Frequency-explicit *a posteriori* error estimates for finite element discretizations of Maxwell's equations

Théophile Chaumont-Frelet and Patrick Vega

Atlantis project-team, Inria Sophia Antipolis - Méditerranée

Atlantis Online Seminar
Sophia Antipolis, France
November 9, 2020

Maxwell's equations

Given $\boldsymbol{J} : \Omega \to \mathbb{C}^3$ and a fixed **non-resonant** frequency $\omega > 0$, we seek $\boldsymbol{E} : \Omega \to \mathbb{C}^3$ such that

$$\left\{ \begin{array}{rl} -\omega^2 \boldsymbol{\varepsilon} \boldsymbol{E} + \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times (\boldsymbol{\mu}^{-1} \boldsymbol{E})) = i\omega \boldsymbol{J} & \text{in } \Omega, \\ \boldsymbol{E} \times \boldsymbol{n} = \boldsymbol{o} & \text{on } \partial\Omega, \end{array} \right.$$

where Ω is a Lipschitz polyhedral domain.

We consider a mesh $\mathcal{T}_h$ that partitions Ω, and a finite element space $\boldsymbol{V}_h$.

We construct an approximation $\boldsymbol{E}_h$ of $\boldsymbol{E}$, being $\boldsymbol{E}_h|_K$ a polynomial function, for each $K \in \mathcal{T}_h$ (Nédélec finite elements), and we are interested in controlling $\boldsymbol{E} - \boldsymbol{E}_h$.

When **local singularities** deteriorates the **overall accuracy** of the numerical approximation, we can refine de the mesh where the solution is less regular.

A posteriori error estimation (main idea)

Where? How to obtain a **balance** between the refined and unrefined regions?

We are interested in controlling $(\boldsymbol{E} - \boldsymbol{E}_h)|_K$ by $\eta_K \in \mathbb{R}$, for each $K \in \mathcal{T}_h$.

η_K depends on the **data** and the **finite element solution**, must be **cheap to compute** and must satisfy two properties:

Reliability:

$$\text{Global Error} \lesssim \left(\sum_{K \in \mathcal{T}_h} \eta_K^2 \right)^{1/2} =: \eta \qquad \text{(error control).}$$

(Local) Efficiency:

$$\eta_K \lesssim \text{Local Error}, \quad \text{for each } K \in \mathcal{T}_h \qquad \text{(choose regions to refine).}$$

Main challenges

We recall

$$-\omega^2 \varepsilon \boldsymbol{E} + \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times (\boldsymbol{\mu}^{-1} \boldsymbol{E})) = i\omega \boldsymbol{J} \quad \text{in } \Omega.$$

Main difficulties in this work:

- High frequency
- $\boldsymbol{\mathcal{H}}(\mathbf{curl})$ context

Given $f : \Omega \to \mathbb{C}$, we consider

$$-\Delta u = f \quad \text{in } \Omega \qquad \text{(Poisson)}$$

$$-\Delta u - \omega^2 u = f \quad \text{in } \Omega \qquad \text{(Helmholtz)}$$

Sketch of the talk:

$$\text{Poisson} \underset{\text{high frequency}}{\longrightarrow} \text{Helmholtz} \underset{\boldsymbol{\mathcal{H}}(\mathbf{curl})}{\longrightarrow} \text{Maxwell}$$

Outline

1 A posteriori error estimates for the Poisson equation

2 A posteriori error estimates for the Helmholtz equation

3 Frequency-explicit a posteriori error estimates for Maxwell's equations
- Settings
- Reliability
- Efficiency

4 Numerical experiments
- Analytical solution in a PEC cavity
- Scattering by a penetrable obstacle

Outline

1. A posteriori error estimates for the Poisson equation

2. A posteriori error estimates for the Helmholtz equation

3. Frequency-explicit a posteriori error estimates for Maxwell's equations
   - Settings
   - Reliability
   - Efficiency

4. Numerical experiments
   - Analytical solution in a PEC cavity
   - Scattering by a penetrable obstacle

Given $f : \Omega \to \mathbb{C}$, we seek $u : \Omega \to \mathbb{C}$ such that

$$\left\{ \begin{array}{rl} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{array} \right.$$

where Ω is a Lipschitz polyhedral domain.

Variational formulation

Assuming $f \in \mathcal{L}^2(\Omega)$, we find $u \in \mathcal{H}_0^1(\Omega)$ such that

$$b(u, v) = (f, v)_\Omega \qquad \forall v \in \mathcal{H}_0^1(\Omega),$$

where $b(u, v) := (\boldsymbol{\nabla} u, \boldsymbol{\nabla} v)_\Omega$.

Finite element discretization

Find $u_h \in V_h \subset \mathcal{H}_0^1(\Omega)$ such that

$$b(u_h, v_h) = (f, v_h)_\Omega \qquad \forall v \in V_h,$$

where $V_h := \{w \in \mathcal{H}_0^1(\Omega) : w|_K \in \mathcal{P}_k(K) \quad \forall K \in \mathcal{T}_h\}$, with $k \in \mathbf{N}^\star$.

Intuitive construction of η_K

Since $f \in \mathcal{L}^2(\Omega)$ and $u \in \mathcal{H}^1(\Omega)$, we have

$$\begin{aligned} -\nabla \cdot (\nabla u) = -\Delta u = f \text{ in } \Omega &\Longrightarrow \nabla u \in \boldsymbol{\mathcal{H}}(\mathrm{div}, \Omega) \\ &\Longrightarrow [\![\nabla u]\!]_F \cdot \boldsymbol{n}_F = 0 \quad \text{for each } F \in \mathcal{F}_h^{\mathrm{i}}. \end{aligned}$$

We note that

$$(f + \Delta u_h)|_K \neq 0 \qquad \text{and} \qquad [\![\nabla u]\!]_F \cdot \boldsymbol{n}_F \neq 0, \quad F \subset \partial K \setminus \partial\Omega,$$

for each $K \in \mathcal{T}_h$.

Then, we propose the estimator

$$\eta_K := h_K \|f + \Delta u_h\|_K + h_K^{1/2} \| [\![\nabla u_h]\!] \cdot \boldsymbol{n} \|_{\partial K \setminus \partial\Omega} \quad \text{for each } K \in \mathcal{T}_h.$$

Here $h_K := \mathrm{diam}(K)$.

Intuitive construction of η_K

$$\eta_K := h_K \|f + \Delta u_h\|_K + h_K^{1/2} \|[\![\nabla u_h]\!] \cdot \boldsymbol{n}\|_{\partial K \setminus \partial\Omega} \quad \text{for each } K \in \mathcal{T}_h.$$

η_K quantifies the mismatching of the strong equation and the trace continuity.

The powers of h_K appears by dimensional and convergence reasons.

The idea of the calculations is to link the error with the estimator.

The role of coercivity in the reliability estimate (error $\lesssim \eta$)

We recall that b is coercive in $\mathcal{H}^1_0(\Omega)$: $\|\boldsymbol{\nabla} v\|^2_\Omega \lesssim b(v,v) \quad \forall v \in \mathcal{H}^1_0(\Omega)$.

Even more

$$\|\boldsymbol{\nabla} v\|^2_\Omega = b(v,v) \qquad \forall v \in \mathcal{H}^1_0(\Omega).$$

Then, if we can establish the bound

$$b(u-u_h,v) \lesssim \eta \|\boldsymbol{\nabla} v\|_\Omega \qquad \forall v \in \mathcal{H}^1_0(\Omega),$$

we obtain $(v = u - u_h)$

$$\|\boldsymbol{\nabla}(u-u_h)\|^2_\Omega = b(u-u_h,u-u_h) \lesssim \eta \|\boldsymbol{\nabla}(u-u_h)\|_\Omega.$$

Tools to prove the reliability estimate

Theorem (Reliability)

The following estimate holds true

$$\|\boldsymbol{\nabla}(u - u_h)\|_\Omega \lesssim \eta.$$

Tools to prove it:

Galerkin orthogonality

$$b(u - u_h, v_h) = 0 \qquad \forall v_h \in V_h.$$

Quasi-interpolation operator

$\mathcal{Q}_h : \mathcal{H}^1_0(\Omega) \to V_h$ such that

$$h_K^{-1}\|v - \mathcal{Q}_h v\|_K + h_K^{-1/2}\|v - \mathcal{Q}_h v\|_{\partial K} \lesssim \|\boldsymbol{\nabla} v\|_{\widetilde{K}} \qquad \forall v \in \mathcal{H}^1_0(\Omega).$$

