TURING MACHINES

DISCRETE MATHEMATICS AND THEORY 2

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GOALS!

1. Our next computational model: The Turing Machine!

2. Analysis of different variations of Turing Machines (including non-deterministic Turing Machines).

3. Formalizing the concept of decidability versus recognizability, algorithms, and an introduction to computability theory.

PART 1: REMINDER OF WHERE WE ARE / CHOMSKY HIERARCHY

OVERVIEW OF THEORY OF COMPUTATION

Defining Computation

$$\begin{tikzcd}\text{Input} \arrow[r] & \text{Computing Machine / Program / Algorithm} \arrow[r] & \text{Output}\end{tikzcd}$$

Computational Models

Circuits < Finite Automata < Pushdown Automata < **Turing Machine** = RAM Model

Computational Complexity

Decidability

P, NP, NP-Hard

P-Space, Co-NP, etc

WHAT IS A TURING MACHINE?

A ***Turing Machine*** (TM), sometimes called a ***Deterministic Turing Machine*** (DTM) is a finite state machine that can read/write from an infinite tape (memory)

Some other features of the Turing Machine:

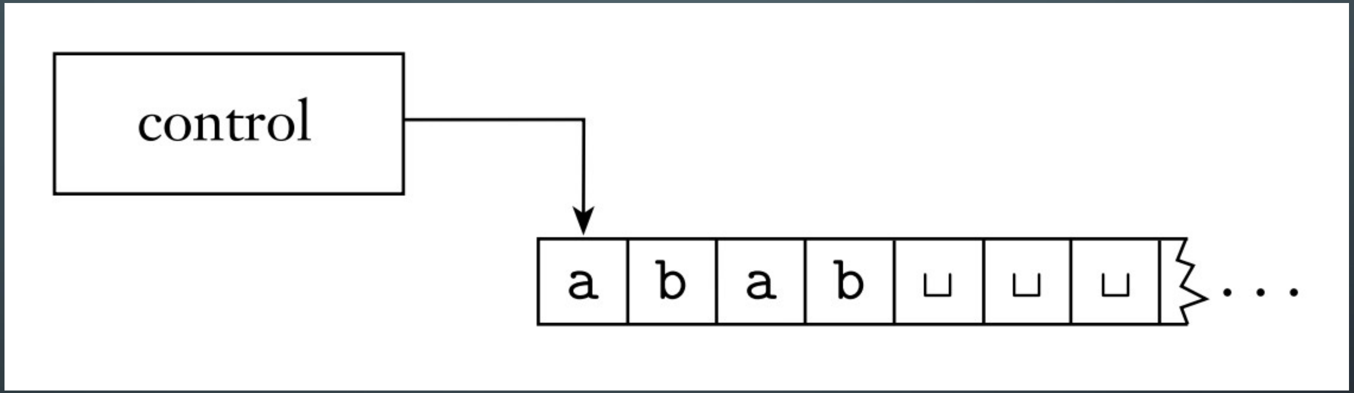
1. A TM can both read and write to/from the tape
2. The TM contains a head that can move left and right along the tape
3. The tape is infinite
4. The special states for accepting / rejecting take effect immediately

INTRODUCING THE TURING MACHINE

WHAT IS A TURING MACHINE?

This ***control*** is a traditional DFA, except the accept/reject states take effect immediately.

The arrow here represents the ***head*** of the machine. It can move left and right and also read/write the symbol at that position.



This ***tape*** contains the input when execution begins.

EXAMPLE TURING MACHINE

Let’s design a Turing Machine to recognize the following language:

Any ideas on how we might do this?

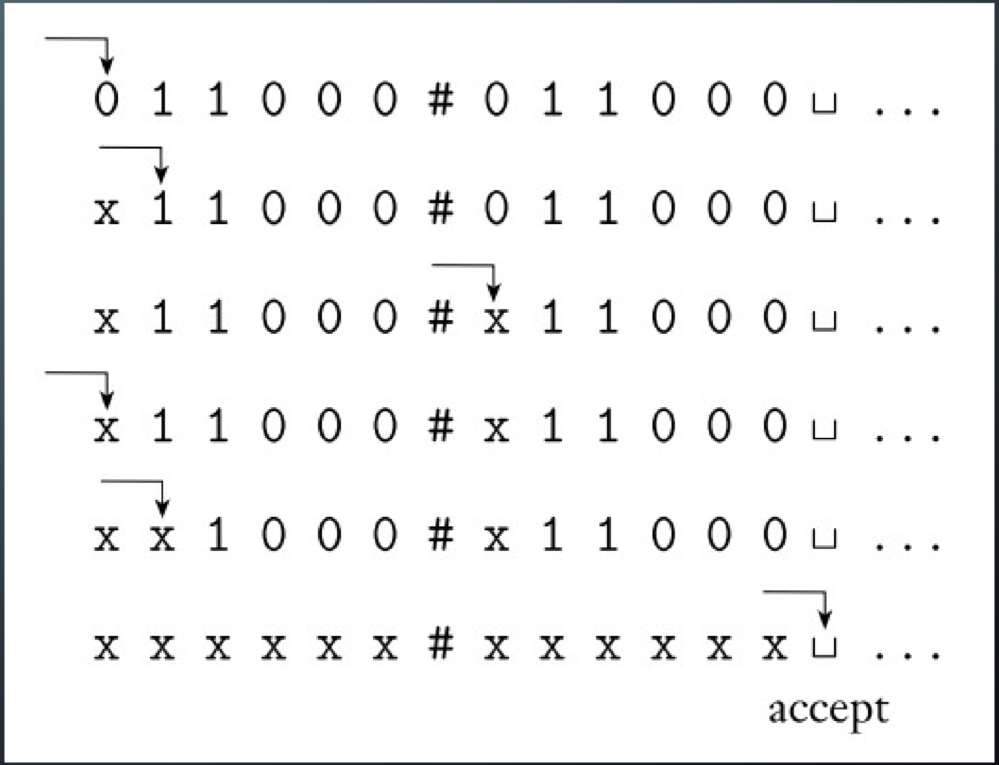
Remember that the proposed string is written on the tape at the beginning of execution, and the head of the tape begins at index 0

EXAMPLE TURING MACHINE

Let’s design a Turing Machine to recognize the following language:

Machine will do the following:

1. Zig-zag across the tape to corresponding positions on either side of the # symbol to check whether these positions contain the same symbol. If they do not or no # is found, ***reject***.
2. When all symbols to the left of the # have been crossed off, check for any remaining symbols to the right of the #. If any exists, ***reject*** otherwise ***accept***.



0 1 1 0 0 0 # 0 1 1 0 0 0 ␣ ...
x 1 1 0 0 0 # 0 1 1 0 0 0 ␣ ...
x 1 1 0 0 0 # x 1 1 0 0 0 ␣ ...
x 1 1 0 0 0 # x 1 1 0 0 0 ␣ ...
x x 1 0 0 0 # x 1 1 0 0 0 ␣ ...
x x x x x x # x x x x x x ␣ ...
accept

FORMAL DEFINITION OF TM

A ***Turing Machine*** is a 7-tuple, , where , , are all finite sets and:

1. is the set of states
2. is the input alphabet not containing the ***blank symbol***
3. is the tape alphabet, where and
4. is the transition function
5. is the start state
6. is the accept state
7. is the reject state, where

Blank symbol often used to mark special cases, end of input, etc.

L, R, S here represent moving the head ***left*** *or* ***right*** *or* ***staying*** *still*

Note that TMs have one accept and one reject state, and they cannot be equal.

