{1}{1-u} du \\ &= \ln|1+u| - \ln|1-u| + C \\ &= \ln|1+\sqrt{1-x}| - \ln|1-\sqrt{1-x}| + C.\end{aligned}$$

4. $$I = \int \ln x dx.$$

   Let $u = \ln x$ so $u' = 1/x$ and let $v' = 1$ so $v = x$. Then applying the integration by parts formula,
$$\begin{aligned}I &= x \ln x - \int dx \\ &= x \ln x - x + C.\end{aligned}$$

1.8.2 Integration techniques

Evaluate the following definite integrals:

1. $\int_0^L xe^{-x}\ dx$
2. $\int_0^{\pi/2} \sin 3\theta \cos\theta\ d\theta$
3. $\int_0^1 \dfrac{x^2+1}{x^3+3x+2}\ \mathrm{d}x$
4. $\int_0^{\pi/2} \dfrac{1}{3+5\cos\theta}\ d\theta$ [use $t = \tan(\frac{1}{2}\theta)$].

In part 1, can you suggest what happens as $L \to \infty$.

1. First we define
$$I(L) = \int_0^L xe^{-x} dx.$$

   Then, by parts, let $u = x$ and $v' = e^{-x}$ so $u' = 1$ and $v = -e^{-x}$ and
$$\begin{aligned}I(L) &= -xe^{-x}|_0^L + \int_0^L e^{-x} dx \\ &= (-xe^{-x})|_0^L + (-e^{-x})|_0^L \\ &= -Le^{-L} - e^{-L} + 1 \\ & -e^{-L}(L+1) + 1.\end{aligned}$$

   As $L \to \infty$, $I(L) \to 1$.

2. First we define

$$I = \int_0^{\pi/2} \sin 3\theta \cos\theta d\theta.$$

Then, by parts, let $u = \sin 3\theta$ and $v' = \cos\theta$ so $u' = 3\cos 3\theta$ and $v = \sin\theta$. Then

$$I = (\sin\theta \sin 3\theta)|_0^{\pi/2} - 3\int \sin\theta \cos 3\theta d\theta.$$

Now, by parts, let $x = \cos 3\theta$ and $y' = \sin\theta$ so that $x' = -3\sin 3\theta$ and $y = -\cos\theta$. Then

$$I = (\sin\theta \sin 3\theta)|_0^{\pi/2} - 3(-\cos\theta\cos 3\theta - 3\int_0^{\pi/2} \cos\theta \sin 3\theta d\theta)$$

$$\begin{aligned}
I &= (\sin\theta \sin 3\theta)|_0^{\pi/2} + (3\cos\theta\cos 3\theta)|_0^{\pi/2} + 9I \\
-8I &= (\sin\theta \sin 3\theta)|_0^{\pi/2} + (3\cos\theta\cos 3\theta)|_0^{\pi/2} \\
&= (-1+0) - (0+3) \\
&= -4. \\
I &= \frac{1}{2}.
\end{aligned}$$

3. First we define

$$I = \int_0^1 \frac{x^2+1}{x^3+3x+2} dx.$$

Let $u = x^3 + 3x + 2$ so that $du = dx(3x^2+3)$. Then

$$\begin{aligned}
I &= \int_2^6 \frac{x^2+1}{u(3x^2+3)} du \\
&= \int_2^6 \frac{1}{3u} du \\
&= (\frac{1}{3}\ln u)|_2^6 \\
&= \frac{1}{3}\ln 3.
\end{aligned}$$

4. First we define

$$I = \int_0^{\pi/2} \frac{1}{3+5\cos\theta}\ d\theta.$$

Then we use the standard t-substitution. That is, $t = \tan(\frac{1}{2}\theta)$ so $d\theta = \cos^2(\frac{1}{2}\theta)dt$. This gives us

$$I = \int_0^1 \frac{1}{3 + \frac{5-5t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\ dt$$

$$= \int_0^1 \frac{2}{3 + 3t^2 + 5 - 5t^2}\ dt$$
$$= \int_0^1 \frac{2}{8 - 2t^2}\ dt$$
$$= \int_0^1 \frac{1}{4 - t^2}\ dt$$
$$= \frac{1}{4}\int_0^1 \frac{1}{2 + t}\ \mathrm{dt} + \frac{1}{4}\int_0^1 \frac{1}{(2 - t)}\ dt$$
$$= \frac{1}{4}\left(\ln 3\right).$$

