# NIM to WIN

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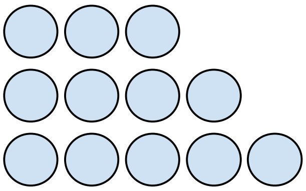
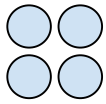
## 1 Introduction

NIM is a mathematical game. Two players take turns removing certain number of objects from just one row. The winning rules of the prime version of this game can is being the one who remove the last object. Although the rules to win can also be forcing the opponent to remove the last object, the principle behind this game is rather the same.



## 2 WINNING Strategy

Suppose there are two players A and B and there are three rows, each have 3, 4 and 5 coins, which is the most popular version of this game.

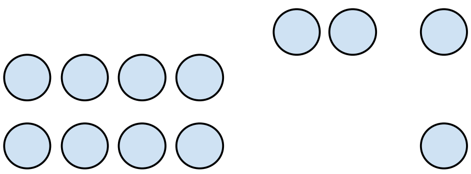


In this game, what we can confirm is that player A's success is destined, as long as in one round, player A can make certain number and certain row of coins left, which is leaving same number n (n>=2) of coins in two rows. The situation which is closest to success is leaving each two coins in two rows.

The above version can be illustrated in a binary form. Since binary digits is representing $2^0$, $2^1$, $2^2$, $2^3$…[1]. For the first row, it can be split into two coins and one coins. For the second row, four coins. For the third row, four coins and one coins. It can be represented as a1= $(011)_2$, a2 = $(100)_2$, a3 = $(101)_2$.

In this case, to win the game, player has to find a position which will definitely lead to the opponent's losing. [2]

Therefore, the conclusion is that the losing position is when the result of XOR all the numbers of coins in each row is zero.

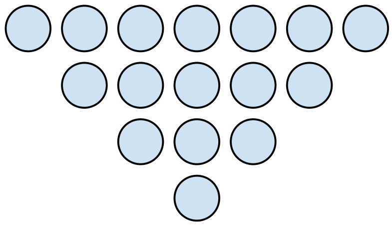


As the winning position which has been mentioned above, it can be rephrased in another way based on binary digits, which is from each column ($2^0$, $2^1$, $2^2$, $2^3$…), there should be even number of rows occupying the space.

So in this version (3,4,5), the best first move would be taking away two coins of the first row.

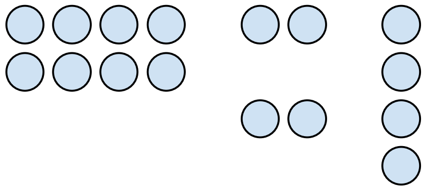
[1] Any number can be represented in binary digits.
[2] Mathematical proofing will be shown later.

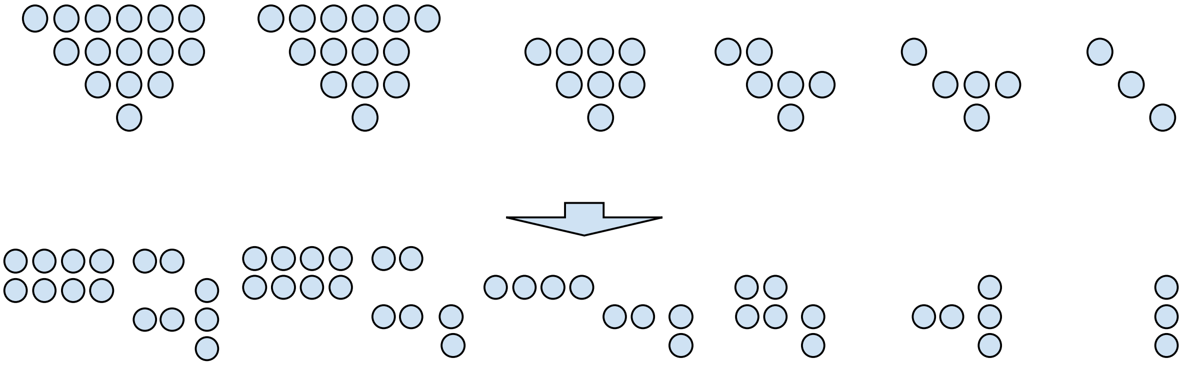
For another example, there are four rows. And each have 1, 3, 5, 7 coins. If using binary digits to represent the, it will be



| | | | |
|---|---|---|---|
| 1st row: | 1 | 1 | 1 |
| 2nd row: | 1 | 0 | 1 |
| 3rd row: | 0 | 1 | 1 |
| 4th row: | 0 | 0 | 1 |

Graphically, it can be modified into the image on the right. Noted the two players are A and B.





From the illustration above, the strategy can be recognized clearly.

To conclude, the optimal strategy to win the game is to make the XOR result be zero after your move.

## 2.1 Mathematical Proof

Based on the background and rules of this game, what can be concluded is that the game is an impartial game, which means the allowable move for the players only depends on the current situation of the game and is not influenced by which player is going to remove the coins. The only difference is which player plays first at the beginning of this game. Each game position gets a number representing the value of the position. The number for a position is the simplest possible value taking into account the moves available to each player - the values of those options having already been computed. (Ted, 2013)

First of all, let's define two terms, "winning position" and "losing position". "Position" means that the current situation that person is going to move from. Therefore, a "losing position" means a player should make the position look like it after his move if he wants to win the game.

If we want to win in nim, the following two propositions have to be satisfied.

