Notes for Math 8302-01
Fall 2014
Introduction to Geometric Measure Theory and Quantitative Rectifiability

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Contents

CHAPTER 1

Intro and Syllabus

The following is basic information about the class from the syllabus.

Institution: University of Missouri at Columbia

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Instructor: Steve Hofmann

Course title: Math 8302 - Introduction to Geometric Measure Theory and Quantitative Rectifiablity.

Description: Geometric Measure Theory (GMT) is largely concerned with the notion of "rectifiability", which describes a sense in which a general set of points (in $\mathbb{R}^n$, say) is approximated by "nice" surfaces. In the past couple of decades, this concept has been sharpened: "uniform (i.e., quantitative) rectifiability" entails approximation of a set by nice surfaces, but in a quanitatively precise way, which turns out to have deep connections with the behavior, on the set, of singular integral operators, square functions, and harmonic measure. We plan to cover the following topics, to the extent that time permits

(1) Review of Hausdorff measure
(2) Bounded variation, sets of finite perimeter, isoperimetric inequality, reduced boundary, measure theoretic boundary, Hauss-Green theorem.
(3) Ahlfors-David Regularity, "Spaces of homogeneous type" and M. Christ's dyadic cube reconstruction.
(4) NTA domains and "big pieces of Lipschitz graphs".
(5) Uniform rectifiablity, Corona decomposition, "Geometric Lemma" and the "Bilateral Weak Geometric Lemma", singular intergrals and square functions.

Prerequisites Familiarity with the elemetntary theory of measure and integration and with the basic subject matter of harmonic analysis: Fourier transform, Hardy-Littlewood maximal function, approximate identities, Whitney decomposition, Littlewood-Paley theory, classical Calderon-Zygmund theory, BMO, Carleson measures, theory of sungular intergrals on Lipschitz graphs.

Text: There is no offical textbook for the course, however, much of the material presented in the course will follow [**1**],[**2**], [**3**], and [**4**]. Other material may be taken directly from the literature.

CHAPTER 2

Review of measure theory

Most of the material in this chapter is from the book "Measure Theory and Fine Properties of Functions" by Evans and Gariepy [3]

DEFINITION 0.1. A *set function* f is a mapping $f : P(x) \to Y$ where Y is a Banach space. Y is usually $\mathbb{R}$ or $\mathbb{C}$.

DEFINITION 0.2. Let X be a set. We let $P(X)$ denote the power set of X i.e. the set of all subsets of X.

DEFINITION 0.3. Let X be a set. A subset $\mathfrak{M} \subseteq P(X)$ is called a σ-algebra on X if $\mathfrak{M}$ has the following properties:

(1) $\emptyset \in \mathfrak{M}$ and $X \in \mathfrak{M}$.
(2) $A \in \mathfrak{M}$ implies $A^c \in \mathfrak{M}$.
(3) $A \in \mathfrak{M}$ whenever A is the countable union of sets $A_1, A_2, \ldots$ in $\mathfrak{M}$

DEFINITION 0.4. Suppose X is a set and $\mathfrak{M}$ is a σ-algebra on X. A set function $\mu : \mathfrak{M} \to [0, \infty]$ is called a *measure* on $(X, \mathfrak{M})$ if μ has the following properties:

(1) $\mu(\emptyset) = 0$
(2) $\mu\left(\sum_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$ whenever $\{A_i\}_{i=1}^{\infty}$ is a countable colelction of disjoint sets in $\mathfrak{M}$.

A slightly more relaxed notion of a measure is what is called an outer measure.

DEFINITION 0.5. A set function $\tilde{\mu} : P(X) \to [0, \infty]$ is called an *outer measure* if

(1) $\mu(\emptyset) = 0$
(2) $\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_k)$ whenever $A \subseteq \bigcup_{i=1}^{\infty} A_k$.

REMARK 0.6. If $\tilde{\mu}$ is an outer measure then we have that $\tilde{\mu}(A) \leq \tilde{\mu}(B)$ whenever $A \subseteq B$ where $A, B \subseteq X$.

Definition 0.7. Given an outer measure $\tilde{\mu}$ a subset $E \subseteq X$ is *$\tilde{\mu}$-measurable* if it satisfies the following *Caratheadory Criterion*:

$$\tilde{\mu}(A) = \tilde{\mu}(A \cap E) + \tilde{\mu}(A \setminus E), \quad \text{for all} \quad A \subseteq X. \tag{0.1}$$

Equivalently we have that E is $\tilde{\mu}$-measurable iff $\tilde{\mu}(A_1 \cap A_2) = \tilde{\mu}(A_1) + \tilde{\mu}(A_2)$ whenever $\tilde{\mu}(A_2 \subseteq E)$ and $A_2 \cap E = \emptyset$.

Proposition 0.8. (exercise) Let $\tilde{\mu}$ be an outer measure on X. If $\tilde{\mu}(Z) = 0$ then Z is $\tilde{\mu}$-measurable.

Note 0.9. This says $\emptyset$ is measurable.