1.9 Differential equations

1.9.1 Seperable first order ODEs

Solve the following differential equation:

$$x\frac{dy}{dx} + (1 - y^2) = 0; \qquad y = 0 \text{ when } x = 1.$$

$$x\frac{dy}{dx} + (1 - y^2) = 0; (1, 0)$$
$$x\frac{dy}{dx} = y^2 - 1$$
$$\int \frac{1}{x} dx = \int \frac{1}{y^2 - 1} dy$$
$$\int \frac{1}{x} dx = \frac{1}{2}\int \frac{1}{y - 1} dy - \frac{1}{2}\int \frac{1}{y + 1} dy$$
$$\ln x + C = \frac{1}{2}\ln|y - 1| - \frac{1}{2}\ln|y + 1|$$
$$Cx = \sqrt{\frac{|y - 1|}{|y + 1|}}.$$

When $x = 1$, $y = 0$. So

$$C = \sqrt{\frac{1}{1}} = 1.$$
$$C = 1.$$
$$x = \sqrt{\frac{|y - 1|}{|y + 1|}}.$$
$$x^2 = \frac{|y - 1|}{|y + 1|}$$

$$x^2y + x^2 = y - 1$$
$$x^2 + 1 = y - x^2y$$
$$y = \frac{1 + x^2}{1 - x^2}.$$

2 Further Mathematics

2.1 Complex mathematics

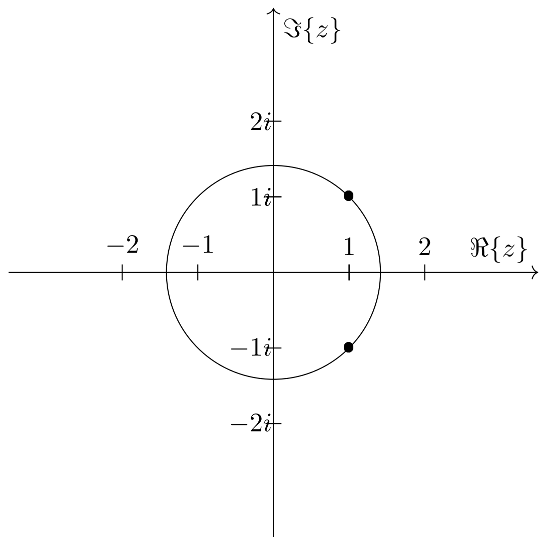
2.1.1 Basic manipulations

1. Determine the real and imaginary parts of
$$\frac{1+i}{2-i}.$$

2. Find the roots of the quadratic equation $z^2 - 2z + 2 = 0$. Determine the modulus and argument of each root. Plot the roots of an Argand diagram.

1.
$$z = \frac{1+i}{2-i} \cdot \frac{2+i}{2+i} = \frac{(1+i)(2+i)}{5}.$$
Hence
$$\mathrm{Re}(z) = \frac{1}{5}.$$
$$\mathrm{Im}(z) = \frac{3}{5}.$$

2.
$$z = 1 \pm i.$$
The arguments are $\pi/4$ (positive) and $-\pi/4$ (negative) and the modulus of both is $\sqrt{2}$ so they lie on a circle of radius $\sqrt{2}$ centered at the origin.

