TTK4190 Guidance and Control of Vehicles

Draft Solution of Assignment 1

Info

Guidelines

- This presentation is only intended as a draft of a possible way to solve the assignment
- There may exist other ways to solve the problems
- It is possible to obtain full score even if the solution deviates from this draft

Grading

- You will most likely receive with feedback within three weeks of the deadline.

Problem 1 - Attitude Control of Satellite

Problem 1.1

The equations of motion for the satellite are

$$\begin{aligned} \dot{\mathbf{q}} &= \mathbf{T}_q(\mathbf{q})\boldsymbol{\omega} \\ \mathbf{I}_{CG}\dot{\boldsymbol{\omega}} - \mathbf{S}(\mathbf{I}_{CG}\boldsymbol{\omega})\boldsymbol{\omega} &= \boldsymbol{\tau} \end{aligned} \tag{1}$$

and we want to identify the equilibrium point corresponding to $\mathbf{q} = [\eta, \varepsilon_1, \varepsilon_2, \varepsilon_3]^\top = [1, 0, 0, 0]^\top$ and $\boldsymbol{\tau} = 0$.

Equilibrium Point

$$\mathbf{x}_0 = [\boldsymbol{\epsilon}_0^\top, \boldsymbol{\omega}_0^\top]^\top = [\mathbf{0}^\top, \mathbf{0}^\top]^\top$$

Linear System

$$\begin{bmatrix} \dot{\boldsymbol{\epsilon}} \\ \dot{\boldsymbol{\omega}} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{0}_{3x3} & \frac{1}{2}\mathbf{I}_{3x3} \\ \mathbf{0}_{3x3} & \mathbf{0}_{3x3} \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} \boldsymbol{\epsilon} \\ \boldsymbol{\omega} \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{0}_{3x3} \\ \mathbf{I}_{CG}^{-1} \end{bmatrix}}_{\mathbf{B}} \boldsymbol{\tau} \tag{2}$$

Problem 1 - Attitude Control of Satellite

Problem 1.2

Inserting the control law

$$\boldsymbol{\tau} = -\mathbf{K}_d\boldsymbol{\omega} - k_p\boldsymbol{\epsilon} = -\underbrace{\begin{bmatrix} k_p\mathbf{I}_{3\times 3} & \mathbf{K}_d \end{bmatrix}}_{\mathbf{K}}\underbrace{\begin{bmatrix} \boldsymbol{\epsilon} \\ \boldsymbol{\omega} \end{bmatrix}}_{\mathbf{x}} \tag{3}$$

gives the following closed-loop dynamics:

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x}$$

$$\Rightarrow \begin{bmatrix} \dot{\boldsymbol{\epsilon}} \\ \dot{\boldsymbol{\omega}} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{3x3} & \frac{1}{2}\mathbf{I}_{3x3} \\ -\mathbf{I}_{CG}^{-1}k_p & -\mathbf{I}_{CG}^{-1}\mathbf{K}_d \end{bmatrix}\begin{bmatrix} \boldsymbol{\epsilon} \\ \boldsymbol{\omega} \end{bmatrix}$$

Poles: $-0.0278 \pm 0.0248i$ (three pairs of *stable* complex-conjugated poles).

Real or Complex-Conjugated Poles?

- Complex-conjugated poles $\Rightarrow$ oscillations.
- Real poles $\Rightarrow$ no oscillations.
- Keep in mind that this analysis only applies to the linearized system and not necessarily to the real nonlinear dynamics

Problem 1 - Attitude Control of Satellite

Problem 1.3

Control input: $\boldsymbol{\tau} = -\mathbf{K}_d\boldsymbol{\omega} - k_p\boldsymbol{\epsilon}$

