

**Exercise 4.2.1.** Let  $f$  and  $g$  be functions defined on some  $A \subseteq \mathbf{R}$  with  $\lim_{x \rightarrow c} f(x) = L$  and  $\lim_{x \rightarrow c} g(x) = M$  for some limit point  $c$  of  $A$ . Then  $\lim_{x \rightarrow c} [f(x) + g(x)] = L + M$ .

(a) Proof using the Sequential Criterion for Functional Limits and the Algebraic Limit Theorem:

*Proof.* By the Sequential Criterion for Functional Limits, we know that for any sequence  $(x_n) \subseteq A$  with  $\lim x_n = c$  and  $x_n \neq c$  for all  $n \in \mathbf{N}$ ,  $\lim f(x_n) = L$  and  $\lim g(x_n) = M$ . Then the Algebraic Limit Theorem tells us that  $\lim [f(x_n) + g(x_n)] = L + M$ . Thus, the Sequential Criterion for Functional Limits implies  $\lim_{x \rightarrow c} [f(x) + g(x)] = L + M$ .  $\square$

(b) Proof using the  $\epsilon - \delta$  definition of functional limits:

*Proof.* Let  $\epsilon > 0$  be arbitrary. There exists a  $\delta_0 > 0$  such that for any  $x \in A$ , if  $0 < |x - c| < \delta_0$ , then  $|f(x) - L| < \epsilon/2$ . Similarly, there exists a  $\delta_1 > 0$  such that for any  $x \in A$ , if  $0 < |x - c| < \delta_1$ , then  $|g(x) - M| < \epsilon/2$ . If  $\delta = \min\{\delta_0, \delta_1\}$  and  $0 < |x - c| < \delta$ , then  $|(f(x) + g(x)) - (L + M)| \leq |f(x) - L| + |g(x) - M| < \epsilon$ . Thus,  $\lim_{x \rightarrow c} [f(x) + g(x)] = L + M$ .  $\square$

(c)  $\lim_{x \rightarrow c} [f(x)g(x)] = LM$ . Proof using the Sequential Criterion for Functional Limits and the Algebraic Limit Theorem:

*Proof.* By the Sequential Criterion for Functional Limits, we know that for any sequence  $(x_n) \subseteq A$  with  $\lim x_n = c$  and  $x_n \neq c$  for all  $n \in \mathbf{N}$ ,  $\lim f(x_n) = L$  and  $\lim g(x_n) = M$ . Then the Algebraic Limit Theorem tells us that  $\lim [f(x_n)g(x_n)] = LM$ . Thus, the Sequential Criterion for Functional Limits implies  $\lim_{x \rightarrow c} [f(x)g(x)] = LM$ .  $\square$

Proof using the  $\epsilon - \delta$  definition of functional limits:

*Proof.* Let  $\epsilon > 0$  be arbitrary. There exists a  $\delta_0 > 0$  such that for any  $x \in A$ , if  $0 < |x - c| < \delta_0$ , then  $|f(x) - L| < \min\{\frac{\epsilon}{2(|M|+1)}, 1\}$ . Similarly, there exists a  $\delta_1 > 0$  such that for any  $x \in A$ , if  $0 < |x - c| < \delta_1$ , then  $|g(x) - M| < \frac{\epsilon}{2(|L|+1)}$ . Then if  $\delta = \min\{\delta_0, \delta_1\}$  and  $0 < |x - c| < \delta$ ,

$$\begin{aligned} |f(x)g(x) - LM| &\leq |M||f(x) - L| + |f(x)||g(x) - M| \\ &< |M||f(x) - L| + (|L| + 1)|g(x) - M| \\ &< \epsilon. \end{aligned}$$

Thus,  $\lim_{x \rightarrow c} [f(x)g(x)] = LM$ .  $\square$

**Exercise 4.2.2.** For each stated limit, find the largest possible  $\delta$ -neighborhood that is a proper response to the given  $\epsilon$  challenge.

(a)  $\lim_{x \rightarrow 3} (5x - 6) = 9$ , where  $\epsilon = 1$ .

$$\delta = \frac{1}{5}$$

(b)  $\lim_{x \rightarrow 4} \sqrt{x} = 2$ , where  $\epsilon = 1$ .

$$\delta = 3$$

(c)  $\lim_{x \rightarrow \pi} \lfloor x \rfloor = 3$ , where  $\epsilon = 1$ .

$$\delta = \pi - 3$$

(d)  $\lim_{x \rightarrow \pi} \lfloor x \rfloor = 3$ , where  $\epsilon = 0.01$ .

$$\delta = \pi - 3$$

**Exercise 4.3.1.** Let  $g(x) = \sqrt[3]{x}$ .

(a) Prove that  $g$  is continuous at  $c = 0$ .