TRANSITION FUNCTION

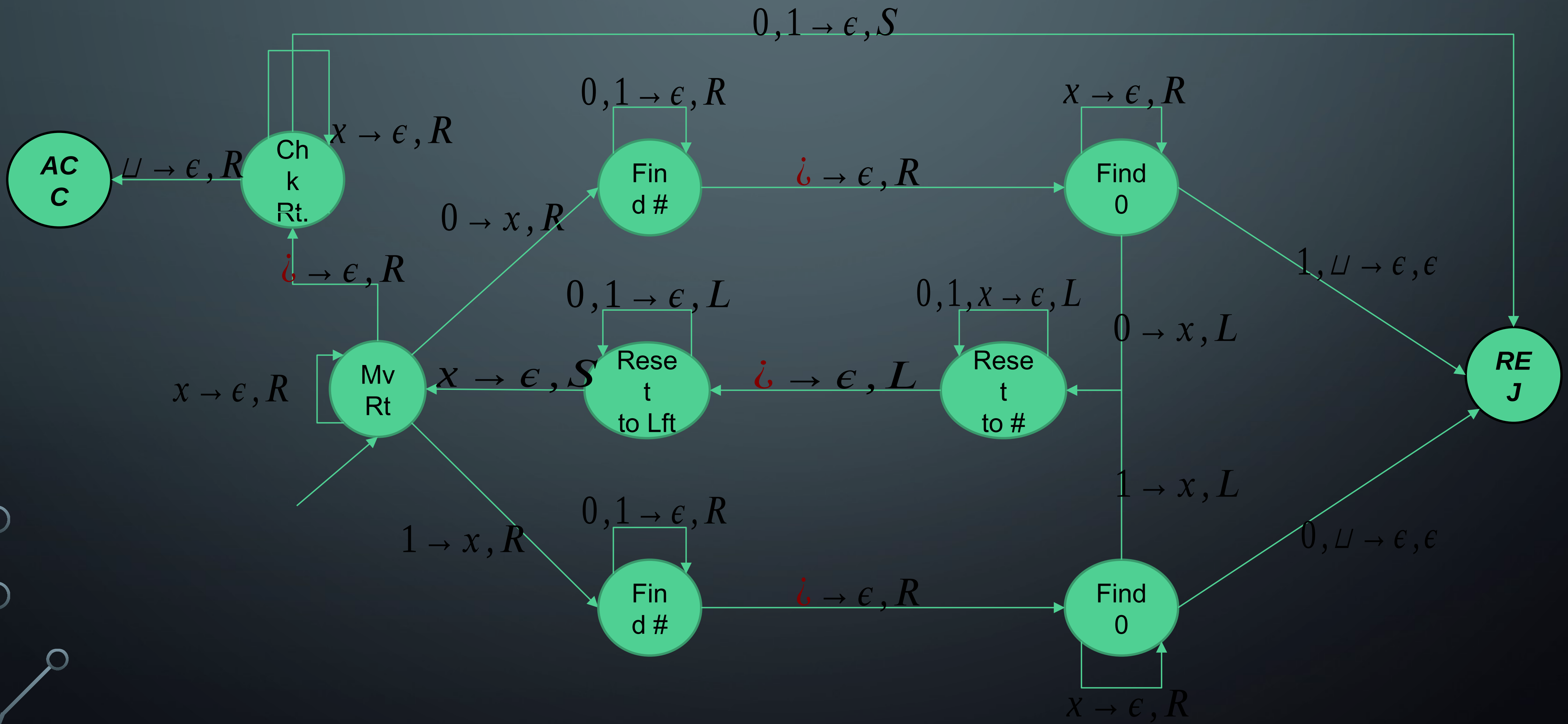
is the transition function

Where the TM transitions depends on this input. What state the machine is in and what symbol is currently on the tape at the head's current position.

Machine will enter a new state (optional)
Machine will write something to the tape (optional)
Head will move Left or Right (or S for staying put)

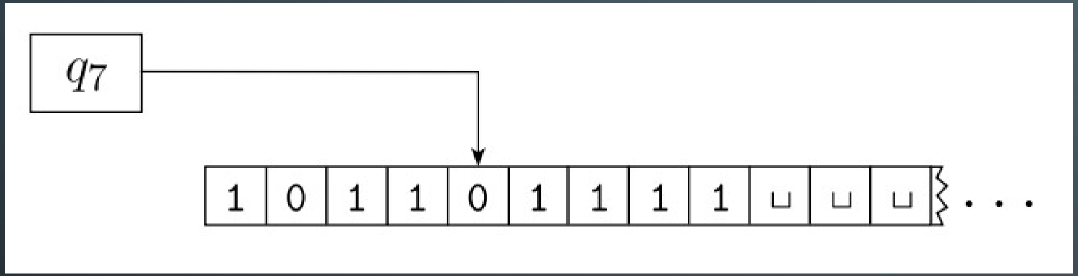
EXAMPLE TURING MACHINE

Let’s design a Turing Machine to recognize the following language:



CONFIGURATIONS OF A TM

A ***configuration*** of a Turing Machine is the complete state the machine is in at any point during execution. This includes the state, the contents of the tape, and the position of the head.

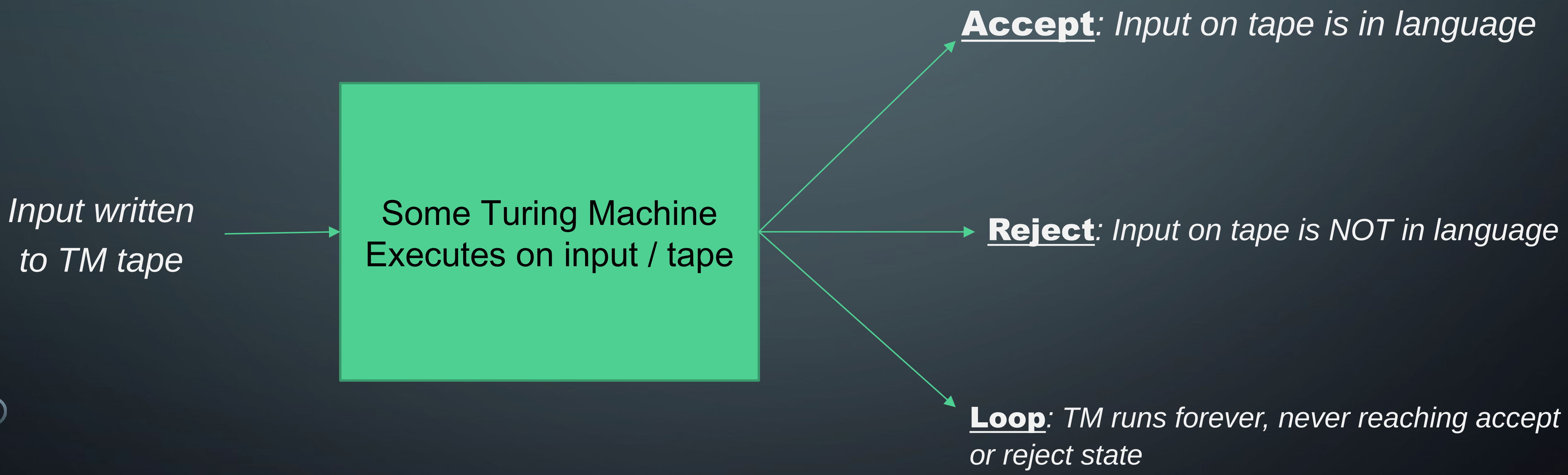


This machine is in state q7, and the contents of the tape / position of the head can be seen in the diagram.

A configuration of a TM can be represented succinctly as a string:

RECOGNIZING VS DECIDING

When a Turing Machine executes, there are three possible outcomes



Input written to TM tape

Some Turing Machine Executes on input / tape

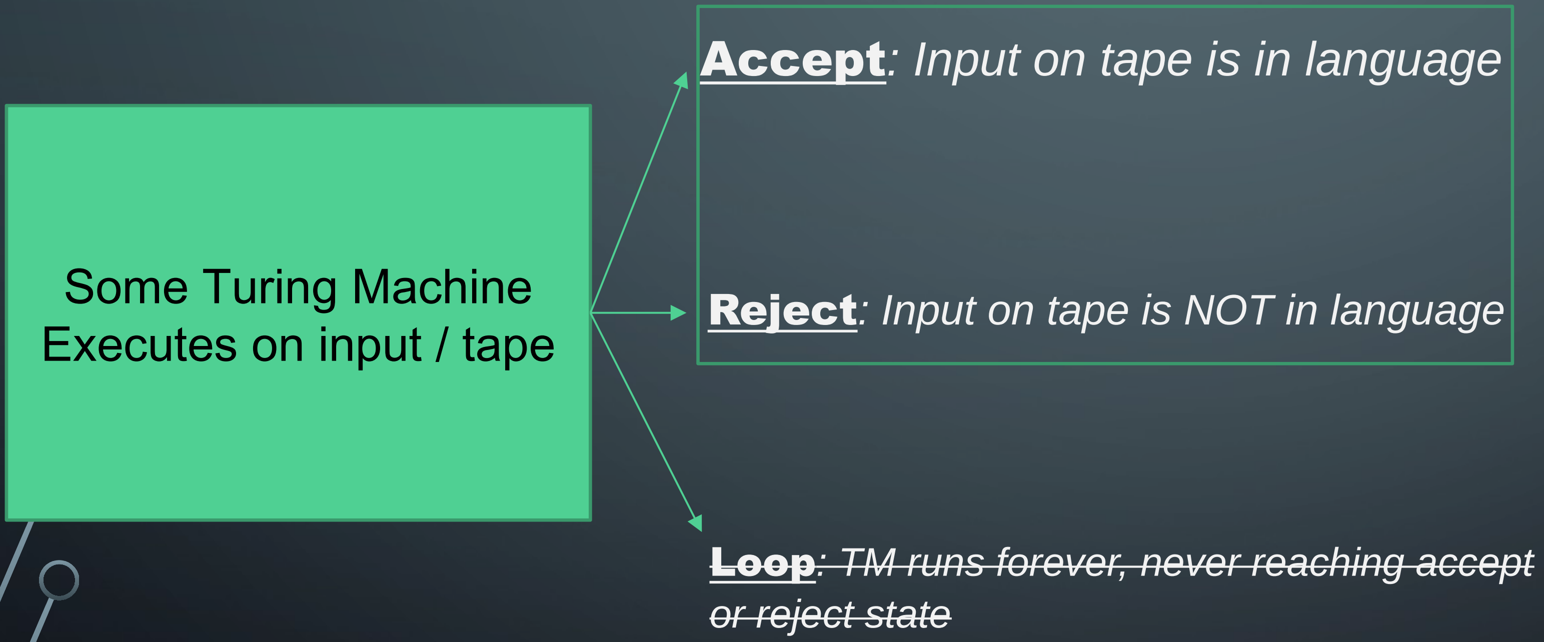
Accept: *Input on tape is in language*

Reject: *Input on tape is NOT in language*

Loop: *TM runs forever, never reaching accept or reject state*

RECOGNIZING VS DECIDING

A Turing Machine ***decides a language (is a decider)*** if it never loops and always correctly accepts or rejects strings for the given language.



Accept: *Input on tape is in language*

Reject: *Input on tape is NOT in language*

~~**Loop**: *TM runs forever, never reaching accept or reject state*~~

A TM ***decides the language*** if it always halts and outputs one of these two possibilities. The language of this TM is said to be a ***decidable language***.

RECOGNIZING VS DECIDING

A Turing Machine ***recognizes a language (is a recognizer)*** if it always accepts strings that are in the language, but might reject or might loop forever on strings that are NOT in the language.

Some Turing Machine Executes on input / tape

Accept: *Input on tape is in language*

A recognizer will always accept when the string IS in the language

Reject: *Input on tape is NOT in language*

Loop: *TM runs forever, never reaching accept or reject state*

However, when the input string is NOT in the language, a recognizer ***might reject*** or it ***might just loop forever***.

Languages that can be recognized are called ***Turing-Recognizable***

EXAMPLES: DESIGNING TURING MACHINES

PRACTICE 1: DESIGN A TM

Construct a Turing Machine that decides the following language:

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Construct a Turing Machine that decides the following language:

<u>*Overall Approach, on input w*</u>:

1. Sweep left to right across the tape, crossing off every other 0.
2. If in stage 1 the tape contained a single 0, accept
3. If in stage 1 the tape contained more than a single 0 and the number of 0s was odd, reject
4. Return the head to the left end of the tape
5. Go to stage 1

PRACTICE 1: DESIGN A TM

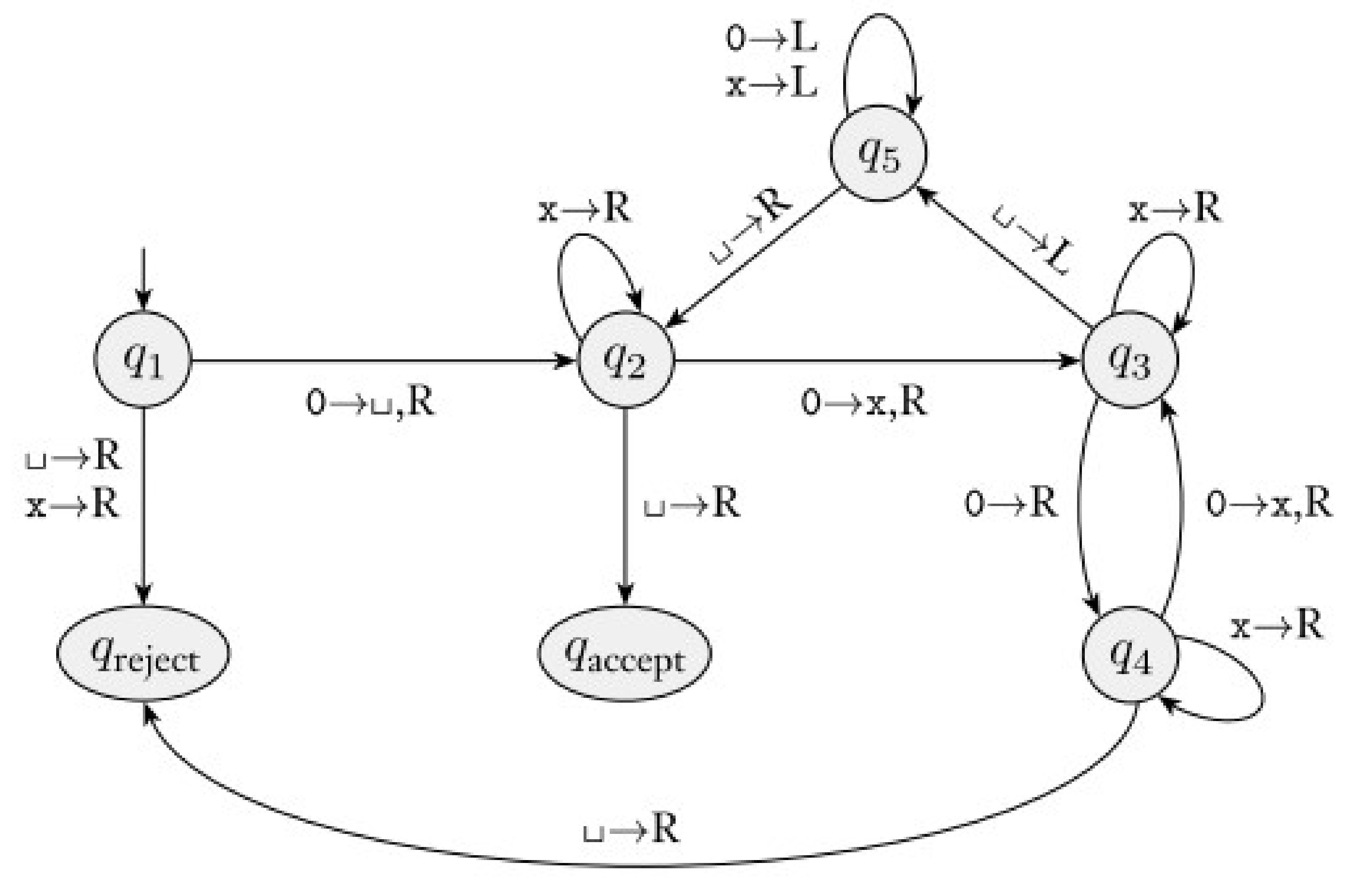
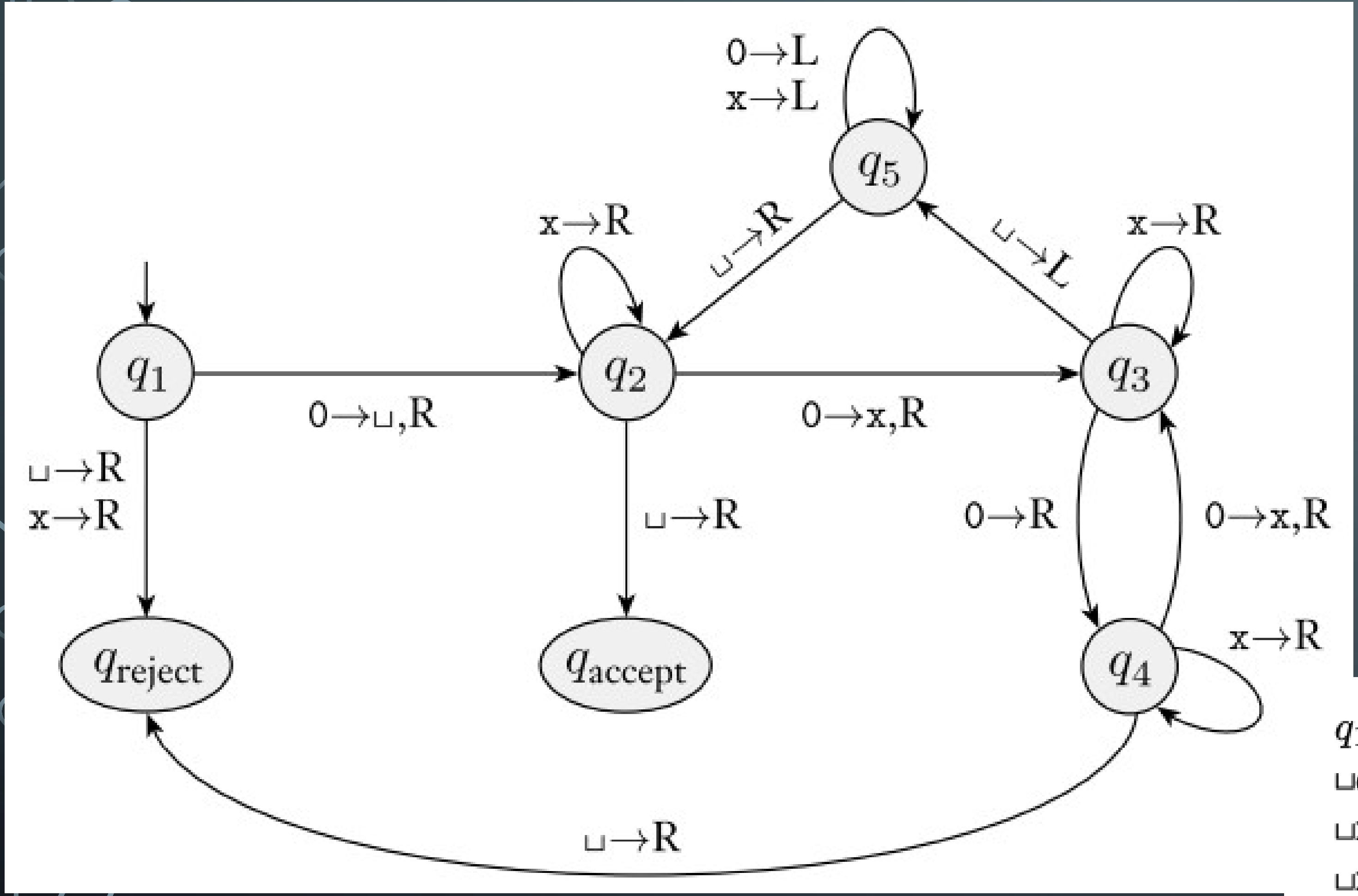