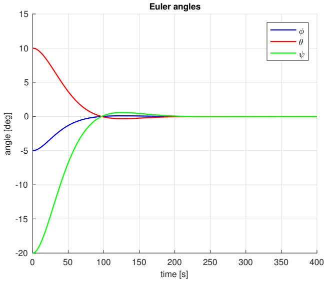

Figure: Euler Angles

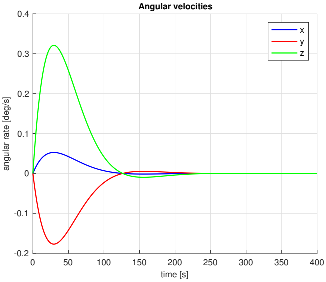

Figure: Angular Velocity

- Converges towards the equilibrium point - expected behaviour.
- The desired state in this case is zero since the reference for $\boldsymbol{\epsilon}$ and $\boldsymbol{\omega}$ in the control law is zero.
- To follow nonzero constant reference signals we use the control law from 1.5: $\boldsymbol{\tau} = -\mathbf{K}_d\boldsymbol{\omega} - k_p\tilde{\boldsymbol{\epsilon}}$

Problem 1 - Attitude Control of Satellite

Problem 1.4

The quaternion error is given by the quaternion product of the current state $\mathbf{q}$ and the desired state $\bar{\mathbf{q}}_d$:

$$
\begin{aligned}
\tilde{\mathbf{q}} &= \bar{\mathbf{q}}_d \otimes \mathbf{q} \\
&= \begin{bmatrix} \eta_d \eta + \boldsymbol{\epsilon}_d^\top \boldsymbol{\epsilon} \\ -\eta \boldsymbol{\epsilon}_d + \eta_d \boldsymbol{\epsilon} - \mathbf{S}(\boldsymbol{\epsilon}_d)\boldsymbol{\epsilon} \end{bmatrix} \\
&= \begin{bmatrix} \eta_d \eta + \epsilon_{d_1}\epsilon_1 + \epsilon_{d_2}\epsilon_2 + \epsilon_{d_3}\epsilon_3 \\ \eta_d \epsilon_1 - \eta \epsilon_{d_1} + \epsilon_{d_3}\epsilon_2 - \epsilon_{d_2}\epsilon_3 \\ \eta_d \epsilon_2 - \eta \epsilon_{d_2} - \epsilon_{d_3}\epsilon_1 + \epsilon_{d_1}\epsilon_3 \\ \eta_d \epsilon_3 - \eta \epsilon_{d_3} + \epsilon_{d_2}\epsilon_1 - \epsilon_{d_1}\epsilon_2 \end{bmatrix}
\end{aligned}
$$

After Convergence

$$
\tilde{\mathbf{q}} = \begin{bmatrix} \eta^2 + \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 \\ \eta\epsilon_1 - \eta\epsilon_1 + \epsilon_3\epsilon_2 - \epsilon_2\epsilon_3 \\ \eta\epsilon_2 - \eta\epsilon_2 - \epsilon_3\epsilon_1 + \epsilon_1\epsilon_3 \\ \eta\epsilon_3 - \eta\epsilon_3 + \epsilon_2\epsilon_1 - \epsilon_1\epsilon_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4}
$$

Problem 1 - Attitude Control of Satellite

Problem 1.5

Control input: $\boldsymbol{\tau} = -\mathbf{K}_d\boldsymbol{\omega} - k_p\tilde{\boldsymbol{\epsilon}}$

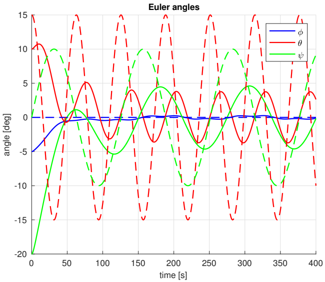

Figure: Euler Angles

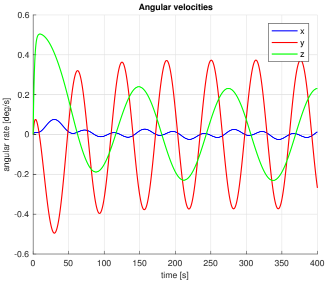

Figure: Angular Velocity

- The Euler angles are pushed towards zero and are not following the time-varying reference.
- The two terms in the controller are working against each other: One term attempts to push the angular velocity to zero, while the other term attempts to track the time-varying reference.

