

# NatSci Maths Booklet

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## 1 Mathematics

### 1.1 Algebra

#### 1.1.1 Powers

$$\begin{aligned} & \frac{x^{-\frac{1}{5}} \times \left(x^{\frac{2}{3}}\right)^6}{x \times \sqrt[2]{x^5} \times \sqrt[5]{x^2}} \\ &= x^{-\frac{1}{5}} \times x^{\frac{12}{3}} \times x^{-1} \times x^{-\frac{5}{2}} \times x^{-\frac{2}{5}} \\ &= x^{\left(-\frac{1}{5} + 4 - 1 - \frac{5}{2} - \frac{2}{5}\right)} \\ &= x^{\left(\frac{-2+40-10-25-4}{10}\right)} \\ &= x^{-\frac{1}{10}}. \end{aligned}$$

#### 1.1.2 Factorisation

1.

$$\begin{aligned} & x^2 - 1 \\ &= (x + 1)(x - 1). \end{aligned}$$

2.

$$\begin{aligned} & a^2 - 4ab + 4b^2 \\ &= (a - 2b)^2. \end{aligned}$$

3.

$$\begin{aligned} & x^3 - 1 \\ &= (x - 1)(x^2 + x + 1). \end{aligned}$$

### 1.1.3 Quadratic equations

1.

$$\begin{aligned} x^2 - 5x + 6 &= 0 \\ \iff (x-3)(x-2) &= 0 \\ \iff x &= 2, 3. \end{aligned}$$

2.

$$\begin{aligned} x^2 + 2x &= 0 \\ \iff (x)(x+2) &= 0 \\ \iff x &= -2, 0. \end{aligned}$$

3.

$$\begin{aligned} x^2 - x - 1 &= 0. \\ \iff x &= \frac{1 \pm \sqrt{5}}{2}. \end{aligned}$$

4.

$$x^4 - 3x^2 + 2 = 0.$$

We let  $y = x^2$  so

$$\begin{aligned} y^2 - 3y + 2 &= 0 \\ \iff (y-2)(y-1) &= 0, \end{aligned}$$

and so

$$y = 1, 2$$

which means

$$x = \pm 1, \pm \sqrt{2}.$$

### 1.1.4 Completing the square

1.

$$\begin{aligned} x^2 - 2x + 6 \\ = (x-1)^2 + 5. \end{aligned}$$

Hence

$$x^2 - 2x + 6 \geq 5.$$

Because  $(x-1)^2$  and therefore  $x^2 - 2x + 6$  is increasing in  $[2, 3]$  the minimum value it attains is attained in this interval when  $x = 2$  and so  $x^2 - 2x + 6 = 6$ .

2.

$$\begin{aligned}x^4 + 2x^2 + 2 \\= (x^2 + 1)^2 + 1.\end{aligned}$$

We know that

$$x^2 \geq 0$$

and consequently

$$x^2 + 1 \geq 1$$

and so

$$(x^2 + 1)^2 \geq 1$$

and hence, finally,

$$x^4 + 2x^2 + 2 \geq 2.$$

### 1.1.5 Inequalities

1.

$$\begin{aligned}x^2 - 3x < 4 \\ \iff x^2 - 3x - 4 < 0 \\ \iff (x - 4)(x + 1) < 0 \\ \iff x \in (-1, 4).\end{aligned}$$

2.

$$\begin{aligned}y^3 > 2y^2 + 3y \\ \iff y^3 - 2y^2 - 3y < 0 \\ \iff y(y - 3)(y + 1) < 0 \\ \iff y \in (0, 3).\end{aligned}$$

### 1.1.6 Factor theorem

1. Divide  $x^3 + 5x^2 - 2x - 24$  by  $(x + 4)$  and hence factorise it completely.

$$x^3 + 5x^2 - 2x - 24 = (x + 4)(x^2 + x - 6) = (x + 4)(x - 2)(x + 3).$$

2. Use the factor theorem to factorise  $t^3 - 7t + 6$ .

We note that  $(1)^3 - 7(1) + 6 = 0$  and so by the factor theorem  $(t - 1)$  is a factor. Then dividing reveals

$$t^3 - 7t + 6 = (t - 1)(t^2 + t - 6) = (t - 1)(t + 3)(t - 2).$$

3. Simplify  $\frac{x^3 + x^2 - 2x}{x^3 + 2x^2 - x - 2}$ .

We see that  $(x - 1)$  is a factor of both the numerator and denominator and proceed from there

$$\begin{aligned} \frac{x^3 + x^2 - 2x}{x^3 + 2x^2 - x - 2} &= \frac{(x - 1)(x^2 + 2x)}{(x - 1)(x^2 + 3x + 2)} \\ &= \frac{(x)(x + 2)}{(x + 2)(x + 1)} \\ &= \frac{x}{x + 1}. \end{aligned}$$

### 1.1.7 Partial fractions

1.

$$\frac{2}{(x + 1)(x - 1)} = \frac{A}{x + 1} + \frac{B}{x - 1}.$$

So we see

$$2 = Ax - A + Bx + B.$$

So

$$A + B = 0 \iff A = -B.$$

$$B - A = 2B = 2.$$

So  $A = -1$ ,  $B = 1$ . Thus

$$\frac{2}{(x + 1)(x - 1)} = \frac{1}{x - 1} - \frac{1}{x + 1}.$$

2.

$$\frac{x + 13}{(x + 1)(x - 2)(x + 3)} = \frac{A}{x + 1} + \frac{B}{x - 2} + \frac{C}{x + 3}.$$

So

$$Ax^2 + Ax - 6A + Bx^2 + 4Bx + 3B + Cx^2 - Cx - 2C.$$

$$A + B + C = 0.$$

$$A + 4B - C = 1.$$

$$-6A + 3B - 2C = 13.$$

Gaussian elimination yields

$$2A + 5B = 1.$$

$$8A + 5B = -11.$$

So  $6A = -12$ , so  $A = -2$ . Hence  $B = 1$  and  $C = 1$  so

$$\frac{x + 13}{(x + 1)(x - 2)(x + 3)} = \frac{1}{x - 2} + \frac{1}{x + 3} - \frac{2}{x + 1}.$$

3.