FIGURE 3.8
State diagram for Turing machine M_2

PRACTICE 1: DESIGN A TM



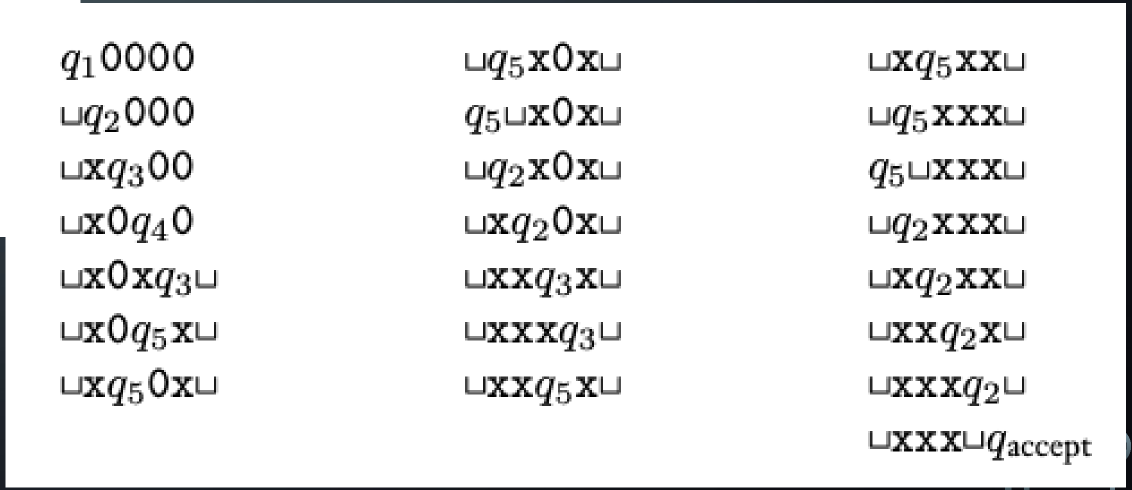
Q1 represents start state. We mark first 0 with blank

Q2 represents we have seen one 0

Q3 represents seen one 0 or even number of 0s

Q4 represents seen an odd number of 0s

Q5 represents moving head back to the front



$q_1 0000$
$\sqcup q_2 000$
$\sqcup x q_3 00$
$\sqcup x 0 q_4 0$
$\sqcup x 0 x q_3 \sqcup$
$\sqcup x 0 q_5 x \sqcup$
$\sqcup x q_5 0 x \sqcup$

$\sqcup q_5 x 0 x \sqcup$
$q_5 \sqcup x 0 x \sqcup$
$\sqcup q_2 x 0 x \sqcup$
$\sqcup x q_2 0 x \sqcup$
$\sqcup x x q_3 x \sqcup$
$\sqcup x x x q_3 \sqcup$
$\sqcup x x q_5 x \sqcup$

$\sqcup x q_5 x x \sqcup$
$\sqcup q_5 x x x \sqcup$
$q_5 \sqcup x x x \sqcup$
$\sqcup q_2 x x x \sqcup$
$\sqcup x q_2 x x \sqcup$
$\sqcup x x q_2 x \sqcup$
$\sqcup x x x q_2 \sqcup$
$\sqcup x x x \sqcup q_{accept}$

EXAMPLE 2!!

Construct a Turing Machine that decides the following language:

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Construct a Turing Machine that decides the following language:

Overall Idea:

On input String *w*:

1. Scan the tape to make sure the input is in the form , if not ***reject***.
2. Return the head to the left end of the tape.
3. Cross off the first a and scan to the right until first b is found. Shuttle between crossing off one b and scanning right to cross off one c until all the b's are gone. If all c's are crossed out, and b's remain, ***reject***.
4. Restore the crossed off b's and repeat stage 3 if there is another a. If all a's are crossed off, determine whether all c's are crossed of. If yes, ***accept***, otherwise ***reject***.

EXAMPLE 3!!

Element Distinctness Problem: Can we decide the language:

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Element Distinctness Problem: Can we decide the language:

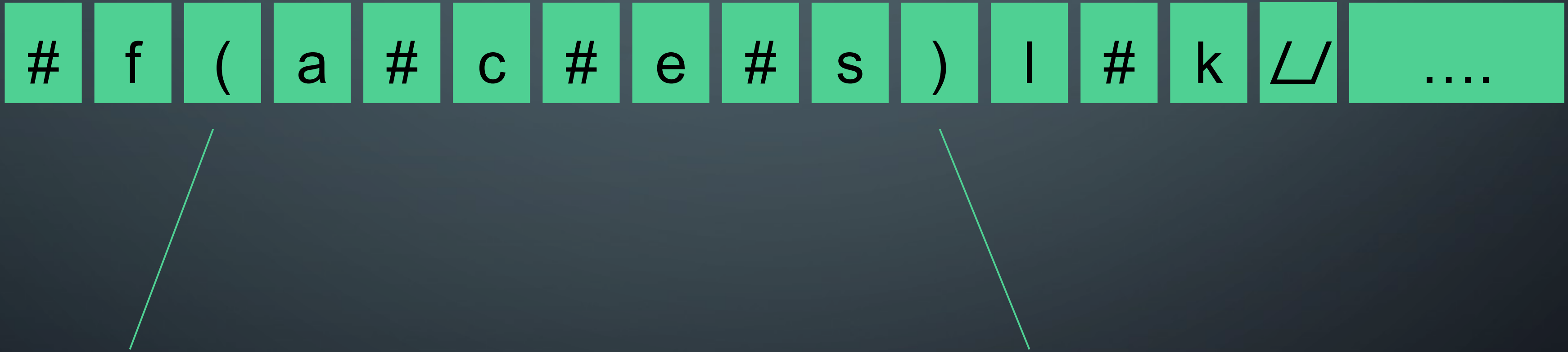
<u>Key ideas we will need for this one</u>:

1. The **tape can be marked to keep track of loops** (which characters are being compared with which characters). This is why the # symbols are useful.
2. How do we actually **compare the characters**? It is annoying but can be done with many states. How do you think that would work?