Reliability (error $\lesssim \eta$). Recall $\eta_K := h_K \|f + \Delta u_h\|_K + h_K^{1/2} \| [\![\nabla u_h]\!] \cdot \boldsymbol{n} \|_{\partial K \setminus \partial\Omega}$

For $v \in \mathcal{H}_0^1(\Omega)$, we have

$$b(u-u_h, v) = b(u-u_h, v-v_h) = (f, v-v_h)_\Omega - (\nabla u_h, \nabla(v-v_h))_\Omega$$

$$= (f + \Delta u_h, v - v_h)_\Omega - \sum_{K \in \mathcal{T}_h} ([\![\nabla u_h]\!] \cdot \boldsymbol{n}, v - v_h)_{\partial K \setminus \partial\Omega}$$

$$\leq \|f + \Delta u_h\|_\Omega \|v - v_h\|_\Omega + \sum_{K \in \mathcal{T}_h} \| [\![\nabla u_h]\!] \cdot \boldsymbol{n} \|_{\partial K \setminus \partial\Omega} \|v - v_h\|_{\partial K \setminus \partial\Omega} \qquad \forall v_h \in V_h.$$

Choosing $v_h = \mathcal{Q}_h v$:

$$b(u-u_h, v) \leq \|f + \Delta u_h\|_\Omega \|(\mathcal{I} - \mathcal{Q}_h) v\|_\Omega + \sum_{K \in \mathcal{T}_h} \| [\![\nabla u_h]\!] \cdot \boldsymbol{n} \|_{\partial K \setminus \partial\Omega} \|(\mathcal{I} - \mathcal{Q}_h) v\|_{\partial K \setminus \partial\Omega}$$

$$\lesssim \left(\sum_{K \in \mathcal{T}_h} h_K \|f + \Delta u_h\|_K + h_K^{1/2} \| [\![\nabla u_h]\!] \cdot \boldsymbol{n} \|_{\partial K \setminus \partial\Omega} \right) \|\nabla v\|_\Omega = \eta \|\nabla v\|_\Omega.$$

Thus ($v = u - u_h$)

$$\|\nabla(u - u_h)\|_\Omega = \frac{b(u - u_h, u - u_h)}{\|\nabla(u - u_h)\|_\Omega} \lesssim \eta.$$

Tools to prove the efficiency estimate ($\eta_K \lesssim$ local error).

Bubble functions

For $K \in \mathcal{T}_h$ and $F \in \mathcal{F}_h$, b_K and b_F are the usual "bubble" functions respectively supported in K and $\widetilde{F}$. We have

$$\|v\|_K \lesssim \|b_K^{1/2} v\|_K \quad \forall v \in \mathcal{P}_{k+1}(K), \qquad \|v\|_F \lesssim \|b_F^{1/2} v\|_F \quad \forall v \in \mathcal{P}_{k+1}(F).$$

Inverse inequality

For each $K \in \mathcal{T}_h$, we have

$$\|\boldsymbol{\nabla}(b_K v)\|_K \lesssim h_K^{-1} \|v\|_K \qquad \forall v \in \mathcal{P}_{k+1}(K).$$

Extension operator

For each $F \in \mathcal{F}_h$, $O_{\mathrm{ext}} : \mathcal{P}_{k+1}(F) \to \mathcal{P}_{k+1}(\widetilde{F})$ such that $O_{\mathrm{ext}}(v)|_F = v$ and

$$\|O_{\mathrm{ext}}(v)\|_{\mathcal{T}_{F,h}} + h_F \|\boldsymbol{\nabla}(O_{\mathrm{ext}}(v))\|_{\mathcal{T}_{F,h}} \lesssim h_F^{1/2} \|v\|_F \qquad \forall v \in \mathcal{P}_{k+1}(F).$$

Efficiency ($\eta_K \lesssim$ local error)

We recall $\eta_K := h_K \|f + \Delta u_h\|_K + h_K^{1/2} \| [\![\boldsymbol{\nabla} u_h]\!] \cdot \boldsymbol{n}\|_{\partial K \setminus \partial\Omega}$.

For $r_K := (f + \Delta u_h)|_K$, with $K \in \mathcal{T}_h$, we have

$$\begin{aligned} \|r_K\|_K^2 &\lesssim (b_K r_K, r_K)_K \\ &= (b_K r_K, f + \Delta u_h)_K \\ &= (\boldsymbol{\nabla}(b_K r_K), \boldsymbol{\nabla}(u - u_h))_K \\ &\lesssim h_K^{-1} \|\boldsymbol{\nabla}(u - u_h)\|_K \|r_K\|_K \end{aligned}$$

and then

$$h_K \|f + \Delta u_h\|_K \lesssim \|\boldsymbol{\nabla}(u - u_h)\|_K \qquad \forall K \in \mathcal{T}_h.$$

Efficiency ($\eta_K \lesssim$ local error)

We recall $\eta_K := h_K \|f + \Delta u_h\|_K + h_K^{1/2} \| [\![\nabla u_h]\!] \cdot \boldsymbol{n}\|_{\partial K \setminus \partial\Omega}$.

For $r_F := [\![\nabla u_h]\!]_F \cdot \boldsymbol{n}_F$, with $F \in \mathcal{F}_K \cap \mathcal{F}_h^{\mathrm{i}}$ and $K \in \mathcal{T}_h$, we have

$$
\begin{aligned}
\|r_F\|_F^2 \lesssim (b_F r_F, r_F)_F &= \sum_{K' \in \mathcal{T}_{F,h}} (O_{\mathrm{ext}}(b_F r_F), \nabla(u_h - u) \cdot \boldsymbol{n})_{\partial K'} \\
&= \sum_{K' \in \mathcal{T}_{F,h}} ((O_{\mathrm{ext}}(b_F r_F), \Delta u_h + f)_{K'} + (\nabla O_{\mathrm{ext}}((b_F r_F)), \nabla(u_h - u))_{K'}) \\
&\lesssim h_F^{-1/2} \sum_{K' \in \mathcal{T}_{F,h}} (h_{K'} \|r_{K'}\|_{K'} + \|\nabla(u - u_h)\|_{K'}) \|r_F\|_F
\end{aligned}
$$

and then

$$
h_K^{1/2} \| [\![\nabla u_h]\!] \cdot \boldsymbol{n}\|_{\partial K \setminus \partial\Omega} \lesssim \|\nabla(u - u_h)\|_{\widetilde{K}} \qquad \forall K \in \mathcal{T}_h.
$$

Summary

We recall

$$\eta_K := h_K \|f + \Delta u_h\|_K + h_K^{1/2} \| [\![\nabla u_h]\!] \cdot \boldsymbol{n} \|_{\partial K \setminus \partial\Omega} \quad \text{for each } K \in \mathcal{T}_h.$$

- $\|\nabla(u - u_h)\|_\Omega^2 = b(u - u_h, u - u_h)$
- $b(u - u_h, v) \lesssim \eta \|v\|_\Omega \qquad \forall v \in \mathcal{H}_0^1(\Omega)$

Theorem (Reliability)

The following estimate holds true

$$\|\nabla(u - u_h)\|_\Omega \lesssim \eta.$$

Theorem (Efficiency)

The following estimate holds true

$$\eta_K \lesssim \|\nabla(u - u_h)\|_{\widetilde{K}} \qquad \forall K \in \mathcal{T}_h.$$

Outline

1 A posteriori error estimates for the Poisson equation

2 A posteriori error estimates for the Helmholtz equation

3 Frequency-explicit a posteriori error estimates for Maxwell's equations
- Settings
- Reliability
- Efficiency

4 Numerical experiments
- Analytical solution in a PEC cavity
- Scattering by a penetrable obstacle

The Helmholtz equation

Given $f : \Omega \to \mathbb{C}$ and a fixed non-resonant frequency $\omega > 0$, we seek $u : \Omega \to \mathbb{C}$ such that

$$\begin{cases} -\Delta u - \omega^2 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a Lipschitz polyhedral domain.

The above problem is ill-posed at high frequencies (or close to resonances).

We want results that are **explicit** in ω.

Variational formulation

Assuming $f \in \mathcal{L}^2(\Omega)$, we find $u \in \mathcal{H}_0^1(\Omega)$ such that

$$b(u, v) = (f, v)_\Omega \qquad \forall v \in \mathcal{H}_0^1(\Omega),$$

where $b(u, v) := (\boldsymbol{\nabla} u, \boldsymbol{\nabla} v)_\Omega - \omega^2 (u, v)_\Omega$.