$$\frac{4x+1}{(x+1)^2(x-2)} = \frac{Ax+B}{(x+1)^2} + \frac{C}{x-2}.$$

So

$$Ax^2 + (B - 2A)x - 2B + Cx^2 + 2Cx + C = 4x + 1.$$

$$A + C = 0.$$

$$-2A + B + 2C = 4.$$

$$-2B + C = 1.$$

Subbing in  $A = -C$ ,

$$B + 4C = 4.$$

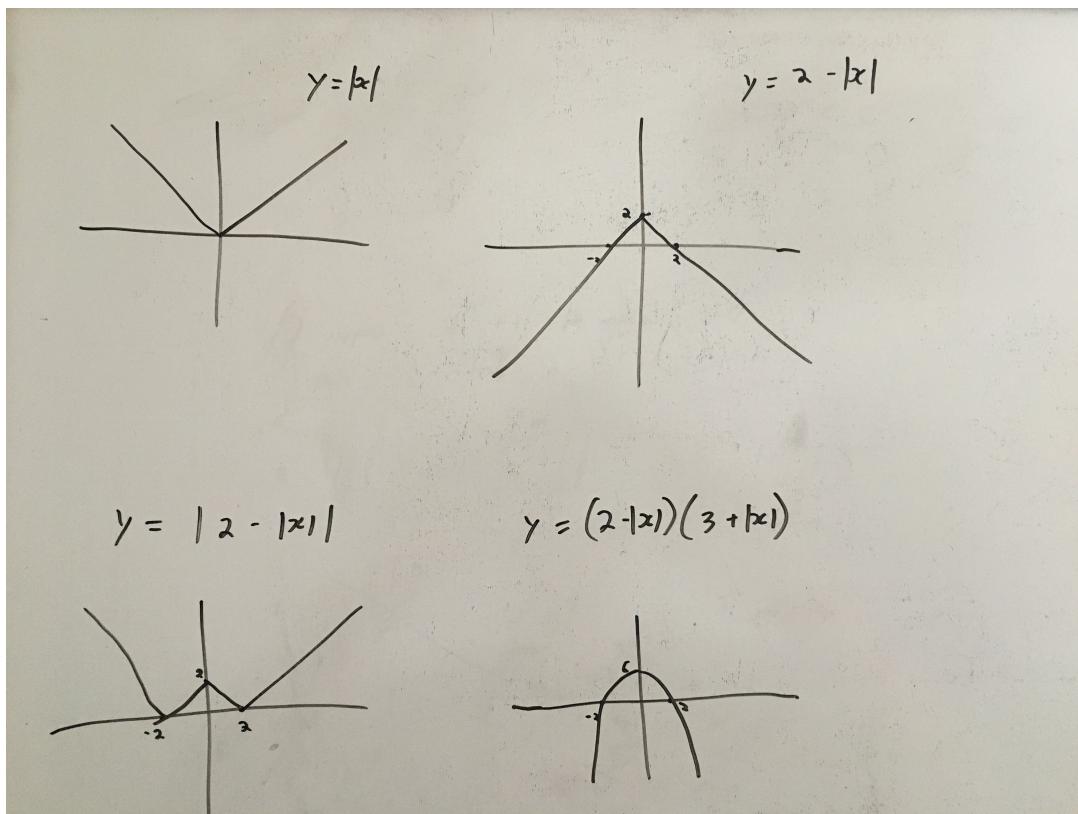
$$-2B + C = 1.$$

So  $9C = 9$ , so  $C = 1$  so  $B = 0$  and  $A = -1$ . Hence

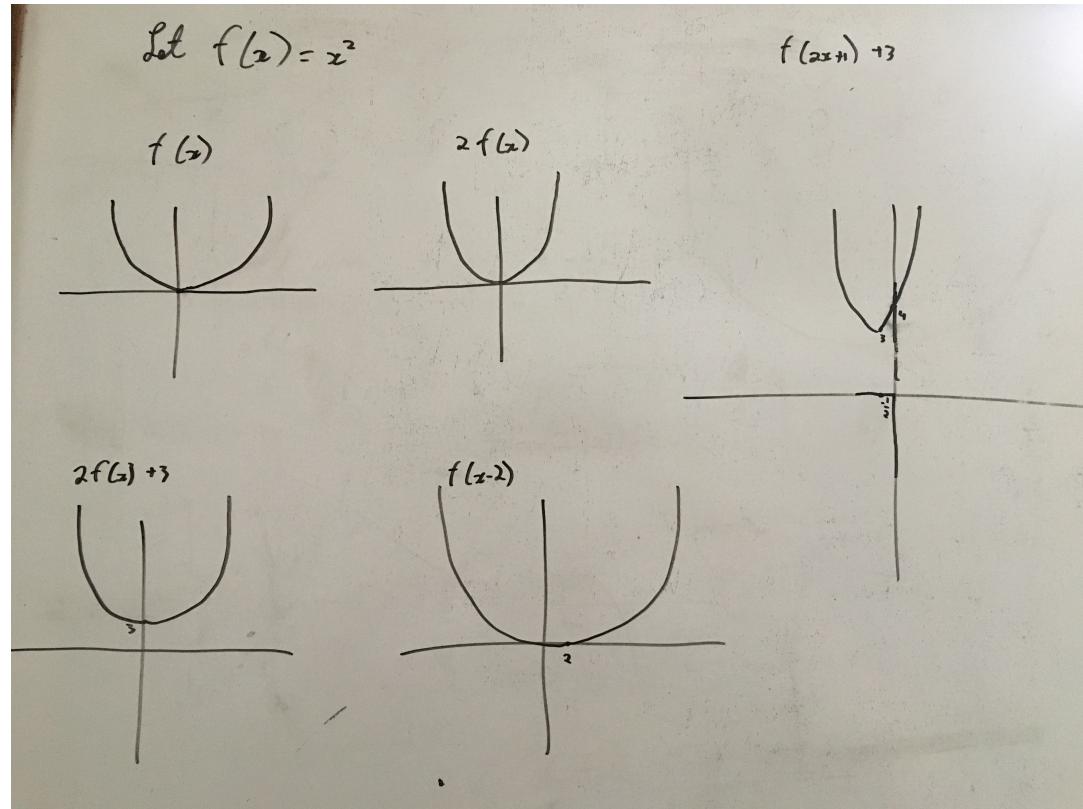
$$\frac{4x+1}{(x+1)^2(x-2)} = \frac{1}{x-2} - \frac{x}{(x+1)^2}.$$

## 1.2 Functions and Curve Sketching

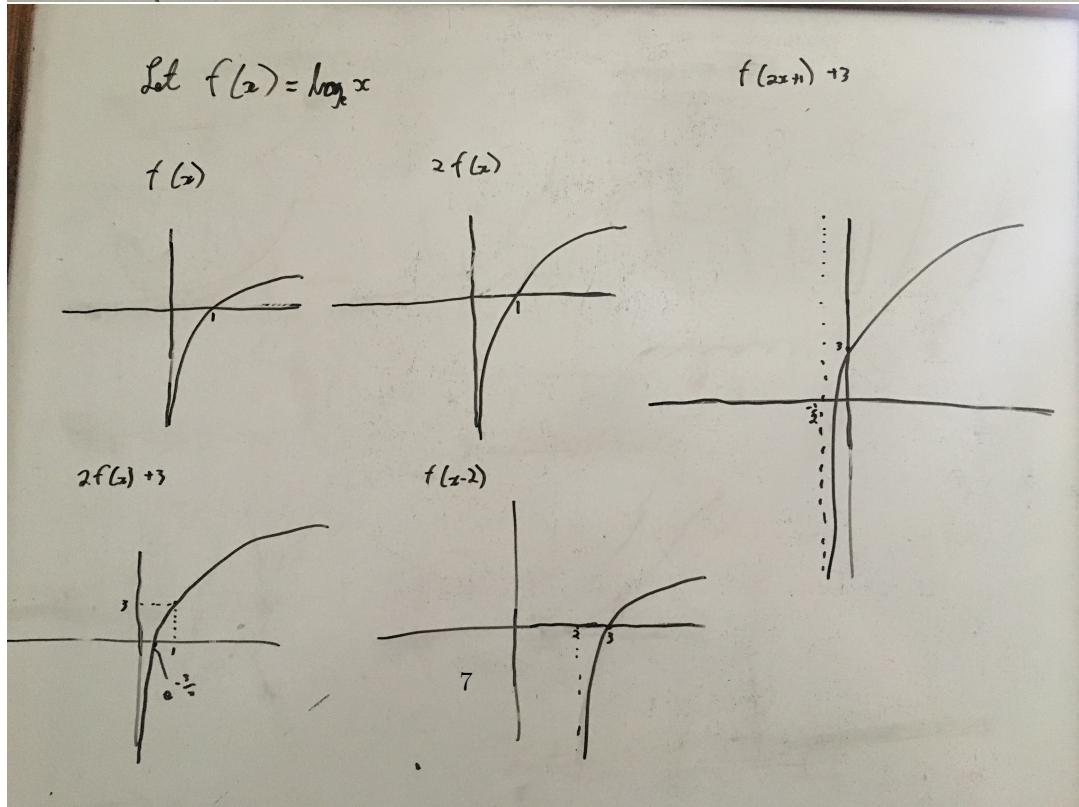
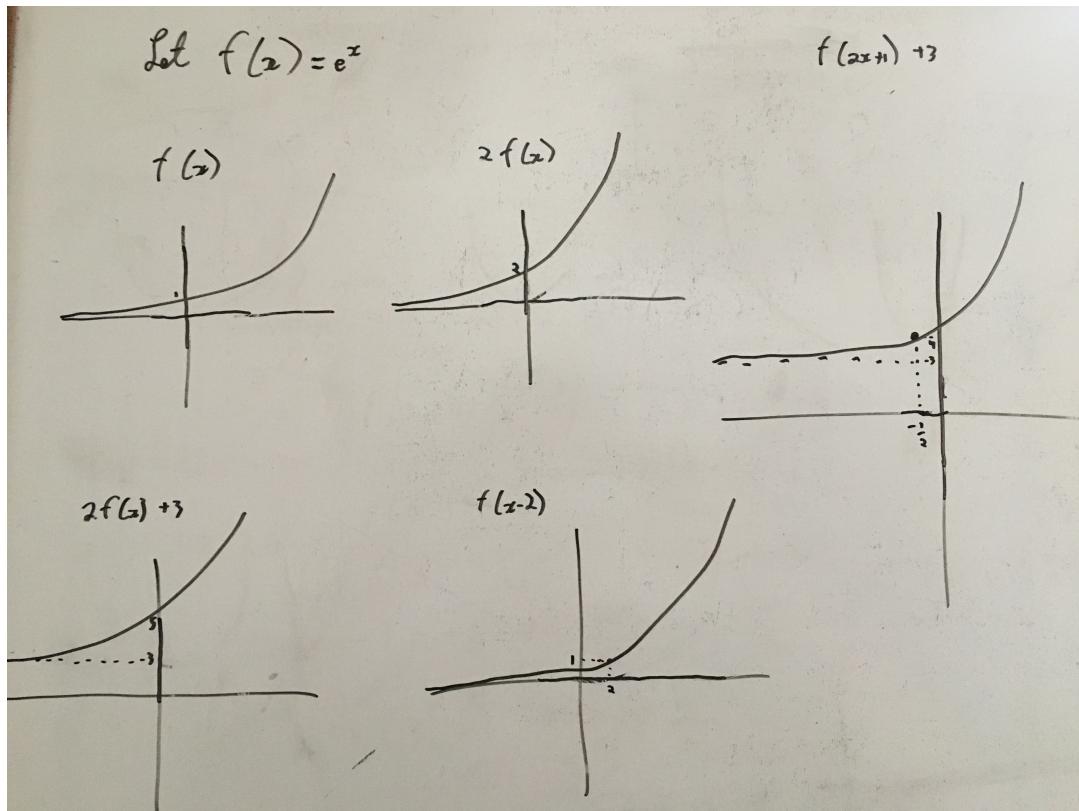
### 1.2.1 Modulus function



