Exercise 4.2.1. Let f and g be functions defined on some $A \subseteq \mathbf{R}$ with $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ for some limit point c of A. Then $\lim_{x\to c}[f(x) + g(x)] = L + M$.

(a) Proof using the Sequential Criterion for Functional Limits and the Algebraic Limit Theorem:

Proof. By the Sequential Criterion for Functional Limits, we know that for any sequence $(x_n) \subseteq A$ with $\lim x_n = c$ and $x_n \neq c$ for all $n \in \mathbf{N}$, $\lim f(x_n) = L$ and $\lim g(x_n) = M$. Then the Algebraic Limit Theorem tells us that $\lim[f(x_n) + g(x_n)] = L + M$. Thus, the Sequential Criterion for Functional Limits implies $\lim_{x\to c}[f(x) + g(x)] = L + M$. $\square$

(b) Proof using the $\epsilon - \delta$ definition of functional limits:

Proof. Let $\epsilon > 0$ be arbitrary. There exists a $\delta_0 > 0$ such that for any $x \in A$, if $0 < |x - c| < \delta_0$, then $|f(x) - L| < \epsilon/2$. Similarly, there exists a $\delta_1 > 0$ such that for any $x \in A$, if $0 < |x - c| < \delta_1$, then $|g(x) - M| < \epsilon/2$. If $\delta = \min\{\delta_0, \delta_1\}$ and $0 < |x - c| < \delta$, then $|(f(x) + g(x)) - (L + M)| \leq |f(x) - L| + |g(x) - M| < \epsilon$. Thus, $\lim_{x\to c}[f(x) + g(x)] = L + M$. $\square$

(c) $\lim_{x\to c}[f(x)g(x)] = LM$. Proof using the Sequential Criterion for Functional Limits and the

Algebraic Limit Theorem:

Proof. By the Sequential Criterion for Functional Limits, we know that for any sequence $(x_n) \subseteq A$ with $\lim x_n = c$ and $x_n \neq c$ for all $n \in \mathbf{N}$, $\lim f(x_n) = L$ and $\lim g(x_n) = M$. Then the Algebraic Limit Theorem tells us that $\lim[f(x_n)g(x_n)] = LM$. Thus, the Sequential Criterion for Functional Limits implies $\lim_{x\to c}[f(x)g(x)] = LM$. $\square$

Proof using the $\epsilon - \delta$ definition of functional limits:

Proof. Let $\epsilon > 0$ be arbitrary. There exists a $\delta_0 > 0$ such that for any $x \in A$, if $0 < |x - c| < \delta_0$, then $|f(x) - L| < \min\{\frac{\epsilon}{2(|M|+1)}, 1\}$. Similarly, there exists a $\delta_1 > 0$ such that for any $x \in A$, if $0 < |x - c| < \delta_1$, then $|g(x) - M| < \frac{\epsilon}{2(|L|+1)}$. Then if $\delta = \min\{\delta_0, \delta_1\}$ and $0 < |x - c| < \delta$,

$$\begin{aligned} |f(x)g(x) - LM| &\leq |M||f(x) - L| + |f(x)||g(x) - M| \\ &< |M||f(x) - L| + (|L| + 1)|g(x) - M| \\ &< \epsilon. \end{aligned}$$

Thus, $\lim_{x\to c}[f(x)g(x)] = LM$. $\square$

Exercise 4.2.2. For each stated limit, find the largest possible δ-neighborhood that is a proper response to the given ϵ challenge.

(a) $\lim_{x\to 3}(5x - 6) = 9$, where $\epsilon = 1$.

$$\delta = \frac{1}{5}$$

(b) $\lim_{x\to 4} \sqrt{x} = 2$, where $\epsilon = 1$.

$$\delta = 3$$

(c) $\lim_{x\to \pi} \lfloor x \rfloor = 3$, where $\epsilon = 1$.

$$\delta = \pi - 3$$

(d) $\lim_{x\to \pi} \lfloor x \rfloor = 3$, where $\epsilon = 0.01$.

$$\delta = \pi - 3$$

Exercise 4.2.3.