TRACKING LOOPS!!

How do we keep track of loops with a Turing Machine?

First, introduce new tape symbols that represent the loop beginning and end. Here we will use (and). Remember that goal is to compare every pair of characters.



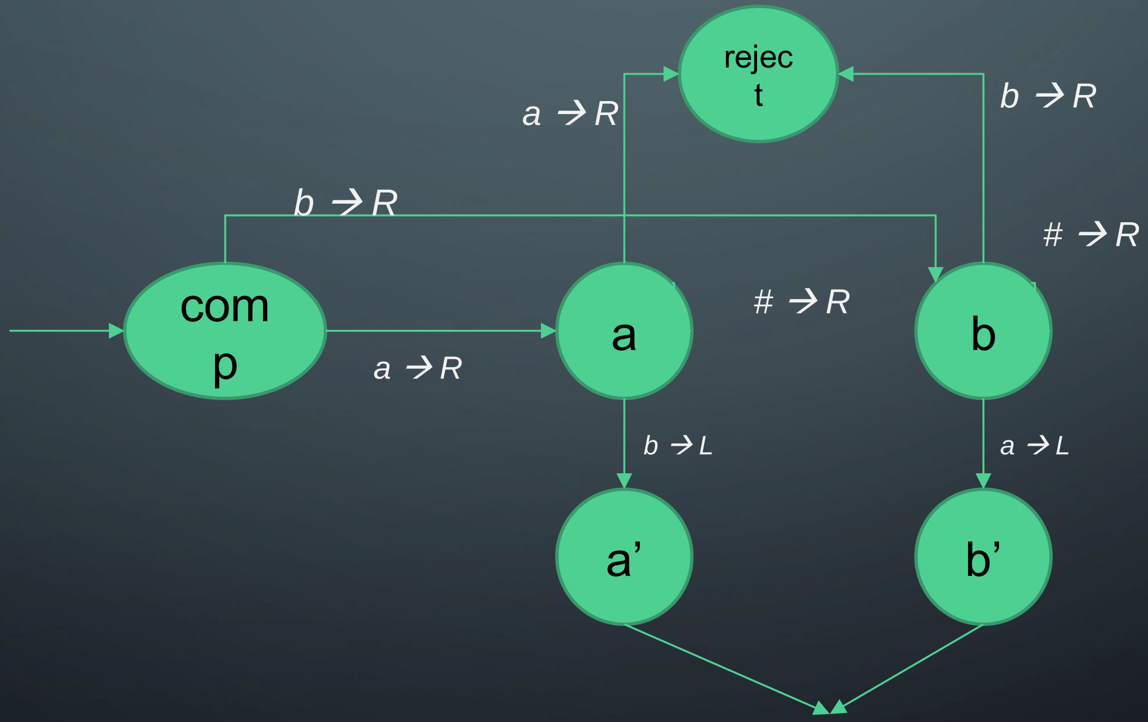
When starting an outer loop, mark the relevant part of the tape with open paren.

Mark closing paren to mark outer loop location if necessary.

COMPARING CHARACTERS

How hard is it to compare characters?

This machine will have to check characters for equivalence. How do we do this? Let's suppose that and tape is pointing at left side of



Back to some other computation

EXAMPLE 3!!

Element Distinctness Problem: Can we decide the language:

On input *w*:

1. Place a mark on top of the leftmost tape symbol. If that symbol was blank, ***<u>accept</u>***. If that symbol was #, continue with the next stage. Otherwise, ***<u>reject</u>***.
2. Scan right to the next # and place a second mark on top of it. If no # is encountered before a blank symbol, only was present, so ***<u>accept</u>***
3. By zig-zagging, compare the two strings to the inside of the marked #s. If equal, ***<u>reject</u>***.
4. Move the rightmost of the two marks to the next # symbol to the right. If no # symbol is encountered before a blank, move the leftmost mark to the next # to its right and the rightmost mark to the # after that. This time, if no # is available for the rightmost mark, all the strings have been compared, so ***<u>accept</u>***.
5. Loop: Go to step 3.

TURING MACHINE VARIANTS

MOTIVATING QUESTION

Can we have different features of TMs that increase / or don't the recognizing power of the traditional TM?

Some we will see:

Turing machines with multiple tapes

Non-deterministic Turing Machines

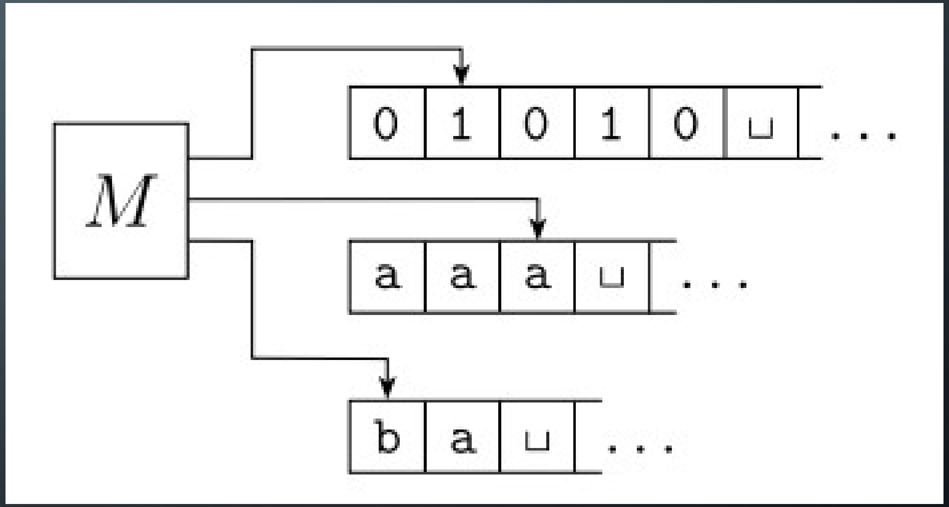
MULTITAPE TURING MACHINE

A ***Multitape Turing Machine*** is like an ordinary TM but it has several tapes instead of one. Each tape has an independent head that can be moved.

The transition function is updated to:

$$\delta : Q \times \Gamma^k \rightarrow Q \times \Gamma^k \times \{L, R, S\}^k$$

Here, k is the number of tapes. So each tape can be read at each step of computation, each head can write / move as well.



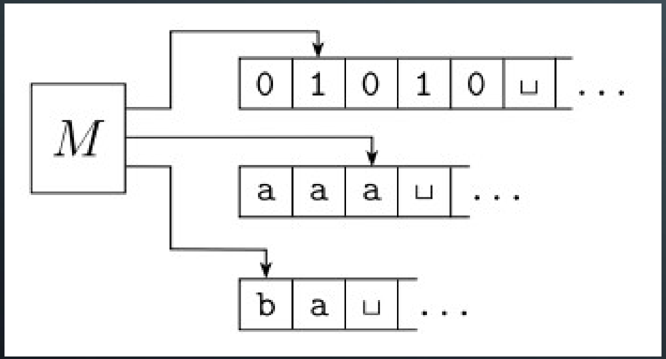
MULTITAPE TURING MACHINE

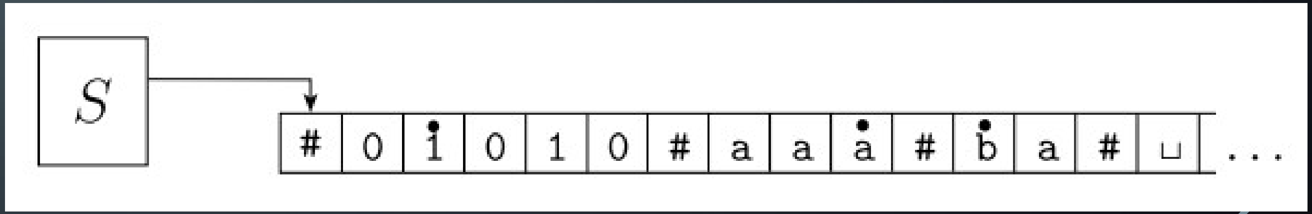
Theorem: Every Multitape Turing Machine has an equivalent single-tape Turing machine.

MULTITAPE TURING MACHINE

Theorem: Every Multitape Turing Machine has an equivalent single-tape Turing machine.