Finite element discretization

Find $u_h \in V_h \subset \mathcal{H}_0^1(\Omega)$ such that

$$b(u_h, v_h) = (f, v_h)_\Omega \qquad \forall v \in V_h,$$

where V_h is the same finite element space.

The lack of coercivity and the reliability estimate (error $\lesssim \eta$)

We need to establish the bound (with Galerkin orthog. and quasi-interpolation)

$$b(u-u_h,v) \lesssim \eta \|\boldsymbol{\nabla} v\|_\Omega \qquad \forall v \in \mathcal{H}^1_0(\Omega),$$

with

$$\eta_K := h_K \|f + \Delta u_h + \omega^2 u_h\|_K + h_K^{1/2} \| [\![\boldsymbol{\nabla} u_h]\!] \cdot \boldsymbol{n}\|_{\partial K \setminus \partial\Omega} \quad \text{for each } K \in \mathcal{T}_h.$$

We can't control the error since b is **not** coercive in $\mathcal{H}^1_0(\Omega)$. We have

$$\|\boldsymbol{\nabla} v\|^2_\Omega = b(v,v) + \omega^2 \|v\|^2_\Omega \qquad \forall v \in \mathcal{H}^1_0(\Omega).$$

We also **need** to control $\omega\|v\|_\Omega$.

Aubin-Nitsche trick: Given $w \in \mathcal{L}^2(\Omega)$, we get $\omega\|w\|_\Omega \lesssim \eta$.

Thus, if $\|\cdot\|_{1,\omega,\Omega} := (\|\boldsymbol{\nabla}\cdot\|^2_\Omega + \omega^2\|\cdot\|^2_\Omega)^{1/2}$

$$\begin{aligned}\|u-u_h\|^2_{1,\omega,\Omega} &= b(u-u_h,u-u_h) + 2\omega^2\|u-u_h\|^2_\Omega \\ &\lesssim \eta\,(\|\boldsymbol{\nabla}(u-u_h)\|_\Omega + \omega\|u-u_h\|_\Omega),\end{aligned}$$

that is

$$\|u-u_h\|_{1,\omega,\Omega} \lesssim \eta.$$

Tools for the Aubin-Nitsche trick: adjoint problem and approximation factor

For $\phi \in \mathcal{L}^2(\Omega)$, $u_\phi^\star \in \mathcal{H}_0^1(\Omega)$ denotes the unique solution of

$$b(w, u_\phi^\star) = (w, \phi)_\Omega \quad \forall w \in \mathcal{H}_0^1(\Omega).$$

Then $(w = u - u_h)$

$$(u - u_h, \phi)_\Omega = b(u - u_h, u_\phi^\star) = b(u - u_h, u_\phi^\star - v_h) \lesssim \eta \|\boldsymbol{\nabla}(u_\phi^\star - v_h)\|_\Omega \quad \forall v_h \in V_h.$$

Approximation factor

$$\sigma_{\mathrm{ba}} := \omega \sup_{\phi \in \mathcal{L}^2(\Omega) \setminus \{0\}} \inf_{v_h \in V_h} \frac{\|u_\phi^\star - v_h\|_{1,\omega,\Omega}}{\|\phi\|_\Omega}.$$

As a consequence: $\omega \inf_{v_h \in V_h} \|u_\phi^\star - v_h\|_{1,\omega,\Omega} \leq \sigma_{\mathrm{ba}} \|\phi\|_\Omega \qquad \forall \phi \in \mathcal{L}^2(\Omega).$

Thus

$$\omega(u - u_h, \phi)_\Omega \lesssim \sigma_{\mathrm{ba}} \eta \|\phi\|_\Omega,$$

that is (choosing $\phi = u - u_h$)

$$\omega \|u - u_h\|_\Omega \lesssim \sigma_{\mathrm{ba}} \eta.$$

Approximation factor and reliability estimate (error $\lesssim \eta$)

$\sigma_{\rm ba} \in \mathbb{R}$ depends on Ω, ω and V_h (indep. of f).

$\sigma_{\rm ba}$ describes the ability of V_h to approximate $u_\phi^\star$.

If Ω is a smooth domain, we show that

$$\sigma_{\rm ba} \leq C(\Omega)\left(\omega h + \frac{\omega}{|\omega - \omega_{\rm r}|}(\omega h)^{k+1}\right).$$

Then, if $\omega h \leq \min\left\{1, \left(\frac{|\omega-\omega_{\rm r}|}{\omega}\right)^{1/(k+1)}\right\}$, $\sigma_{\rm ba}$ has a bound independent of ω.

Chaumont-Frelet T., Ern A. and Vohralík M., *On the derivation of guaranteed and p-robust a posteriori error estimates for the Helmholtz equation* (2019), hal-02202233v2.

Theorem (Reliability)

The following estimate holds true

$$\|u - u_h\|_{1,\omega,\Omega} \lesssim (1 + \sigma_{\rm ba})\eta.$$

If $\omega h \to 0$, then $\sigma_{\rm ba} \to 0$, and thus

$$\|u - u_h\|_{1,\omega,\Omega} \lesssim \eta.$$

Efficiency ($\eta_K \lesssim$ local error)

We recall $\eta_K := h_K \|f + \Delta u_h + \omega^2 u_h\|_K + h_K^{1/2} \| [\![\nabla u_h]\!] \cdot \boldsymbol{n} \|_{\partial K \setminus \partial \Omega}$.

For $r_K := (f + \Delta u_h + \omega^2 u_h)|_K$, with $K \in \mathcal{T}_h$, we have

$$\begin{aligned}
\|r_K\|_K^2 &\lesssim (b_K r_K, r_K)_K \\
&= (b_K r_K, f + \Delta u_h + \omega^2 u_h)_K \\
&= (\nabla (b_K r_K), \nabla (u - u_h))_K - \omega^2 (b_K r_K, u - u_h)_K \\
&\lesssim h_K^{-1} (1 + \omega h_K) (\|\nabla (u - u_h)\|_K + \omega \|u - u_h\|_K) \|r_K\|_K
\end{aligned}$$

and then

$$h_K \|f + \Delta u_h + \omega^2 u_h\|_K \lesssim (1 + \omega h_K) \|u - u_h\|_{1,\omega,K} \qquad \forall K \in \mathcal{T}_h.$$

Efficiency ($\eta_K \lesssim$ local error)

We recall $\eta_K := h_K \|f + \Delta u_h + \omega^2 u_h\|_K + h_K^{1/2} \| [\![\nabla u_h]\!] \cdot \boldsymbol{n} \|_{\partial K \setminus \partial \Omega}$.

For $r_F := [\![\nabla u_h]\!]_F \cdot \boldsymbol{n}_F$, with $F \in \mathcal{F}_K \cap \mathcal{F}_h^{\mathrm{i}}$ and $K \in \mathcal{T}_h$, we have

$$\|r_F\|_F^2 \lesssim (b_F r_F, r_F)_F = \sum_{K' \in \mathcal{T}_{F,h}} (\mathcal{O}_{\mathrm{ext}}(b_F r_F), \nabla(u_h - u) \cdot \boldsymbol{n})_{\partial K'}$$

$$= \sum_{K' \in \mathcal{T}_{F,h}} ((\mathcal{O}_{\mathrm{ext}}(b_F r_F), \Delta u_h + f)_{K'} + (\nabla \mathcal{O}_{\mathrm{ext}}((b_F r_F)), \nabla(u_h - u))_{K'})$$

$$\lesssim h_F^{-1/2} \sum_{K' \in \mathcal{T}_{F,h}} (h_{K'} \|r_{K'}\|_{K'} + \|\nabla(u - u_h)\|_{K'}) \|r_F\|_F$$

and then

$$h_K^{1/2} \| [\![\nabla u_h]\!] \cdot \boldsymbol{n} \|_{\partial K \setminus \partial \Omega} \lesssim (1 + \omega h_K) \|u - u_h\|_{1,\omega,\widetilde{K}} \qquad \forall K \in \mathcal{T}_h.$$

Summary

We recall

$$\eta_K := h_K \|f + \Delta u_h + \omega^2 u_h\|_K + h_K^{1/2} \| [\![\nabla u_h]\!] \cdot \boldsymbol{n} \|_{\partial K \setminus \partial\Omega} \quad \text{for each } K \in \mathcal{T}_h.$$

- $\|\nabla(u - u_h)\|_\Omega^2 = b(u - u_h, u - u_h) + \omega^2 \|u - u_h\|_\Omega^2$
- $b(u - u_h, v) \lesssim \eta \|v\|_\Omega \qquad \forall v \in \mathcal{H}_0^1(\Omega)$
- $\omega \|u - u_h\|_\Omega \leq \sigma_{\mathrm{ba}} \eta$