Exercise 4.3.1. Let $g(x) = \sqrt[3]{x}$.

(a) Prove that g is continuous at $c = 0$.

Proof. If $|x| < \epsilon^3$, then $|\sqrt[3]{x}| < \epsilon$. $\square$

(b) Prove that g is continuous at a point $c \neq 0$.

Proof. Let $|x - c| < \min c^{2/3}\epsilon, |c|$. Then

$$|g(x) - g(c)| = |x^{1/3} - c^{1/3}| = \frac{|x-c|}{|x^{2/3} + x^{1/3}c^{1/3} + c^{2/3}|} < \frac{|x-c|}{c^{2/3}} < \epsilon.$$

$\square$

Exercise 4.3.2. Let f be a function defined on all of $\mathbf{R}$.

(a) f is *onetinuous* at c if $\forall \epsilon > 0, |x - c| < 1 \implies |f(x) - f(c)| < \epsilon$.

$f(x) = k$ for some $k \in \mathbf{R}$ is *onetinuous*.

(b) f is *equaltinuous* at c if $\forall \epsilon > 0, |x - c| < \epsilon \implies |f(x) - f(c)| < \epsilon$.

$f(x) = mx + b$ where $m, b \in \mathbf{R}$ and $0 < |m| \leq 1$ is *equaltinuous* and not *onetinuous*.

(c) f is *lesstinuous* at c if $\forall \epsilon > 0, \exists \delta > 0, \delta < \epsilon, |x - c| < \delta \implies |f(x) - f(c)| < \epsilon$.

$f(x) = mx + b$ where $m, b \in \mathbf{R}$ and $|m| < 1$ is *lesstinuous* and not *equaltinuous*.

(d) Is every *lesstinuous* function continuous? Is every continuous function *lesstinuous*?

lesstinuous functions are continuous because there exists a $\delta > 0$ for every $\epsilon > 0$. Continuous functions are *lesstinuous* because if there exists a $\delta > 0$ satisfying a $\epsilon > 0$, then any smaller δ also satisfies that ϵ.

Exercise 4.3.3. Let $f : A \to \mathbf{R}$ and $g : B \to \mathbf{R}$ with $f(A) \subseteq B$ so that $g \circ f$ is defined on A. If f is continuous at $c \in A$ and g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c.

(a) Prove this using the ϵ-δ characterization of continuity.

Proof. Let $\epsilon > 0$ be arbitrary. g is continous at $f(c)$ so there exists some $\delta' > 0$ such that for all $y \in B$, if $|y - f(c)| < \delta'$, then $|g(y) - g(f(c))| < \epsilon$. f is continuous at c so there exists some $\delta > 0$ such that for all $x \in A$, if $|x - c| < \delta$, then $|f(x) - f(c)| < \delta'$. This implies that if $|x - c| < \delta$, then $|g(f(x)) - g(f(c))| < \epsilon$. Thus $g \circ f$ is continuous at c. $\square$

(b) Prove this using the sequential characterization of continuity.

Proof. g is continuous at $f(c)$ so for any sequence $(y_n) \subseteq B$ with $(y_n) \to f(c)$, we have $g(y_n) \to g(f(c))$. Similarly, f's continuity at c implies that for any sequence $(x_n) \subseteq A$ with $(x_n) \to c$, we know $f(x_n) \to f(c)$. $f(x_n) \subseteq B$ so g's continuity implies $g(f(x_n)) \to g(f(c))$. Thus $g \circ f$ is continuous at c. $\square$

Exercise 4.4.1.

(a) Show that $f(x) = x^3$ is continuous on all of $\mathbf{R}$.