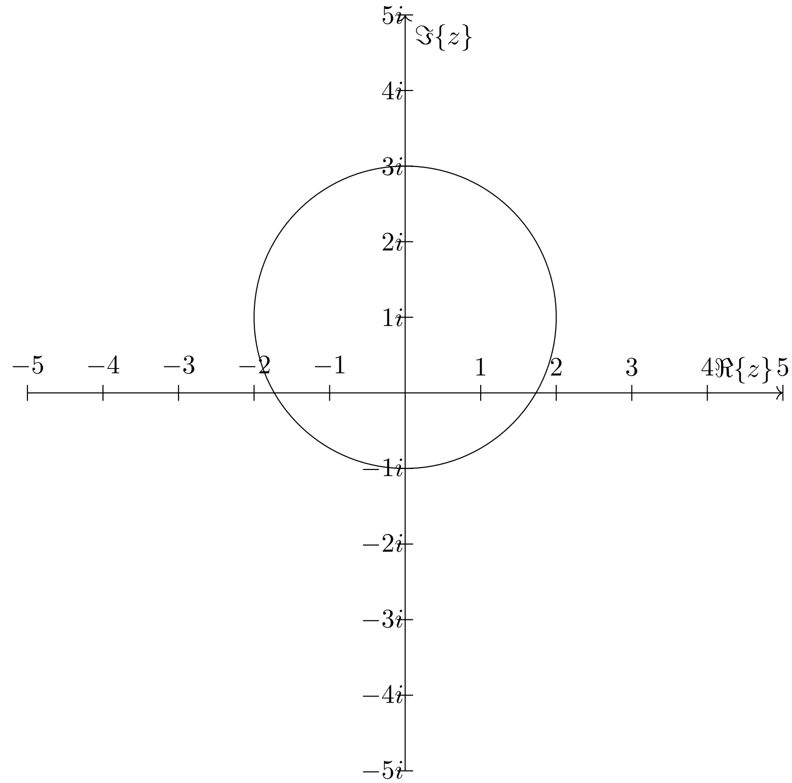


2.1.2 Further properties

1. Use de Moivre's theorem to express $\cos 5\theta$ in terms of powers of $\sin\theta$ and $\cos\theta$.
2. Sketch the loci $|z-i|=2$ and $|z+i|=|z-2|$.

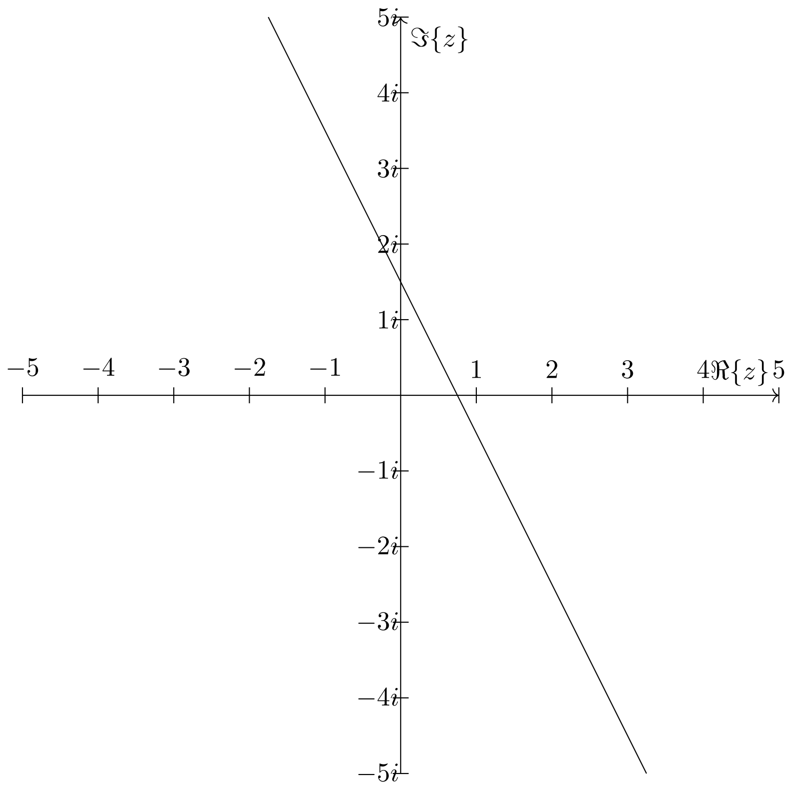
1.
$$\begin{aligned}\mathrm{Re}(e^{5i\theta}) &= \mathrm{Re}((\mathrm{cis}\theta)^5)\\ &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta.\end{aligned}$$

