Exercise 3.4.6. (Sequential Characterization of Connected Sets) A set $E \subseteq \mathbf{R}$ is connected if and only if, for all nonempty disjoint sets A and B satisfying $E = A \cup B$, there always exists a convergent sequence $(x_n) \to x$ with (x_n) contained in one of A or B, and x an element of the other.

Proof. Let $E \subseteq \mathbf{R}$ be a connected set. If $E = A \cup B$ where A and B are nonempty disjoint sets, then either $\overline{A} \cap B$ or $A \cap \overline{B}$ is nonempty. Without loss of generality, suppose $\overline{A} \cap B$ is nonempty. Then there exists some $x \in \overline{A} \cap B$ where $x \notin A$. $x \in A$ and $x \notin \overline{A}$ implies x is a limit point of A so there exists some $(x_n) \subseteq A$ with $(x_n) \to x$ and $x \in B$.

Now we'll prove the converse. Suppose that whenever $A \cup B = E$ for nonempty disjoint sets A and B, there exists some $(x_n) \to x$ with (x_n) contained in either A or B and x contained in the other. Without loss of generality, suppose $(x_n) \subseteq A$. Then $x \in \overline{A}$ so $x \in \overline{A} \cap B$. Thus E is connected. □

Exercise 4.2.1. Let f and g be functions defined on some $A \subseteq \mathbf{R}$ with $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ for some limit point c of A. Then $\lim_{x\to c}[f(x) + g(x)] = L + M$.

(a) Proof using the Sequential Criterion for Functional Limits and the Algebraic Limit Theorem:

Proof. By the Sequential Criterion for Functional Limits, we know that for any sequence $(x_n) \subseteq A$ with $\lim x_n = c$ and $x_n \neq c$ for all $n \in \mathbf{N}$, $\lim f(x_n) = L$ and $\lim g(x_n) = M$. Then the Algebraic Limit Theorem tells us that $\lim[f(x_n) + g(x_n)] = L + M$. Thus, the Sequential Criterion for Functional Limits implies $\lim_{x\to c}[f(x) + g(x)] = L + M$. □

(b) Proof using the $\epsilon - \delta$ definition of functional limits:

Proof. Let $\epsilon > 0$ be arbitrary. There exists a $\delta_0 > 0$ such that for any $x \in A$, if $0 < |x - c| < \delta_0$, then $|f(x) - L| < \epsilon/2$. Similarly, there exists a $\delta_1 > 0$ such that for any $x \in A$, if $0 < |x - c| < \delta_1$, then $|g(x) - M| < \epsilon/2$. If $\delta = \min\{\delta_0, \delta_1\}$ and $0 < |x - c| < \delta$, then $|(f(x) + g(x)) - (L + M)| \leq |f(x) - L| + |g(x) - M| < \epsilon$. Thus, $\lim_{x\to c}[f(x) + g(x)] = L + M$. □

(c) $\lim_{x\to c}[f(x)g(x)] = LM$.

Proof using the Sequential Criterion for Functional Limits and the Algebraic Limit Theorem:

Proof. By the Sequential Criterion for Functional Limits, we know that for any sequence $(x_n) \subseteq A$ with $\lim x_n = c$ and $x_n \neq c$ for all $n \in \mathbf{N}$, $\lim f(x_n) = L$ and $\lim g(x_n) = M$. Then the Algebraic Limit Theorem tells us that $\lim[f(x_n)g(x_n)] = LM$. Thus, the Sequential Criterion for Functional Limits implies $\lim_{x\to c}[f(x)g(x)] = LM$. □

Proof using the $\epsilon - \delta$ definition of functional limits:

Proof. Let $\epsilon > 0$ be arbitrary. There exists a $\delta_0 > 0$ such that for any $x \in A$, if $0 < |x - c| < \delta_0$, then $|f(x) - L| < \min\{\frac{\epsilon}{2(|M|+1)}, 1\}$. Similarly, there exists a $\delta_1 > 0$ such that for any $x \in A$, if $0 < |x - c| < \delta_1$, then $|g(x) - M| < \frac{\epsilon}{2(|L|+1)}$. Then if $\delta = \min\{\delta_0, \delta_1\}$ and $0 < |x - c| < \delta$,

$$\begin{aligned}|f(x)g(x) - LM| &\leq |M||f(x) - L| + |f(x)||g(x) - M| \\ &< |M||f(x) - L| + (|L| + 1)|g(x) - M| \\ &< \epsilon.\end{aligned}$$

Thus, $\lim_{x\to c}[f(x)g(x)] = LM$. □

Exercise 4.2.2. For each stated limit, find the largest possible δ-neighborhood that is a proper response to the given ϵ challenge.