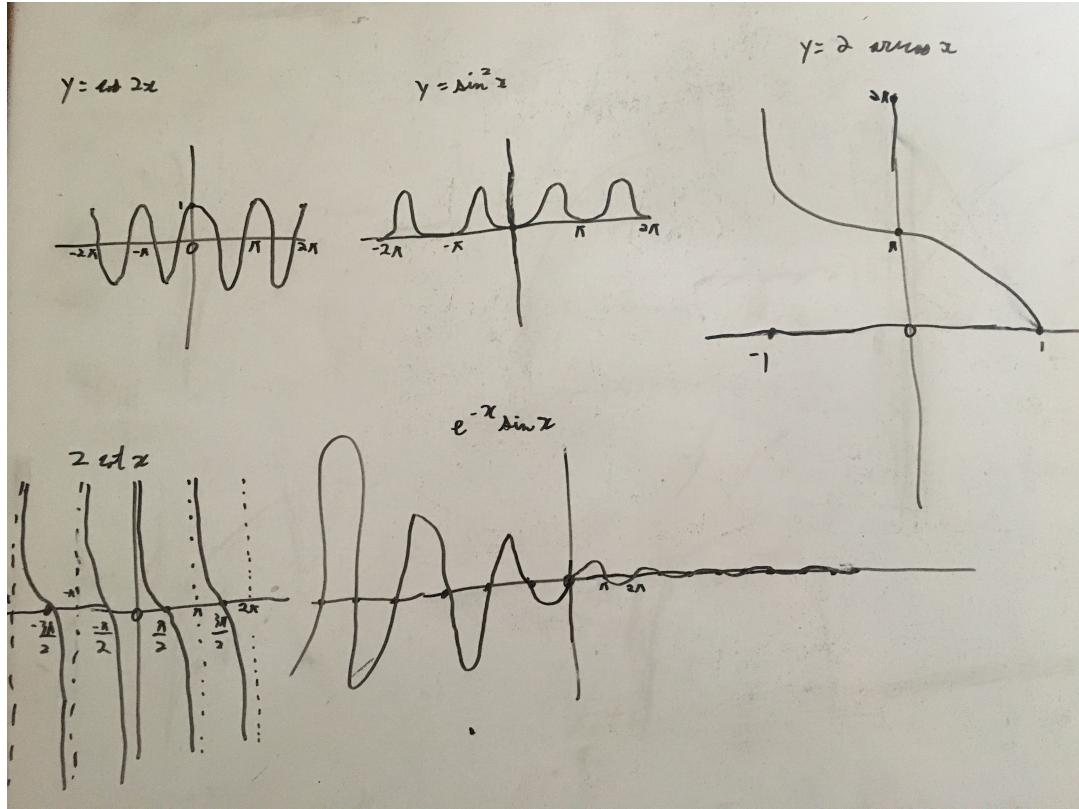
### 1.2.2 Transformations of functions



### 1.2.3 Transformations of functions



#### 1.2.4 Trig and inverse trig functions



#### 1.2.5 Logarithms

1. If  $3 = 9^{-x}$ , find  $x$ .

$$3 = 9^{-x}$$

$$\log_9 3 = -x$$

$$\frac{1}{2} = -x$$

$$x = \frac{1}{2}.$$

2. If  $\log_a b = c$ , show that  $c = \frac{\log_\alpha b}{\log_\alpha a}$  for any base  $\alpha$ .

$$\log_a b = c$$

$$b = a^c$$

$$\log_{\alpha} b = \log_{\alpha}(a^c)$$

$$\log_{\alpha} b = c \log_{\alpha} a$$

$$c = \frac{\log_{\alpha} b}{\log_{\alpha} a}.$$

3. Find  $x$  if  $16 \log_x 3 = \log_3 x$ .

$$16 \log_x 3 = \log_3 x$$

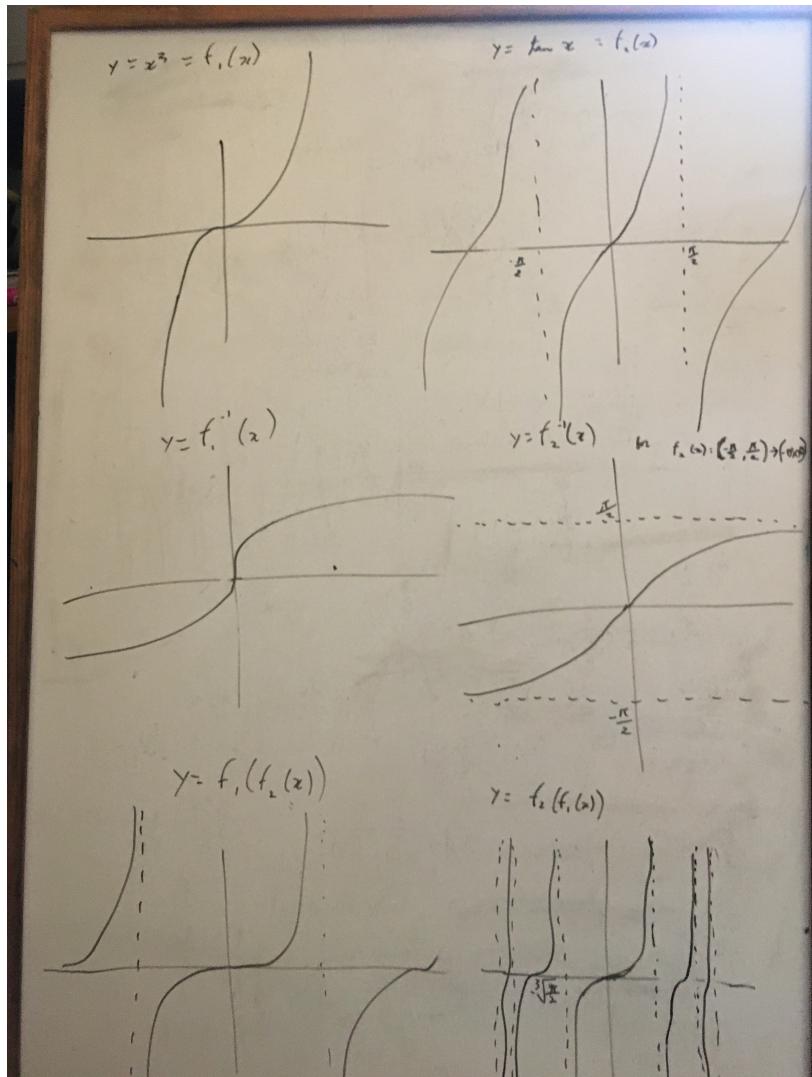
$$16 \frac{\ln 3}{\ln x} = \frac{\ln x}{\ln 3}$$

$$16(\ln 3)^2 = (\ln x)^2.$$

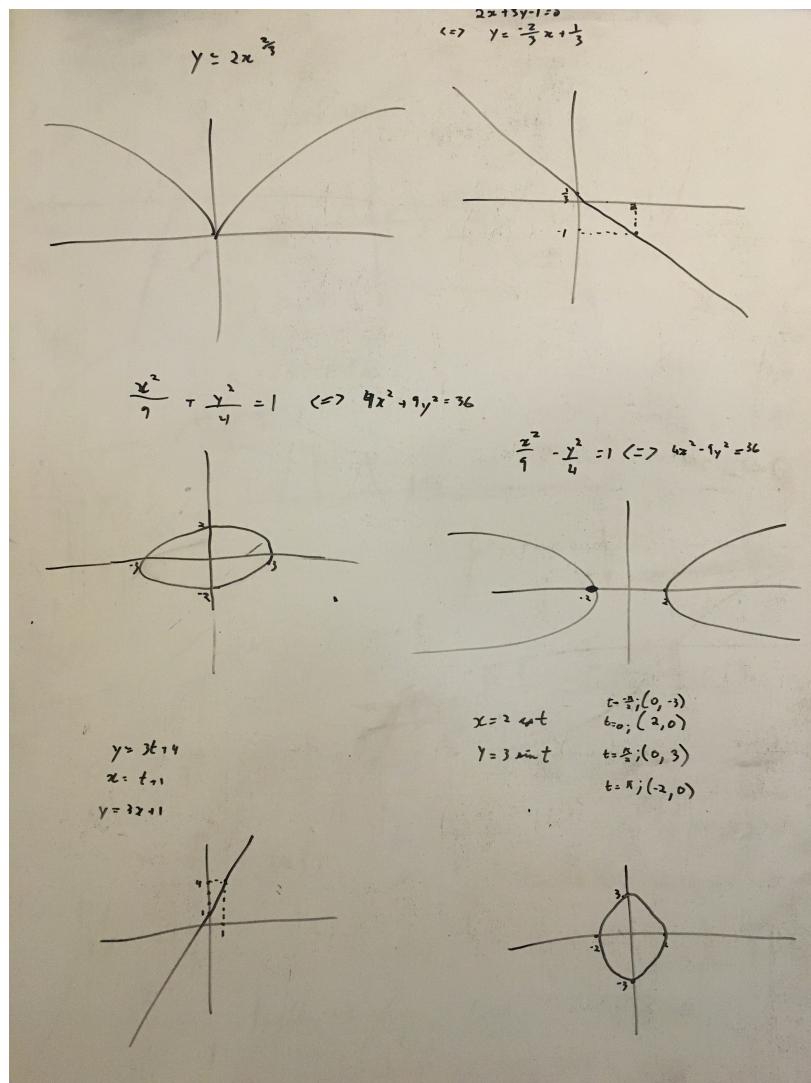
$$\ln x = \pm 4 \ln 3$$

$$\begin{aligned}x &= e^{\pm 4 \ln 3} \\&= \frac{1}{81}, 81.\end{aligned}$$

### 1.2.6 Composition of functions



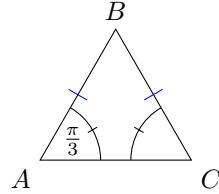
### 1.2.7 Curve sketching



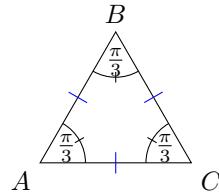
## 1.3 Geometry

### 1.3.1 Triangles

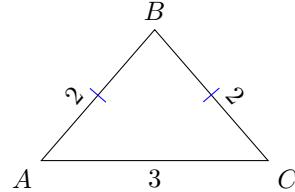
- In triangle  $ABC$ ,  $AB = 1$ ,  $BC = 1$ , and  $\angle A = \frac{\pi}{3}$  radians. Find  $CA$  and  $\angle B$ .



Since we know the triangle is isosceles since  $BC = AB$ , we see that  $\angle C = \frac{\pi}{3}$  radians. But that means that  $\angle B = \frac{\pi}{3}$  radians as well (since the three internal angles must sum to  $\pi$ ) and so the triangle is equilateral. Consequently every side has a length of 1, and so  $AC = 1$ . In fact the picture should look like the following.



2. In triangle  $ABC$ ,  $AB = 2$ ,  $BC = 2$  and  $AC = 3$ . Find the angles of the triangle.



By the law of cosines we have

$$9 = 8 - 8 \cos(\angle B),$$

or equivalently

$$\cos(\angle B) = -\frac{1}{8}.$$

$$\angle B \approx 1.696 \text{ radians.}$$

Then since  $\angle A = \angle C$  and the three angles sum to  $\pi$ , we have

$$\angle A = \angle C \approx 0.723 \text{ radians.}$$

### 1.3.2 Circles

Find, for a sector of angle  $\frac{\pi}{3}$  radians of a disc of radius 3:

1. the length of the perimeter; and
2. the area.