Goal: Given an arbitrary MTTM, convert it into an equivalent TM





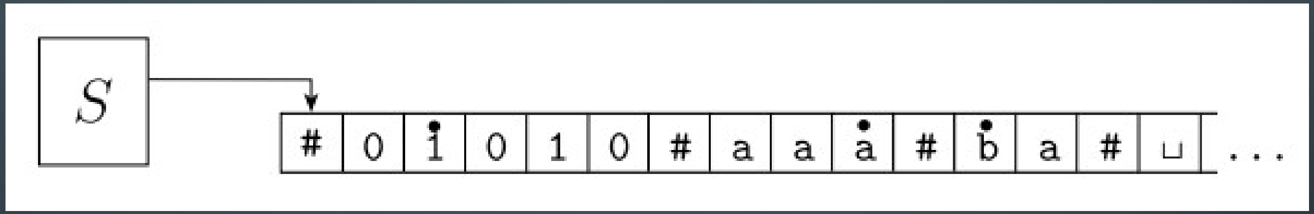
***Need to argue that S simulates M's computation exactly in all scenarios*

MULTITAPE TURING MACHINE

Two major issues we need to overcome for this proof:

Issue 1: How to simulate multiple tapes / heads with only one tape / head?

Issue 2: How to handle space constraints fitting k tapes on just 1?

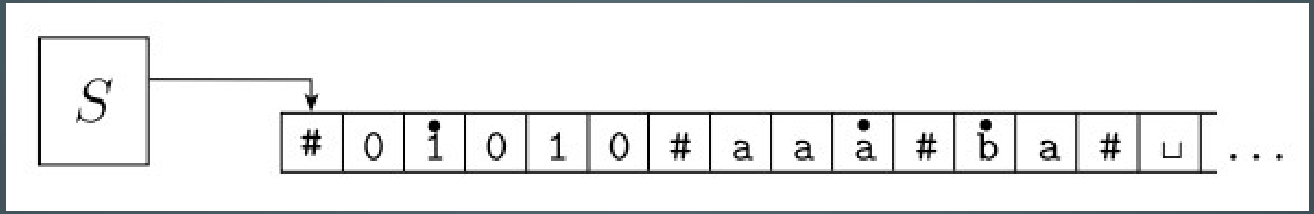


For issue 1, we make special versions of each tape symbol that represent a “virtual head” being positioned at that location.

For issue 2, we separate each tape by a special # symbol and create a subroutine that shifts the contents of everything to the right down when we need more space on any individual tape.

MULTITAPE TURING MACHINE

Summary of how to simulate M with S



S = on input :

1. *First S puts its tape into the format representing all k tapes of M, separated by the # symbol. The special "virtual head" symbol is used at the left most symbol on each of the k "sections" of the tape.*
2. *To simulate a single step, S first steps across the whole tape to see which symbols are under each of the virtual heads (reading each of the tapes), then S simulates writing and updating the "head" of the tape for each individual section according to Ms original transition function.*
3. *If at any point S runs out of space on any tape, we run a subroutine that shifts everything on the tape over by one, creating one more cell of space on this particular tape.*

MULTITAPE TURING MACHINE

Theorem: Every Multitape Turing Machine has an equivalent single-tape Turing machine.

Corrollary: A language is Turing-recognizable iff some Multitape Turing Machine recognizes it.

Proof:
Direction 1: A Turing recognizable language is recognized by some Turing Machine, and any TM is also a valid multitape TM.

Direction 2: If a multitape TM recognizes a language, then it can be converted into an equivalent single tape TM as described earlier.

NON-DETERMINISTIC TURING MACHINES (NTM)

A non-deterministic Turing Machine (NTM) can branch into many possibilities. Transition function has form:

The tape, state, etc. are all branched to an independent machine. No resources are shared across the branches.

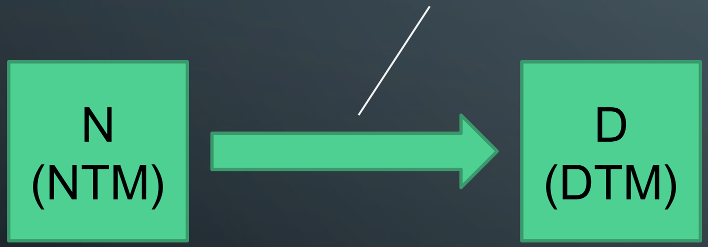
NTM VERSUS DTM

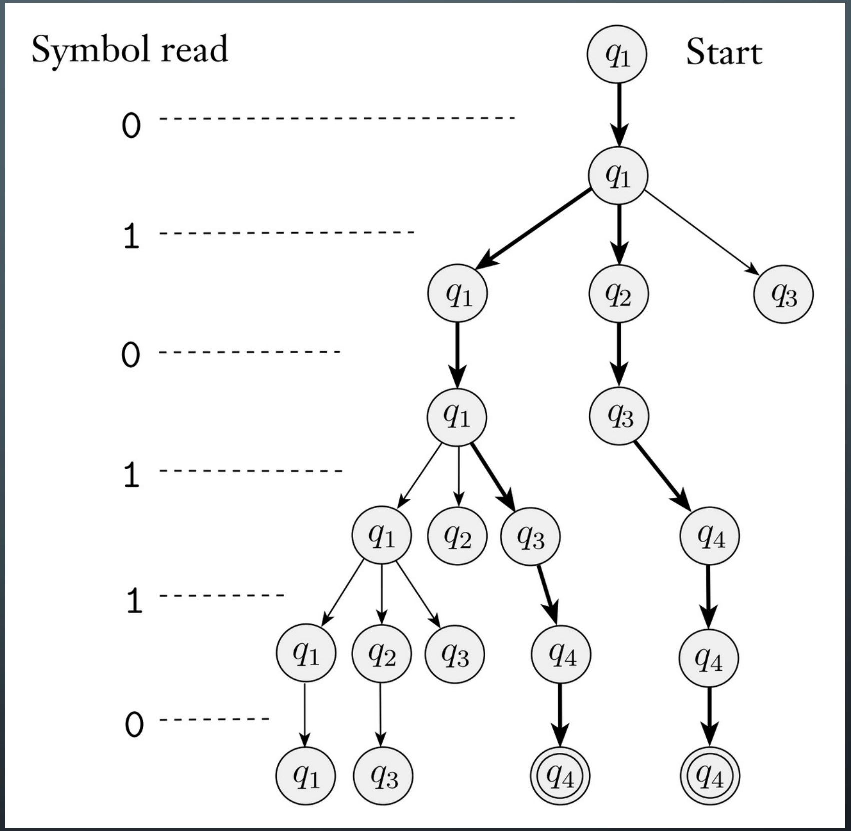
Theorem: Every NTM has an equivalent DTM

NTM VERSUS DTM

Theorem: Every NTM has an equivalent DTM

Using D, simulate N's execution exactly





__Idea__: Machine N has branching computations. Let's search this computation tree in a breadth-first manner.

Issues:

How to switch between branches?

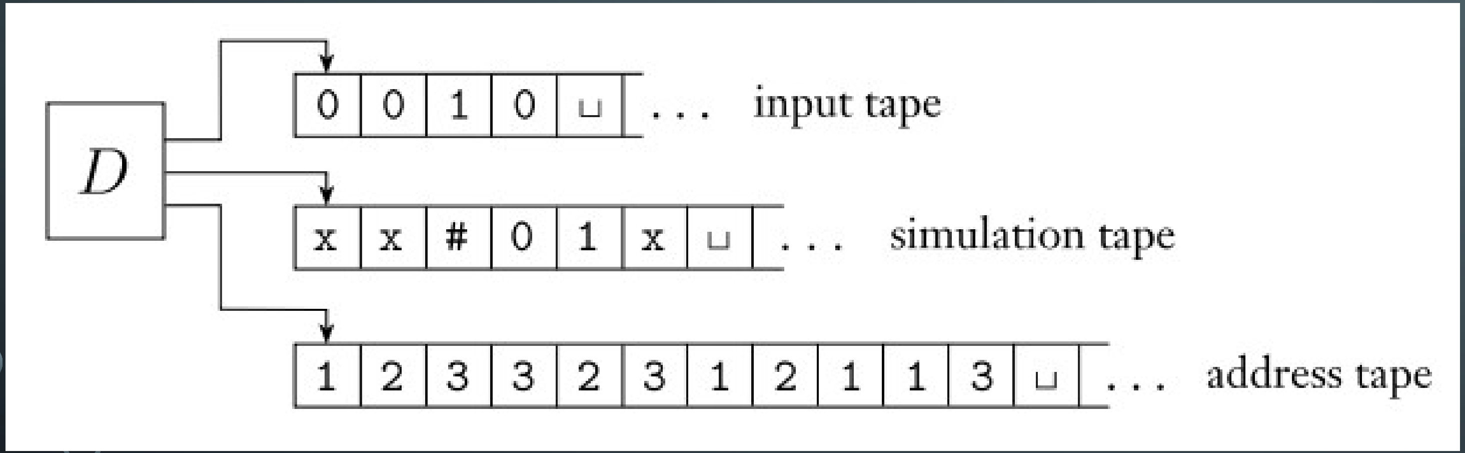
Why BFS instead of DFS?