Theorem (Reliability)

The following estimate holds true

$$\|u - u_h\|_{1,\omega,\Omega} \lesssim (1 + \sigma_{\mathrm{ba}})\eta.$$

Theorem (Efficiency)

The following estimate holds true

$$\eta_K \lesssim (1 + \omega h_K) \|u - u_h\|_{1,\omega,\widetilde{K}} \qquad \forall K \in \mathcal{T}_h.$$

If $\omega h \to 0$, then $\sigma_{\mathrm{ba}} \to 0$, and thus

$$\|u - u_h\|_{1,\omega,\Omega} \lesssim \eta \qquad \text{and} \qquad \eta_K \lesssim \|u - u_h\|_{1,\omega,\widetilde{K}} \qquad \forall K \in \mathcal{T}_h.$$

Outline

1 A posteriori error estimates for the Poisson equation

2 A posteriori error estimates for the Helmholtz equation

3 Frequency-explicit a posteriori error estimates for Maxwell's equations
- Settings
- Reliability
- Efficiency

4 Numerical experiments
- Analytical solution in a PEC cavity
- Scattering by a penetrable obstacle

Outline

1 A posteriori error estimates for the Poisson equation

2 A posteriori error estimates for the Helmholtz equation

3 Frequency-explicit a posteriori error estimates for Maxwell's equations
- Settings
- Reliability
- Efficiency

4 Numerical experiments
- Analytical solution in a PEC cavity
- Scattering by a penetrable obstacle

Maxwell's equations

Given $\boldsymbol{J} : \Omega \to \mathbb{C}^3$ and a fixed non-resonant frequency $\omega > 0$, we seek $\boldsymbol{E} : \Omega \to \mathbb{C}^3$ such that

$$\left\{ \begin{aligned} -\omega^2 \boldsymbol{E} + \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) &= i\omega \boldsymbol{J} && \text{in } \Omega, \\ \boldsymbol{E} \times \boldsymbol{n} &= \boldsymbol{o} && \text{on } \partial\Omega, \end{aligned} \right.$$

where Ω is a Lipschitz polyhedral domain.

The above problem is ill-posed at high frequencies (or close to resonances).

We want results that are **explicit** in ω.

Variational formulation

Assuming $\boldsymbol{J} \in \boldsymbol{\mathcal{H}}(\mathrm{div}, \Omega)$, we find $\boldsymbol{E} \in \boldsymbol{\mathcal{H}}_0(\mathbf{curl}, \Omega)$ such that

$$b(\boldsymbol{E}, \boldsymbol{v}) := -\omega^2 (\boldsymbol{E}, \boldsymbol{v})_\Omega + (\boldsymbol{\nabla} \times \boldsymbol{E}, \boldsymbol{\nabla} \times \boldsymbol{v})_\Omega = i\omega (\boldsymbol{J}, \boldsymbol{v})_\Omega \qquad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{H}}_0(\mathbf{curl}, \Omega).$$

Finite element discretization

Find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ such that

$$b(\boldsymbol{E}, \boldsymbol{v}_h) = i\omega (\boldsymbol{J}, \boldsymbol{v}_h)_\Omega \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h,$$

where $\boldsymbol{V}_h := \{ \boldsymbol{v}_h \in \boldsymbol{\mathcal{H}}_0(\mathbf{curl}, \Omega) : \boldsymbol{v}_h|_K \in \boldsymbol{\mathcal{N}}_k(K) \quad \forall K \in \mathcal{T}_h \}$ with

$$\boldsymbol{\mathcal{N}}_k(K) := \boldsymbol{\mathcal{P}}_k(K) + \boldsymbol{x} \times \boldsymbol{\mathcal{P}}_k(K).$$

Strong equation: $-\omega^2 \boldsymbol{E} + \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) = i\omega \boldsymbol{J}$ in Ω.

Hidden equation: $\boldsymbol{\nabla} \cdot (\boldsymbol{J} - i\omega \boldsymbol{E}) = 0$ in Ω.

Trace continuity: $[\![\boldsymbol{\nabla} \times \boldsymbol{E}]\!] \times \boldsymbol{n} = \boldsymbol{o}$ and $[\![\boldsymbol{E}]\!] \cdot \boldsymbol{n} = 0$ for each $F \in \mathcal{F}_h^{\mathrm{i}}$.

A posteriori error estimators

For $K \in \mathcal{T}_h$, we set

$$\eta_K^2 := \eta_{\mathrm{div},K}^2 + \eta_{\mathbf{curl},K}^2$$

with

$$\eta_{\mathrm{div},K} := h_K \|\boldsymbol{\nabla} \cdot (\boldsymbol{J} - i\omega \boldsymbol{E}_h)\|_K + \omega h_K^{1/2} \| [\![\boldsymbol{E}_h]\!] \cdot \boldsymbol{n}\|_{\partial K \setminus \partial\Omega}$$

and

$$\eta_{\mathbf{curl},K} := h_K \|i\omega \boldsymbol{J} + \omega^2 \boldsymbol{E}_h - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}_h)\|_K + h_K^{1/2} \| [\![\boldsymbol{\nabla} \times \boldsymbol{E}_h]\!] \times \boldsymbol{n}\|_{\partial K \setminus \partial\Omega}.$$

We also set

$$\eta^2 := \sum_{K \in \mathcal{T}_h} \eta_K^2, \qquad \eta_{\mathrm{div}}^2 := \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{div},K}^2, \qquad \eta_{\mathbf{curl}}^2 := \sum_{K \in \mathcal{T}_h} \eta_{\mathbf{curl},K}^2.$$

Beck R., Hiptmair R., Hoppe R. H. W. and Wohlmuth B., *Residual based a posteriori error estimators for eddy current computation*. ESAIM Math. Model. Numer. Anal., **34** (2000), pp. 159-182.

Nicaise S. and Creusé E., *A posteriori error estimation for the heterogeneous Maxwell equations on isotropic and anisotropic meshes*. Calcolo, **40** (2003), pp. 249-271.

Finite element discretization and tools for the reliability estimate (error $\lesssim \eta$)

Tools to prove the reliability estimate:

Galerkin orthogonality

$$b(\boldsymbol{E} - \boldsymbol{E}_h, \boldsymbol{v}_h) = 0 \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

Quasi-interpolation operator

$\mathcal{S}_h : \boldsymbol{\mathcal{H}}^1(\Omega) \to \boldsymbol{V}_h$ such that

$$h_K^{-1} \|\boldsymbol{w} - \mathcal{S}_h \boldsymbol{w}\|_K + h_K^{-1/2} \|(\boldsymbol{w} - \mathcal{S}_h \boldsymbol{w}) \times \boldsymbol{n}\|_{\partial K} \lesssim \|\boldsymbol{\nabla} \boldsymbol{w}\|_{\widetilde{K}} \quad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{H}}^1(\Omega).$$

Gradient extraction

For all $\boldsymbol{\theta} \in \boldsymbol{\mathcal{H}}_0(\mathbf{curl}, \Omega)$, there exist $\boldsymbol{\phi} \in \boldsymbol{\mathcal{H}}^1(\Omega) \cap \boldsymbol{\mathcal{H}}_0(\mathbf{curl}, \Omega)$ and $r \in \mathcal{H}_0^1(\Omega)$ such that $\boldsymbol{\theta} = \boldsymbol{\phi} + \boldsymbol{\nabla} r$ with

$$\|\boldsymbol{\nabla} \boldsymbol{\phi}\|_\Omega \lesssim \|\boldsymbol{\nabla} \times \boldsymbol{\theta}\|_\Omega \quad \text{and} \quad \|\boldsymbol{\nabla} r\|_\Omega \lesssim \|\boldsymbol{\theta}\|_{\mathbf{curl},\omega,\Omega}.$$

COSTABEL M., DAUGE M. AND NICAISE S., *Singularities of Maxwell interface problems*. ESAIM Math. Model. Numer. Anal., **33** (1999), pp. 627-649.