(a) $\lim_{x\to 3}(5x - 6) = 9$, where $\epsilon = 1$.

$$\delta = \frac{1}{5}$$

(b) $\lim_{x\to 4} \sqrt{x} = 2$, where $\epsilon = 1$.

$$\delta = 3$$

(c) $\lim_{x\to\pi} \lfloor x \rfloor = 3$, where $\epsilon = 1$.

$$\delta = \pi - 3$$

(d) $\lim_{x\to\pi} \lfloor x \rfloor = 3$, where $\epsilon = 0.01$.

$$\delta = \pi - 3$$

Exercise 4.3.1. Let $g(x) = \sqrt[3]{x}$.

(a) Prove that g is continuous at $c = 0$.

Proof. If $|x| < \epsilon^3$, then $|\sqrt[3]{x}| < \epsilon$. □

(b) Prove that g is continuous at a point $c \neq 0$.

Proof. Let $|x - c| < \min c^{2/3}\epsilon, |c|$. Then

$$|g(x) - g(c)| = |x^{1/3} - c^{1/3}| = \frac{|x-c|}{|x^{2/3} + x^{1/3}c^{1/3} + c^{2/3}|} < \frac{|x-c|}{c^{2/3}} < \epsilon.$$

□

Exercise 4.3.2. Let f be a function defined on all of $\mathbf{R}$.

(a) f is *onetinuous* at c if $\forall \epsilon > 0, |x - c| < 1 \implies |f(x) - f(c)| < \epsilon$.

$f(x) = k$ for some $k \in \mathbf{R}$ is *onetinuous*.

(b) f is *equaltinuous* at c if $\forall \epsilon > 0, |x - c| < \epsilon \implies |f(x) - f(c)| < \epsilon$.

$f(x) = mx + b$ where $m, b \in \mathbf{R}$ and $0 < |m| \leq 1$ is *equaltinuous* and not *onetinuous*.

(c) f is *lesstinuous* at c if $\forall \epsilon > 0, \exists \delta > 0, \delta < \epsilon, |x - c| < \delta \implies |f(x) - f(c)| < \epsilon$.

$f(x) = mx + b$ where $m, b \in \mathbf{R}$ and $|m| < 1$ is *lesstinuous* and not *equaltinuous*.

(d) Is every *lesstinuous* function continuous? Is every continuous function *lesstinuous*?

lesstinuous functions are continuous because there exists a $\delta > 0$ for every $\epsilon > 0$. Continuous functions are *lesstinuous* because if there exists a $\delta > 0$ satisfying a $\epsilon > 0$, then any smaller δ also satisfies that ϵ.

Exercise 4.4.1.

(a) Show that $f(x) = x^3$ is continuous on all of $\mathbf{R}$.

Proof. Given some $\epsilon > 0$, if $|x| < \sqrt[3]{\epsilon}$, then $|x^3| < \epsilon$. Thus f is continuous at 0. If $c \neq 0$ and $|x - c| < \min\{|c|, \frac{\epsilon}{7c^2}\}$, then $|x| < 2|c|$ so

$$|x^3 - c^3| < |x - c||x^2 + xc + c^2| < |x - c|(4c^2 + 2c^2 + c^2) = 7c^2|x - c| < \epsilon$$

Thus f is continuous at $c \neq 0$. □

(b) Show that f is not uniformly continuous on $\mathbf{R}$.

Proof. Let $x_n = n$ and $y_n = n + \frac{1}{n}$. $\lim |x_n - y_n| = \lim |\frac{1}{n}| = 0$ but $|f(x) - f(y)| = 3n + \frac{3}{n} + \frac{1}{n^3} > 3$ so f is not uniformly continuous. □

(c) Show that f is uniformly continuous on any bounded subset of $\mathbf{R}$.

Proof. Let $A \subset \mathbf{R}$ be bounded. Then there exists some $M > 0$ such that for all $x \in A$, $|x| \leq M$.

$$|x^3 - y^3| = |x - y||x^2 + xy + y^2| \leq |x - y|(x^2 + |xy| + y^2) \leq |x - y|3M^2.$$

If we choose $\delta < \frac{\epsilon}{3M^2}$ and $|x - y| < \delta$, then $|x^3 - y^3| < \epsilon$. Thus, f is uniformly continuous on A. □

Exercise 4.4.2.

(a) Is $f(x) = \frac{1}{x}$ uniformly continuous on $(0, 1)$?