1. Since the sector has an angle of  $\frac{\pi}{3}$  it is a sixth of the circle. Consequently its arc length is

$$\frac{1}{6}2\pi \times 3 = \pi.$$

In addition, it has two sides that are radii and so have lengths of 3 each. Hence the perimeter is  $6 + \pi$ .

2. Again, the area is a sixth and so is equal to

$$\frac{1}{6} \times 9\pi = \frac{3}{2}\pi \text{ units squared.}$$

## 1.4 Sequences And Series

### 1.4.1 Arithmetic progressions

An arithmetic progression has third term  $\alpha$  and ninth term  $\beta$ . Find the sum to thirty terms.

If the first term is  $u$  and the common difference is  $d$ , then we have

$$\alpha = u + 2d$$

$$\beta = u + 8d$$

and the sum of the overall series is

$$15(2u + 29d) = 30u + 435d.$$

We thus wish to write

$$a\alpha + b\beta = 30u + 435d.$$

Expanding gives

$$(a + b)u + (2a + 8b)d = 30u + 435d.$$

We thus have a two variable system of equations:

$$a + b = 30.$$

$$2a + 8b = 435.$$

So

$$b = \frac{125}{2}$$

and

$$a = \frac{-65}{2}.$$

So we have that  $\Sigma = -32.5\alpha + 62.5\beta$ .

### 1.4.2 Binomial expansions

Expand the following expressions, using the binomial expansion, as far as the fourth term:

1.  $(1 + x)^3$

2.  $(2 + x)^4$

3.  $\left(2 + \frac{3}{x}\right)^5$ .

1.

$$(1 + x)^3 = 1 + 3x + 3x^2 + x^3.$$

2.

$$(2 + x)^4 = 16 + 32x + 24x^2 + 8x^3 + (x^4).$$

3.

$$\left(2 + \frac{3}{x}\right)^5 = 32 + \frac{240}{x} + \frac{720}{x^2} + \frac{1080}{x^3} + \frac{810}{x^4} + \frac{243}{x^5}.$$

### 1.4.3 Arithmetic and geometric progressions

Prove that  $\sum_1^N n = \frac{1}{2}N(N + 1)$ .

Evaluate:

1. the sum of the odd integers from 11 to 99 inclusive.

2.  $\sum_{n=1}^5 (3n + 2)$

3.  $\sum_{n=0}^N (an + b)$  ( $a$  and  $b$  are constants)

4.  $\sum_{r=0}^{10} 2^r$

5.  $\sum_{n=0}^N ar^{2n}$  ( $a$  and  $r$  are constants).

*Proof.* We have

$$\sum_{n=1}^N n = 1 + 2 + 3 + \dots + N.$$

This is a sum of  $N$  terms. The mean of these terms, since they are consecutive numbers, is the middle term. That is half the  $(N+1)^{\text{th}}$  term. So we have a sum of  $N$  terms, which have a mean of  $\frac{1}{2}(N+1)$ . Hence their sum is the product of the mean and the number of terms:

$$\frac{1}{2}N(N + 1).$$

□

1.

$$\sum_{n=0}^{44} 11 + 2n = 45(55) = 2475.$$

2.

$$\sum_{n=1}^5 (3n + 2) = 5 + 8 + 11 + 14 + 17 + 20 = 75.$$

3.

$$\sum_{n=0}^N (an + b) = \frac{1}{2}(aN + 2b)(N).$$

4.

$$\sum_{r=0}^{10} 2^r = 2^{11} - 1 = 2047.$$

5.

$$\sum_{n=0}^N ar^{2n} = \frac{a(r^{2N} - 1)}{r^2 - 1}.$$

#### 1.4.4 Iterative sequences

The sequence  $u_n$  satisfies  $u_{n+1} = ku_n$  where  $k$  is a fixed number, and  $u_0 = 1$ . Express  $u_n$  in terms of  $k$ . Describe the behaviour of  $u_n$  for large  $n$  in the different cases that arise according to the value of  $k$ .

$$u_{n+1} = ku_n.$$

$$u_0 = 1.$$

$$u_n = k^n.$$

- If  $k > 1$ , then  $u_n \rightarrow \infty$ .
- If  $k = 1$ , then  $u_n = 1$  for all  $n$ .
- If  $0 < k < 1$  then  $u_n \rightarrow 0$ .
- If  $k = 0$  then  $u_n = 0$ ,  $n \neq 0$ .
- If  $-1 < k < 0$  then  $u_n \rightarrow 0$ .
- If  $k = -1$  then  $u_{2n} = 1$  and  $u_{2n+1} = -1$ .
- If  $k < -1$  then  $u_{2n} \rightarrow \infty$  and  $u_{2n+1} \rightarrow -\infty$ .

### 1.4.5 Binomial expansion for rational powers

Find the first four terms in the expansions in ascending powers of  $x$  of the following expressions, stating for what values of  $x$  the expansion is valid in each case:

$$1. (1+x)^{\frac{1}{2}}$$

$$2. (2+x)^{\frac{2}{5}}$$

$$3. \frac{(1+2x)^{\frac{1}{2}}}{(2+x)^{\frac{1}{3}}}$$

1.

$$\begin{aligned} (1+x)^{\frac{1}{2}} &\approx 1 + \frac{1}{2}x + \frac{1}{2!} \cdot \frac{1}{2} \cdot \frac{-1}{2}x^2 + \frac{1}{3!} \cdot \frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2}x^3 \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3, |x| < 1. \end{aligned}$$

2.

$$\begin{aligned} (2+x)^{\frac{2}{5}} &= 2^{\frac{2}{5}}(1+\frac{x}{2})^{\frac{2}{5}} \approx 2^{\frac{2}{5}} \left( 1 + \frac{2}{5} \cdot \frac{x}{2} + \frac{1}{2!} \cdot \frac{2}{5} \cdot \frac{-3}{5} \cdot \frac{x^2}{4} + \frac{1}{3!} \cdot \frac{2}{5} \cdot \frac{-3}{5} \cdot \frac{-8}{5} \cdot \frac{x^3}{8} \right), |x| < 2. \\ &= 2^{\frac{2}{5}} \left( 1 + \frac{1}{5}x - \frac{3}{100}x^2 + \frac{1}{125}x^3 \right) \end{aligned}$$

3.

$$\begin{aligned} &\frac{(1+2x)^{\frac{1}{2}}}{(2+x)^{\frac{1}{3}}} \\ &= (1+2x)^{\frac{1}{2}}(2+x)^{-\frac{1}{3}} \\ &= 2^{-\frac{1}{3}}(1+2x)^{\frac{1}{2}} \left( 1 + \frac{x}{2} \right)^{-\frac{1}{3}} \\ &\approx 2^{-\frac{1}{3}} \left( 1 + \frac{1}{2}2x + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{-1}{2}4x^2 + \frac{1}{3!} \cdot \frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2}8x^3 \right) \times \dots \\ &\dots \left( 1 - \frac{1}{3} \cdot \frac{x}{2} + \frac{1}{2!} \cdot \frac{-1}{3} \cdot \frac{-4}{3} \cdot \frac{x^2}{4} + \frac{1}{3!} \cdot \frac{-1}{3} \cdot \frac{-4}{3} \cdot \frac{-7}{3} \frac{x^3}{8} \right) \\ &= 2^{-\frac{1}{3}} \left( 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 \right) \left( 1 - \frac{x}{6} + \frac{x^2}{18} - \frac{7x^3}{324} \right) \\ &= 2^{-\frac{1}{3}} \left( 1 + \frac{5}{6}x - \frac{11}{18}x^2 + \frac{50}{81}x^3 \right), |x| < \frac{1}{2}. \end{aligned}$$

#### 1.4.6 Composition of approximations

Given that, for small  $\theta$ ,  $\sin \theta \approx \theta - \frac{1}{6}\theta^3$  and  $\cos \theta \approx 1 - \frac{1}{2}\theta^2$ , find an approximation, ignoring powers of  $\theta$  greater than 3, for  $\sin(\frac{1}{2}\theta) \cos \theta + \sec 2\theta$ .

Substituting the approximations, we obtain:

$$\begin{aligned} & \left( \frac{\theta}{2} - \frac{\theta^3}{48} \right) \left( 1 - \frac{\theta^2}{2} \right) + (1 - 2\theta^2)^{-1} \\ & \approx \frac{\theta}{2} - \frac{13\theta^3}{48} + (1 - 2\theta^2)^{-1} \\ & \approx \frac{\theta}{2} - \frac{13\theta^3}{48} + (1 + 2\theta^2) \\ & = 1 + \frac{\theta}{2} + 2\theta^2 - \frac{13\theta^2}{48}. \end{aligned}$$

### 1.5 Trigonometry

#### 1.5.1 Solving trig equations

Find the four values of  $\theta$  in the range  $0$  to  $2\pi$  that satisfy the equation  $2\sin^2 \theta = 1$ .

$$\begin{aligned} 2\sin^2 \theta &= 1 \\ \sin^2 \theta &= \frac{1}{2} \\ \sin \theta &= \pm \frac{\sqrt{2}}{2} \\ \theta &= \pi/4, 3\pi/4, 5\pi/4, 7\pi/4. \end{aligned}$$

#### 1.5.2 Trig identities

Prove that  $\frac{\cot^2 x + \sin^2 x}{\cos x + \cosec x} = \cosec x - \cos x$ .