An approach based on gradient extraction

Then, if we write $\boldsymbol{w} \in \boldsymbol{\mathcal{H}}_0(\mathbf{curl}, \Omega)$ as

$$\boldsymbol{w} = \boldsymbol{\phi} + \boldsymbol{\nabla} r,$$

with $\boldsymbol{\phi} \in \boldsymbol{\mathcal{H}}^1(\Omega) \cap \boldsymbol{\mathcal{H}}_0(\mathbf{curl}, \Omega)$ and $r \in \mathcal{H}^1_0(\Omega)$, we can use $\mathcal{S}_h$ to get

$$b(\boldsymbol{E} - \boldsymbol{E}_h, \boldsymbol{\phi}) \lesssim \eta_{\mathbf{curl}} \|\boldsymbol{\nabla} \boldsymbol{\phi}\|_\Omega \qquad \forall \boldsymbol{\phi} \in \boldsymbol{\mathcal{H}}^1(\Omega) \cap \boldsymbol{\mathcal{H}}_0(\mathbf{curl}, \Omega),$$

following the idea that we used for Helmholtz and recalling that

$$\eta_{\mathbf{curl},K} := h_K \|i\omega \boldsymbol{J} + \omega^2 \boldsymbol{E}_h - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}_h)\|_K + h_K^{1/2} \| [\![\boldsymbol{\nabla} \times \boldsymbol{E}_h]\!] \times \boldsymbol{n}\|_{\partial K \setminus \partial \Omega}.$$

On the other hand,

$$\begin{aligned} b(\boldsymbol{E} - \boldsymbol{E}_h, \boldsymbol{\nabla} r) &= i\omega(\boldsymbol{J} - i\omega \boldsymbol{E}_h, \boldsymbol{\nabla} r)_\Omega \\ &= -i\omega \left((\boldsymbol{\nabla} \cdot (\boldsymbol{J} - i\omega \boldsymbol{E}_h), r)_\Omega - i\omega \langle [\![\boldsymbol{E}_h]\!] \cdot \boldsymbol{n}, r \rangle_{\mathcal{F}_h^{\mathrm{i}}} \right), \end{aligned}$$

and using $\mathcal{Q}_h$, we get

$$b(\boldsymbol{E} - \boldsymbol{E}_h, \boldsymbol{\nabla} r) \lesssim \omega \eta_{\mathrm{div}} \|\boldsymbol{\nabla} r\|_\Omega \qquad \forall r \in \mathcal{H}^1_0(\Omega),$$

recalling that

$$\eta_{\mathrm{div},K} := h_K \|\boldsymbol{\nabla} \cdot (\boldsymbol{J} - i\omega \boldsymbol{E}_h)\|_K + \omega h_K^{1/2} \| [\![\boldsymbol{E}_h]\!] \cdot \boldsymbol{n}\|_{\partial K \setminus \partial \Omega}.$$

Outline

1 A posteriori error estimates for the Poisson equation

2 A posteriori error estimates for the Helmholtz equation

3 Frequency-explicit a posteriori error estimates for Maxwell's equations
- Settings
- Reliability
- Efficiency

4 Numerical experiments
- Analytical solution in a PEC cavity
- Scattering by a penetrable obstacle

Quasi-interpolation operator and gradient extraction

Quasi-interpolation operator

$\mathcal{S}_h : \boldsymbol{\mathcal{H}}^1(\Omega) \to \boldsymbol{V}_h$ such that

$$h_K^{-1}\|\boldsymbol{w} - \mathcal{S}_h\boldsymbol{w}\|_K + h_K^{-1/2}\|(\boldsymbol{w} - \mathcal{S}_h\boldsymbol{w}) \times \boldsymbol{n}\|_{\partial K} \lesssim \|\boldsymbol{\nabla}\boldsymbol{w}\|_{\widetilde{K}} \qquad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{H}}^1(\Omega) \cap \boldsymbol{\mathcal{H}}_0(\mathbf{curl}, \Omega).$$

Gradient extraction (improved)

For all $\boldsymbol{\theta} \in \boldsymbol{\mathcal{H}}_0(\mathbf{curl}, \Omega)$, there exist $\boldsymbol{\phi} \in \boldsymbol{\mathcal{H}}^1(\Omega) \cap \boldsymbol{\mathcal{H}}_0(\mathbf{curl}, \Omega)$ and $r \in \mathcal{H}^1_0(\Omega)$ such that $\boldsymbol{\theta} = \boldsymbol{\phi} + \boldsymbol{\nabla} r$ with

$$\|\boldsymbol{\nabla}\boldsymbol{\phi}\|_\Omega \lesssim \|\boldsymbol{\nabla} \times \boldsymbol{\theta}\|_\Omega \quad \text{and} \quad \|\boldsymbol{\nabla} r\|_\Omega \lesssim \|\boldsymbol{\theta}\|_\Omega.$$

Novelty: In contrast to standard results, we include the $\boldsymbol{\mathcal{L}}^2(\Omega)$-norm of $\boldsymbol{\theta}$ instead of their $\boldsymbol{\mathcal{H}}_0(\mathbf{curl}, \Omega)$-norm to get correct scalings with respect to ω.

Chaumont-Frelet T. and Vega P., *Frequency-explicit a posteriori error estimates for finite element discretizations of Maxwell's equations* (2020), hal-02943386.

Reliability (error $\lesssim \eta$)

Lemma (Control of the residual)

The estimates

$$b(\boldsymbol{E} - \boldsymbol{E}_h, \boldsymbol{\nabla} q) \lesssim \omega \eta_{\mathrm{div}} \|\boldsymbol{\nabla} q\|_\Omega \quad \text{and} \quad b(\boldsymbol{E} - \boldsymbol{E}_h, \boldsymbol{\phi}) \lesssim \eta_{\mathbf{curl}} \|\boldsymbol{\nabla} \boldsymbol{\phi}\|_\Omega$$

hold true for all $q \in \mathcal{H}^1_0(\Omega)$ *and* $\boldsymbol{\phi} \in \boldsymbol{\mathcal{H}}^1(\mathcal{T}_h) \cap \boldsymbol{\mathcal{H}}_0(\mathbf{curl}, \Omega)$.

Lemma (General control of the residual)

The estimate

$$b(\boldsymbol{E} - \boldsymbol{E}_h, \boldsymbol{\theta}) \lesssim \eta \|\boldsymbol{\theta}\|_{\mathbf{curl},\omega,\Omega}$$

hold true for all $\boldsymbol{\theta} \in \boldsymbol{\mathcal{H}}_0(\mathbf{curl}, \Omega)$.

Sketch of the proof: Gradient extraction + Control of the residual.

Chaumont-Frelet T. and Vega P., *Frequency-explicit a posteriori error estimates for finite element discretizations of Maxwell's equations* (2020), hal-02943386.

We recall that b is **not** coercive in $\boldsymbol{\mathcal{H}}_0(\mathbf{curl}, \Omega)$. We have

$$\|\boldsymbol{\nabla} \times \boldsymbol{v}\|^2_\Omega = b(\boldsymbol{v}, \boldsymbol{v}) + \omega^2 \|\boldsymbol{v}\|^2_\Omega \qquad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{H}}_0(\mathbf{curl}, \Omega).$$

We will need to use the Aubin-Nitsche trick!

Approximation factor

For $\boldsymbol{j} \in \boldsymbol{\mathcal{L}}^2(\Omega)$, $\boldsymbol{e}_{\boldsymbol{j}}^\star \in \boldsymbol{\mathcal{H}}_0(\mathbf{curl}, \Omega)$ denotes the unique solution of

$$b(\boldsymbol{w}, \boldsymbol{e}_{\boldsymbol{j}}^\star) = \omega(\boldsymbol{w}, \boldsymbol{j})_\Omega \quad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{H}}_0(\mathbf{curl}, \Omega).$$

Approximation factor

$$\sigma_{\rm ba} := \omega \sup_{\boldsymbol{j} \in \boldsymbol{\mathcal{H}}(\mathrm{div}^0, \Omega)} \inf_{\boldsymbol{e}_h \in \boldsymbol{V}_h} \frac{\|\boldsymbol{e}_{\boldsymbol{j}}^\star - \boldsymbol{e}_h\|_{\mathbf{curl}, \omega, \Omega}}{\|\boldsymbol{j}\|_\Omega}.$$

If we consider $\boldsymbol{j} \in \boldsymbol{\mathcal{L}}^2(\Omega)$ instead of $\boldsymbol{j} \in \boldsymbol{\mathcal{H}}(\mathrm{div}^0, \Omega)$, we get $\sigma_{\rm ba} \gtrsim 1$.