*Proof.* If  $|x| < \epsilon^3$ , then  $|\sqrt[3]{x}| < \epsilon$ .  $\square$

- (b) Prove that  $g$  is continuous at a point  $c \neq 0$ .

*Proof.* Let  $|x - c| < \min c^{2/3}\epsilon, |c|$ . Then

$$|g(x) - g(c)| = |x^{1/3} - c^{1/3}| = \frac{|x - c|}{|x^{2/3} + x^{1/3}c^{1/3} + c^{2/3}|} < \frac{|x - c|}{c^{2/3}} < \epsilon.$$

□

**Exercise 4.3.2.** Let  $f$  be a function defined on all of  $\mathbf{R}$ .

- (a)  $f$  is *onetiniuous* at  $c$  if  $\forall \epsilon > 0, |x - c| < 1 \implies |f(x) - f(c)| < \epsilon$ .

$f(x) = k$  for some  $k \in \mathbf{R}$  is *onetiniuous*.

- (b)  $f$  is *equaltiniuous* at  $c$  if  $\forall \epsilon > 0, |x - c| < \epsilon \implies |f(x) - f(c)| < \epsilon$ .

$f(x) = mx + b$  where  $m, b \in \mathbf{R}$  and  $0 < |m| \leq 1$  is *equaltiniuous* and not *onetiniuous*.

- (c)  $f$  is *lesstiniuous* at  $c$  if  $\forall \epsilon > 0, \exists \delta > 0, \delta < \epsilon, |x - c| < \delta \implies |f(x) - f(c)| < \epsilon$ .

$f(x) = mx + b$  where  $m, b \in \mathbf{R}$  and  $|m| < 1$  is *lesstiniuous* and not *equaltiniuous*.

- (d) Is every *lesstiniuous* function continuous? Is every continuous function *lesstiniuous*?

*lesstiniuous* functions are continuous because there exists a  $\delta > 0$  for every  $\epsilon > 0$ . Continuous functions are *lesstiniuous* because if there exists a  $\delta > 0$  satisfying a  $\epsilon > 0$ , then any smaller  $\delta$  also satisfies that  $\epsilon$ .

**Exercise 4.4.1.**

- (a) Show that  $f(x) = x^3$  is continuous on all of  $\mathbf{R}$ .

*Proof.* Given some  $\epsilon > 0$ , if  $|x| < \sqrt[3]{\epsilon}$ , then  $|x^3| < \epsilon$ . Thus  $f$  is continuous at 0. If  $c \neq 0$  and  $|x - c| < \min\{|c|, \frac{\epsilon}{7c^2}\}$ , then  $|x| < 2|c|$  so

$$|x^3 - c^3| < |x - c||x^2 + xc + c^2| < |x - c|(4c^2 + 2c^2 + c^2) = 7c^2|x - c| < \epsilon$$

Thus  $f$  is continuous at  $c \neq 0$ .

□

- (b) Show that  $f$  is not uniformly continuous on  $\mathbf{R}$ .

*Proof.* Let  $x_n = n$  and  $y_n = n + \frac{1}{n}$ .  $\lim |x_n - y_n| = \lim |\frac{1}{n}| = 0$  but  $|f(x) - f(y)| = 3n + \frac{3}{n} + \frac{1}{n^3} > 3$  so  $f$  is not uniformly continuous. □

- (c) Show that  $f$  is uniformly continuous on any bounded subset of  $\mathbf{R}$ .

*Proof.* Let  $A \subset \mathbf{R}$  be bounded. Then there exists some  $M > 0$  such that for all  $x \in A, |x| \leq M$ .

$$|x^3 - y^3| = |x - y||x^2 + xy + y^2| \leq |x - y|(x^2 + |xy| + y^2) \leq |x - y|3M^2.$$

If we choose  $\delta < \frac{\epsilon}{3M^2}$  and  $|x - y| < \delta$ , then  $|x^3 - y^3| < \epsilon$ . Thus,  $f$  is uniformly continuous on  $A$ . □

**Exercise 4.4.2.**

- (a) Is  $f(x) = \frac{1}{x}$  uniformly continuous on  $(0, 1)$ ?

No. Let  $x_n = \frac{1}{n}$  and  $y_n = \frac{1}{n+1}$ .  $|x_n - y_n| = \frac{1}{n^2+n}$  so  $\lim |x_n - y_n| = 0$  but  $|f(x_n) - f(y_n)| = 1$  so by the sequential criterion for the absence of uniform continuity,  $f$  is not uniformly continuous.

- (b) Is  $g(x) = \sqrt{x^2 + 1}$  uniformly continuous on  $(0, 1)$ ?

Yes.  $g$  is continuous on  $[0, 1]$  which is compact so  $g$  is uniformly continuous on  $[0, 1]$ .  $(0, 1)$  is a subset of  $[0, 1]$  so  $g$  is uniformly continuous on  $(0, 1)$ .

(c) Is  $h(x) = x \sin(\frac{1}{x})$  uniformly continuous on  $(0, 1)$ ?