Problem 1 - Attitude Control of Satellite

Problem 1.5

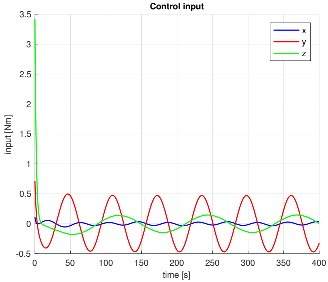

Figure: Control Input

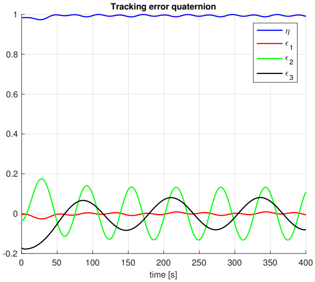

Figure: Tracking Error

Problem 1 - Attitude Control of Satellite

Problem 1.6

Find the desired angular velocity $\rightarrow \boldsymbol{\tau} = -\mathbf{K}_d \tilde{\boldsymbol{\omega}} - k_p \tilde{\boldsymbol{\epsilon}}$

$$
\begin{aligned}
\boldsymbol{\omega}_d &= \mathbf{T}_{\Theta_d}^{-1}(\Theta_d)\dot{\Theta}_d \\
&= \begin{bmatrix} 1 & 0 & -\sin(\theta_d) \\ 0 & \cos(\phi_d) & \cos(\theta_d)\sin(\phi_d) \\ 0 & -\sin(\phi_d) & \cos(\theta_d)\cos(\phi_d) \end{bmatrix} \begin{bmatrix} 0 \\ -1.5\sin(0.1t) \\ 0.5\cos(0.05t) \end{bmatrix}
\end{aligned}
$$

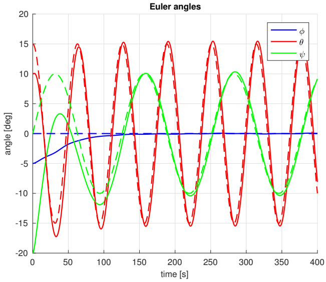

Figure: Euler Angles

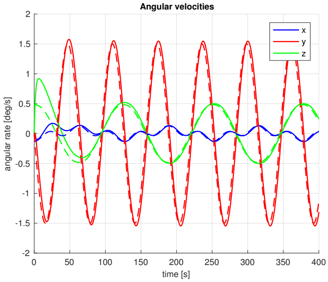

Figure: Angular Velocity

Problem 1 - Attitude Control of Satellite

Problem 1.6

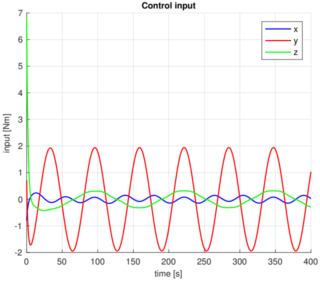

Figure: Control Input

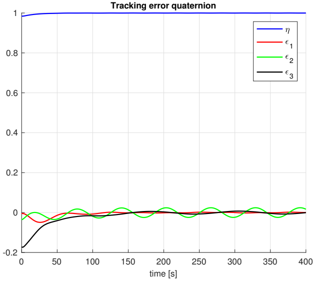

Figure: Tracking Error

- The tracking error is still not perfect. Including a reference model with higher order derivatives in the control law would increase the accuracy.

Problem 1 - Attitude Control of Satellite

Problem 1.7

$$V = \frac{1}{2}\tilde{\boldsymbol{\omega}}^\top \mathbf{I}_{CG}\tilde{\boldsymbol{\omega}} + 2k_p(1-\tilde{\eta}), \quad \dot{\tilde{\eta}} = -\frac{1}{2}\tilde{\boldsymbol{\epsilon}}^\top\tilde{\boldsymbol{\omega}}, \quad \mathbf{I}_{CG}\dot{\boldsymbol{\omega}} = \mathbf{S}(\mathbf{I}_{CG}\boldsymbol{\omega})\boldsymbol{\omega} + \boldsymbol{\tau}$$

Positive and radially Unbounded

- The Lyapunov function is positive because the first term is quadratic and always greater or equal to zero and the second term is always positive as long as k_p is greater than zero $(1 - \tilde{\eta} \geq 0)$.
- The Lyapunov function is radially unbounded because it increases with $\tilde{\boldsymbol{\omega}}$ without an upper bound. $\tilde{\eta}$ cannot grow towards infinity by definition.