Proposition 0.10. (exercise) Given an outer measure $\tilde{\mu}$, the collection of all $\tilde{\mu}$-measurable sets forms a σ-algebra.

Proposition 0.11. (exercise) Any outer measure $\tilde{\mu}$ restricted to its measurable sets is a measure.

Theorem 0.12. (exercise) Suppose $\tilde{\mu}$ is an outer measure and that $\{A_k\}_{k=1}^{\infty}$ is a collection of $\tilde{\mu}$-measurable sets.

(1) If $\{A_k\}_{k=1}^{\infty}$ are pairwise disjoint then $\mu\left(\cup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \tilde{\mu}(A_k)$.
(2) If $A_1 \subseteq A_2 \subseteq \ldots$ then $\lim_{k\to\infty} \tilde{\mu}(A_k) = \tilde{\mu}\left(\cup_{k=1}^{\infty} A_k\right)$.
(3) If $A_1 \supseteq A_2 \supseteq \ldots$ and $\tilde{\mu}(A_1) < \infty$ then $\lim_{k\to\infty} \tilde{\mu}(A_k) = \tilde{\mu}(\cap_{k=1}^{\infty} A_k)$.

Definition 0.13. Suppose X is a topological space. The *Borel σ-algebra*, denoted $\mathfrak{B}_X$, is the smallest σ-algebra that contains all the open set. This can be constructed by taking the intersection of all σ-algebras on X which contain the all open sets of X.

Definition 0.14. A subset $A \subseteq X$ is called σ-finite w.r.t. $\tilde{\mu}$ if $A = \bigcup_{k=1}^{\infty} B_k$ where B_k is a $\tilde{\mu}$-measurable set and $\tilde{\mu}(B_k) < \infty, \forall k$.

Definition 0.15. Let X be an arbitrary space and $\tilde{\mu}$ an outer measure on X.

(1) $\tilde{\mu}$ is *regular* if $\forall A \subseteq X$ there exists a $\tilde{\mu}$-measurable set B so that $A \subseteq B$ and $\tilde{\mu}(A) = \tilde{\mu}(B)$.
(2) $\tilde{\mu}$ is a *Borel* measure if every Borel set is $\tilde{\mu}$-measurable.
(3) $\tilde{\mu}$ is *Borel regular* if $\tilde{\mu}$ is Borel and $\forall A \subseteq \mathbb{R}$ (or X) there exists a Borel set B with $A \subseteq B$ and $\tilde{\mu}(A) = \tilde{\mu}(B)$.
(4) $\tilde{\mu}$ is a *Radon* measure if $\tilde{\mu}$ is Borel and if $\tilde{\mu}(K) < \infty$ for every compact set K.

THEOREM 0.16. [3, thm 2] Let $\tilde{\mu}$ be a regular outer measure on X. If $A_1 \subseteq A_2 \subseteq \ldots$ then $\lim_{k\to\infty} \tilde{\mu}(A_k) = \tilde{\mu}\left(\cup_{k=1}^{\infty} A_k.\right)$

REMARK 0.17. The significance of this theorem is that we are not requiring the A_k to be $\tilde{\mu}$-measurable.

THEOREM 0.18. [3, thm 3] Suppose $\tilde{\mu}$ is a Borel regular outer measure on $\mathbb{R}^n$ and $A \subseteq \mathbb{R}^n$ is $\tilde{\mu}$-measurable with $\tilde{\mu}(A) < \infty$. Then $\tilde{\mu}|_A$ is a Radon measure. Here $\tilde{\mu}|_A$ is defined as $\tilde{\mu}|_A(B) = \tilde{\mu}(A \cap B)$ for ever $B \subseteq X$.

LEMMA 0.19. [3, lemma 1] Let $\tilde{\mu}$ be a Borel regular outer measure on $\mathbb{R}^n$. and let $B \in \mathfrak{B}_X$.

(1) If $\tilde{\mu} < \infty$ then for every $\varepsilon > 0$ there exists a closed set C_ε so that $C_\varepsilon \subseteq B$ and $\tilde{\mu}(B \setminus C_\varepsilon) < \varepsilon$.
(2) If $\tilde{\mu}$ is a Radon outer measure, then for ever $\varepsilon > 0$ there exists an open set U_ε so that $B \subseteq U_\varepsilon$ and $\tilde{\mu}(B \setminus U_\varepsilon) < \varepsilon$.

THEOREM 0.20. [3, thm 4](Approximation by open and compact sets)
Let $\tilde{\mu}$ be a Radon outer measure on $\mathbb{R}^n$. Then

(1) for each set $A \subseteq \mathbb{R}^n$
$$\tilde{\mu}(A) = \inf\left\{\tilde{\mu}(U) : A \subseteq U, U \text{ is open}\right\},$$
and
(2) for each $\tilde{\mu}$-measurable set $A \subseteq \mathbb{R}^n$
$$\tilde{\mu}(A) = \sup\left\{\tilde{\mu}(K) : K \subseteq A, K \text{ is compact}\right\}.$$

Bibliography

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