No. Let $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n+1}$. $|x_n - y_n| = \frac{1}{n^2+n}$ so $\lim |x_n - y_n| = 0$ but $|f(x_n) - f(y_n)| = 1$ so by the sequential criterion for the absence of uniform continuity, f is not uniformly continuous.

(b) Is $g(x) = \sqrt{x^2 + 1}$ uniformly continuous on $(0, 1)$?

Yes. g is continuous on $[0, 1]$ which is compact so g is uniformly continuous on $[0, 1]$. $(0, 1)$ is a subset of $[0, 1]$ so g is uniformly continuous on $(0, 1)$.

(c) Is $h(x) = x \sin(\frac{1}{x})$ uniformly continuous on $(0, 1)$?

Yes. We can extend h over $[0, 1]$ by letting $h(0) = 0$. h is continuous on compact set $[0, 1]$ so h is uniformly continuous on $[0, 1]$. $(0, 1)$ is a subset of $[0, 1]$ so h is uniformly continuous on $(0, 1)$.

Exercise 4.5.1. Show how the Intermediate Value Theorem follows as a corollary to the preservation of connected sets.

Proof. Let $f : [a, b] \to \mathbf{R}$ be a continuous function and L be a point satisfying either $f(a) < L < f(b)$ or $f(a) > L > f(b)$. $[a, b]$ is connected and f is continuous so $f([a, b])$ is connected. Theorem 3.4.7 implies that because $L \in f([a, b])$, there exists some $c \in [a, b]$ such that $f(c) = L$. □

Exercise 4.5.2. Provide an example of each of the following, or explain why the request is impossible.

(a) A continuous function defined on an open interval with range equal to a closed interval.

$f(x) = \sin x$ over $(0, 2\pi)$. f is continuous and has range $[-1, 1]$.

(b) A continuous function defined on a closed interval with range equal to an open interval.

This is not possible. Closed intervals are compact and continuous functions preserve compactness.

(c) A continuous function defined on an open interval with range equal to an unbounded closed set different from $\mathbf{R}$.

$f(x) = \sec x$ over $(-\frac{\pi}{2}, \frac{\pi}{2})$. f is continuous on this interval and has range $[1, \infty)$.

(d) A continuous function defined on all of $\mathbf{R}$ with range equal to $\mathbf{Q}$.

This is not possible because it violates the Intermediate Value Theorem. Pick any two points $x, y \in \mathbf{R}$ such that $x < y$ and $f(x) \neq f(y)$. The density of the irrationals in the reals implies there exists some $l \notin \mathbf{Q}$ such that $l \in (f(x), f(y))$ (or $l \in (f(y), f(x))$). The Intermediate Value Theorem implies the existence of some $z \in (x, y)$ such that $f(z) = l$. Thus f's range is not $\mathbf{Q}$.

Exercise 4.6.1. Construct a function $f : \mathbf{R} \to \mathbf{R}$ so that

(a) $D_f = \mathbf{Z}^c$.

$$f(x) = \begin{cases} x, & x \in \mathbf{Q} \\ \lfloor x + 0.5 \rfloor, & x \notin \mathbf{Q} \end{cases}$$

(b) $D_f = \{x : 0 < x \leq 1\}$.

$$f(x) = \begin{cases} x, & x \leq 0 \text{ or } x > 1 \text{ or } x \in \mathbf{Q} \\ 0, & 0 < x < 1 \text{ and } x \notin \mathbf{Q} \end{cases}$$

Exercise 4.6.2.

Exercise 5.3.2. Let f be differentiable on an interval A. If $f'(x) \neq 0$ on A, show that f is one-to-one on A. Provide an example to show that the converse statement need not be true.

Proof. Suppose f is not one-to-one on interval A. Then there exist two points $x, y \in A$ with $f(x) = f(y)$. By the Mean Value Theorem, there exists some point $z \in (x, y)$ such that

$$f'(z) = \frac{f(x) - f(y)}{x - y} = 0.$$

Thus, if $f'(x) \neq 0$ for all $x \in A$, then f is one-to-one. □

As an example that shows the converse is not necessarily true, consider $f(x) = x^2$ on $[0, \infty)$. f is one-to-one and $f'(0) = 0$.

Exercise 5.3.3. Let h be a differentiable function defined on the interval $[0, 3]$, and assume that $h(0) = 1$, $h(1) = 2$, and $h(3) = 2$.

(a) Argue that there exists a point $d \in [0, 3]$ where $h(d) = d$. **TODO**