Notes for Math 8302-01
Fall 2014
Introduction to Geometric Measure Theory and Quantitative Rectifiability

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Contents

CHAPTER 1

Intro and Syllabus

The following is basic information about the class from the syllabus.

Institution: University of Missouri at Columbia

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Instructor: Steve Hofmann

Course title: Math 8302 - Introduction to Geometric Measure Theory and Quantitative Rectifiablity.

Description: Geometric Measure Theory (GMT) is largely concerned with the notion of "rectifiability", which describes a sense in which a general set of points (in $\mathbb{R}^n$, say) is approximated by "nice" surfaces. In the past couple of decades, this concept has been sharpened: "uniform (i.e., quantitative) rectifiability" entails approximation of a set by nice surfaces, but in a quanitatively precise way, which turns out to have deep connections with the behavior, on the set, of singular integral operators, square functions, and harmonic measure. We plan to cover the following topics, to the extent that time permits

(1) Review of Hausdorff measure
(2) Bounded variation, sets of finite perimeter, isoperimetric inequality, reduced boundary, measure theoretic boundary, Hauss-Green theorem.
(3) Ahlfors-David Regularity, "Spaces of homogeneous type" and M. Christ's dyadic cube reconstruction.
(4) NTA domains and "big pieces of Lipschitz graphs".
(5) Uniform rectifiablity, Corona decomposition, "Geometric Lemma" and the "Bilateral Weak Geometric Lemma", singular intergrals and square functions.

Prerequisites Familiarity with the elemetntary theory of measure and integration and with the basic subject matter of harmonic analysis: Fourier transform, Hardy-Littlewood maximal function, approximate identities, Whitney decomposition, Littlewood-Paley theory, classical Calderon-Zygmund theory, BMO, Carleson measures, theory of sungular intergrals on Lipschitz graphs.

Text: There is no offical textbook for the course, however, much of the material presented in the course will follow [**1**],[**2**], [**3**], and [**5**]. Other material may be taken directly from the literature.

CHAPTER 2

Review of measure theory

1. Outer Measures

Most of the material in this chapter is from the book "Measure Theory and Fine Properties of Functions" by Evans and Gariepy [3]

DEFINITION 1.1. *A set function f is a mapping $f : P(x) \to Y$ where Y is a Banach space. Y is usually $\mathbb{R}$ or $\mathbb{C}$.*

DEFINITION 1.2. *Let X be a set. We let $P(X)$ denote the power set of X i.e. the set of all subsets of X.*

DEFINITION 1.3. *Let X be set. A non-empty subset $\mathfrak{M} \subseteq P(X)$ is called* an algebra *on X if $\mathfrak{M}$ has the following properties:*

(1) $A \in \mathfrak{M}$ implies $A^c \in \mathfrak{M}$.
(2) If $n \in \mathbb{N}$ and $A_1, \ldots, A_n \in \mathfrak{M}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathfrak{M}$

A σ-algebra *is an algebra which is closed under countable unions instead of finite unions.*

REMARK 1.4. *Notice that if $\mathfrak{M}$ is an algebra or a σ-algebra on X then $\emptyset \in \mathfrak{M}$ and $X \in \mathfrak{M}$ because if $E \in \mathfrak{M}$ then $\emptyset = E \cap E^c$ and $X = E \cup E^c$.*

DEFINITION 1.5. *Suppose X is a set and $\mathfrak{M}$ is a σ-algebra on X. A set function $\mu : \mathfrak{M} \to [0, \infty]$ is called* a measure *on $(X, \mathfrak{M})$ if μ has the following properties:*

(1) $\mu(\emptyset) = 0$
(2) $\mu\left(\sum_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$ whenever $\{A_i\}_{i=1}^{\infty}$ is a countable colelction of disjoint sets in $\mathfrak{M}$.

A slightly more relaxed notion of a measure is what is called an outer measure.