We start by multiplying the right hand side by a carefully chosen form of 1 and proceed from there:

$$\begin{aligned} & (\cosec x - \cos x) \frac{\cosec x + \cos x}{\cosec x + \cos x} \\ &= \frac{\frac{1}{\sin^2 x} - \cos^2 x}{\cosec x + \cos x} \\ &= \frac{\frac{1}{\sin^2 x} - (1 - \sin^2 x)}{\cosec x + \cos x} \\ &= \frac{\frac{1-\sin^2 x}{\sin^2 x} + \sin^2 x}{\cosec x + \cos x} \\ &= \frac{\cot^2 x + \sin^2 x}{\cosec x + \cos x}. \end{aligned}$$

### 1.5.3 Trig identities

By writing  $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$ , use trigonometric identities to evaluate:

$$1. \cos \frac{\pi}{12}$$

$$2. \sin \frac{\pi}{12}$$

$$3. \cot \frac{\pi}{12}$$

1.

$$\begin{aligned}\cos \frac{\pi}{12} &= \cos \left( \frac{\pi}{3} - \frac{\pi}{4} \right) \\ &= \cos \frac{\pi}{3} \cos \frac{\pi}{4} + \sin \frac{\pi}{3} \sin \frac{\pi}{4} \\ &= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{6} + \sqrt{2}}{4}.\end{aligned}$$

2.

$$\begin{aligned}\sin \frac{\pi}{12} &= \sin \left( \frac{\pi}{3} - \frac{\pi}{4} \right) \\ &= \sin \frac{\pi}{3} \cos \frac{\pi}{4} - \sin \frac{\pi}{4} \cos \frac{\pi}{3} \\ &= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \\ &= \frac{\sqrt{6} - \sqrt{2}}{4}.\end{aligned}$$

3.

$$\begin{aligned}\cot \frac{\pi}{12} &= \frac{\cos \frac{\pi}{12}}{\sin \frac{\pi}{12}} \\ &= \frac{\sqrt{6} + \sqrt{2}}{\sqrt{6} - \sqrt{2}} \\ &= \frac{8 + 4\sqrt{3}}{4} \\ &= 2 + \sqrt{3}.\end{aligned}$$

#### 1.5.4 Trig identities

If  $t = \tan \frac{1}{2}\theta$ , express the following in terms of  $t$ :

1.  $\cos \theta$
2.  $\sin \theta$
3.  $\tan \theta$

1. We first note that

$$t^2 + 1 = \frac{\sin^2 \frac{\theta}{2}}{\cos^2 \frac{\theta}{2}} + 1 = \frac{\sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2}}{\cos^2 \frac{\theta}{2}} = \frac{1}{\cos^2 \frac{\theta}{2}}.$$

Hence

$$\cos^2 \frac{\theta}{2} = \frac{1}{1+t^2}.$$

Then

$$\begin{aligned} \cos \theta &= 2 \cos^2 \frac{\theta}{2} - 1 \\ &= \frac{2}{1+t^2} - 1 \\ &= \frac{1-t^2}{1+t^2}. \end{aligned}$$

2. Similarly,

$$\begin{aligned} \sin \theta &= 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \\ &= 2t \cos^2 \frac{\theta}{2} \\ &= \frac{2t}{1+t^2}. \end{aligned}$$

3. Finally, from the tan double angle identity,

$$\tan \theta = \frac{2t}{1-t^2}.$$

#### 1.5.5 Trig identities

Simplify  $\tan(\arctan \frac{1}{3} + \arctan \frac{1}{4})$ .

We proceed via the tan sum-angle identity and obtain:

$$\begin{aligned} \tan \left( \arctan \frac{1}{3} + \arctan \frac{1}{4} \right) &= \frac{\frac{1}{3} + \frac{1}{4}}{1 - \frac{1}{3} \cdot \frac{1}{4}} \\ &= \frac{7}{11}. \end{aligned}$$

### 1.5.6 Trig identities

If  $A, B$  and  $C$  are the angles of a triangle, prove that

$$\cos\left(\frac{B-C}{2}\right) - \sin\left(\frac{A}{2}\right) = 2 \sin\left(\frac{B}{2}\right) \sin\left(\frac{C}{2}\right).$$

We first apply a compound angle identity backwards to the right hand side to obtain

$$2 \sin\left(\frac{B}{2}\right) \sin\left(\frac{C}{2}\right) = \cos\left(\frac{B-C}{2}\right) - \cos\left(\frac{B+C}{2}\right).$$

Since  $A, B$  and  $C$  are three internal angles in a triangle we know that  $A+B+C = \pi$  and consequently  $B+C = \pi - A$ , which, when substituted gives us

$$2 \sin\left(\frac{B}{2}\right) \sin\left(\frac{C}{2}\right) = \cos\left(\frac{B-C}{2}\right) - \cos\left(\frac{\pi-A}{2}\right).$$

But,  $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$  and so  $\cos\left(\frac{\pi-A}{2}\right) = \sin\left(\frac{A}{2}\right)$ , which when substituted gives us, as desired,

$$2 \sin\left(\frac{B}{2}\right) \sin\left(\frac{C}{2}\right) = \cos\left(\frac{B-C}{2}\right) - \sin\left(\frac{A}{2}\right).$$

### 1.5.7 Solving trig equations

Write  $\sqrt{3}\sin\theta + \cos\theta$  in the form  $A\sin(\theta+\alpha)$ , where  $A$  and  $\alpha$  are to be determine. Working backwards, we see

$$A\sin(\theta + \alpha) = A(\sin\theta \cos\alpha + \sin\alpha \cos\theta).$$

From here we equate coefficients and see that

$$A\cos\alpha = \sqrt{3}$$

$$A\sin\alpha = 1.$$

Dividing these two equations by each other yields

$$\tan\alpha = \frac{1}{\sqrt{3}}$$

and so one solution is

$$\alpha = \frac{\pi}{6}.$$

From there we can substitute this into either of the equations to obtain

$$A = 2.$$

So

$$\sqrt{3}\sin\theta + \cos\theta = 2\sin\left(\theta + \frac{\pi}{6}\right).$$

### 1.5.8 Solving trig equations

Find the values of  $\theta$  in the range  $0$  to  $2\pi$  which satisfy the equation

$$\cos \theta + \cos 3\theta = \sin \theta + \sin 3\theta.$$

We apply the compound angle identities repeatedly and obtain

$$\begin{aligned} \cos \theta + \cos \theta \cos(2\theta) - \sin \theta \sin(2\theta) &= \sin \theta + \sin \theta \cos(2\theta) + \sin(2\theta) \cos \theta \\ \iff \cos \theta + \cos \theta (\cos^2 \theta - \sin^2 \theta) - \sin \theta (2 \sin \theta \cos \theta) &= \sin \theta + \sin \theta (\cos^2 \theta - \sin^2 \theta) + \cos \theta (2 \sin \theta \cos \theta) \\ \iff \cos \theta + \cos^3 \theta - \cos \theta \sin^2 \theta - 2 \sin^2 \theta \cos \theta &= \sin \theta + \cos^2 \theta \sin \theta - \sin^3 \theta + 2 \cos^2 \theta \sin \theta. \end{aligned}$$

Now we try and factor cleverly and look for where we can replace  $1 - \sin^2 \theta$  with  $\cos^2 \theta$ .

$$\begin{aligned} \cos \theta (1 + \cos^2 \theta - \sin^2 \theta - 2 \sin^2 \theta) &= \sin \theta + \cos^2 \theta \sin \theta - \sin^3 \theta + 2 \cos^2 \theta \sin \theta \\ \cos \theta (2 \cos^2 \theta - 2 \sin^2 \theta) &= \sin \theta (1 + \cos^2 \theta - \sin^2 \theta + 2 \cos^2 \theta) \\ \cos \theta (2 \cos^2 \theta - 2 \sin^2 \theta) &= \sin \theta (4 \cos^2 \theta) \end{aligned}$$

Before we divide on both sides by  $\cos \theta$ , we must check if  $\cos \theta$  could be equal to zero. If it were then  $\theta = \frac{\pi}{2}$  or  $\frac{3\pi}{2}$ . If  $\theta = \frac{\pi}{2}$ , this would mean that  $\cos \theta + \cos 3\theta = 0$  while  $\sin \theta + \sin 3\theta = 0$ . Hence  $\theta = \frac{\pi}{2}$  is a valid solution. Similarly, checking reveals that so too is  $\frac{3\pi}{2}$ . Now we can divide:

$$\begin{aligned} 2 \cos^2 \theta - 2 \sin^2 \theta &= 4 \sin \theta \cos \theta \\ 2 \cos(2\theta) &= 2 \sin(2\theta) \\ \cos(2\theta) &= \sin(2\theta) \\ \tan(2\theta) &= 1. \end{aligned}$$

Hence:

$$\theta = \frac{\pi}{8}, \frac{\pi}{2}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{3\pi}{2}, \frac{13\pi}{8}.$$

## 1.6 Vectors

### 1.6.1 Vectors in 3D

Consider the four vectors

$$\mathbf{A} = \begin{pmatrix} 16 \\ -6 \\ 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 4 \\ 14 \\ -9 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} -15 \\ 7 \\ 4 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 12 \\ 12 \\ 1 \end{pmatrix}.$$

1. Order the vectors by magnitude.
2. Calculate the distance between the points with position vectors  $\mathbf{A}$  and  $\mathbf{B}$ .

1.

$$|\mathbf{A}| = \left| \begin{pmatrix} 16 \\ -6 \\ 1 \end{pmatrix} \right| = \sqrt{16^2 + 6^2 + 1^2} \\ = \sqrt{293}.$$

$$|\mathbf{B}| = \left| \begin{pmatrix} 4 \\ 14 \\ -9 \end{pmatrix} \right| = \sqrt{4^2 + 14^2 + 9^2} \\ = \sqrt{293}.$$

$$|\mathbf{C}| = \left| \begin{pmatrix} -15 \\ 7 \\ 4 \end{pmatrix} \right| = \sqrt{15^2 + 7^2 + 4^2} \\ = \sqrt{290}.$$

$$|\mathbf{D}| = \left| \begin{pmatrix} 12 \\ 12 \\ 1 \end{pmatrix} \right| = \sqrt{12^2 + 12^2 + 1^2} \\ = \sqrt{288}.$$

Consequently

$$|\mathbf{A}| = |\mathbf{B}| > |\mathbf{C}| > |\mathbf{D}|.$$

2.

$$|\mathbf{A} - \mathbf{B}| = \left| \begin{pmatrix} 12 \\ 20 \\ 10 \end{pmatrix} \right| = \sqrt{12^2 + 20^2 + 10^2} \\ = \sqrt{644}.$$

## 1.7 Differentiation

### 1.7.1 Stationary points

Find the stationary points of the following functions, stating whether they are local maxima, minima or points of inflexion:

1.  $y = x^2 + 2$
2.  $y = x^3 - 3x + 3$
3.  $y = x^3 - 3x^2 + 3x$
4.  $y = x^3 + 3x + 3$ .