Finding the derivative ($\tilde{\boldsymbol{\omega}} = \boldsymbol{\omega} - \boldsymbol{\omega}_d = \boldsymbol{\omega}$):

$$\begin{aligned}
\dot{V} &= \boldsymbol{\omega}^\top \mathbf{I}_{CG}\dot{\boldsymbol{\omega}} - 2k_p\dot{\tilde{\eta}} \\
&= \boldsymbol{\omega}^\top(\boldsymbol{\tau} + \mathbf{S}(\mathbf{I}_{CG}\boldsymbol{\omega})\boldsymbol{\omega}) - 2k_p(-\frac{1}{2}\tilde{\boldsymbol{\epsilon}}^\top\tilde{\boldsymbol{\omega}}) \\
&= \boldsymbol{\omega}^\top(-\mathbf{K}_d\boldsymbol{\omega} - k_p\tilde{\boldsymbol{\epsilon}} + \mathbf{S}(\mathbf{I}_{CG}\boldsymbol{\omega})\boldsymbol{\omega}) + k_p\boldsymbol{\omega}^\top\tilde{\boldsymbol{\epsilon}} \\
&= \boldsymbol{\omega}^\top(-\mathbf{K}_d\boldsymbol{\omega} - k_p\tilde{\boldsymbol{\epsilon}} + \mathbf{S}(\mathbf{I}_{CG}\boldsymbol{\omega})\boldsymbol{\omega} + k_p\tilde{\boldsymbol{\epsilon}}) \\
&= -\boldsymbol{\omega}^\top\mathbf{K}_d\boldsymbol{\omega} = -\boldsymbol{\omega}^\top k_d\mathbf{I}\boldsymbol{\omega} = -k_d\boldsymbol{\omega}^\top\boldsymbol{\omega} \leq 0 \text{ (if } k_d > 0)
\end{aligned} \tag{5}$$

Problem 1 - Attitude Control of Satellite

Problem 1.7

Lyapunov's direct method

Let $\mathbf{x}_e = \mathbf{0}$ be an equilibrium point for $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that

$$
\begin{aligned}
&(i)\ V(\mathbf{0}) = 0 \quad \text{and} \quad V(\mathbf{x}) > 0, \quad \forall \mathbf{x} \neq 0 \\
&(ii)\ ||\mathbf{x}|| \to \infty \quad \Rightarrow \quad V(\mathbf{x}) \to \infty \\
&(iii)\ \dot{V}(\mathbf{x}) < 0, \quad \forall \mathbf{x} \neq 0
\end{aligned}
$$

then $\mathbf{x_e} = 0$ is globally asymptotically stable (GAS).

- Condition (i) and (ii) are satisfied on the previous slide.
- However, $\dot{V}(x) = -k_d \boldsymbol{\omega}^\top \boldsymbol{\omega} \leq 0 \Rightarrow$ (iii) not satisfied.
- We *cannot* use Lyapunov's direct method to conclude GAS (or AS).

Problem 1 - Attitude Control of Satellite

Problem 1.7

We need to use LaSalle's theorem to investigate AS since Lyapunovs direct method was not conclusive.

The Krasovskii-LaSalles's Theorem

Let V : $\mathbb{R}^n \rightarrow \mathbb{R}$ be a continuously differentiable positive definite function such that

$$\begin{aligned} &||\mathbf{x}|| \rightarrow \infty \quad \Rightarrow \quad V(\mathbf{x}) \rightarrow \infty \\ &\dot{V}(\mathbf{x}) \leq 0, \quad \forall \mathbf{x} \end{aligned} \tag{6}$$

Let Ω be the set of all points where $\dot{V}(\mathbf{x}) = 0$, i.e.,

$$\Omega = \{\mathbf{x} \in \mathbb{R}^n | \dot{V}(\mathbf{x}) = 0\} \tag{7}$$

and M be the largest invariant set in Ω. Then all solutions $\mathbf{x}(t)$ converge to M. If $M = \{\mathbf{x}_e\}$ then the equilibrium point $\mathbf{x}_e$ is globally asymptotically stable.