Proof. Given some $\epsilon > 0$, if $|x| < \sqrt[3]{\epsilon}$, then $|x^3| < \epsilon$. Thus f is continuous at 0. If $c \neq 0$ and $|x - c| < \min\{|c|, \frac{\epsilon}{7c^2}\}$, then $|x| < 2|c|$ so

$$|x^3 - c^3| < |x - c||x^2 + xc + c^2| < |x - c|(4c^2 + 2c^2 + c^2) = 7c^2|x - c| < \epsilon$$

Thus f is continuous at $c \neq 0$. $\square$

(b) Show that f is not uniformly continuous on $\mathbf{R}$.

Proof. Let $x_n = n$ and $y_n = n + \frac{1}{n}$. $\lim |x_n - y_n| = \lim |\frac{1}{n}| = 0$ but $|f(x) - f(y)| = 3n + \frac{3}{n} + \frac{1}{n^3} > 3$ so f is not uniformly continuous. $\square$

(c) Show that f is uniformly continuous on any bounded subset of $\mathbf{R}$.

Proof. Let $A \subset \mathbf{R}$ be bounded. Then there exists some $M > 0$ such that for all $x \in A$, $|x| \leq M$.

$$|x^3 - y^3| = |x-y||x^2+xy+y^2| \leq |x-y|(x^2+|xy|+y^2) \leq |x-y|3M^2.$$

If we choose $\delta < \frac{\epsilon}{3M^2}$ and $|x-y| < \delta$, then $|x^3 - y^3| < \epsilon$. Thus, f is uniformly continuous on A. $\square$

Exercise 4.4.2.

(a) Is $f(x) = \frac{1}{x}$ uniformly continuous on $(0,1)$?

No. Let $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n+1}$. $|x_n - y_n| = \frac{1}{n^2+n}$ so $\lim |x_n - y_n| = 0$ but $|f(x_n) - f(y_n)| = 1$ so by the sequential criterion for the absence of uniform continuity, f is not uniformly continuous.

(b) Is $g(x) = \sqrt{x^2+1}$ uniformly continuous on $(0,1)$?

Yes. g is continuous on $[0,1]$ which is compact so g is uniformly continuous on $[0,1]$. $(0,1)$ is a subset of $[0,1]$ so g is uniformly continuous on $(0,1)$.

(c) Is $h(x) = x\sin(\frac{1}{x})$ uniformly continuous on $(0,1)$?

Yes. We can extend h over $[0,1]$ by letting $h(0) = 0$. h is continuous on compact set $[0,1]$ so h is uniformly continuous on $[0,1]$. $(0,1)$ is a subset of $[0,1]$ so h is uniformly continuous on $(0,1)$.

Exercise 4.4.3. Show that $f(x) = \frac{1}{x^2}$ is uniformly continuous on the set $[1,\infty)$ but not on the set $(0,\infty)$.

First, we'll show that f is uniformly continuous on $[1,\infty)$.

$$|\frac{1}{x^2} - \frac{1}{y^2}| = |\frac{y^2-x^2}{x^2y^2}| = \frac{(x+y)|x-y|}{x^2y^2} = (\frac{1}{xy^2} + \frac{1}{x^2y})|x-y|$$

With $x, y \in [1,\infty)$, we have $x, y > 1$ so

$$|\frac{1}{x^2} - \frac{1}{y^2}| \leq 2|x-y|.$$

If $\delta = \frac{\epsilon}{2}$ and $|x-y| < \delta$, $|\frac{1}{x^2} - \frac{1}{y^2}| < \epsilon$. Given any ϵ, the associated δ only relies on ϵ and not the specific values of x or y. Thus, f is uniformly continuous on $[1,\infty)$.

Next, we'll show that f is not uniformly continuous on $(0,1]$. Let $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n+1}$. $|x_n - y_n| = |\frac{1}{n(n+1)}|$ so $\lim|x_n - y_n| = 0$ but $|f(x_n) - f(y_n)| = |2n+1| \geq 2$ for all $n \in \mathbf{N}$. By the sequential criterion for the absence of uniform continuity, f is not uniformly continuous on $(0,1]$.