Sketch the graphs of the functions.

1.

$$\frac{dy}{dx} = 2x.$$

Consequently, this graph has a minimum at  $(0, 2)$ .

2.

$$\frac{dy}{dx} = 3x^2 - 3.$$

Consequently this graph has turning points at

$$x = \pm 1.$$

So there is a local maximum at  $(-1, 5)$  and a local minimum at  $(1, 1)$ .

3.

$$\frac{dy}{dx} = 3x^2 - 6x + 3.$$

Consequently this graph has turning points at

$$x = 1 \pm \sqrt{36 - 36}.$$

Hence this graph has its only turning point at  $(1, 1)$ . The second derivative

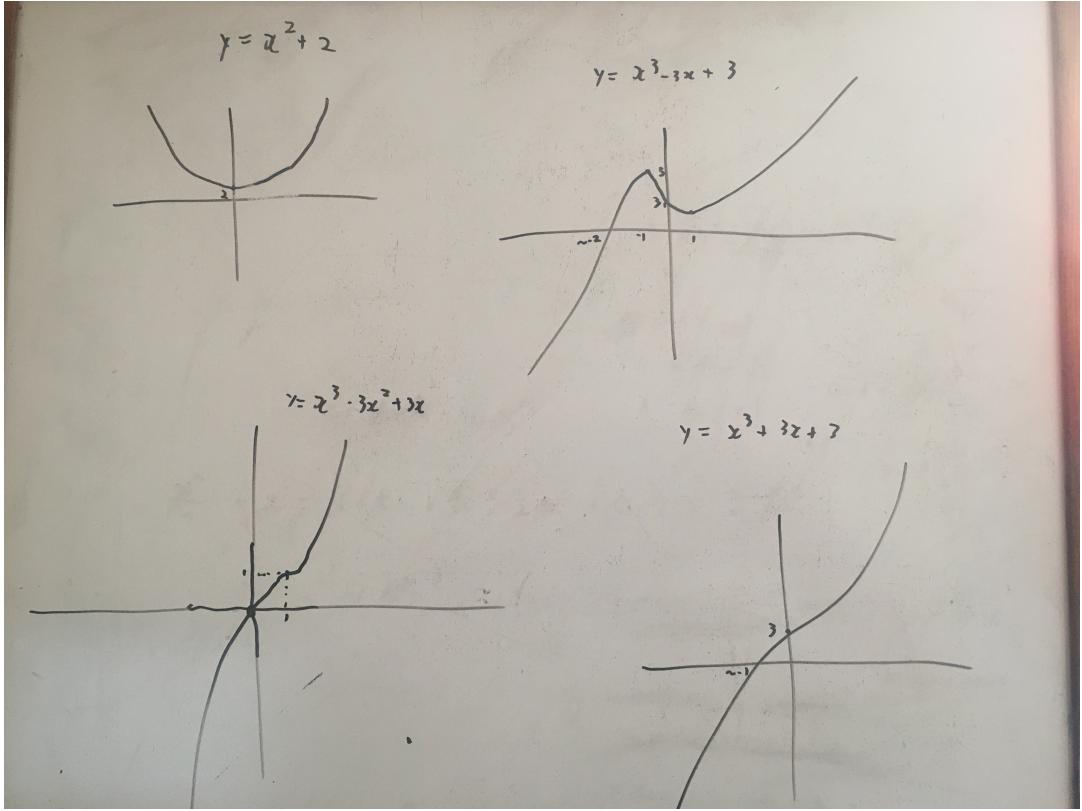
$$y''(1) = 6(1) - 6 = 0$$

and so this is an inflection point (which was expected since its the only turning point in a cubic anyway).

4.

$$\frac{dy}{dx} = 3x^2 + 3.$$

Since the derivative is strictly positive (from the trivial inequality), there are no turning points in this graph.



### 1.7.2 Differentiation from first principles

Calculate the derivative of  $y = x^2 + 1$  from first principles (i.e. by considering the derivative of a function as the limit of the gradient of a chord).

We do as requested and note that the gradient would be given by the change in  $y$  over the change in  $x$  where we let the change in  $x$  tend very small.

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{[(x+h)^2 + 1] - (x^2 + 1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \rightarrow 0} 2x + h \\ &= 2x. \end{aligned}$$

### 1.7.3 Chain rule and product rule

Using the chain and product rules etc., find the derivatives of:

1.  $y = \sin(x^2)$

2.  $y = a^x$  (hint: take logs)

3.  $y = \ln(x^a + x^{-a})$

4.  $y = x^x$

5.  $y = \sin^{-1} x$

where  $a$  is a positive constant.

1.

$$\frac{dy}{dx} = 2x \cos x^2$$

2.

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a} = a^x \ln a.$$

3.

$$\frac{dy}{dx} = \frac{ax^{a-1} - ax^{-a-1}}{x^a + x^{-a}}$$

4.

$$y = x^x.$$

$$\ln y = x \ln x$$

$$\frac{1}{y} \frac{dy}{dx} = \ln x + 1$$

$$\frac{dy}{dx} = \ln(x)x^x + x^x.$$

5.

$$\sin y = x$$

$$\cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$= \frac{1}{\sqrt{1 - x^2}}.$$

#### 1.7.4 Implicit differentiation

If  $y + e^y = x + x^3 + 1$ , find  $\frac{dy}{dx}$  in terms of  $y$  and  $x$ .

$$y + e^y = x + x^3 + 1$$

$$\frac{dy}{dx} + e^y \frac{dy}{dx} = 3x^2 + 1.$$

$$\frac{dy}{dx} = \frac{3x^2 + 1}{1 + e^y}.$$

### 1.7.5 Implicit differentiation

If  $y = \frac{t+1}{t-2}$  and  $x = \frac{2t+1}{t-3}$ , find  $\frac{dy}{dx}$  when  $t = 1$ .

$$y = \frac{t+1}{t-2}$$

$$x = \frac{2t+1}{t-3}$$

$$\frac{dy}{dt} = \frac{-3}{(t-2)^2}$$

$$\frac{dx}{dt} = \frac{-7}{(t-3)^2}.$$

Consequently, when  $t = 1$ , at  $(-1/2, -2)$ ,

$$\frac{dy}{dx} = \frac{-3(1-3)^2}{-7(1-2)^2} = \frac{12}{7}.$$

## 1.8 Integration

### 1.8.1 Integration techniques

Find the following indefinite integrals (stating the values of  $x$  for which the integrand is a real function):

1.  $\int \frac{1}{2+x^2} dx$  (set  $x = \sqrt{2} \tan \theta$ )
2.  $\int \frac{1}{\sqrt{3+2x-x^2}} dx$  (set  $x-1=2\sin\theta$ )
3.  $\int \frac{1}{x\sqrt{1-x}} dx$
4.  $\int \ln x dx.$

1.

$$I = \int \frac{1}{2+x^2} dx.$$

Let  $x = \sqrt{2} \tan \theta$  so  $dx = \sqrt{2} \sec^2 \theta d\theta$ . Then

$$\begin{aligned} I &= \int \frac{\sqrt{2} \sec^2 \theta}{2 + 2 \tan^2 \theta} d\theta \\ &= \int \frac{\sqrt{2}}{2 \cos^2 \theta + 2 \sin^2 \theta} d\theta \\ &= \int \frac{\sqrt{2}}{2} d\theta \end{aligned}$$

$$\begin{aligned}
&= \frac{\sqrt{2}}{2} \theta \\
&= \frac{\sqrt{2}}{2} \arctan \frac{x}{\sqrt{2}} + C.
\end{aligned}$$

2.

$$I = \int \frac{1}{\sqrt{3+2x-x^2}} dx$$

Let  $x-1 = 2 \sin \theta$  so  $dx = 2 \cos \theta d\theta$ .

$$\begin{aligned}
I &= \int \frac{1}{\sqrt{-(x-1)^2 + 4}} \\
&= \int \frac{2 \cos \theta}{\sqrt{-4 \sin^2 \theta + 4}} d\theta \\
&= \int \frac{\cos \theta}{\sqrt{1 - \sin^2 \theta}} d\theta \\
&= \int d\theta \\
&= \theta \\
&= \arcsin \frac{x-1}{2}.
\end{aligned}$$

3.

$$I = \int \frac{1}{x\sqrt{1-x}} dx.$$

Let  $u = \sqrt{1-x}$  so  $x = 1 - u^2$  and  $du = \frac{-1}{2u}$ . Then

$$\begin{aligned}
I &= \int \frac{2u}{u(1-u^2)} du \\
&= 2 \int \frac{1}{1-u^2} du.
\end{aligned}$$

Now we try to use partial fractions and write

$$\frac{1}{1-u^2} = \frac{A}{1+u} + \frac{B}{1-u}.$$

Hence

$$1 = A - uA + B + uB.$$

So  $A - B = 0$  and  $A + B = 1$ . Hence

$$\frac{1}{1-u^2} = \frac{1}{2(1+u)} + \frac{1}{2(1-u)}.$$

Hence

$$\begin{aligned} I &= \int \frac{1}{1+u} + \frac{1}{1-u} du \\ &= \ln|1+u| - \ln|1-u| + C \\ &= \ln|1+\sqrt{1-x}| - \ln|1-\sqrt{1-x}| + C. \end{aligned}$$

4.

$$I = \int \ln x dx.$$

Let  $u = \ln x$  so  $u' = 1/x$  and let  $v' = 1$  so  $v = x$ . Then applying the integration by parts formula,

$$\begin{aligned} I &= x \ln x - \int dx \\ &= x \ln x - x + C. \end{aligned}$$

### 1.8.2 Integration techniques

Evaluate the following definite integrals:

1.  $\int_0^L xe^{-x} dx$
2.  $\int_0^{\pi/2} \sin 3\theta \cos \theta d\theta$
3.  $\int_0^1 \frac{x^2 + 1}{x^3 + 3x + 2} dx$
4.  $\int_0^{\pi/2} \frac{1}{3 + 5 \cos \theta} d\theta$  [use  $t = \tan(\frac{1}{2}\theta)$ ].