DEFINITION 1.6. *A set function $\tilde{\mu} : P(X) \to [0, \infty]$ is called* an outer measure *if*

(1) $\tilde{\mu}(\emptyset) = 0$
(2) $\tilde{\mu}(A) \le \tilde{\mu}(B)$ if $A \subseteq B \subseteq X$,

(3) $\tilde{\mu}\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \tilde{\mu}(A_k)$ *whenever* $\{A_i\}_{i=1}^{\infty} \subseteq P(X)$.

DEFINITION 1.7. *Given an outer measure* $\tilde{\mu}$ *a subset* $E \subseteq X$ *is* $\tilde{\mu}$-measurable *if it satisfies the following* Caratheadory Criterion*:*

$$\tilde{\mu}(A) = \tilde{\mu}(A \cap E) + \tilde{\mu}(A \setminus E), \quad \text{for all} \quad A \subseteq X. \tag{1.1}$$

Equivalently we have that E *is* $\tilde{\mu}$*-measurable iff* $\tilde{\mu}(A_1 \cap A_2) = \tilde{\mu}(A_1) + \tilde{\mu}(A_2)$ *whenever* $\tilde{\mu}(A_2 \subseteq E)$ *and* $A_2 \cap E = \emptyset$.

PROPOSITION 1.8. *(exercise) Let* $\tilde{\mu}$ *be an outer measure on* X*. If* $\tilde{\mu}(Z) = 0$ *then* Z *is* $\tilde{\mu}$*-measurable.*

NOTE 1.9. *This says* $\emptyset$ *is measurable.*

PROPOSITION 1.10. *(exercise) Given an outer measure* $\tilde{\mu}$*, the collection of all* $\tilde{\mu}$*-measurable sets forms a* σ*-algebra.*

DEFINITION 1.11. *Denote the* σ*-algebra of* $\tilde{\mu}$*-mesaurable sets as* $\mathfrak{M}_{\tilde{\mu}}$.

PROPOSITION 1.12. *(exercise) Any outer measure* $\tilde{\mu}$ *restricted to its measurable sets is a measure.*

THEOREM 1.13. *(exercise) Suppose* $\tilde{\mu}$ *is an outer measure and that* $\{A_k\}_{k=1}^{\infty}$ *is a collection of* $\tilde{\mu}$*-measurable sets.*

(1) If $\{A_k\}_{k=1}^{\infty}$ *are pairwise disjoint then* $\mu\left(\cup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \tilde{\mu}(A_k)$.
(2) If $A_1 \subseteq A_2 \subseteq \ldots$ *then* $\lim_{k\to\infty} \tilde{\mu}(A_k) = \tilde{\mu}\left(\cup_{k=1}^{\infty} A_k\right)$.
(3) If $A_1 \supseteq A_2 \supseteq \ldots$ *and* $\tilde{\mu}(A_1) < \infty$ *then* $\lim_{k\to\infty} \tilde{\mu}(A_k) = \tilde{\mu}(\cap_{k=1}^{\infty} A_k)$.

DEFINITION 1.14. *Suppose* X *is a topological space. The* Borel σ-algebra*, denoted* $\mathfrak{B}_X$*, is the smallest* σ*-algebra that contains all the open set. This can be constructed by taking the intersection of all* σ*-algebras on* X *which contain the all open sets of* X*.*

DEFINITION 1.15. *A subset* $A \subseteq X$ *is called* σ*-finite w.r.t.* $\tilde{\mu}$ *if* $A = \bigcup_{k=1}^{\infty} B_k$ *where* $\{B_i\}_{i=1}^{\infty} \subseteq \mathfrak{M}_{\tilde{\mu}}$ *and* $\tilde{\mu}(B_i) < \infty, \forall i$.

