Exercise 3.4.6. (Sequential Characterization of Connected Sets) A set $E \subseteq \mathbf{R}$ is connected if and only if, for all nonempty disjoint sets A and B satisfying $E = A \cup B$, there always exists a convergent sequence $(x_n) \to x$ with (x_n) contained in one of A or B, and x an element of the other.

Proof. Let $E \subseteq \mathbf{R}$ be a connected set. If $E = A \cup B$ where A and B are nonempty disjoint sets, then either $\overline{A} \cap B$ or $A \cap \overline{B}$ is nonempty. Without loss of generality, suppose $\overline{A} \cap B$ is nonempty. Then there exists some $x \in \overline{A} \cap B$ where $x \notin A$. $x \in A$ and $x \notin \overline{A}$ implies x is a limit point of A so there exists some $(x_n) \subseteq A$ with $(x_n) \to x$ and $x \in B$.

Now we'll prove the converse. Suppose that whenever $A \cup B = E$ for nonempty disjoint sets A and B, there exists some $(x_n) \to x$ with (x_n) contained in either A or B and x contained in the other. Without loss of generality, suppose $(x_n) \subseteq A$. Then $x \in \overline{A}$ so $x \in \overline{A} \cap B$. Thus E is connected. $\square$