In part 1, can you suggest what happens as  $L \rightarrow \infty$ .

1. First we define

$$I(L) = \int_0^L xe^{-x} dx.$$

Then, by parts, let  $u = x$  and  $v' = e^{-x}$  so  $u' = 1$  and  $v = -e^{-x}$  and

$$\begin{aligned} I(L) &= -xe^{-x}|_0^L + \int_0^L e^{-x} dx \\ &= (-xe^{-x})|_0^L + (-e^{-x})|_0^L \\ &= -Le^{-L} - e^{-L} + 1 \\ &\quad -e^{-L}(L+1) + 1. \end{aligned}$$

As  $L \rightarrow \infty$ ,  $I(L) \rightarrow 1$ .

2. First we define

$$I = \int_0^{\pi/2} \sin 3\theta \cos \theta d\theta.$$

Then, by parts, let  $u = \sin 3\theta$  and  $v' = \cos \theta$  so  $u' = 3 \cos 3\theta$  and  $v = \sin \theta$ . Then

$$I = (\sin \theta \sin 3\theta)|_0^{\pi/2} - 3 \int \sin \theta \cos 3\theta d\theta.$$

Now, by parts, let  $x = \cos 3\theta$  and  $y' = \sin \theta$  so that  $x' = -3 \sin 3\theta$  and  $y = -\cos \theta$ . Then

$$\begin{aligned} I &= (\sin \theta \sin 3\theta)|_0^{\pi/2} - 3(-\cos \theta \cos 3\theta) - 3 \int_0^{\pi/2} \cos \theta \sin 3\theta d\theta \\ &= (\sin \theta \sin 3\theta)|_0^{\pi/2} + (3 \cos \theta \cos 3\theta)|_0^{\pi/2} + 9I \\ -8I &= (\sin \theta \sin 3\theta)|_0^{\pi/2} + (3 \cos \theta \cos 3\theta)|_0^{\pi/2} \\ &= (-1 + 0) - (0 + 3) \\ &= -4. \\ I &= \frac{1}{2}. \end{aligned}$$

3. First we define

$$I = \int_0^1 \frac{x^2 + 1}{x^3 + 3x + 2} dx.$$

Let  $u = x^3 + 3x + 2$  so that  $du = dx(3x^2 + 3)$ . Then

$$\begin{aligned} I &= \int_2^6 \frac{x^2 + 1}{u(3x^2 + 3)} du \\ &= \int_2^6 \frac{1}{3u} du \\ &= \left(\frac{1}{3} \ln u\right)|_2^6 \\ &= \frac{1}{3} \ln 3. \end{aligned}$$

4. First we define

$$I = \int_0^{\pi/2} \frac{1}{3 + 5 \cos \theta} d\theta.$$

Then we use the standard  $t$ -substitution. That is,  $t = \tan(\frac{1}{2}\theta)$  so  $d\theta = \cos^2(\frac{1}{2}\theta)dt$ . This gives us

$$I = \int_0^1 \frac{1}{3 + \frac{5-5t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt$$

$$\begin{aligned}
&= \int_0^1 \frac{2}{3 + 3t^2 + 5 - 5t^2} dt \\
&= \int_0^1 \frac{2}{8 - 2t^2} dt \\
&= \int_0^1 \frac{1}{4 - t^2} dt \\
&= \frac{1}{4} \int_0^1 \frac{1}{2+t} dt + \frac{1}{4} \int_0^1 \frac{1}{(2-t)} dt \\
&= \frac{1}{4} (\ln 3).
\end{aligned}$$

## 1.9 Differential equations

### 1.9.1 Seperable first order ODEs

Solve the following differential equation:

$$x \frac{dy}{dx} + (1 - y^2) = 0; \quad y = 0 \text{ when } x = 1.$$

$$\begin{aligned}
&x \frac{dy}{dx} + (1 - y^2) = 0; (1, 0) \\
&x \frac{dy}{dx} = y^2 - 1 \\
&\int \frac{1}{x} dx = \int \frac{1}{y^2 - 1} dy \\
&\int \frac{1}{x} dx = \frac{1}{2} \int \frac{1}{y-1} dy - \frac{1}{2} \int \frac{1}{y+1} dy \\
&\ln x + C = \frac{1}{2} \ln |y-1| - \frac{1}{2} \ln |y+1| \\
&Cx = \sqrt{\frac{|y-1|}{|y+1|}}.
\end{aligned}$$

When  $x = 1, y = 0$ . So

$$C = \sqrt{\frac{1}{1}} = 1.$$

$$C = 1.$$

$$x = \sqrt{\frac{|y-1|}{|y+1|}}.$$

$$x^2 = \frac{|y-1|}{|y+1|}$$

$$\begin{aligned}x^2y + x^2 &= y - 1 \\x^2 + 1 &= y - x^2y \\y &= \frac{1 + x^2}{1 - x^2}.\end{aligned}$$

## 2 Further Mathematics

### 2.1 Complex mathematics

#### 2.1.1 Basic manipulations

1. Determine the real and imaginary parts of

$$\frac{1+i}{2-i}.$$

2. Find the roots of the quadratic equation  $z^2 - 2z + 2 = 0$ . Determine the modulus and argument of each root. Plot the roots of an Argand diagram.

1.

$$z = \frac{1+i}{2-i} \cdot \frac{2+i}{2+i} = \frac{(1+i)(2+i)}{5}.$$

Hence

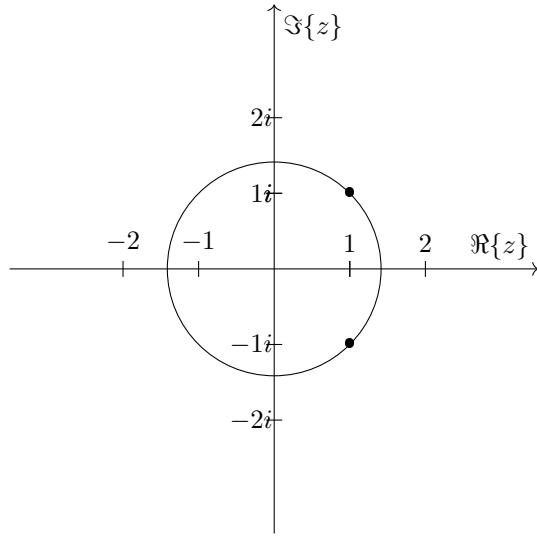
$$\operatorname{Re}(z) = \frac{1}{5}.$$

$$\operatorname{Im}(z) = \frac{3}{5}.$$

2.

$$z = 1 \pm i.$$

The arguments are  $\pi/4$  (positive) and  $-\pi/4$  (negative) and the modulus of both is  $\sqrt{2}$  so they lie on a circle of radius  $\sqrt{2}$  centered at the origin.



### 2.1.2 Further properties

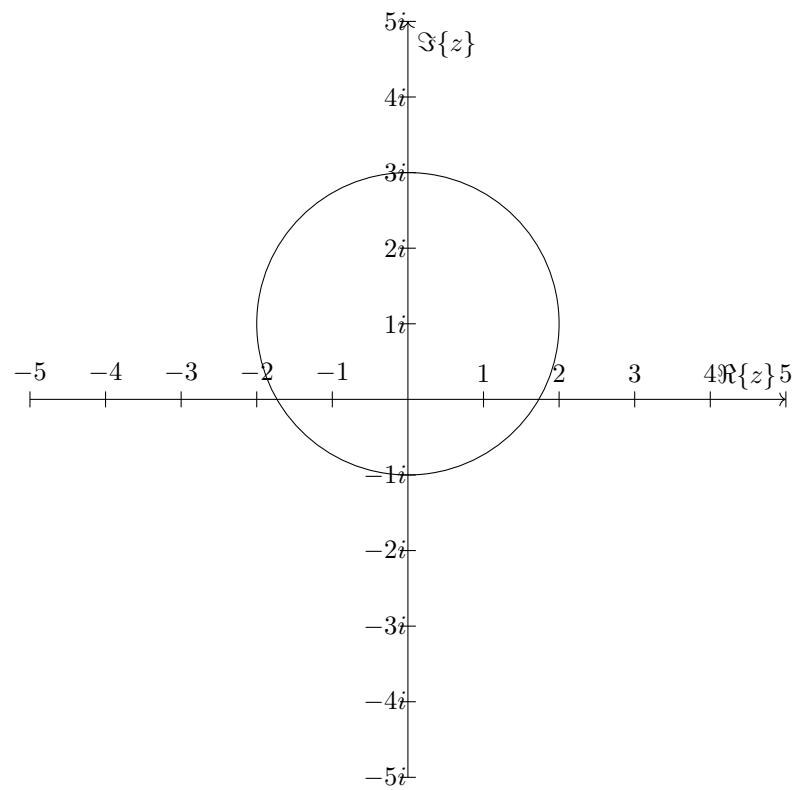
1. Use de Moivre's theorem to express  $\cos 5\theta$  in terms of powers of  $\sin \theta$  and  $\cos \theta$ .
2. Sketch the loci  $|z - i| = 2$  and  $|z + i| = |z - 2|$ .

