[thm]Definition

Notes 8302-01 Harmonic Analysis Seminar - Steve Hofmann

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Contents

1. Monday, August 25, 2014

1.1. Review of measure theory.

Definition 1.1. A *set function* is a mapping $\mu : P(x) \to Y$ where Y is a Banach space. Y is usually $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.2. A set function is an *outer measure* if

(i) $\mu(A) \geq 0$, $\forall A \subseteq X, \mu(\emptyset) = 0$
(ii) (monotonicity) If $A \subseteq B$ then $\mu(A) \leq \mu(B)$.
(iii) (Subaditivity) Given a countable collection $\{A_k\}_{k=1}^{\infty}$ of subsets of X,

$$\mu(\cup_{k=1}^{\infty} A_k) \leq \mu\left(\sum_{k=1}^{\infty} A_k\right)$$

Fact 1.3. Every outer measure includes a measure in the following way.

Definition 1.4. Given an outmeasure μ a subset $E \subseteq X$ is *measurable* if it satisfies the following *Caratheadory Criterion*: $\forall A \subseteq X$ $\mu(A) = \mu(A \cap E) + \mu(A \setminus E)$. Notice A does not have to necessarily be μ-measurable.

Equivalently we have that E is μ-measurable iff $\mu(A_1 \cap A_2) = \mu(A_1) + \mu(A_2)$ whenever $\mu(A_2 \subseteq E)$ and $A_2 \cap E = \emptyset$.

Proposition 1.5. (exercise) Let μ be an outer measure on X. If $\mu(Z) = 0$ then Z is μ-measurable.

Note 1.6. This says $\emptyset$ is measurable. This also says the E^c is measurable.

Proposition 1.7. (exercise) Given an outer measure μ the collection of all μ-measurable sets forms a σ-algebra.

Definition 1.8. An outer measure μ is a *mesaure* when restricted to its measurable sets.

Theorem 1.9. (exercise) Suppose μ is an outer measure and that $\{A_k\}_{k=1}^{\infty}$ is a collection of μ-measurable sets.

(1) If $\{A_k\}_{k=1}^{\infty}$ are pairwise disjoint then $\sigma\left(\cup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$.
(2) If $A_1 \subseteq \ldots A_k \subseteq A_{k+1} \subseteq \ldots$ then $\lim_{k\to\infty} \mu(A_k) = \mu\left(\cup_{k=1}^{\infty} A_k\right)$.
(3) If $A_1 \supseteq A_2 \supseteq \ldots$ and $\mu(A_1) < \infty$ then $\lim_{k\to\infty} \mu(A_k) = \mu(\cap_{k=1}^{\infty} A_k)$.

Definition 1.10. Suppose X is a topological space. The *Borel σ-algebra* is the smallest σ-alg. that contains all the open set.

DEFINITION 1.11. A subest $A \subseteq X$ is σ-finite w.r.t. μ if $A = \Cup_{k=1}^{\infty} B_k$ where B_k is a μ-measurable set and $\mu(B_k) < \infty, \forall k$.

DEFINITION 1.12.
(1) μ is *regular* if $\forall$A$\subseteq X$ there exists a μ-measurable set B so that $A \subseteq B$ and $\mu(A) = \mu(B)$.
(2) μ is a *Borel* measure if every Borel set is μ-measurable.
(3) μ is *Borel regular* if μ is Borel and $\forall A \subseteq \mathbb{R}$ (or X) there exists a Borel set B with $A \subseteq B$ and $\mu(A) = \mu(B)$.
(4) μ is a *Radon* measure if μ is Borel and if $\mu(K) < \infty$ for every cpt set K.

THEOREM 1.13. Let μ be a regular measure on X. If $A_1 \subseteq A_2 \subseteq \dots$ then $\lim_{k\to\infty} \mu(A_k) = \mu\left(\cup_{k=1}^{\infty} A_k.\right)$

the significance of this theorem is that we are not requiring the A_k to be μ-measurable.

THEOREM 1.14. Suppose μ is a Borel regular on $\mathbb{R}^n$ and $A \subseteq \mathbb{R}^n$ is μ-measurable with $\mu(A) < \infty$. Then $\mu|_A$ is a Radon measure. Here $\mu|_A$ is defined as $\mu|_A(B) = \mu(A \cap B)$.

Bibliography