Exercise 4.5.1. Show how the Intermediate Value Theorem follows as a corollary to the preservation of connected sets.

Proof. Let $f : [a,b] \to \mathbf{R}$ be a continuous function and L be a point satisfying either $f(a) < L < f(b)$ or $f(a) > L > f(b)$. $[a,b]$ is connected and f is continuous so $f([a,b])$ is connected. Theorem 3.4.7 implies that because $L \in f([a,b])$, there exists some $c \in [a,b]$ such that $f(c) = L$. $\square$

Exercise 4.5.2. Provide an example of each of the following, or explain why the request is impossible.

(a) A continuous function defined on an open interval with range equal to a closed interval.

$f(x) = \sin x$ over $(0,2\pi)$. f is continuous and has range $[-1,1]$.

(b) A continuous function defined on a closed interval with range equal to an open interval.

This is not possible. Closed intervals are compact and continuous functions preserve compactness.

(c) A continuous function defined on an open interval with range equal to an unbounded closed set different from $\mathbf{R}$.

$f(x) = \sec x$ over $(-\frac{\pi}{2}, \frac{\pi}{2})$. f is continuous on this interval and has range $[1,\infty)$.

(d) A continuous function defined on all of $\mathbf{R}$ with range equal to $\mathbf{Q}$.

This is not possible because it violates the Intermediate Value Theorem. Pick any two points $x, y \in \mathbf{R}$ such that $x < y$ and $f(x) \neq f(y)$. The density of the irrationals in the reals implies there exists some $l \notin \mathbf{Q}$ such that $l \in (f(x), f(y))$ (or $l \in (f(y), f(x))$). The Intermediate Value Theorem implies the existence of some $z \in (x, y)$ such that $f(z) = l$. Thus f's range is not $\mathbf{Q}$.

Exercise 4.6.1. Construct a function $f : \mathbf{R} \to \mathbf{R}$ so that

(a) $D_f = \mathbf{Z}^c$.

$$f(x) = \begin{cases} x, & x \in \mathbf{Q} \\ \lfloor x + 0.5 \rfloor, & x \notin \mathbf{Q} \end{cases}$$

(b) $D_f = \{x : 0 < x \leq 1\}$.

$$f(x) = \begin{cases} x, & x \leq 0 \text{ or } x > 1 \text{ or } x \in \mathbf{Q} \\ 0, & 0 < x < 1 \text{ and } x \notin \mathbf{Q} \end{cases}$$

Exercise 4.6.2. Given a countable set $A = \{a_1, a_2, a_3, ...\}$, define $f(a_n) = \frac{1}{n}$ and $f(x) = 0$ for all $x \in A$. Find D_f.

$D_f = A$.

Proof. For any $a_n \in A$, let $\epsilon = \frac{1}{n}$. Any $V_\delta(a_n)$ is uncountable so there exists some $x \in V_\delta(a_n)$, $x \notin A$ which implies $f(x) = 0 \notin V_\epsilon(f(a_n))$. Thus f is not continuous at any $a_n \in A$.

For $c \notin A$, we'll consider the cases when c is and is not a limit point of A separately. If c is not a limit point of A, then there exists some $V_\delta(c)$ such that $V_\delta(c) \cap A = \emptyset$. The for any $V_\epsilon(f(c))$, we can choose $\delta \leq \delta_0$ and have

$$x \in V_\delta(c) \implies x \notin A \implies f(x) = 0 \in V_\epsilon(f(c)).$$

Thus f is continuous at any point c that is not a limit point of A.

If c is a limit point of A, then given some $V_\epsilon(f(c))$, choose $N \in \mathbf{N}$ such that $\frac{1}{N} < \epsilon$. Let $\delta = \min\{|c - a_n| \mid a_n \in A \text{ and } n < N\}$. We know such a minimum exists because there are a finite number of terms a_n with $n < N$. Then for any $x \in V_\delta(c)$, we have $f(x) < \frac{1}{N}$ so $f(x) \in V_\epsilon(f(c))$. $\square$