1) From a losing position, all moves will lead to winning position.
2) From a winning position, at least one move will lead to losing position

Among the seven basic logical gates, based on the properties, only XOR and XNOR gates are suitable, which is by changing either bit of the current situation, the result would be different from the previous one. Since XNOR is actually a negation of XOR, the principle is the same.

Applying XOR operation in this strategy. Let's write @ for XOR for simplicity.

1) From a position where $n_1@n_2@n_3...@n_k= 0$, all moves will lead to a position that $n_1@n_2@n_3...@n_k \neq 0$.
2) From a positon that $n_1@n_2@n_3...@n_k \neq 0$, there is at least one move will lead to a position that $n_1@n_2@n_3...@n_k= 0$.

Proof for the XOR statements:

1) According to the game rules, we can only decrease one number among all. Notice that if you only change one bit in a XOR calculation, the result will change. Therefore, if you are in a position that $n_1@n_2@n_3...@n_k= 0$ and only change one $n_i$, the final result definitely will change. Then your opponent will face a $n_1@n_2@n_3...@n_k \neq 0$ position.
2) Let's declare N as $n_1@n_2@n_3...@n_k$ , and represent all the numbers in binary later. N will have at least one "1" digit if $N \neq 0$. Let's find the leftmost "1" digit in N and find a $n_i$ that also has a "1" digit in the same position.[3] Then flip all the digits in $n_i$ corresponding to the "1" digits in N.[4] Then N'(N after current move) is 0.

In conclusion, we can treat "$n_1@n_2@n_3...@n_k= 0$" as a "losing position" and "$n_1@n_2@n_3...@n_k \neq 0$" as a "winning position". We only need to make sure that a losing position occurred after our current move. Then whatever moves our opponent choose, there will always be a winning position waiting for us.

# 3 New Game using XOR

Let's add some new rules on our Nim game.

The only one rule is added on the previous version is that a pile must have at least the same number of token as the pile on its left[5].

It means, initially every pile has at least as many tokens as its left pile. And this property must be maintained throughout the game. On each turn, at least one token must be removed from a single pile. Two players take turn playing this game. The last person who take the last move win the game.

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[3] If no $n_i$ can be found, it is just k "0"s doing a XOR operation then this digit in N should be "0" instead of "1". Therefore, there exist at least one $n_i$ that satisfy the condition.

[4] Since the digit in $n_i$ corresponding to the leftmost "1" digit in N is also "1". After flipping, "1" will turn into "0" and it is the most significant digit that have been flipped. Therefore, the value of $n_i$ decrease, which is a legal Nim operation.

[5] This is the 'StoneGameStrategist' problem on the topcoder website. URL: https://community.topcoder.com/stat?c=problem_statement&pm=6239

Similar to Nim, a winning strategy also exist in this game.

## 3.1 Mathematical Proof for winning strategy of new game

I. Let's consider some simple cases first.
   1) First of all, the case (1,1) is a losing position obviously. Since you can only take the first token from the first pile and then your opponent will take the last one. Similar reason, for a constant $C_1$, the case ($C_1$, $C_1$) is a losing case. Your opponent only need to force you to move tokens from the first pile and just copying your move.
   2) For the same reason, ($C_1$, $C_1$, $C_2$, $C_2$, … $C_n$, $C_n$) is also a losing case

II. Let consider more general cases. For simplicity, we assume the number of pile is n and the number of tokens a pile has is $a_i$ (1 <= i <= n). Therefore, the case is that ($a_1$, $a_2$, $a_3$, $a_4$, … $a_{n-1}$, $a_n$).
   1) If n is even, then we can group the adjacent two piles together, for example, grouping $a_1$ and $a_2$ together, $a_3$ and $a_4$ together. Taking the difference of each group, then we get a new sequence, ($a_2$-$a_1$, $a_4$-$a_3$, …, $a_n$-$a_{n-1}$). If we only take the even indexed piles' tokens, then once the new sequence become (0, 0, … 0), then the piles become into 2) situation, which is a losing case. Therefore, you only need to use the general Nim's strategy to play with new sequence. Then your opponent will face the losing situation. Thus, you win the game[6].
   2) If n is odd, then we first insert a pile of 0 token into the game. Then the n becomes even. Using the strategy in II. 1), we can also win the game.

# 4 Application of Nim game

The basic version of NIM game is actually rudimentary to the Sprague–Grundy theorem. It states that every impartial game under normal play convention is equivalent to the number of heaps. Games like chess, tic-tac-tos are not impartial games. The impartial game focuses on the situation which is either the current player or the next player will win.

One of the interesting game similar like Nim is a popular game called "The 21 game". Players take turns to say a number or at most three numbers. For example, if the first player say one, then the next player can say at most three number 2, 3, 4 increasingly. The one who is forced to say 21 lose. The principle behind it is similar to NIM game. Therefore, the winning strategy will be to always say a multiple of 4. That ensures the opponent will say 21. That also means, if the first player only say one, then its loss is guaranteed.

Reference:


1. Ted (http://math.stackexchange.com/users/15012/ted), Why XOR operator works?, URL (version: 2013-06-10): http://math.stackexchange.com/q/416078
2. Topcoder, Problem statement of 'StoneGameStrategist' URL: https://community.topcoder.com/stat?c=problem_statement&pm=6239


[6] Of course you need to start with a situation with nim-sum = 0, otherwise you have no winning strategy of winning nim.