Project 3: Systems of ODEs

Due Date: 19 Nov 2015 (Thursday)
50 Points

In Project 2, we learned about how numerical methods can be used to approximate the solution of a single first order ODE. Unfortunately, real life rarely results in such simple mathematics. Instead, most applications use systems of differential equations or differential equations of an order higher than one. In this project, we will explore both of these situations.

Using Python to Solve Systems of ODEs

It is possible to use Python's `odeint()` to solve systems of first order differential equations, much like we used it to solve an individual differential equation in Project 2. `Intro to Python 3.ipynb` on Blackboard (also available as a `.html` file) walks you through how to do this.

Reducing Higher Order ODEs to a System

Numerical methods to solve higher order differential equations are actually built using a bit of "trickery". In particular, it's possible to reduce a higher order ODE to a system of first order ODEs. Let's see how this works through an example.

Recall the spring question from our midterm exam. This resulted in the differential equation:

$$u'' + 5u' + 6u = 4\cos(2t)$$

Using the characteristic equation of the corresponding homogeneous ODE to find the complementary solution and the Method of Undetermined Coefficients to find the particular solution, we determined the general solution of this is of the form:

$$u(t) = c_1 e^{-2t} + c_2 e^{-3t} + \frac{1}{13}\cos(2t) + \frac{5}{13}\sin(2t)$$

And with some initial conditions, we can determine:

$$u(0) = 1 \text{ and } u'(0) = 0 \quad \Rightarrow \quad c_1 = 2 \text{ and } c_2 = -\frac{14}{13}$$

So that the analytic solution of the initial value problem is:

$$\phi(t) = 2e^{-2t} - \frac{14}{13}e^{-3t} + \frac{1}{13}\cos(2t) + \frac{5}{13}\sin(2t)$$

However suppose we wanted to find a numerical approximation of this (or some other second order ODE). In order to do this, let's be a little clever. We define:

$$x_1 := u \quad \text{and} \quad x_2 := u'$$

So that we can rewrite the given ODE:

$$\begin{aligned} u'' + 5u' + 6u &= 4\cos(2t) \\ (u')' + 5(u') + 6(u) &= 4\cos(2t) \\ x_2' + 5x_2 + 6x_1 &= 4\cos(2t) \\ x_2' &= 4\cos(2t) - 6x_1 - 5x_2 \end{aligned}$$

Thus we have an equation that determines the derivative of x_2. However by definition, we also have:

$$x_2 = u' = (x_1)' = x_1'$$

which is an equation that determines the derivative of x_1. More importantly, these two equations together form a system of first order ODEs. In particular:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= 4\cos(2t) - 6x_1 - 5x_2 \end{aligned}$$

And our initial conditions become:

$$u(0) = 1 \quad \Rightarrow \quad x_1(0) = 1 \quad \text{and} \quad u'(0) = 0 \quad \Rightarrow \quad x_2(0) = 0$$

So that this can be solved with a numerical method!

To see exactly how this is done using `odeint()`, look towards the end of `Intro to Python 3`.

Goals

By the end of this project, you should...

- be able to use the numerical ODE solver `odeint()` from SciPy to solve a system of first-order ordinary differential equations.
- have some feeling for how to determine the end behavior of solutions to systems of ODEs by analyzing a phase portrait.
- know how to reduce ODEs of order higher than one to a system of first order ODEs.

Instructions

First, work through the `Intro to Python 3.ipynb` file on Blackboard (also available as an `.html` file). Included in this file is the code to define the command `phase()`, which will allow you to plot a phase plane. (You may also write your own code to build the phase diagram if you wish. Doing so would allow you to put the arrows and solutions in the same figure like we've seen in class. This could be handy, but it is not required.)

1. Consider the following system of differential equations:

$$\mathbf{x}' = \begin{bmatrix} 2 & -\frac{5}{2} \\ \frac{9}{5} & -1 \end{bmatrix} \mathbf{x} \quad \text{where } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(a) Use `phase()` to plot a phase plane for $-2 \le x_1 \le 2$ and $-2 \le x_2 \le 2$.

(b) Use `odeint()` to solve the system for $0 \le t \le 7$ and several different initial conditions (your choice). Then plot their time series on the same axes.

(c) Looking at the phase plane and the selection of time series, describe the long term behavior of the system. Find the eigenvalues of the matrix (either by hand or by using Python's `linalg` as explained in `Intro to Python 3.ipynb`), and use them to corroborate your intuitive understanding from the phase plane and the set of solutions.

2. Consider the following system of differential equations:

$$\begin{aligned} \frac{dx_1}{dt} &= -(x_1 - x_2)(1 - x_1 - x_2) \\ \frac{dx_2}{dt} &= x_1(2 + x_2) \end{aligned}$$

(a) Use `phase()` to plot a phase plane for $-5 \le x_1 \le 5$ and $-4 \le x_2 \le 6$.

(b) Use `odeint()` to solve the system for $0 \le t \le 10$ and each of the following initial conditions:

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1.5 \\ -0.5 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3.1 \\ -2.1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \text{ and } \begin{bmatrix} -4 \\ 1 \end{bmatrix}.$$

Then plot them on the same phase plane. (In other words, plot them on an x_1, x_2-plane just without the arrows. It's probably a good idea to set the axes by using the command `plt.axis([-5,5,-4,6])` so it matches the axes on your phase plane.)

(c) Looking at the phase plane and the selection of solutions, describe the long term behavior of the system.

3. Consider the following differential equation:

$$u'' + u'(5 + tu) = 0.75\sin(0.5t) \quad \text{where } u(0) = 0 \text{ and } u'(0) = 0.5$$

(a) Write the given second order ODE as a system of two first order ODEs. Also rewrite the initial conditions for the system.

(b) Solve the system of ODEs using `odeint()` for $0 \le t \le 75$. Then plot the solution to the second order ODE as a time series (i.e., plot u with respect to t).

(c) What appears to be the long term behavior of this solution?