$\sigma_{\rm ba} \in \mathbb{R}$ depends on Ω, ω and $\boldsymbol{V}_h$ (indep. of f) and is such that

$$\inf_{\boldsymbol{e}_h \in \boldsymbol{V}_h} \|\boldsymbol{e}_{\boldsymbol{j}}^\star - \boldsymbol{e}_h\|_{\mathbf{curl}, \omega, \Omega} \leq \sigma_{\rm ba} \|\boldsymbol{j}\|_\Omega \qquad \forall \boldsymbol{j} \in \boldsymbol{\mathcal{H}}(\mathrm{div}^0, \Omega).$$

Moreover, the condition $\boldsymbol{j} \in \boldsymbol{\mathcal{H}}(\mathrm{div}^0, \Omega)$ gives us enough regularity to show that

$$\sigma_{\rm ba} \leq C(\Omega) \left(\omega h + \frac{\omega}{|\omega - \omega_{\rm r}|} (\omega h)^{k+1} \right).$$

Then, for ωh small enough, $\sigma_{\rm ba}$ has a bound indep. of ω (as for Helmholtz).

The Aubin-Nitsche trick

Lemma (Aubin-Nitsche)

We have

$$\omega\|\boldsymbol{E}-\boldsymbol{E}_h\|_\Omega \lesssim (1+\sigma_{\mathrm{ba}})\eta.$$

Sketch of the proof:

Since $\boldsymbol{E}-\boldsymbol{E}_h \notin \boldsymbol{\mathcal{H}}(\mathrm{div}^0,\Omega)$, we can't use it as a rhs.

- Helmholtz decomposition: $\boldsymbol{E}-\boldsymbol{E}_h = \boldsymbol{\nabla} p + \boldsymbol{\theta}$, with $p \in H^1_0(\Omega)$ such that
$$(\boldsymbol{\nabla} p, \boldsymbol{\nabla} v) = (\boldsymbol{E}-\boldsymbol{E}_h, \boldsymbol{\nabla} v) \quad \forall v \in H^1_0(\Omega)$$
and $\boldsymbol{\theta} \in \boldsymbol{\mathcal{H}}(\mathrm{div}^0,\Omega)$.

- $\omega^2\|\boldsymbol{\nabla} p\|^2_\Omega = \omega^2(\boldsymbol{E}-\boldsymbol{E}_h, \boldsymbol{\nabla} p) = -b(\boldsymbol{E}-\boldsymbol{E}_h, \boldsymbol{\nabla} p) \lesssim \omega\eta_{\mathrm{div}}\|\boldsymbol{\nabla} p\|_\Omega$.

Chaumont-Frelet T. and Vega P., *Frequency-explicit a posteriori error estimates for finite element discretizations of Maxwell's equations* (2020), hal-02943386.

Aubin-Nitsche trick

Lemma (Aubin-Nitsche)

We have

$$\omega \|\boldsymbol{E} - \boldsymbol{E}_h\|_\Omega \lesssim (1 + \sigma_{\rm ba})\eta.$$

Sketch of the proof (cont'd): We recall

$$\omega \|\nabla p\|_\Omega \lesssim \eta_{\rm div}.$$

- Let be $\boldsymbol{\xi} \in \boldsymbol{\mathcal{H}}_0(\mathbf{curl}, \Omega)$ such that $b(\boldsymbol{w}, \boldsymbol{\xi}) = \omega(\boldsymbol{w}, \boldsymbol{\theta}) \quad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{H}}_0(\mathbf{curl}, \Omega)$.
- For all $\boldsymbol{\xi}_h \in \boldsymbol{V}_h$, we have
$$\begin{aligned}\omega(\boldsymbol{\theta}, \boldsymbol{\theta}) &= \omega(\boldsymbol{E} - \boldsymbol{E}_h, \boldsymbol{\theta}) - \omega(\nabla p, \boldsymbol{\theta}) = b(\boldsymbol{E} - \boldsymbol{E}_h, \boldsymbol{\xi}) - \omega(\nabla p, \boldsymbol{\theta}) \\ &= b(\boldsymbol{E} - \boldsymbol{E}_h, \boldsymbol{\xi} - \boldsymbol{\xi}_h) - \omega(\nabla p, \boldsymbol{\theta}) \lesssim \eta \|\boldsymbol{\xi} - \boldsymbol{\xi}_h\|_{\mathbf{curl}, \omega, \Omega} + \omega \|\nabla p\|_\Omega \|\boldsymbol{\theta}\|_\Omega.\end{aligned}$$
- $\omega \|\boldsymbol{\theta}\|_\Omega^2 = \omega(\boldsymbol{\theta}, \boldsymbol{\theta}) \lesssim (\sigma_{\rm ba}\eta + \omega\|\nabla p\|_\Omega) \|\boldsymbol{\theta}\|_\Omega \lesssim (1 + \sigma_{\rm ba})\eta \|\boldsymbol{\theta}\|_\Omega$.
- $\omega \|\boldsymbol{E} - \boldsymbol{E}_h\|_\Omega \leq \omega \|\nabla p\|_\Omega + \omega \|\boldsymbol{\theta}\|_\Omega \lesssim (1 + \sigma_{\rm ba})\eta$.

Chaumont-Frelet T. and Vega P., *Frequency-explicit a posteriori error estimates for finite element discretizations of Maxwell's equations* (2020), hal-02943386.

Reliability (error $\lesssim \eta$)

"Gårding inequality"

We have
$$\|\boldsymbol{v}\|^2_{\mathbf{curl},\omega,\Omega} = b(\boldsymbol{v},\boldsymbol{v}) + 2\omega^2\|\boldsymbol{v}\|^2_{\Omega} \qquad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{H}}_0(\mathbf{curl},\Omega).$$

Theorem (Reliability)

The following estimate holds true
$$\|\boldsymbol{E} - \boldsymbol{E}_h\|_{\mathbf{curl},\omega,\Omega} \lesssim (1 + \sigma_{\mathrm{ba}})\eta.$$

Sketch of the proof:

- $b(\boldsymbol{E} - \boldsymbol{E}_h, \boldsymbol{E} - \boldsymbol{E}_h) \lesssim \eta\|\boldsymbol{E} - \boldsymbol{E}_h\|_{\mathbf{curl},\omega,\Omega}$ (General control of the residual)
- $\|\boldsymbol{E} - \boldsymbol{E}_h\|^2_{\mathbf{curl},\omega,\Omega} \lesssim b(\boldsymbol{E} - \boldsymbol{E}_h, \boldsymbol{E} - \boldsymbol{E}_h) + \omega^2\|\boldsymbol{E} - \boldsymbol{E}_h\|^2_{\Omega}$ (Gårding ineq.)

$+$ Aubin-Nitsche

Chaumont-Frelet T. and Vega P., *Frequency-explicit a posteriori error estimates for finite element discretizations of Maxwell's equations* (2020), hal-02943386.

Outline

1 A posteriori error estimates for the Poisson equation

2 A posteriori error estimates for the Helmholtz equation

3 Frequency-explicit a posteriori error estimates for Maxwell's equations
- Settings
- Reliability
- Efficiency

4 Numerical experiments
- Analytical solution in a PEC cavity
- Scattering by a penetrable obstacle

Efficiency ($\eta_K \lesssim$ local error)

Lemma (Upper bound for $\eta_{\mathrm{div},K}$)

We have

$$\eta_{\mathrm{div},K} \lesssim \omega \|\boldsymbol{E} - \boldsymbol{E}_h\|_{\widetilde{K}} \qquad \forall K \in \mathcal{T}_h.$$

Sketch of the proof: Bubble functions properties + integration by parts + identities $\boldsymbol{\nabla} \cdot (\boldsymbol{J}_h - i\omega \boldsymbol{E}_h) = \boldsymbol{\nabla} \cdot (\boldsymbol{J} - \boldsymbol{J}_h) - i\omega \boldsymbol{\nabla} \cdot (\boldsymbol{E} - \boldsymbol{E}_h)$ and $i\omega \boldsymbol{\nabla} \cdot \boldsymbol{E} = \boldsymbol{J}$.

Lemma (Upper bound for $\eta_{\mathbf{curl},K}$)

We have

$$\eta_{\mathbf{curl},K} \lesssim (1 + \omega h_K) \|\boldsymbol{E} - \boldsymbol{E}_h\|_{\mathbf{curl},\omega,\widetilde{K}} \qquad \forall K \in \mathcal{T}_h.$$

Sketch of the proof: Similar to the case of the Helmholtz equation.

Theorem (Efficiency)

The following estimate holds true

$$\eta_K \lesssim (1 + \omega h_K) \|\boldsymbol{E} - \boldsymbol{E}_h\|_{\mathbf{curl},\omega,\widetilde{K}} \qquad \forall K \in \mathcal{T}_h.$$

CHAUMONT-FRELET T. AND VEGA P., *Frequency-explicit a posteriori error estimates for finite element discretizations of Maxwell's equations* (2020), hal-02943386.