2.
$$|z-i| = 2$$



$\Im\{z\}$
$\Re\{z\}$
$5i$
$4i$
$3i$
$2i$
$1i$
$-1i$
$-2i$
$-3i$
$-4i$
$-5i$
-5 -4 -3 -2 -1 1 2 3 4 5

$|z+i| = |z-2|$



2.2 Vectors

2.2.1 Vector equation of lines

Show that the points with position vectors

$$\begin{pmatrix}1\\0\\1\end{pmatrix}, \quad \begin{pmatrix}2\\1\\0\end{pmatrix}, \quad \begin{pmatrix}0\\-1\\2\end{pmatrix}$$

lie on a straight line and give the equation of the line in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$.

$$\mathbf{a} = \begin{pmatrix}1\\0\\1\end{pmatrix}, \mathbf{b} = \begin{pmatrix}2\\1\\0\end{pmatrix} \mathbf{c} = \begin{pmatrix}0\\-1\\2\end{pmatrix}.$$

$$\mathbf{a}-\mathbf{b} = \begin{pmatrix}-1\\-1\\1\end{pmatrix}; \quad \mathbf{b}-\mathbf{c} = \begin{pmatrix}2\\2\\-2\end{pmatrix} = -2(\mathbf{a}-\mathbf{b}).$$

Since the line segments joining A and B and joining B and C are scalar multiples of each other and both go through point B with position vector $\mathbf{b}$, the three points are colinear, lying on the line $\mathbf{r} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

2.3 Matrices

2.3.1 Basic properties

Calculate $\mathbf{A} + \mathbf{B}$, $\mathbf{AB}$ and $\mathbf{BA}$ for

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix}.$$

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} -1 & 1 \\ 7 & 6 \end{bmatrix}.$$

$$\mathbf{AB} = \begin{bmatrix} 6 & 3 \\ 10 & 5 \end{bmatrix}; \mathbf{BA} = \begin{bmatrix} -5 & -8 \\ 10 & 16 \end{bmatrix}.$$

2.3.2 Non-commutativity

Find matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{AB} = 0$ and $\mathbf{BA} \neq 0$.
An example of two such matrices is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

2.3.3 Transformations

A linear transformation is described by the matrix

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Show that this transformation is the composition of a rotation and a scaling.
Note that

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

and so this is the composition of scaling by factor $\sqrt{2}$ and rotating $\pi/4$ radians.

2.4 Series

2.4.1 Summation of series

Sum the following series

$$\sum_{r=1}^{n} r^2 \qquad \sum_{r=1}^{n} r(r^2+2).$$

$$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{r=1}^{n} r(r^2+2) = \frac{n^4+5n^3+2n^2+n}{4}$$

2.4.2 Method of differences

Use partial fractions to sum the series

$$\sum_{r=1}^{n} \frac{1}{r(r+1)}.$$

$$\frac{1}{r(r+1)} = \frac{A}{r+1} + \frac{B}{r}.$$

So $Ar + Br + B = 1$. Hence $A + B = 0$ and $B = 1$. So $A = -1$. So

$$\frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}$$

and so

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = (\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \ldots + -\frac{1}{n+1}$$

which telescopes and so

$$= 1 - \frac{1}{n+1}$$

$$= \frac{n}{n+1}.$$

2.5 Mathematical induction

2.5.1 Sequences

If $a_{n+1} = 3a_n + 4$ and $a_1 = 1$ then deduce a formula for a_n for any $n \geq 1$. Use mathematical induction to prove your result.
Computing a few values gives

$$a_2 = 7$$
$$a_3 = 25$$
$$a_4 = 79$$

It looks like $a_n = 3^n - 2$. Hence let $P(n)$ be the statement that $a_n = 3^n - 2$. P_1 is true since $a_1 = 1 = 3^1 - 2$. Now assume $P(k)$ is true for some natural k. Now consider $P(k+1)$.

$$a_{k+1} = 3(a_k) + 4$$

but since $P(k)$ has been assumed to be true that means

$$a_{k+1} = 3(3^k - 2) + 4 = 3^{k+1} - 6 + 4 = 3^{k+1} - 2.$$

Hence $P(k+1)$ is also true. Since $P(k)$ implies $P(k+1)$ and since $P(1)$ is true, by mathematical induction, $P(n)$ is true for all natural $n \geq 1$.