Keeping track of tree location?

NTM VERSUS DTM

Theorem: Every NTM has an equivalent DTM

Proof Idea: Simulate the NTM with a DTM (multi-tape)


D
0 0 1 0 ␣ . . . input tape
x x # 0 1 x ␣ . . . simulation tape
1 2 3 3 2 3 1 2 1 1 3 ␣ . . . address tape

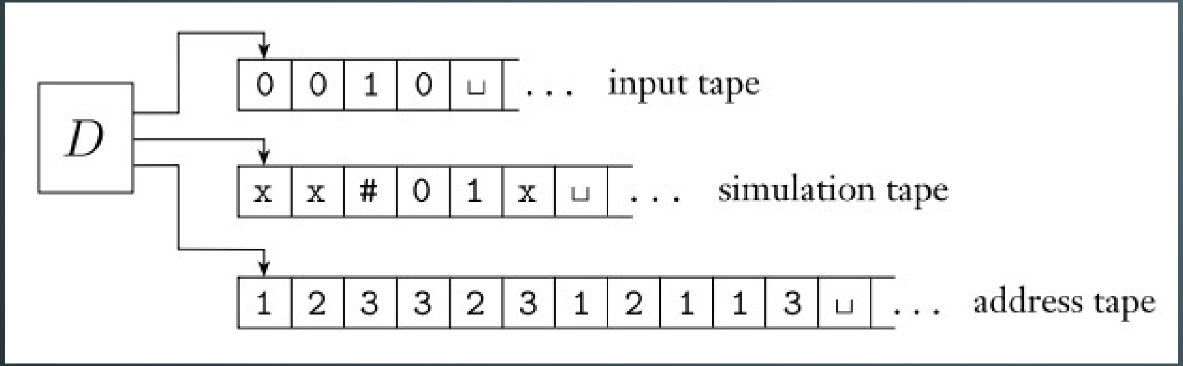
Tape 1 contains the input and will never be altered.

Tape 2 contains N’s tape at that particular branch of execution as we simulate

Tape 3 is the address tape, it tells us which branch of computation to simulate at each step (more about this on the next slide).

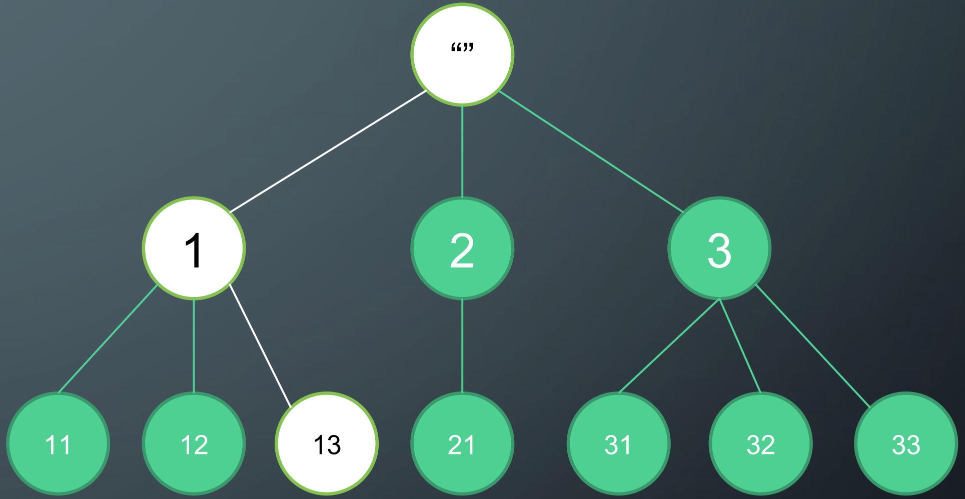
NTM VERSUS DTM

Theorem: Every NTM has an equivalent DTM



D
0 0 1 0 ␣ . . . input tape
x x # 0 1 x ␣ . . . simulation tape
1 2 3 3 2 3 1 2 1 1 3 ␣ . . . address tape

Let's look at Tape 3 for a moment



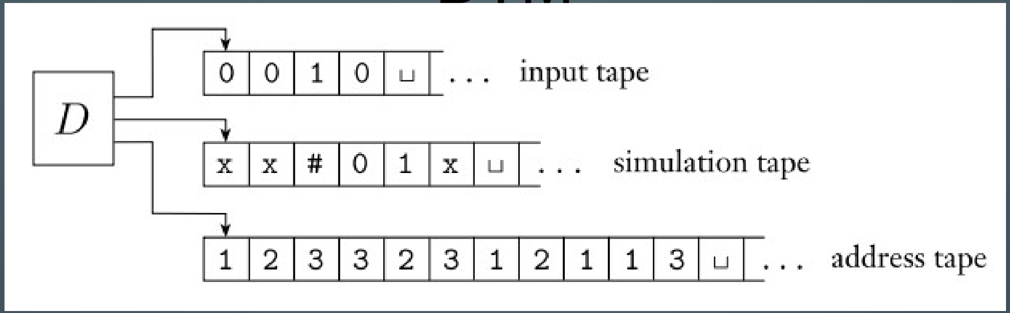
Tape 3 tells you which direction down tree to go each time you make a step in your computation. Each cell is at most b, where b is the max number of branches possible for that machine.

For example, if tape 3 contains "13", we will simulate the highlighted (white) path above and terminate (even if there is more computation possible).

Tape 3 will try "", 1, 2, 3, 11, 12, 13, 21, 31, 32, 33, ….

NTM VERSUS DTM

Theorem: Every NTM has an equivalent DTM



D
0 0 1 0 ⊔ . . . input tape
x x # 0 1 x ⊔ . . . simulation tape
1 2 3 3 2 3 1 2 1 1 3 ⊔ . . . address tape

Given Machine N, Machine D Proceeds As Follows:

1. *Tape 1 contains the input, tapes 2 and 3 are empty.*
2. *Copy tape 1 onto tape 2 to initialize the simulation.*
3. *Use tape 2 to simulate N on input. At every stage of computation, look at tape 3 to determine which non-deterministic branch to follow (which next state out of many choices to take). If we reach an accept state, then accept! If we run out of branches to follow on tape 3, or we reach a reject state, or this configuration is invalid (the choice on tape 3 is not a valid choice), then abort this branch and go to step 4.*
4. *Replace the string on tape 3 with the next string in the string ordering (e.g., “11” might become “12”). Simulate the next branch by going to step 2.*
5. *This machine rejects if all branches are enumerated and no accept state is found (note the machine could loop forever).*

NTM VERSUS DTM

Theorem: A language is decidable if and only if some non-deterministic Turing machine decides it (same holds for recognizing).

This follows from our previous proof. Any NTM can be simulated with a DTM so these machines are equivalent for deciding languages!

LAST CONCLUSION

Any computational model satisfying reasonable requirements to that of a TM is equivalent in power to a TM

Most notably:

1. Access to unlimited memory
2. The ability to perform a finite amount of work in a single step

THE DEFINITION OF AN ALGORITHM

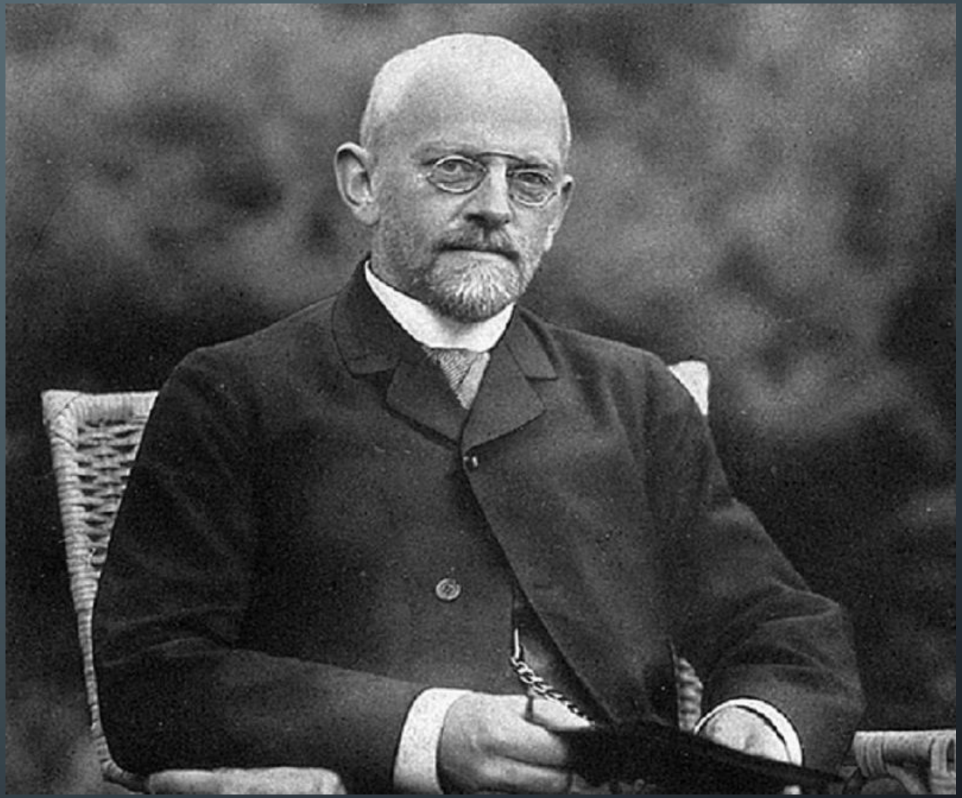
SHORT DISCUSSION

What is an ***<u>Algorithm</u>***? Does seeing the Turing Machine change your perspective on this at all?

***It was not until the 20th century that the notion of an algorithm was defined precisely.*

HILBERT'S PROBLEMS

David Hilbert: German Mathematician

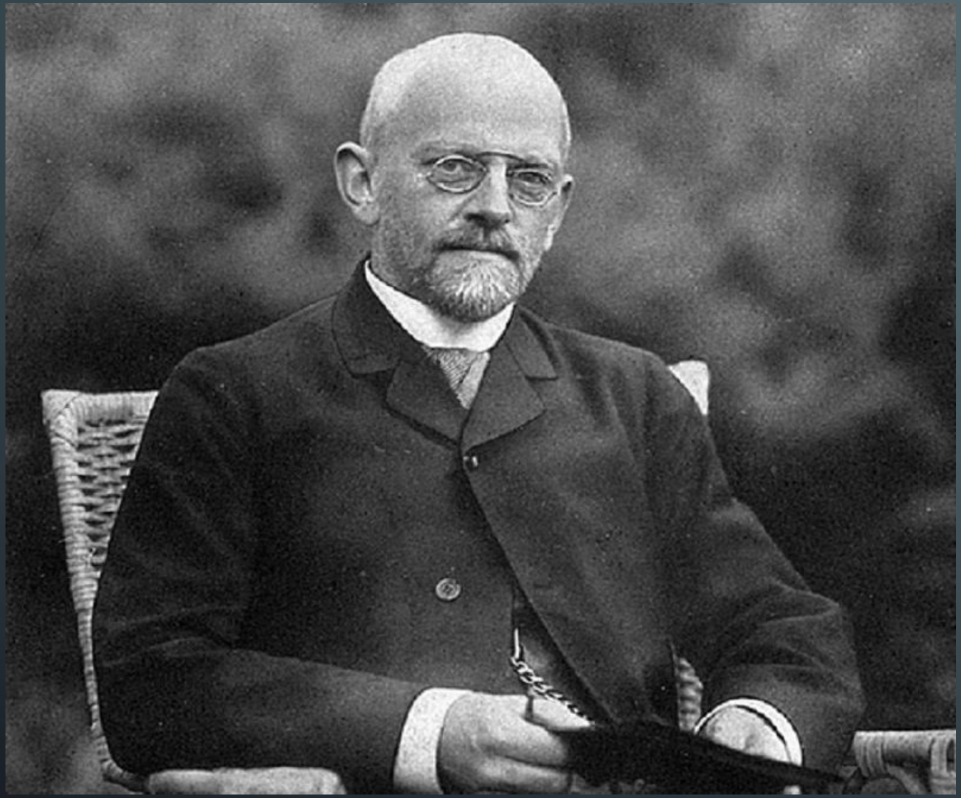


In 1900, Hilbert gave a now famous address at the International Congress of Mathematicians in Paris. He identified 23 mathematical problems and posed them as a challenge for the coming century. Hilbert's 10th problem concerned algorithms.

What was Hilbert's 10th problem*: Devise an algorithm that tests whether a polynomial has an integral root (he did not use the word "algorithm").*

HILBERT’S PROBLEMS

David Hilbert: German Mathematician



Hilbert used the word “devise”, implying that some algorithm / process must exist (he was wrong about that part).

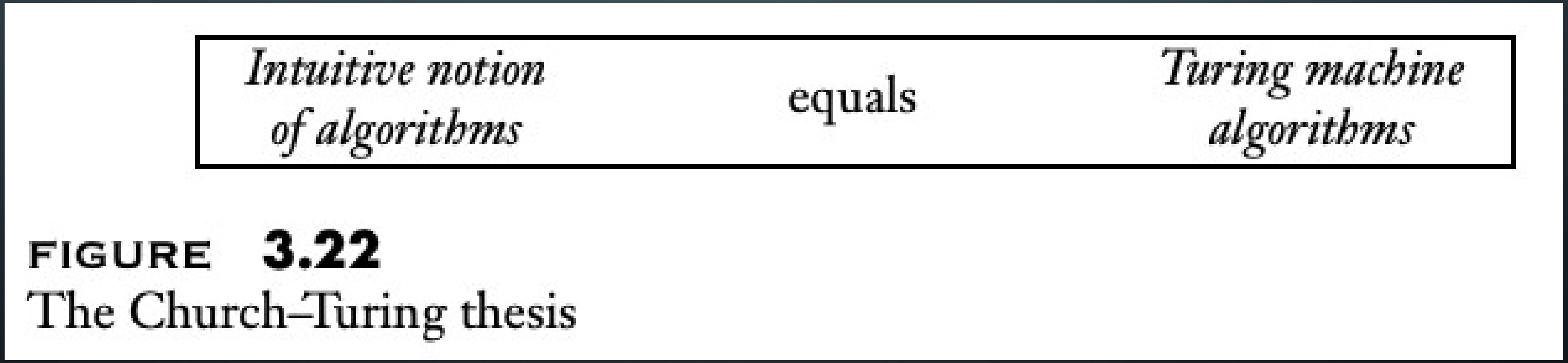
The 10th problem is unsolvable, but mathematicians in 1900 only had an intuitive understanding of algorithm.

The world needed a formal definition of an “algorithm” in order to prove that this particular problem was unsolvable…

THE CHURCH-TURING THESIS

The Church-Turing Thesis (1936): Lambda-Calculus (Alonzo Church) and Turing Machines (Alan Turing) provide the mechanism for formally defining an algorithm.

So…an algorithm is any process that can be programmed on a Turing Machine in order to recognize or decide some function (or language).



Intuitive notion of algorithms	equals	*Turing machine algorithms*

FIGURE 3.22
The Church–Turing thesis

FINDING POLYNOMIAL ROOTS IS RECOGNIZABLE

Problem: Can you recognize (not decide) the following language:

*Also recall that this language is **recognizable** if you will can find the root if it exists, but you can loop forever if it does not exist.*

Note that the polynomial is guaranteed to be over just one variable.

FINDING POLYNOMIAL ROOTS IS RECOGNIZABLE

Problem: Can you recognize (not decide) the following language:

Solution:

```
On input <p>:
    1. Evaluate p with x set successfully to the values 0, 1, -1, 2, -2, 3, -3, …
    2. Accept if at any point the polynomial evaluates to 0
```

Notice that this solution loops forever anytime the answer should be "reject". It is possible to turn this into a decider (see book for details)

FINDING POLYNOMIAL ROOTS IS RECOGNIZABLE

Problem: Can you recognize (not decide) the following language:

*This problem is **undecidable**, meaning there does not exist an algorithm (Turing Machine) that decides it. Now that we understand Turing Machines, we can begin to formulate how some problems cannot be decided.*

WHAT DID WE LEARN IN THIS DECK

1. Definition of Turing Machines, both deterministic and non-deterministic along with other variants
2. Simple design of algorithms using Turing Machines
3. Definitions of recognizable vs decidable languages
4. Practice with designing Turing Machines for simple problems.