DEFINITION 1.16. *Let* X *be an arbitrary space and* $\tilde{\mu}$ *an outer measure on* X*.*

(1) $\tilde{\mu}$ *is* regular *if* $\forall A \subseteq X$ *there exists a* $B \in \mathfrak{M}_{\tilde{\mu}}$ *so that* $A \subseteq B$ *and* $\tilde{\mu}(A) = \tilde{\mu}(B)$.
(2) $\tilde{\mu}$ *is a* Borel measure *if every Borel set is* $\tilde{\mu}$*-measurable.*
(3) $\tilde{\mu}$ *is* Borel regular *if* $\tilde{\mu}$ *is Borel and* $\forall A \subseteq \mathbb{R}$ *(or* X*) there exists a Borel set* B *with* $A \subseteq B$ *and* $\tilde{\mu}(A) = \tilde{\mu}(B)$.

(4) $\tilde{\mu}$ *is a Radon measure if* $\tilde{\mu}$ *is Borel regular and if* $\tilde{\mu}(K) < \infty$ *for every compact set* K.

THEOREM 1.17. [3, sec 1.1] *Let* $\tilde{\mu}$ *be a regular outer measure on* X. *If* $A_1 \subseteq A_2 \subseteq \dots$ *then* $\lim_{k\to\infty} \tilde{\mu}(A_k) = \tilde{\mu}\left(\cup_{k=1}^{\infty} A_k.\right)$

REMARK 1.18. *The significance of this theorem is that we are not requiring the* A_k *to be* $\tilde{\mu}$*-measurable.*

DEFINITION 1.19. *Let* $\tilde{\mu}|_A$ *be defined as* $\tilde{\mu}|_A(B) = \tilde{\mu}(A \cap B)$ *for ever* $B \subseteq X$.

THEOREM 1.20. [3, sec 1.1] *Suppose* $\tilde{\mu}$ *is a Borel regular outer measure on* $\mathbb{R}^n$ *and* $A \subseteq \mathbb{R}^n$ *is* $\tilde{\mu}$*-measurable with* $\tilde{\mu}(A) < \infty$. *Then* $\tilde{\mu}|_A$ *is a Radon measure.*

LEMMA 1.21. [3, sec 1.1] *Let* $\tilde{\mu}$ *be a Borel regular outer measure on* $\mathbb{R}^n$. *and let* $B \in \mathfrak{B}_X$.

(1) *If* $\tilde{\mu} < \infty$ *then for every* $\varepsilon > 0$ *there exists a closed set* C_ε *so that* $C_\varepsilon \subseteq B$ *and* $\tilde{\mu}(B \setminus C_\varepsilon) < \varepsilon$.
(2) *If* $\tilde{\mu}$ *is a Radon outer measure, then for ever* $\varepsilon > 0$ *there exists an open set* U_ε *so that* $B \subseteq U_\varepsilon$ *and* $\tilde{\mu}(B \setminus U_\varepsilon) < \varepsilon$.

THEOREM 1.22. [3, sec 1.1]*(Approximation by open and compact sets) Let* $\tilde{\mu}$ *be a Radon outer measure on* $\mathbb{R}^n$. *Then*

(1) *for each set* $A \subseteq \mathbb{R}^n$

$$\tilde{\mu}(A) = \inf\left\{\tilde{\mu}(U) : A \subseteq U, U \text{ is open}\right\},$$

and

(2) *for each* $\tilde{\mu}$*-measurable set* $A \subseteq \mathbb{R}^n$

$$\tilde{\mu}(A) = \sup\left\{\tilde{\mu}(K) : K \subseteq A, K \text{ is compact}\right\}.$$

THEOREM 1.23. [3, sec 1.1]*(Caratheodory's Criterion) Let* $\tilde{\mu}$ *be an outer measure on* $\mathbb{R}^n$. *If* $\tilde{\mu}(A \cup B) = \tilde{\mu}(A) + \tilde{\mu}(B)$ *for all sets* $A, B \subseteq \mathbb{R}^n$ *with dist*$(A, B) > 0$, *then* $\tilde{\mu}$ *is a Borel outer measure.*

