Exercise 5.3.2. Let f be differentiable on an interval A. If $f'(x) \neq 0$ on A, show that f is one-to-one on A. Provide an example to show that the converse statement need not be true.

Proof. Suppose f is not one-to-one on interval A. Then there exist two points $x, y \in A$ with $f(x) = f(y)$. By the Mean Value Theorem, there exists some point $z \in (x, y)$ such that

$$f'(z) = \frac{f(x) - f(y)}{x - y} = 0.$$

Thus, if $f'(x) \neq 0$ for all $x \in A$, then f is one-to-one. □

As an example that shows the converse is not necessarily true, consider $f(x) = x^2$ on $[0, \infty)$. f is one-to-one and $f'(0) = 0$.

Exercise 5.3.3. Let h be a differentiable function defined on the interval $[0, 3]$, and assume that $h(0) = 1$, $h(1) = 2$, and $h(3) = 2$.

(a) Argue that there exists a point $d \in [0, 3]$ where $h(d) = d$. **TODO**