2.5.2 Integration

Use mathematical induction to prove that, for a non-negative integer n,

$$\int_0^\infty x^n e^{-x} dx = n!.$$

Let $P(n)$ be the statement that $\int_0^\infty x^n e^{-x} dx = n!$. $P(0)$ is true since

$$\int_0^\infty e^{-x} dx = -e^{-x}|_0^\infty = 0 - (-1) = 1 = 0!.$$

Now assume $P(k)$ is true for some natural k. Now consider $P(k+1)$. We are interested in computing

$$\int_0^\infty x^{k+1} e^{-x} dx.$$

By parts, let $u = x^{k+1}$ so $u' = (k+1)x^k$ and let $v' = e^{-x}$ so $v = -e^{-x}$. Then

$$\int_0^\infty x^{k+1} e^{-x} dx = (-x^{k+1} e^{-x})|_0^\infty + (k+1) \int_0^\infty x^k e^{-x} dx.$$

But since we have assumed $P(k)$ is true, this means that this can be simplified to

$$\begin{aligned} &= (-x^{k+1} e^{-x})|_0^\infty + (k+1)k! = 0 + (k+1)k! \\ &= (k+1)!. \end{aligned}$$

Hence $P(k+1)$ is true if $P(k)$ is true. Since $P(k)$ implies $P(k+1)$ and since $P(0)$ is true, $P(n)$ is true for all non-negative integers n by mathematical induction.

2.6 Hyperbolic functions

2.6.1 Basic properties

State the definitions of sinh x and cosh x. Prove that

$$\cosh^2 x - \sinh^2 x = 1 \qquad \sinh 2x = 2\sinh x \cosh x$$

$\sinh x$ is the odd part of the exponential function and $\cosh x$ is the even part. We thus have

$$\sinh x = \frac{e^x - e^{-x}}{2}$$
$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

A simplie computation reveals

$$\cosh^2 x - \sinh^2 x = (\cosh x - \sinh x)(\cosh x + \sinh x)$$
$$e^{-x}e^x = 1.$$

Similarly,

$$\begin{aligned} 2\sinh x \cosh x &= 2\left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^x - e^{-x}}{2}\right) \\ &= \frac{(e^x + e^{-x})(e^x - e^{-x})}{2} \\ &= \frac{e^{2x} - e^{-2x}}{2} = \sinh 2x. \end{aligned}$$

2.6.2 Differentiation

Prove that

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x.$$

$$\begin{aligned} \tanh x &= \frac{\sinh x}{\cosh x} \\ &= \frac{e^x - e^{-x}}{e^x + e^{-x}}. \end{aligned}$$

$$\frac{d}{dx}\tanh x = \frac{d}{dx}(e^x - e^{-x})(e^x + e^{-x})^{-1}$$
$$= (e^x + e^{-x})(e^x + e^{-x})^{-1} + (e^x - e^{-x}) - 1(e^x + e^{-x})^{-2}(e^x - e^{-x})$$
$$= \frac{e^x + e^{-x}}{e^x + e^{-x}} - \frac{(e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

$$= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2}$$

$$= \frac{[(e^x + e^{-x}) + (e^x - e^{-x})][(e^x + e^{-x}) - (e^x - e^{-x})]}{(e^x + e^{-x})^2}$$

$$= \frac{(2e^x)(2e^{-x})}{(e^x + e^{-x})^2}$$

$$= \frac{4}{(e^x + e^{-x})^2}$$

$$= \frac{1}{(\frac{e^x + e^{-x}}{2})^2}$$

$$= \frac{1}{\cosh^2 x}$$

$$= \mathrm{sech}^2 x.$$