2. Covering Lemmas

THEOREM 2.1 (Vitali's covering lemma). [3, sec 1.5.1] *Let* $\mathcal{F}$ *be an collection of nondegenerate closed balls in* $\mathbb{R}^n$ *with* $\sup\left\{diam(B) | B \in \mathcal{F}\right\} < \infty$. *Then there exists a countable family* $\mathcal{G}$ *of disjoint balls in* $\mathcal{F}$ *such that*

$$\bigcup_{B\in\mathcal{F}} B \subseteq \bigcup_{B\in\mathcal{G}} 5B.$$

If $\tilde{\mu}$ is an arbitrary Radon outer measure on $\mathbb{R}^n$, we can't always control $\tilde{\mu}(5B)$ in terms of $\tilde{\mu}(B)$. So instead of enlarging the balls to cover the space we try to control the number of overlaps by a constant which depends only on the dimension. This is the content of the next theorem.

THEOREM 2.2. *(Besicovitch's covering theorem)*[**3**, sec 1.5.2] *There exists a constant $M = M_n \in \mathbb{N}$ depending only on n, with the following property: If $\mathcal{F}$ is any collection of nondegenerate closed balls in $\mathbb{R}^n$ with*

$$\sup \{ diam(B) | B \in \mathcal{F} \} < \infty$$

and if $\mathcal{A}$ is the set of centers of balls in $\mathcal{F}$, then there exist $\mathcal{G}_1, \ldots, \mathcal{G}_M \subseteq \mathcal{F}$ such that $\mathcal{G}_i$ $(i = 1, \ldots, M)$ is a countable collection of disjoint balls in $\mathcal{F}$ and

$$\mathcal{A} \subseteq \bigcup_{i=1}^{M} \bigcup_{B \in \mathcal{G}_i} B.$$

The following decompostion/covering theorem says you can cover an arbitrary nonempty open set $\mathbb{R}^n$ with closed cubes whose side lengths are proportional to the cube's distance to the boundaries of the set. Also the number of neighboring cubes of any given cube is bounded according to the cube's side length.

THEOREM 2.3. *(Whitney decomposition)*[**4**, appendix J] *Let Ω be an open, nonempty, proper subset of $\mathbb{R}^n$. Then there exists family of closed cubes $\{Q_j\}_j$ such that*

(1) $\bigcup_j Q_j = \Omega$ and the Q_j's have disjoint interiors.
(2) $\sqrt{n}\ell(Q_j) \leq dist(Q_j, \Omega^c) \leq 4\sqrt{n}\ell(Q_j)$.
(3) If the boundaries of two cubes Q_j and Q_k touch then

$$\frac{1}{4} \leq \frac{\ell(Q_j)}{\ell(Q_k)} \leq 4.$$

(4) For a given Q_j there exists at most 12^n Q_k's that touch it.

3. Hausdorff Measure

This section follows section of 2.1 of [**?**].

DEFINITION 3.1.
(1) Let $A \subseteq \mathbb{R}^n$, $0 \leq s < \infty$, $\delta \leq \infty$. Define

$$\mathcal{H}^s_\delta := \inf \left\{ \sum_{j=1}^{\infty} \alpha(s) \left(\frac{diam(C_j)}{2} \right)^s : A \subseteq \bigcup_{j=1}^{\infty} C_j, diam(C_j) \leq \delta \right\}$$

where

$$\alpha(s) := \frac{\pi^{s/2}}{\Gamma\left(\frac{s}{2}+1\right)}.$$

Here Γ *is the Gamma function of complex analysis.*

(2) *For* A *and* s *as above, define*

$$\mathcal{H}^s := \lim_{\delta \to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta > 0} \mathcal{H}^s_\delta(A).$$

We call $\mathcal{H}^s$ *the* s*-dimensional Hausdorff measure.*

REMARK 3.2.

(1) *Notice that* $\mathcal{L}^n\left(B(x,r)\right) = \alpha(n)r^n$ *for all balls* $B(x,r) \subseteq \mathbb{R}^n$*. We will see later that if* $s = k$ *is an integer* $\mathcal{H}^k$ *agrees with the ordinary "k-dimensional surface area" on nice sets. This is the reason we include the normalizing constant* $\alpha(s)$*, in the definition.*

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