Summary (We recall: If $\omega h \to 0$, then $\sigma_{\rm ba} \to 0$)

- $\|\nabla \times (\boldsymbol{E} - \boldsymbol{E}_h)\|_\Omega^2 = b(\boldsymbol{E} - \boldsymbol{E}_h, \boldsymbol{E} - \boldsymbol{E}_h) + \omega^2 \|\boldsymbol{E} - \boldsymbol{E}_h\|_\Omega^2$
- $b(\boldsymbol{E} - \boldsymbol{E}_h, \boldsymbol{\theta}) \lesssim \eta \|\boldsymbol{\theta}\|_\Omega \qquad \forall \boldsymbol{\theta} \in \boldsymbol{\mathcal{H}}_0(\mathbf{curl}, \Omega)$
- $\omega \|\boldsymbol{E} - \boldsymbol{E}_h\|_\Omega \leq (1 + \sigma_{\rm ba})\eta$

Theorem (Reliability)

The following estimate holds true

$$\|\boldsymbol{E} - \boldsymbol{E}_h\|_{\mathbf{curl},\omega,\Omega} \lesssim (1 + \sigma_{\rm ba})\eta.$$

Theorem (Efficiency)

The following estimate holds true

$$\eta_K \lesssim (1 + \omega h_K)\|\boldsymbol{E} - \boldsymbol{E}_h\|_{\mathbf{curl},\omega,\widetilde{K}} \qquad \forall K \in \mathcal{T}_h.$$

General coefficients: We can consider the problem

Find $\boldsymbol{E} : \Omega \to \mathbb{C}^3$ such that

$$\left\{ \begin{array}{rl} -\omega^2 \boldsymbol{\varepsilon} \boldsymbol{E} + \nabla \times \left(\boldsymbol{\mu}^{-1} \nabla \times \boldsymbol{E}\right) = i\omega \boldsymbol{J} & \text{in } \Omega, \\ \boldsymbol{E} \times \boldsymbol{n} = \boldsymbol{o} & \text{on } \partial\Omega, \end{array} \right.$$

where $\boldsymbol{\varepsilon}$, $\boldsymbol{\mu}$ are symmetric tensor-valued functions (useful for PML conditions).

Outline

1 A posteriori error estimates for the Poisson equation

2 A posteriori error estimates for the Helmholtz equation

3 Frequency-explicit a posteriori error estimates for Maxwell's equations
- Settings
- Reliability
- Efficiency

4 Numerical experiments
- Analytical solution in a PEC cavity
- Scattering by a penetrable obstacle

Effectivity index

$$\text{Effectivity index} := \frac{\text{error}}{\eta}.$$

Recalling the reliability and efficiency results

$$\text{error} \leq C_{\mathrm{rel}}\eta \qquad \text{and} \qquad \eta \leq C_{\mathrm{eff}}\text{error},$$

we can write

$$\frac{1}{C_{\mathrm{eff}}} \leq \text{Effectivity index} \leq C_{\mathrm{rel}}.$$

We expect the effective index to be independent of ω and h for sufficiently refined meshes.

Outline

1 A posteriori error estimates for the Poisson equation

2 A posteriori error estimates for the Helmholtz equation

3 Frequency-explicit a posteriori error estimates for Maxwell's equations
- Settings
- Reliability
- Efficiency

4 Numerical experiments
- Analytical solution in a PEC cavity
- Scattering by a penetrable obstacle

Analytical solution in a PEC cavity

We set $\Omega := (-1,1)^2$, $\boldsymbol{\varepsilon} := \boldsymbol{I}$, $\mu := 1$ and $\boldsymbol{J} := \boldsymbol{e}_1$. Thus

$$\boldsymbol{E}(\boldsymbol{x}) = \frac{1}{\omega}\left(\frac{\cos(\omega \boldsymbol{x}_2)}{\cos \omega} - 1\right)\boldsymbol{e}_1$$

Since the problem doesn't feature absorption, there are resonance frequencies for which it is not well-posed:

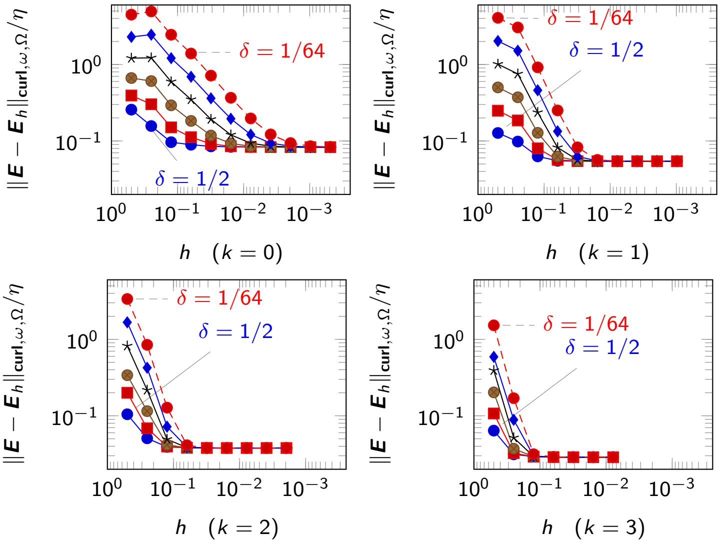
$$\omega = m\pi/2, \;\; m \in \mathbb{N}^\star.$$

We consider two sequences of frequencies to illustrate the influence of σ_{ba}:

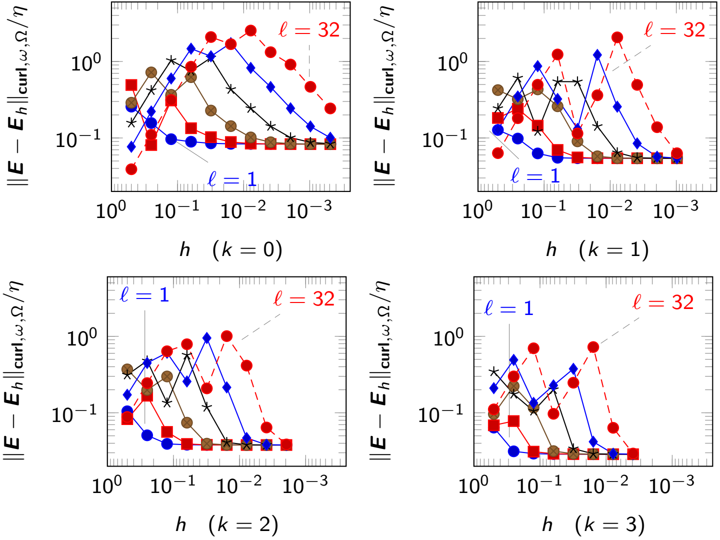
- Frequencies tending towards the resonance frequency $\omega_{\mathrm{r}} = 3\pi/2$: $\omega_\delta := \omega_{\mathrm{r}} + \delta(\pi/2)$ with $\delta = 1/2, 1/4, 1/8, 1/16, 1/32, 1/64$.
- Increasing frequencies uniformly separated from the resonance set: $\omega_\ell := (\ell + 3/10) \times 2\pi$, for $\ell := 1, 2, 4, 8, 16, 32$.

We use structured meshes.

Frequencies tending towards the resonance frequency



Increasing frequencies



Analytical solution in a PEC cavity

For a fixed polynomial degree k:

- When the frequency gets closer to $\omega_{\rm r}$ or is increased, the effectivity index is larger for coarse meshes.
- When $h \to 0$, the same effectivity index is achieved for all frequencies.

The effectivity index is independent of the frequency if the mesh is sufficiently refined (highlighting our key theoretical finding).

Comparing different values of k:

- The effectivity index decreases when k increases.
- The "asymptotic" effectivity index is achieved faster for higher values of k.

The effectivity index of residual estimators depends on k.
For a fixed mesh, $\sigma_{\rm ba}$ decreases when k is increased.

Outline

1 A posteriori error estimates for the Poisson equation

2 A posteriori error estimates for the Helmholtz equation

3 Frequency-explicit a posteriori error estimates for Maxwell's equations
- Settings
- Reliability
- Efficiency

4 Numerical experiments
- Analytical solution in a PEC cavity
- Scattering by a penetrable obstacle

Scattering by a penetrable obstacle

Penetrable obstacle: $G := (-1/4, 1/4)^2$.

Ω consist in $\Omega_0 := (-1,1)^2$, surrounded with a PML of length $\frac{1}{4}$.