Problem 1 - Attitude Control of Satellite

Problem 1.7

Define set $\Omega = \{\mathbf{x} \in \mathbb{R}^n | \dot{V}(\mathbf{x}) = 0\}$

Find all points such that

$$\dot{V}(\tilde{\epsilon}, \boldsymbol{\omega}) = -k_d \boldsymbol{\omega}^\top \boldsymbol{\omega} = 0 \quad \Rightarrow \boldsymbol{\omega} = \mathbf{0} \tag{8}$$

The set Ω of all points $\mathbf{x}$ where $\dot{V} = 0$ therefore becomes

$$\Omega = \{\boldsymbol{\omega} = \mathbf{0}, \quad \tilde{\mathbf{q}}(\tilde{\epsilon}) \in Q\} \tag{9}$$

where Q is defined as the set of unit quaternions.

Find largest invariant set M

Inserting $\boldsymbol{\omega}(t) \equiv \mathbf{0} \Rightarrow \dot{\boldsymbol{\omega}}(t) \equiv \mathbf{0}$ into the dynamical system we find that

$$\dot{\boldsymbol{\omega}}(t) = \mathbf{I}_{CG}^{-1} \boldsymbol{\tau}(t) \equiv \mathbf{0} \Rightarrow -\mathbf{K}_d \boldsymbol{\omega}(t) - k_p \tilde{\epsilon}(t) \equiv \mathbf{0} \Rightarrow \tilde{\epsilon}(t) \equiv \mathbf{0}$$

In other words, the angular velocity can only be zero when the attitude error is zero, i.e., after convergence.

Problem 1 - Attitude Control of Satellite

Problem 1.7

Find largest invariant set M

The largest invariant set M in Ω comprises *two* equlibrium points corresponding to two different unit quaternions (recall that $\tilde{\eta}^2 + \tilde{\boldsymbol{\epsilon}}^\top \tilde{\boldsymbol{\epsilon}} = 1$)

$$M = \{\boldsymbol{\omega} = \mathbf{0}, \quad \tilde{\boldsymbol{\epsilon}} = \mathbf{0}, \quad \tilde{\eta} = \pm 1\} \tag{10}$$

The existence of multiple equilibrium points implies that that only local asymptotic stability can be proven. Hence, we must investigate the stability of each equilibrium individually.

Stability analysis of $\tilde{\eta} = 1$

The steady-state value of the Lyapunov function, $V = \frac{1}{2}\tilde{\boldsymbol{\omega}}^\top \mathbf{I}_{CG} \tilde{\boldsymbol{\omega}} + 2k_p(1 - \tilde{\eta})$, is in this case

$$V_{ss} = V(\tilde{\eta} = 1) = 0$$

Perturbing the system by a small amount such that $\tilde{\eta} = 1 - \delta$ where $\delta > 0$, results in a steady-state value

$$V_{ss}^{'} = V(\tilde{\eta} = 1 - \delta) = 2k_p\delta$$

Problem 1 - Attitude Control of Satellite

Problem 1.7

Since $V_{ss}^{'} > V_{ss}$ and V decreases monotonically for $\tilde{\eta} \neq \pm 1$, the system will return to $V_{ss} = 0$. Thus, the equilibrium point corresponding to $\tilde{\eta} = 1$ is locally AS.

Stability analysis of $\tilde{\eta} = -1$

The steady-state value of the Lyapunov function, $V = \frac{1}{2}\tilde{\boldsymbol{\omega}}^\top \mathbf{I}_{CG}\tilde{\boldsymbol{\omega}} + 2k_p(1 - \tilde{\eta})$, is in this case

$$V_{ss} = V(\tilde{\eta} = -1) = 4k_p$$

Perturbing the system by a small amount such that $\tilde{\eta} = -1 + \delta$ where $\delta > 0$, results in a steady-state value

$$V_{ss}^{'} = V(\tilde{\eta} = 1 - \delta) = 4k_p - 2k_p\delta$$

Since $V_{ss}^{'} < V_{ss}$ and V decreases monotonically for $\tilde{\eta} \neq \pm 1$, the system will never return to $V_{ss} = 4k_p$. Thus, the equilibrium point corresponding to $\tilde{\eta} = -1$ is unstable.

For Further Reading I

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