Yes. We can extend  $h$  over  $[0, 1]$  by letting  $h(0) = 0$ .  $h$  is continuous on compact set  $[0, 1]$  so  $h$  is uniformly continuous on  $[0, 1]$ .  $(0, 1)$  is a subset of  $[0, 1]$  so  $h$  is uniformly continuous on  $(0, 1)$ .

**Exercise 4.5.1.** Show how the Intermediate Value Theorem follows as a corollary to the preservation of connected sets.

*Proof.* Let  $f : [a, b] \rightarrow \mathbf{R}$  be a continuous function and  $L$  be a point satisfying either  $f(a) < L < f(b)$  or  $f(a) > L > f(b)$ .  $[a, b]$  is connected and  $f$  is continuous so  $f([a, b])$  is connected. Theorem 3.4.7 implies that because  $L \in f([a, b])$ , there exists some  $c \in [a, b]$  such that  $f(c) = L$ .  $\square$

**Exercise 4.5.2.** Provide an example of each of the following, or explain why the request is impossible.

(a) A continuous function defined on an open interval with range equal to a closed interval.

$f(x) = \sin x$  over  $(0, 2\pi)$ .  $f$  is continuous and has range  $[-1, 1]$ .

(b) A continuous function defined on a closed interval with range equal to an open interval.

This is not possible. Closed intervals are compact and continuous functions preserve compactness.

(c) A continuous function defined on an open interval with range equal to an unbounded closed set different from  $\mathbf{R}$ .

$f(x) = \sec x$  over  $(-\frac{\pi}{2}, \frac{\pi}{2})$ .  $f$  is continuous on this interval and has range  $[1, \infty)$ .

(d) A continuous function defined on all of  $\mathbf{R}$  with range equal to  $\mathbf{Q}$ .

This is not possible because it violates the Intermediate Value Theorem. Pick any two points  $x, y \in \mathbf{R}$  such that  $x < y$  and  $f(x) \neq f(y)$ . The density of the irrationals in the reals implies there exists some  $l \notin \mathbf{Q}$  such that  $l \in (f(x), f(y))$  (or  $l \in (f(y), f(x))$ ). The Intermediate Value Theorem implies the existence of some  $z \in (x, y)$  such that  $f(z) = l$ . Thus  $f$ 's range is not  $\mathbf{Q}$ .

**Exercise 4.6.1.** Construct a function  $f : \mathbf{R} \rightarrow \mathbf{R}$  so that

(a)  $D_f = \mathbf{Z}^c$ .

$$f(x) = \begin{cases} x, & x \in \mathbf{Q} \\ \lfloor x + 0.5 \rfloor, & x \notin \mathbf{Q} \end{cases}$$

(b)  $D_f = \{x : 0 < x \leq 1\}$ .

$$f(x) = \begin{cases} x, & x \leq 0 \text{ or } x > 1 \text{ or } x \in \mathbf{Q} \\ 0, & 0 < x < 1 \text{ and } x \notin \mathbf{Q} \end{cases}$$

**Exercise 4.6.2.** Given a countable set  $A = \{a_1, a_2, a_3, \dots\}$ , define  $f(a_n) = \frac{1}{n}$  and  $f(x) = 0$  for all  $x \in A$ . Find  $D_f$ .

$D_f = A$ .

*Proof.* For any  $a_n \in A$ , let  $\epsilon = \frac{1}{n}$ . Any  $V_\delta(a_n)$  is uncountable so there exists some  $x \in V_\delta(a_n)$ ,  $x \notin A$  which implies  $f(x) = 0 \notin V_\epsilon(f(a_n))$ . Thus  $f$  is not continuous at any  $a_n \in A$ .

For  $c \notin A$ , we'll consider the cases when  $c$  is and is not a limit point of  $A$  separately. If  $c$  is not a limit point of  $A$ , then there exists some  $V_\delta(c)$  such that  $V_\delta(c) \cap A = \emptyset$ . Then for any  $V_\epsilon(f(c))$ , we can choose  $\delta \leq \delta_0$  and have

$$x \in V_\delta(c) \implies x \notin A \implies f(x) = 0 \in V_\epsilon(f(c)).$$

Thus  $f$  is continuous at any point  $c$  that is not a limit point of  $A$ .

If  $c$  is a limit point of  $A$ , then given some  $V_\epsilon(f(c))$ , choose  $N \in \mathbf{N}$  such that  $\frac{1}{N} < \epsilon$ . Let  $\delta = \min\{|c - a_n| \mid a_n \in A \text{ and } n < N\}$ . We know such a minimum exists because there are a finite number of terms  $a_n$  with  $n < N$ . Then for any  $x \in V_\delta(c)$ , we have  $f(x) < \frac{1}{N}$  so  $f(x) \in V_\epsilon(f(c))$ .  $\square$