$\boldsymbol{E}_{\mathrm{aux}}$ is a plane wave:

- travelling in the direction $\boldsymbol{d} := (\cos\phi, \sin\phi)$,
- polarized along $\boldsymbol{p} := (\sin\phi, -\cos\phi)$, with $\phi = \pi/12$,
- injected through a cut-off function supported in the ring $0.8 \leq |\boldsymbol{x}| \leq 0.9$.

$\boldsymbol{J} := -\omega^2 \boldsymbol{E}_{\mathrm{aux}} + \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{E}_{\mathrm{aux}}$, is supported in the ring $0.8 \leq |\boldsymbol{x}| \leq 0.9$.

We set $\varepsilon := \begin{pmatrix} 8 & 0 \\ 0 & 32 \end{pmatrix}$ and $\mu := \frac{1}{4}$ in G.

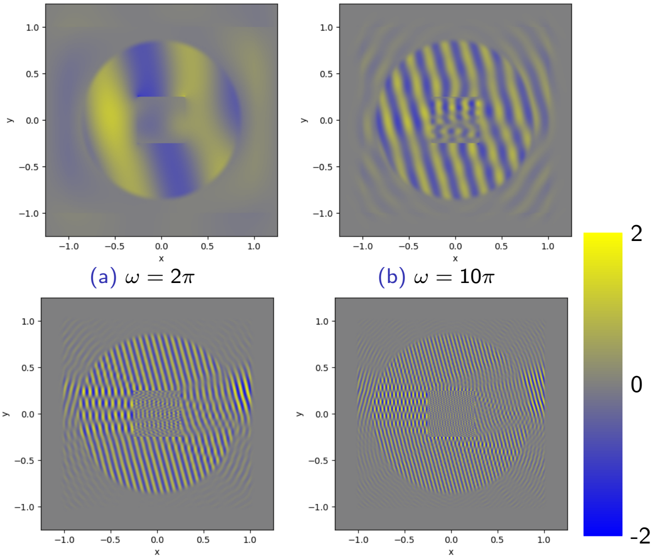
Here, the analytical solution is unavailable. Given $\boldsymbol{E}_h$ we then compute errors compared to $\widetilde{\boldsymbol{E}}$, where $\widetilde{\boldsymbol{E}}$ is computed on the same mesh than $\boldsymbol{E}_h$ with $k = 6$.

Scattering by a penetrable obstacle

We analyze the ability of our estimator to drive a mesh adaptive algorithm.

- We consider general unstructured meshes (MMG software package).
- This package takes as input an already existing mesh, and a set of maximal mesh sizes associated with each vertex of the input mesh.
- The output is a new mesh, locally refined respecting prescribed mesh sizes.
- Using the MMG package and Dörfler's marking, refine the mesh iteratively.

Re $\widetilde{\boldsymbol{E}}_2$ computed at the last iteration of the adaptive algorithm



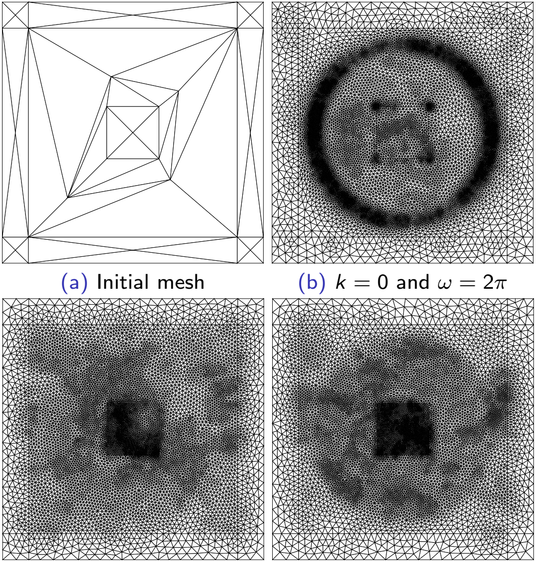
(a) $\omega = 2\pi$

(b) $\omega = 10\pi$

(c) $\omega = 30\pi$

(d) $\omega = 50\pi$

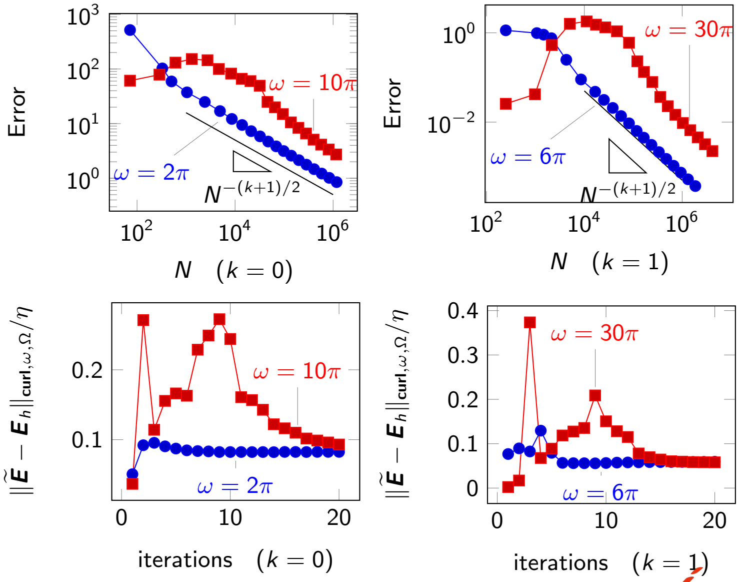
Initial mesh and mesh obtained at iteration 10 of the algorithm



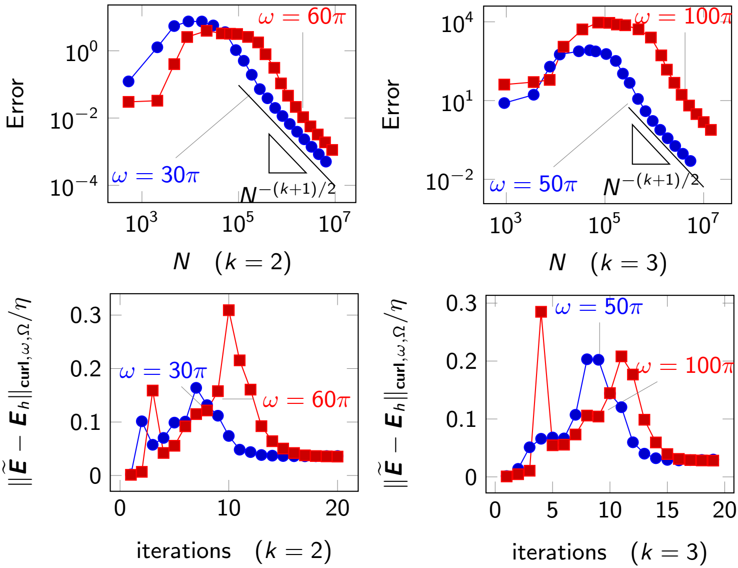
(a) Initial mesh (b) $k = 0$ and $\omega = 2\pi$

(c) $k = 1$ and $\omega = 30\pi$ (d) $k = 2$ and $\omega = 60\pi$

Error vs. number of dofs / Effectivity index vs. number of iterations



Error vs. number of dofs / Effectivity index vs. number of iterations



Error vs. number of dofs / Effectivity index vs. number of iterations

$\mathrm{Error} := 100\frac{\|\widetilde{\boldsymbol{E}}-\boldsymbol{E}_h\|_{\mathbf{curl},\omega,\Omega}}{\|\widetilde{\boldsymbol{E}}\|_{\mathbf{curl},\omega,\Omega}}$ vs. N:

- Asymptotically: $\mathrm{Error} \sim O(N^{-(k+1)/2})$ (optimal rate).

The produced meshes are adequately refined for all ω and k considered.

Effectivity index vs. iterations:

- The error is underestimated on coarse meshes, and this under estimation is more pronounced for higher frequencies.
- Asymptotically, the effectivity index becomes independent of the frequency.

The curves are similar to the one presented for uniform meshes.

Conclusion

- We analyzed residual-based *a posteriori* error estimators for the discretization of time-harmonic Maxwell's equations in heterogeneous media.
  Novelty: We derive frequency-explicit reliability and efficiency estimates.
- Our findings generalize previous results for scalar wave propagation problems.
- The reliability and efficiency constants are independent of the frequency for sufficiently refined meshes.
- We presented numerical experiments including interior problems and scattering problems with PMLs. In all cases, the behavior of the estimator fits our theoretical predictions.
- The estimator was used to drive an adaptive refinement. We got optimal convergence rates, and therefore our estimator is suited for adaptivity purposes.