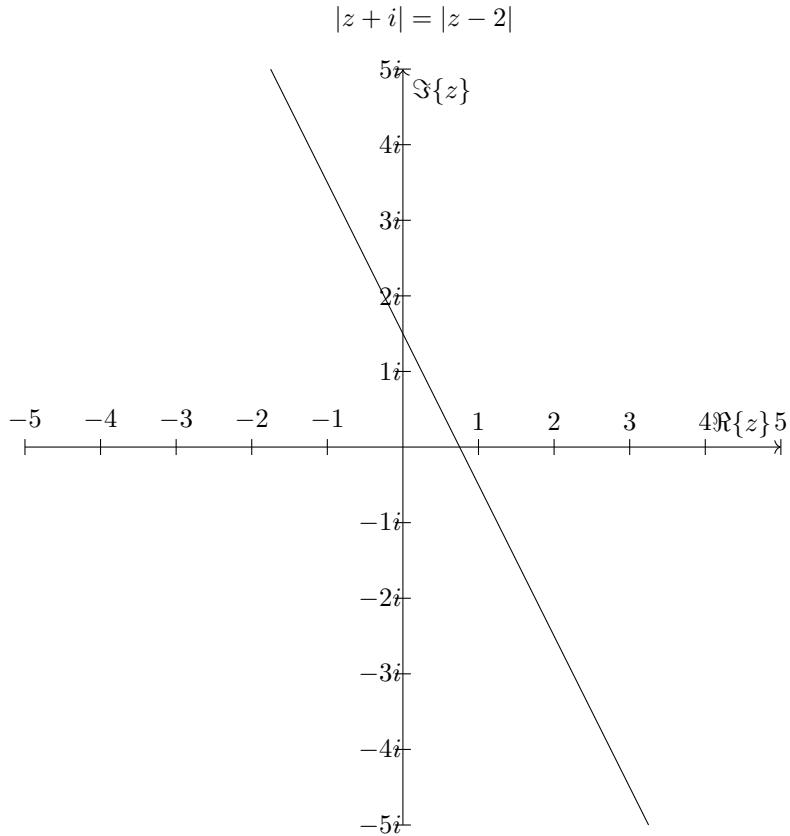
1.

$$\begin{aligned}\operatorname{Re}(e^{5i\theta}) &= \operatorname{Re}((\operatorname{cis}\theta)^5) \\ &= \cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta.\end{aligned}$$

2.

$$|z - i| = 2$$





## 2.2 Vectors

### 2.2.1 Vector equation of lines

Show that the points with position vectors

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

lie on a straight line and give the equation of the line in the form  $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ .

$$\begin{aligned} \mathbf{a} &= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}, \\ \mathbf{a} - \mathbf{b} &= \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}; \quad \mathbf{b} - \mathbf{c} = \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} = -2(\mathbf{a} - \mathbf{b}). \end{aligned}$$

Since the line segments joining A and B and joining B and C are scalar multiples of each other and both go through point B with position vector  $\mathbf{b}$ , the three points are colinear, lying on the line  $\mathbf{r} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ .

## 2.3 Matrices

### 2.3.1 Basic properties

Calculate  $\mathbf{A} + \mathbf{B}$ ,  $\mathbf{AB}$  and  $\mathbf{BA}$  for

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix}.$$

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} -1 & 1 \\ 7 & 6 \end{bmatrix}.$$

$$\mathbf{AB} = \begin{bmatrix} 6 & 3 \\ 10 & 5 \end{bmatrix}; \mathbf{BA} = \begin{bmatrix} -5 & -8 \\ 10 & 16 \end{bmatrix}.$$

### 2.3.2 Non-commutativity

Find matrices  $\mathbf{A}$  and  $\mathbf{B}$  such that  $\mathbf{AB} = 0$  and  $\mathbf{BA} \neq 0$ .

An example of two such matrices is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

### 2.3.3 Transformations

A linear transformation is described by the matrix

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Show that this transformation is the composition of a rotation and a scaling.

Note that

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

and so this is the composition of scaling by factor  $\sqrt{2}$  and rotating  $\pi/4$  radians.

## 2.4 Series

### 2.4.1 Summation of series

Sum the following series

$$\sum_{r=1}^n r^2 \quad \sum_{r=1}^n r(r^2 + 2).$$

$$\begin{aligned}\sum_{r=1}^n r^2 &= \frac{n(n+1)(2n+1)}{6} \\ \sum_{r=1}^n r(r^2 + 2) &= \frac{n^4 + 5n^3 + 2n^2 + n}{4}\end{aligned}$$

### 2.4.2 Method of differences

Use partial fractions to sum the series

$$\sum_{r=1}^n \frac{1}{r(r+1)}.$$

$$\frac{1}{r(r+1)} = \frac{A}{r+1} + \frac{B}{r}.$$

So  $Ar + Br + B = 1$ . Hence  $A + B = 0$  and  $B = 1$ . So  $A = -1$ . So

$$\frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}$$

and so

$$\sum_{r=1}^n \frac{1}{r(r+1)} = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(-\frac{1}{n+1}\right)$$

which telescopes and so

$$\begin{aligned}&= 1 - \frac{1}{n+1} \\ &= \frac{n}{n+1}.\end{aligned}$$

## 2.5 Mathematical induction

### 2.5.1 Sequences

If  $a_{n+1} = 3a_n + 4$  and  $a_1 = 1$  then deduce a formula for  $a_n$  for any  $n \geq 1$ . Use mathematical induction to prove your result.

Computing a few values gives

$$a_2 = 7$$

$$a_3 = 25$$

$$a_4 = 79$$

It looks like  $a_n = 3^n - 2$ . Hence let  $P(n)$  be the statement that  $a_n = 3^n - 2$ .  $P_1$  is true since  $a_1 = 1 = 3^1 - 2$ . Now assume  $P(k)$  is true for some natural  $k$ . Now consider  $P(k+1)$ .

$$a_{k+1} = 3(a_k) + 4$$

but since  $P(k)$  has been assumed to be true that means

$$a_{k+1} = 3(3^k - 2) + 4 = 3^{k+1} - 6 + 4 = 3^{k+1} - 2.$$

Hence  $P(k+1)$  is also true. Since  $P(k)$  implies  $P(k+1)$  and since  $P(1)$  is true, by mathematical induction,  $P(n)$  is true for all natural  $n \geq 1$ .

### 2.5.2 Integration

Use mathematical induction to prove that, for a non-negative integer  $n$ ,

$$\int_0^\infty x^n e^{-x} dx = n!.$$

Let  $P(n)$  be the statement that  $\int_0^\infty x^n e^{-x} dx = n!$ .  $P(0)$  is true since

$$\int_0^\infty e^{-x} dx = -e^{-x}|_0^\infty = 0 - (-1) = 1 = 0!.$$

Now assume  $P(k)$  is true for some natural  $k$ . Now consider  $P(k+1)$ . We are interested in computing

$$\int_0^\infty x^{k+1} e^{-x} dx.$$

By parts, let  $u = x^{k+1}$  so  $u' = (k+1)x^k$  and let  $v' = e^{-x}$  so  $v = -e^{-x}$ . Then

$$\int_0^\infty x^{k+1} e^{-x} dx = (-x^{k+1} e^{-x})|_0^\infty + (k+1) \int_0^\infty x^k e^{-x} dx.$$

But since we have assumed  $P(k)$  is true, this means that this can be simplified to

$$\begin{aligned} &= (-x^{k+1} e^{-x})|_0^\infty + (k+1)k! = 0 + (k+1)k! \\ &= (k+1)!. \end{aligned}$$

Hence  $P(k+1)$  is true if  $P(k)$  is true. Since  $P(k)$  implies  $P(k+1)$  and since  $P(0)$  is true,  $P(n)$  is true for all non-negative integers  $n$  by mathematical induction.

## 2.6 Hyperbolic functions

### 2.6.1 Basic properties

State the definitions of  $\sinh x$  and  $\cosh x$ . Prove that

$$\cosh^2 x - \sinh^2 x = 1 \quad \sinh 2x = 2\sinh x \cosh x$$

$\sinh x$  is the odd part of the exponential function and  $\cosh x$  is the even part. We thus have

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

A simple computation reveals

$$\begin{aligned} \cosh^2 x - \sinh^2 x &= (\cosh x - \sinh x)(\cosh x + \sinh x) \\ e^{-x} e^x &= 1. \end{aligned}$$

Similarly,

$$\begin{aligned} 2 \sinh x \cosh x &= 2 \left( \frac{e^x + e^{-x}}{2} \right) \left( \frac{e^x - e^{-x}}{2} \right) \\ &= \frac{(e^x + e^{-x})(e^x - e^{-x})}{2} \\ &= \frac{e^{2x} - e^{-2x}}{2} = \sinh 2x. \end{aligned}$$

### 2.6.2 Differentiation

Prove that

$$\frac{d}{dx} \tanh x = \operatorname{sech}^2 x.$$

$$\begin{aligned} \tanh x &= \frac{\sinh x}{\cosh x} \\ &= \frac{e^x - e^{-x}}{e^x + e^{-x}}. \\ \frac{d}{dx} \tanh x &= \frac{d}{dx} (e^x - e^{-x})(e^x + e^{-x})^{-1} \\ &= (e^x + e^{-x})(e^x + e^{-x})^{-1} + (e^x - e^{-x}) - 1(e^x + e^{-x})^{-2}(e^x - e^{-x}) \\ &= \frac{e^x + e^{-x}}{e^x + e^{-x}} - \frac{(e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2} \end{aligned}$$

$$\begin{aligned}
&= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} \\
&= \frac{[(e^x + e^{-x}) + (e^x - e^{-x})][(e^x + e^{-x}) - (e^x - e^{-x})]}{(e^x + e^{-x})^2} \\
&= \frac{(2e^x)(2e^{-x})}{(e^x + e^{-x})^2} \\
&= \frac{4}{(e^x + e^{-x})^2} \\
&= \frac{1}{(\frac{e^x + e^{-x}}{2})^2} \\
&= \frac{1}{\cosh^2 x} \\
&= \operatorname{sech}^2 x.
\end{aligned}$$